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Large deviations and phase separation in the two-dimensional Ising model. *

C.E. Pfister
Departement of mathematics E.P.F.-L.
CH-1015 Lausanne Switzerland

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1 Introduction.

In 1967 Minlos and Sinai published a remarkable paper on the Ising model [M.S.1]. Many important ideas, which were later on developed in Statistical Mechanics were in germ in it. In their paper the phenomenon of phase separation or phase segregation is explained, at a mathematical level, on the basis of the first principles of Statistical Mechanics. In 1988 Dobrushin, Kotecky and Shlosman [D.K.S] announced new important results: the phenomenological theory of Wulff, which gives the shape of the spatial region occupied by one phase immersed in the other one, is derived within Statistical Mechanics.

This paper is based on a series of lectures delivered at Troisieme Cycle de la Physique en Suisse Romande in February 1991. The aim of these lectures was to expose part of the work of Minlos and Sinai by incorporating the main features of the recent developments of Dobrushin, Kotecky and Shlosman. The mathematical aspects of the problem were emphasized in the lectures and not the physical aspects, which are relevant, as the wetting phenomenon for example [F.P.2]. I tried to get the main results, but not in the sharpest form, in order to keep the analysis as simple as possible. In particular I chose to use a constraint ensemble where the magnetization does not have a fixed value. (See (1.8) and comments at the end of the introduction.) One lecture was devoted to an exposition of the method of the cluster expansion, which plays an important role in the analysis and which replaces the method of equations for correlation functions used by Minlos and Sinai [M.S.2] (see section 3). Only the two-dimensional Ising model at low temperature was treated in these lectures, since the results of Dobrushin, Kotecky and Shlosman are restricted to this case.

Let us summarize the main points of the theory of Gibbs states and large deviations of the magnetization for the two-dimensional Ising model. The free energy $p(h,\beta)$, and the Gibbs states depend on two parameters, the external magnetic field h and the inverse temperature β, since the coupling constant J of the interaction can be chosen equal to one without restricting the generality. (In this introduction the

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free energy is normalized as in Physics, by dividing the logarithm of the partition function by the inverse temperature. This normalization is not used in the rest of the paper.) The Gibbs states describe the equilibrium states in the thermodynamic limit. They are solutions of the D-L-R equations. For the two-dimensional Ising model all solutions of these equations are known. The set of solutions of the D-L-R equations is a convex set which has only one element for all nonzero values of h and for $h = 0$ and $\beta \leq \beta^c$, where β^c, is the critical inverse temperature. For $h = 0$ and $\beta > \beta^c$ there are exactly two extremal solutions of the D-L-R equations, denoted below by μ^+ and μ^-, all other solutions are $\mu = a\mu^+ + (1-a)\mu^-$, $0 \leq a \leq 1$. The Gibbs states μ^+ and μ^- describe the pure phases, and the measure $a\mu^+ + (1-a)\mu^-$ describes an equilibrium state which is a mixture of the two pure phases. There is a criterion for the unicity of the solution of the D-L-R equations, which is related to a smoothness property of the function $p(h,\beta)$. Let $\sigma(t)$ be the spin variable at $t \in \mathbb{Z}^2$. We suppose that the random variables $\sigma(t)$ are distributed according to the Gibbs measure $\mu^+(h,\beta)$ for some fixed values of the parameters h and β. The expectation value of $\sigma(t)$ is independent on t and is written $m^+(h,\beta)$. Similarly we define $m^-(h,\beta)$, which is the expectation value of $\sigma(t)$ with respect to the Gibbs measure $\mu^-(h,\beta)$. There is a unique solution of the D-L-R equations if and only if $m^+(h,\beta) = m^-(h,\beta)$ and this happens if and only if the thermodynamical function $p(h,\beta)$ is differentiable with respect to the magnetic field at (h,β). Indeed, this function is convex in h and the right derivative (left derivative) with respect to the magnetic field is equal to $m^+(h,\beta)$ $(m^-(h,\beta))$.

Let $\Lambda = \Lambda(L)$ be a finite subset of $\mathbb{Z}^2$, which we suppose to be a square. The cardinality of $\Lambda(L)$ is $|\Lambda| = L^2$. An important variable is $X(\Lambda)$,

$$X(\Lambda) = \frac{1}{|\Lambda|}\sum_{t\in\Lambda}\sigma(t) \tag{1.1}$$

which is the mean magnetization inside the region Λ. The extremal Gibbs states μ^+ and μ^- are ergodic measures with respect to the group of translations of the lattice $\mathbb{Z}^2$. We have

$$\lim_{L\to\infty}\frac{1}{|\Lambda(L)|}\sum_{t\in\Lambda(L)}\sigma(t) = m^+(h,\beta) \quad \mu^+-a.s. \tag{1.2}$$

A similar result holds for the measure μ^-. The study of the distribution of the random variable $X(\Lambda)$ is related to the large deviations of the magnetization inside Λ, i.e. to the estimation of $\mathrm{Prob}(\{X(\Lambda)\in A\})$ for some subset A. Let us suppose that the random variables $X(\Lambda)$ are distributed according to the measure $\mu^+(h,\beta)$. Then, these variables obey a large deviation principle with rate function $I(m|h,\beta)$ (see e.g. [E]). The rate function is equal to

$$I(m|h,\beta) = \beta(\sup_t(m\cdot t - p(h+t,\beta)) + p(h,\beta)) = \tag{1.3}$$
$$\beta(\sup_t(m\cdot t - p(t,\beta)) + p(h,\beta) - m\cdot h)$$

If A is an open set, then

$$-\inf_{m\in A} I(m|h,\beta) \leq \liminf_L \frac{1}{|\Lambda(L)|}\ln \mathrm{Prob}(\{X(\Lambda(L))\in A\}) \tag{1.4}$$

with the probability computed with the measure $\mu^+(h,\beta)$. If A is closed set, then

$$-\inf_{m\in A} I(m|h,\beta) \geq \limsup_L \frac{1}{|\Lambda(L)|}\ln \mathrm{Prob}(\{X(\Lambda(L))\in A\}) \tag{1.5}$$

The rate function is nonnegative and it is equal to the sum of the Legendre transform of p and the affine function of m, $p(h,\beta) - m\cdot h$. It is a convex function, and it is essentially equal to the thermodynamical function associated with a constraint ensemble with given specific magnetization m. On the other hand, p is associated with an unconstraint ensemble. The results (1.4) and (1.5) are independent on the choice of the Gibbs measure (for the same values of h and β). The phase transition region of the model in the (h,β)-plane corresponds to the region where several solutions of the D-L-R equations exist. It is also characterized by the non-differentiability of the function $p(h,\beta)$ with respect to the magnetic field h. This non-differentiability of p implies, via the Legendre transform, the existence of a non trivial affine part in the graph of the rate function. Let us choose $h = 0$ and $\beta > \beta^c$. Then $p(h,\beta)$ is non differentiable at $h = 0$ and the left-derivative of p at $h = 0$ is equal to $m^-(0,\beta) \equiv -m^*(\beta) < 0$ and the right-derivative is equal to $m^*(\beta) > 0$. In this case the graph of the rate function has an horizontal part : $I(m|0,\beta) = 0$ for all $-m^*(\beta) \leq m \leq m^*(\beta)$. Consequently the statements (1.4) and (1.5) become trivial for any set A included in the interval $[-m^*, +m^*]$.

The summary above shows that the theory of Gibbs measures in the thermodynamical limit is unadequate for describing the coexistence of phases in the sense that any Gibbs measure is of the form

$$\mu = a\mu^+ + (1-a)\mu^- \quad, 0 \leq a \leq 1 \tag{1.6}$$

The Gibbs measures are related to the equilibrium states of an unconstraint ensemble (the value of the magnetization is not given a priori). A measure like μ describes a mixture of two phases in a statistical sense only, the coefficient a being the fraction of the pure phase which is associated with the measure μ^+. In order to study the coexistence of the phases we work with a system defined in a finite box and we use a constraint ensemble with given magnetization. The physical situation which is described in these lectures is the coexistence of the two phases when one of the phase is attracted by the boundary of the box and the other one is repulsed. We choose the + boundary condition. The above results on the large deviations of the magnetization in the interval $[-m^*, +m^*]$ are trivial, because large deviations in the presence of several phases is a less rare event than in the region of a single phase. Indeed, the probability that

$$\sum_{t\in\Lambda}\sigma(t) = m|\Lambda| \quad, -m^* < m < +m^* \tag{1.7}$$

is now $\exp(O(|\Lambda|^{1/2}))$ and not anymore $\exp(O(|\Lambda|))$ (see [S]). One main purpose of these lectures is to show the relation between this behaviour of the large deviations of the magnetization when there is coexistence of phases and the phenomenon of phase separation. The two themes are intimately related, and in a mean field version of the model it is easy to see that we have no phase separation, an equilibrium state with given magnetization is always an homogeneous state, and the above theory of large deviations is not trivial. Notice that the rate function is not convex.

Let us consider the model in a finite square box $\Lambda(L)$. We always choose the + boundary condition for the box. The parameters of the models are chosen so that there is no magnetic field and the inverse temperature β is large enough. Let σ be a configuration. We define

$$A(m) = A(m; c, c_0) = \{\sigma : |\sum_{t\in\Lambda} \sigma(t) - m|\Lambda|| \leq c_0|\Lambda| \cdot L^{-c}\} \tag{1.8}$$

where $-m^*(\beta) < m < m^*(\beta)$ (with m not too small) and $0 < c < 1/2$. All configurations σ of A have a total magnetization of order $O(|\Lambda|) \cdot m$. We define a constraint model by considering only configurations in A. Therefore, for finite $\Lambda(L)$, the equilibrium state of the constraint model is described by the conditional measure $\mu_\Lambda^+(\,\cdot\,|A)$ where μ_Λ^+ is the Gibbs measure in Λ with + boundary condition. Our purpose is to find a set of typical configurations for $\mu_{\Lambda(L)}^+(\,\cdot\,|A)$ for large values of L. The main result, which were proven by Minlos and Sinai, is that there exists a set of typical configurations which can be roughly described as follows. We can partition this set into subsets, each of these subsets being characterized by a spatial region $\mathcal{R}$, so that inside $\mathcal{R}$ and not too close to the boundary of the region $\mathcal{R}$ we have typical configurations of the measure μ^- (restricted to $\mathcal{R}$) and in $\Lambda\backslash\mathcal{R}$, and not too close to the boundary of $\Lambda\backslash\mathcal{R}$, we have typical configurations of the measure μ^+ (restricted to the region $\Lambda\backslash\mathcal{R}$). The volume of the regions $\mathcal{R}$ is

$$\text{vol}(\mathcal{R}) = V(m) + O(|\Lambda|^{3/4}) , \; V(m) \equiv \alpha(m)|\Lambda| \equiv \frac{m^* - m}{2m^*}|\Lambda| \tag{1.9}$$

Dobrushin, Kotecky and Shlosman give a better estimate of the volume of the regions $\mathcal{R}$, and show that the shape of $\mathcal{R}$ is given by the Wulff's variational principle. Before reviewing this principle let us state the results on the large deviations of the magnetization, which are a direct consequence of these phase separation results:

$$\lim_{L\to\infty} -\frac{1}{L} \ln \text{Prob}_{\Lambda(L)}^+(A(m; c, c_O)) = 2(|W_\tau| \cdot \alpha(m))^{1/2} \tag{1.10}$$

where $|W_\tau|$ is a constant, which depends on β and which is equal to the volume of the Wulff crystal (see below), and $\alpha(m)$ is defined in (1.9). We have

$$\lim_{\beta\to\infty} \frac{|W_\tau|}{16\beta^2} = 1 \tag{1.11}$$

It is important to notice that in this case the result (1.10) *depends* on the choice of the conditional Gibbs state and also on the shape of the box Λ. The result e.g. for periodic boundary conditions is different (see [D.K.S] and [Sh]). This is in fact very natural since we have here a surface phenomenon and we cannot expect that the boundary of Λ, or the boundary condition for this set, do not play a dominant role. The phenomenon of wetting is of course important although we do not discuss this topic. We only mention that the result (1.10) reflects the fact that there is a repulsion of the negatively magnetized phase by the boundary of Λ, as a consequence

of our choice of the boundary condition ([F.P.2]).

The Wulff's theory predicts the shape of a crystal in equilibrium with its vapor on the basis of a simple variational argument. This is a phenomenological macroscopic theory. Let us consider only the two-dimensional version. We suppose that a possible shape of the crystal is described by a simple closed curve c and that the crystal is inside c. Let n be a unit vector of $\mathbf{R}^2$ and $\tau(n)$ the surface tension (per unit length) of an interface perpendicular to n and separating the crystal and the vapor. The total surface free energy associated with the shape c is,

$$\int_c \tau(n(s))ds \tag{1.12}$$

where $n(s)$ is the unit normal vector exterior to c at $c(s)$. The shape of the crystal of volume V is given by the solution which minimizes (1.12) over "all sufficiently regular" simple curves V which are the boundaries of regions of volume V. If we have two fluid phases in coexistence, then the same argument applies. In particular if the surface tension is isotropic (1.12) is proportional to the length of the curve c and the equilibrium shape is a disc, as a consequence of the classical isoperimetric inequality. The variational problem to solve is a generalization of the classical isoperimetric problem. When τ is positive, which is the case here, we can intrepret (1.12) as a new length of c (τ-length). Let $\tau : \mathbf{R}^2 \to \mathbf{R}$ be a positively homogeneous function of degree one. (We can always extend in this way the definition of the surface tension to $\mathbf{R}^2$.) We define the Wulff crystal

$$\begin{aligned} W_\tau &= \{x^* \in \mathbf{R}^2 : \langle x|x^*\rangle \leq \tau(x) , \text{ for every } x \in \mathbf{R}^2\} \\ &= \{x^* \in \mathbf{R}^2 : \tau^*(x^*) \leq 0\} \end{aligned} \tag{1.13}$$

The Lebesgue measure of the set W_τ is $|W_\tau|$. For example when τ is the Euclidean norm, then $|W_\tau| = \pi$, and when $\tau(x) = |x_1| + |x_2|$ then W_τ is a square of volume 4. In (1.13) $\langle\,\cdot\,|\,\cdot\,\rangle$ is the Euclidean scalar product in $\mathbf{R}^2$, and τ^* is the Legendre transform of τ. Let $s \in [0,1] \mapsto c(s) = (c_1(s), c_2(s)) \in R^2$ be a parametrized closed curve (which is sufficiently regular). We define the Wulff functional by

$$\tau(c) := \int_0^1 \tau(c_2'(s), -c_1'(s))ds \tag{1.14}$$

and

$$\text{vol}(c) := \frac{1}{2}\int_0^1 (c_2'(s)c_1(s) - c_1'(s)c_2(s))ds \tag{1.15}$$

Notice that vol(c) is the Lebesgue measure of the set enclosed by c, when c is a simple closed curve. We have the following theorem, which generalized the classical isoperimetric inequality, and which gives the solution of the variational problem:

$$(\tau(c))^2 \geq 4|W_\tau| \cdot \text{vol}(c) \tag{1.16}$$

Equality holds if and only if c is the boundary of a region which is obtained by a dilatation and translation of the Wulff crystal. Inequality (1.16) has been proven several times under various conditions. Although inequality (1.16) plays an important role in the analysis we do not prove it here. There is a d-dimensional version of it. For this version there is a proof based on Brunn-Minkowski inequality (see in particular [D], [T]). Recently new proofs were published, see [F], [D.P]. A curve which realizes almost the minimum is almost a Wulff crystal of volume vol(c) (see e.g. [D.P]). This stability of the variational problem is best expressed by the generalized Bonnesen's inequalities (see [D.K.S]). We use these inequalities only at the very end of the analysis.

A precise formulation of the results on the phase separation is given in theorems 9.2, 9.3, 9.4 and in the conclusion of section 9. Let us outline the content of the next sections. The typical configurations of the conditional probability $\mu_\Lambda^+(\,\cdot\,|A)$ are described in terms of large contours and small contours. One needs an estimate of the large deviations of the magnetization computed with the probability which is obtained by conditioning the Gibbs measure μ_Λ^+ with respect to the event that there are only small contours. To simplify the analysis the constraint ensemble is defined by the event A, which specifies the magnetization up to a term of order $O(|\Lambda|\cdot L^{-c})$. Therefore it is sufficient to prove Bernstein's inequality for these large deviations. Such an inequality is derived in section 5, and follows easily from the results of section 4. (From the results of section 4 one can get stronger results, and prove local limit theorems analogous to those obtained by Richter for independent random variables [R].) There are two main estimates in the analysis of the typical configurations of the measure $\mu_\Lambda^+(\,\cdot\,|A)$. One of them is a lower bound for the probability of the event A computed with the Gibbs measure μ_Λ^+. This estimate is done in section 7. The second estimate is an upper bound for events described in terms of geometrical objects called droplets. The droplets are defined in section 8, and the estimate is established in the same section. It is essential that both estimates are expressed in terms of surface tension, and for that purpose we introduce an intermediate scale in the analysis, following an idea of Dobrushin, Kotecky and Shlosman. The method of proof for these estimates differs from the one used by Dobrushin, Kotecky and Shlosman. There is a convenient way of studying the surface tension, which is suggested by duality. It is known that the two-dimensional Ising model is self-dual, and that the surface tension is equal to the mass-gap of the two-point function of the same model at the dual inverse temperature. Duality and surface tension are the subjects of section 6, which also contains two basic simple estimates, which are important for sections 7 and 8. One of these estimates expresses the fact that a complicated large contour has a small probability and it is proved by the cluster expansion. The second estimate is based on monotonicity properties of the expectation value of the spin variables with respect to the size of the system. This second estimate greatly simplifies the analysis. Moreover such an approach is also possible for higher dimensions. Correlation inequalities are used for proving the lower bound of section 7 by mimicking the method of reflection of the theory of random walks. Here we use monotonicity properties of the two-point function with respect to the position of the spins. What is really needed to know about the surface tension corresponds to prove in the dual model that the two-point function has an Ornstein-Zernicke behaviour. Section 9 contains the main theorems

and the conclusions of the analysis. A lemma due to Minlos and Sinai plays an important role, when we prove the separation of phases. Section 2 is devoted to some basic definitions and notations. The correlation inequalities which we use are quoted there. The method of the cluster expansion is explained in section 3.

Remarks.
1) In the definition of the constraint ensemble we allow some fluctuations of the magnetization. As a consequence in the set of typical configurations we always have some contours of intermediate size (which are still small contours for our definition.) To study such contours, one must investigate the intermediate fluctuations of the magnetization. This important subject is treated in [D.K.S].
2) In [D.K.S] the authors use a definition of contours, which is very particular. This is not the case in these lectures, and our approach is better for generalizations. However the geometry of the (large) contours is more complicated in our case (see section 8). We could avoid these complications by using the definition of contours of [D.K.S]. This brings a non trivial simplification at the expense of generality. This simplified approach is discussed in section 10.
3) The main steps of the analysis are summarized below.

- Lemma 6.3 which gives the relation between the surface tension and the mass-gap of the two-point function of the dual model.
- Theorem 7.1 which gives the lower bound on the probability of the set $A(m;c,c_0)$.
- Theorem 8.2 which gives an estimate of the total volume (and the total length) of the large contours.
- The definition of the droplets and lemma 8.8 which gives an upper bound on the probability of a family of droplets. This lemma is proved by using the basic estimate of lemma 6.7.
- Theorem 8.4 which describes a typical set of configurations in terms of droplets.

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2 Ising model, notations.

We set up the main notations in section 2.1 and recall some basic properties of the model in 2.2. Finally we state in 2.3 the correlation inequalities which we use later on.

2.1 Notations.

2.1.1 The lattice.

The model is defined on $\mathbb{Z}^2$ or on some bounded part of $\mathbb{Z}^2$,

$$\mathbb{Z}^2 = \{t = (t(1), t(2)) : t(i) \in \mathbb{Z}, i = 1, 2\} \tag{2.1}$$

Another lattice, the *dual lattice* is important. In our case the dual lattice is $\mathbb{Z}^2_*$,

$$\mathbb{Z}^2_* = \{t = (t(1), t(2)) : t(i) + 1/2 \in \mathbb{Z}, i = 1, 2\} \tag{2.2}$$

We also think of the lattice in a more geometrical way, as a cell-complex. The lattice is the set of all elements of $\mathbb{Z}^2$, called *sites* (0-dim. cells), all *edges* e, $e = [t, t']$, which are horizontal and vertical segments of $\mathbb{R}^2$ with endpoints $t \in \mathbb{Z}^2$, $t' \in \mathbb{Z}^2$ and $|t(1) - t'(1)| + |t(2) - t'(2)| = 1$ (1-dim. cells), and all *plaquettes* p , which are the 2-dim. squares of unit area of $\mathbb{R}^2$ with corners belonging to $\mathbb{Z}^2$. When we consider the lattice with this structure we denote it by $\mathbb{L}$. Similarly we introduce $\mathbb{L}^*$, the dual cell -complex. We have the important geometrical relations :

- each site t of $\mathbb{L}$ is the center of a unique plaquette p^* of $\mathbb{L}^*$
- each edge e of $\mathbb{L}$ is crossed by a unique edge e^* of $\mathbb{L}^*$
- each plaquette p of $\mathbb{L}$ has a unique site t^* of $\mathbb{L}^*$ as center.

The *boundary of an edge* $e = [t, t']$ is by definition $\delta e = \{t, t'\}$. We extend the notion of boundary for subsets γ of edges. By definition $\delta\gamma$, the *boundary of* γ, is the set of sites which belong to an <u>odd</u> number of edges of γ. The *boundary of a plaquette* p is the set δp formed by the four edges of its boundary (as set of $\mathbb{R}^2$). The sites have no boundary.

The *cardinality* of a set $\Lambda \subset \mathbb{Z}^2$ is denoted by $|\Lambda|$. We use two distances. The distance d_1,

$$d_1(t, t') = \sum_{i=1}^{2} |t(i) - t'(i)| \tag{2.3}$$

and the Euclidean distance

$$d_2(t, t') = (\sum_{i=1}^{2} |t(i) - t'(i)|^2)^{1/2} \tag{2.4}$$

As usual the distance of a point t to a set A is

$$d_i(t, A) = \inf_{t' \in A} d_i(t, t') \quad i = 1, 2. \tag{2.5}$$

Let Λ be a bounded set of $\mathbb{Z}^2$. We also use the notation Λ for the following subset of $\mathbb{L}$: all sites of Λ are the elements t of Λ (as subset of $\mathbb{Z}^2$); all edges of Λ are the edges of $\mathbb{L}$, $e = [t, t']$ with t, t' sites of Λ; all plaquettes of Λ are all plaquettes of $\mathbb{L}$, p , such that $\delta p = \{e_1, e_2, e_3, e_4\}$ with all e_i edges of Λ. We write $\Lambda \subset \mathbb{L}$. With each $\Lambda \subset \mathbb{L}$ we associate a dual subset Λ^* of $\mathbb{L}^*$: all plaquettes of Λ^* are the plaquettes $p^*(t)$ of $\mathbb{L}^*$ whose centers $t \in \Lambda$; all edges of Λ^* are all edges of the boundaries of these plaquettes; all sites of Λ^* are all sites which are in the boundaries of the edges of Λ^*. A *path* on $\mathbb{Z}^2$ is an ordered sequence of sites and edges, $t_0, e_0, t_1, e_1, \ldots, t_n$ with $\delta e_i = \{t_i, t_{i-1}\}$, all i. The site t_0 is the *initial point* of the path and t_n is the *final point*. The path is *self-avoiding* if $t_i \neq t_j$ for all $i \neq j$. It is *closed* if $t_0 = t_n$. A subset $\Lambda \subset \mathbb{Z}^2$ is *connected* if for any pair of points $t, t' \in \Lambda$, there is a path with initial point t and final point t' which contains only sites of Λ. A subset $\Lambda \subset \mathbb{Z}^2$ is *simply connected* if the set of $\mathbb{R}^2$ which is the union of all plaquettes $p^*(t), t \in \Lambda$, is a simply connected set of $\mathbb{R}^2$. A subset Γ of edges of $\mathbb{L}$ is *connected* if the set of $\mathbb{R}^2$ which is the union of the edges of Γ is connected in $\mathbb{R}^2$. Finally, for any finite set $\Lambda \subset \mathbb{Z}^2$ we set

$$\overline{\Lambda} = \{t \in \mathbb{Z}^2 : \max_{i=1,2} |t(i) - t'(i)| \leq 1, \text{all } t' \in \Lambda\} \tag{2.6}$$

2.1.2 The configurations.

A configuration σ of the model is an element of the product space

$$\mathsf{X} = \{-1, 1\}^{\mathbb{Z}^2} \tag{2.7}$$

When the model is defined on Λ the set of configurations is

$$\mathsf{X}(\Lambda) = \{-1, 1\}^{\Lambda} \tag{2.8}$$

An element of this set is usualy denoted by σ but sometimes we write σ_Λ when we want to specify the set Λ. There is a natural action of $\mathbb{Z}^2$ on the set X as group of translations : to each $t \in \mathbb{Z}^2$, T_t is a map $\mathsf{X} \longrightarrow \mathsf{X}$,

$$(T_t\sigma)(t') := \sigma(t - t') \tag{2.9}$$

where $\sigma(t)$ is the value of the configuration at t. For each subset $\Lambda \subset \mathbb{Z}^2$ we introduce $\mathsf{F}(\Lambda)$ as the σ-algebra of X generated by the cylinder sets with bases in Λ. We write F for $\mathsf{F}(\mathbb{Z}^2)$. By definition we can decide whether a configuration σ belongs to some cylinder set A with base Λ if and only if we know all its values $\sigma(t), t \in \Lambda$.

Let Λ be a finite subset of $\mathbb{Z}^2$. We say that we have specified a *boundary condition for* Λ when we have chosen one particular configuration $\sigma' \in \mathsf{X}$. When a set Λ is given with a boundary condition (b.c) then we can extend uniquely any configuration σ_Λ of $\mathsf{X}(\Lambda)$ to a configuration $\sigma \in \mathsf{X}$,

$$\begin{aligned} \sigma(t) &:= \sigma_\Lambda(t) \quad if\ t \in \Lambda \\ \sigma(t) &:= \sigma'(t) \quad if\ t \notin \Lambda \end{aligned} \tag{2.10}$$

Two boundary conditions are fundamental : the + *boundary condition* (+ b.c.) and the − *boundary condition* (− b.c). The + b.c. (resp. − b.c.) corresponds to the choice of $\sigma(t) \equiv 1$ (resp $\sigma(t) \equiv -1$). Let Λ be given with + b.c. All configurations σ which are compatible with this b.c. (i.e. $\sigma(t) = 1,\ t \notin \Lambda$) can be described geometrically as follows : we consider the set

$$\{t \in \mathbb{Z}^2 : \sigma(t) = -1\} \subset \Lambda \tag{2.11}$$

and then the set

$$\bigcup_{t:\sigma(t)=-1} p^*(t) \tag{2.12}$$

where p^* is the plaquette of $\mathbb{L}^*$ with center t. As subset of $\mathbb{R}^2$ the set (2.12) has a boundary, which is composed of edges of $\mathbb{L}^*$. We decompose the boundary into maximal connected components $\gamma_1, \dots, \gamma_n$. Connected sets of edges of $\mathbb{L}^*$ are called *contours*. All contours in a configuration are disjoint two by two and have no boundary, $\delta\gamma_i = \emptyset$ for all i. This last property can be verified by noticing that $\delta(\gamma_1 \cup \dots \cup \gamma_n)$ is the boundary of the boundary of the set (2.12). We say that the contour γ_i is *closed* if $\delta\gamma_i = \emptyset$. We define two notions of compatibility for contours : a family $\underline{\gamma} = (\gamma_1, \dots, \gamma_n)$ of connected subsets of edges of Λ^* is Λ^*-*compatible* if

- $\delta\gamma_i = \emptyset$ for all i
- γ_i and γ_j are disjoint, all $i \neq j$

A family $\underline{\gamma} = (\gamma_1, \dots, \gamma_n)$ of connected subsets of edges of Λ^* is Λ^+-*compatible* if

- there is a configuration $\sigma \in X$ which is compatible with the + b.c. such that the family $\underline{\gamma}$ is exactly the set of contours of the configuration σ

The Λ^+-compatibility is introduced in order that there is a one-to-one correspondence between all configurations $\sigma \in X$ compatible with the + b.c. of Λ and all Λ^+-compatible families of contours $\underline{\gamma} = (\gamma_1, \dots, \gamma_n)$. On the other hand the notion of Λ^*-compatibility is purely geometrical and does not refer to a configuration σ or a boundary condition. The following fact is very important, and can be checked easily on examples : when Λ is a simply connected set, a family of contours $\underline{\gamma} = (\gamma_1, \dots, \gamma_n)$ is Λ^+-compatible if and only if it is Λ^*-compatible. In general only the implication Λ^+-compatibility $\Rightarrow$ Λ^*-compatibility is true. Similarly we introduce the notion of Λ^--compatibility.

Let γ be a contour. Then there is a unique configuration σ_γ which has γ as unique contour. We define the *interior of* γ , intγ, as the set of all $t \in \mathbb{Z}^2$, $\sigma(t) = -1$ and $d_1(t, \gamma) \geq 1$. Notice that $\overline{\text{int}\gamma}$ is exactly the set of all $t \in \mathbb{Z}^2$ with $\sigma(t) = -1$. In general $\overline{\text{int}\gamma}$ has several connected components see figure 1. The *volume* of γ is the cardinality of $\overline{\text{int}\gamma}$, vol$\gamma$ = $|\overline{\text{int}\gamma}|$. We also use the notation $\overline{\text{int}\gamma}$ for the closed subset of $\mathbb{R}^2$

$$\bigcup_{t:\sigma(t)=-1} p^*(t) \tag{2.13}$$

Notice that intγ is a simply connected set.

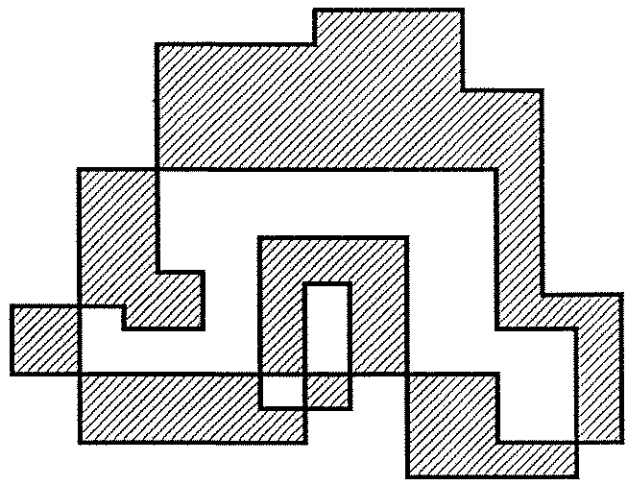


Figure 1: $\overline{\text{int}\gamma}$

2.2 The model.

2.2.1 Definition.

The model is a spin model. For each $t \in \mathbb{Z}^2$ we have a spin variable which takes the values + 1 or − 1. We also use the notation $\sigma(t)$ for the spin variables. Thus $\sigma(t)$ may denote two different but intimately related quantities : the value of a configuration σ at t or the function "spin at t" defined on X and whose value at σ is the value of the configuration at t. In this case $\sigma(t)$ is a random variable on (X , F) indexed by t. The energy of a configuration is the sum of one-body interactions $-\ h(t)\sigma(t)$, $h(t) \in \mathbb{R}$, and two-body interactions $-\ J\sigma(t)\sigma(t')$, $d_1(t,t') = 1$. We always consider the ferromagnetic case $J > 0$. On the other hand the magnetic field h, $t \mapsto h(t)$, may be inhomogeneous. Let Λ be some finite subset of $\mathbb{Z}^2$. Let σ and σ' be two configurations $\in$ X. *The energy in* Λ *of the configuration* σ *given* σ' is by definition

$$H_\Lambda(\sigma|\sigma') \equiv H_\Lambda(\sigma) + \Delta H_\Lambda(\sigma|\sigma') = \\ -J/2 \sum_{\substack{t\in\Lambda, t'\in\Lambda \\ d_1(t,t')=1}} \sigma(t)\sigma(t') - \sum_{t\in\Lambda} h(t)\sigma(t) - J \sum_{\substack{t\in\Lambda, t'\notin\Lambda \\ d_1(t,t')=1}} \sigma(t)\sigma'(t) \tag{2.14}$$

Notice that $\sigma \mapsto H_\Lambda(\sigma|\sigma')$ defines a function on X which depends only on the part of the configuration in Λ. This function is thus F(Λ)-measurable and we may consider that it is defined on X(Λ) when necessary. On the other hand, the function $\sigma' \mapsto H_\Lambda(\sigma|\sigma')$ is F($\overline{\Lambda}\backslash\Lambda$)-measurable. In the first part of the lectures we consider the model as defined above, characterized by a coupling constant J and a magnetic field

h. In this case we do not introduce explicitely the temperature. In the last part of the lectures we consider the case where the magnetic field is zero. Here we introduce explicitely the inverse temperature β by setting $J = \beta$ (i.e. by taking a coupling constant J equal to one). The corresponding expression (2.14) is interpreted as the energy at inverse temperature β.

2.2.2 The equilibrium states.

We study mainly finite volume Gibbs states. The theory of Gibbs states or Gibbs measures is exposed in Georgii's book [Ge] and in Sinai's book [Si]. The two books are different and complementary. We simply recall some basic facts.

On (X, F) we define the counting measure λ, as reference measure. Let Λ be a finite subset of $\mathbb{Z}^2$ and let σ be a configuration of X. Let

$$\mu_\Lambda^\sigma(\sigma') = \begin{cases} (\mathbb{Z}^\sigma(\Lambda))^{-1} \cdot \exp(-H_\Lambda(\sigma'|\sigma)) & \text{if } \sigma'(t) = \sigma(t), \text{ all } t \notin \Lambda \\ 0 & \text{otherwise} \end{cases} \tag{2.15}$$

The factor $\mathbb{Z}^\sigma(\Lambda)$ is a normalization factor,

$$\mathbb{Z}^\sigma(\Lambda) = \sum_{\sigma'_\Lambda \in \mathsf{X}(\Lambda)} \exp(-H_\Lambda(\sigma'_\Lambda|\sigma_\Lambda)) \tag{2.16}$$

so that the sum of $\mu_\Lambda^\sigma(\sigma')$ over all σ' is equal to one. We define a probability measure

$$d\mu_\Lambda^\sigma(\sigma') := \mu_\Lambda^\sigma(\sigma') d\lambda(\sigma') \tag{2.17}$$

and we often denote expectation value of f with respect to this measure by

$$\langle f \rangle^\sigma(\Lambda) = \int_{\mathsf{X}} f(\sigma') d\mu_\Lambda^\sigma(\sigma') = \sum_{\sigma' \in \mathsf{X}} f(\sigma') \mu_\Lambda^\sigma(\sigma') \tag{2.18}$$

The measure (2.17) is the *finite Gibbs measure* on Λ with b.c. σ. For any measurable function f the function $\sigma \mapsto \langle f \rangle^\sigma(\Lambda)$ is $\mathsf{F}(\mathbb{Z}^2\backslash\Lambda)$-measurable. Moreover, if f is $\mathsf{F}(\Lambda)$-measurable, then the function $\sigma \mapsto \langle f \rangle^\sigma(\Lambda)$ is $\mathsf{F}(\overline{\Lambda}\backslash\Lambda)$-measurable. It is easy to verify that for any finite set Ω containing $\overline{\Lambda}$ and for any $\mathsf{F}(\Lambda)$-measurable function f the conditional expectation value of f computed with μ_Ω^σ, given $\mathsf{F}(\mathbb{Z}^2\backslash\Lambda)$, is

$$\mathsf{E}(f|\mathsf{F}(\mathbb{Z}^2\backslash\Lambda)(\sigma) = \langle f \rangle^\sigma(\Lambda) \tag{2.19}$$

Definition :

A probability measure μ on (X, F) is an *equilibrium state* or *Gibbs measure* if for all finite subsets Λ of $\mathbb{Z}^2$, all bounded measurable functions f, the conditional expectation value of f given $\mathsf{F}(\mathbb{Z}^2\backslash\Lambda)$ with respect to μ is $\mathsf{E}_\mu(f|\mathsf{F}(\mathbb{Z}^2\backslash\Lambda))(\sigma) = \langle f \rangle^\sigma(\Lambda)$ μ-a.s.

In our case, equation (2.19) holds for any $\mathsf{F}(\Lambda)$-measurable function when μ is a Gibbs measure: the Gibbs measures of the Ising model have the *local Markov property*. A Gibbs measure μ is translation invariant if

$$\mu(f \circ T_t) = \mu(f) \quad \text{all } t \in \mathbb{Z}^2 \tag{2.20}$$

Let $\mathcal{E}_2(J, h)$ be the set of Gibbs measures of the model.

Theorem 2.1

For the two-dimensional Ising model with coupling constant J and homogeneous magnetic field h the following results hold:

1) The set $\mathcal{E}_2(J,h)$ is a convex set and all Gibbs measures are translation invariant. Either $\mathcal{E}_2(J,h)$ contains a unique element or all elements of $\mathcal{E}_2(J,h)$ are convex combinations of two extremal elements μ^+ and μ^-. The latter situation occurs if and only if $h = 0$ and $J > J_c$, $\sinh(2J_c) = 1$.

