An explicit solution to a system
of implicit differential equations

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Abstract

In this article we give an *explicit* solution to the vectorial differential inclusion where we have to find a map $u : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^2$ with $u = 0$ on $\partial\Omega$ and

$$Du(x) \in O(2), \text{ a.e. } x \in \Omega.$$

Titre: Une solution explicite d'un système implicite d'équations différentielles

Résumé: Nous construisons une solution *explicite* d'une inclusion différentielle où on cherche une application $u : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^2$ avec $u = 0$ sur $\partial\Omega$ et

$$Du(x) \in O(2), \text{ p.p. } x \in \Omega.$$

1 Introduction

In the last few years methods of finding *almost everywhere solutions* of partial differential equations of *implicit type* have been developed; the theory can be applied also to *nonlinear systems* of pdes, for which the notion of *viscosity solution* seem not to be appropriate, mostly because of the lack of *maximum principle*. Some references in this field are the article [7] and the monograph [8] by Dacorogna and Marcellini, where the *Baire category method* is a crucial step in the proof, and the method of *convex integration* by Gromov as in Müller and Sverak [14]. These methods are not constructive, i.e., they give existence of solutions but they do not give a rule to compute them.

Cellina introduced the use of the Baire category method in the study of differential equations (see [3], in the simple but pioneering context of one single ode). Cellina in [4], [5] and Friesecke [11] gave in the scalar case, for a single pde, an explicit solution, which in the monograph we named *pyramid* (see Section 2.3.1 in [8]). Cellina and Perrotta [6] also considered a genuine 3×3 *system* of pdes of *implicit type* and proposed an explicit solution for the associated Dirichlet problem. The solution is described through an iterative scheme: the set Ω is divided into a countable number of cells and the value of the vector-valued function u at $x \in \Omega$ is defined taking into account values of u previously defined

in other cells. The work of Cellina-Perrotta inspired the research presented in this paper.

As we said, the aim of this paper is to introduce a method of computing an explicit solution to the Dirichlet problem for nonlinear systems of pdes. The approach that we propose seems flexible enough to be compatible with a class of nonlinear systems, for an unknown mapping $u : \Omega \subset \mathbb{R}^n \longrightarrow \mathbb{R}^n$, of the form

$$\begin{cases} \lambda_i(Du(x)) = \gamma_i \,, & \text{a.e. } x \in \Omega, \ \ i = 1, 2, \cdots, n, \\ u(x) = \varphi(x) \,, & x \in \partial\Omega \,, \end{cases} \tag{1}$$

where $\{\lambda_i(A)\}_{i=1,2,\cdots n}$ are the *singular values* of the matrix A, i.e., the eigenvalues of $(A^t A)^{1/2}$, and $\{\gamma_i\}_{i=1,2,\cdots,n}$ $(0 < \gamma_1 \leq \cdots \leq \gamma_i \leq \cdots \leq \gamma_n)$ are given real numbers.

We limit ourselves here to the bidimensional case $n = 2$, with the homogenous boundary datum $\varphi = 0$. To describe one of the difficulties, due to the vectorial nature of the problem, consider the system

$$\begin{cases} \lambda_1(Du) = \lambda_2(Du) = 1 \,, & \text{a.e. in } \Omega, \\ u = 0 \,, & \text{on } \partial\Omega \end{cases} \tag{2}$$

and its scalar counter part

$$\begin{cases} |Du| = 1 \,, & \text{a.e. in } \Omega, \\ u = 0 \,, & \text{on } \partial\Omega \,. \end{cases} \tag{3}$$

The scalar problem (3) has $u(x) = \text{dist}(x, \partial\Omega)$ as the unique viscosity solution and it is therefore, close to the boundary, as smooth as the boundary itself. On the contrary, in the vector valued case (2), the solution is Lipschitz continuous but never of class C^1 at any point of the boundary. Indeed (2) is equivalent to

$$\begin{cases} |Du|^2 = 2\,|\det Du| = 2 \,, & \text{a.e. in } \Omega, \\ u = 0 \,, & \text{on } \partial\Omega, \end{cases} \tag{4}$$

