On non quasiconvex problems of the calculus of variations

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Contents

Abstract

We study existence of minimizers for problems of the type

$$\inf \quad \int_\Omega f\left(Du\left(x\right)\right)dx : u = u_{\xi_0} \text{ on } \partial\Omega$$

where f is non quasiconvex and u_{ξ_0} is an affine function. Applying some new results on differential inclusions, we get sufficient conditions. We also study necessary conditions. We then consider some examples.

1 Introduction

We discuss the existence of minimizers for the problem

$$(P) \qquad \inf\left\{\int_\Omega f\left(Du\left(x\right)\right)dx : u = u_{\xi_0} \text{ on } \partial\Omega\right\}.$$

where $\Omega \subset \mathbb{R}^n$ is a bounded open set, $u : \Omega \to \mathbb{R}^m$ (if $m = 1$ or, by abuse of language, if $n = 1$, we will say that it is *scalar* valued while if $m, n \geq 2$, we will speak of the *vector* valued case) and Du denotes its Jacobian matrix, i.e. $Du = \left(\frac{\partial u_i}{\partial x_j}\right)$ seen as a real matrix of size $m \times n$, $f : \mathbb{R}^{m\times n} \to \mathbb{R}$ is lower semicontinuous and u_{ξ_0} is a given affine map (i.e., $Du_{\xi_0} = \xi_0$, where $\xi_0 \in \mathbb{R}^{m\times n}$ is a fixed matrix).

If the function f is *quasiconvex*, meaning that

$$\int_U f\left(\xi + D\varphi\left(x\right)\right) dx \geq f\left(\xi\right) \text{ meas}(U)$$

for every bounded domain $U \subset \mathbb{R}^n$, $\xi \in \mathbb{R}^{m\times n}$, and $\varphi \in W_0^{1,\infty}\left(U; \mathbb{R}^m\right)$, then the problem (P) trivially has u_{ξ_0} as a minimizer. We also recall that in the scalar case ($n = 1$ or $m = 1$), quasiconvexity and ordinary convexity are equivalent.

The aim of the article is to study the case where f fails to be quasiconvex. The first step in dealing with such problems is the *relaxation theorem*, first established in the vectorial case by Dacorogna (see [5]) and then further generalized by many others. It has as a direct consequence (cf. Lemma 11) that (P) has a solution $\overline{u} \in u_{\xi_0} + W_0^{1,\infty}(\Omega; \mathbb{R}^m)$ if and only if

$$f\left(D\overline{u}\left(x\right)\right) = Qf\left(D\overline{u}\left(x\right)\right), \text{ a.e. } x \in \Omega$$

$$\int_\Omega Qf\left(D\overline{u}\left(x\right)\right) dx = Qf\left(\xi_0\right) \text{ meas}\,\Omega$$

where Qf is the *quasiconvex envelope* of f, namely

$$Qf = \sup\left\{g \leq f : g \text{ quasiconvex}\right\}.$$

The problem is then to discuss the existence or non existence of a $\overline{u}$ satisfying the two equations. The two equations are not really of the same nature. The first one is what is called an *implicit partial differential equation*, which has recently received a lot of attention and we refer to Dacorogna-Marcellini [7] for some bibliographical and historical comments. The second one is more geometric in nature and has to do with some *"quasiaffinity"* of the quasiconvex envelope Qf.

The scalar case ($n = 1$ or $m = 1$) has been intensively studied by many authors including: Aubert-Tahraoui, Bauman-Phillips, Buttazzo-Ferone-Kawohl, Celada-Perrotta, Cellina, Cellina-Colombo, Cesari, Cutri, Dacorogna, Ekeland, Friesecke, Fusco-Marcellini-Ornelas, Giachetti-Schianchi, Klötzler, Marcellini, Mascolo, Mascolo-Schianchi, Monteiro Marques-Ornelas, Ornelas, Raymond, Sychev, Tahraoui, Treu and Zagatti. For precise references see [6] and [7].

The vectorial case has been investigated for some special examples notably by Allaire-Francfort [1], Cellina-Zagatti [4], Dacorogna-Ribeiro [10], Dacorogna-Tanteri [11], Mascolo-Schianchi [17], Müller-Sverak [18] and Raymond [20]. A more systematic study was achieved by Dacorogna-Marcellini in [6], [7] and [8]. Building on [6] and owing to the recent developments in the treatment of implicit partial differential equations, we will obtain at the same time simpler and more general existence theorems. Several examples can be treated as a direct consequence of the general and simple theorem obtained in Section 3. We will concentrate on two of them (in Subsection 5.1 and 5.2) and we will mention very briefly those classical examples that could be treated in the same way. We will also devote some attention to necessary conditions.

2 Preliminaries

We recall the main notations that we will use throughout the article and we refer, if necessary, for more details to Dacorogna [5] and Dacorogna-Marcellini [7].

We start with one notation for matrices.

Notation 1 *For $\xi \in \mathbb{R}^{m\times n}$ we let*

$$T(\xi) = (\xi, adj_2\xi, \ldots, adj_{m\wedge n}\xi) \in \mathbb{R}^{\tau}$$

where $adj_s\xi$ stands for the matrix of all $s\times s$ subdeterminants of the matrix ξ, $1 \leq s \leq m\wedge n = \min\{m,n\}$ and where

$$\tau = \tau(m,n) = \sum_{s=1}^{m\wedge n} \begin{pmatrix} m \\ s \end{pmatrix} \begin{pmatrix} n \\ s \end{pmatrix} \text{ and } \begin{pmatrix} m \\ s \end{pmatrix} = \frac{m!}{s!\,(m-s)!}.$$

In particular if $m = n = 2$, then $T(\xi) = (\xi, \det\xi)$.