2) Let Λ_n be any sequence of finite subsets of $\mathbb{Z}^2$ with the properties : a) $\Lambda_n \subset \Lambda_{n+1}$, b) for any finite set $A \subset \mathbb{Z}^2$, there exists $n(A)$ such that $\Lambda_n \supset A$ for all $n \geq n(A)$. Let $\mu_{\Lambda_n}^+$ resp. $\mu_{\Lambda_n}^-$ be the finite Gibbs measures with + b.c. resp. − b.c. Then the Gibbs measures μ^+ and μ^- of 1) are the weak limits of $\mu_{\Lambda_n}^+$ and $\mu_{\Lambda_n}^-$ as n tends to infinity.

3) If $h = 0$, then there are several Gibbs measures if and only if $m^*(J) := \mu^+(\sigma(t)) > 0$. When h=0, then $\mu^-(\sigma(t)) = -\mu^+(\sigma(t))$.

Remarks.

1) The first statement of theorem 2.1 is an important result of Aizenman [A] and Higuchi [H]. It is not true for higher dimensions.

2) In general it is difficult to determine all extremal translation invariant Gibbs measures. However, for ferromagnetic models, with spins taking their values in a compact abelian metrizable group, all extremal translation invariant measures can be classified in terms of the notion of symmetry breakdown in "generic" situations [Pf.2].

3) The statement of point 3) indicates that $m^*(J)$ is an order parameter. The value of $m^*(J)$ was given by Onsager.

2.3 Correlation inequalities.

We state three lemmas which summarize the correlation inequalities which are used in the next sections.

Lemma 2.1 (Griffiths' inequalities, [Gr])

Let A be a finite subset of $\mathbb{Z}^2$ and let $\sigma(A) = \prod_{t \in A} \sigma(t)$ (as random variable). Then for any $J > 0$, $h(t) \geq 0$

$$\langle \sigma(A) \rangle^+ (\Lambda|J,h) \geq 0 \tag{2.21}$$

and

$$\langle \sigma(A) \cdot \sigma(B) \rangle^+ (\Lambda|J,h) \geq \langle \sigma(A) \rangle^+ (\Lambda|J,h) \cdot \langle \sigma(B) \rangle^+ (\Lambda|J,h) \tag{2.22}$$

Let $n(t)$ be the random variable equal to one if $\sigma(t) = 1$ and 0 otherwise. It is an example of an *increasing* function. In general we say that $\sigma_1 \leq \sigma_2$, $\sigma_i \in X$, if and only if $\sigma_1(t) \leq \sigma_2(t)$ for all t. A function $f : X \to \mathbb{R}$ is *increasing* if

$$\sigma_1 \leq \sigma_2 \quad \Rightarrow f(\sigma_1) \leq f(\sigma_2) \tag{2.23}$$

Let A be a finite subset of $\mathbb{Z}^2$ and $n(A) = \prod_{t\in A} n(t)$.

Lemma 2.2 (Fortuin-Kasteleyn-Ginibre inequalities, [F.K.G])

Let $J > 0$ and $h(t)$ be arbitrary. Then $\sigma \mapsto \langle n(A)\rangle^\sigma(\Lambda)$ is an increasing function of σ. The function $h \mapsto \langle n(A)\rangle^\sigma(\Lambda|J,h)$ is an increasing function of h. Moreover,

$$\langle n(A)\cdot n(B)\rangle^\sigma(\Lambda) \geq \langle n(A)\rangle^\sigma(\Lambda)\cdot\langle n(B)\rangle^\sigma(\Lambda) \tag{2.24}$$

and

$$\langle n(A)\rangle^+(\Lambda_1) \geq \langle n(A)\rangle^+(\Lambda_2) \quad \Lambda_1 \subset \Lambda_2 \tag{2.25}$$

We introduce the notion of *free boundary condition* (f-b.c.). Let Λ be some finite subset of $\mathbb{Z}^2$. We define a measure on $X(\Lambda)$ as before, but we replace $H_\Lambda(\sigma'|\sigma)$ by $H_\Lambda(\sigma)$,

$$\mu_\Lambda^f(\sigma'_\Lambda) = \left(Z^f(\Lambda)\right)^{-1}\cdot\exp(-H_\Lambda(\sigma'_\Lambda)) \tag{2.26}$$

with

$$Z^f(\Lambda) = \sum_{\sigma_\Lambda\in X(\Lambda)} \exp(-H_\Lambda(\sigma_\Lambda)) \tag{2.27}$$

Expectation value of g with respect to μ_Λ^f is denoted by $\langle g\rangle^f(\Lambda)$. It follows from lemma 2.1 that for any finite sets A, Λ_1, Λ_2 with $\Lambda_1 \supset \Lambda_2$,

$$\langle\sigma(A)\rangle^f(\Lambda_1) \geq \langle\sigma(A)\rangle^f(\Lambda_2), \quad \Lambda_1 \supset \Lambda_2 \tag{2.28}$$

whenever $J > 0$ and $h(t) \geq 0$. Let $h(t) \equiv h$. Then for any sequence Λ_n as in theorem 2.1 point 2,

$$\lim_n \langle\sigma(A)\rangle^f(\Lambda_n) = \langle\sigma(A)\rangle^f \tag{2.29}$$

exists. Therefore there exists a measure μ^f on X such that

$$\langle\sigma(A)\rangle^f = \mu^f(\sigma(A)) \tag{2.30}$$

Moreover, μ^f is a translation invariant Gibbs measure.

Lemma 2.3 (Simon's inequality [Sim])

Let $J > 0$, $h = 0$. Let $t_1 \in \mathbb{Z}^2$, $t_2 \in \mathbb{Z}^2$ and let B be a finite connected subset of $\mathbb{Z}^2$, such that $\mathbb{Z}^2\backslash B$ has two connected components, one containing t_1 the other containing t_2 (B separates t_1 and t_2). Then

$$\langle\sigma(t_1)\sigma(t_2)\rangle^f \leq \sum_{t\in B}\langle\sigma(t_1)\sigma(t)\rangle^f\cdot\langle\sigma(t)\sigma(t_2)\rangle^f \tag{2.31}$$

Finally, we mention some monotonicity properties of the two-point correlation function. These properties have been proven in [M.M]. Let $u = (u_1, u_2)$ and $v = (v_1, v_2)$ be two points of the lattice. Let l be the half-line passing through u and v, with end-point u.

Lemma 2.4

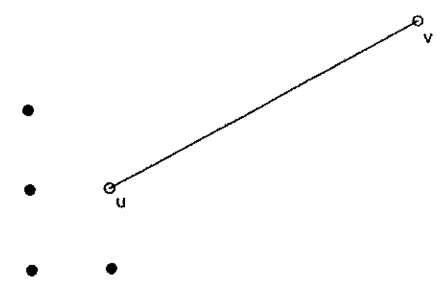
At the thermodynamic limit we have

$$\langle\sigma(u)\sigma(v)\rangle^f \geq \langle\sigma(\overline{u})\sigma(v)\rangle^f \tag{2.32}$$

in the following three cases :

- $\overline{u_2} = u_2$, $|\overline{u_1} - u_1| = 1$ *and the vertical line separating $\overline{u}$ and u does not cut l.*
- $\overline{u_1} = u_1$, $|\overline{u_2} - u_2| = 1$ *and the horizontal line separating $\overline{u}$ and v does not cut l.*
- $|\overline{u_1} - u_1| = 1$ *and* $|\overline{u_2} - u_2| = 1$ *and the diagonal line separating u and $\overline{u}$ does not cut l.*

In the next figure we have marked by • the points $\overline{u}$ for which lemma 2.4 applies.



3 The cluster expansion.

The cluster expansion is one of the oldest tool of Statistical Mechanics. It was introduced by Ursell (1927), Yvon (1935), Mayer and collaborators (1937). We expose the basic elements of this method in essentially the original form, following Brydges' lectures [Br]. We do not discuss more recent approaches. The third volume of "Phase Transitions and Critical Phenomena" [D.G] is devoted to this topics and also related topics. Chapter 4 of Ruelle's book [Ru] is also a good reference for our purposes and we refer to the book of Glimm and Jaffe [G.J] and to the thesis of Pordt [Po] for applications to Quantum Field Theory.

The exposition below is sufficient to handle many interesting models. We need only a convergence theorem. In paragraph 3.1 we define the cluster expansion in an abstract way and give in lemmas 3.1, 3.2 and 3.3 the general properties of the coefficients of the expansion. We have written this section having in mind applications for lattice systems. In the second part, sections 3.2 and 3.3, we treat the problem of the convergence of the expansion using the so-called "tree-graph bound". We follow here a paper by Cammarota [C] and Brydges' lectures. We do not treat the most general case but give a sufficiently general exposition which covers the case of "polymer expansions". Polymer expansion were introduced by Kunz in [K].

3.1 Definition of the cluster expansion.

Let Ω be some set. The elements of Ω are for example the positions of the particles of a one-component fluid or the contours of an Ising model and so on. For each integer n, $n \geq 1$, let g_n be a symmetric function of n variables $x_1, \ldots, x_n$, defined on $\Omega \times \cdots \times \Omega$ (n factors). Since g_n is symmetric we also use the notation $g(X)$, instead of $g_n(x_1, \ldots, x_n)$, with $X = \{x_1, \ldots, x_n\}$. We suppose that for each n we have an average,

$$\langle g_n \rangle = \sum_{x_1 \in \Omega} \cdots \sum_{x_n \in \Omega} g_n(x_1, \ldots, x_n) \tag{3.1}$$

We have sums in (3.1) because we have in mind lattice models. But for a classical system of n particles in a box Λ, $\langle g_n \rangle$ is given by

$$\langle g_n \rangle = \int_\Lambda dx_1 \cdots \int_\Lambda dx_n \, g_n(x_1, \ldots, x_n) \tag{3.2}$$

Lemma 3.1

If

$$\sum_{n\geq 1} \frac{1}{n!} \sum_{x_1 \in \Omega} \cdots \sum_{x_n \in \Omega} |g_n(x_1, \ldots, x_n)| < \infty \tag{3.3}$$

then the following identity is true

$$\exp\left(\sum_{n\geq 1} \frac{1}{n!} \langle g_n \rangle\right) = 1 + \sum_{n\geq 1} \frac{1}{n!} \langle G_n \rangle \tag{3.4}$$

with

$$G_n(x_1, \ldots, x_n) \equiv G(X) = \tag{3.5}$$
$$\sum_{k\geq 1}^{n} \frac{1}{k!} \sum_{\substack{X_1 \subset X \cdots X_k \subset X: \\ X_i \cap X_j = \emptyset \\ \cup_i X_i = X}} g(X_1) \cdots g(X_k)$$

The average $\langle \cdot \rangle$ is defined by (3.1).

Proof.

Since (3.3) holds, we have

$$\exp\left(\sum \frac{1}{n!} \langle g_n \rangle\right) = \tag{3.6}$$
$$1 + \sum_{k\geq 1} \frac{1}{k!} \left(\sum_{n_1 \geq 1} \frac{1}{n_1!} \langle g_{n_1} \rangle\right) \cdots \left(\sum_{n_k \geq 1} \frac{1}{n_k!} \langle g_{n_k} \rangle\right) =$$
$$1 + \sum_{n\geq 1} \frac{1}{n!} \sum_{k=1}^{n} \frac{1}{k!} \sum_{\substack{n_1 \geq 1 \cdots n_k \geq 1 \\ \sum_{i=1}^k n_i = n}} \frac{n!}{n_1! \cdots n_k!} \langle g_{n_1} \rangle \cdots \langle g_{n_k} \rangle$$

The sum in (3.6) over $n_1, \ldots, n_k$ can be evaluated in the following way: we consider the term indexed by $n_1 \leq \ldots \leq n_k$ with $n_1 = \ldots = n_{m_1}$, $n_{m_1+1} = \ldots = n_{m_1+m_2}, \ldots, n_{m_1+\cdots+m_{s-1}+1} = \ldots = n_{m_1+\cdots+m_s} = n_k$; there are $k!/m_1! \cdots m_s!$ terms in the sum which give the same contribution as this term. On the other hand there are (see [B])

$$\frac{n!}{(n_1!)^{m_1} \cdots (n_{m_1+\cdots+m_{s-1}+1}!)^{m_s}} \cdot \frac{1}{m_1! \cdots m_s!} \tag{3.7}$$

partitions of the set $X = \{x_1, \ldots, x_n\}$ of n elements with m_1 sets of n_1 elements, $\ldots$ m_s sets of $n_{m_1+\cdots m_{s-1}+1}$ elements. Therefore (3.6) is equal to

$$1 + \sum_{n\geq 1} \frac{1}{n!} \sum_{\substack{\text{partitions of } X \\ \text{into k subsets}}} \langle g(X_1) \rangle \cdots \langle g(X_k) \rangle = \tag{3.8}$$
$$1 + \sum_{n\geq 1} \frac{1}{n!} \sum_{k=1}^{n} \frac{1}{k!} \sum_{\substack{X_1 \subset X \cdots X_k \subset X \\ X_i \cap X_j = \emptyset \\ \cup_i X_i = X}} \langle g(X_1) \rangle \cdots \langle g(X_k) \rangle$$

By comparing with

$$1 + \sum_{n\geq 1} \frac{1}{n!} \langle G_n \rangle \tag{3.9}$$

we get formula (3.5).

In Statistical Mechanics we study partition functions. Sometimes the partition function Z is an expression of the form

$$Z = 1 + \sum_{n\geq 1} \frac{1}{n!}(G_n) \tag{3.10}$$

For example, in the theory of classical fluids, with activity z and in a box Λ, the grand canonical partition function is

$$1 + \sum_{n\geq 1} \frac{z^n}{n!} \int_\Lambda dx_1 \ldots \int_\Lambda dx_n \exp(-\beta V(x_1,\ldots,x_n)) \tag{3.11}$$

where $V(x_1,\ldots,x_n)$ is the (potential) energy of the particles. The functions G_n are given and we determine the functions g_n recursively by the formulas (3.5) :

$$g_1(x_1) = G_1(x_1) \tag{3.12}$$

$$g_2(x_1,x_2) + g_1(x_1)\cdot g_1(x_2) = G_2(x_1,x_2) \tag{3.13}$$

$$\begin{aligned} &g_3(x_1,x_2,x_3) + g_2(x_1,x_2)\cdot g_1(x_3) + g_2(x_1,x_3)\cdot g_1(x_2) \\ &g_2(x_2,x_3)\cdot g_1(x_1) + g_1(x_1)\cdot g_1(x_2)\cdot g_1(x_3) = \\ &G_3(x_1,x_2,x_3) \end{aligned} \tag{3.14}$$

We now give an explicit form of the function g_n in a special case, which is sufficient to treat problems with two-body interactions or hard-core conditions. Let $\mathcal{G}$ be an *unoriented graph.* The set of vertices of $\mathcal{G}$ is $V(\mathcal{G})$ and the set of edges of $\mathcal{G}$ is $E(\mathcal{G})$. All graphs below are unoriented *simple* graphs (i.e. without loop and with at most one edge between two different vertices). We denote by $\mathcal{G}_n$ the graph with n vertices and with one edge $e(i,j)$ between each pair of vertices $i \neq j$. ($\mathcal{G}_n$ is called the *complete* graph with n vertices). With each vertex k of $\mathcal{G}_n$ we associate a *variable* x_k and we suppose that $G_n(x_1,\ldots,x_n)$ is given by the expression (we write $e(i,j)\in\mathcal{G}_n$ instead of $e(i,j)\in E(\mathcal{G}_n)$)

$$G_n(x_1,\ldots,x_n) = \prod_{i=1}^n z(x_i) \prod_{e(i,j)\in\mathcal{G}_n} (1+\varphi_2(x_i,x_j)) \tag{3.15}$$

where $z(x)$ is a function of one variable and $\varphi_2(x,y)$ is a symmetric function of two variables. If we consider again the example of a classical fluid, with two-body interactions $\psi(x,y)$ between particles at x and y, then

$$\begin{aligned} &\exp(-\beta V(x_1,\ldots,x_n) = \\ &\prod_{e(i,j)\in\mathcal{G}_n} \exp(-\beta\psi(x_i,x_j)) \equiv \prod_{e(i,j)\in\mathcal{G}_n} (1+\varphi_2(x_i,x_j)) \end{aligned} \tag{3.16}$$

with

$$\varphi_2(x_i,x_j) = \exp(-\beta\psi(x_i,x_j)) - 1 \tag{3.17}$$

Let us consider the second factor in (3.15).By definition a *partial graph* $\mathcal{G}'$ of a graph $\mathcal{G}$ is a graph with the *same set of vertices* as $\mathcal{G}$, $V(\mathcal{G}') = V(\mathcal{G})$, and whose set of edges is a subset of $E(\mathcal{G})$, $E(\mathcal{G}')\subset E(\mathcal{G})$. We write $\mathcal{G}'\subset\mathcal{G}$ if $\mathcal{G}'$ is a partial graph of $\mathcal{G}$. We decompose any partial graph $\mathcal{G}'$ of $\mathcal{G}_n$ in (3.15) into connected components $\mathcal{C}_1,\ldots,\mathcal{C}_p$, each connected component being a connected graph, $V(\mathcal{C}_i)\subset V(\mathcal{G}')$ and $E(\mathcal{C}_i)\subset E(\mathcal{G}')$, so that

$$V(\mathcal{C}_i)\cap V(\mathcal{C}_j) = \emptyset \ , \ i\neq j\,. \tag{3.18}$$

and

$$\bigcup_i V(\mathcal{C}_i) = V(\mathcal{G}') \ , \ \bigcup_i E(\mathcal{C}_i) = E(\mathcal{G}') \tag{3.19}$$

Let $\mathcal{C}$ be a connected component with $V(\mathcal{C}) = \{1,\ldots,n\}$. We define

$$\tilde\varphi(\mathcal{C}) = \begin{cases} 1 & \text{, if } |V(\mathcal{C})| = 1 \\ \prod_{e(i,j)\in\mathcal{C}} \varphi_2(x_i,x_j) & \text{, if } |V(\mathcal{C})| \geq 2 \end{cases} \tag{3.20}$$

and

$$\varphi_n^T(x_1,\ldots,x_n) \equiv \varphi^T(X) = \sum_{\substack{\mathcal{C}:\text{connected}\\ \mathcal{C}\subset\mathcal{G}_n}} \tilde\varphi(\mathcal{C}) \tag{3.21}$$

The function φ^T is called *Ursell function of order n* or *truncated function.*

Lemma 3.2

Let $\mathcal{G}_n$ be the complete graph on $\{1,\ldots,n\}$. For each vertex i let x_i be a variable and let $G_n(x_1,\ldots,x_n)$ be defined by (3.15). Then

$$g_n(x_1,\ldots,x_n) = \prod_{i=1}^n z(x_i)\cdot\varphi_n^T(x_1,\ldots,x_n) \tag{3.22}$$

with

$$\varphi_n^T(x_1,\ldots,x_n) = \sum_{\substack{\mathcal{C}:\text{connected}\\ \mathcal{C}\subset\mathcal{G}_n}} \tilde\varphi(\mathcal{C}) \tag{3.23}$$

and $\tilde\varphi(\mathcal{C})$ defined by (3.20).

Proof.

We compute

$$\prod_{e(i,j)\in\mathcal{G}_n} (1+\varphi_2(x_i,x_j)) = \sum_{\mathcal{G}':\,\mathcal{G}'\subset\mathcal{G}_n} \prod_{e(i,j)\in\mathcal{G}'} \varphi_2(x_i,x_j) \tag{3.24}$$

Let $X_1,\ldots,X_k$ be a partition of $X = \{1,\ldots,n\}$ into k subsets ($1\leq k\leq n$). We group together all terms of the sum (3.24) which are represented by partial graphs $\mathcal{G}'$ with k connected components $\mathcal{C}_1,\ldots,\mathcal{C}_k$ having as sets of vertices $V(\mathcal{C}_i) = X_i$. Then we sum over all possible partitions of X. Thus (3.24) is equal to

$$\sum_{\substack{\text{partitions of } X\\ X=X_1+\cdots+X_k\\ 1\leq k\leq n}} \prod_{i=1}^k \varphi^T(X_i) = \sum_{k=1}^n \frac{1}{k!} \sum_{\substack{X_1\subset X\cdots X_k\subset X\\ X_i\cap X_j=\emptyset\\ \cup_i X_i = X}} \varphi^T(X_1)\cdots\varphi^T(X_k) \tag{3.25}$$

We can identify $\varphi^T(X_i)$ with $g(X_i)$ of (3.5) because the functions g_n are uniquely defined.

We emphasize that the n vertices of the graph $\mathcal{G}_n$ in lemma 3.2 are in one-to-one correspondence with the n *variables* $x_1,\dots,x_n$ *independently* of their values. Let now fix the values of the variables $x_1,\dots,x_n$. We introduce a *new graph with n vertices* which depends explicitely on the *values* of $x_1,\dots,x_n$. The vertices of this graph, $\mathcal{G}_n^T(x_1,\dots,x_n)$, are $1,\dots,n$. The vertex i corresponds to the variable x_i and we have an edge $e(i,j)$ between vertices i and j if and only if $\varphi_2(x_i,x_j)\neq 0$. Clearly, if in (3.23) the variables x_i have given values, then only the connected partial graphs of $\mathcal{G}_n^T(x_1,\dots,x_n)$ contribute to (3.23). Consequently, if $\mathcal{G}_n^T(x_1,\dots,x_n)$ is not connected, then $\varphi_n^T(x_1,\dots,x_n)=0$ for those values of $x_1,\dots,x_n$.

Lemma 3.3

Let $\hat{x}_1,\dots,\hat{x}_n$ be a sequence of n fixed elements of Ω, not necessarily different. Let $\mathcal{G}_n^T(\hat{x}_1,\dots,\hat{x}_n)$ be the graph with n vertices, the vertex i for the element $\hat{x}_i$ of the sequence, and whose edges are all edges $e(i,j)$ for which $\varphi_2(\hat{x}_i,\hat{x}_j)\neq 0$. If $\mathcal{G}_n^T(\hat{x}_1,\dots,\hat{x}_n)$ is not connected, then

$$\varphi_n^T(x_1=\hat{x}_1,\dots,x_n=\hat{x}_n)=0 \tag{3.26}$$

We finish this section by an example, the Ising model, with no magnetic field. Notice that the partition function Z is not given directly as

$$Z=1+\sum_{n\geq 1}\frac{1}{n!}\langle G_n\rangle \tag{3.27}$$

One of the nontrivial steps in the study of a model is often to write Z as in (3.27). One method is to try to write Z as the partition function of a system of polymers. In the case of the Ising model, at low temperature and in absence of a magnetic field an expression like (3.27) for the partition function is well-known. Here the basic objects are the contours which describe the configurations of the model.

Let Λ be a finite subset of $\mathbb{Z}^2$ with + b.c. We suppose that Λ is *simply connected* so that each family of closed disjoint contours $\underline{\gamma}=(\gamma_1,\dots,\gamma_n)$ on Λ^*, i.e. each Λ^*-compatible family of contours, is also a Λ^+-compatible family of contours. The main point here is that there is a one-to-one correspondence between the set of all configurations σ compatible with the + b.c. for Λ and the set of all families of Λ^*-compatible contours in Λ^*. This is important, because we can check *locally* whether $\underline{\gamma}=(\gamma_1,\dots,\gamma_n)$ is Λ^*-compatible : we need only to check that $\delta\gamma_i=\emptyset$ $i=1,\dots n$, and γ_i, γ_j are disjoint for all $i\neq j$. Because of this property we can write the function G_n using the following (local) hard-core potential. Let Ω be the set of all closed contours in Λ^*. The hard-core potential $\varphi_2(\gamma,\gamma')$ is defined on $\Omega\times\Omega$ by

$$\varphi_2(\gamma,\gamma')=\begin{cases}0 & \text{if } \gamma,\gamma' \text{ disjoint}\\ -1 & \text{if } \gamma\cap\gamma' \text{ not disjoint}\end{cases} \tag{3.28}$$

The energy of a configuration, compatible with the + b.c., is equal to (up to a constant)

$$-J/2\sum_t\sum_{t'}(\sigma(t)\sigma(t')-1)=\sum_{i=1}^n 2J|\gamma_i(\sigma)| \tag{3.29}$$

where $(\gamma_1(\sigma),\dots,\gamma_n(\sigma))$ is the family of all contours in σ. Let

$$z(\gamma)=\exp(-2J|\gamma|) \tag{3.30}$$

(we recall that $|\gamma|$ is the number of edges of γ and represents its length). We define

$$G_n(\gamma_1,\dots,\gamma_n)=\begin{cases}\prod_{i=1}^n z(\gamma_i) & \text{if } (\gamma_1,\dots,\gamma_n) \text{ is } \Lambda^*\text{-compatible}\\ 0 & \text{otherwise}\end{cases} \tag{3.31}$$

We can express G_n as

$$G_n(\gamma_1,\dots,\gamma_n)=\prod_{i=1}^n z(\gamma_i)\prod_{i<j}(1+\varphi_2(\gamma_i,\gamma_j)) \tag{3.32}$$

and the partition function, up to a constant, is equal to

$$1+\sum_{n\geq 1}\frac{1}{n!}\sum_{\gamma_1\in\Omega}\cdots\sum_{\gamma_n\in\Omega}G_n(\gamma_1,\dots,\gamma_n) \tag{3.33}$$

From lemma 3.3 we see that a necessary condition for $\varphi_n^T(\gamma_1,\dots,\gamma_n)$ to be nonzero is that

$$\bigcup_{i=1}^n\gamma_i \text{ is a connected subset} \tag{3.34}$$

Indeed, if this is not the case we can partition the sequence $\gamma_1,\dots,\gamma_n$ into two subsequences $\gamma_1,\dots,\gamma_k$ and $\gamma_{k+1},\dots,\gamma_n$ (by labelling the contours conveniently) so that each contour of the first subsequence is disjoint from each contour of the second subsequence. This implies that the graph $\mathcal{G}_n^T(\gamma_1,\dots,\gamma_n)$ is not connected.

3.2 The tree-graph bound.

We suppose that $x_1,\dots,x_n$ have given fixed values. Let $\mathcal{G}_n^T(x_1,\dots,x_n)$ be the graph defined in lemma 3.3. We have

$$\varphi_n^T(x_1,\dots,x_n)=\sum\tilde{\varphi}(\mathcal{C}) \tag{3.35}$$

where in (3.35) we sum over all connected partial graphs of $\mathcal{G}_n^T(x_1,\dots,x_n)$. There is a distinguished class of connected partial graphs of $\mathcal{G}_n^T(x_1,\dots,x_n)$: the trees. A *tree* is a connected graph without closed path (cycle). The following three definitions are equivalent. A *tree* is a connected graph such that if we delete one edge then the resulting graph is not connected. A *tree* is a graph without cycle such that each time we add one edge then the resulting graph has exactly one cycle. Finally a *tree* with n vertices is a connected graph with $n-1$ edges. This class of graphs is relatively easy to handle and this is why in Statistical Mechanics a problem is often solved in the "tree-graph approximation" which simply means that the sum (3.35) is restricted to the trees (see the article of Domb in [D.G]). Our goal is to have a theorem giving sufficient conditions for the convergence of the cluster expansion. This is achieved by proving the tree-graph bound on φ_n^T which we now explain in details.

Let C be a connected partial graph of $\mathcal{G}_n^T(x_1,\ldots,x_n)$, $x_1,\ldots,x_n$ having fixed values. We associate with C a specific tree $\mathcal{T}=\mathcal{T}(C)$ following a paper by Penrose [P]. The graph C has n vertices $1,\ldots,n$, and we define a weight $w(i)$ for each vertex:

$$w(1)=0 \tag{3.36}$$

and

$$w(k)=\begin{cases}\text{minimal length of a path}\\ \text{in } C \text{ with endpoints 1 and k}\end{cases} \tag{3.37}$$

(the *length* of a path is the number of edges which compose the path). Since C is connected, $w(k)\geq 1$ for $k\geq 2$. We construct a tree $\mathcal{T}$ by a two-step construction.

- We delete all edges $e(i,j)$ of C with $w(i)=w(j)$

After that operation, we get a connected graph C' with the same weights. Moreover, all edges $e(i,j)$ of C' are such that

$$|w(i)-w(j)|=1 \tag{3.38}$$

- Each vertex $i\neq 1$ of C' is connected by an edge to one or more vertices j with $w(j)=w(i)-1$. We delete all these edges except the one with j minimal.

The resulting graph is still connected and clearly has no cycle because (3.38) holds. It is the tree $\mathcal{T}(C)$. Notice that the weights $w(i)$ of $\mathcal{T}$ are equal to those of C. Conversely, given a tree $\hat{\mathcal{T}}$ and its weights, we can reconstruct all C such that $\mathcal{T}(C)=\hat{\mathcal{T}}$. It is not difficult to prove that among all graphs $C\subset\mathcal{G}_n^T$, with $\mathcal{T}(C)=\hat{\mathcal{T}}$, there is a maximal graph $C^*(\hat{\mathcal{T}})$ with respect to the "partial graph" relation $\subset$. This maximal graph is obtained from $\hat{\mathcal{T}}$ as follows. Let i be a vertex of the tree, with weight $w(i)$ and which is connnected with the vertex k, with weight $w(k)=w(i)-1$ by an edge $e(i,k)$. We add all edges $e(i,j)$ of $\mathcal{G}_n^T$ to the tree, with $j>k$ and $w(j)=w(i)-1$ and all edges $e(i,j)$ of $\mathcal{G}_n^T$ with $w(j)=w(i)$. We do this construction for all vertices. (Of course an edge is added only once.) We have

$$\{C\ :\ \mathcal{T}(C)=\hat{\mathcal{T}}\}=\{C\ :\ \hat{\mathcal{T}}\subset C\subset C^*(\hat{\mathcal{T}})\} \tag{3.39}$$

and we can write

$$\begin{aligned}\varphi_n^T(x_1,\ldots,x_n) &= \sum_{\substack{C\subset\mathcal{G}_n^T:\\ C\text{ connected}}}\tilde{\varphi}(C) \\ &= \sum_{\substack{\hat{\mathcal{T}}:\text{tree}\\ \hat{\mathcal{T}}\subset\mathcal{G}_n^T}}\tilde{\varphi}(\hat{\mathcal{T}})\sum_{\substack{C:\\ \mathcal{T}(C)=\hat{\mathcal{T}}}}\prod_{e(i,j)\in E(C)\setminus E(\hat{\mathcal{T}})}\varphi_2(x_i,x_j) \\ &= \sum_{\substack{\hat{\mathcal{T}}:\text{tree}\\ \hat{\mathcal{T}}\subset\mathcal{G}_n^T}}\tilde{\varphi}(\hat{\mathcal{T}})\prod_{\substack{e(i,j)\in\\ E(C^*(\hat{\mathcal{T}}))\setminus E(\hat{\mathcal{T}})}}(1+\varphi_2(x_i,x_j))\end{aligned} \tag{3.40}$$

The expression (3.40) indicates how we can estimate $\varphi_n^T(x_1,\ldots,x_n)$. We estimate the factor (e.g. using the stability of the potential)

$$\prod_{\substack{e(i,j)\in\\ E(C^*(\hat{\mathcal{T}}))\setminus E(\hat{\mathcal{T}})}}(1+\varphi_2(x_i,x_j)) \tag{3.41}$$

and then we must only consider a sum indexed by *trees*. This is the key point for proving the convergence of the cluster expansion (see lemma 3.5). Notice that (3.41) is particularly easy to estimate when $-1\leq\varphi_2(x_i,x_j)\leq 0$, since we can replace the factor (3.41) by one. In the case of a hard-core condition, where $\varphi_2(x_i,x_j)=-1$ or 0, as in (3.28), the product (3.41) is zero, except when $C^*(\hat{\mathcal{T}})=\hat{\mathcal{T}}$. In this case we have

$$\varphi_n^T(x_1,\ldots,x_n)=\sum_{\substack{\mathcal{T}\text{ tree }\subset\mathcal{G}_n^T:\\ C^*(\mathcal{T})=\mathcal{T}}}\tilde{\varphi}(\mathcal{T}) \tag{3.42}$$

We shall use this result in the example at the end of the chapter. However, this identity is in general not very useful because we need to know the structure of $\mathcal{G}_n^T$ explicitely in order to write (3.42). Before stating lemma 3.4, which gives the tree-graph bound, we recall that the *incidence number* $d(i)$ of a vertex i is the number of edges of the graph which have the vertex i as endpoint.

Lemma 3.4

1) Let $-1\leq\varphi_2(x,y)\leq 0$. Then

$$0\leq(-1)^{n-1}\varphi_n^T(x_1,\ldots,x_n)\leq\sum_{\substack{\mathcal{T}:\text{tree}\\ \mathcal{T}\subset\mathcal{G}_n^T(x_1,\ldots,x_n)}}|\tilde{\varphi}(\mathcal{T})| \tag{3.43}$$

2) If $\varphi_2(x,y)=1$ or 0, then

$$0\leq(-1)^{n-1}\varphi_n^T(x_1,\ldots,x_n)\leq n^{n-2} \tag{3.44}$$

3) The number $T(n;\ d(1),\ldots,d(n))$ of trees with vertices $1,\ldots,n$ and incidence numbers $d(1),\ldots,d(n)$ is equal to

$$T(n;\ d(1),\ldots,d(n))=\binom{n-2}{d(1)-1,\ldots,d(n)-1} \tag{3.45}$$

Proof.

The bound n^{n-2} is simply the number of trees with n vertices. Statement 3) is Cayley formula. For a proof see e.g. [B].

3.3 Convergence of the cluster expansion.

We use lemma 3.4 in order to prove the convergence of the cluster expansion. Since we need only to consider the case of a hard-core potential in these lectures, we consider this case, and to be specific we consider the cluster expansion for the Ising model at low temperature. Other situations are treated almost identically.

We first prove a lemma for the set Ω of all contours on $\mathbb{L}^*$, the dual lattice. To each $\gamma \in \Omega$ we have a weight $z(\gamma)$ which can be complex. We suppose that there is an upper bound

$$|z(\gamma)| \le w(\gamma) \tag{3.46}$$

such that $w(\gamma) = w(\gamma')$ for any contour γ' obtained by a translation of γ. The hard-core condition is expressed by the function $\varphi_2(\gamma, \gamma')$ (see (3.28) and (3.32)). For any γ, there is a finite subset $i(\gamma)$ such that

$$(\gamma \text{ and } \gamma' \text{ not disjoint}) \Rightarrow (\gamma' \cap i(\gamma) \neq \emptyset) \tag{3.47}$$

For the 2-dim. Ising model $i(\gamma)$ is the set $\mathbb{Z}^2_* \cap \gamma$, and $|i(\gamma)| \le |\gamma|$.

Lemma 3.5

Under the above condition, if

$$C := \sum_{\gamma:\, \gamma \ni t^*} w(\gamma)\exp(|i(\gamma)|) < \infty \tag{3.48}$$

(where t^* is any site of the dual lattice $\mathbb{L}^*$) then

$$\sum_{\gamma_1 \ni t^*} \sum_{\gamma_2} \cdots \sum_{\gamma_n} |\varphi_n^T(\gamma_1, \ldots, \gamma_n)| \prod_{k=1}^{n} w(\gamma_k) \le (n-1)!\, C^n \tag{3.49}$$

Proof.