since $\lambda_1(Du)\lambda_2(Du) = |\det Du|$ and $\lambda_1^2(Du) + \lambda_2^2(Du) = |Du|^2$. Let us argue by contradiction: if u is smooth in a neighborhood O of the boundary, say of class $C^1(\overline{O}; \mathbb{R}^2)$, then, since $\det Du = \pm 1$, the Jacobian has a sign, say $\det Du = 1$. Therefore the equation $|Du|^2 = 2 \det Du$ can be easily transformed into the system

$$\begin{cases} v_x - w_y = 0 \,, \\ v_y + w_x = 0 \,, & \text{in } \overline{O}, \\ |Dv| = |Dw| = 1 \end{cases} \tag{5}$$

with the notation $u(x, y) = \binom{v(x,y)}{w(x,y)}$ for the components of u. Thus in $\overline{O} \cap \partial\Omega$ we have, denoting by τ, ν respectively the tangent and normal unit vectors and up to a sign,

$$\begin{cases} \langle Dv; \tau \rangle = \langle Dw; \nu \rangle \\ \langle Dw; \tau \rangle = \langle Dv; \nu \rangle \,. \end{cases}$$

Since $v = w = 0$ on $\partial\Omega$, we also obtain $Dv = Dw = 0$ in $\partial\Omega \cap \overline{O}$, which contradicts the fact that $|Dv| = |Dw| = 1$ in $\overline{O}$.

Therefore any solution to the differential problem (2) is Lipschitz continuous and not C^1 near the boundary; thus it assumes in a *fractal way* the homogenous boundary datum $\varphi = 0$.

Our vector-valued solution u of (2) will be explicitly defined at every $(x, y) \in \Omega$ and it will be piecewise affine, with infinitely many pieces (to be in accord with its fractal nature near the boundary of Ω). Moreover one component of $u(x, y) = \left(\begin{smallmatrix} v(x,y) \\ w(x,y) \end{smallmatrix}\right)$, say $v(x, y)$, will be defined in an elementary way, having in mind the scalar *pyramid* by Cellina and Friesecke, as described above; on the contrary, the other component $w(x, y)$ needs more care and, at the end, matches perfectly to form a solution $u(x, y)$ to the Dirichlet problem (2). This is one of the main differences with the work by Cellina-Perrotta, where the components mix at every step, one component being dependent from the values of all the other components at the previous cells. Therefore it seems that we have a better degree of flexibility.

After completing our work, we learnt from V. Sverak that R.D. James [12] had some hints on how to find an explicit solution for the problem we are considering. His construction uses 6 matrices and his basic cells are composed of triangles, trapezoids and pentagons all fitted in an hexagon.

Ball-James [1] pointed out the importance of such questions in nonlinear elasticity. Existence results for solutions to nonlinear systems of pdes of *implicit type* can be found in [8], [14], see also [2], [9], [10]; in particular Kirchheim [13] considers differential inclusions with finitely many gradients not necessarily differing by rank one. However, for vector-valued solutions to differential systems of pdes no general criterion is known to select one, among the infinitely many solutions, with special characteristics, such as some optimality (pointwise maximum, optimality in some energy norm, etc.). In this paper we try to give a method to compute a *canonical solution*, a kind of *vectorial pyramid*, analogously to the *pyramid* solution for the scalar case.

2   Formulation of the problem

We now formulate more precisely our problem. We want to find an explicit solution $u \in W_0^{1,\infty}\left(\Omega; \mathbb{R}^2\right)$ for the Dirichlet problem involving an implicit system of partial differential equations of the form

$$\lambda_1(Du(x,y)) = \gamma_1 \,, \ \lambda_2(Du(x,y)) = \gamma_2 \ \text{ a.e. } (x,y) \in \Omega, \tag{6}$$

where $0 < \gamma_1 \le \gamma_2$ and $\Omega \subset \mathbb{R}^2$ is a bounded open set that will later be chosen.