We next define the main notions of convexity used throughout the article.

Definition 2 *(i) A function $f : \mathbb{R}^{m\times n} \to \overline{\mathbb{R}} = \mathbb{R}\cup\{+\infty\}$ is said to be* polyconvex *if*

$$f\left(\sum_{i=1}^{\tau+1} t_i\xi_i\right) \leq \sum_{i=1}^{\tau+1} t_i f(\xi_i)$$

whenever $t_i \geq 0$ and

$$\sum_{i=1}^{\tau+1} t_i = 1,\ T\left(\sum_{i=1}^{\tau+1} t_i\xi_i\right) = \sum_{i=1}^{\tau+1} t_i T(\xi_i).$$

(ii) A Borel measurable function $f : \mathbb{R}^{m\times n} \to \mathbb{R}$ is said to be quasiconvex *if*

$$\int_U f(\xi + D\varphi(x))\,dx \geq f(\xi)\ \text{meas}(U)$$

for every bounded domain $U \subset \mathbb{R}^n$, $\xi \in \mathbb{R}^{m\times n}$, and $\varphi \in W_0^{1,\infty}(U;\mathbb{R}^m)$.

(iii) A function $f : \mathbb{R}^{m\times n} \to \overline{\mathbb{R}} = \mathbb{R}\cup\{+\infty\}$ is said to be rank one convex *if*

$$f(t\xi_1 + (1-t)\xi_2) \leq t\,f(\xi_1) + (1-t)\ f(\xi_2)$$

for every $\xi_1, \xi_2 \in \mathbb{R}^{m\times n}$ *with* $\operatorname{rank}\{\xi_1 - \xi_2\} = 1$ *and every* $t \in [0,1]$.

(iv) A Borel measurable function $f : \mathbb{R}^{m\times n} \to \mathbb{R}$ *is said to be* quasiaffine *(or equivalently polyaffine or rank one affine) if both* f *and* $-f$ *are quasiconvex.*

(v) The different envelopes of a given function f *are defined as*

$$\begin{aligned}
Cf &= \sup\{g \leq f : g \text{ convex}\}, \\
Pf &= \sup\{g \leq f : g \text{ polyconvex}\}, \\
Qf &= \sup\{g \leq f : g \text{ quasiconvex}\}, \\
Rf &= \sup\{g \leq f : g \text{ rank one convex}\}.
\end{aligned}$$

As well known we have that the following implications hold

$$f \text{ convex} \implies f \text{ polyconvex} \implies f \text{ quasiconvex} \implies f \text{ rank one convex}$$

and thus

$$Cf \leq Pf \leq Qf \leq Rf \leq f.$$

Remark 3 *An equivalent characterization of polyconvexity can be given in terms of Hahn-Banach theorem (cf. Theorem 1.3 page 107 in Dacorogna [5]). A function* $f : \mathbb{R}^{m\times n} \longrightarrow \mathbb{R}$ *is polyconvex if and only if for every* $\xi \in \mathbb{R}^{m\times n}$ *there exists* $\lambda = \lambda(\xi) \in \mathbb{R}^{\tau(m,n)}$ *so that*

$$f(\xi + \eta) - f(\xi) - \langle \lambda; T(\xi+\eta) - T(\xi)\rangle \geq 0, \text{ for every } \eta \in \mathbb{R}^{m\times n}. \tag{1}$$

We now give an important example that concerns singular values.

Example 4 *Let* $0 \leq \lambda_1(\xi) \leq \ldots \leq \lambda_n(\xi)$ *denote the* singular values *of a matrix* $\xi \in \mathbb{R}^{n\times n}$*, which are defined as the eigenvalues of the matrix* $(\xi\xi^t)^{1/2}$*. The functions*

$$\xi \to \sum_{i=\nu}^{n} \lambda_i(\xi) \quad \text{and} \quad \xi \to \prod_{i=\nu}^{n} \lambda_i(\xi), \ \nu = 1, \ldots, n,$$

are respectively convex and polyconvex (note that $\prod_{i=1}^{n} \lambda_i(\xi) = |\det \xi|$*). In particular the function* $\xi \to \lambda_n(\xi)$ *is convex and in fact is the operator norm.*

We finally recall the notations for various convex hulls of sets.

Notation 5 *We let, for* $E \subset \mathbb{R}^{m\times n}$,

$$\begin{aligned}
\overline{\mathcal{F}}_E &= \left\{f : \mathbb{R}^{m\times n} \to \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\} : f|_E \leq 0\right\} \\
\mathcal{F}_E &= \left\{f : \mathbb{R}^{m\times n} \to \mathbb{R} : f|_E \leq 0\right\}.
\end{aligned}$$

We then have respectively, the convex, polyconvex, rank one convex *and (closure of the)* quasiconvex hull *defined by*

$$\begin{aligned}
\operatorname{co} E &= \left\{\xi \in \mathbb{R}^{m\times n} : f(\xi) \leq 0, \text{ for every convex } f \in \overline{\mathcal{F}}_E\right\} \\
\operatorname{Pco} E &= \left\{\xi \in \mathbb{R}^{m\times n} : f(\xi) \leq 0, \text{ for every polyconvex } f \in \overline{\mathcal{F}}_E\right\} \\
\operatorname{Rco} E &= \left\{\xi \in \mathbb{R}^{m\times n} : f(\xi) \leq 0, \text{ for every rank one convex } f \in \overline{\mathcal{F}}_E\right\} \\
\overline{\operatorname{Qco}} E &= \left\{\xi \in \mathbb{R}^{m\times n} : f(\xi) \leq 0, \text{ for every quasiconvex } f \in \mathcal{F}_E\right\}.
\end{aligned}$$