By lemma 3.4 we have

$$\sum_{\gamma_1 \ni t^*} \sum_{\gamma_2} \cdots \sum_{\gamma_n} |\varphi_n^T(\gamma_1, \ldots, \gamma_n)| \prod_{k=1}^{n} |z(\gamma_k)| \le \tag{3.50}$$

$$\sum_{\mathcal{T} \subset \mathcal{G}_n} \sum_{\gamma_1 \ni t^*} \sum_{\gamma_2} \cdots \sum_{\gamma_n} \prod_{e(i,j) \in \mathcal{T}} |\varphi_2(\gamma_i, \gamma_j)| \prod_{k=1}^{n} |z(\gamma_k)|$$

The last sum in (3.50) is over all trees of the complete graph with n vertices $1, \ldots, n$. Let $\mathcal{T}$ be a fixed tree with incidence numbers $d(1), \ldots, d(n)$. The summation is done in the following order. We first sum over all γ_k, $k \ge 2$ such that $d(\gamma_k) = 1$. Such values of k correspond to extremities of the tree $\mathcal{T}$. Let $\sum_\gamma^*$ denote the sum over all γ which contain a fixed point t^*. Since the upper bound on $|z(\gamma)|$, $w(\gamma)$, is independent on the position of the contour γ,

$$\sum\nolimits_\gamma^* |i(\gamma)|^p\, |w(\gamma)| \tag{3.51}$$

is independent on the fixed point t^*. Let $k \ge 2$ be such that $d(k) = 1$ and let j be the (unique) vertex which is connected to k in $\mathcal{T}$. We have $\gamma_j \cap \gamma_k \neq \emptyset$ and we get by summing over γ_k a contribution which is smaller than

$$|i(\gamma_j)| \sum\nolimits_{\gamma_k}^* w(\gamma_k) = |i(\gamma_j)| \sum\nolimits_{\gamma_k}^* |i(\gamma_k)|^{d(k)-1} w(\gamma_k) \tag{3.52}$$

We do the summation for all k with $d(k) = 1$ and then delete from $\mathcal{T}$ all edges containing such points. We get a new tree $\mathcal{T}'$ and we sum over all γ_j such that $j \ge 2$ and j is an extremity of $\mathcal{T}'$. The summation over γ_j gives a contribution bounded by

$$|i(\gamma_i)| \sum\nolimits_{\gamma_j}^* |i(\gamma_j)|^{d(j)-1} w(\gamma_j) \tag{3.53}$$

where $d(j)$ is the incidence number of j for the initial tree $\mathcal{T}$ and i is the unique vertex connected to j in the new tree $\mathcal{T}'$. Therefore,

$$\sum_{\gamma_1 \ni t^*} \sum_{\gamma_2} \cdots \sum_{\gamma_n} |\varphi_2(\gamma_i, \gamma_j)| \prod_{k=1}^{n} |z(\gamma_k)| \le \tag{3.54}$$

$$\sum\nolimits_{\gamma_1}^* |i(\gamma_1)|^{d(1)} w(\gamma_1) \prod_{k=2}^{n} \sum\nolimits_{\gamma_k}^* |i(\gamma_k)|^{d(k)-1} w(\gamma_k)$$

The sum over the trees is easy since

$$T(n, d(1), \ldots, d(n)) = \frac{(n-2)!}{(d(1)-1)! \cdots (d(n)-1)!} \tag{3.55}$$

$$\le \frac{(n-1)!}{(d(1)!(d(2)-1)! \cdots (d(n)-1)!}$$

From (3.54) and (3.55), we get by summing over $d(i)$ the bound $(n-1)!\, C^n$ for (3.50).

Theorem 3.1

Let Λ be a simply connected finite subset of $\mathbb{Z}^2$. Let Λ^* be the dual of Λ (as cell complex) and let the hypothesis of lemma 3.5 be satisfied,

$$\sum_{\gamma:\gamma \ni t^*} w(\gamma)\exp(|i(\gamma)|) \le C \tag{3.56}$$

with a constant $C < 1$.

1) The partition function for the Ising model, with + b.c. is given by

$$Z^+(\Lambda) \equiv 1 + \sum_{n \ge 1} \frac{1}{n!} \sum_{\gamma_1 \subset \Lambda^*} \cdots \sum_{\gamma_n \subset \Lambda^*} \prod_{k=1}^{n} z(\gamma_k) \tag{3.57}$$

$$= \exp\left(\sum_{n \ge 1} \frac{1}{n!} \sum_{\gamma_1 \subset \Lambda^*} \cdots \sum_{\gamma_n \subset \Lambda^*} \varphi_n^T(\gamma_1, \ldots \gamma_n) \prod_{k=1}^{n} z(\gamma_k) \right)$$

The series in the argument of the exponential function is absolutely convergent. It is the cluster expansion of $\ln Z^+(\Lambda)$.

2) If $z(\gamma) = z(\gamma')$ for all γ' which are obtained by a translation of γ, and if for each p Λ_p is a square, $\Lambda_{p+1} \supset \Lambda_p$, such that eventually any finite subset $A \subset \mathbb{Z}^2$ is in Λ_p, then

$$\lim_{p\to\infty} \frac{1}{|\Lambda_p \cap \mathbb{Z}^2|} \ln Z^+(\Lambda_p) = \sum_{n\geq 1} \frac{1}{n!} \sum_{\gamma_1 \ni t^*} \sum_{\gamma_2} \cdots \sum_{\gamma_n} \varphi_n^T(\gamma_1, \ldots \gamma_n) \prod_{k=1}^n z(\gamma_k) \tag{3.58}$$

and the series is absolutely convergent. If for each γ, $z(\gamma)$ is a function of some parameter θ, $\theta \mapsto z(\gamma|\theta)$, which is analytic in θ for $\theta \in D$, some domain in D, then the function defined by (3.58) is analytic in θ, $\theta \in D$. In the above formula t^* is any point of the dual lattice.

Remark
Part 2) of Theorem 3.1 is still true if the sequence Λ_p tends to $\mathbb{Z}^2$ in the sense of van Hove when p tends to infinity (see [Ru] p. 14).

Proof.

The condition

$$\sum_{\gamma\ni t^*} \exp(|i(\gamma)|)|z(\gamma)| \leq \sum_{\gamma \ni t^*} \exp(|i(\gamma)|)|w(\gamma)| < 1 \tag{3.59}$$

leads immediately to the absolute convergence of the cluster expansion. Indeed from lemma 3.5, we see that condition (3.3) of lemma 3.1 is verified. Part 2) is a consequence of the absolute convergence.

We finish the section by an example for the readers which are not familiar with the cluster expansion. Let $Z = 1 + z$. We can think of Z as the partition function of a system of particles with hard-core interaction only, activity z, in a zero-dimensional space ! Applying the results above, we have

$$1 + z = \exp\left(\sum_{n\geq 1} \frac{1}{n!} \varphi_n^T(1, \ldots, n) z^n\right) \tag{3.60}$$

for $|z|$ sufficiently small. Here $\varphi_n^T(1, \ldots, n)$ is given by

$$\varphi_n^T(1, \ldots, n) = \sum_{\substack{C:\text{connected} \\ C\subset \mathcal{G}_n}} \bar{\varphi}(C) \tag{3.61}$$

with $\mathcal{G}_n$ the complete graph with n vertices. The function $\bar{\varphi}(C)$ is

$$\bar{\varphi}(C) = (-1)^{|E(C)|} \tag{3.62}$$

It is instructive to write down explicitely some terms of the cluster expansion. Already for $n = 4$, $\varphi_4^T(1, 2, 3, 4)$ is a sum of 38 terms. On the other hand, the number of connected C, $C \subset \mathcal{G}_4$, which are trees is only 16. The condition $C < 1$ is equivalent to $|z| \cdot e = 1$. Thus, from theorem 3.1, the cluster expansion converges for $|z| < 1/e$.

However, since in (3.3) $\mathcal{G}_n$ is the complete graph with n vertices, it is not difficult to show that in the identity (3.42), only the trees which are chains starting at 1 contribute to $\varphi_n^T(1, \ldots, n)$. Since there are $(n-1)!$ such chains we get

$$\sum_{n\geq 1} \frac{1}{n!} \varphi_n^T(1, \ldots, n) z^n = \sum \frac{(-1)^{n-1}}{n} z^n \tag{3.63}$$

as it should be ! Notice that the convergence radius of the cluster expansion is one since there is a "non-physical" singularity at $z = -1$. The physical values of z are positive, and for those values Z is analytic.

4 The phase of small contours.

In this section we study the Ising model at low temperature,with + b.c. and when we take into account only the spin configurations in which all contours have a given maximal size. The phase obtained in this way is called the *phase of small contours*. When we restrict the size of the contours appearing in the positively magnetized phase of the Ising model, then it is possible to continue analytically the corresponding free energy in the magnetic field h up to some negative value $-h^*$ depending on the maximal size of the allowed contours. On the other hand, we know that this is not possible for the Ising model : the free energy has an essential singularity at $h = 0$ [I.1], [I.2]. The phase of small contours has a positive magnetization for negative values of the magnetic field h, $-h^* < h < 0$. For these values of the magnetic field this phase has been proposed as a possible metastable phase by Capocaccia, Cassandro and Olivieri [C.C.O], and it is essentially the *unstable phase* introduced by Zahradnik in its formulation of Pirogov-Sinai theory [Z]. Our main purpose here is to get a precise estimation of h^*. From such an information we get useful results on large deviations of the magnetization in this phase at $h = 0$. This is the subject of section 5.

4.1 Ising model with an inhomogeneous magnetic field.

We consider the model with coupling constant J and inhomogeneous magnetic field h. The inverse temperature is not introduced explicitely. It is convenient to normalize the Hamiltonian according to the boundary condition which is chosen. Let Λ be some finite subset of $\mathbb{Z}^2$ which is simply connected, and let us consider the + b.c. for Λ, i.e. $\sigma(t) = 1$ if $t \notin \Lambda$. We normalize the Hamiltonian so that the configuration $\sigma(t) \equiv 1$ has energy zero,

$$H_\Lambda^+ = -J/2 \sum_{\substack{t,t':\\ d_1(t,t')=1}} (\sigma(t)\sigma(t') - 1) - \sum_{t\in\Lambda} h(t)(\sigma(t) - 1) \tag{4.1}$$

If we have − b.c. we normalize the energy so that

$$H_\Lambda^- = -J/2 \sum_{\substack{t,t':\\ d_1(t,t')=1}} (\sigma(t)\sigma(t') - 1) - \sum_{t\in\Lambda} h(t)(\sigma(t) + 1) \tag{4.2}$$

The corresponding partition functions are $Z^+(\Lambda)$ and $Z^-(\Lambda)$.

Let γ be a (low-temperature) contour and let σ_γ be the configuration on $\mathbb{Z}^2$ which is specified by γ, and the + b.c. At the end of section 2.1.2. we have defined $\text{int}\gamma$ and $\overline{\text{int}\gamma}$. We recall that $\overline{\text{int}\gamma}$ is the set of all t such that $\sigma_\gamma(t) = -1$ and the volume of γ, $\text{vol}(\gamma)$, is equal to the cardinality of $\overline{\text{int}\gamma}$. All spins at $t \in \overline{\text{int}\gamma}\setminus\text{int}\gamma$ have the same value in any configuration which has γ as one of its contours. We say that γ is of *type* +, resp. *type* −, if the value of these spins is + 1, resp. − 1. The type of a contour depends on the whole spin configuration and the choice of the boundary condition. The pair, which is constituted by a closed contour and the type of the contour, is a *signed contour*. We say that γ is an *outer contour* if it is not contained

in the interior of another contour. For outer signed contours the type depends only on the b.c. If we have + b.c., resp. − b.c., then an outer signed contour is of type −, resp. +.

We define a function $\xi(\gamma)$ for *signed contour* :

$$\xi(\gamma) = \begin{cases} \exp\left(-2J|\gamma| - 2\sum_{t\in\overline{\text{int}(\gamma)}} h(t)\right) &, \ \gamma \text{ of type } - \\ \exp\left(-2J|\gamma| + 2\sum_{t\in\overline{\text{int}(\gamma)}} h(t)\right) &, \ \gamma \text{ of type } + \end{cases} \tag{4.3}$$

We can write

$$Z^+(\Lambda) = 1 + \sum_{n\geq 1} \frac{1}{n!} \sum_{\substack{\gamma_1,\ldots,\gamma_n:\\ \text{compatible}}} \prod_{k=1}^n \xi(\gamma_k) \tag{4.4}$$

respectively

$$Z^-(\Lambda) = 1 + \sum_{n\geq 1} \frac{1}{n!} \sum_{\substack{\gamma_1,\ldots,\gamma_n:\\ \text{compatible}}} \prod_{k=1}^n \xi(\gamma_k) \tag{4.5}$$

All contours in (4.4) and (4.5) are signed contours and the notion of compatibility is the Λ^+-compatibility, resp. the Λ^--compatibility. Notice that the weight $\xi(\gamma)$ depends explicitely on the type of the contour, and therefore we cannot apply directly the method of the cluster expansion, since the notion of compatibility is not local. The way to solve this difficulty has been indicated by Minlos and Sinai. In (4.4) (or in (4.5)) we resum over all contours which are not outer contours. A simple computation leads to the identity

$$Z^+(\Lambda) = 1 + \sum_{n\geq 1} \frac{1}{n!} \sum_{\substack{\gamma_1,\ldots,\gamma_n:\\ \text{outer contours}\\ \text{compatible}}} \prod_{k=1}^n \xi(\gamma_k)\cdot Z^-(\text{int}\gamma_k) \tag{4.6}$$

and a similar expression holds for $Z^-(\Lambda)$. Since all contours are outer contours they have the same type. We write the product in (4.6) as

$$\prod_{k=1}^n \xi(\gamma_k)\cdot Z^-(\text{int}\gamma_k) = \prod_{k=1}^n \frac{Z^-(\text{int}\gamma_k)}{Z^+(\text{int}\gamma_k)}\cdot Z^+(\text{int}\gamma_k) \tag{4.7}$$

and put

$$z(\gamma_k) := \xi(\gamma_k)\cdot \frac{Z^-(\text{int}\gamma_k)}{Z^+(\text{int}\gamma_k)} \tag{4.8}$$

so that we get for (4.6)

$$Z^+(\Lambda) = 1 + \sum_{n\geq 1} \frac{1}{n!} \sum_{\substack{\gamma_1,\ldots,\gamma_n:\\ \text{outer contours}\\ \text{compatible}}} \prod_{k=1}^n z(\gamma_k)\cdot Z^+(\text{int}\gamma_k) \tag{4.9}$$

Since $Z^+(\text{int}\gamma_k)$ is the partition function of a system with + b.c. we can express it in terms of outer contours inside $\text{int}\gamma_k$, as in (4.9). Iterating this procedure we get

$$Z^+(\Lambda) = 1 + \sum_{n\geq 1} \frac{1}{n!} \sum_{\substack{\gamma_1,\ldots,\gamma_n: \\ \text{contours of type} - \\ \text{compatible}}} \prod_{k=1}^{n} z(\gamma_k) \tag{4.10}$$

where in (4.10) only contours of type − occur and the compatibility condition is purely geometrical and local. A similar expression holds for $Z^-(\Lambda)$.

The isoperimetric inequality on the lattice is

$$|\gamma|^2 \geq 16 \cdot \text{vol}(\gamma) \tag{4.11}$$

Definition :

Let s be some positive number. A contour is *s-small* (or *small* if the value of s is fixed) whenever $\text{vol}(\gamma) \leq s^2$. The class of s-small contours is denoted by $\Omega(s)$.

An important property of this definition is that a contour γ is small if it is contained in the interior of a small contour. Let γ be a small contour. Let $\text{Re}J > 0$ and $h^* = \sup|h(t)|$. We have

$$\begin{aligned} |\xi(\gamma)| &\leq \exp(-2\text{Re}J|\gamma| + 2\,h^*\text{vol}(\gamma)) \\ &\leq \exp\left(-2\text{Re}J\left(1 - \frac{h^*\text{vol}(\gamma)}{|\gamma|\text{Re}J}\right)|\gamma|\right) \\ &\leq \exp\left(-2\text{Re}J\left(1 - \frac{h^*(\text{vol}(\gamma))^{1\backslash 2}}{4\text{Re}J}\right)|\gamma|\right) \\ &\leq \exp\left(-2\text{Re}J\left(1 - \frac{h^*\,s}{4\text{Re}J}\right)|\gamma|\right) \end{aligned} \tag{4.12}$$

Theorem 4.1

Let $J \in \mathbb{C}$, $\text{Re}J > 0$, be the coupling constant of the 2-dimensional Ising model. Let $h(t) \in \mathbb{C}$ be an inhomogeneous magnetic field and let $\Omega(s)$, $s \in \mathbb{N}$, be the class of all s-small contours γ, i.e. $\text{vol}(\gamma) \leq s^2$. Let

$$\frac{h^*\,s}{4\text{Re}J} \equiv \theta < 1 \;,\quad h^* = \sup_t |h(t)| \,. \tag{4.13}$$

If $\text{Re}J \geq J_0$, J_0 is given in (4.22), then the cluster expansion for s-small contours of one particular type,

$$\sum_{n\geq 1} \frac{1}{n!} \sum_{\substack{\gamma_1\in\Omega(s) \\ \gamma_1 \ni t^*}} \sum_{\gamma_2\in\Omega(s)} \cdots \sum_{\gamma_n\in\Omega(s)} \varphi_n^T(\gamma_1,\ldots,\gamma_n) \prod_{k=1}^{n} z(\gamma_k) \tag{4.14}$$

is absolutely convergent. All contours are of type + (resp. −) if we have − (resp. +) boundary condition.

Remarks.

1) If Λ is a simply connected finite subset and $\Omega(\Lambda, s)$ is the set of all s- small contours in Λ, then under the same hypothesis

$$Z^+(\Lambda, s) = \exp\left(\sum_{n\geq 1} \frac{1}{n!} \sum_{\gamma_1\in\Omega(\Lambda,s)} \cdots \sum_{\gamma_n\in\Omega(\Lambda,s)} \varphi_n^T(\gamma_1,\ldots,\gamma_n) \prod_{k=1}^{n} z(\gamma_k)\right) \tag{4.15}$$

where in (4.15) we sum over all small contours of type −.

2) We can still apply (4.15) in the following situation. Let Λ be a bounded set. For each connected component of Λ we have either + b.c. or − b.c., and the Hamiltonian is normalized so that the configuration with no contour has energy zero. We suppose that

- there is a one-to-one correspondence between the set of all allowed configurations in Λ compatible with the boundary condition on Λ and the set of all families of Λ^*-compatible s-small contours.

The corresponding partition function is denoted by $Z(\Lambda, s)$ and

$$Z(\Lambda, s) = \exp\left(\sum_{n\geq 1} \frac{1}{n!} \sum_{\gamma_1\in\Omega(\Lambda,s)} \cdots \sum_{\gamma_n\in\Omega(\Lambda,s)} \varphi_n^T(\gamma_1,\ldots,\gamma_n) \prod_{k=1}^{n} z(\gamma_k)\right) \tag{4.16}$$

3) The same theorem holds if we replace the definition of s-small contour by the following one: a contour γ is *s-small* if any connected component of $\overline{\text{int}\gamma}$ has a volume smaller than s^2. This generalization is used in section 8.

Proof.

We apply the results of section 3. Let $K \geq 0$ be large enough so that

$$\alpha(K) := \sum_{\substack{\gamma:\,\gamma\in\Omega(s) \\ \gamma\ni t^*}} |\gamma|^2 \exp(-K|\gamma|) \cdot \exp(|i(\gamma)|) \tag{4.17}$$

is convergent. Notice that

$$|\gamma|/4 \leq |i(\gamma)| \leq |\gamma| \tag{4.18}$$

The function $\alpha(K)$ behaves essentially like $\exp(-4K)$ for large K. We verify condition (3.48) of lemma 3.5 with a constant C smaller than one, so that we can apply theorem 3.1. It is sufficient to find a function $w(\gamma)$, invariant by translation, such that

$$\sum_{\substack{\gamma:\,\gamma\in\Omega(s) \\ \gamma\ni t^*}} w(\gamma) \exp|i(\gamma)| < 1 \tag{4.19}$$

The proof of the existence of $w(\gamma)$ is done inductively. We say that $\gamma \in \Omega(s)$ is of class one if its interior cannot contain any contour. We say that γ is of class two when its interior can contain only contours of class one. Inductively we define contours of class q. Let K be large enough so that $\alpha(K) < 1$ and $\lambda(K) < 1$, with

$$\lambda(K) := \frac{\alpha(K)}{1-\alpha(K)} \tag{4.20}$$

We choose $K_0 \geq K$ and so that

$$\frac{sh^*}{4\mathrm{Re}J} \cdot \frac{1}{1-\lambda(K_0)} = \frac{\theta}{1-\lambda(K_0)} \equiv \theta^* < 1 \tag{4.21}$$

We define

$$J_0 := \frac{K_0}{2(1-\theta^*)} \tag{4.22}$$

Lemma 4.1

Let J and h satisfy the conditions of theorem 4.1. Let γ be a contour of class smaller or equal to q. We suppose that

$$\frac{z(\gamma|h)}{z(\gamma|0)} \equiv \exp(f(\gamma|h)) \ , \ |f(\gamma|h)| \leq 2h_q \mathrm{vol}(\gamma) \tag{4.23}$$

with

$$\frac{h_q\, s}{4\mathrm{Re}J} \leq \theta^* \tag{4.24}$$

so that

$$|z(\gamma|h)| \leq \exp(-K_0|\gamma|) \tag{4.25}$$

Then for all contours $\hat{\gamma}$ of class $(q+1)$ we have

$$|z(\hat{\gamma}|h)| \leq \exp(-K_0|\hat{\gamma}|) \tag{4.26}$$

and

$$\frac{z(\hat{\gamma}|h)}{z(\hat{\gamma}|0)} \equiv \exp(f(\hat{\gamma}|h)) \ , \ |f(\hat{\gamma}|h)| \leq 2h_{q+1} \mathrm{vol}(\hat{\gamma}) \tag{4.27}$$

with

$$h_{q+1} = h^* + \lambda(K_0)\cdot h_q \tag{4.28}$$

and (4.24) holds with h_{q+1} instead of h_q.

Proof.

Let Λ be a bounded simply connected set and let $Z_q^+(\Lambda|h)$ be the partition function for contours of class $\leq q$ with $+$ boundary condition for Λ. All contours appearing

in the expression of $Z_q^+(\Lambda|h)$ are of type $-$. Since $\alpha(K_0) < 1$ the cluster expansion of $Z_q^+(\Lambda|h)$ is absolutely convergent. We estimate the quotient

$$\frac{Z_q^+(\Lambda|h)}{Z_q^+(\Lambda|0)} = \exp\left(\sum_{n\geq 1}\frac{1}{n!}\sum_{\substack{\gamma_1\\ \text{class}\leq q}}\cdots\sum_{\substack{\gamma_n\\ \text{class}\leq q}} \varphi_n^T(\gamma_1,\ldots,\gamma_n)\left(\prod_{k=1}^n z(\gamma_k|h) - \prod_{k=1}^n z(\gamma_k|0)\right)\right) \tag{4.29}$$

If x is a complex number, then

$$|e^x - 1| = |x\int_0^1 e^{tx}dt| \leq |x|e^{|x|} \tag{4.30}$$

We have by hypothesis (4.23) and the isoperimetric inequality

$$\begin{aligned} &|\prod_{k=1}^n z(\gamma_k|h) - \prod_{k=1}^n z(\gamma_k|0)| = \\ &\prod_{k=1}^n |z(\gamma_k|0)|\,|\prod_{k=1}^n \frac{z(\gamma_k|h)}{z(\gamma_k|0)} - 1| \leq \\ &\prod_{k=1}^n |z(\gamma_k|0)|(\sum_{k=1}^n |f(\gamma_k|h)|)\exp(\sum_{k=1}^n |f(\gamma_k|h)|) \leq \\ &\prod_{k=1}^n |z(\gamma_k|0)|(\sum_{k=1}^n 2h_q\mathrm{vol}(\gamma_k))\exp(\sum_{k=1}^n 2h_q\mathrm{vol}(\gamma_k)) \leq \\ &h_q\prod_{k=1}^n |z(\gamma_k|0)|\prod_{k=1}^n |\gamma_k|^2\exp(2h_q\mathrm{vol}(\gamma_k)) \end{aligned} \tag{4.31}$$

By hypothesis (4.24) and the identity $z(\gamma_k|0) = \xi(\gamma_k|0)$

$$\begin{aligned} |z(\gamma_k|0)|\exp(2h_q\mathrm{vol}(\gamma_k)) &\leq \exp\left(-2\mathrm{Re}J\left(1-\frac{h_q\, s}{4\mathrm{Re}J}\right)|\gamma|\right) \\ &\leq \exp(-2\mathrm{Re}J(1-\theta^*)|\gamma|) \\ &\leq \exp(-K_0|\gamma|) \end{aligned} \tag{4.32}$$

and therefore we get (following the proof of lemma 3.5)

$$\begin{aligned} &\sum_{n\geq 1}\frac{1}{n!}\sum_{\substack{\gamma_1\\ \text{class}\leq q}}\cdots\sum_{\substack{\gamma_n\\ \text{class}\leq q}} |\varphi_n^T(\gamma_1,\ldots,\gamma_n)||\prod_{k=1}^n z(\gamma_k|h) - \prod_{k=1}^n z(\gamma_k|0)| \leq \\ &\sum_{n\geq 1}\frac{h_q}{n!}\sum_{\substack{\gamma_1\\ \text{class}\leq q}}\cdots\sum_{\substack{\gamma_n\\ \text{class}\leq q}} |\varphi_n^T(\gamma_1,\ldots,\gamma_n)|\prod_{k=1}^n |\gamma_k|^2 z(\gamma_k|0)\exp(2h_q\mathrm{vol}(\gamma_k)) \leq \\ &h_q|\Lambda^*|\sum_{n\geq 1}\alpha(K_0)^n = \\ &h_q|\Lambda^*|\lambda(K_0) \end{aligned} \tag{4.33}$$

Exactly the same result holds for the $-$ b.c. Using the identiy

$$Z_q^+(\Lambda|0) = Z_q^-(\Lambda|0) \tag{4.34}$$

we can write for any contour γ of class $q+1$, say of type +,(since $\mathrm{int}(\gamma)$ is simply connected)

$$\begin{aligned} |z(\gamma)| &= |\xi(\gamma)\cdot\frac{Z^+(\mathrm{int}(\gamma)|h)}{Z^-(\mathrm{int}(\gamma)|h)}| \qquad (4.35)\\ &= |\xi(\gamma)|\cdot|\frac{Z^+(\mathrm{int}(\gamma)|h)}{Z^+(\mathrm{int}(\gamma)|0)}\frac{Z^-(\mathrm{int}(\gamma)|0)}{Z^-(\mathrm{int}(\gamma)|h)}| \\ &\le \exp(-2\mathrm{Re}J|\gamma|+2(h^*+h_q\lambda(K_0))\mathrm{vol}(\gamma)) \end{aligned}$$

Thus we have

$$h_{q+1}=h^*+h_q\lambda(K_0) \qquad (4.36)$$

and

$$\begin{aligned} \frac{sh^*}{4\mathrm{Re}J}+\frac{sh_q\lambda(K_0)}{4\mathrm{Re}J} &\le \theta+\theta^*\lambda(K_0) \qquad (4.37)\\ &= \theta+\frac{\theta\lambda(K_0)}{1-\lambda(K_0)}\\ &= \theta^* \end{aligned}$$

Formula (4.26) follows since by hypothesis

$$2\mathrm{Re}J(1-\theta^*)\ge K_0 \qquad (4.38)$$

Theorem 4.1 can now be proved without difficulty. For contours of class 1 we have

$$\begin{aligned} |z(\gamma)| &= |\xi(\gamma)|\le\exp(-2\mathrm{Re}J|\gamma|+2h^*\mathrm{vol}(\gamma)) \qquad (4.39)\\ &\le \exp(-2\mathrm{Re}J|\gamma|(1-\frac{sh^*}{4\mathrm{Re}J})) \end{aligned}$$

and the hypothesis of lemma 4.1 are fulfilled with $h_1=h$. Thus for all contours of class 2 the hypothesis of lemma 4.1 are fulfilled with

$$h_2=h^*(1+\lambda(K_0))>h_1 \qquad (4.40)$$

By induction the hypothesis of lemma 4.1 are fufilled for contours of class $q+1$ with

$$h_{q+1}=h^*(1+\lambda(K_0)+\cdots+(\lambda(K_0))^q)>h_q \qquad (4.41)$$

and therefore the bound (4.25) holds for all small contours. The cluster expansion is absolutely convergent.

4.2 Remarks on the phase of small contours.

In this section we always suppose that Λ is a finite set with the property

- there is a one-to-one correspondence between the set of all allowed configurations in Λ compatible with the boundary condition on Λ and the set of all families of Λ^*-compatible s-small contours.

The partition function of the phase of small contours is denoted by $Z(\Lambda,s)$. Since Λ is not necessarily a connected set we may have different boundary conditions on the different connected components of Λ. However, these boundary conditions are either + b.c. or − b.c. The set of small contours in Λ is denoted by $\Omega(\Lambda,s)$. If the hypothesis of theorem 4.1 are fulfilled, then we have a cluster expansion for $Z(\Lambda)$ (see remark 2 following theorem 4.1) :

$$Z(\Lambda,s)=\exp\left(\sum_{n\ge1}\frac{1}{n!}\sum_{\gamma_1\in\Omega(\Lambda,s)}\cdots\sum_{\gamma_n\in\Omega(\Lambda,s)}\varphi_n^T(\gamma_1,\ldots,\gamma_n)\prod_{k=1}^n z(\gamma_k)\right) \qquad (4.42)$$

where in (4.42) the type of the contours is +, resp. −, if the contour is contained in a component of Λ with − b.c., resp. + b.c. The free energy of the phase, $P_s(\Lambda)$, is given by the formula

$$\exp(|\Lambda|P_s(\Lambda)):=Z(\Lambda,s) \qquad (4.43)$$

The statistical properties of the phase of small contours are described by the measure $\langle\cdot\rangle(\Lambda,s)$ which is obtained by conditioning the Gibbs measure defined on Λ with respect to the set of configurations which contain only s-small contours.

1. Let γ be a small contour of type −.

$$\begin{aligned} z(\gamma) &= \xi(\gamma)\cdot\frac{Z^-(\mathrm{int}\gamma)}{Z^+(\mathrm{int}\gamma)} = \qquad (4.44)\\ &\exp\left(-2\left(J|\gamma|+\sum_{t\in\overline{\mathrm{int}\gamma}\setminus\mathrm{int}\gamma}h(t)\right)\right)\cdot\frac{\exp\left(-\sum_{t\in\mathrm{int}\gamma}h(t)\right)Z^-(\mathrm{int}\gamma)}{\exp\left(+\sum_{t\in\mathrm{int}\gamma}h(t)\right)Z^+(\mathrm{int}\gamma)} \equiv \\ &\exp\left(-2\left(J|\gamma|+\sum_{t\in\overline{\mathrm{int}\gamma}\setminus\mathrm{int}\gamma}h(t)\right)\right)\cdot\frac{\hat{Z}^-(\mathrm{int}\gamma)}{\hat{Z}^+(\mathrm{int}\gamma)} \end{aligned}$$

where $\hat{Z}^-$ resp. $\hat{Z}^+$ are the partition functions for the Hamiltonian

$$H=-J/2\sum_{\substack{t,t'\\ d_1(t,t')=1}}(\sigma(t)\sigma(t')-1)-\sum_t h(t)\sigma(t) \qquad (4.45)$$

with − b.c., resp. + b.c. A similar expression holds for contours of type +.

Lemma 4.2

Let $J>0$, h real and $K=2J(1-\theta^*)$. If the hypothesis of theorem 4.1 are fulfilled and if K is so large such that

$$\sum_{p\ge0}|p^6|3^p e^{-(K-1)p}<1 \qquad (4.46)$$

then there exists a function χ, independent on the magnetic field h and of Λ, so that

$$|\frac{d^2P_s(\Lambda)}{dh^2}|\le\chi(K) \qquad (4.47)$$

For large K, we have

$$\chi(K) = O(\exp(-4K)) \tag{4.48}$$

Proof.

We compute for a contour of type −

$$\frac{dz}{dh} = -2|\overline{\mathrm{int}\gamma}\backslash\mathrm{int}\gamma|\cdot z + \left\langle\sum_{t\in\mathrm{int}\gamma}\sigma(t)\right\rangle^{-}(\mathrm{int}\gamma|h)\cdot z - \left\langle\sum_{t\in\mathrm{int}\gamma}\sigma(t)\right\rangle^{+}(\mathrm{int}\gamma|h)\cdot z \tag{4.49}$$

and

$$\frac{d^2z}{dh^2} = \frac{dz}{dh}\left(-2|\overline{\mathrm{int}\gamma}\backslash\mathrm{int}\gamma| + \left\langle\sum_{t\in\mathrm{int}\gamma}\sigma(t)\right\rangle^{-}(\mathrm{int}\gamma|h) - \left\langle\sum_{t\in\mathrm{int}\gamma}\sigma(t)\right\rangle^{+}(\mathrm{int}\gamma|h)\right) + z\sum_{t\in\mathrm{int}\gamma}\sum_{t'\in\mathrm{int}\gamma}\left(\langle\sigma(t);\sigma(t')\rangle^{-}(\mathrm{int}\gamma|h) - \langle\sigma(t);\sigma(t')\rangle^{+}(\mathrm{int}\gamma|h)\right) \tag{4.50}$$

where

$$\langle\sigma(t);\sigma(t')\rangle := \langle\sigma(t)\cdot\sigma(t')\rangle - \langle\sigma(t)\rangle\cdot\langle\sigma(t')\rangle \tag{4.51}$$

Therefore

$$\left|\frac{dz}{dh}\right| \le 2|z|\cdot\mathrm{vol}\gamma \tag{4.52}$$

and

$$\left|\frac{d^2z}{dh^2}\right| \le 6|z|(\mathrm{vol}\gamma)^2 \le |z||\gamma|^4 \tag{4.53}$$

The free energy $P_s(\Lambda)$ is given by the series in (4.42) divided by $|\Lambda|$. We may derive it term by term since (4.46) holds. The lemma follows easily from the estimate (4.25) of lemma 4.1 and (4.46).

2. We give an expression of the expectation value of the local observable $\sigma(A)$, when the cluster expansion is convergent. We consider for example the state $\langle\cdot\rangle^{+}(\Lambda,s)$ of the phase of small contours with + b.c. We use a simple trick, which we learn from Kunz and Souillard. We define

$$\sigma_\gamma(t) = \begin{cases} -1 & \text{if } t\in\overline{\mathrm{int}\gamma} \\ +1 & \text{if } t\notin\overline{\mathrm{int}\gamma}\end{cases} \tag{4.54}$$

Let A be given. We introduce new weights for the (signed) contours,

$$\xi'(\gamma) = \prod_{t\in A}\sigma_\gamma(t)\xi(\gamma) \tag{4.55}$$

with $\xi(\gamma)$ given by (4.3). We can write the numerator of $\langle\sigma(A)\rangle^{+}(\Lambda,s)$ as

$$Z_A^{+}(\Lambda,s) = 1 + \sum_{n\ge1}\frac{1}{n!}\sum_{\substack{\gamma_1,\dots,\gamma_n:\\ \text{compatible}}}\prod_{k=1}^{n}\xi'(\gamma_k) \tag{4.56}$$

Notice that we have $\xi(\gamma) = \xi'(\gamma)$ if $A\cap\overline{\mathrm{int}\gamma} = \emptyset$. The weights are modified only locally. We have also a convergent cluster expansion with the new weights. Therefore

$$\langle\sigma(A)\rangle^{+}(\Lambda,s) = \exp\left(\sum_{n\ge1}\frac{1}{n!}\sum_{\substack{\gamma_1\\ \text{small}}}\cdots\sum_{\substack{\gamma_n\\ \text{small}}}\varphi_n^T(\gamma_1,\dots,\gamma_n)\left(\prod_{k=1}^{n}z'(\gamma_k) - \prod_{k=1}^{n}z(\gamma_k)\right)\right) \tag{4.57}$$

In the expression (4.57) all terms cancel in the sum unless there is a γ_i such that $A\cap\overline{\mathrm{int}\gamma}\ne\emptyset$. From (4.57) we see immediately that in the phase of small contours the expectation value $\langle\sigma(A)\rangle^{+}(\Lambda,s)$ is analytic in the magnetic field. When there is no magnectic field we have a similar expression for $\langle\sigma(A)\rangle^{+}(\Lambda)$, but in this case we do not need the restriction that the contours are small. It is very easy to compare the ratio of the expectation values computed with or without this restriction, since all terms in (4.57) appear in the analogous expression for $\langle\sigma(A)\rangle^{+}(\Lambda)$.

Lemma 4.3

Let Λ be a simply connected set, and let A be a finite subset of Λ. If there is no magnetic field and if J is large enough, then

$$|\langle\sigma(A)\rangle^{+}(\Lambda,s) - \langle\sigma(A)\rangle^{+}(\Lambda)| \le |A|O(exp(-8JL^s))\cdot\langle\sigma(A)\rangle^{+}(\Lambda) \tag{4.58}$$

5 An estimate of the large deviations of the magnetization in the phase of small contours.

We study the total magnetization in the phase of small contours for a system in a box Λ. The results presented here are based on lemma 4.2. We show how Chebyshev's inequality allows to control the large deviations and leads to Bernstein's inequality.

5.1 Chebyshev's inequality and large deviations.

Let $\sigma(t)$ be a *real-valued random variable* indexed by $t \in \Lambda$, Λ a finite set. The expectation value for these random variables is denoted by $\mathsf{E}(\cdot)$. The generating function of the cumulants of the random variables is $P(\Lambda|\mu)$,

$$\exp\left(|\Lambda|P(\Lambda|\mu)\right) := \mathsf{E}\left(\exp(\mu\sum_{t\in\Lambda}\sigma(t))\right) \tag{5.1}$$

In the rest of the paragraph we suppose that this function is well-defined and finite in some interval containing $\mu = 0$ as interior point. For those values of μ we can define a new probability law for the random variables $\sigma(t)$, by setting for an event A

$$\mathsf{E}(A|\mu) := \frac{\mathsf{E}\left(A\exp(\mu\sum_{t\in\Lambda}\sigma(t))\right)}{\mathsf{E}\left(\exp(\mu\sum_{t\in\Lambda}\sigma(t))\right)} \tag{5.2}$$

Of course $\mathsf{E}(\cdot|\mu=0) = \mathsf{E}(\cdot)$. By formal differentiation with respect to μ, we get the identities

$$|\Lambda|\frac{d}{d\mu}P(\Lambda|\mu) = \mathsf{E}(\sum_{t\in\Lambda}\sigma(t)|\mu) \tag{5.3}$$

and

$$|\Lambda|\frac{d^2}{d\mu^2}P(\Lambda|\mu) = \mathsf{E}\left(\left(\sum_{t\in\Lambda}\sigma(t) - \mathsf{E}(\sum_{t\in\Lambda}\sigma(t)|\mu)\right)^2 |\mu\right) \geq 0 \tag{5.4}$$

which are the mean value and the variance of the random variables with respect to the new probability law indexed by μ.