We solve (6) by studying a more restrictive problem, namely

$$Du(x,y) \in \left\{ \pm \begin{pmatrix} \gamma_1 & 0 \\ 0 & \gamma_2 \end{pmatrix}, \pm \begin{pmatrix} \gamma_1 & 0 \\ 0 & -\gamma_2 \end{pmatrix}, \pm \begin{pmatrix} 0 & \gamma_2 \\ \gamma_1 & 0 \end{pmatrix}, \pm \begin{pmatrix} 0 & \gamma_2 \\ -\gamma_1 & 0 \end{pmatrix} \right\}.$$
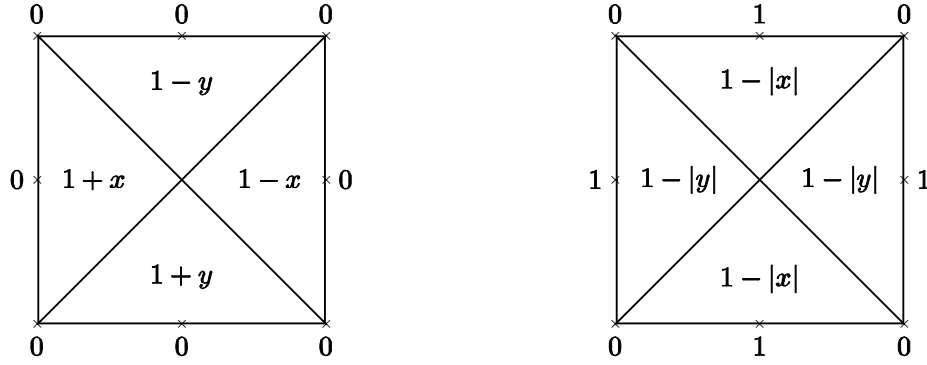
Figure 1: Pyramids a and b

By the simple linear transformation $u(x,y) = v(\gamma_1 x, \gamma_2 y)$ we can reduce the problem to the case where $\gamma_1 = \gamma_2 = 1$. We will therefore solve

$$Du(x,y) \in E \quad \text{a.e.} \ (x,y) \in \Omega, \tag{7}$$

where

$$E := \left\{ \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \pm \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \pm \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \pm \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}$$

and where $\Omega := (-2,2) \times (-2,2)$. We notice that more general sets Ω can be considered by using Vitali's covering argument (see [8] for details).

3   The vectorial pyramid

We start to introduce a *canonical solution* which we call a *vectorial pyramid*, see Section 2.3.1 of [8] for the scalar case.

3.1   The six basic pyramids

We define the basic pyramids.

1) We start by defining the following function (see Figure 1)

$$a\,(x,y) = 1 - \frac{|x+y| + |x-y|}{2} = \min\{1 \pm x, 1 \pm y\}$$

and observe that $a \in W^{1,\infty}((-1,1)^2)$, $a\,(x,y) = 0$ on the boundary and

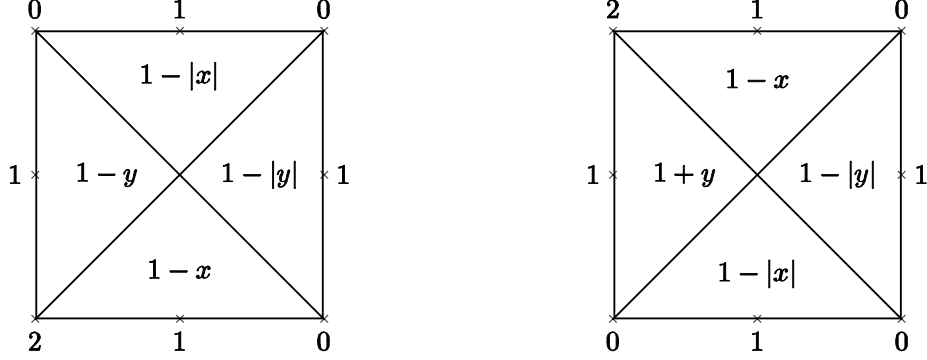$$Da \in \{\pm\,(1,0)\,, \pm\,(0,1)\} \quad \text{a.e. in } (-1,1)^2.$$