We should point out that by replacing $\overline{\mathcal{F}}_E$ by $\mathcal{F}_E$ in the definitions of $\operatorname{co} E$ and $\operatorname{Pco} E$ we get their closures denoted by $\overline{\operatorname{co}} E$ and $\overline{\operatorname{Pco}} E$. However if we do so in the definition of $\operatorname{Rco} E$ we get a larger set than the closure of $\operatorname{Rco} E$. We should also draw the attention that some authors call the set

$$\left\{\xi \in \mathbb{R}^{m\times n} : f(\xi) \leq 0, \text{ for every rank one convex } f \in \overline{\mathcal{F}}_E\right\}$$

the lamination convex hull, while they reserve the name of rank one convex hull to the set

$$\left\{\xi \in \mathbb{R}^{m\times n} : f(\xi) \leq 0, \text{ for every rank one convex } f \in \mathcal{F}_E\right\}.$$

We think however that our terminology is more consistent with the classical definition of convex hull.

In general we have, for any set $E \subset \mathbb{R}^{m\times n}$,

$$E \subset \operatorname{Rco} E \subset \operatorname{Pco} E \subset \operatorname{co} E$$

$$\overline{E} \subset \overline{\operatorname{Rco}} E \subset \overline{\operatorname{Qco}} E \subset \overline{\operatorname{Pco}} E \subset \overline{\operatorname{co}} E.$$

We now turn our attention to differential inclusions. We will need the following definition introduced by Dacorogna-Marcellini (cf. [7]), which is the key condition to get existence of solutions.

Definition 6 (Relaxation property) *Let $E, K \subset \mathbb{R}^{m\times n}$. We say that K has the* relaxation property *with respect to E if for every bounded open set $\Omega \subset \mathbb{R}^n$, for every affine function u_ξ satisfying*

$$Du_\xi(x) = \xi \in K,$$

there exist a sequence $u_\nu \in Aff_{piec}\left(\overline{\Omega}; \mathbb{R}^m\right)$ (the set of piecewise affine maps)

$$u_\nu \in u_\xi + W_0^{1,\infty}(\Omega; \mathbb{R}^m),\ Du_\nu(x) \in E \cup K,\ a.e.\ in\ \Omega$$

$$u_\nu \overset{*}{\rightharpoonup} u_\xi\ in\ W^{1,\infty},\ \int_\Omega \operatorname{dist}(Du_\nu(x); E)\, dx \to 0\ as\ \nu \to \infty.$$

The main theorem, established in Dacorogna-Pisante [9], is then.

Theorem 7 *Let $\Omega \subset \mathbb{R}^n$ be open and bounded. Let $E, K \subset \mathbb{R}^{m\times n}$ be such that E is compact and K is bounded. Assume that K has the relaxation property with respect to E. Let $\varphi \in Aff_{piec}\left(\overline{\Omega}; \mathbb{R}^m\right)$ be such that*

$$D\varphi(x) \in E \cup K,\ a.e.\ in\ \Omega.$$

Then there exists (a dense set of) $u \in \varphi + W_0^{1,\infty}(\Omega; \mathbb{R}^m)$ such that

$$Du(x) \in E,\ a.e.\ in\ \Omega.$$

Remark 8 *This theorem was first proved by Dacorogna-Marcellini (see Theorem 6.3 in [7]) under the further hypothesis that*

$$E = \left\{ \xi \in \mathbb{R}^{m \times n} : F_i(\xi) = 0,\ i = 1, 2, ..., I \right\}$$

where $F_i : \mathbb{R}^{m \times n} \to \mathbb{R}, i = 1, 2, ..., I,$ *are quasiconvex. This hypothesis was later removed by Sychev in [22] (see also Müller and Sychev [19]). Kirchheim in [14] pointed out that using a classical result of function theory then the proof of Dacorogna-Marcellini was still valid without the extra hypothesis on* E*; it is this idea combined with the original proof of Dacorogna-Marcellini that is used by Dacorogna-Pisante in [9].*

We next give a sufficient condition that ensures the relaxation property. In concrete examples this condition is usually much easier to check than the relaxation property. We start with a definition.

Definition 9 (Approximation property) *Let* $E \subset K(E) \subset \mathbb{R}^{m \times n}$*. The sets* E *and* $K(E)$ *are said to have the approximation property if there exists a family of closed sets* E_δ *and* $K(E_\delta)$*,* $\delta > 0$*, such that*

(1) $E_\delta \subset K(E_\delta) \subset \operatorname{int} K(E)$ *for every* $\delta > 0$;

(2) for every $\epsilon > 0$ *there exists* $\delta_0 = \delta_0(\epsilon) > 0$ *such that* $\operatorname{dist}(\eta; E) \leq \epsilon$ *for every* $\eta \in E_\delta$ *and* $\delta \in [0, \delta_0]$*;*

(3) if $\eta \in \operatorname{int} K(E)$ *then* $\eta \in K(E_\delta)$ *for every* $\delta > 0$ *sufficiently small.*

We therefore have the following theorem (cf. Theorem 6.14 in [7] and for a slightly more flexible one see Theorem 6.15).