Lemma 5.1

Let $\sigma(t)$, $t \in \Lambda$, be a family of real-valued random variables indexed by the elements of the finite set Λ. Let $P(\Lambda|\mu)$ be defined on the interval $\mathrm{I} = [\mu_1, \mu_2]$, which contains the point $\mu = 0$. We suppose that $P(\Lambda|\mu)$ is of class $C^2(\mathrm{I})$ and that

$$\sup_{\mu\in\mathrm{I}}\frac{d^2}{d\mu^2}P(\Lambda|\mu) \leq C(\Lambda) < \infty \tag{5.5}$$

Let $M(\Lambda) = \mathsf{E}\left(\sum_{t\in\Lambda}\sigma(t)\right)$.

- *If $x/(|\Lambda|C(\Lambda)) \in \mathrm{I}$ and $-x/(|\Lambda|C(\Lambda)) \in \mathrm{I}$, then*

$$\mathsf{E}\left(\{|\sum_{t\in\Lambda}\sigma(t) - M(\lambda)| \geq x\}\right) \leq 2\exp\left(-\frac{x^2}{2|\Lambda|C(\Lambda)}\right). \tag{5.6}$$

- *If $x/(|\Lambda|C(\Lambda)) \notin \mathrm{I}$ or $x/(|\Lambda|C(\Lambda)) \notin \mathrm{I}$, then*

$$\mathsf{E}\left(\{|\sum_{t\in\Lambda}\sigma(t) - M(\lambda)| \geq x\}\right) \leq 2\exp\left(-\mu^* x + \frac{|\Lambda|C(\Lambda)}{2}(\mu^*)^2\right) \tag{5.7}$$

with $\mu^* = \min\{|\mu_1|, \mu_2\}$

Proof.

We estimate

$$Q_1 = \mathrm{Prob}(\{\sum_{t\in\Lambda}\sigma(t) - M(\Lambda) \geq x\}) \tag{5.8}$$

If $\mu \geq 0$, then we get by Chebyshev's inequality

$$\begin{aligned} Q_1 &\leq \exp(-\mu(M(\Lambda)+x))\cdot\mathsf{E}\left(\exp\mu\sum_{t\in\Lambda}\sigma(t)\right) \\ &= \exp(-\mu(M(\Lambda)+x) + |\Lambda|P(\Lambda|\mu)) \end{aligned} \tag{5.9}$$

We may write

$$\begin{aligned} P(\Lambda|\mu) &= P(\Lambda|0) + \frac{d}{d\mu}P(\Lambda|\mu=0)\cdot\mu \\ &+ 1/2\frac{d}{d\mu^2}P(\Lambda|\mu=\mu')\cdot\mu^2 \end{aligned} \tag{5.10}$$

for some μ', $0 \leq \mu' \leq \mu$. Here $P(\Lambda|0) = 0$ and $|\Lambda|\frac{d}{d\mu}P(\Lambda|\mu=0) = M(\Lambda)$. Thus

$$Q_1 \leq \exp(-\mu x + 1/2\mu^2|\Lambda|C(\Lambda)) \tag{5.11}$$

We look for the best choice of μ. In the first case the best choice is $\mu = x/(|\Lambda|C(\Lambda))$ and in the second case the best choice is μ_2. Similarly we estimate

$$\begin{aligned} Q_2 &= \mathrm{Prob}(\{\sum_{t\in\Lambda}\sigma(t) - M(\Lambda) < -x\}) \\ &\leq \exp(-\mu(M(\Lambda)-x))\cdot\mathsf{E}\left(\exp\mu\sum_{t\in\Lambda}\sigma(t)\right), \quad \mu \leq 0 \end{aligned} \tag{5.12}$$

and we get

$$Q_2 \leq \exp(\mu x + 1/2\mu^2|\Lambda|C(\Lambda)) \tag{5.13}$$

The best choice of μ in the first case is $-x/(|\Lambda|C(\Lambda))$ and in the second case, $\mu = \mu_1$.

We apply lemma 5.1 to the random variables $\sigma(t)$ of the Ising model when $\mathsf{E}(\cdot)$ is the expectation value corresponding to the probability measure of the phase of s-small contours without magnetic field. We express the results by introducing explicitely the inverse temperature β : We replace J by β which is equivalent to choose the coupling constant of the two-body interaction equal to one. Let Λ be a subset of $\mathbf{Z}^2$. For each connected component of Λ we have either + b.c. or − b.c. The only hypothesis on Λ, as in section 4.2, is that there is a bijection between the set of configurations of the model, compatible with the boundary conditions on Λ, and the set of all Λ^*- compatible families of small contours in Λ^*. This hypothesis allows to use the method of the cluster expansion and to apply lemma 4.2. The expectation value $\mathsf{E}(\cdot)$ is $\langle\cdot\rangle(\Lambda,s)$ and the expectation value $\mathsf{E}(\cdot|\mu)$ is $\langle\cdot\rangle(\Lambda,s|\mu)$, which is the expectation value in the phase of small contours with $J=\beta$ and $h=\mu$. We estimate

$$\text{Prob}\left(\{|\sum_{t\in\Lambda}\sigma(t)-\langle\sigma(t)\rangle(\Lambda,s)|\geq\epsilon|\Lambda|\}\right) \tag{5.14}$$

If β is large enough, then we can continue analytically the function $P(\Lambda,\mu)$ from $\mu=0$ up to $|\mu|\leq\mu^*$, with

$$\mu^* = \frac{4\beta}{s}\theta\ ,\ 0<\theta<1 \tag{5.15}$$

where θ is some fixed number. The constant $C(\Lambda)$ of lemma 5.1 is estimated using lemma 4.2,

$$C(\Lambda) = \overline{\chi}(\beta) \tag{5.16}$$

with

$$\overline{\chi}(\beta) = O(\exp(-8(1-\theta^*)\beta)) \tag{5.17}$$

For any ϵ we have

$$\frac{\epsilon|\Lambda|}{|\Lambda|C(\Lambda)} = \frac{\epsilon}{\overline{\chi}(\beta)} \tag{5.18}$$

and the value of this quotient is smaller or greater than μ^*, depending on the value of β. Notice that for large β it is always greater than μ^*

Theorem 5.1

Let Λ be a bounded set. For each connected component of Λ we have either + b.c. or − b.c. and we suppose that there is a one-to-one correspondence between the set of configurations of the model, compatible with the boundary condition for Λ, and the set of all Λ^*- compatible families of small contours in Λ^*. Let $\langle\cdot\rangle(\Lambda,s)$ be the measure of the phase of s-small contours at inverse temperature β and without magnetic field. Let θ, $0<\theta<1$ be given and let $\overline{\chi}(\beta)$ be the function of (5.17). There exists β_0,

independent on Λ and s, such that for $\beta>\beta_0$ the following statements are true.

- *If $\epsilon/\overline{\chi}(\beta)\leq 4\beta\theta/s$, then*

$$\text{Prob}\left(\{|\sum_{t\in\Lambda}\sigma(t)-\langle\sigma(t)\rangle(\Lambda,s)|\geq\epsilon|\Lambda|\}\right)\leq 2\exp\left(-\frac{\epsilon^2}{2\overline{\chi}(\beta)}|\Lambda|\right) \tag{5.19}$$

- *If $\epsilon/\overline{\chi}(\beta)>4\beta\theta/s$, then*

$$\text{Prob}\left(\{|\sum_{t\in\Lambda}\sigma(t)-\langle\sigma(t)\rangle(\Lambda,s)|\geq\epsilon|\Lambda|\}\right)\leq \tag{5.20}$$
$$2\exp\left(-\epsilon\frac{4\beta\theta}{s}|\Lambda|\left(1-2\frac{\beta\overline{\chi}(\beta)\theta}{s}\right)\right)$$

All probabilities are computed with the measure $\langle\cdot\rangle(\Lambda,s)$.

Remarks.

1) The same results hold if we choose the other definition of small contours mentioned in remark 3 following theorem 4.1.

2) In the next sections we apply the second part of this theorem when $|\Lambda|=L^2$, $s=L^a$ and $\epsilon=C_1/L^c$, with $c=1-a$. Let θ' be some fixed number, $0<\theta'<1$. If $1<c<1/2$, then there exist β_0 and L_0 such that for all $\beta>\beta_0$ and $L>L_0$,

$$\text{Prob}\left(\{|\sum_{t\in\Lambda}\sigma(t)-\langle\sigma(t)\rangle(\Lambda,s)|\geq C_1/L^c\cdot|\Lambda|\}\right)\leq \tag{5.21}$$
$$2\exp(-4C_1\beta\theta' L)$$

If $1/2<c<1$, then there exist β_0 and $L_0(\beta)$ such that for all $\beta>\beta_0$ and $L>L_0(\beta)$,

$$\text{Prob}\left(\{|\sum_{t\in\Lambda}\sigma(t)-\langle\sigma(t)\rangle(\Lambda,s)|\geq C_1/L^c\cdot|\Lambda|\}\right)\leq \tag{5.22}$$
$$2\exp\left(-\frac{C_1^2}{2\overline{\chi}(\beta)L^{2c}}|\Lambda|\right)$$

6 Surface tension.

The main topic of this section is the study of the surface tension for the 2- dim. Ising model. The surface tension is a basic thermodynamical quantity when there is coexistence of several phases. It determines in particular the shape of a macroscopic droplet of one phase in presence of the other phase. The surface tension is non zero only in the coexistence region of the phase diagram. Consequently we always suppose that there is no magnetic field in this and subsequent sections. We introduce explicitely the inverse temperature β and choose $J = 1$ for the coupling constant of the model. The main tools which we use are correlation inequalities and duality. The notion of duality was introduced by Krammer and Wannier and used to determine the critical temperature of the Ising model. In this form the duality is the statement that some properties of the two-dimensional Ising model below the critical temperature are related to other properties at high-temperature. Later on Wegner [W] introduced the modern notion of duality for spin systems defined on a cell-complex. In these lectures we need only Krammer-Wannier duality which is defined in section 6.1. We define the surface tension in section 6.2 and proves its existence in section 6.3. The last section 6.4 contains two estimates on probabilities of events, which are expressed in terms of large contours. The estimates in 6.4.1 and 6.4.2 are basic estimates for sections 7 and 8.

6.1 Duality transformation.

Let Λ be some finite box with + b.c. We define $Z^+(\Lambda)$ as in section 4, by normalizing the energy so that the configuration $\sigma(t) \equiv 1$ has energy zero. The configurations compatible with + b.c. are uniquely described by sets $\underline{\gamma}$ of closed contours, which are Λ^+-compatible (see section 2.1.2). Here we do not introduce the signed contours because we have no magnetic field. The partition function is

$$Z^+(\Lambda) = \sum_{\substack{\underline{\gamma}: \\ \Lambda^+-\text{compatible}}} \exp(-2\sum_{\gamma\in\underline{\gamma}} |\gamma|) \tag{6.1}$$

Lemma 6.1

Let Λ be simply connected. Then any family of Λ^*-compatible contours is Λ^+-compatible and vice-versa.

Proof.

Let $\underline{\gamma}$ be a Λ^*-compatible family. We first consider the outer contours of $\underline{\gamma}$, say $\gamma_1, \ldots, \gamma_p$. We construct a spin configuration $\hat{\sigma}$. Since Λ is simply connected $\overline{\text{int}\gamma_i} \cap \Lambda = \overline{\text{int}\gamma_i}$, and we can define a configuration σ which is compatible with the + b.c. by setting $\sigma(t) = +1$ if $t \notin \overline{\text{int}\gamma_i}$, $i = 1, \ldots p$ and $\sigma(t) = -1$ if $t \in \overline{\text{int}\gamma_i}$, $i = 1, \ldots p$. The value of $\hat{\sigma}$ is $\hat{\sigma}(t) = \sigma(t)$ for all $t \notin \overline{\text{int}\gamma_i}$, $i = 1, \ldots p$. Then we consider the contours, if any, which are in the interior of γ_1. If there is no contour then the value of the configuration $\hat{\sigma}$ is $\hat{\sigma}(t) = \sigma(t)$ for $t \in \overline{\text{int}\gamma_1}$. If there are some contours in $\text{int}\gamma_1$, say $\gamma_{p+1}, \ldots, \gamma_q$, then we consider the outer contours among these contours, say $\gamma_{p+1}, \ldots, \gamma_r$. We define $\hat{\sigma}(t) = \sigma(t)$ for all $t \in \text{int}\gamma_1 \setminus \cup_{j=p+1}^r \text{int}\gamma_j$ and we change the sign of $\sigma(t)$ for all $t \in \cup_{j=p+1}^r \text{int}\gamma_j$. Iterating this procedure we finally get the configuration $\hat{\sigma}$.

Given Λ, we construct Λ^* as in section 2.1.1. On this set we define the Ising model with *free boundary condition* (f-b.c.) by

$$H^f_{\Lambda^*} = -1/2 \sum_{\substack{t,t'\in\Lambda^* \\ d_1(t,t')=1}} \sigma(t)\sigma(t') \tag{6.2}$$

The inverse temperature of this model is β^*. The *high-temperature expansion* for the partition function of this model is equal to

$$\sum_{\text{spin conf}} \prod_{\substack{\{t,t'\} \\ d_1(t,t')=1}} \exp(\beta^*\sigma(t)\sigma(t')) = \tag{6.3}$$

$$\sum\prod(\cosh\beta^* + \sigma(t)\sigma(t')\sinh\beta^*) =$$

$$(\cosh\beta^*)^{\sharp\text{edges}}\sum\prod(1+\sigma(t)\sigma(t')\tanh\beta^*)$$

We expand the product in (6.3). Each term of the expansion is labelled by a set of edges on Λ^*, which we decompose into connected components $\gamma_1, \gamma_2, \ldots$. If a term is such that the corresponding components $\gamma_1, \ldots$ have no boundary (see section 2.1.1), then each spin variable of the term occurs an even number of times. Since $\sigma(t)^2 = 1$, the contribution of this term to (6.3) is

$$(\tanh\beta^*)^{\sum_i|\gamma_i|} \tag{6.4}$$

The summation over the configurations is trivial in this case and equal to $2^{|\Lambda^*|}$. All other terms do not contribute to the sum because at least one spin variable occurs an odd number of times. Let us normalize the partition function

$$Z^f(\Lambda^*|\beta^*) := \sum_{\substack{\underline{\gamma}: \\ \text{compatible on } \Lambda^*}} (\tanh\beta^*)^{\sum_{\gamma\in\underline{\gamma}}|\gamma|} \tag{6.5}$$

Here the notion of compatibility is the Λ^*-compatibility. We can state the duality theorem of Krammer-Wannier.

Theorem 6.1

Let Λ be a finite set of $\mathbb{Z}^2$ which is simply connected. Let Λ^* be the dual set of Λ and let $Z^+(\Lambda|\beta)$, resp. $Z^f(\Lambda^*|\beta^*)$, be the normalized partition functions defined above. If β and β^* are in duality i.e. if

$$\tanh\beta^* = e^{-2\beta} \tag{6.6}$$

then

$$Z^+(\Lambda) = Z^f(\Lambda^*) \tag{6.7}$$

The theorem is a direct corollary of lemma 6.1.

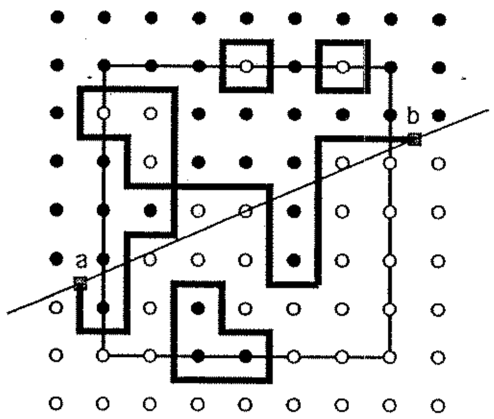


Figure 2: the n-boundary condition

6.2 Surface tension.

Physically the surface tension is the contribution to the free energy coming from the coexistence of phases. We refer to the review [Pf.1] for more informations, in particular on the relations between surface tension, structure of interfaces and non translation invariant Gibbs states.

Let us consider a box $\Lambda(L,M)$ on $\mathbb{Z}^2$,

$$\Lambda(L,M) = \{t = (t_1,t_2) : -L < t_1 \leq L, -M < t_2 \leq M\} \tag{6.8}$$

with a new kind of boundary condition : n - b.c., where n is a unit vector of $\mathbb{R}^2$. Let $l(n)$ be the straight line of $\mathbb{R}^2$ passing through $(1/2, 1/2)$ and perpendicular to n. The n - b.c. is

$$\sigma(t) = \begin{cases} +1 & \text{if } t \notin \Lambda(L,M),\ t \text{ above or on } l(n) \\ -1 & \text{if } t \notin \Lambda(L,M),\ t \text{ below } l(n) \end{cases} \tag{6.9}$$

The idea behind this choice of boundary condition is simple. Let us suppose for simplicity that $l(n)$ passes through two points a and b of the dual lattice as in figure 2. We consider the ground states of the model in $\Lambda(L,M)$. They are characterized as follows. Let λ be a line on $\mathbb{Z}^2$ passing through a and b and of minimal length. All spins above λ have value $+1$ and all spins below λ have value -1. If the energy of the ground state for the $+$ b.c. is zero, then the energy of the ground states in $\Lambda(L,M)$ for the n - b.c. is $-2|\lambda|$. In general, there are several ground states, because there are several lines λ of minimal length. We expect that the typical

configurations for the n - b.c. are locally those of the $+$ phase or $-$ phase with some " interface" separating these regions, as λ separates the spins $\sigma(t) = 1$ and $\sigma(t) = -1$ in a ground-state configuration. It is easy to prove that all configurations in $\Lambda(L,M)$ with n - b.c. are in one-to-one correspondence with a set of disjoint contours such that

- there is a unique contour, say λ, which is <u>not</u> closed and going from a to b
- all other contours are closed.

Let

$$F^+(L,M) = -\ln Z^+(\Lambda(L,M)) \tag{6.10}$$

be the free energy of the model in $\Lambda(L,M)$ with $+$ b.c.. Then

$$F^+(L,M) = f \cdot |\Lambda(L,M)| + g^+ \cdot |\partial\Lambda(L,M)| + h^+(L,M) \tag{6.11}$$

where f is the bulk free energy which is *independent* on the choice of the boundary condition, g^+ is a surface free energy which depends strongly on the choice of the boundary condition, h^+ is a correction term, and $|\partial\Lambda(L,M)|$ is the length of the boundary of $\Lambda(L,M)$. The important fact is that

$$\lim_{\substack{L\to\infty \\ M\to\infty}} \frac{1}{|\partial\Lambda(L,M)|} |h^+(L,M)| = 0 \tag{6.12}$$

Similarly we have

$$F^-(L,M) = f \cdot |\Lambda(L,M)| + g^- \cdot |\partial\Lambda(L,M)| + h^-(L,M) \tag{6.13}$$

However, by symmetry

$$g^- = g^+ \text{ and } h^-(L,M) = h^+(L,M) \tag{6.14}$$

On the other hand if we choose the n-b.c.

$$F^n(L,M) = f \cdot |\Lambda(L,M)| + g^n \cdot |\partial\Lambda(L,M)| + h^n(L,M) \tag{6.15}$$

We do not expect that g^n is equal to g^+ or g^-, but since $g^+ = g^-$ we expect that the difference between g^n and g^+ is due only to the presence of the interface which is induced by the n - b.c. This is precisely what is called the surface tension, and we define

$$\tau(n|\Lambda(L,M)) := \frac{-1}{d_2(a,b)} \ln \frac{Z^n(\Lambda(L,M))}{Z^+(\Lambda(L,M))} \tag{6.16}$$

(d_2 is the Euclidean distance) and

$$\tau(n) = \lim_{\substack{L\to\infty \\ M\to\infty}} \tau(n|\Lambda(L,M)) \tag{6.17}$$

Notice that we do not divide (6.16) by $1/\beta$. The limits $L \to \infty$ and $M \to \infty$ in (6.17) can be taken in any order (see [F.P.1]). This is a non trivial fact because the interface is not rigid in dimension two, but fluctuates.

6.3 Existence of the surface tension.

In this paragraph we prove that $\tau(n)$ is well-defined. This is done via a basic identity which relates $\tau(n)$ to the mass-gap of the two-point function of the model at the dual temperature. For this reason, we consider more closely the two-point correlation function and more generally even-point correlation functions of the model with free boundary condition.

It is convenient to introduce a *contour model*. Let Λ be some (finite) subset of the lattice and let Λ^* be the dual set of Λ. A *configuration* of the model is a family of *disjoint* contours $\underline{\gamma} = (\gamma_1, \ldots, \gamma_n)$, not necessarily closed. We put

$$\delta\underline{\gamma} = \bigcup_{\gamma\in\underline{\gamma}} \delta\gamma \tag{6.18}$$

if $\underline{\gamma}$ is a configuration. The weight of a configuration is

$$w(\underline{\gamma}) = \prod_{\gamma\in\underline{\gamma}} (\tanh\beta^*)^{|\gamma|} \tag{6.19}$$

Let $\underline{\gamma}'$ be a configuration. We put

$$Z(\Lambda^*|\underline{\gamma}') := \sum_{\substack{\underline{\gamma}:\delta\underline{\gamma}=\emptyset \\ \underline{\gamma}\cup\underline{\gamma}' \text{ compatible}}} w(\underline{\gamma}) \tag{6.20}$$

and $Z(\Lambda^*) = Z(\Lambda^*|\emptyset)$. The notion of compatibility means here that the contours of $\underline{\gamma}\cup\underline{\gamma}'$ are disjoint two by two.

We study the even-correlation functions of the model with free boundary condition defined on Λ^*. Let A be a subset of sites of Λ^*, $|A|$ even. Let $\sigma(A) = \prod_{t\in A}\sigma(t)$. We consider the numerator of $\langle\sigma(A)\rangle^f(\Lambda^*)$,

$$\sum_{\text{spin conf.}} \sigma(A)\exp\left(\beta^*/2 \sum_{\substack{t,t'\in\Lambda^*: \\ d_1(t,t')=1}} \sigma(t)\sigma(t')\right) \tag{6.21}$$

Up to a constant factor, $(\cosh\beta^*)^{\sharp(\text{edges})}\cdot 2^{\sharp(\text{sites})}$, (6.21) is equal to

$$\sum_{\substack{\underline{\gamma}: \\ \delta\underline{\gamma}=A}} w(\underline{\gamma})Z(\Lambda^*|\underline{\gamma}) \tag{6.22}$$

The proof of (6.22) is similar to the proof of (6.5). Notice that $Z^f(\Lambda^*) = Z(\Lambda^*)$. Thus

$$\langle\sigma(A)\rangle^f(\Lambda^*) = Z(\Lambda^*)^{-1}\cdot \sum_{\substack{\underline{\gamma}: \\ \delta\underline{\gamma}=A}} w(\underline{\gamma})Z(\Lambda^*|\underline{\gamma}) \tag{6.23}$$

Remark.
If $|A|$ is odd, then $\langle\sigma(A)\rangle^f(\Lambda^*) = 0$ because the number of points of $\delta\gamma$ is always even, so that there is no γ with $\delta\gamma = A$. If Λ^* is not connected, and if in a connected component of Λ^* there is an odd number of points of A, then again $\langle\sigma(A)\rangle^f(\Lambda^*) = 0$.

Lemma 6.2

Let Λ be a subset of $\mathbf{Z}^2$ and let Λ^* be its dual set. Let $A \subset \Lambda^*$ be an even subset of Λ^*. Then the correlation function $\langle\sigma(A)\rangle^f(\Lambda^*)$ of the Ising model on Λ^* with free b.c. is expressed in the contour model by

$$\langle\sigma(A)\rangle^f(\Lambda^*) = Z(\Lambda^*)^{-1}\cdot \sum_{\substack{\underline{\gamma}: \\ \delta\underline{\gamma}=A}} w(\underline{\gamma})Z(\Lambda^*|\underline{\gamma}) \tag{6.24}$$

We introduce the notion of massgap $\alpha(m)$ for the two-point function. Let m be some unit vector of $\mathbf{R}^2$ and let $l^*(m)$ be the straight line passing through $(1/2, 1/2)$ and of the direction m. We suppose that m is such that $l^*(m)$ contains at least two points of the dual lattice. By Griffiths' inequalities

$$\lim_{\Lambda^*\uparrow\mathbf{Z}^2_*} \langle\sigma(A)\rangle^f(\Lambda^*) = \langle\sigma(A)\rangle^f \tag{6.25}$$

exists, because $\langle\sigma(A)\rangle^f(\Lambda^*)$ is a monotonous function of Λ^*,

$$\langle\sigma(A)\rangle^f(\Lambda_1^*) \le \langle\sigma(A)\rangle^f(\Lambda_2^*) \quad \Lambda_1^*\subset\Lambda_2^* \tag{6.26}$$

Let q_0 be the point $(1/2, 1/2)$ and q be another point of the dual lattice on $l^*(m)$. The *massgap* $\alpha(m)$ is by definition

$$\alpha(m) := \lim_{d_2(q_0,q)\to\infty} -\frac{1}{d_2(q_0,q)}\ln\langle\sigma(q_0)\sigma(q)\rangle^f \tag{6.27}$$

Let q_1 be a point of $\mathbf{Z}^2_*$ and of $l^*(m)$, which is at a minimal distance from q_0. We can write $q_1 = q_0 + p_1$ with $p_1 \in \mathbf{Z}^2$. Let p_r be the point obtained by multiplying the coordinates of p_1 by the positive integer r. We set $q_r = q_0 + p_r$. We have

$$\begin{aligned} \alpha(m) &= \lim_{r\to\infty} -\frac{\ln\langle\sigma(q_0)\sigma(q_r)\rangle^f}{r d_2(q_0,q_1)} \\ &\equiv \lim_{r\to\infty}\frac{1}{r}G(r) \end{aligned} \tag{6.28}$$

By Griffiths' inequalities and translation invariance the function G is subadditive,

$$G(r_1+r_2) \le G(r_1) + G(r_2) \tag{6.29}$$

Indeed,

$$\begin{aligned} \langle\sigma(q_0)\sigma(q_{r_1+r_2})\rangle^f &= \langle\sigma(q_0)\sigma(q_{r_1})\sigma(q_{r_1})\sigma(q_{r_1+r_2})\rangle^f \\ &\ge \langle\sigma(q_0)\sigma(q_{r_1})\rangle^f\cdot\langle\sigma(q_{r_1})\sigma(q_{r_1+r_2})\rangle^f \\ &= \langle\sigma(q_0)\sigma(q_{r_1})\rangle^f\cdot\langle\sigma(q_0)\sigma(q_{r_2})\rangle^f \end{aligned} \tag{6.30}$$

Therefore the mass-gap $\alpha(m)$ is well-defined,

$$\lim_{r\to\infty}\frac{1}{r}G(r) = \inf_r \frac{1}{r}G(r) \tag{6.31}$$

and in particular for any r

$$\alpha(m) \leq \frac{1}{r} G(r) \tag{6.32}$$

Lemma 6.3

Let n be some unit vector of $\mathbb{R}^2$, such that the line $l(n)$ contains at least two points of $\mathbb{Z}^2_*$. Let n^* be a unit vector of $\mathbb{R}^2$ such that $l^*(n^*) = l(n)$. Then

$$\tau(n) = \alpha(n^*) \tag{6.33}$$

For any p, q on the dual lattice

$$\langle \sigma(p)\sigma(q)\rangle^f \leq \exp(-d_2(p,q) \cdot \alpha(n^*_{p,q})) \tag{6.34}$$

where $n^*_{p,q}$ is the unit vector giving the direction of the straight line passing through p and q.

Proof.

We follow the proof of [B.L.P.1]. The definition of $\tau(n)$ is given in (6.17). Let us suppose, that the points q_r are defined as above and that $q_{-r} = -p_r + q_0$. We also suppose that the points a and b in figure 2 are q_{-1} and q_1. We set

$$\Lambda_1 = \Lambda = \{t : -L < t_1 \leq L, -M < t_2 \leq M\} \tag{6.35}$$

and

$$\Lambda_r = \{t : -rL < t_1 \leq rL, -rM < t_2 \leq rM\} \tag{6.36}$$

We prove that

$$\lim_{r\to\infty} -\frac{1}{d_2(q_{-r}, q_r)} \ln \frac{Z^n(\Lambda_r)}{Z^+(\Lambda_r)} = \alpha(n^*) \tag{6.37}$$

We can write, using lemma 6.2,

$$\frac{Z^n(\Lambda_r)}{Z^+(\Lambda_r)} = \langle \sigma(q_{-r})\sigma(q_r)\rangle^f (\Lambda^*_r) \tag{6.38}$$

By Griffiths' inequalities

$$\langle \sigma(q_{-r})\sigma(q_r)\rangle^f (\Lambda^*_r) \leq \langle \sigma(q_{-r})\sigma(q_r)\rangle^f \tag{6.39}$$

so that

$$\liminf_{r\to\infty} -\frac{1}{r d_2(q_{-1}, q_1)} \ln \frac{Z^n(\Lambda_r)}{Z^+(\Lambda_r)} \geq \alpha(n^*) \tag{6.40}$$

On the other hand if $s \in \mathbb{N}$, we have

$$\begin{aligned}
\langle \sigma(q_{-sr})\sigma(q_{sr})\rangle^f (\Lambda^*_{sr}) &= \left\langle \prod_{i=-r+1}^{r} \sigma(q_{(i-1)s})\sigma(q_{is}) \right\rangle^f (\Lambda^*_{sr}) \\
&\geq \prod_{i=-r+1}^{r} \left\langle \sigma(q_{(i-1)s})\sigma(q_{is}) \right\rangle^f (\Lambda^*_{sr}) \\
&\equiv \exp(-\sum_i G(i,r,s))
\end{aligned} \tag{6.41}$$

Thus, for any $r' = sr + t$, $0 \leq t < s$,

$$\begin{aligned}
-\ln \langle \sigma(q_{-r'})\sigma(q_{r'})\rangle^f (\Lambda^*_{r'}) &\leq -\ln \langle \sigma(q_{-r'})\sigma(q_{-r'+t})\rangle^f (\Lambda^*_{r'}) \\
&\quad -\ln \langle \sigma(q_{r'-t})\sigma(q_{r'})\rangle^f (\Lambda^*_{r'}) + \sum_i G(i,r,s)
\end{aligned} \tag{6.42}$$

and

$$\begin{aligned}
\limsup_{r'\to\infty} -\frac{1}{r' d_2(q_{-1}, q_1)} \ln \langle \sigma(q_{-r'})\sigma(q_{r'})\rangle^f (\Lambda^*_{r'}) &\leq \\
\lim_{r\to\infty} \frac{1}{2r} \sum_{i=-r+1}^{r} \frac{G(i,r,s)}{d_2(q_{-0}, q_s)}
\end{aligned} \tag{6.43}$$

Let $\epsilon > 0$ be given. Then from (6.26) there exists $\delta > 0$ such that

$$\left| \left\langle \sigma(q_{is})\sigma(q_{(i-1)s}) \right\rangle^f (\Lambda^*_{sr}) - \left\langle \sigma(q_{is})\sigma(q_{(i-1)s}) \right\rangle^f \right| \leq \epsilon \tag{6.44}$$

provided $d_2(q_{is}, \partial\Lambda^*_{sr}) \geq \delta$ and $d_2(q_{(i-1)s}, \partial\Lambda^*_{sr}) \geq \delta$. Since $\left\langle \sigma(q_{is})\sigma(q_{(i-1)s}) \right\rangle^f = \left\langle \sigma(q_0)\sigma(q_{(s)}) \right\rangle^f$ and $\left\langle \sigma(q_0)\sigma(q_{(s)}) \right\rangle^f > 0$ we get for small ϵ

$$\begin{aligned}
\limsup_{r'\to\infty} -\frac{1}{r' d_2(q_{-1}, q_1)} \ln \langle \sigma(q_{-r'})\sigma(q_{r'})\rangle^f (\Lambda^*_{r'}) &\leq \\
-\frac{1}{d_2(q_0, q_s)} \left(\ln \langle \sigma(q_0)\sigma(q_s)\rangle^f + O(\epsilon) \right)
\end{aligned} \tag{6.45}$$

Since ϵ is arbitrarily small we have proven the existence of $\tau(n)$ for a special sequence of boxes. We do not prove here that the limits $L \to \infty$ and $M \to \infty$ in (6.17) can be done in any manner . The proof (for a similar case) is given in [F.P.1] and uses again in an essential way Griffiths' inequalities. The second statement of lemma 6.3 is simply (6.32).

Lemma 6.4

There exists a constant C_1 such that for any L, M, and unit vectors n, n'

$$|\tau(n|\Lambda(L,M)) - \tau(n'|\Lambda(L,M))| \leq C_1 |\varphi(n,n')| \tag{6.46}$$

where $\varphi(n,n')$ is the interior angle between n and n'.

Proof.

By inspection, the difference of energy of any spin configuration in $\Lambda(L,M)$ computed with the n- b.c. and the n'-b.c. is smaller than

$$C_1'|\varphi(n,n')|\cdot L \qquad (6.47)$$

with C_1' independent on L, M and the configuration. From (6.47) the result follows easily.

Lemma 6.5

For any unit vector n the limit

$$\lim_{\substack{L\to\infty \\ M\to\infty}} \tau(n|\Lambda(L,M)) = \tau(n) \qquad (6.48)$$

exists and is a continuous function of n.

If we extend τ as a function defined on $\mathbb{R}^2$ by setting

$$\tau(x) = |x|\tau(x/|x|) \qquad (6.49)$$

then τ is a norm for $\beta > \beta_c$

Proof.

Lemma 6.4 allows to define $\tau(n)$ for any n by continuity using the fact that for a dense set of n $\tau(n)$ exists (lemma 6.3) The second part of lemma 6.5 is a consequence of Griffiths' inequalities. Let x_1 and x_2 be two fixed vectors of $\mathbb{R}^2$. We have

$$\begin{aligned}\langle\sigma(0)\sigma(x_1+x_2)\rangle^f &\geq \langle\sigma(0)\sigma(x_1)\rangle^f\cdot\langle\sigma(x_1)\sigma(x_1+x_2)\rangle^f \\ &= \langle\sigma(0)\sigma(x_1)\rangle^f\cdot\langle\sigma(0)\sigma(x_2)\rangle^f\end{aligned} \qquad (6.50)$$

Let r be some positive number. Then

$$\begin{aligned}&\tau(x_1+x_2) = \lim_{r\to\infty} -\frac{1}{r}\ln\langle\sigma(0)\sigma(rx_1+rx_2)\rangle^f \leq \\ &\lim_{r\to\infty} -\frac{1}{r}\ln\langle\sigma(0)\sigma(rx_1)\rangle^f + \lim_{r\to\infty} -\frac{1}{r}\ln\langle\sigma(0)\sigma(rx_2)\rangle^f = \\ &\tau(x_1)+\tau(x_2)\end{aligned} \qquad (6.51)$$

For the positivity of $\tau(x)$ see comment 3) below.

Comments.

1) The surface tension of the two-dimensional case is related to the behavior of a random line. We can study by the same method the surface tension of the three-dimensional model. Here the role of the contour model is played by the $\mathbb{Z}_2$-gauge model which is a model of random surfaces. Such models are more difficult to analyze, however since Griffiths' inequalities are still valid the proof of the existence of the surface tension is essentially the same as the one given above [Pf.3].

2) If we want to study with more details the surface tension then we must analyze the statistical properties of the random line λ passing through a and b (see figure 2). When a and b are on the same horizontal line this analysis has been done by Gallavotti [G] and extended in [B.L.P.2] and in [B.F]. When a and b are not on the same horizontal or vertical line then a similar analysis can be done, but this is more difficult. This analysis is part of the work of Dobrushin, Kotecky, Shlosman . In section 7, we need one result of their analysis, which is quoted in lemma 7.1.

3) We mention that it can be proven that the surface tension $\tau(n) = \tau(n|\beta)$ is non negative and positive if and only if $\beta > \beta_c$, where β_c is the critical point of the model which is given by the self-duality relation $\tanh\beta_c = \exp(-2\beta_c)$. (See [L.P] and the review [Pf.1]. The proof in [L.P] is given for a special case, but can be extended to the general case using (6.51) and the monotonicity properties of the two-point function.)

6.4 Two basic estimates.

We discuss two types of estimates, which play an essential role in the next sections

6.4.1.