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Figure 2: Pyramids c and d

2) We then let (see Figures 1, 2, 3)

$$b(x,y) = \max\{1-|x|, 1-|y|\} = \begin{cases} 1-|x| & \text{if } |x| \le |y| \\ 1-|y| & \text{if } |y| \le |x| \end{cases}$$

$$c(x,y) = \begin{cases} 1-|x| & \text{if } |x| \le y \\ 1-y & \text{if } |y| \le -x \\ 1-x & \text{if } |x| \le -y \\ 1-|y| & \text{if } |y| \le x \end{cases} \qquad d(x,y) = \begin{cases} 1-x & \text{if } |x| \le y \\ 1+y & \text{if } |y| \le -x \\ 1-|x| & \text{if } |x| \le -y \\ 1-|y| & \text{if } |y| \le x \end{cases}$$

$$e(x,y) = \begin{cases} 1+x & \text{if } |x| \le y \\ 1-|y| & \text{if } |y| \le -x \\ 1-|x| & \text{if } |x| \le -y \\ 1+y & \text{if } |y| \le x \end{cases} \qquad f(x,y) = \begin{cases} 1-|x| & \text{if } |x| \le y \\ 1-|y| & \text{if } |y| \le -x \\ 1+x & \text{if } |x| \le -y \\ 1-y & \text{if } |y| \le x \end{cases}$$

Contrary to a, the functions b, c, d, e and f are not identically equal to zero at the boundary of $(-1,1)^2$.

3) Note that

$$D\binom{a}{b}, \ D\binom{a}{c}, \ D\binom{a}{d}, \ D\binom{a}{e}, \ D\binom{a}{f} \in E \quad \text{a.e. in } (-1,1)^2. \tag{8}$$

In fact in every piece where $a = 1 \pm x$, that is $Da = (\pm 1, 0)$, then Db, Dc, Dd, De, Df are equal either to $(0,1)$ or to $(0,-1)$. Similarly when $a = 1 \pm y$, that is $Da = (0, \pm 1)$, then Db, Dc, Dd, De, Df are equal either to $(1,0)$ or to $(-1,0)$.

4) Next, for $k \in \mathbb{N}$, we let

$$a_k(x,y) = 2^{-k} a\left(2^k x, 2^k y\right)$$
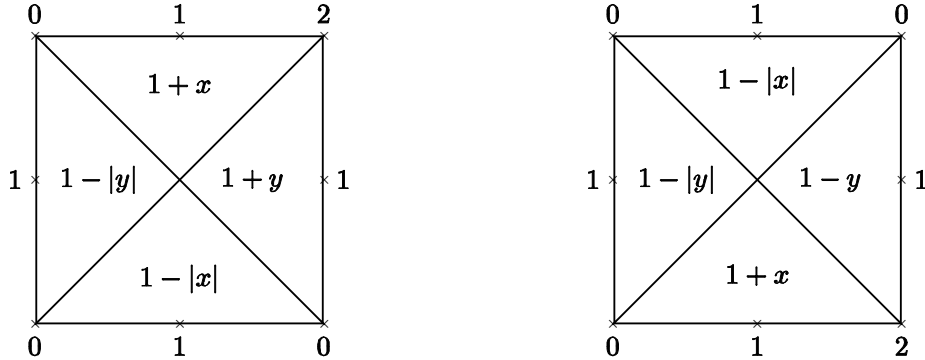
and similarly for b, c, d, e, f.

Figure 3: Pyramids e and f

3.2 The appropriate net

The net must be appropriately chosen in order to guarantee the continuity of the solution $u(x,y) = \binom{v(x,y)}{w(x,y)}$ across two adjacent cells of the net. This will be easily achieved for v, since it will be constructed through dilations and translations of the pyramid a, which is equal to 0 everywhere at the boundary of each cell and thus it matches continuously. On the contrary, the second component w, by (8), must be defined through the pyramids b, c, d, e, f and they are not identically equal to 0 at the boundary. The details for the continuity of w across the boundary of the cells are given in the proof of the theorem.

We first define the *net* on the set $T := \{(x,y) \in \mathbb{R}^2 : x \geq y \geq 0\}$ in the following manner (see Figure 4).