Theorem 10 *Let* $E \subset \mathbb{R}^{m \times n}$ *be compact and* $\operatorname{Rco} E$ *has the approximation property with* $K(E_\delta) = \operatorname{Rco} E_\delta$*, then* $\operatorname{int} \operatorname{Rco} E$ *has the relaxation property with respect to* E*.*

3 Sufficient conditions

The problem under consideration is

$$(P) \qquad \inf \left\{ I(u) = \int_\Omega f(Du(x))\, dx : \ u \in u_{\xi_0} + W_0^{1,\infty}(\Omega; \mathbb{R}^m) \right\}$$

where Ω is a bounded open set of $\mathbb{R}^n$, u_{ξ_0} is affine, i.e. $Du_{\xi_0} = \xi_0$. We will assume throughout the article that $f : \mathbb{R}^{m \times n} \longrightarrow \mathbb{R}$ is lower semicontinuous, locally bounded and satisfies

$$f(\xi) \geq \langle \alpha; T(\xi) \rangle + \beta, \ \forall \, \xi \in \mathbb{R}^{m \times n}$$

and for some $\alpha \in \mathbb{R}^{\tau(m,n)}$ and $\beta \in \mathbb{R}$.

With the help of the relaxation theorem and of Theorem 7 we are now in a position to discuss some existence results for the problem (P). The following lemma (cf. [6]) is elementary and gives a necessary and sufficient condition for existence of minima.

Lemma 11 *Let Ω, f and u_{ξ_0} be as above, in particular $Du_{\xi_0} = \xi_0$. The problem (P) has a solution if and only if there exists $\overline{u} \in u_{\xi_0} + W_0^{1,\infty}(\Omega; \mathbb{R}^m)$ such that*

$$f\left(D\overline{u}\left(x\right)\right) = Qf\left(D\overline{u}\left(x\right)\right), \text{ a.e. } x \in \Omega \tag{2}$$

$$\int_\Omega Qf\left(D\overline{u}\left(x\right)\right) dx = Qf\left(\xi_0\right) \operatorname{meas}\Omega. \tag{3}$$

Proof. By the relaxation theorem (cf. [5]) and since u_{ξ_0} is affine, we have

$$\inf\left(P\right) = \inf\left(QP\right) = Qf\left(\xi_0\right) \operatorname{meas}\Omega.$$

Moreover, since we always have $f \geq Qf$ and we have a solution of (2) satisfying (3), we get that $\overline{u}$ is a solution of (P). The fact that (2) and (3) are necessary for the existence of a minimum for (P) follows in the same way. ■

The previous lemma explains why the set

$$K = \left\{\xi \in \mathbb{R}^{m\times n} : Qf\left(\xi\right) < f\left(\xi\right)\right\}$$

plays a central role in the existence theorems that follow. In order to ensure (2) we will have to consider differential inclusions of the form studied in the previous section, namely: find $\overline{u} \in u_{\xi_0} + W_0^{1,\infty}(\Omega; \mathbb{R}^m)$ such that

$$D\overline{u}\left(x\right) \in \partial K, \text{ a.e. } x \in \Omega.$$

In order to deal with the second condition (3) we will have to impose some hypotheses of the type "Qf is *quasiaffine* on K".

The main abstract theorem is the following.

Theorem 12 *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, $\xi_0 \in \mathbb{R}^{m\times n}$, $f : \mathbb{R}^{m\times n} \longrightarrow \mathbb{R}$ a lower semicontinuous function and let*

$$K = \left\{\eta \in \mathbb{R}^{m\times n} : Qf\left(\eta\right) < f\left(\eta\right)\right\}.$$

Assume that there exists $K_0 \subset K$ such that

- $\xi_0 \in K_0$,
- *K_0 is bounded and has the relaxation property with respect to $\overline{K}_0 \cap \partial K$,*
- *Qf is quasiaffine on $\overline{K}_0$.*

Let $u_{\xi_0}\left(x\right) = \xi_0 x$. Then the problem

$$(P) \qquad \inf\left\{I\left(u\right) = \int_\Omega f\left(Du\left(x\right)\right) dx : \ u \in u_{\xi_0} + W_0^{1,\infty}(\Omega; \mathbb{R}^m)\right\}$$

has a solution $\bar{u} \in u_{\xi_0} + W_0^{1,\infty}(\Omega; \mathbb{R}^m)$.

Proof. Since $\xi_0 \in K_0$ and K_0 is bounded and has the relaxation property with respect to $\overline{K}_0 \cap \partial K$, we can find, appealing to Theorem 7, a map $\bar{u} \in u_{\xi_0} + W_0^{1,\infty}(\Omega;\mathbb{R}^m)$ satisfying

$$D\bar{u} \in \overline{K}_0 \cap \partial K, \text{ a.e. in } \Omega,$$

which means that (2) of Lemma 11 is satisfied. Moreover, since Qf is quasiaffine on $\overline{K}_0$, we have that (3) of Lemma 11 holds and thus the claim. ■

The second hypothesis in the theorem is clearly the most difficult to verify, nevertheless there are some cases when it is automatically satisfied. For example if K is bounded we can prove that K has the relaxation property with respect to ∂K.

We will see that, in many applications, the set K turns out to be unbounded and in order to apply Theorem 12 we need to find some weaker conditions on K that guarantees the existence of a subset K_0 of K satisfying the requested properties. With this aim in mind we give the following notations and definitions.

Notation 13 *Let $K \subset \mathbb{R}^{m\times n}$ be open and $\lambda \in \mathbb{R}^{m\times n}$.*

(i) For $\xi \in K$, we denote by $L_K(\xi,\lambda)$ the largest segment of the form $[\xi + t\lambda, \xi + s\lambda]$, $t < 0 < s$, so that $(\xi + t\lambda, \xi + s\lambda) \subset K$.