We consider the following situation. Let γ^* be some closed contour, which is fixed. Let γ be another closed contour which contains γ^* as a connected subset. Any contour γ of this type can be uniquely decomposed into γ^* and a family of closed disjoint contours $\eta_1,\ldots,\eta_k$ such that each η_i has at least one site in common with γ^*, but has no edge in common with γ^*. Conversely γ^* and any family $(\eta_1,\ldots,\eta_k)$ with the above properties define a contour γ, which is the union of γ^* and of the contours η. We denote by $C(\gamma^*)$ the set of all such contours. We also denote by $C(\gamma^*|q)$ the subset of $C(\gamma^*)$ with $|\eta|\leq q$ for all η. We define

$$\mathrm{Prob}_\Lambda^+(C(\gamma^*)) = \sum_{\gamma\in C(\gamma^*)} \mathrm{Prob}_\Lambda^+(\gamma) \qquad (6.52)$$

and similarly $\mathrm{Prob}_\Lambda^+(C(\gamma^*|q))$.

Lemma 6.6

Let Λ be a simply connected and finite set. Let $C(\gamma^*)$ and $C(\gamma^*|q)$ be as above. Then for β large enough

$$\begin{aligned}&\mathrm{Prob}_\Lambda^+(C(\gamma^*)) = \\ &\mathrm{Prob}_\Lambda^+(C(\gamma^*|q))\cdot\exp(|\gamma^*|O_{\Lambda,\gamma^*}(e^{-2\beta q}))\end{aligned} \qquad (6.53)$$

with the function $O_{\Lambda,\gamma^*}(e^{-2\beta q})$ such that

$$\sup_{\Lambda,\gamma^*}|O_{\Lambda,\gamma^*}(e^{-2\beta q})|\cdot e^{2\beta q} \leq \mathrm{Const} \qquad (6.54)$$

Proof.

Let $\gamma = (\gamma^*, \eta_1, \dots, \eta_k)$ be an element of $\mathcal{C}(\gamma^*)$. Then

$$\mathrm{Prob}^+_\Lambda(\gamma) = \exp\left(-2\beta(|\gamma^*| + \sum |\eta|)\right) \frac{Z(\gamma)}{Z^+(\Lambda)} \tag{6.55}$$

where $Z(\gamma) \equiv Z(\gamma^*; \eta_1, \dots, \eta_k)$ is the partition function obtained by summing over the following subset of spin configurations which are conveniently described by contours : each spin configuration is in one-to-one correspondence with the set of compatible families of contours $(\theta_1, \dots, \theta_n)$ such that $(\gamma, \theta_1, \dots, \theta_n)$ is still a compatible family. If we take the union of these sets over all possible $\underline{\eta} = (\eta_1, \dots, \eta_k)$, k arbitrary, then we get a set $\mathcal{E}(\gamma^*)$ of configurations which is in one-to-one correspondence with the set of families of contours $(\eta_1, \dots, \eta_k, \theta_1, \dots, \theta_n)$ such that

- $\eta_1, \dots, \eta_k$ are disjoint two by two.
- the union of γ^*, $\eta_1, \dots, \eta_k$ is a single contour γ
- $\{\gamma, \theta_1, \dots, \theta_n\}$ is a Λ^*-compatible family of contours.

Notice that necessarily $(\eta_1, \dots, \eta_k, \theta_1, \dots, \theta_n)$ is a Λ^*-compatible family of closed contours. The partition function which we get by summing over the configurations of $\mathcal{E}(\gamma^*)$ is denoted by $\hat{Z}(\gamma^*)$. Similarly $\hat{Z}(\gamma^*|q)$ is the partition function which we get by summing over the configurations of $\mathcal{E}(\gamma^*)$ with all contours η such that $|\eta| \le q$. We have

$$\mathrm{Prob}^+_\Lambda(\mathcal{C}(\gamma^*)) = \exp(-2\beta|\gamma^*|) \cdot \frac{\hat{Z}(\gamma^*)}{Z^+_\Lambda} \tag{6.56}$$

and we can apply a cluster expansion for $\hat{Z}(\gamma^*)$,

$$\hat{Z}(\gamma^*) = \sum_{(\eta_1,\dots,\eta_k)} e^{-2\beta|\eta_1|} \cdots e^{-2\beta|\eta_k|} Z(\gamma^*; \eta_1, \dots, \eta_k) = \tag{6.57}$$

$$\exp\left(\sum_{n\ge1} \frac{1}{n!} \sum_{\lambda_1 \in \Omega(\gamma^*)} \cdots \sum_{\lambda_n \in \Omega(\gamma^*)} \varphi^T_n(\lambda_1, \dots, \lambda_n) e^{-2\beta|\lambda_1|} \cdots e^{-2\beta|\lambda_n|}\right)$$

where $\Omega(\gamma^*)$ is the family of possible contours appearing in the configurations of $\mathcal{E}(\gamma^*)$. (A contour λ is in $\Omega(\gamma^*)$ if and only if the union of γ^* and λ forms a single contour or γ^* and λ are disjoint.) We have a similar expression for $\hat{Z}(\gamma^*|q)$. It is easy to take the ratio of $\hat{Z}(\gamma^*)$ and $\hat{Z}(\gamma^*|q)$: all terms in the arguments of the exponential functions cancel except those which contain at least one η with $|\eta| > q$. Thus

$$\frac{\hat{Z}(\gamma^*)}{\hat{Z}(\gamma^*|q)} = \exp(|\gamma^*| O_{\Lambda,\gamma^*}(e^{-2\beta q})) \tag{6.58}$$

Remark.
We can extend this result to a family of disjoint closed contours $\gamma_1^*, \dots, \gamma_p^*$. Let $\mathcal{C}(\gamma_1^*, \dots, \gamma_p^*)$ be the set of all families of p compatible contours $(\gamma_1, \dots, \gamma_p)$ such that $\gamma_i \supset \gamma_i^*$. Then

$$\mathrm{Prob}^+_\Lambda(\mathcal{C}(\gamma_1^*, \dots, \gamma_p^*)) = \tag{6.59}$$
$$\mathrm{Prob}^+_\Lambda(\mathcal{C}(\gamma_1^*, \dots, \gamma_p^*|q)) \cdot \exp((\sum_{i=1}^p |\gamma_i^*|) O_{\Lambda,\gamma_i^*}(e^{-2\beta q}))$$

In this case the set $\Omega(\gamma^*)$ appearing in (6.57) is replaced by $\Omega(\gamma_1^*, \dots, \gamma_p^*)$. A contour λ is in $\Omega(\gamma_1^*, \dots, \gamma_p^*)$ if and only if λ is disjoint from $\gamma_1^*, \dots, \gamma_p^*$ or there exists a γ_i^* so that the union of λ and γ_i^* is a closed contour, which is disjoint from all contours γ_j^*, $j \ne i$.

6.4.2.

We introduce two notions.

1. Let λ be an open contour and let A be a subset of the boundary of λ, $A \subset \delta\lambda$. We say that

- λ is *reductible at A* if we can decompose λ into λ', such that $\delta\lambda' = \delta\lambda$, and a closed contour γ with the property that $\lambda' \cap \gamma \subset A$ and $\lambda' \cup \gamma = \lambda$. If λ is not reductible at A it is called *irreductible at A*.

Remark.
If each point of A has incidence number one, then λ is necessarily irreductible at A.

2. Let λ be a contour with boundary $\delta\lambda = \{t_0^*, t_n^*\}$. We say that λ has a *decomposition with cutting points* $t_1, \dots, t_{n-1}$ if the following conditions are verified:

- there are n open contours $\lambda_1, \dots, \lambda_n$ with $\delta\lambda_i = \{t_{i-1}^*, t_i^*\}$, $i = 1, \dots, n$ and all points t_i are distinct
- $\lambda_i \cap \lambda_{i+1} = \{t_i^*\}$ and $\lambda_i \cap \lambda_j = \emptyset$ if $|i - j| > 1$.
- $\lambda = \lambda_1 \cup \cdots \cup \lambda_n$
- λ_i is irreductible at t_{i-1}^* for all $i = 2, \dots, n$.

Remark.
The last condition is important. It prevents to have overcounting problems in the proof below.

We also have a *decomposition with cutting points* for closed contours. The first three conditions are the same, with the obvious modifications in order to take into account that now $t_0 = t_n$. The last condition reads

- λ_i is irreductible at t^*_{i-1} for all $i = 2, \ldots, n-1$, and λ_n is irreductible at $\{t_{n-1}, t_n\}$.

Notice that there is no irreductibility condition on λ_1. As in section 6.3. we define

$$Z(\Lambda^*|\lambda) = \sum_{\substack{\underline{\gamma}:\delta\underline{\gamma}=\emptyset \\ \underline{\gamma}\cup\lambda \text{ comp.}}} \prod_{\gamma\in\underline{\gamma}} (\tanh\beta^*)^{|\gamma|} \tag{6.60}$$

Lemma 6.7

Let $t^*_0, t^*_1, \ldots, t^*_n$ be $n+1$ distinct points and let $\lambda = (\lambda_1, \ldots, \lambda_n)$ be a decomposition with cutting points $t^*_1, \ldots, t^*_{n-1}$ of the open contour λ such that $\delta\lambda = \{t^*_0, t^*_n\}$. Then

$$(Z(\Lambda^*))^{-1} \cdot \sum_{\substack{\lambda:\delta\lambda=\{t^*_0,t^*_n\} \\ t^*_1,\ldots,t^*_{n-1} \text{ cutting points}}} Z(\Lambda^*|\lambda)(\tanh\beta^*)^{|\lambda|} \leq \tag{6.61}$$

$$\prod_{k=1}^{n} \left\langle \sigma(t^*_{k-1})\sigma(t^*_k)\right\rangle^f (\Lambda^*|\beta^*) \leq \prod_{k=1}^{n} \left\langle \sigma(t^*_{k-1})\sigma(t^*_k)\right\rangle^f (\beta^*)$$

where in the last expression we have taken the thermodynamic limit. The same result holds if λ is closed, i.e. $t^*_0 = t^*_n$.

Proof.

Let $\lambda = \lambda(\lambda_1, \ldots, \lambda_n)$ be given. We suppose that $\lambda_2, \ldots, \lambda_n$ are kept fixed for the moment. From (6.23) we have

$$\sum_{\lambda_1} (\tanh\beta^*)^{|\lambda_1|} Z(\Lambda^*|\lambda_1, \ldots, \lambda_n) \leq \tag{6.62}$$

$$Z^f(\Lambda^*(\lambda_2, \ldots, \lambda_n)) \left\langle \sigma(t^*_0)\sigma(t^*_1)\right\rangle^f (\Lambda^*(\lambda_2, \ldots, \lambda_n))$$

where $Z^f(\Lambda^*(\lambda_2, \ldots, \lambda_n))$ is the partition function of the Ising model with free b.c. defined on the set $\Lambda^*(\lambda_2, \ldots, \lambda_n)$, which is obtained (as set of sites) by removing all sites of $\lambda_2, \ldots, \lambda_n$, except the point t^*_1. By Griffiths' inequalities

$$\begin{aligned} \left\langle \sigma(t^*_0)\sigma(t^*_1)\right\rangle^f (\Lambda^*(\lambda_2, \ldots, \lambda_n)) &\leq \left\langle \sigma(t^*_0)\sigma(t^*_1)\right\rangle^f (\Lambda^*) \\ &\leq \left\langle \sigma(t^*_0)\sigma(t^*_1)\right\rangle^f \end{aligned} \tag{6.63}$$

Therefore, we can put forward in the sum the factor $\left\langle \sigma(t^*_0)\sigma(t^*_1)\right\rangle^f (\Lambda^*)$ which is independent on $\lambda_2, \ldots, \lambda_n$. Let us sum over λ_2

$$\sum_{\lambda_2} Z^f(\Lambda^*(\lambda_2, \ldots, \lambda_n))(\tanh\beta^*)^{|\lambda_2|} \tag{6.64}$$

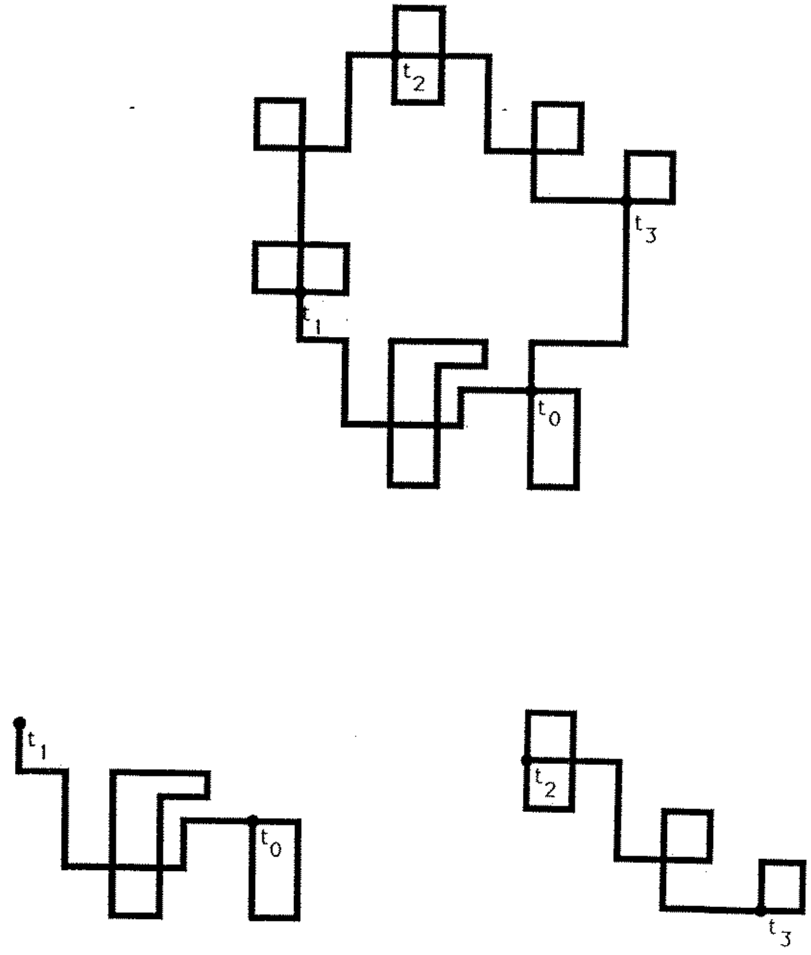


Figure 3: Decomposition of λ into four open contours $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ with cutting points t_1, t_2, t_3, t_4.

when $\lambda_3, \ldots, \lambda_n$ are fixed. Let $(\gamma_1, \ldots, \gamma_p)$ be a configuration of contours contributing to $Z^f(\Lambda^*(\lambda_2, \ldots, \lambda_n))$. All these contours are disjoint two by two and one of them at most may touch only at t_1^* the contour formed by the union of $\lambda_2, \ldots, \lambda_n$. For any λ_2 occuring in (6.64) we can interpret the union of $\gamma_1, \ldots, \gamma_p$ and λ_2 as a set of high-temperature contours contributing to

$$Z^f(\Lambda^*(\lambda_3, \ldots, \lambda_n)) \langle \sigma(t_1^*)\sigma(t_2^*)\rangle^f (\Lambda^*(\lambda_3, \ldots, \lambda_n)) \tag{6.65}$$

If one of the contours γ touches λ_2 we suppose that this is the contour γ_1. Thus we have $p-1$ closed contours, $\gamma_2, \ldots, \gamma_p$ and one open contour λ_2', which is the union of λ_2 and γ_1. In that case the open contour λ_2' is reductible at t_1^*. Therefore the contour λ_2' cannot occur in the sum (6.64), since all contours λ_2 in this sum are irreductible at t_1. Thus we can bound (6.64) by (6.65),

$$\begin{aligned} &\sum_{\lambda_2} Z^f(\Lambda^*(\lambda_2, \ldots, \lambda_n))(\tanh\beta^*)^{|\lambda_2|} \leq \\ &Z^f(\Lambda^*(\lambda_3, \ldots, \lambda_n)) \langle \sigma(t_1^*)\sigma(t_2^*)\rangle^f (\Lambda^*(\lambda_3, \ldots, \lambda_n)) \leq \\ &Z^f(\Lambda^*(\lambda_3, \ldots, \lambda_n)) \langle \sigma(t_1^*)\sigma(t_2^*)\rangle^f (\Lambda^*) \end{aligned} \tag{6.66}$$

By repeating this argument we get the proof of the lemma. For the second part of the lemma we have a similar proof, except that in the first step, and therefore in the subsequent steps, $\Lambda^*(\lambda_2, \ldots, \lambda_n)$ contains also the site t_0^*. The presence of a spin at t_0 modifies the proof only in the last step when we sum over λ_n. This is why we require that λ_n is irreducible at $\{t_{n-1}, t_0\}$.

Remark.
Lemma 6.7 can of course easily be generalized to the case where we have several disjoint contours, each having a decomposition with cutting points. In section 8 we also have to consider the following situation: two (or more) closed contours have a decomposition with cutting points, say $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ with cutting points t_1^*, t_2^*, t_3^*, and $\theta = (\theta_1, \theta_2, \theta_3)$ with cutting points s_1^*, s_2^*, s_3^* but they are not disjoint: λ and θ must go though a fixed common point p^*. Then summing over all decompositions (the cutting points are also fixed), we still get the upper bound

$$\begin{aligned} &\langle \sigma(s_1^*)\sigma(s_2^*)\rangle^f \cdot \langle \sigma(s_2^*)\sigma(s_3^*)\rangle^f \cdot \langle \sigma(s_3^*)\sigma(s_1^*)\rangle^f \cdot \\ &\langle \sigma(t_1^*)\sigma(t_2^*)\rangle^f \cdot \langle \sigma(t_2^*)\sigma(t_3^*)\rangle^f \cdot \langle \sigma(t_3^*)\sigma(t_1^*)\rangle^f \end{aligned} \tag{6.67}$$

Indeed, we can sum first over the decompositions of λ. The argument of the above proof is valid. Then we must sum over the decompositions of θ. Let us suppose that p^* is not one of the cutting point of θ, and that p^* belongs to θ_2. We sum over θ_1 and then we sum over θ_2,

$$\sum_{\theta_2} Z^f(\Lambda^*(\theta_2, \theta_3))(\tanh\beta^*)^{|\theta_2|} \tag{6.68}$$

Since p^* was a point of the contour λ, the set $\Lambda^*(\theta_2, \theta_3)$ contains the point p^*. Let $(\gamma_1, \ldots, \gamma_p)$ be a configuration of contours contributing to $Z^f(\Lambda^*(\theta_2, \theta_3))$. All these contours are disjoint two by two and one of them at most may touch θ_2 at p^*, and one of them at most may touch the contour θ_2 at s_2^*. It is possible that the same

contour touches the contour θ_2 at s_2^* and at p^*. For any θ_2 we can interpret the union of $\gamma_1, \ldots, \gamma_p$ and θ_2 as a set of high-temperature contours contributing to

$$Z^f(\Lambda^*(\theta_3)) \langle \sigma(s_2^*)\sigma(s_3^*)\rangle^f (\Lambda^*(\theta_3,)) \tag{6.69}$$

If one or two contours γ touch θ_2 we suppose that these are the contours γ_1 or γ_2. In this case we have $p-1$ or $p-2$ closed contours, $\gamma_k, \ldots, \gamma_p$, $k=2$ or $k=3$, and one open contour θ_2', which is the union of θ_2 and the contours γ_1 or γ_2. The open contour θ_2' is reductible at $\{s_2^*, p^*\}$. But, since the contours λ_2 and θ_2 had the point p^* in common, the contour θ_2 is irreductible at p^* and at s_2^* by definition. Thus we can apply the argument of the proof of lemma 6.7. If p^* is one cutting point, say s_2, then we use the fact that θ_1 is necessarily irreductible at p^*.

when $\lambda_3, \ldots, \lambda_n$ are fixed. Let $(\gamma_1, \ldots, \gamma_p)$ be a configuration of contours contributing to $Z^f(\Lambda^*(\lambda_2, \ldots, \lambda_n))$. All these contours are disjoint two by two and one of them at most may touch only at t_1^* the contour formed by the union of $\lambda_2, \ldots, \lambda_n$. For any λ_2 occuring in (6.64) we can interpret the union of $\gamma_1, \ldots, \gamma_p$ and λ_2 as a set of high-temperature contours contributing to

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By repeating this argument we get the proof of the lemma. For the second part of the lemma we have a similar proof, except that in the first step, and therefore in the subsequent steps, $\Lambda^*(\lambda_2, \ldots, \lambda_n)$ contains also the site t_0^*. The presence of a spin at t_0 modifies the proof only in the last step when we sum over λ_n. This is why we require that λ_n is irreductible at $\{t_{n-1}, t_0\}$.

Remark.
Lemma 6.7 can of course easily be generalized to the case where we have several disjoint contours, each having a decomposition with cutting points. In section 8 we also have to consider the following situation: two (or more) closed contours have a decomposition with cutting points, say $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ with cutting points t_1^*, t_2^*, t_3^*, and $\theta = (\theta_1, \theta_2, \theta_3)$ with cutting points s_1^*, s_2^*, s_3^* but they are not disjoint: λ and θ must go though a fixed common point p^*. Then summing over all decompositions (the cutting points are also fixed), we still get the upper bound

$$\begin{aligned}&\langle \sigma(s_1^*)\sigma(s_2^*)\rangle^f \cdot \langle \sigma(s_2^*)\sigma(s_3^*)\rangle^f \cdot \langle \sigma(s_3^*)\sigma(s_1^*)\rangle^f \cdot \\ &\langle \sigma(t_1^*)\sigma(t_2^*)\rangle^f \cdot \langle \sigma(t_2^*)\sigma(t_3^*)\rangle^f \cdot \langle \sigma(t_3^*)\sigma(t_1^*)\rangle^f\end{aligned} \tag{6.67}$$

Indeed, we can sum first over the decompositions of λ. The argument of the above proof is valid. Then we must sum over the decompositions of θ. Let us suppose that p^* is not one of the cutting point of θ, and that p^* belongs to θ_2. We sum over θ_1 and then we sum over θ_2,

$$\sum_{\theta_2} Z^f(\Lambda^*(\theta_2, \theta_3))(\tanh \beta^*)^{|\theta_2|} \tag{6.68}$$

Since p^* was a point of the contour λ, the set $\Lambda^*(\theta_2, \theta_3)$ contains the point p^*. Let $(\gamma_1, \ldots, \gamma_p)$ be a configuration of contours contributing to $Z^f(\Lambda^*(\theta_2, \theta_3))$. All these contours are disjoint two by two and one of them at most may touch θ_2 at p^*, and one of them at most may touch the contour θ_2 at s_2^*. It is possible that the same

contour touches the contour θ_2 at s_2^* and at p^*. For any θ_2 we can interpret the union of $\gamma_1, \ldots, \gamma_p$ and θ_2 as a set of high-temperature contours contributing to

$$Z^f(\Lambda^*(\theta_3)) \langle \sigma(s_2^*)\sigma(s_3^*)\rangle^f (\Lambda^*(\theta_3,)) \tag{6.69}$$

If one or two contours γ touch θ_2 we suppose that these are the contours γ_1 or γ_2. In this case we have $p-1$ or $p-2$ closed contours, $\gamma_k, \ldots, \gamma_p$, $k=2$ or $k=3$, and one open contour θ_2', which is the union of θ_2 and the contours γ_1 or γ_2. The open contour θ_2' is reductible at $\{s_2^*, p^*\}$. But, since the contours λ_2 and θ_2 had the point p^* in common, the contour θ_2 is irreductible at p^* and at s_2^* by definition. Thus we can apply the argument of the proof of lemma 6.7. If p^* is one cutting point, say s_2, then we use the fact that θ_1 is necessarily irreductible at p^*.

7 Lower bound on the probability of a large deviation of the magnetization.

We prove a lower bound for the probability of the event $A(m) = A(m; c, c_0)$

$$A(m; c, c_0) = \{\sigma : |\sum_{t\in\Lambda} \sigma(t) - m|\Lambda|\,| \leq c_0|\Lambda| \cdot L^{-c}\} \tag{7.1}$$

where Λ is a square box, $|\Lambda| = L^2$, with + b.c. and m is some fixed number,

$$-m^*(\beta) < m < m^*(\beta) \tag{7.2}$$

($m^*(\beta)$ is the spontaneous magnetization in the + phase). The parameter c is such that

$$0 < c < 1/2 \tag{7.3}$$

The probability is computed with the measure $\langle \cdot \rangle^+(\Lambda)$. We introduce an intermediate scale in the analysis, which allows to bound the probability of the event $A(m)$ in terms of the surface tension. This essential idea of Dobrushin, Kotecky and Shlosman gives an improvement of the work of Minlos and Sinai. Notice that we do not fix the total magnetization here. This is very natural from the point of view of Physics and simplifies slightly the mathematical analysis. Let W_τ be the Wulff crystal,

$$W_\tau = \{x \in \mathbb{R}^2 : (n|x) = \sum_{i=1}^{2} n_i x_i \leq \tau(n)\} \tag{7.4}$$

The volume of W_τ (in $\mathbb{R}^2$) is $|W_\tau|$. By a dilatation of W_τ we construct a Wulff droplet $W_\tau(m)$ of total volume

$$V(m) = \frac{m^* - m}{2m^*}|\Lambda| \equiv \alpha(m)|\Lambda| \tag{7.5}$$

The value of the Wulff functional for the Wulff droplet is $T^* = T^*(m)$ and is equal to

$$(T^*(m))^2 = 4|W_\tau| \cdot V(m) \tag{7.6}$$

Remark.
We suppose that the Wulff droplet $W_\tau(m)$ can be put inside the square box of volume $|\Lambda|$. It could happen that for small values of m satisfying (7.2) the Wulff droplet could not be put inside the square box. In this case we could take a box which has the Wulff shape and a volume L^2 in order that the results of this section remain true. Indeed, if the square box cannot contain the Wulff droplet $W_\tau(m)$, then the constant $T^*(m)$ must be modified in theorem 7.1 (the value of the constant is larger). We do not consider this possibility in these lectures.

Theorem 7.1

Let $-m^*(\beta) < m < m^*(\beta)$, $0 < c < 1/2$ and $c_0 > 0$. Let ϵ be given, $0 < \epsilon < 1$. Then there exist $\beta(\epsilon, c_0, c)$, $L(\epsilon, c_0, c)$ such that for all $\beta > \beta(\epsilon, c_0, c)$, $L > L(\epsilon, c_0, c)$

$$\begin{aligned} &\mathrm{Prob}(A(m)) = \\ &\mathrm{Prob}\left(\left\{|\sum_{t\in\Lambda}\sigma(t) - m|\Lambda|\,| \leq c_0|\Lambda| \cdot L^{-c}\right\}\right) \geq \\ &(1-\epsilon)\exp\left(-T^*(m)(1 + O(c_0 \cdot L^{-c}))\right) \end{aligned} \tag{7.7}$$

where $T^*(m)$ is the value of the Wulff functional for a Wulff droplet of total volume $V(m) = (m^* - m)/2m^* \cdot |\Lambda|$.

Proof.

1. The rest of the section is devoted to the proof of the theorem. In a first step we get a lower bound on $\mathrm{Prob}(A(m))$ by choosing suitably a subset of $A(m)$, and by estimating its probability (see (7.22). Let $\Gamma(m)$ be the contour defined by the configuration $\sigma(t) = -1$ if $t \in \mathrm{int}W_\tau(m)$ and $\sigma(t) = 1$ if $t \notin \mathrm{int}W_\tau(m)$. (We suppose that $W_\tau(m)$ is "in the middle" of Λ). The contour $\Gamma(m)$ is a simple closed line on Λ^*. We approximate $\Gamma(m)$ by a convex polygon $P(m)$ in $\mathbb{R}^2$, whose vertices are sites of $\Gamma(m)$ and the Euclidean length of the edges of the polygon is $\hat{c}_0 L^{1-c}$ with $\hat{c}_0 \leq c_0$. The value of $\hat{c}_0$ is chosen later. The vertices of the polygon are denoted by $t_1, \ldots, t_N$. For each edge we construct a square box, whose sides are horizontal and vertical, and which is divided by the edge in two parts of equal volume, the extremities of the edge being on the sides of the box (see figure 4).

Let Γ be a closed contour passing through $t_1, \ldots, t_N$ and entirely inside the boxes which we have constructed. We also suppose that there is some constant such that the length of Γ satisfies $|\Gamma| \leq \mathrm{const} \cdot L$. (The value of the constant is specified later on). Let $B(\Gamma)$ be the set of configurations which have the contour Γ and such that all other contours γ have a volume smaller than $L^{2(1-c)}$, i.e. they are s-small with $s = L^{1-c}$.

$$\begin{aligned} \mathrm{Prob}(A(m)) &\geq \sum_{\Gamma} \mathrm{Prob}(A(m) \cdot B(\Gamma)) \\ &= \sum_{\Gamma} \mathrm{Prob}(A(m)|B(\Gamma)) \cdot \mathrm{Prob}(B(\Gamma)) \end{aligned} \tag{7.8}$$

where the sums are restricted to the contours Γ above.

$$\begin{aligned} &\mathrm{Prob}(A(m)|B(\Gamma)) = \\ &1 - \mathrm{Prob}\left(\{|\sum_{t\in\Lambda}\sigma(t) - m|\Lambda|\,| \geq c_0|\Lambda| \cdot L^{-c}\}|B(\Gamma)\right) \end{aligned} \tag{7.9}$$

The volume of Γ is such that

$$|V(m) - \mathrm{vol}(\Gamma)| \leq |\Gamma(m)|\hat{c}_0 L^{1-c} \leq 4L\hat{c}_0 L^{1-c} = 4\hat{c}_0|\Lambda| \cdot L^{-c} \tag{7.10}$$

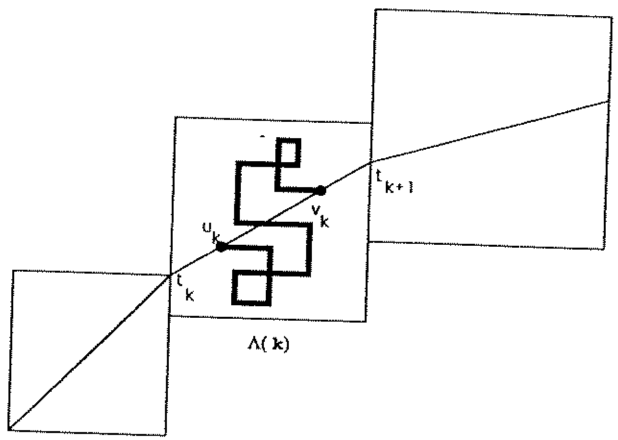

Figure 4: Part of the polygon $P(m)$ and the square boxes.

Therefore, if Γ is fixed,

$$\sum_{t\in\Lambda}\sigma(t)-m|\Lambda| = \left(\sum_{t\in\Lambda}\sigma(t)-\left\langle\sum_{t\in\Lambda}\sigma(t)|B(\Gamma)\right\rangle^+(\Lambda)\right) + \left(\left\langle\sum_{t\in\Lambda}\sigma(t)|B(\Gamma)\right\rangle^+(\Lambda)-m|\Lambda|\right) \tag{7.11}$$

and

$$\left\langle\sum_{t\in\Lambda}\sigma(t)|B(\Gamma)\right\rangle^+(\Lambda)=m^*(|\Lambda|-\mathrm{vol}(\Gamma))-m^*\mathrm{vol}(\Gamma)+O(L) \tag{7.12}$$

The term $O(L)=O(1/L)|\Lambda|$ takes into account the boundary effects, which are of order $O(L)$ since $|\Gamma|\le \mathrm{const}\cdot L$, and the fact that all contours except Γ are small, which gives a correction of order $O(\exp(-8\beta L^{1-c}))$ (see section 4). From (7.10), (7.11) and (7.12) we get for any configuration $\sigma\notin A(m)$,

$$|\sum_{t\in\Lambda}\sigma(t)-\left\langle\sum_{t\in\Lambda}\sigma(t)|B(\Gamma)\right\rangle^+(\Lambda)|\ge \tag{7.13}$$
$$|\Lambda|\,(c_0/L^c-8m^*\hat{c}_0/L^c-O(1/L))\ge 1/2|\Lambda|c_0\cdot L^{-c}$$

provided we choose $\hat{c}_0$ small enough. Theorem 5.1 implies that

$$\mathrm{Prob}\left(\{|\sum_{t\in\Lambda}\sigma(t)-m|\Lambda|\,|\ge c_0|\Lambda|\cdot L^{-c}|B(\Gamma)\}\right)\le \tag{7.14}$$
$$2\exp(-1/2c_0\beta\theta'\cdot L)$$

for some fixed θ', β and L being large enough. Therefore

$$\mathrm{Prob}(A(m))\ge(1-1/2\exp(-2c_0\beta\theta'\cdot L))\,\mathrm{Prob}(B) \tag{7.15}$$

where $B=\cup_\Gamma B(\Gamma)$. We can replace in (7.15) the set B by a subset of B, which we choose as follows. Let us consider the part of $\Gamma(m)$ inside one of the small boxes, which we introduced above, say the box denoted by $\Lambda(k)$, which contains the points t_k and t_{k+1}. Let u_k, resp. v_k, be the sites of $\Gamma(m)\cap\Lambda(k)$ which are at a distance L^δ from t_k, resp. t_{k+1}, $0<\delta<1-c$. We cut $\Gamma(m)$ at u_k and v_k and remove the part of $\Gamma(m)$ between u_k and v_k. Let γ_k be an open contour, entirely inside $\Lambda(k)$ and such that $\delta\gamma_k=\{u_k,v_k\}$, and not touching the remaining part of $\Gamma(m)$. Moreover we suppose that $|\gamma_k|\le\mathrm{const}\cdot L^{1-c}$. We glue together γ_k and the remaining part of $\Gamma(m)$ at u_k and v_k, and repeat this operation for each box. In this way we get a closed contour, passing through $t_1,\dots,t_N$. The set of all closed contours passing through $t_1,\dots,t_N$, which are constructed as above, is denoted by H. Then

$$\mathrm{Prob}(B)\ge\sum_{\Gamma\in H}\mathrm{Prob}(B(\Gamma)) \tag{7.16}$$

and

$$\mathrm{Prob}(B(\Gamma))=\left(Z^+(\Lambda)\right)^{-1}e^{-2\beta|\Gamma|}\sum_{\substack{\underline{\eta}:\ \text{all } |\eta_i| \text{ small}\\ (\Gamma,\underline{\eta})\ \text{compatible}}}\exp\left(-2\beta\sum_{\eta\in\underline{\eta}}|\eta|\right) \tag{7.17}$$

If we remove the constraint $|\eta_i|$ small, then we get $\mathrm{Prob}^+_\Lambda(\Gamma)$. $\mathrm{Prob}^+_\Lambda(\Gamma)$ can be written

$$\mathrm{Prob}^+_\Lambda(\Gamma)=e^{-2\beta|\Gamma|}\langle n(\Gamma)\rangle^+(\Lambda)(1+O(e^{-\beta O(L)})) \tag{7.18}$$

with

$$n(\Gamma)=\prod_{\substack{t:\\ d_1(t,\Gamma)\le 1}}n(t)\ ,\quad n(t)=\frac{1}{2}(1+\sigma(t)) \tag{7.19}$$

since for any subset Ω, $Z^+(\Omega)=Z^-(\Omega)$ by symmetry. Therefore we divide and multiply by

$$\sum_{\substack{\underline{\eta}:\\ (\Gamma,\underline{\eta})\ \text{compatible}}}\exp\left(-2\beta\sum_{\eta\in\underline{\eta}}|\eta|\right) \tag{7.20}$$

and we have

$$\begin{aligned}&\mathrm{Prob}(B(\Gamma))\ge \\ &\exp\left(-O(e^{-8\beta L^{1-c}})\right)\cdot\mathrm{Prob}^+_\Lambda(\Gamma)\ge \\ &\left(1-O(\exp(-8\beta L^{1-c}))\right)e^{-2\beta|\Gamma|}\langle n(\Gamma)\rangle^+(\Lambda)\ge \\ &\left(1-O(\exp(-8\beta L^{1-c}))\right)e^{-2\beta|\Gamma|}\left\langle n(\overline{\Gamma})\right\rangle^+\prod_k\langle n(\gamma_k)\rangle^+\end{aligned} \tag{7.21}$$

The first inequality is proven by using the cluster expansion and the last line is a consequence of F.K.G. inequalities. In (7.21) $\overline{\Gamma}$ is the part of $\Gamma(m)$ which is common to all $\Gamma \in H$. Since $\Gamma = (\overline{\Gamma}, \gamma_1, \ldots, \gamma_N)$ we get

$$\sum_{\Gamma \in H} \mathrm{Prob}(B(\Gamma)) \;\geq\; \left(1 - O(\exp(-8\beta L^{1-\epsilon}))\right) e^{-2\beta|\overline{\Gamma}|} \left\langle n(\overline{\Gamma})\right\rangle^+ \cdot \prod_{k=1}^{N} \left(\sum\nolimits^*_{\gamma_k} e^{-2\beta|\gamma_k|} \langle n(\gamma_k)\rangle^+\right) \tag{7.22}$$

the index * means that we sum only over the allowed contours γ_k.