Let $x_0 = 0$ and

$$x_k = \sum_{j=0}^{k-1} 2^{-j} = 2 - \frac{1}{2^{k-1}} , \ k \in \mathbb{N},$$

so that $x_1 = 1$ and $\lim_{k \to \infty} x_k = 2$. We next let $y_0 = 0$ and, for every $k \in \mathbb{N}$,

$$y_k^i = \frac{i}{2^{k-1}}, \ i = 0, 1, \cdots, 2^k - 1;$$

so that $0 \leq y_k^i \leq x_k = y_k^{2^k - 1}$, for $i = 0, 1, \cdots, 2^k - 1$. Now each cell of the net is a square described by two indices (k, i), namely

$$Q_{k,i} := (x_{k-1}, x_k) \times \left(y_k^i, y_k^{i+1}\right), \ k \geq 1, \ i = 0, 1, \cdots, 2^k - 2.$$

3.3 The solution

We now start our construction of a solution to (7). We will give the precise construction in the theorem below before in $\Omega \cap T$, where

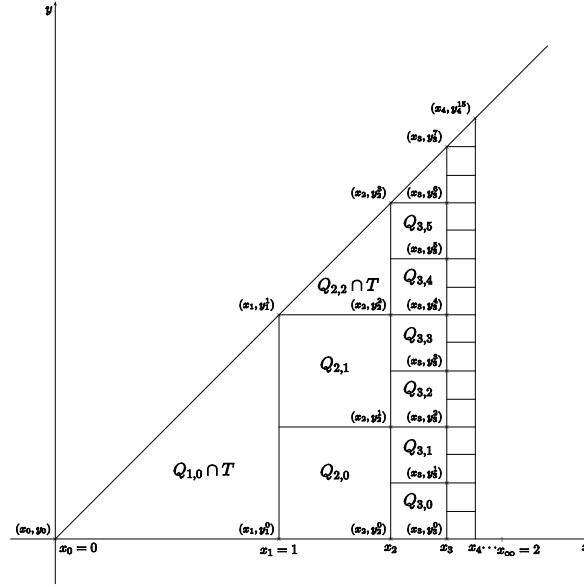$$T := \left\{(x,y) \in \mathbb{R}^2 : x \geq y \geq 0\right\}.$$

Figure 4: The sets $Q_{k,i}$

To obtain the function on the whole of $\Omega = (-2, 2) \times (-2, 2)$, we proceed as follows.

1) We first extend it by symmetry around the axis $\{y = x\}$ to

$$\Omega \cap \left\{ (x, y) \in \mathbb{R}^2 : x, y \geq 0 \right\};$$

2) then by symmetry around the axis $\{y = 0\}$ to $\Omega \cap \left\{ (x, y) \in \mathbb{R}^2 : x \geq 0 \right\}$;

3) and finally by symmetry around the axis $\{x = 0\}$ to the whole of Ω.

With the aim to construct an explicit solution $u(x, y) = \binom{v(x,y)}{w(x,y)}$ in T to (7) we first give the values of u at the nodes $\left(x_k, y_{k+1}^i \right)$ in the triangle $T = \left\{ (x, y) \in \mathbb{R}^2 : x \geq y \geq 0 \right\}$. We define $u(0, 0) = \binom{0}{0}$ and, for every $k \geq 1$ and $i = 0, 1, \cdots, 2^{k+1} - 2$,

$$v\left(x_k, y_{k+1}^i \right) = 0 \quad \text{and} \quad w\left(x_k, y_{k+1}^i \right) = \left\{ \begin{array}{ll} 0 & \text{if } i \text{ is even} \\ 2^{-k} & \text{if } i \text{ is odd} \end{array} \right. .$$

We then extend u to T as a piecewise linear function and, by symmetry as explained above, to $\Omega = (-2, 2) \times (-2, 2)$. Since the values $u\left(x_k, y_{k+1}^i \right)$ tend to zero as $k \to +\infty$, then $u(x, y)$ converges to zero uniformly as (x, y) tend to a boundary point. Thus we can extend u by continuity at $\partial\Omega$ with the zero value. It turns out that $Du \in E$ almost everywhere.
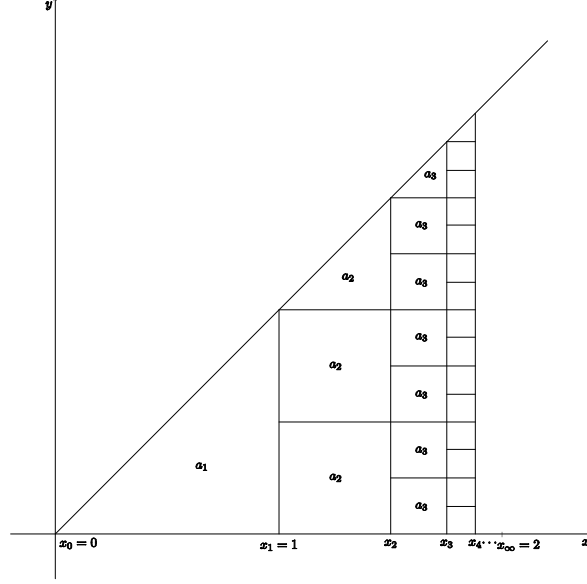
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Figure 5: The component v