(ii) If $L_K(\xi,\lambda)$ is bounded, we denote by $t_-(\xi) < 0 < t_+(\xi)$ the elements so that $L_K(\xi,\lambda) = [\xi + t_-\lambda, \xi + t_+\lambda]$. They therefore satisfy

$$\xi + t_\pm\lambda \in \partial K \quad \text{and} \quad \xi + t\lambda \in K \ \ \forall t \in (t_-, t_+).$$

(iii) If $H \subset K$, we let

$$L_K(H,\lambda) = \bigcup_{\xi \in H} L_K(\xi,\lambda).$$

Definition 14 *(Boundedness and stable boundedness in a direction λ). Let $K \subset \mathbb{R}^{m\times n}$ be open, $\xi_0 \in K$ and $\lambda \in \mathbb{R}^{m\times n}$.*

(i) We say that K is bounded at ξ_0 in the direction λ *if $L_K(\xi_0,\lambda)$ is bounded.*

(ii) We say that K is stably bounded at ξ_0 in the rank-one direction $\lambda = \alpha\otimes\beta$ *(with $\alpha \in \mathbb{R}^m$ and $\beta \in \mathbb{R}^n$) if there exists $\epsilon > 0$ so that $L_K(\xi_0 + \alpha\otimes B_\epsilon, \lambda)$ is bounded, where we have denoted by*

$$\xi_0 + \alpha \otimes B_\epsilon = \left\{\xi \in \mathbb{R}^{m\times n} : \xi = \xi_0 + \alpha \otimes b \text{ with } |b| < \epsilon\right\}.$$

Clearly a bounded set K is bounded at every point $\xi \in K$ and in any direction λ and consequently it is also stably bounded.

We now give an example of a globally unbounded set which is bounded in certain directions.

Example 15 *Let $m = n = 2$ and*

$$K = \left\{\xi \in \mathbb{R}^{2\times 2} : \alpha < \det\xi < \beta\right\}.$$

The set K is clearly unbounded.

(i) If $\xi_0 = I$ then K is bounded, and even stably bounded, at ξ_0, in a direction of rank one, for example with

$$\lambda = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \text{ or } \lambda = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

(ii) However if $\xi_0 = 0$, then K is unbounded in any rank one direction, but is bounded in any rank two direction.

In the following result we deal with sets K that are bounded in one rank-one direction only. This corollary says, roughly speaking, that if K is bounded at ξ_0 in a rank-one direction λ and this boundedness (in the same direction) is preserved under small perturbations of ξ_0 along rank-one λ-compatible directions, then we can ensure the relaxation property required in the main existence theorem.

Corollary 16 *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, $f : \mathbb{R}^{m\times n} \longrightarrow \mathbb{R}$ a lower semicontinuous function and let $\xi_0 \in K$ where*

$$K = \left\{\xi \in \mathbb{R}^{m\times n} : Qf(\xi) < f(\xi)\right\}.$$

If there exist a rank-one direction $\lambda \in \mathbb{R}^{m\times n}$ such that

(i) K is stably bounded at ξ_0 in the direction $\lambda = \alpha \otimes \beta$,

(ii) Qf is quasiaffine on the set (cf. Definition 14) $L_K(\xi_0 + \alpha \otimes \overline{B}_\epsilon, \lambda)$,

then the problem

$$(P) \qquad \inf\left\{ I(u) = \int_\Omega f(Du(x))\, dx : \ u \in u_{\xi_0} + W_0^{1,\infty}(\Omega;\mathbb{R}^m) \right\}$$

has a solution $\bar{u} \in u_{\xi_0} + W_0^{1,\infty}(\Omega;\mathbb{R}^m)$.

To prove the corollary we will need the following result. It is due to Müller-Sychev [19] and is a refinement of a classical result.

Lemma 17 (Approximation lemma) *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set. Let $t \in [0,1]$ and $A, B \in \mathbb{R}^{m\times n}$ such that*

$$A - B = a \otimes b$$

with $a \in \mathbb{R}^m$ and $b \in \mathbb{R}^n$. Let $b_3, \ldots, b_k \in \mathbb{R}^n$, $k \geq n$, such that $0 \in \operatorname{int}\operatorname{co}\{b, -b, b_3, \ldots, b_k\}$. Let φ be an affine map such that

$$D\varphi(x) = \xi = tA + (1-t)B, \ x \in \overline{\Omega}$$

(i.e. $A = \xi + (1-t)\,a \otimes b$ and $B = \xi - ta \otimes b$). Then, for every $\varepsilon > 0$, there exists a piecewise affine map u and there exist disjoint open sets $\Omega_A, \Omega_B \subset \Omega$,

such that

$$\begin{cases} |\text{meas } \Omega_A - t \text{ meas } \Omega|, \ |\text{meas } \Omega_B - (1-t) \text{ meas } \Omega| \le \varepsilon \\ u(x) = \varphi(x), \ x \in \partial\Omega \\ |u(x) - \varphi(x)| \le \varepsilon, \ x \in \Omega \\ Du(x) = \begin{cases} A & in\ \Omega_A \\ B & in\ \Omega_B \end{cases} \\ Du(x) \in \xi + \{(1-t)\, a \otimes b, -ta \otimes b, a \otimes b_3, \ldots, a \otimes b_k\}, \ a.e. \ in\ \Omega. \end{cases}$$

Proof. (Corollary 16). We divide the proof into two steps.