2. We must bound from below the sum

$$\sum\nolimits^*_{\gamma} e^{-2\beta|\gamma|} \langle n(\gamma)\rangle^+ \tag{7.23}$$

where γ is a contour inside of a square box as in figure 4 , $|\gamma| \leq \mathrm{const} \cdot L^{1-\epsilon}$. We observe that the sum in (7.23), when we remove the constraint on the contours γ, is equal (essentially) to a two-point function of the dual model. Let Ω be some big square box containing γ. We consider

$$\sum\nolimits^*_{\gamma} e^{-2\beta|\gamma|} \langle n(\gamma)\rangle^+ (\Omega) \tag{7.24}$$

and at the end of the estimation, we take the limit $\Omega \uparrow Z^2$. We have

$$\langle n(\gamma)\rangle^+ (\Omega) = \frac{\hat{Z}(\Omega^*|\gamma)}{Z(\Omega^*)} \tag{7.25}$$

where the partition function of the numerator differs from $Z(\Omega^*|\gamma)$ of (6.20) by the fact that some families of contours appearing in (6.20) do not appear here, namely those families which contain an odd number of closed contours surrounding γ. However, the cluster expansion gives

$$\frac{\hat{Z}(\Omega^*|\gamma)}{Z(\Omega^*|\gamma)} \geq \exp\left(-O(e^{-\beta L^{1-\epsilon}})\right) \tag{7.26}$$

with $O(e^{-\beta L^{1-\epsilon}})$ independent on Ω and γ. Thus

$$\sum\nolimits^*_{\gamma} e^{-2\beta|\gamma|} \langle n(\gamma)\rangle^+ (\Omega) \geq \exp\left(-O(e^{-\beta L^{1-\epsilon}})\right) \cdot \sum\nolimits^*_{\gamma} e^{-2\beta|\gamma|} \frac{Z(\Omega^*|\gamma)}{Z(\Omega^*)} \tag{7.27}$$

and

$$\sum\nolimits^*_{\gamma} e^{-2\beta|\gamma|} \frac{Z(\Omega^*|\gamma)}{Z(\Omega^*)} = \sum_{\substack{\text{all } \gamma \\ \text{through u and v}}} e^{-2\beta|\gamma|} \frac{Z(\Omega^*|\gamma)}{Z(\Omega^*)} - \sum_{\substack{\text{all } \gamma \\ \text{forbidden}}} e^{-2\beta|\gamma|} \frac{Z(\Omega^*|\gamma)}{Z(\Omega^*)} = \langle \sigma(u)\sigma(v)\rangle^f (\Omega^*) - \sum_{\substack{\text{all } \gamma \\ \text{forbidden}}} e^{-2\beta|\gamma|} \frac{Z(\Omega^*|\gamma)}{Z(\Omega^*)} \tag{7.28}$$

where u and v are the extremities of γ.

3. The problem now is to get an upper bound of the following type

$$\sum_{\substack{\text{all } \gamma \\ \text{forbidden}}} e^{-2\beta|\gamma|} \frac{Z(\Omega^*|\gamma)}{Z(\Omega^*)} \leq \epsilon' \langle \sigma(u)\sigma(v)\rangle^f (\Omega^*) \tag{7.29}$$

with $\epsilon' < 1$, so that we can estimate (7.28) in terms of $\langle \sigma(u)\sigma(v)\rangle^f (\Omega^*)$. The forbidden contours γ are divided into different classes and each class is estimated separately. Among the forbidden contours γ there are those which have a length $|\gamma| \geq \mathrm{const} \cdot L^{1-\epsilon}$. From lemma 6.3 we know that

$$\langle \sigma(u)\sigma(v)\rangle^f (\Omega^*) \leq \exp(-\tau(u,v)) \tag{7.30}$$

where $\tau(u,v) = d_2(u,v)\alpha(n^*_{u,v}) = \beta O(L^{1-\epsilon})$, $n^*_{u,v}$ being the unit vector giving the direction of the straight line through u and v. If the constant in $|\gamma| \geq \mathrm{const} \cdot L^{1-\epsilon}$ is large enough then the contribution of these contours is negligible.

4. The other forbidden contours γ have a length $|\gamma| \leq \mathrm{const} \cdot L^{1-\epsilon}$ and must touch the boundary of the small box, denoted by Λ, or must touch $\overline{\Gamma}$. Let

$$C = \left\{\gamma : |\gamma| \leq \mathrm{const} \cdot L^{1-\epsilon},\ \ \delta\gamma = \{u,v\}\right\} \tag{7.31}$$

Let $\gamma \in C$ and $G(\gamma)$ be the set of all shortest paths in γ from u to v. Such paths are simple and we choose one of them, denoted by $g(\gamma)$. We list all simple paths $\gamma \in C$. We define $g(\gamma)$ as the path of $G(\gamma)$ which is the first one in the list. Once $g(\gamma)$ is chosen, we can describe uniquely the contour γ by $\gamma = (g; \eta_1, \ldots, \eta_k)$ where $\eta_1, \ldots, \eta_k$ are closed disjoint contours such that the union of g and any η_i is a connected set and g and η_i have no common edge. In other words the union of g and the contours η_i is an open contour with boundary points u and v. This is the type of decomposition considered in section 6.4.1. Conversely, given a simple open contour g with endpoints u and v, and a family of closed contours η, we say that $(g; \eta_1, \ldots, \eta_k)$ is *weakly-admissible* if the union of g and the contours η is an element of C, and we say that $(g; \eta_1, \ldots, \eta_k)$ is *strongly-admissible* if the union of g and the contours η is a contour $\gamma' \in C$ such that $g(\gamma') = g$.

5. We consider the contours $\gamma = (g; \eta_1, \ldots, \eta_k)$ which contain at least one contour η such that $|\eta| \geq \ln L$. The estimation below is done in the spirit of section 6.4.1. Let C^* be the subset of C of all $(g; \eta_1, \ldots, \eta_k)$ with $|\eta_i| \leq \ln L$, $i = 1, \ldots, k$. We define a map Θ on C with values in C^*. Let $\gamma = (g; \eta_1, \ldots, \eta_k, \eta_{k+1}, \ldots, \eta_q) \in C$ with $|\eta_i| \leq \ln L$, $i = 1, \ldots, k$, and $|\eta_j| > \ln L$ $j = k+1, \ldots, q$. By definition

$$\Theta(\gamma) := (g; \eta_1, \ldots, \eta_k) \tag{7.32}$$

In order that Θ be well-defined, we must verify that γ', which is the union of g and $\eta_1, \ldots, \eta_k$, has the decomposition $(g; \eta_1, \ldots, \eta_k)$. Since γ is an element of C, γ' is also an element of C, and thus $(g; \eta_1, \ldots, \eta_k)$ is weakly- admissible. Moreover, since we have removed some contours η,

$$G(\gamma) \supset G(\gamma') \tag{7.33}$$

But g is an element of $G(\gamma)$ and also of $G(\gamma')$. We must have $g(\gamma') = g$ and therefore $(g;\eta_1,\ldots,\eta_k)$ is strongly admissible and Θ is well-defined. Let $\gamma^* \in C^*$. Then

$$\sum_{\gamma\in\Theta^{-1}(\gamma^*)\setminus\gamma^*} e^{-2\beta|\gamma|}\frac{Z(\Omega^*|\gamma)}{Z(\Omega^*)} \leq \tag{7.34}$$
$$\sum_{(\eta'_1,\ldots,\eta'_k)} e^{-2\beta|\gamma|}\frac{Z(\Omega^*|\gamma)}{Z(\Omega^*)}\cdot\frac{Z(\Omega^*|\gamma^*)}{Z(\Omega^*|\gamma^*)} \leq$$
$$e^{-2\beta|\gamma^*|}\frac{Z(\Omega^*|\gamma^*)}{Z(\Omega^*)}\sum_{(\eta'_1,\ldots,\eta'_k)} e^{-2\beta\sum_{i=1}^k|\eta'_i|}\frac{Z(\Omega^*|\gamma)}{Z(\Omega^*|\gamma^*)}$$

where in (7.34) we sum over all non empty families $(\eta'_1,\ldots,\eta'_k)$ with $|\eta'_i| > \ln L$, such that the union of γ^* and $\eta'_1,\ldots,\eta'_k$ is a contour $\gamma \in \Theta^{-1}(\gamma^*)\setminus\gamma^*$. The last factor in (7.34) is smaller than one since $\gamma^* \subseteq \gamma$. Therefore the last sum in (7.34) can be estimated using the method of the cluster expansion,

$$\sum_{n\geq 1}\frac{1}{n!}\sum_{\substack{\eta'_1,\ldots,\eta'_n:\\ \text{comp.},|\eta'_i|>\ln L\\ \text{connected to } g}} \exp\left(-2\beta(|\eta'_1|+\cdots+|\eta'_n|)\right) = \tag{7.35}$$
$$\exp\left(\sum_{n\geq 1}\frac{1}{n!}\sum_{\substack{\eta'_1,\ldots,\eta'_n:\\ |\eta'_i|>\ln L,\ \text{conn. to } g}}\varphi_n^T(\eta'_1,\ldots,\eta'_n)\prod_{i=1}^n e^{-2\beta|\eta'_i|}\right) - 1 \leq$$
$$\exp\left(O(|g|\cdot e^{-2\beta\ln L})\right) - 1$$

Since $|g| \leq \text{const}\,L^{1-c}$, we get by combining (7.34), (7.35) and summing over γ^*

$$\sum_{\gamma\in C\setminus C^*} e^{-2\beta|\gamma|}\frac{Z(\Omega^*|\gamma)}{Z(\Omega^*)} \leq \tag{7.36}$$
$$\left(\exp\left(O(L^{1-c}\cdot e^{-2\beta\ln L})\right)-1\right)\cdot\sum_{\gamma^*\in C^*} e^{-2\beta|\gamma^*|}\frac{Z(\Omega^*|\gamma^*)}{Z(\Omega^*)} \leq$$
$$\left(\exp\left(O(L^{1-c}\cdot e^{-2\beta\ln L})\right)-1\right)\cdot\langle\sigma(u)\sigma(v)\rangle^f(\Omega^*)$$

6. It remains to consider only the contours $\gamma \in C^*$ such that

- either the distance of $g(\gamma)$ to the boundary of Λ is less than $\ln L$
- or γ touches $\overline{\Gamma}\cap\Lambda$.

Let us examine with more details the structure of a contour $\gamma = (g;\eta_1,\ldots,\eta_k) \in C^*$. The line g going from u to v is simple. We parametrize it with unit speed, $s \in [0,|g|] \mapsto g(s)$. Let η be some closed contour in the decomposition of γ. We define

- $s_1(\eta)$: the smallest value of s such that $g(s)$ is a point of η

- $s_2(\eta)$: the largest value of s such that $g(s)$ is a point of η

Since η is closed, there is a path on η going from s_1 to s_2 with length smaller than $1/2\ln L$. Therefore we must have

$$|s_2 - s_1| \leq 1/2\ln L \tag{7.37}$$

otherwise we could make the path g shorter. The next question which we consider is the existence of cutting points for γ (see section 6.4.2). If there is no η in the decomposition of γ such that $s_1(\eta) \leq s$ and $s_2(\eta) \geq s$, then we can decompose γ with a cutting point at $g(s)$. The next estimation is useful when we look for such a situation. (Not all $\gamma \in C^*$ have cutting points.) This estimation is similar to the estimation (7.36). Let I be some interval of $[0,|g|]$. Let $\gamma = (g;\eta_1,\ldots,\eta_k;\eta_{k+1},\ldots,\eta_q)$ where we have distinguished in the notation the contours η such that $s_1(\eta)$ or $s_2(\eta) \in$ I, which are denoted by $\eta_{k+1},\ldots,\eta_q$. For each γ we define $\overline{\gamma} = (g;\eta_1,\ldots,\eta_k)$. We have

$$\sum_{\gamma:\overline{\gamma}(\gamma)=\overline{\gamma}} e^{-2\beta|\gamma|}\frac{Z(\Omega^*|\gamma)}{Z(\Omega^*)} \leq \tag{7.38}$$
$$\sum_{\eta_{k+1},\ldots,\eta_q}\prod_{i=k+1}^{q} e^{-2\beta|\eta_i|}e^{-2\beta|\overline{\gamma}|}\frac{Z(\Omega^*|\overline{\gamma})}{Z(\Omega^*)} \leq$$
$$e^{-2\beta|\overline{\gamma}|}\frac{Z(\Omega^*|\overline{\gamma})}{Z(\Omega^*)}\exp\left(|\mathrm{I}|O(e^{-2\beta})\right)$$

We have used the inequality

$$Z(\Omega^*|\overline{\gamma}) \geq Z(\Omega^*|\gamma) \tag{7.39}$$

and the cluster expansion for going from the second line to the third line.

7. We first consider the contours $\gamma \in C^*$ such that $d_2(g(\gamma),\partial\Lambda) \leq \ln L$. The estimation below is done using the reflection principle of the theory of random walks and the results of section 6.4.2. Let Δ be the set of points of Λ which are at a distance less than $2\ln L$ from the boundary of Λ. There exists for each γ an interval I of length $\ln L$, such that all points $g(s)$, $s \in$ I, are in Δ. Let s' be the middle point of I and $t = g(s')$. Let $\overline{\gamma}$ be the contour constructed in the preceding paragraph with the above interval I. Then t is a cutting point for $\overline{\gamma}$. Let us suppose that t is as in figure 5. Let l be the horizontal line through t. Let p and u^* be the points of figure 5 on the vertical line through u, such that $d_2(u^*,p) = d_2(p,u)$. Finally let $\overline{u}$ be the point obtained by a symmetry of axis l. We have $d_2(\overline{u},u) > d_2(u^*,u)$, and the line through u^* and v has a slope equal to one. By the results of section 6.4.2 we get

$$\sum_{\substack{\overline{\gamma}:\\ t(\overline{\gamma})=t}} e^{-2\beta|\overline{\gamma}|}\frac{Z(\Omega^*|\overline{\gamma})}{Z(\Omega^*)} \leq \langle\sigma(u)\sigma(t)\rangle^f\cdot\langle\sigma(t)\sigma(v)\rangle^f \tag{7.40}$$
$$= \langle\sigma(\overline{u})\sigma(t)\rangle^f\cdot\langle\sigma(t)\sigma(v)\rangle^f$$
$$\leq \langle\sigma(\overline{u})\sigma(v)\rangle^f$$

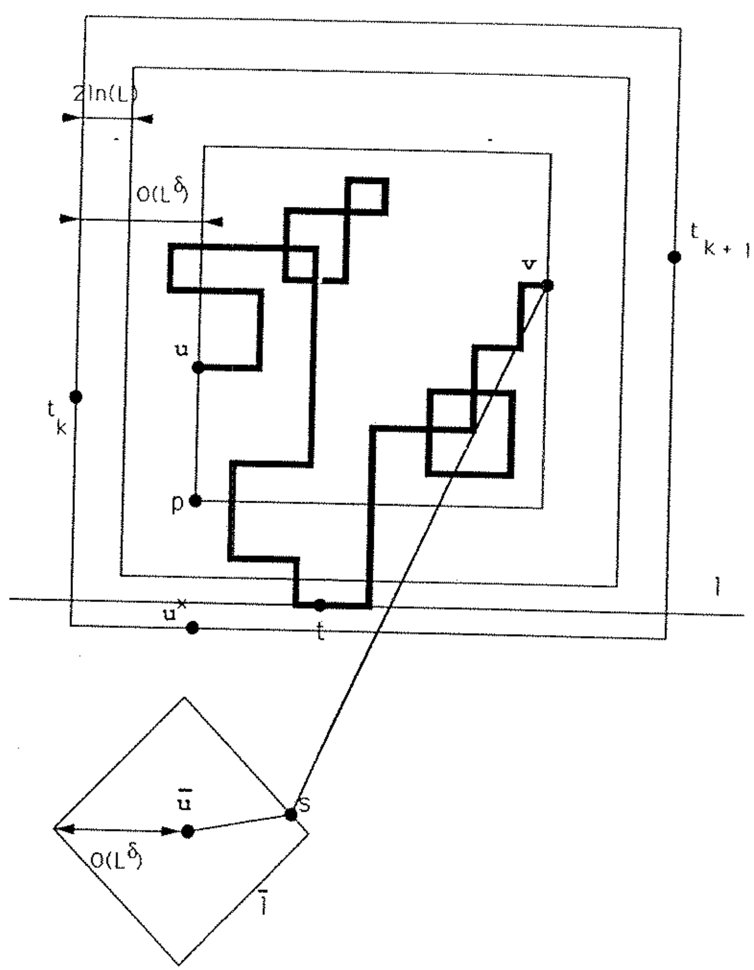

Figure 5: The point p is at a corner of the square passing through u and v. The points u, v, u^* and $\overline{u}$ are on the same vertical line, and $d_2(u,p) = d_2(p,u^*)$. The point $\overline{u}$ is obtained from u by a reflection of axis l.

where we have used the symmetry properties of the two-point function in the thermodynamic limit and Griffiths' inequality. The monotonicity properties of the correlation functions (lemma 2.4) imply that

$$\begin{aligned}\langle\sigma(u)\sigma(v)\rangle^f &\geq \langle\sigma(u^*)\sigma(v)\rangle^f \\ &\geq \langle\sigma(s)\sigma(v)\rangle^f\end{aligned} \tag{7.41}$$

for all points s of the polygonal line $\bar{l}$ of figure 5. Simon's inequality and (7.41) imply

$$\begin{aligned}\langle\sigma(\overline{u})\sigma(v)\rangle^f &\leq \sum_{s\in\bar{l}}\langle\sigma(\overline{u})\sigma(s)\rangle^f\cdot\langle\sigma(s)\sigma(v)\rangle^f \\ &\leq \langle\sigma(u)\sigma(v)\rangle^f\cdot\sum_{s\in\bar{l}}\langle\sigma(\overline{u})\sigma(s)\rangle^f\end{aligned} \tag{7.42}$$

Therefore

$$\sum_{\substack{\gamma\in C^*:\\ d_1(g(\gamma),\partial\Lambda)\leq\ln L}} e^{-2\beta|\gamma|}\frac{Z(\Omega^*|\gamma)}{Z(\Omega^*)} \leq \tag{7.43}$$
$$O(L^{1-\epsilon}\ln L)\cdot\exp\left(O(e^{-2\beta})\ln L\right)\cdot\exp\left(-\beta O(L^\delta)\right)\cdot\langle\sigma(u)\sigma(v)\rangle^f$$

since $|\Delta| = O(L^{1-\epsilon}\ln L)$ and we can choose the line $\bar{l}$ so that $d_2(s,\overline{u}) \leq O(L^\delta)$ for each point s of the line $\bar{l}$.

8. Finally we consider the case when γ touches $\overline{\Gamma}\cap\Lambda$. Let $\gamma = (g;\eta_1,\ldots,\eta_k)$ and let l be the part of $\overline{\Gamma}\cap\Lambda$ containing the point u. We order linearly the points of l, starting with u. Let $t(\gamma)$ be the point of l, belonging to γ and which is the first one in l. We first suppose that $t(\gamma)$ is a point of $g(\gamma) = g$, and we denote by $\overline{g}(\gamma)$ the part of g going from u to $t(\gamma)$. We decompose uniquely γ into (γ_1,γ_2) where γ_1 is the union of $\overline{g}$ and all contours η of γ with $g(s_1(\eta)) \in \overline{g}$. The contour γ_2 (as set of edges) is $\gamma_2 = \gamma\backslash\gamma_1$. We fix γ_1 and sum over γ_2. We get

$$\sum_{\substack{\gamma_2:\\ t(\gamma)=t}} e^{-2\beta(|\gamma_1|+|\gamma_2|)}\frac{Z(\Omega^*|\gamma_1\cup\gamma_2)}{Z(\Omega^*)} = \tag{7.44}$$
$$\sum_{\substack{\gamma_2:\\ t(\gamma)=t}} e^{-2\beta|\gamma_2|}\frac{Z(\Omega^*|\gamma_2)}{Z(\Omega^*)}\cdot e^{-2\beta|\gamma_1|}\frac{Z(\Omega^*|\gamma_1\cup\gamma_2)}{Z(\Omega^*|\gamma_2)}$$

The last quotient is estimated using the cluster expansion,

$$\frac{Z(\Omega^*|\gamma_1\cup\gamma_2)}{Z(\Omega^*|\gamma_2)} = \tag{7.45}$$
$$\exp\left(-\sum_{n\geq1}\frac{1}{n!}\sum_{\substack{\exists\lambda_i:\\ \lambda_i\cap\gamma_1\neq\emptyset}}\varphi^T(\lambda_1,\ldots,\lambda_n)\prod_{i=1}^n e^{-2\beta|\lambda_i|}\right) =$$
$$\exp\left(|\gamma_1|O(e^{-2\beta})\right) \tag{7.46}$$

Therefore we get (see (6.23)) the upper bound

$$\langle\sigma(t)\sigma(v)\rangle^f(\Omega^*)\cdot\exp\left(-(2\beta-O(e^{-2\beta}))|\gamma_1|\right)\leq \tag{7.47}$$
$$\langle\sigma(u)\sigma(v)\rangle^f(\Omega^*)\cdot\exp\left(-(2\beta-O(e^{-2\beta}))|\gamma_1|\right)$$

since $t=t(\gamma)$ and γ_2 is an open contour with $\delta(\gamma_2)=\{t,v\}$. Then we sum over all γ_1 and all $t(\gamma)$. Thus we can bound (7.44) by

$$O(e^{-2\beta})\cdot\langle\sigma(u)\sigma(v)\rangle^f \tag{7.48}$$

The last case is when $t=t(\gamma)\notin g(\gamma)$. In this case there exists a unique η, say η_1, of $\gamma=(g,\eta_1,\dots,\eta_k)$ such that $t(\gamma)\in\eta_1$. Let $\overline{g}(\gamma)$ be a shortest path from t to u in γ. We choose this path as follows. Starting at t, this path is first in η_1 until it reaches a point of g, say t^*, of parameter s^*, $g(s^*)=t^*$. If there are several possibilities we choose a path with s^* minimal. From t^* the path is given by the part of g going from t^* to u. If there are still several paths satisfying the above requirements then we choose the first one in a list of all paths from t to u. We define γ_1 as the union of $\overline{g}(\gamma)$ and all contours η of γ which are different from η_1 and with $s_1(\eta)<s^*$. The contour γ_2 (as set of edges) is $\gamma_2=\gamma\backslash\gamma_1$. The contour γ is uniquely decomposed into (γ_1,γ_2) and we can repeat the above argument. (See remark of 6.4.1.)

9. Let $\epsilon>0$. Then there exist $L(\epsilon)$ and $\beta(\epsilon)$, such that for $L>L(\epsilon)$, $\beta>\beta(\epsilon)$

$$\text{Prob}(A(m))\geq(1-\epsilon)e^{-2\beta|\overline{\Gamma}|}\left\langle n(\overline{\Gamma})\right\rangle^+\prod_{k=1}^{N}\langle\sigma(u_k)\sigma(v_k)\rangle^f \tag{7.49}$$

By F.K.G. inequalities

$$\left\langle n(\overline{\Gamma})\right\rangle^+\geq\prod_{e:\text{edges of }\overline{\Gamma}}\langle n(e^*)\rangle^+=\exp\left(-|\overline{\Gamma}|O(e^{-2\beta})\right) \tag{7.50}$$

with $n(e^*)=n(t)n(t')$, $e^*=\{t,t'\}$ being the edge dual to e. Notice that $|\overline{\Gamma}|=O(L^{c+\delta})$. We finish the proof of theorem 7.1 by using lemma 7.1.

Lemma 7.1 ([D.K.S])

For β high enough

$$\langle\sigma(u)\sigma(v)\rangle^f=e^{-\tau(u,v)}O\left(\frac{1}{d_2(u,v)^{1/2}}\right) \tag{7.51}$$

Summarizing all the results, we have

$$\text{Prob}(A(m))\geq \tag{7.52}$$
$$(1-\epsilon)\exp(-\sum_{k=1}^{N}\tau(u_k,v_k))\exp\left(-(2\beta+O(e^{-2\beta}))\cdot O(L^{c+\delta})\right) \tag{7.53}$$

Since the points t_k,t_{k+1},u_k,v_k are on the same straight line

$$\tau(P(m))=\sum_{k=1}^{N}\tau(t_k,t_{k+1})=\sum_{k=1}^{N}\tau(u_k,v_k)+\beta O(L^{c+\delta}) \tag{7.54}$$

But

$$|T^*-\tau(P(m))|\leq\hat{c}_0O(L^{1-c})\beta\leq c_0O(L^{1-c})\beta \tag{7.55}$$

and we may choose $\delta>0$, as small as we want, so that $1-c>c+\delta$. This ends the proof of theorem 7.1.

Remark.
Lemma 7.1 expresses the fact that the two-point function at high temperature has an Ornstein-Zernicke behaviour. It would be sufficient for our purpose to have a constant α instead of $1/2$ in this lemma.

8 Droplets.

The model is defined on Λ, a square of volume L^2 in $\mathbb{Z}^2$. We have + b.c. and no magnetic field. This hypothesis holds for the whole section. Prob(E) is the probability of the event E computed with the measure $\langle\cdot\rangle^+(\Lambda)$. We distinguish in each configuration between small contours and large contours. In section 8.1 we define these notions and a set of configurations E such that

$$\frac{\langle E\cdot A(m)\rangle^+}{\langle A(m)\rangle^+}\geq 1-O\left(e^{-O(\beta L)}\right) \tag{8.1}$$

In the rest of the section we give another description of the set E, introducing the notion of droplet. We partition the set E into subsets $E(S_1,\dots,S_k)$ indexed by geometrical objects, called droplets. A droplet is defined at the scale L^b with $c<b<1-c$, it has a volume, and the length of its boundary is measured by the Wulff functional. We estimate the probability of $E(S_1,\dots,S_k)$ in terms of the Wulff functional. The introduction of an intermediate scale is essential for this estimation.

8.1 A typical set of configurations for a large deviation of the magnetization.

We define the notion of large contours. We proceed in several steps. We first make concrete the idea that a "complicated" contour is not important because its probability is small.

1. Let γ be some arbitrary closed contour, and σ_γ be the unique configuration which has only this contour. The subset of $\mathbb{R}^2$, which is the union of all plaquettes $p^*(t)$, $t\in\mathbb{Z}^2$, such that $\sigma_\gamma(t)=-1$, is bounded. The complement of this set in $\mathbb{R}^2$ has a unique connected component of infinite volume. The *exterior enveloppe* $e(\gamma)$ of γ is the boundary of this infinite component. It is a connected subset of γ. The exterior enveloppe $e(\gamma)$ divides the plane into several connected components. Each bounded component has a boundary which is a simple closed contour, called *cycle*. We can decompose $e(\gamma)$ into cycles $e(\gamma)=(e_1(\gamma),\dots,e_k(\gamma))$. The contours $e_i(\gamma)$ and $e_j(\gamma)$, as sets of edges, are disjoint. By definition $\mathrm{Int}e_i(\gamma)$ is the bounded closed set of $\mathbb{R}^2$ whose boundary is the cycle $e_i(\gamma)$, and

$$\mathrm{Int}\gamma:=\bigcup_{\substack{e_i \text{ cycles}\\ \text{of } e(\gamma)}}\mathrm{Int}e_i(\gamma) \tag{8.2}$$

Notice that $\mathrm{Int}\gamma$ does not coincide with the set $\overline{\mathrm{int}\gamma}$, but we have

$$\mathrm{Int}\gamma\supset\overline{\mathrm{int}\gamma}\ ,\ \mathrm{vol}(\mathrm{Int}\gamma)\geq\mathrm{vol}(\gamma)\ \text{ and }\ |e(\gamma)|\leq|\gamma| \tag{8.3}$$

2. We decompose uniquely γ into $e,\eta_1,\dots,\eta_p$ and $\xi_1,\dots,\xi_q$ where $e=e(\gamma)$ is the exterior enveloppe and the contours η and ξ are closed disjoint contours, which have at least one point, but no edge, in common with e (see figure 6). This is the kind

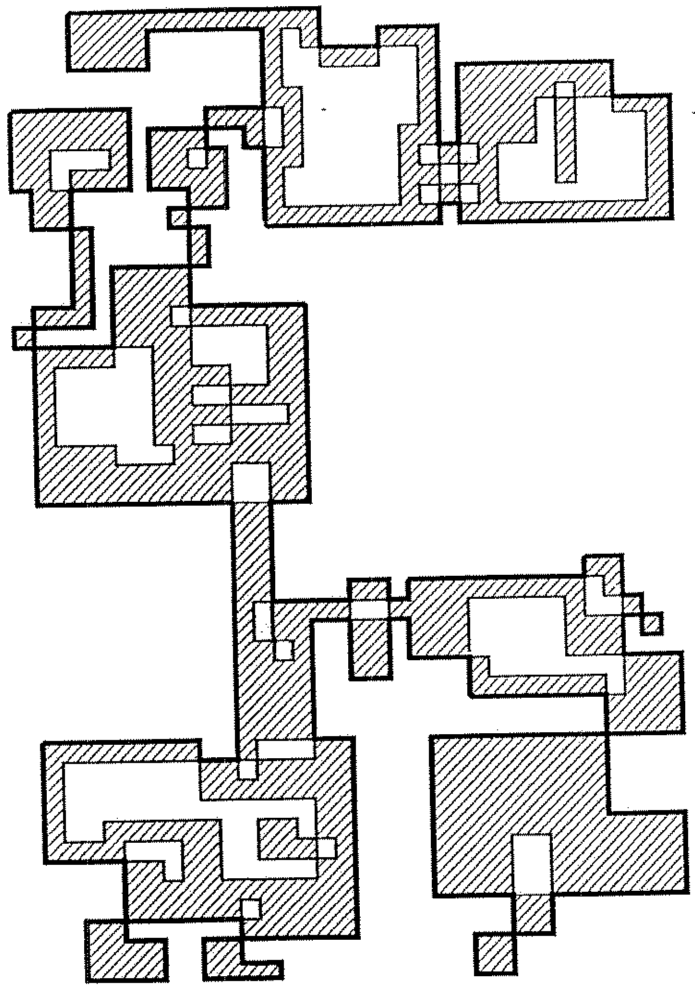

Figure 6: Decomposition of γ into the exterior enveloppe e and the contours η and ξ. The exterior enveloppe has three large cycles and ten small cycles. There are six contours ξ. Compare with figure 7.

of decomposition of section 6.4.1. By definition $|\eta_i| \leq \ln L$ and $|\xi| > \ln L$. We write $\gamma = \gamma(e; \eta_1, \ldots, \eta_p; \xi_1, \ldots, \xi_q)$. We define a map F_1, on the set of contours :

$$F_1(\gamma(e; \eta_1, \ldots, \eta_p; \xi_1, \ldots, \xi_q) := \gamma(e; \eta_1, \ldots, \eta_p) \tag{8.4}$$

From lemma 6.6 and the remark following its proof we get

Lemma 8.1

For large β,

$$\begin{aligned} &\text{Prob}\,(F_1(\gamma_1) = \tilde{\gamma}_1, \ldots, F_1(\gamma_k) = \tilde{\gamma}_k) \leq \\ &\exp\left((\sum_{i=1}^{k} |e(\tilde{\gamma}_i)|) \cdot O(1/L^{2\beta})\right) \cdot \text{Prob}(F_1(\tilde{\gamma}_1), \ldots, F_1(\tilde{\gamma}_k)) \end{aligned} \tag{8.5}$$

3. The intersection points of the exterior enveloppe e and the closed contours η or ξ are necessarily corner points of e and of η or ξ. This implies that the subsets Intη and Intξ are disjoint two by two. Therefore if $\gamma = \gamma(e; \eta_1, \ldots, \eta_p; \xi_1, \ldots, \xi_q)$, then

$$\text{Int}\gamma = \text{Int}F_1(\gamma) \tag{8.6}$$

because we do not modify the exterior enveloppe by the map F_1, but we have

$$\overline{\text{int}F_1(\gamma)} \supset \overline{\text{int}\gamma} \tag{8.7}$$

Indeed, $\overline{\text{int}\gamma}$, as set of $\mathbb{R}^2$, is composed of the closure of the set

$$\text{Int}\gamma \setminus \left(\bigcup_{\eta_i} \text{Int}\eta_i \bigcup_{\xi_j} \text{Int}\xi_j\right) \tag{8.8}$$

and of subsets of Intη_i or Intξ_j. By the mapping F_1 we do not modify the structure of γ inside Intη_i and if remove the sets Intξ_j, then we can only increase $\overline{\text{int}\gamma}$.

4. We recall that the set $A(m)$ is

$$A(m) = \{\sigma : |\sum_{t \in \Lambda} \sigma(t) - m|\Lambda|| \leq c_0|\Lambda| \cdot L^{-c}\} \tag{8.9}$$

with c a parameter, $0 < c < 1/2$.

Definition:

A contour γ is *small* if all connected components of $\overline{\text{int}F_1(\gamma)}$ have a volume $\leq L^{2a}$, $a = 1 - c$. All other contours are *large*.

In a configuration the small contours are denoted by $\gamma_1, \ldots, \gamma_n$ and the large contours by $\Gamma_1, \ldots, \Gamma_k$. The isoperimetric inequality on the lattice is

$$16 \cdot \text{vol}(\gamma) \leq |\gamma|^2 \tag{8.10}$$

Therefore all large contours have a length larger or equal to $4L^a$. Notice that all connected components of $\overline{\text{int}\gamma}$ have a volume smaller than L^{2a} when γ is a small contour. For small contours we can apply the results of sections 4 and 5.

We estimate the total length of the large contours.

Lemma 8.2

For β and L large enough,

$$\begin{aligned} &\text{Prob}\,(\{\text{total length of the large contours is equal x}\}) \leq \\ &q(x)\exp(-x(2\beta - \ln 4)) \end{aligned} \tag{8.11}$$

where $q(x)$ is the number of solutions of $1 \leq \alpha_1 \leq \cdots \leq \alpha_k \leq x$, $\alpha_i \in \mathbb{N}$ and $\sum_{i=1}^k \alpha_i = x$, k arbitrary. For large x

$$q(x) \sim \frac{1}{4\sqrt{3}x} \exp\left(2\pi\sqrt{x/6}\right) \tag{8.12}$$

Proof.

Let $q(x,k)$ be the number of solutions of $1 \leq \alpha_1 \leq \cdots \leq \alpha_k \leq x$, $\sum_{i=1}^k \alpha_i = x$, k fixed. Let $\Gamma_1, \ldots, \Gamma_k$, be the k large contours of a configuration. We have at most

$$k_{\max} = \frac{x}{4L^a} \tag{8.13}$$

large contours. The number of contours, which have a length $|\Gamma|$ and contain a fixed point of $\mathbb{L}^*$ is smaller than $3^{|\Gamma|}$. Therefore

$$\begin{aligned} \text{Prob}\left(\{\sum_i |\Gamma| = x\}\right) &\leq \sum_{k=1}^{k_{\max}} q(x,k) e^{-2\beta x} 3^x (L^2)^k \\ &\leq \sum_{k=1}^{k_{\max}} q(x,k) e^{-2\beta x} 3^x (L^2)^{k_{\max}} \\ &\leq q(x) \exp\left(-x(2\beta - \ln 3 - 1/2L^{-a} \cdot \ln L)\right) \\ &\leq q(x) \exp(-x(2\beta - \ln 4)) \end{aligned} \tag{8.14}$$

provided that L is large enough.

As a corollary of lemma 8.2 we have

Theorem 8.1

Let $\delta > 0$ and $c_2 = (\alpha(m)\cdot|W_\tau|)^{1/2}\cdot\beta^{-1}+\delta$, with $\alpha(m) = (m^* - m)/2m^*$. Then for β and L large enough

$$\mathrm{Prob}\left(\sum_i |\Gamma_i| \geq c_2 L | A(m)\right) \leq \exp(-\beta\delta L) \tag{8.15}$$

Proof.

We recall that T^* is defined by

$$(T^*)^2 = 4|W_\tau|\cdot V(m) \tag{8.16}$$

and

$$V(m) = \alpha(m)L^2 \tag{8.17}$$

Therefore

$$T^* = 2(\alpha(m)\cdot|W_\tau|)^{1/2}L \tag{8.18}$$

We have

$$\mathrm{Prob}\left(\sum_i|\Gamma_i| = x|A(m)\right) = \tag{8.19}$$

$$\mathrm{Prob}\left(A(m)|\sum_i|\Gamma_i| = x\right)\frac{\mathrm{Prob}(\sum_i|\Gamma_i| = x)}{\mathrm{Prob}(A(m))}$$

But by theorem 7.1

$$\mathrm{Prob}(A(m)) \geq (1-\epsilon)\exp(-T^*(1+O(c_0L^{-c})) \tag{8.20}$$

where $0 < \epsilon < 1$, and ϵ can be chosen as small as we want provided that β and L are large enough. By lemma 8.2

$$\mathrm{Prob}\left(\sum_i|\Gamma_i| = x\right) \leq \exp(-2x(\beta - \ln 5)) \tag{8.21}$$

provided that β and L are large enough. The condition on the total length of large contours implies

$$2\beta x \geq 2c_2\beta L = T^* + 2\beta\delta L \tag{8.22}$$

and hence the lemma is proved.