We now write explicitly the function

$$u(x,y) = \begin{pmatrix} v(x,y) \\ w(x,y) \end{pmatrix}. \tag{9}$$

As explained above it is enough to define the map only on $Q_{k,i} \cap T$. Our construction privileges the first component (see Figure 5) which takes the very simple form

$$v(x,y) := a_k \left(x - \frac{x_{k-1} + x_k}{2}, y - \frac{y_k^i + y_k^{i+1}}{2} \right), \text{ if } (x,y) \in Q_{k,i} \cap T.$$

Because we are interested in describing the construction of the second component (see Figure 6) only in $\Omega \cap T$, we only need the pyramids b, c, d. For every $(x,y) \in Q_{k,i} \cap T$, we set

$$w(x,y) := \begin{cases} d_k \left(x - \frac{x_{k-1}+x_k}{2}, y - \frac{y_k^i+y_k^{i+1}}{2} \right) & \begin{array}{l} \text{if } i \text{ is even} \\ \text{and } i \in \{0, \cdots, 2^k - 4\} \end{array} \\[2ex] c_k \left(x - \frac{x_{k-1}+x_k}{2}, y - \frac{y_k^i+y_k^{i+1}}{2} \right) & \begin{array}{l} \text{if } i \text{ is odd} \\ \text{and } i \in \{1, \cdots, 2^k - 3\} \end{array} \\[2ex] b_k \left(x - \frac{x_{k-1}+x_k}{2}, y - \frac{y_k^i+y_k^{i+1}}{2} \right) & \text{if } i = 2^k - 2 \,. \end{cases}$$

We finally extend u by symmetry from $\Omega \cap T$ to Ω and up to the boundary of Ω by continuity.
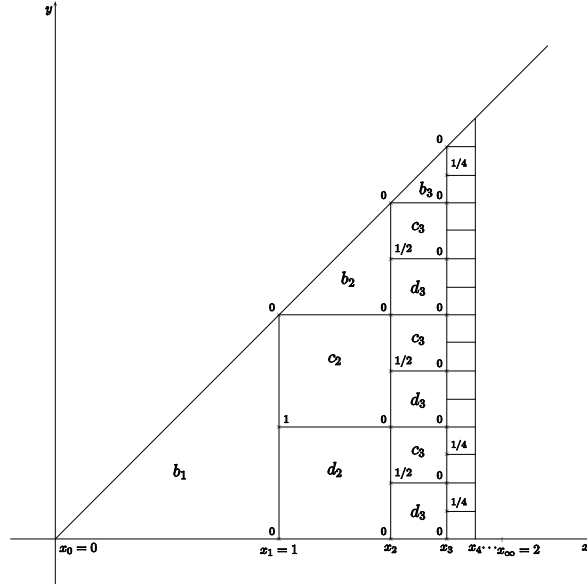
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Figure 6: The component w

For the sake of illustration we write the function u in $[0, 3/2] \times [0, 3/2] \cap T$.
Recall that $x_0 = y_0 = 0$ and

$$\begin{cases} x_1 = 1 \quad \text{and} \quad y_1^0 = 0, \ y_1^1 = 1 \\ Q_{1,0} = (x_0, x_1) \times (y_1^0, y_1^1) = (0, 1) \times (0, 1) \end{cases}$$

$$\begin{cases} x_2 = 3/2 \quad \text{and} \quad y_2^0 = 0, \ y_2^1 = 1/2, \ y_2^2 = 2/2, \ y_2^3 = 3/2 \\ Q_{2,0} = (x_1, x_2) \times (y_2^0, y_2^1), \ Q_{2,1} = (x_1, x_2) \times (y_2^1, y_2^2), \ Q_{2,2} = (x_1, x_2) \times (y_2^2, y_2^3). \end{cases}$$