Step 1. Assume that $|\beta| = 1$, otherwise replace it by $\beta / |\beta|$, and, for $j \le k$ and some $k \ge n$, let $\beta_j \in \mathbb{R}^n$, with $|\beta_j| = 1$, be such that

$$0 \in H := \operatorname{int} \operatorname{co}\{\beta, -\beta, \beta_3, \ldots, \beta_k\} \subset B_1(0) = \{x \in \mathbb{R}^n : |x| < 1\}.$$

Let then, for $\epsilon > 0$ as in the hypothesis,

$$K_0 := (\xi_0 + \alpha \otimes \epsilon H) \cup \left[\partial K \cap L_K(\xi_0 + \alpha \otimes \epsilon \overline{H}, \lambda)\right].$$

We therefore have that $\xi_0 \in K_0$ and, by hypothesis, that K_0 is bounded, since

$$K_0 \subset \overline{K}_0 \subset L_K(\xi_0 + \alpha \otimes \overline{B}_\epsilon, \lambda).$$

Furthermore we have

$$\overline{K}_0 \cap \partial K = \partial K \cap L_K(\xi_0 + \alpha \otimes \epsilon \overline{H}, \lambda).$$

In order to deduce the corollary from Theorem 12, we only need to show that K_0 has the relaxation property with respect to $\overline{K}_0 \cap \partial K$. This will be achieved in the next step.

Step 2. We now prove that K_0 has the relaxation property with respect to $\overline{K}_0 \cap \partial K$. Let $\xi \in K_0$ and let us find a sequence $u_\nu \in Aff_{piec}\left(\overline{\Omega}; \mathbb{R}^m\right)$ so that

$$\begin{aligned} & u_\nu \in u_\xi + W_0^{1,\infty}(\Omega; \mathbb{R}^m), \ Du_\nu(x) \in \left(\overline{K}_0 \cap \partial K\right) \cup K_0, \text{ a.e. in } \Omega \\ & u_\nu \overset{*}{\rightharpoonup} u_\xi \text{ in } W^{1,\infty}, \ \int_\Omega \operatorname{dist}\left(Du_\nu(x); \overline{K}_0 \cap \partial K\right) dx \to 0 \text{ as } \nu \to \infty. \end{aligned} \tag{4}$$

If $\xi \in \partial K \cap L_K(\xi_0 + \alpha \otimes \epsilon \overline{H}, \lambda)$, nothing is to be proved; so we assume that $\xi \in \xi_0 + \alpha \otimes \epsilon H$. By hypothesis (i), we can find $t_-(\xi) < 0 < t_+(\xi)$ so that

$$\xi_\pm := \xi + t_\pm \lambda \in \partial K \ \text{ and } \ \xi + t\lambda \in K \ \ \forall t \in (t_-, t_+)$$

and hence $\xi_\pm \in \overline{K}_0 \cap \partial K$. We moreover have that

$$\xi = \frac{-t_-}{t_+ - t_-}\xi_+ + \frac{t_+}{t_+ - t_-}\xi_- \ \text{ with } \ \xi_\pm \in \overline{K}_0 \cap \partial K.$$

Furthermore, since $\xi \in \xi_0 + \alpha \otimes \epsilon H$, we can find $\gamma \in \epsilon H$ such that

$$\xi = \xi_0 + \alpha \otimes \gamma.$$

The set H being open we have that $\overline{B}_\delta(\gamma) \subset \epsilon H$, for every sufficiently small $\delta > 0$. Moreover since for every $\delta > 0$, we have

$$0 \in \delta H = \operatorname{int} \operatorname{co}\{\pm\delta\beta, \delta\beta_3, \ldots, \delta\beta_k\}$$

and since for every sufficiently small $\delta > 0$, we have

$$\pm\delta\beta \in \operatorname{co}\{\pm(t_+ - t_-)\beta\} \subset \operatorname{co}\{\pm(t_+ - t_-)\beta, \delta\beta_3, \ldots, \delta\beta_k\},$$

we get that

$$0 \in \delta H = \operatorname{int} \operatorname{co}\{\pm\delta\beta, \delta\beta_3, \ldots, \delta\beta_k\} \subset \operatorname{int} \operatorname{co}\{\pm(t_+ - t_-)\beta, \delta\beta_3, \ldots, \delta\beta_k\}.$$

We are therefore in a position to apply Lemma 17 to

$$\begin{aligned}
&a = \alpha,\ b = (t_+ - t_-)\beta,\ b_j = \delta\beta_j \text{ for } j = 3, ..., k,\ t = \frac{-t_-}{t_+ - t_-},\\
&A = \xi_+ = \xi + \frac{t_+}{t_+ - t_-}\alpha \otimes (t_+ - t_-)\beta = \xi + (1-t)\, a \otimes b,\\
&B = \xi_- = \xi + \frac{t_-}{t_+ - t_-}\alpha \otimes (t_+ - t_-)\beta = \xi - t a \otimes b
\end{aligned}$$

and find $u_\delta \in Aff_{piec}\left(\overline{\Omega}; \mathbb{R}^m\right)$, open sets $\Omega_+, \Omega_- \subset \Omega$, such that

$$\begin{cases}
|\text{meas}\ (\Omega_+) - t \,\text{meas}\ \Omega|\,,\ |\text{meas}\ (\Omega_-) - (1-t)\,\text{meas}\ \Omega| \leq \delta\\
u_\delta(x) = u_\xi(x),\ x \in \partial\Omega\\
|u_\delta(x) - u_\xi(x)| \leq \delta,\ x \in \Omega\\
Du_\delta(x) = \xi_\pm \ \text{ a.e. in } \ \Omega_\pm\\
Du_\delta(x) \in \xi + \{t_+\alpha \otimes \beta, t_-\alpha \otimes \beta, \alpha \otimes \delta\beta_3, \ldots, \alpha \otimes \delta\beta_k\},\ \text{a.e. in } \Omega.
\end{cases} \tag{5}$$