A small contour cannot surround a large contour, and the type of Γ_i is uniquely determined by the collection of large contours of the configuration. Let $\underline{\Gamma}$ be the set of all large contours of a configuration σ. Then $\alpha(\underline{\Gamma})$ is defined by the identity

$$\sum_{\substack{\Gamma_i:\text{type}-\\ \Gamma_i\in\underline{\Gamma}}} \mathrm{vol}(\Gamma_i) - \sum_{\substack{\Gamma_i:\text{type}+\\ \Gamma_i\in\underline{\Gamma}}} \mathrm{vol}(\Gamma_i) \equiv \alpha(\underline{\Gamma})\cdot|\Lambda| \tag{8.23}$$

We estimate the probability that the random variable $\alpha(\underline{\Gamma})$ has a value different from $\alpha(m)$.

Lemma 8.3

Let $\underline{\Gamma} = (\Gamma_1,\dots,\Gamma_k)$ be fixed and such that

$$|\alpha(\underline{\Gamma}) - \alpha(m)| \geq \frac{c_0 + c_3}{2m^*(\beta)}\cdot\frac{1}{L^c} \tag{8.24}$$

with c_0 and c the constants appearing in the definition of $A(m)$ and $c_3 > 0$. We also suppose that $\sum|\Gamma_i| \leq c_2L$, c_2 being the constant given in theorem 8.1. Let $B(\underline{\Gamma})$ be the set of all configurations having the collection $\underline{\Gamma}$ of large contours. Let θ' be a constant, $0 < \theta' < 1$. Then

$$\mathrm{Prob}(A(m)|B(\underline{\Gamma})) \leq 2\exp(-4\beta\theta' c_3 L) \tag{8.25}$$

provided that β and L are large enough.

Proof.

Let $\Lambda(\underline{\Gamma})$ be the set

$$\Lambda(\underline{\Gamma}) = \Lambda\backslash\{t\in\Lambda : d_1(t,\cup_i\Gamma_i) \leq 1\} \tag{8.26}$$

Let σ be the configuration in Λ compatible with the + b.c. and having exactly the contours of $\underline{\Gamma}$. We choose the σ b.c. for the set $\Lambda(\underline{\Gamma})$. Then

$$\mathrm{Prob}(A(m)|B(\underline{\Gamma})) = \langle A(m)\rangle^*(\Lambda(\underline{\Gamma})) \tag{8.27}$$

The index * means that the configurations in $\Lambda(\underline{\Gamma})$ have only small contours and that the boundary condition is σ. Theorem 5.1 applies with $s = L^a$. We have

$$\begin{aligned}\sum_{t\in\Lambda}\sigma(t) - m|\Lambda| \;&=\; \left(\sum_{t\in\Lambda}\sigma(t) - \left\langle\sum_{t\in\Lambda}\sigma(t)\right\rangle^*(\Lambda(\underline{\Gamma}))\right)\\ &+\; \left(\left\langle\sum_{t\in\Lambda}\sigma(t)\right\rangle^*(\Lambda(\underline{\Gamma})) - m|\Lambda|\right)\end{aligned} \tag{8.28}$$

On the other hand

$$\begin{aligned}&\left\langle\sum_{t\in\Lambda}\sigma(t)\right\rangle^*(\Lambda(\underline{\Gamma})) = \\ &m^*(\beta)(|\Lambda| - \alpha(\underline{\Gamma})|\Lambda|) - m^*(\beta)\alpha(\underline{\Gamma})|\Lambda| + O(\frac{1}{L})|\Lambda| = \\ &m^*(\beta)|\Lambda|(1 - 2\alpha(\underline{\Gamma})) + O(\frac{1}{L})|\Lambda|\end{aligned} \tag{8.29}$$

The term $O(1/L)$ takes into account the error we make when we replace $\langle\sigma(t)\rangle^*(\Lambda(\underline{\Gamma}))$ by $\langle\sigma(t)\rangle^*$ (infinite volume limit) and then $\langle\sigma(t)\rangle^*$ by $m^*(\beta)$. The error is of the order of the length of the boundary of $\Lambda(\underline{\Gamma})$, which is $O(L)$ since $\sum|\Gamma_i| \leq c_2L$. Notice that we have

$$m|\Lambda| = m^*(\beta)(1 - 2\alpha(m))|\Lambda| \tag{8.30}$$

Consequently, for any $\sigma \in A(m)$,

$$\left|\sum_{t\in\Lambda}\sigma(t) - \left\langle\sum_{t\in\Lambda}\sigma(t)\right\rangle^{*}(\Lambda(\underline{\Gamma}))\right| \geq c_3 L^{-c}|\Lambda| \tag{8.31}$$

if L is large enough. The lemma follows from theorem 5.1.

We can now state the first main result on the typical configurations of a large deviation of the magnetization in the Gibbs state $\langle\cdot\rangle^{+}$. This theorem is analogous to the theorem of Minlos and Sinai p.365 in [M.S.1]. However, our definition of small contours is different and we do not fix the exact value of the magnetization.

Theorem 8.2

Let m, $-m^*(\beta) < m < m^*(\beta)$ be given. Let $\underline{\Gamma}(\sigma) = (\Gamma_1(\sigma),\ldots,\Gamma_k(\sigma))$ be the collection of all large contours in a configuration σ. Let

$$A(m) = \{\sigma : |\sum_{t\in\Lambda}\sigma(t) - m|\Lambda|| \leq c_0|\Lambda|\cdot L^{-c}\} \tag{8.32}$$

and

$$E = \{\sigma : \sum_i |\Gamma_i(\sigma)| \leq c_2 L,\ |\alpha(\underline{\Gamma}) - \alpha(m)| \leq c_4 L^{-c}\} \tag{8.33}$$

The constant c_2 is

$$c_2 = (\alpha(m)\cdot|W_\tau|)^{1/2}\cdot\beta^{-1} + \delta \tag{8.34}$$

and $\alpha(\underline{\Gamma})$ is defined in (8.23). If

$$c_4 \geq \frac{c_0}{2m^*(\beta)} + \frac{1}{4}(\alpha(m)\cdot|W_\tau|)^{1/2}\cdot(\beta m^*(\beta))^{-1}\cdot\kappa \tag{8.35}$$

with $\kappa > 1$, then for any θ', $o < \theta' < 1$ and $\theta'\cdot\kappa > 1$, there exist $L(\theta')$, $\beta(\theta')$ such that for all $L > L(\theta')$, $\beta > \beta(\theta')$

$$\mathrm{Prob}(E|A(m)) \geq 1 - \exp(-\beta\delta L) - \exp(-T^* 1/2(\theta'\kappa - 1)) \tag{8.36}$$

The probability in (8.36) is computed with the Gibbs measure $\langle\cdot\rangle^{+}(\Lambda)$ of an Ising model in Λ, with + b.c., coupling constant $J = 1$, no magnetic field and at inverse temperature β. The constant $T^* = 2(\alpha(m)\cdot|W_\tau|)^{1/2}\cdot L$.

Proof.

We estimate the complementary event E^c,

$$\begin{aligned} &\mathrm{Prob}(E^c|A(m)) \leq \\ &\mathrm{Prob}\left(\sum|\Gamma_i| \geq c_2 L|A(m)\right) + \\ &\mathrm{Prob}\left(\sum|\Gamma_i| \leq c_2 L,\ |\alpha(\underline{\Gamma}) - \alpha(m)| \geq c_4 L^{-c}|A(m)\right) \leq \\ &\exp(-\beta\delta L) + \sum_{\substack{\underline{\Gamma}:\sum|\Gamma_i|\leq c_2 L \\ |\alpha(\underline{\Gamma})-\alpha(m)|\geq c_4 L^{-c}}} \mathrm{Prob}(B(\underline{\Gamma})|A(m)) \end{aligned} \tag{8.37}$$

Now

$$\mathrm{Prob}(B(\underline{\Gamma})|A(m)) = \mathrm{Prob}(A(m)|B(\underline{\Gamma}))\cdot\frac{\mathrm{Prob}(B(\underline{\Gamma}))}{\mathrm{Prob}(A(m))} \tag{8.38}$$

and

$$\left(c_4 - \frac{c_0}{2m^*}\right)\cdot 4\beta\theta' L\cdot 2m^* \geq T^*\kappa\theta' \tag{8.39}$$

Therefore the theorem follows from lemma 8.3 and theorem 7.1.

8.2 Large contours and droplets.

We consider configurations of the set E of theorem 8.2. The total length of large contours is bounded by $c_2\cdot L$ and the total volume of large contours is at least

$$\begin{aligned} \sum \mathrm{vol}(\Gamma_i) &\geq \alpha(\underline{\Gamma})|\Lambda| \\ &\geq (\alpha(m) - c_4\cdot L^{-c})|\Lambda| \\ &= V(m)\left(1 - \frac{c_4}{\alpha(m)L^c}\right) \end{aligned} \tag{8.40}$$

We study the geometrical structure of the large contours of E and define the notion of droplet. We proceed in several steps.

1. We distinguish between *small* cycles and *large* cycles in the decomposition of a large contour Γ into an exterior enveloppe and closed contours. By definition a small cycle e_i is such that the volume of $\mathrm{Int}e_i$ is smaller than L^{2b}, with $c < b < a = 1 - c$. (We cannot choose b too small for entropy reasons.)

Lemma 8.4

In a configuration $\sigma \in E$ the total volume of all small cycles is smaller than

$$L\cdot L^b\cdot(c_2/2 + 2L^b/L) \tag{8.41}$$

Proof.

We enumerate in some way all small cycles. We collect the cycles into families. There is only one family if the total volume of the small cycles is less than $2L^{2b}$. Otherwise the first family contains all cycles e_j, $j = 1,\ldots,m$, such that

$$\sum_{j=1}^{m-1}\mathrm{vol}(\mathrm{Int}e_j) < L^{2b} \tag{8.42}$$

and

$$L^{2b} \leq \sum_{j=1}^{m}\mathrm{vol}(\mathrm{Int}e_j) < 2L^{2b} \tag{8.43}$$

Then we define a second family and so on. If there are more than one family, the total sum of the lengths of the cycles in a family is larger than $4L^b$ as a consequence of the isoperimetric inequality (the last family may be an exception). Thus there are at most $(1/4 \cdot c_2 L^{1-b} + 1)$ families of small cycles, so that the total volume of small cycles is smaller than

$$2L^{2b}(1/4 \cdot c_2 L^{1-b}) + 2L^{2b} \tag{8.44}$$

2. Up to this point we have described the contours from the "outside", using as basic geometrical object the exterior enveloppe. Now we describe the large contours from the "inside". This new description is done for the contours Γ such that $F_1(\Gamma) = \Gamma$, since $F_1(\Gamma)$ is the important part of the contour. We introduce the notions of inner component and of inner boundary. Let $\Gamma = (e; \eta_1, \ldots, \eta_k)$, and let e_j be one large cycle of the exterior enveloppe e of Γ. Let $(e_j; \eta_1, \ldots, \eta_r)$ be the part of Γ which is composed of the cycle e_j and all contours η with a point in common with e_j, i.e. all contours η with $\eta \subset \mathrm{Int}e_j$. Since $|\eta| \leq \ln L$

$$\mathrm{vol}(\mathrm{Int}\eta) \leq (\ln L)^2 \tag{8.45}$$

We have

$$\left(\mathrm{Int}e_j \backslash \bigcup_{i=1}^{r} \mathrm{Int}\eta_i\right) \subset \overline{\mathrm{int}\Gamma} \tag{8.46}$$

and we decompose the set

$$\mathrm{Int}e_j \backslash \bigcup_{i=1}^{r} \mathrm{Int}\eta_i \tag{8.47}$$

into connected components. A connected component is *large* (*small*) if its volume is larger than L^{2b} (smaller than L^{2b}). The *large* components are denoted by $D_i = D_i(\Gamma)$, and are called *inner components*. We extend this notion to an arbitrary large contour and define an *inner boundary*.

- An *inner component* of Γ is by definition an inner component of $F_1(\Gamma)$.
- An *inner boundary* is the boundary of an inner component.

Lemma 8.5

In a configuration $\sigma \in E$ the total volume of the inner components is greater than

$$V(m)\left(1 - \frac{c_4}{\alpha(m)L^c} - O(\frac{1}{L^{1-b}})\right) \tag{8.48}$$

Proof.

Let $\Gamma_1, \ldots, \Gamma_k$ be the large contours of the configuration $\sigma \in E$. We have

$$\begin{aligned} \sum \mathrm{vol}(\Gamma_i) &\geq \alpha(\underline{\Gamma})|\Lambda| \\ &= V(m)\left(1 - \frac{c_4}{\alpha(m)L^c}\right) \end{aligned} \tag{8.49}$$

and

$$\mathrm{Int}\Gamma_i = \mathrm{Int}F_1(\Gamma_i)\,,\ \mathrm{vol}(\mathrm{Int}\Gamma_i) \geq \mathrm{vol}(\Gamma_i) \tag{8.50}$$

Therefore

$$\sum_i \mathrm{vol}(\mathrm{Int}F_1(\Gamma_i)) \geq \sum_i \mathrm{vol}(\Gamma_i) \geq V(m)\left(1 - \frac{c_4}{\alpha(m)L^c}\right) \tag{8.51}$$

The total volume of small components of a contour $\Gamma_i = F_1(\Gamma_i)$ is estimated like the total volume of small cycles, and is smaller than

$$L \cdot L^b \cdot (c_2/2 + 2L^b/L) = V(m)\left(\frac{L^b c_2}{2L\alpha(m)} + \frac{2L^{2b}}{L^2\alpha(m)}\right) \tag{8.52}$$

The total volume of the sets $\mathrm{Int}\eta$ is smaller than

$$c_2 L \cdot \frac{1}{\ln L} \cdot \frac{1}{16}(\ln L)^2 = V(m)\left(\frac{c_2}{16\alpha(m)} \cdot \frac{\ln L}{L}\right) \tag{8.53}$$

Thus, the total volume of the inner components is greater than

$$\begin{aligned} \sum_{\Gamma_i} \mathrm{vol}(\mathrm{Int}F_1(\Gamma_i)) &- V(m)\left(\frac{L^b c_2}{2L\alpha(m)} + \frac{2L^{2b}}{L^2\alpha(m)}\right) \\ &- V(m)\left(\frac{c_2}{16\alpha(m)} \cdot \frac{\ln L}{L}\right) \\ &\geq V(m)\left(1 - \frac{c_4}{L^c\alpha(m)} - O(\frac{L^b}{L})\right) \end{aligned} \tag{8.54}$$

Remark.
An inner component of a contour Γ is a connected component of $\overline{\mathrm{int}F_1(\Gamma)}$. Each large contour has at least one inner component, since by definition $\overline{\mathrm{int}F_1(\Gamma)}$ has a connected component with a volume larger than L^{2a}.

3. In this paragraph we give a more precise description of an inner boundary δD. By definition δD is the boundary of an inner component D, which is inside Int$e$, where e is a large cycle of the exterior enveloppe. We consider e as a unit-speed parametrized curve $s' \in [0, |e|] \mapsto e(s')$, where $|e|$ is the length of e. The orientation of the curve is counterclockwise so that each $\mathrm{Int}\eta \subset \mathrm{Int}e$ are at the left of the curve. We also consider δD as a unit-speed parametrized curve $s \in [0, |\delta D|] \mapsto \delta D(s)$.

Contrary to the cycle e, this curve may have points of multiplicity two. (A point t has multiplicity two if there exist s_1 and $s_2 \neq s_1$ such that $\delta D(s_1) = \delta D(s_2) = t$.) In order to specify the parametrization we always make a left turn at each point of multiplicity two, and the orientation is counterclockwise. This implies that each $\mathrm{Int}\eta$, which is connected to the component D, is at the right of the curve δD. The parametrization is uniquely fixed by choosing the starting point. The curve is not a simple curve since points of multiplicity two may exist. (However such points are never crossing points, and by a slight perturbation at those points we get a simple curve.) The inner boundary is decomposed into maximal connected sets of edges of the exterior enveloppe, and maximal connected sets of edges of the contours η. A maximal connected set of last type is necessarily a subset of a single contour η, since the contours η are disjoint. We always have at least one connected set of edges of the exterior enveloppe, since $|\delta D| \geq 4L^b$ and $|\eta| \leq \ln L$. Let us suppose that $\delta D(0) = e(0)$. Let us also suppose that s_1 is the first time such that $\delta D(s_1+1)$ is not a point of e. Up to that time, both parametrized curves δD and e are the same. At that time we make a left turn if we follow δD, and a right turn if we follow e. Let us follow δD and let s_2 be the first time greater than s_1 such that $\delta D(s_2)$ is a point of e, $\delta D(s_2) = e(s_2')$. (We may have $s_1 = s_2'$.) All edges between s_1 and s_2 are edges of a single contour η, and the set $\mathrm{Int}\eta$ is in the interior of the parametrized curve which is given by the curve $e(s')$ with $s_1' = s_1 \leq s' \leq s_2'$ and then by $\delta D(s)$ with $s_2 \geq s \geq s_1$. (We go backward along δD.) From that it follows that two components of the inner boundary which are composed of edges of contours η are necessarily subsets of two different contours η. In other words, the intersection of the inner boundary and of a contour η is always a subset of the form

$$\{\delta D(s) | s_1 \leq s \leq s_2\} \tag{8.55}$$

4. We describe the relative position of the inner components in a contour Γ. Let $\Gamma = (e;\eta_1,\ldots,\eta_k)$ with $|\eta_i| \leq \ln L$. Let $D_1,\ldots,D_p$ be the inner components of Γ. We decompose (the set of $\mathbb{R}^2$)

$$\mathrm{Int}\Gamma \setminus \bigcup_{i=1}^{p} D_i \tag{8.56}$$

into connected components. Let $(B_1,\ldots,B_{q'})$ be the closures of these components. We add to this collection all points of the exterior enveloppe e which belong to two different inner boundaries (see figure 7). The resulting collection is $(B_1,\ldots,B_q)$, $q \geq q'$, and these sets are called *blocks*. We consider the intersections of these blocks with the inner boundaries, and we decomposed these sets into maximal connected sets, which we call *gluing sets*. We have two kinds of gluing sets. We have the gluing sets, which are composed of a single point of the exterior enveloppe where two cycles meet. We have gluing sets, which are intersections of inner boundaries and contours η. Indeed, if a contour η has a nonempty intersection with a block B then

$$\mathrm{Int}\eta \subset B \tag{8.57}$$

because

$$\mathrm{Int}\Gamma \setminus \bigcup_{i=1}^{p} D_i \supset \bigcup_{\eta} \mathrm{Int}\eta \tag{8.58}$$

and $\mathrm{Int}\eta$ is a connected subset of $\mathbb{R}^2$. We construct a graph. The vertices of the graph are in one-to-one correspondence with the inner components and the blocks. We have edges between an inner component and a block, if the block has a nonempty intersection with the inner boundary of the inner component. We draw between these vertices as many edges as there are gluing sets in the intersection.

Lemma 8.6

The above graph is a tree

Proof.

We first prove that the graph is simple, i.e. there is at most one edge between two vertices. Let us suppose the converse. Then there is an inner component, say D, and a block, say B, such that $B \cap \delta D$ has (at least) two gluing sets. Let u be a point of one gluing set, and v be a point of the other gluing set. We can find in $\mathbb{R}^2$ a simple closed curve going from u to v inside the inner component, and then back to u in the block. Let A be the bounded set encircled by this path. We go along the inner boundary δD from u to v by a path inside A. This path necessarily contains an edge of the exterior enveloppe, since the gluing sets are disjoint. On the other hand the set A is in $\mathrm{Int}\Gamma$. This a contradiction. The graph is a connected graph, because Γ is a connected set. Let us suppose that we have a cycle in the graph. Let $p_1,\ldots,p_n$ be the vertices which represent the blocks, and $q_1,\ldots,q_n$ be the vertices which represent the inner components of this cycle. Going around the cycle we go through $q_1,p_1,q_2,p_2,\ldots,q_n,p_n$ and then to q_1. All vertices of the cycle are different. As above we construct in $\mathbb{R}^2$ a closed simple curve entirely contained in the union of the sets represented by the vertices of the cycle, and we get a contradiction since this curve encircles an edge of the exterior enveloppe.

We call *external blocks* the blocks which are represented by vertices of incidence number one in the graph, and we call *internal blocks* the other blocks.

5. We now describe a large contour $\Gamma = F_1(\Gamma)$ by taking as basic geometrical objects the inner boundaries of Γ. The inner components of Γ are $D_1(\Gamma),\ldots,D_r(\Gamma)$, and the inner boundaries of Γ are $\delta D_1(\Gamma),\ldots,\delta D_r(\Gamma)$. We decompose the contour Γ into $\Gamma = (\delta D_1(\Gamma),\ldots,\delta D_r(\Gamma),\lambda_1,\ldots,\lambda_n)$. The contours $\lambda_1,\ldots,\lambda_n$ are closed contours and disjoint two by two. There are contours λ, which are connected to a unique inner boundary, and which we call *external contours*. They are the parts of Γ inside the external blocks. There are contours λ, which are connected to at least two disjoint inner boundaries, and which we call *internal contours*. The internal contours are the parts of Γ which are inside the internal blocks. Since the external contours do not play a special role, it is better to consider them together with the inner boundaries

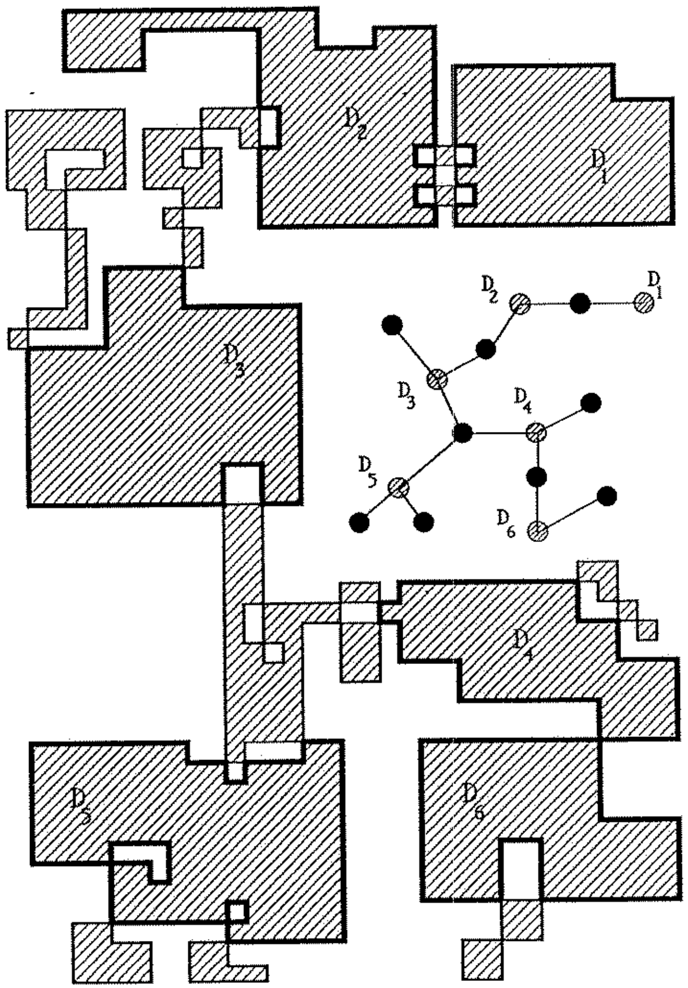

Figure 7: The inner boundaries and the tree associated with Γ. There are six inner components, five external blocks and three internal blocks. One internal block is composed of a point of multiplicity two of the exterior enveloppe.

to which they are connected.

Definition:

A *bare droplet* of a contour Γ is a closed connected set of edges of $\mathbb{L}^*$ which is the union of an inner boundary δD of the contour $F_1(\Gamma)$ and all external closed contours λ of $F_1(\Gamma)$ which are connected to δD. A bare droplet is given by $\mathcal{D} = (\delta D; \lambda_1, \ldots, \lambda_p)$.

6. We define the droplets. Let $\mathcal{D} = (\delta D; \lambda_1, \ldots, \lambda_p)$ be a bare droplet. We make a coarse graining description of $\mathcal{D}$, following an idea of Dobrushin, Kotecky and Shlosman. To each $\mathcal{D}$ we associate a unique sequence of points in the following way. We introduce some total order for the sites of Λ^*. Let us choose a parametrization of δD so that $\delta D(0)$ is the first point, for the order in Λ^*, with the property that $[\delta D(0), \delta D(1)]$ is an edge of the exterior enveloppe. Let $t_0 = \delta D(1)$ be the initial point of the sequence which we are defining. The next point t_1 is chosen as follows. Let c_5 be some small fixed number. Let s' be the first value of the parameter such that

$$d_2(\delta D(1), \delta D(s')) > L^b \cdot c_5 \tag{8.59}$$

If $[\delta D(s'), \delta D(s'+1)]$ is an edge of the exterior enveloppe, then $t_1 := \delta D(s')$. If not, we define $t_1 := \delta D(s_1)$, with s_1 the first value of the parameter which is greater than s' and such that $[\delta D(s_1), \delta D(s_1+1)]$ is an edge of the exterior enveloppe. Notice that

$$c_5 L^b - \ln L \le d_2(t_0, t_1) \le c_5 L^b + \ln L \tag{8.60}$$

since for all contours η, $|\eta| \le \ln L$. Then we define t_2 as above and so on. For any δD we have a unique sequence $S(\delta D)$ of points $t_0, \ldots, t_n$, with

$$c_5 L^b - \ln L \le d_2(t_i, t_{i+1}) \le c_5 L^b + \ln L \tag{8.61}$$

for $i = 0, \ldots, n-1$. The distance between t_n and t_0 may be smaller than $c_5 L^b - \ln L$.

We say that two bare droplets $\mathcal{D}$ and $\mathcal{D}'$ (of different configurations) are *equivalent* if the sequences $S(\mathcal{D})$ and $S(\mathcal{D}')$ are the same.

Definition:

A *droplet* is a sequence of points $\hat{S}$ such that there exists a configuration of E having a bare droplet $\mathcal{D}$ with $S(\mathcal{D}) = \hat{S}$. An *arrangement of droplets* is a family of sequences $S_1, \ldots, S_k$ such that there exists a configuration of E with bare droplets $\mathcal{D}_1, \ldots, \mathcal{D}_k$ such that $S(\mathcal{D}_i) = S_i$, $i = 1, \ldots, k$.

To each droplet we associate a closed polygonal line going from t_0 to t_1, from t_1 to t_2, ..., from t_n to t_0. This closed polygonal line divides the plane into a finite number of bounded connected sets and one unbounded connected set. The *volume*, vol(S), of a droplet is the sum of the volume of these bounded connected sets of $\mathbb{R}^2$. The *boundary of a droplet* S is the polygonal line defined above.

Theorem 8.3

Let $E(S_1,\dots,S_k)$ be the set of configurations of E which are compatible with the arrangement of droplets $S_1,\dots,S_k$. The set E can be partitioned into subsets $E(S_1,\dots,S_k)$ such that the total volume of an arrangement of droplets $S_1,\dots,S_k$ is bounded by

$$\sum_{i=1}^{k}\mathrm{vol}(S_i)\geq V(m)\left(1-\frac{c_4}{L^c\alpha(m)}-O(\frac{1}{L^{1-b}})\right) \tag{8.62}$$

Proof.

The total volume of an arrangement of droplets is greater than the total volume of the large components minus

$$\sum_{i=1}^{k}\frac{|\delta D_i|}{L^b c_5-\ln L}\cdot\pi(L^b c_5+\ln L)^2\leq O(L^{1+b}) \tag{8.63}$$

8.3 Estimation of the probability of an arrangement of droplets.

Let $\underline{\Gamma}=(\Gamma_1,\dots,\Gamma_k)$ be the set of large contours in a configuration $\sigma\in E$. We define a map F_2

$$F_2(\underline{\Gamma})=(\mathcal{D}_1,\dots,\mathcal{D}_q) \tag{8.64}$$

where $(\mathcal{D}_1,\dots,\mathcal{D}_q)$ is the arrangement of bare droplets in the configuration σ. Notice that the bare droplets are closed contours, but they are not necessarily disjoint : it is possible that they can meet at points of multiplicity two of the exterior enveloppes. However, the union of all bare droplets forms a compatible family of closed contours, which we still denote by $(\mathcal{D}_1,\dots,\mathcal{D}_q)$

Lemma 8.7

Let $(\hat{\mathcal{D}}_1,\dots,\hat{\mathcal{D}}_q)$ be an admissible arrangement of bare droplets. Then, for β large enough,

$$\mathrm{Prob}\left(\{\underline{\Gamma}:F_2(\underline{\Gamma}=(\hat{\mathcal{D}}_1,\dots,\hat{\mathcal{D}}_q)\}\right)\leq \\ \exp(q\cdot O(\ln L))\cdot\mathrm{Prob}\left(\{\hat{\mathcal{D}}_1,\dots,\hat{\mathcal{D}}_q\}\right) \tag{8.65}$$

where Prob $\left(\{\hat{\mathcal{D}}_1,\dots,\hat{\mathcal{D}}_q\}\right)$ *is the probability of the family of closed contours whose union is $\hat{\mathcal{D}}_1\cup\dots\cup\hat{\mathcal{D}}_q$.*

Proof. Let σ be a configuration, and $(\Gamma_1(\sigma),\dots,\Gamma_k(\sigma))$ be the set of large contours of σ. Let $(\Gamma_1',\dots,\Gamma_k')$ be a set of compatible large contours. The set $(\Gamma_1',\dots,\Gamma_k')$ is mapped by F_1 into $(\Gamma_1",\dots,\Gamma_k")$. By definition $F_2(\Gamma_i')=F_2(\Gamma_i")$. Using lemma 8.1 we can write

$$\mathrm{Prob}\left(\{\underline{\Gamma}:F_2(\underline{\Gamma})=(\hat{\mathcal{D}}_1,\dots,\hat{\mathcal{D}}_q)\}\right)= \\ \sum_{\substack{\underline{\Gamma}":\\ F_2(\underline{\Gamma}")=(\hat{\mathcal{D}}_1,\dots,\hat{\mathcal{D}}_q)}}\cdot\sum_{\substack{\underline{\Gamma}':\\ F_1(\underline{\Gamma}')=\underline{\Gamma}"}}\cdot\sum_{\substack{\sigma:\\ \underline{\Gamma}(\sigma)=\underline{\Gamma}'}}\mathrm{Prob}(\sigma)\leq \\ \exp\left(c_2L\cdot O(1/L^{2\beta})\right)\cdot\sum_{\substack{\underline{\Gamma}":\\ F_2(\underline{\Gamma}")=(\hat{\mathcal{D}}_1,\dots,\hat{\mathcal{D}}_q)}}\mathrm{Prob}(\underline{\Gamma}") \tag{8.66}$$

Each contour $\Gamma_i"$ is uniquely decomposed into a family of bare droplets and internal closed contours, since $F_1(\Gamma_i")=\Gamma_i"$. By definition all these internal closed contours are subsets of the internal blocks. By lemma 8.6 there are at most q internal blocks. Indeed, if we remove from the tree all vertices which represent the external blocks, then all vertices of incidence number one represent inner components. Since there are q inner components there are at most q internal blocks. All internal contours are inside the internal blocks. They meet the inner boundaries at the gluing sets, which have lengths $\leq\ln L$. Let us denote by I a gluing set. If we resum in (8.66) over all internal contours which are connected to I, then we get a factor

$$\exp(O(\mathrm{e}^{-2\beta})|\mathrm{I}|)\leq\exp(O(\mathrm{e}^{-2\beta})\ln L) \tag{8.67}$$

Therefore, we can bound (8.66) by

$$(L^{2q}3^{q\ln L})\cdot\exp\left(c_2L\cdot O(1/L^{2\beta})\right)\cdot\exp(qO(\mathrm{e}^{-2\beta})\ln L) \\ \mathrm{Prob}\left(\hat{\mathcal{D}}_1,\dots,\hat{\mathcal{D}}_q\right) \tag{8.68}$$

the factor $(L^2 3^{\ln L})^q$ giving a very rough bound on the number of possible choices of the gluing sets of type I.

We can now estimate the probability of an arrangement of droplets. Let $S=(t_1,\dots,t_n)$ be a sequence of points defining a droplet. We define

$$\tau(S)=\sum_{i=1}^{n}\tau(t_i,t_{i+1}) \tag{8.69}$$

with $t_{n+1}\equiv t_1$ and $\tau(t_i,t_{i+1})$ as in (7.30).

Lemma 8.8

Let $\{S_1,\dots,S_p\}$ be an admissible arrangement of droplets. If β is large enough, then

$$\mathrm{Prob}\left(\{S_1,\dots,S_p\}\right)\leq\left(\prod_{i=1}^{p}\mathrm{e}^{-\tau(S_i)}\right)\cdot\exp(p\cdot O(\ln L)) \tag{8.70}$$

Proof.

In any configuration compatible with the arrangement of droplets there are p bare droplets, one for each S_i. The lemma is proven using lemmas 8.7, and 6.7. We use lemma 8.7 in order to reduce the estimation to the estimation of the probability of the family of p bare droplets. Let us consider the case where we have a unique droplet $S = (t_0, \ldots, t_{n-1})$. We can decompose any bare droplet $\mathcal{D}$, $S(\mathcal{D}) = S$, into n pieces θ_i, $i = 1, \ldots, n$, with cutting points $t_1, \ldots, t_n = t_0$. Let $\mathcal{D} = (\delta D; \lambda_1, \ldots, \lambda_k)$ and s_i be the value of the parameter s such that $\delta D(s_i) = t_i$. For $i = 2, \ldots, n$ we define θ_i as the open contour which is formed by the union of the part of the inner boundary $\{\delta D(s) : s_{i-1} \leq s \leq s_i\}$ and all external contours λ which are connected to that part of the inner boundary except at the point t_{i-1}. Since $[\delta D(s_{i-1}), \delta D(s_{i-1} + 1)]$ is an edge of the exterior enveloppe, the contour θ_i is irreductible at t_{i-1}, when $i = 2, \ldots, n-1$. The contour θ_n is irreductible at $\{t_0, t_{n-1}\}$ since $[\delta D(0), \delta D(1)]$ is an also edge of the exterior enveloppe. The contour θ_1 is defined as the union of the part of the inner boundary $\{\delta D(s) : 1 \leq s \leq s_1\}$ and all external contours λ which are connected to that part of the inner boundary. Since all external contours are inside external blocks it follows from the tree-graph structure of a large contour that all other requirements for a decomposition with cutting points are satisfied. Therefore we can apply lemma 6.7. When several bare droplets are present it is possible that they are not disjoint. The only possibility is that they are joined by an internal block consisting of a single point of the exterior enveloppe where two large cycles meet. By the tree-graph structure of a large contour there are at most p such points in a configuration of p bare droplets. Using the remark following the proof of lemma 6.7, and the fact there are at most $2^p \cdot (L^2)^p$ different families of at most p points, we get an extra factor to the estimate of lemma 6.7, which is

$$2^p \cdot (L^2)^p = \exp(pO(\ln L)) \tag{8.71}$$

Lemma 8.9

Let β be large enough. Then for any δ, $0 < \delta < b$, and L large enough

$$\text{Prob}\left(\sum_i \tau(S_i) \geq \hat{T}\right) \leq \exp\left(-\hat{T}(1 - O(1/\beta L^{b-\delta}))\right) \tag{8.72}$$

Proof.