We therefore have in $[0, 1] \times [0, 1]$

$$v(x, y) = a_1 \left(x - \frac{x_0 + x_1}{2}, y - \frac{y_1^0 + y_1^1}{2} \right)$$
$$= a_1 \left(x - \frac{1}{2}, y - \frac{1}{2} \right) = \frac{1}{2} a \left(2x - 1, 2y - 1 \right)$$

$$w(x, y) = b_1 \left(x - \frac{x_0 + x_1}{2}, y - \frac{y_1^0 + y_1^1}{2} \right)$$
$$= b_1 \left(x - \frac{1}{2}, y - \frac{1}{2} \right) = \frac{1}{2} b \left(2x - 1, 2y - 1 \right)$$

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(note that since v is symmetric with respect to $\{y = x\}$ the function u is already defined around the diagonal and in the first iteration it is in fact already defined on the whole of $[0,1] \times [0,1]$). In $[1, 3/2] \times [0, 3/2] \cap T$, we have

$$v(x, y) = \begin{cases} a_1 (x - 1/2, y - 1/2) & \text{if } (x, y) \in [0, 1] \times [0, 1] \\ a_2 (x - 5/4, y - 1/4) & \text{if } (x, y) \in [1, 3/2] \times [0, 1/2] \\ a_2 (x - 5/4, y - 3/4) & \text{if } (x, y) \in [1, 3/2] \times [1/2, 1] \\ a_2 (x - 5/4, y - 5/4) & \text{if } (x, y) \in [1, 3/2] \times [1, 3/2] \cap T \end{cases}$$

$$w(x, y) = \begin{cases} b_1 (x - 1/2, y - 1/2) & \text{if } (x, y) \in [0, 1] \times [0, 1] \\ d_2 (x - 5/4, y - 1/4) & \text{if } (x, y) \in [1, 3/2] \times [0, 1/2] \\ c_2 (x - 5/4, y - 3/4) & \text{if } (x, y) \in [1, 3/2] \times [1/2, 1] \\ b_2 (x - 5/4, y - 5/4) & \text{if } (x, y) \in [1, 3/2] \times [1, 3/2] \cap T. \end{cases}$$

3.4 The theorem

By the construction in the previous section we have all the elements to state and prove the following result.

Theorem 1 *The map u in (9) is a $W_0^{1,\infty}(\Omega; \mathbb{R}^2)$ solution to the Dirichlet problem (7).*

Proof. It is clear that

1) $Du \in E$, since $D\binom{a_k}{b_k}$, $D\binom{a_k}{c_k}$, $D\binom{a_k}{d_k} \in E$, by (8).

2) $v(x_k, y) = 0$ for every $y \in [0, x_k = y_k^{2^k - 1}]$ and for every $k \geq 1$; thus, as $k \to \infty$, we obtain $v(2, y) = 0$, for every $y \in [0, 2]$.

3) $w(x_k, y_k^i) = 0$ for every $i \in \{0, \cdots, 2^k - 1\}$ and for every $k \geq 1$; thus, as $k \to \infty$, we obtain $w(2, y) = 0$, for every $y \in [0, 2]$.

4) u is continuous. This is clear for the v component, because it is 0 on the boundary of each cell. The w component is also continuous, since by alternating pyramids d_{k+1} and c_{k+1}, we continuously match (see Figure 4) with pyramids d_k, c_k, b_k at the nodes (x_k, y_{k+1}^i), $i = 0, 1, \cdots, 2^k - 1$, with values

$$0, 2^{-k}, 0, 2^{-k}, \cdots, 0, 2^{-k}, 0.$$

This explains the continuity with respect to x. The continuity with respect to y follows, since (see Figure 4) the pyramids d_k, c_k, b_k are vertically piled so that they match continuously.

Thus u is Lipschitz continuous, since it is continuous and its gradient belongs to E almost everywhere. ∎

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