Since $\xi_\pm \in \overline{K}_0 \cap \partial K$ and

$$\xi + \alpha \otimes \delta\beta_j \in \xi + \alpha \otimes \delta\overline{H} = \xi_0 + \alpha \otimes \left(\gamma + \delta\overline{H}\right) \subset \xi_0 + \alpha \otimes \epsilon H \subset K_0 \ \text{ for } j = 3, ..., k,$$

we deduce, by choosing $\delta = 1/\nu$ as $\nu \to \infty$, from (5), the relaxation property (4). This achieves the proof of Step 2 and thus of the corollary. ∎

We finally want to point out that as a particular case of Corollary 16 we find the existence theorem (Theorem 3.1) proved by Dacorogna-Marcellini in [6].

4 Necessary conditions

Recall that we are considering the minimization problem

$$(P) \qquad \inf\left\{I(u) = \int_\Omega f(Du(x))\, dx :\ u \in u_{\xi_0} + W_0^{1,\infty}(\Omega; \mathbb{R}^m)\right\}$$

where Ω is a bounded open set of $\mathbb{R}^n$, u_{ξ_0} is affine, i.e. $Du_{\xi_0} = \xi_0$ and $f : \mathbb{R}^{m\times n} \longrightarrow \mathbb{R}$ is a lower semicontinuous function. In order to avoid the trivial case we will always assume that

$$Qf(\xi_0) < f(\xi_0)\,.$$

Most non existence results for problem (P) follow by showing that the relaxed problem (QP) has a *unique* solution, namely u_{ξ_0}, which is by hypothesis not a solution of (P). This approach was strongly used in Marcellini [16] and Dacorogna-Marcellini [6]. We will here extend this idea in order to handle more general cases. However we should point out that we will give an example (see Proposition 36 in Section 5.2) related to minimal surfaces, where non existence occurs, while the relaxed problem has infinitely many solutions, none of them being a solution of (P).

The right notion in order to have uniqueness of the relaxed problem is

Definition 18 *A quasiconvex function $f : \mathbb{R}^{m\times n} \longrightarrow \mathbb{R}$ is said to be* strictly quasiconvex *at $\xi_0 \in \mathbb{R}^{m\times n}$, if for some bounded domain $U \subset \mathbb{R}^n$ and every $\varphi \in W_0^{1,\infty}(U;\mathbb{R}^m)$ such that*

$$\int_U f(\xi_0 + D\varphi(x))\, dx = f(\xi_0)\ \mathrm{meas}(U)$$

then $\varphi \equiv 0$.

We will see below some sufficient conditions that can ensure strict quasiconvexity, but let us start with the elementary following non existence theorem.

Theorem 19 *Let $f : \mathbb{R}^{m\times n} \longrightarrow \mathbb{R}$ be lower semicontinuous, $\xi_0 \in \mathbb{R}^{m\times n}$ with $Qf(\xi_0) < f(\xi_0)$ and Qf be strictly quasiconvex at ξ_0. Then the relaxed problem (QP) has a unique solution, namely u_{ξ_0}, while (P) has no solution.*

Proof. The fact that (QP) has only one solution follows by definition of the strict quasiconvexity of Qf and the fact that the definition of strict quasiconvexity is independent of the choice of the domain U. Assume for the sake of contradiction that (P) has a solution $\overline{u} \in u_{\xi_0} + W_0^{1,\infty}(\Omega;\mathbb{R}^m)$. We should have from Lemma 11 that (writing $\overline{u}(x) = \xi_0 x + \varphi(x)$)

$$f(\xi_0 + D\varphi(x)) = Qf(\xi_0 + D\varphi(x)),\ \text{a.e. } x \in \Omega$$

$$\int_\Omega Qf(\xi_0 + D\varphi(x))\ dx = Qf(\xi_0)\ \mathrm{meas}\,\Omega.$$

Since Qf is strictly quasiconvex at ξ_0, we deduce from the last identity that $\varphi \equiv 0$. Hence we have, from the first identity, that $Qf(\xi_0) = f(\xi_0)$, which is in contradiction with the hypothesis. ■

We now want to give some criteria that can ensure the strict quasiconvexity of a given function. The first one has been introduced by Dacorogna-Marcellini in [6].

Definition 20 *A convex function $f : \mathbb{R}^{m\times n} \longrightarrow \mathbb{R}$ is said to be* strictly convex at $\xi_0 \in R^{m\times n}$ in at least m directions *if there exists $\alpha = \left(\alpha^i\right)_{1\le i\le m} \in \mathbb{R}^{m\times n}$, $\alpha^i \neq 0$ for every $i = 1, ..., m$, such that: if for some $\eta \in \mathbb{R}^{m\times n}$ the identity*

$$\frac{1}{2} f(\xi_0 + \eta) + \frac{1}{2} f(\xi_0) = f\left(\xi_0 + \frac{1}{2}\eta\right)$$

holds, then necessarily

$$\left\langle \alpha^i ; \eta^i \right\rangle = 0,\ i = 1, ..., m.$$

In order to understand better the generalization of this notion to polyconvex functions (cf. Proposition 26), it might be enlightening to state the definition in the following way.