The proof is similar to the proof of lemma 8.2. Let S_i be a droplet and $n_i = n(S_i)$ be the number of points of the sequence S_i. Let us suppose that we have k droplets, $S_1, \ldots, S_k$ and that $n_1 \leq \cdots \leq n_k$ and $\sum n_i = N$. Let $T = \sum \tau(S_i)$. We have

$$\begin{aligned} &\text{Prob}\left(\{S_1, \ldots, S_k\}\right) \leq \\ &\exp(-T)\exp(kO(\ln L)) = \\ &\exp(-T + kL^\delta)\exp(-kL^\delta + kO(\ln L)) \end{aligned} \tag{8.73}$$

Clearly $k \leq N$ and N is such that

$$N \leq \frac{T}{c_5 L^b + \ln L} \cdot \frac{1}{\tau^*} = O(L^{1-b}) \tag{8.74}$$

with $\tau^* = \max \tau(n) = O(\beta)$. Therefore

$$\begin{aligned} &\text{Prob}\left(\{S_1, \ldots, S_k\}\right) \leq \\ &\exp\left(-T + O(L^{1-b})L^\delta\right) \cdot \exp\left(-N(L^\delta - O(\ln L))\right) \end{aligned} \tag{8.75}$$

We estimate the number of droplets with $n_1 + \cdots + n_k = N$. There are at most $(L^2)^k$ choices for choosing the first points of the k droplets. If we have chosen the first i points of a droplet, then there are at most $2 \ln L(c_5 L^b + \ln L)$ choices for the next point. The number of droplets with $n_1 + \cdots + n_k = N$ is smaller than

$$\begin{aligned} &\sum_k q(N,k) \cdot \left(2 \ln L(c_5 L^b + \ln L)\right)^N \cdot (L^2)^k \leq \\ &\sum_k q(N,k) \cdot \left(2 \ln L(c_5 L^b + \ln L) L^2\right)^N = \\ &\exp(NO(\ln L)) \end{aligned} \tag{8.76}$$

Therefore, we have (for L large enough)

$$\begin{aligned} &\text{Prob}\left(\sum_i T(S_i) \geq \hat{T}\right) \leq \\ &\exp\left(-\hat{T}\left(1 - O(1/\beta L^{b-\delta})\right)\right) \cdot \sum_{N \geq 1} \exp\left(-N(L^\delta - O(\ln L))\right) \leq \\ &\exp\left(-\hat{T}\left(1 - O(1/\beta L^{b-\delta})\right)\right) \end{aligned} \tag{8.77}$$

Theorem 8.4

Let b be such that $c < b < 1 - c = a$. Let $\mathcal{E}_1$ be the event which is the intersection of the set of configurations such that the sum of the lengths of the large contours is smaller than $c_2 \cdot L$ and the set

$$\mathcal{E}_1' = \left\{\sigma : \sum_{S_i \in \underline{S}(\sigma)} \text{vol}(S_i) \geq V(m)\left(1 - \frac{c_4}{\alpha(m) L^c} - O\left(\frac{L^b}{L}\right)\right)\right\} \tag{8.78}$$

with $O(L^b/L)$ as in theorem 8.3. Let $\mathcal{E}_2$ be the event

$$\mathcal{E}_2 = \left\{\sigma : \sum_{S_i \in \underline{S}(\sigma)} \tau(S_i) \leq T^*(m)\left(1 + 2O\left(\frac{c_0}{L^c}\right)\right)\right\} \tag{8.79}$$

with $O(c_0/L^c)$ the function appearing in the estimation of Prob$(A(m))$ *in theorem 7.1. If β is large enough, then*

$$\begin{aligned} \text{Prob}\left(\{\mathcal{E}_1 \cap \mathcal{E}_2 | A(m)\}\right) &\geq 1 - 2\exp\left(-1/2 \cdot T^*(m) \cdot O\left(\frac{c_0}{L^c}\right)\right) \\ &= 1 - 2\exp\left(-\beta \cdot O(L^{1-c})\right) \end{aligned} \tag{8.80}$$

Proof.

By theorem 8.3 $\mathcal{E}_1$ contains the subset of E, and therefore $\mathrm{Prob}(\mathcal{E}_1^c|A(m))$ can be estimated by theorem 8.2. We estimate the complementary event $\mathcal{E}_2^c$ of $\mathcal{E}_2$ under the condition $A(m)$.

$$\begin{aligned}\mathrm{Prob}(\mathcal{E}_2^c|A(m)) &= \mathrm{Prob}(A(m)|\mathcal{E}_2^c)\frac{\mathrm{Prob}(\mathcal{E}_2^c)}{\mathrm{Prob}(A(m))} \\ &\leq \frac{\mathrm{Prob}(\mathcal{E}_2^c)}{\mathrm{Prob}(A(m))}\end{aligned} \tag{8.81}$$

By lemma 8.9 we have

$$\mathrm{Prob}\left(\sum \tau(S_i) \geq T^*(1+2O(\frac{c_0}{L^c}))\right) \leq \tag{8.82}$$
$$\exp\left(-T^*\left(1+2O(\frac{c_0}{L^c}) - O(\frac{1}{\beta L^{b-\delta}}) - O(\frac{1}{\beta L^{c+b-\delta}})\right)\right)$$

and by theorem 7.1 we have

$$\mathrm{Prob}(A(m)) \geq (1-\epsilon)\exp\left(-T^*(1+O(\frac{c_0}{L^c}))\right) \tag{8.83}$$

We choose b, $c<b<1-c$, and δ so that $b-\delta>c$. Therefore, if L is large enough

$$\mathrm{Prob}(\mathcal{E}_2^c|A(m)) \leq \exp\left(-1/2\cdot T^*(m)O(\frac{c_0}{L^c})\right) \tag{8.84}$$

We have (see theorem 8.2)

$$\begin{aligned}\mathrm{Prob}((\mathcal{E}_1\cap\mathcal{E}_2)^c|A(m)) &\leq \mathrm{Prob}(\mathcal{E}_1^c|A(m)) + \mathrm{Prob}(\mathcal{E}_2^c|A(m)) \\ &\leq 2\exp\left(-1/2\cdot T^*(m)O(\frac{c_0}{L^c})\right)\end{aligned} \tag{8.85}$$

Remark.
The factor 2 in (8.79) can be replaced by any factor strictly larger than 1. Then in (8.80) the factor 1/2 is replaced by $(\nu-1)/2$.

9 Large deviations and phase separation.

We come to the last step of the analysis. From the above results we know a set of typical configurations $\mathcal{E}_1\cap\mathcal{E}_2$, which is the union of subsets $E(S_1,\ldots,S_k)$ where $S_1,\ldots,S_k$ is an arrangement of droplets. We first prove that all arrangements of droplets of configurations of $\mathcal{E}_1\cap\mathcal{E}_2$ have only a single droplet of volume larger than L^{2a} if β is large enough. This is a consequence of a lemma due to Minlos and Sinai. The phase separation and the large deviations results follow then easily.

Lemma 9.1 ([M.S.1])

Let d_i, $i=1,\ldots,r$, be r positive numbers such that

$$\sum_{i=1}^r d_i \leq 1+\epsilon_1 \tag{9.1}$$

and

$$\sum_{i=1}^r d_i^2 \geq 1-\epsilon_2 \tag{9.2}$$

ϵ_1 and ϵ_2 positive. Let $d_{\max}=\max(d_i : i=1,\ldots,r)$. If ϵ_1, ϵ_2 are sufficiently small, then there exists a positive function, $\epsilon_3(\epsilon_1,\epsilon_2)$, such that ϵ_3 tends to zero when ϵ_1 and ϵ_2 tend to zero, and such that

$$d_{\max} \geq 1-\epsilon_3 \ , \quad \sum_{d_i\neq d_{\max}} d_i \leq \epsilon_1+\epsilon_3 \tag{9.3}$$

Proof.([M.S.1])

Let $r=2$ and let us suppose that $d_1\geq d_2$. Thus $d_{\max}=d_1$ and we have

$$d_1+d_2 \leq 1+\epsilon_1 \ , \quad d_1^2+d_1^2 \geq 1-\epsilon_2 \tag{9.4}$$

Therefore

$$(d_1+d_2)^2 = d_1^2+d_2^2+2d_1d_2 \leq (1+\epsilon_1)^2 \tag{9.5}$$

and

$$2d_1d_2 \leq 1+\epsilon_1^2+2\epsilon_1-1+\epsilon_2 = \epsilon_1^2+2\epsilon_1+\epsilon_2 \tag{9.6}$$

We have from (9.4) that

$$d_1 \geq \left(\frac{1-\epsilon_2}{2}\right)^{1/2} \tag{9.7}$$

and from (9.6) and (9.7)

$$d_2 \leq \frac{1}{\sqrt{2}}\cdot\frac{\epsilon_1^2+2\epsilon_1+\epsilon_2}{(1-\epsilon_2)^{1/2}} \equiv \epsilon_4(\epsilon_1,\epsilon_2) \tag{9.8}$$

and

$$d_1 \geq (1-\epsilon_2-d_2^2)^{1/2} \geq (1-\epsilon_2-\epsilon_4^2)^{1/2} \equiv 1-\epsilon_3 \tag{9.9}$$

Let us consider the general case. We divide $d_1,\dots,d_r$ in two groups, $d_{i_1},\dots,d_{i_k}$ and $d_{j_1},\dots,d_{j_m}$, and set

$$\tilde{d}_1 = d_{i_1}+\cdots+d_{i_k} \ , \ \tilde{d}_2 = d_{j_1}+\cdots+d_{j_m} \tag{9.10}$$

Clearly

$$\tilde{d}_1+\tilde{d}_2 \leq 1+\epsilon_1 \ , \ \tilde{d}_1^2+\tilde{d}_2^2 \geq 1-\epsilon_2 \tag{9.11}$$

Therefore $\tilde{d}_{\max} \geq 1-\epsilon_3$, $\tilde{d}_{\min} \leq \epsilon_4$ and

$$|\tilde{d}_1-\tilde{d}_2| \geq 1-\epsilon_3-\epsilon_4 \tag{9.12}$$

Let us consider the case where in one group we take only one element $d_{\max}$ which we label by d_1. Let us prove that $d_{\max} = \tilde{d}_{\min}$ is impossible. Indeed, $d_{\max} \leq \epsilon_4$ implies that $d_i \leq \epsilon_4$ for all $i \geq 2$. In this case we can always divide $d_1,\dots,d_r$ into two new groups such that

$$|\tilde{d}_1-\tilde{d}_2| \leq 2\epsilon_4 \tag{9.13}$$

which is in contradiction with (9.12) if ϵ_1 and ϵ_2 are small enough. We have therefore proven that

$$d_1 \equiv d_{\max} \geq 1-\epsilon_3 \ , \ \sum_{i\geq 2} d_i \leq \epsilon_4 \tag{9.14}$$

We notice that

$$\sum_{i\geq 2} d_i = \sum_{i\geq 1} d_i - d_1 \leq \epsilon_1+\epsilon_3 \tag{9.15}$$

Let us call *large droplet* any droplet whose volume is larger than L^{2a}.

Theorem 9.1

Let $\mathcal{E}_1 \cap \mathcal{E}_2$ be the set of theorem 8.4. If β is large enough then there is a single large contour in any configurations σ of $\mathcal{E}_1 \cap \mathcal{E}_2$. In all arrangements of droplets of configurations σ of $\mathcal{E}_1 \cap \mathcal{E}_2$ there is a single large droplet $S(\sigma)$ such that

$$\mathrm{vol}(S) \geq V(m)\left(1-\frac{c_4}{\alpha(m)L^c}-O(\frac{L^b}{L})\right) \tag{9.16}$$

and

$$\tau(S) \leq T^*(m)\left(1+2O(\frac{c_0}{L_c})\right) \tag{9.17}$$

Proof.

From theorem 8.4 we know that a typical set of configurations, under the condition $A(m)$, is the set of all configurations which contain droplets $S_1,\dots,S_k$ such that

$$\sum_{i=1}^{k} \tau(S_i) \leq T^*(m)\cdot\left(1+2O(\frac{c_0}{L^c})\right) \tag{9.18}$$

and

$$\sum_{i=1}^{k} \mathrm{vol}(S_i) \geq V(m)\cdot\left(1-\frac{c_4}{\alpha(m)L^c}-O(\frac{L^b}{L})\right) \tag{9.19}$$

For droplets the isoperimetric inequality is

$$\tau(S_i)^2 \geq 4\cdot|W_\tau|\cdot\mathrm{vol}(S_i) \tag{9.20}$$

and therefore we have

$$\begin{aligned}\sum_{i=1}^{k}\tau(S_i)^2 &\geq 4\cdot|W_\tau|\cdot\sum_{i=1}^{k}\mathrm{vol}(S_i)\\ &\geq 4\cdot|W_\tau|\cdot V(m)\cdot\left(1-\frac{c_4}{\alpha(m)L^c}-O(\frac{L^b}{L})\right)\end{aligned} \tag{9.21}$$

But, the relation between $T^*(m)$ and $V(m)$ is precisely

$$(T^*(m))^2 = 4\cdot|W_\tau|\cdot V(m) \tag{9.22}$$

and

$$V(m) = \alpha(m)|\Lambda| \ , \ \alpha(m) = \frac{m^*-m}{2m^*} \tag{9.23}$$

Therefore, by putting $d_i = \tau(S_i)/T^*(m)$ we get

$$\sum_{i=1}^{k} d_i \leq 1+\epsilon_1 \ \text{ and } \ \sum_{i=1}^{k} d_i^2 \geq 1-\epsilon_2 \tag{9.24}$$

with

$$\epsilon_1 = 2O(\frac{c_0}{L^c}) \ , \ \epsilon_2 = \frac{c_4}{\alpha(m)L^c}+O(\frac{L^b}{L}) \tag{9.25}$$

and we may choose (see theorem 8.2)

$$c_4 = \frac{c_0}{2m^*}+1/4(\alpha(m)|W_\tau|)^{1/2}\frac{\kappa}{m^*\beta} \ , \ \kappa > 1 \tag{9.26}$$

From the lemma 9.1, we know that there exists a large droplet, say S_1, such that

$$\tau(S_1) \geq (1-\epsilon_3)\cdot T^*(m) \tag{9.27}$$

and otherwise

$$\sum_{i\geq 2}\tau(S_i) \leq (\epsilon_1+\epsilon_3)\cdot T^*(m) \tag{9.28}$$

Let us examine this last inequality. We recall that $0 < c < 1/2$ and $c < b < 1 - c$. Therefore the dominant term in ϵ_2 is the first term and we can neglect the second term for large L. We set

$$\epsilon := \frac{1}{4\alpha(m)} \cdot \frac{\kappa}{\beta m^*} \cdot (\alpha(m)|W_\tau|)^{1/2} \cdot \frac{1}{L^c} \tag{9.29}$$

and we have

$$\epsilon_2 = \epsilon + \frac{1}{2m^*\alpha(m)} \cdot \frac{c_0}{L^c} \tag{9.30}$$

We can choose c_0 as small as we want so that we have for c_0 very small

$$\epsilon_2 \approx \epsilon \text{ and } \epsilon_3 \approx \epsilon_2/2 \approx \epsilon/2 \tag{9.31}$$

because $|W_\tau| = O(\beta^2)$. Therefore

$$\sum_{i\geq 2} \mathrm{vol}(S_i) \leq \frac{1}{4|W_\tau|}\sum_{i\geq 2}\tau(S_i)^2 \leq \tag{9.32}$$

$$\frac{1}{4|W_\tau|}\left(\sum_{i\geq 2}\tau(S_i)\right)^2 \approx \frac{1}{4|W_\tau|}\cdot\frac{\epsilon^2}{4}(T^*(m))^2 =$$

$$\frac{\kappa^2}{4(m^*)^2}\cdot\frac{|W_\tau|}{16\beta^2}\cdot L^{2a}$$

When β tends to infinity the Wulff crystal is a square of side 2, if we normalize the surface tension by dividing by β. In our case

$$\lim_{\beta\to\infty}\frac{|W_\tau|}{16\beta^2} = 1 \tag{9.33}$$

Thus, for large β, the total volume of the droplets $S_2, \ldots, S_k$ is at most $1/4L^{2a}$ since $m^* \approx 1$ and we can choose $\kappa > 1$ as small as we want. This implies the existence of a single large droplet. We have two possibilities: either there is only one large contour, and each droplet is associated with this large contour, or there are several large contours. This second possibility is excluded for large enough β. The total length of the boundaries of the droplets $S_i, i \geq 2$ is at most of order $O(L^{1-c})$. Therefore the total number of the points of the sequences $S_2, \ldots, S_k$ is of order at most $O(L^{1-c-b})$. The total volume of the droplets which are not linked to S_1 by the same large contour is at least

$$L^{2a} - O(L^{1-c+b}) \tag{9.34}$$

Indeed, each large contour has an inner component of volume larger than L^{2a}. But since $c < b < 1 - c = a$ we have $1 - c + b < 2a$, and we get a contradiction because from (9.32) we know that the total volume of the droplets $S_2, \ldots, S_k$ is at most $1/4L^{2a}$ when β is large. We conclude that there is a unique large contour in any configuration of the set $\mathcal{E}_1 \cap \mathcal{E}_2$.

We recall the following definitions. Let $\tau(n)$ be the surface tension of an interface perpendicular to the unit vector n of $\mathbb{R}^2$. Let W_τ be the Wulff crystal,

$$W_\tau = \{x \in \mathbb{R}^2 : \langle n|x\rangle \leq \tau(n)\} \tag{9.35}$$

where $\langle\cdot|\cdot\rangle$ is the Euclidean scalar product. The volume of W_τ is denoted by $|W_\tau|$. By a dilatation of the Wulff crystal we define a set $W_\tau(m)$ of volume $V(m) = \alpha(m)|\Lambda|$, with $\alpha(m) = (m^* - m)/2m^*$. The value of the Wulff functional for this set is $T^*(m)$, $(T^*(m))^2 = 4|W_\tau|\cdot V(m)$. Let $\Lambda = \Lambda(L)$ be a square box of volume L^2 and $A(m) = A(m; c, c_0)$ be the set

$$A(m; c, c_0) = \{\sigma : |\sum_{t\in\Lambda}\sigma(t) - m|\Lambda|| \leq c_0|\Lambda|\cdot L^{-c}\} \tag{9.36}$$

with $0 < c < 1/2$, and c_0 not too large.

Theorem 9.2 (Phase separation)

Let $-m^* < m < +m^*$, m not too small (see remark below). Let E be the subset of all configurations σ such that

- *there is a single large contour Γ of length $|\Gamma| \leq c_2L$*
- *the volume of Γ is such that $|\mathrm{vol}(\Gamma) - V(m)| \leq c_4|\Lambda|\cdot L^{-c}$*

The constants c_2, c_4 are defined in theorem 8.2. If β and L are large enough, then

$$\mathrm{Prob}^+_\Lambda(E|A(m)) \geq 1 - \exp(-\beta O(L^{1-c})) \tag{9.37}$$

The conditional probability is computed with the finite Gibbs measure μ^+_Λ.

Theorem 9.3 (Phase separation)

Let $-m^* < m < +m^*$, m not too small (see remark below). Let S be a large droplet, $E(S)$ be the set of all configurations σ having only one large droplet S, and $\mathcal{E}$ be the union of the sets $E(S)$ such that

- *the τ-length of S is such that $|\tau(S) - T^*(m)| \leq T^*(m)\cdot O(L^{-c})$*
- *the volume of S is such that $|\mathrm{vol}S - V(m)| \leq |\Lambda|\cdot O(L^{-c})$*

If β and L are large enough, then

$$\mathrm{Prob}^+_\Lambda(\mathcal{E}|A(m)) \geq 1 - \exp(-\beta O(L^{1-c})) \tag{9.38}$$

The conditional probability is computed with the finite Gibbs measure μ^+_Λ.

Proof.

The lower bound on $\tau(S)$ is a consequence of the isoperimetric inequality and of the lower bound on the volume of $\tau(S)$.

Theorem 9.4 (Large deviations)

Let $-m^* < m < +m^*$, m not too small (see remark below). If β is large enough, then

$$\lim_{L\to\infty} -\frac{1}{L} \ln \mathrm{Prob}^+_{\Lambda(L)}(A(m;c,c_0)) = 2(|W_\tau| \cdot \alpha(m))^{1/2} \qquad (9.39)$$

The probability is computed with the finite Gibbs measure μ^+_Λ.

Remark.
The value of m must be such that the square box Λ contains a set isometric to $W_\tau(m)$. If not, the results above are not correct. They remain correct if we choose a box Λ which is obtained by a dilatation of the Wulff crystal.

Proof.

Theorem 7.1 gives an upper bound. For the other bound we may consider only the case $0 < c < 1/2$. We have

$$\begin{aligned} \mathrm{Prob}(A) &= \mathrm{Prob}(A \cap \mathcal{E}) + \mathrm{Prob}(A \cap \mathcal{E}^c) \qquad (9.40)\\ &= \mathrm{Prob}(A \cap \mathcal{E}) + \mathrm{Prob}(\mathcal{E}^c|A)\mathrm{Prob}(A)\\ &\leq \mathrm{Prob}(\mathcal{E}) + \mathrm{Prob}(\mathcal{E}^c|A)\mathrm{Prob}(A) \end{aligned}$$

But

$$\mathrm{Prob}(\mathcal{E}) \leq \mathrm{Prob}(\{\sigma : \tau(\underline{S}(\sigma)) \geq T^*(m)(1 + O(L^{-c})\}) \qquad (9.41)$$

Where $\underline{S}(\sigma)$ is the arrangement of droplets of the configuration σ. The theorem follows from lemma 8.9

Conclusion.

The single large contour Γ of each configuration of the set $E \cap \mathcal{E}$ can be decomposed into its inner boundary δD whose length is $O(L)$ and a family of closed contours θ, which are disjoint two by two and which have at least one side in common with Γ, but no edge in common with Γ. Let $\mathcal{E}^*$ be the subset of configurations σ of $E \cap \mathcal{E}$ which are characterized by the fact that each small contour γ has a length $|\gamma| \leq CL^a$ and each closed contour θ of the decomposition of Γ has a length $|\theta| \leq CL^a$. The constant a satisfies $1/2 < a < 1$. From the theorems on the phase separation, the results of 6.41 and theorem 7.1 there exist β_0, L_0 and a constant C such that for all $\beta_0 > \beta_0$ and $L > L_0$,

- $\mathrm{Prob}^+_\Lambda(\mathcal{E}^*|A(m)) \geq 1 - \exp(-\beta O(L^a))$

There is another picture of the typical configurations at a scale of order $O(L^a)$. Let S be a droplet of $\mathcal{E}^*$ and $E(S)$ be the subset of all configurations of $\mathcal{E}^*$ associated with the droplet S. Let $\Delta(\Lambda)$ be the set of all $t \in \Lambda$ which are at a distance smaller than $C_1 L^a$ from the boundary of Λ, and $\Delta(S)$ be the set of all $t \in \Lambda$ which are at a distance smaller than $C_1 L^a$ from a point t_i of S. There exists a constant C_1 so that

- the unique large contour Γ of any configuration of $E(S)$ is in $\Delta(S)$.

The set $\Lambda \backslash (\Delta(\Lambda) \cup \Delta(S))$ has two connected components, $\Lambda_+(S)$ and $\Lambda_-(S)$, which are serarated by $\Delta(S)$. Let A be a finite subset of $\mathbb{Z}^2$. By choosing the constant C_1 large enough we get the following results (see section 4 and also appendix A in [B.L.P.2]):

- if A is in Λ_+, then

$$|\langle \sigma(A)|E(S)\rangle^+_\Lambda - \langle \sigma(A)\rangle^+| \leq O(\exp(-\beta L^a))\langle \sigma(A)\rangle^+$$

- if A is in Λ_-, then

$$|\langle \sigma(A)|E(S)\rangle^+_\Lambda - \langle \sigma(A)\rangle^-| \leq O(\exp(-\beta L^a))\langle \sigma(A)\rangle^-$$

Let $W_\tau(m)$ be the Wulff droplet of volume $V(m)$, and $\Delta(m)$ be the subset

$$\Delta(m) = \{t \in \mathbb{R}^2 : d_2(t, W_\tau(m) \leq C_2 L^{\frac{1+a}{2}}\}$$

There exists a constant C_2 so that

- any polygon constructed with the vertices of a droplet of $\mathcal{E}^*$ is covered by a set isometric to $\Delta(m)$.

This statement is a consequence of the generalization of the classical geometric inequalities of Bonnesen to the case of the τ-length (see [D.K.S]). Let c be a rectifiable Jordan curve, which is the boundary of a region G of volume $\mathrm{vol}(c)$. Let

$$\begin{aligned} r(c) &= \sup\{r : r \cdot W_\tau + x \subset G \text{ for some } x \in \mathbb{R}^2\}\\ R(c) &= \inf\{R : R \cdot W_\tau + x \supset G \text{ for some } x \in \mathbb{R}^2\} \end{aligned}$$

where $\rho \cdot W_\tau = \{y = \rho \cdot x : x \in W_\tau\}$. Bonnesen's inequalities are

$$\frac{\tau(c) - \sqrt{\tau(c)^2 - 4|W_\tau|\mathrm{vol}(c)}}{2|W_\tau|} \leq r(c) \leq R(c) \leq \frac{\tau(c) + \sqrt{\tau(c)^2 - 4|W_\tau|\mathrm{vol}(c)}}{2|W_\tau|}$$

10 A shorter proof of the main results.

It is possible to get a shorter proof of the main results, if another notion of contours is introduced. Let Λ be a finite subset of $\mathbb{Z}^2$, and let σ be a spin configuration in Λ compatible with the + b.c. for Λ. The configuration is uniquely specified by a family of Λ^+-compatible contours $(\gamma_1, \ldots, \gamma_n)$. We say that a point $t^* \in \mathbb{Z}^2_*$ is a crossing point of a contour γ if there are four edges of γ which contain the point t^*. We modify the family of contours as follows :

- at each crossing point we change the contours according to rule a) of figure 8
- we round off the corners of the contours according to rules b), c) of figure 8.

After these transformations we get a family of closed simple disjoint lines, $\gamma'_1, \ldots, \gamma'_m$, which we call *simple contours.* We define for the simple contours the notions of $\mathrm{int}\gamma'$, $\overline{\mathrm{int}\gamma'}$, Λ^*-compatibility and Λ^+-compatibility as before. We also use an equivalent description, which is more convenient for the duality. We introduce some order for the points of the dual lattice and we orient each simple contour so that the interior of the simple contour is at the left-hand side. We deform the simple contours so that they are again drawn on the dual lattice and we consider these new lines as unit-speed parametrized curves, the origin of the curve being the first point of the curve for the order of the dual lattice. We call these lines *parametrized contours.* We still denote the parametrized contours by $\gamma'_1, \ldots, \gamma'_m$. The length of γ' is by definition the length of the parametrized contour γ'. A simple contour, or a parametrized contour is,

- *small* if $\mathrm{vol}\gamma' \leq L^{2a}$, $1/2 < a < 1$,
- *large* if $\mathrm{vol}\gamma' > L^{2a}$.

The great advantage of the simple contours is that their geometry is trivial. Therefore we can avoid a large part of the discussion of section 8. The main steps of the analysis are summarized in remark 3 of the introduction. The first three steps are proven in essentially the same way when we adopt the new definition of contours. The proof of theorem 7.1 is in fact simpler, since we do not need the discussion of points 4, 5 and 6 of the previous proof (see below). We concentrate the discussion on the changes which occur in section 8. We prove a lemma replacing lemma 8.8. Once this lemma is established, then theorem 8.4 is proved as before, and hence also theorems 9.2, 9.3 and 9.4.

As in section 6.4.2 we use the duality and correlation inequalities. It is therefore convenient to work with parametrized contours. Let Γ' be a large parametrized contour. We define the notion of droplet. (This is essentially the skeleton of [D.K.S].) A droplet is specified by an ordered sequence of points of Γ'. The first point of the sequence is the origin of Γ', $t_1 = \Gamma'(s = 0)$. The next point, t_2, is $\Gamma'(s')$ so that s' is the first value of the parameter $s \in \mathbb{N}$ such that $d_2(\Gamma'(0), \Gamma'(s)) \geq L^b$. As before $c < b < a = 1 - c$. The next point is defined similarly and so on. In this way we can associate with each large parametrized contour a sequence of points

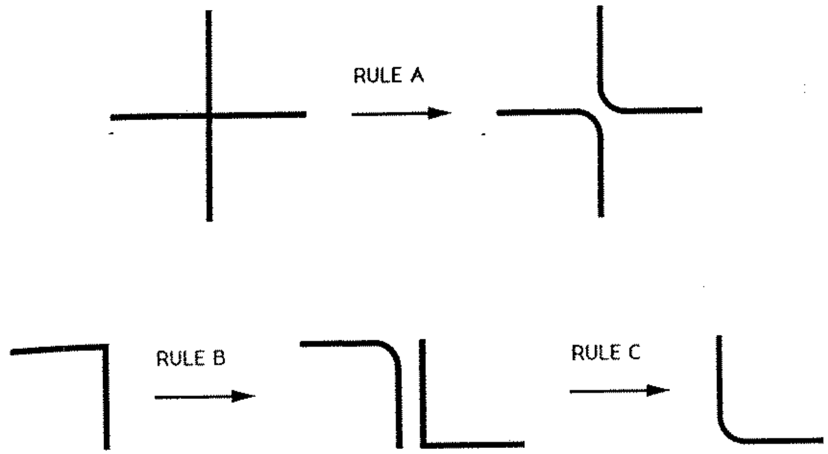

Figure 8: Modification rules for the contours.

$S(\Gamma') = \{t_1, \ldots, t_n\}$, which we call a *droplet.* (It is possible that we have $t_i = t_j$ for $i \neq j$ since the parametrized contours may have points of multiplicity two). An arrangement of droplets $S_1, \ldots, S_k$ is defined as in section 8 and $E(S_1, \ldots, S_k)$ is the set of all configurations associated with the arrangement of droplets $S_1, \ldots, S_k$. (It is possible that the same point t of the dual lattice occurs in two different droplets since the parametrized contours are not disjoint.) For any droplet $S = \{t_1, \ldots, t_n\}$ we define its τ-length,

$$\tau(S) := \sum_{j=1}^{n} \tau(t_j, t_{j+1}) \, , \; t_{n+1} \equiv t_1 \tag{10.1}$$

Here $\tau(t_j, t_{j+1})$ is defined as in (7.30). We prove a lemma replacing lemma 8.8.

Lemma 10.1

Let Λ be a simply connected finite set of $\mathbb{Z}^2$. Let $S_1, \ldots, S_k$ be an arrangement of k droplets, the droplet S_i having n_i points. If β is large enough, then

$$\mathrm{Prob}(E(S_1, \ldots, S_k)) \leq \prod_{j=1}^{k} \exp\left(n_j O(e^{-2\beta})\right) \prod_{j=1}^{k} e^{-\tau(S_j)} \tag{10.2}$$

The probability is computed with the Gibbs measure μ_Λ^+.

Proof.

To simplify the notations we consider the case $k = 1$ of a single droplet $S = \{t_1, \ldots, t_n\}$. Let Γ' be a large contour such that $S(\Gamma') = S$. We decompose the parametrized contour Γ' into n open parametrized contours, $\lambda_1, \ldots, \lambda_n$. By definition

$$\lambda_i := \{\Gamma'(s) | s_i \leq s \leq s_{i+1}\} \tag{10.3}$$

where s_i is defined by $t_i = \Gamma'(s_i)$. We must estimate

$$\begin{aligned} \mathrm{Prob}(E(S)) &= \sum_{\substack{\Gamma': \\ S(\Gamma')=S}} \mathrm{Prob}(\Gamma') \\ &= \sum_{\substack{\lambda_1,\ldots,\lambda_n: \\ \delta\lambda_i = \{t_i, t_{i+1}\}}} \prod_{k=1}^{n} (\tanh\beta^*)^{|\lambda_k|} \frac{Z(\Lambda^*(\lambda_1, \ldots, \lambda_n))}{Z(\Lambda^*)} \end{aligned} \tag{10.4}$$

where the partition function $Z(\Lambda^*(\lambda_1, \ldots, \lambda_n))$ is the partition function of an Ising model with free boundary condition, at inverse temperature β^*, in a finite box $\Lambda^*(\lambda_1, \ldots, \lambda_n)$. This box contains all spins of Λ^* which are not on the parametrized contours $\lambda_1, \ldots, \lambda_n$ and all spins on these parametrized contours which are at corners which are modified by rules b) and c). We *remove* all coupling constants of the model between two spins which are on these parametrized curves. The partition functions are normalized as in (6.5), see also (6.23). We fix for the moment $\lambda_2, \ldots, \lambda_n$ and sum over λ_1. Let $Z(\Lambda^*(\lambda_2, \ldots, \lambda_n))$ be the partition function of an Ising model with free boundary condition, at inverse temperature β^*, in the finite box $\Lambda^*(\lambda_2, \ldots, \lambda_n)$. This box contains all spins of Λ^* which are not on the parametrized contours $\lambda_2, \ldots, \lambda_n$ and all spins on these parametrized contours which are at corners which are modified by rules b) and c), as well as the spins t_1 and t_2 which are on the boundary of the union of these parametrized contours. Again, we *remove* all coupling constants between two spins which are on these parametrized contours. We can interpret

$$e^{-2\beta|\lambda_1|} \frac{Z(\Lambda^*(\lambda_1, \ldots, \lambda_n))}{Z(\Lambda^*(\lambda_2, \ldots, \lambda_n))} \tag{10.5}$$

as a contribution to the expectation value $\langle \sigma(t_1)\sigma(t_2) \rangle^f (\Lambda^*(\lambda_2, \ldots, \lambda_n))$. Therefore by Griffiths'inequalities we can bound the sum over λ_1 by $\langle \sigma(t_1)\sigma(t_2) \rangle^f$. We get

$$\sum_{\substack{\lambda_1,\ldots,\lambda_n: \\ \delta\lambda_i = \{t_i, t_{i+1}\}}} \prod_{k=1}^{n} (\tanh\beta^*)^{|\lambda_k|} \frac{Z(\Lambda^*(\lambda_2, \ldots, \lambda_n))}{Z(\Lambda^*)} \leq \langle \sigma(t_1)\sigma(t_2) \rangle^f \cdot \tag{10.6}$$

$$\sum_{\substack{\lambda_2,\ldots,\lambda_n: \\ \delta\lambda_i = \{t_i, t_{i+1}\}}} \prod_{k=2}^{n} (\tanh\beta^*)^{|\lambda_k|} \frac{Z(\Lambda^*(\lambda_2, \ldots, \lambda_n))}{Z(\Lambda^*(\lambda_3, \ldots, \lambda_n))} \cdot \frac{Z(\Lambda^*(\lambda_3, \ldots, \lambda_n))}{Z(\Lambda^*)}$$

We sum over λ_2 when the other parametrized contours are fixed. Let $\eta_1, \ldots, \eta_m$ be m compatible closed parametrized contours of a configuration contributing to $Z(\Lambda^*(\lambda_2, \ldots, \lambda_n))$. Then the family of parametrized of contours $\lambda_2, \eta_1, \ldots, \eta_m$ is

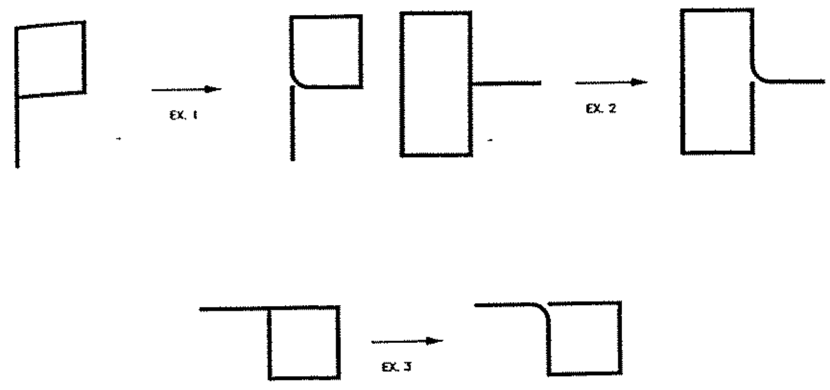

Figure 9: Examples of rule a) for an open contour.

a compatible family of parametrized contours contributing to the numerator of $\langle \sigma(t_2)\sigma(t_3) \rangle^f (\Lambda^*(\lambda_3, \ldots, \lambda_n))$. If one of the parametrized contours η_i touches λ_2 at t_2, then we suppose, without restricting the generality, that this is the parametrized contour η_1. The union of λ_2 and η_1 is denoted by λ_2'. It may happen that λ_2' must be considered as a single parametrized contour. Indeed, we must extend the rule a) to the case of an open contour when three edges have a common point. Examples of this situation are given in figure 9. Whenever this situation occurs we may get the same family of compatible parametrized contours in two different ways : either it is the family $\lambda_2', \eta_2, \ldots, \eta_m$ or it is the family $\lambda_2, \eta_1, \eta_2, \ldots, \eta_m$. In figure 9 this is the case for the third example. To avoid this problem we multiply and divide by the partition function $\hat{Z}(\Lambda^*(\lambda_2, \ldots, \lambda_n))$ which is the partition function for an Ising model defined on $\Lambda^*(\lambda_2, \ldots, \lambda_n) \backslash \{t_2\}$. (As before we have no coupling constant between two spins on the parametrized contours $\lambda_2, \ldots, \lambda_n$.) We have

$$\sum_{\lambda_2} (\tanh\beta^*)^{|\lambda_2|} \frac{\hat{Z}(\Lambda^*(\lambda_2, \ldots, \lambda_n))}{Z(\Lambda^*(\lambda_3, \ldots, \lambda_n))} \leq \langle \sigma(t_2)\sigma(t_3) \rangle^f \tag{10.7}$$

and by the cluster expansion

$$\frac{Z(\Lambda^*(\lambda_2, \ldots, \lambda_n))}{\hat{Z}(\Lambda^*(\lambda_2, \ldots, \lambda_n))} \leq \exp\left(O(e^{-2\beta})\right) \tag{10.8}$$

By repeating this argument we prove lemma 10.1.

Remark.
The bound of lemma 8.8 is better than the bound of lemma 10.1. However, the extra factor is of an order comparable with the entropy estimate of a collection of droplets and therefore the proofs of lemma 8.9 and theorem 8.4 remain the same.

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On the Critical Behavior of the First Passage Time in d ≥ 3

L. Chayes*

Department of Mathematics, University of California
Los Angeles California 90024

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Abstract The first passage times for the Bernoulli percolation problems on the d-dimensional hypercubic lattices are investigated. For all d (and hence for $d \geq 3$) it is rigorously established that, in the critical region, the first passage times tend to zero with the same scaling behavior as the decay rate for correlations in the associated percolation problem.

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