Proposition 21 *Let $f : \mathbb{R}^{m\times n} \longrightarrow \mathbb{R}$ be a convex function and, for $\xi \in \mathbb{R}^{m\times n}$, denote by $\partial f(\xi)$ the subdifferential of f at ξ. The two following conditions are then equivalent:*

(i) f is strictly convex at $\xi_0 \in \mathbb{R}^{m\times n}$ in at least m directions

(ii) there exists $\alpha = \left(\alpha^i\right)_{1\le i\le m} \in \mathbb{R}^{m\times n}$ with $\alpha^i \neq 0$ for every $i = 1, ..., m$, so that whenever

$$f(\xi_0 + \eta) - f(\xi_0) - \langle \lambda ; \eta \rangle = 0$$

for some $\eta \in \mathbb{R}^{m\times n}$ and for some $\lambda \in \partial f(\xi_0)$, then

$$\left\langle \alpha^i ; \eta^i \right\rangle = 0,\ i = 1, ..., m.$$

Proof. *Step 1.* We start with a preliminary observation that if

$$\frac{1}{2} f(\xi_0 + \eta) + \frac{1}{2} f(\xi_0) = f\left(\xi_0 + \frac{1}{2}\eta\right) \tag{6}$$

then, for every $t \in [0, 1]$, we have

$$tf(\xi_0 + \eta) + (1 - t) f(\xi_0) = f(\xi_0 + t\eta). \tag{7}$$

Let us show this under the assumption that $t > 1/2$ (the case $t < 1/2$ is handled similarly). We can therefore find $\alpha \in (0, 1)$ such that

$$\frac{1}{2} = \alpha t + (1 - \alpha)\, 0 = \alpha t.$$

From the convexity of f and by hypothesis, we obtain

$$\frac{1}{2} f(\xi_0 + \eta) + \frac{1}{2} f(\xi_0) = f\left(\xi_0 + \frac{1}{2}\eta\right) \le \alpha f(\xi_0 + t\eta) + (1 - \alpha) f(\xi_0).$$

Assume, for the sake of contradiction, that

$$f(\xi_0 + t\eta) < tf(\xi_0 + \eta) + (1 - t) f(\xi_0).$$

Combine then this inequality with the previous one to get

$$\tfrac{1}{2} f(\xi_0 + \eta) + \tfrac{1}{2} f(\xi_0) <$$

$$\alpha \left[tf(\xi_0 + \eta) + (1 - t) f(\xi_0)\right] + (1 - \alpha) f(\xi_0)$$

$$= \tfrac{1}{2} f(\xi_0 + \eta) + \tfrac{1}{2} f(\xi_0)$$

which is clearly a contradiction. Therefore the convexity of f and the above contradiction implies (7). This also implies that

$$f'(\xi_0, \eta) := \lim_{t \to 0^+} \frac{f(\xi_0 + t\eta) - f(\xi_0)}{t} = f(\xi_0 + \eta) - f(\xi_0).$$

Applying Theorem 23.4 in Rockafellar [21], combined with the fact that $\partial f(\xi_0)$ is non empty and compact, we get that there exists $\lambda \in \partial f(\xi_0)$ so that $f(\xi_0 + \eta) - f(\xi_0) = \langle \lambda; \eta \rangle$ and hence

$$f(\xi_0 + t\eta) - f(\xi_0) - t\langle \lambda; \eta \rangle = 0, \ \forall t \in [0, 1]. \tag{8}$$

We have therefore proved that (6) implies (8). Since the converse is obviously true, we conclude that they are equivalent.

Step 2. Let us show the equivalence of the two conditions.

(i) $\Longrightarrow$ (ii): We first observe that for any $\mu \in \mathbb{R}^{m \times n}$ we have

$$\begin{gathered} \tfrac{1}{2} f(\xi_0 + \eta) + \tfrac{1}{2} f(\xi_0) - f\left(\xi_0 + \tfrac{1}{2}\eta\right) = \\ \tfrac{1}{2}\left[f(\xi_0 + \eta) - f(\xi_0) - \langle \mu; \eta \rangle\right] - \left[f\left(\xi_0 + \tfrac{1}{2}\eta\right) - f(\xi_0) - \tfrac{1}{2}\langle \mu; \eta \rangle\right]. \end{gathered} \tag{9}$$

Assume that, for $\lambda \in \partial f(\xi_0)$, we have

$$f(\xi_0 + \eta) - f(\xi_0) - \langle \lambda; \eta \rangle = 0.$$

From (9) applied to $\mu = \lambda$, from the definition of $\partial f(\xi_0)$ and from the convexity of f, we have

$$\begin{aligned} 0 &\le \tfrac{1}{2} f(\xi_0 + \eta) + \tfrac{1}{2} f(\xi_0) - f\left(\xi_0 + \tfrac{1}{2}\eta\right) \\ &= -\left[f\left(\xi_0 + \tfrac{1}{2}\eta\right) - f(\xi_0) - \tfrac{1}{2}\langle \lambda; \eta \rangle\right] \le 0. \end{aligned}$$

Using the above identity, we then are in the framework of (i) and we deduce that $\left\langle \alpha^i; \eta^i \right\rangle = 0$, $i = 1, ..., m$, and thus (ii).

(ii) $\Longrightarrow$ (i): Assume now that we have (6), namely

$$\frac{1}{2} f(\xi_0 + \eta) + \frac{1}{2} f(\xi_0) - f\left(\xi_0 + \frac{1}{2}\eta\right) = 0$$

which, by Step 1, implies that there exists $\lambda \in \partial f(\xi_0)$ so that

$$f(\xi_0 + t\eta) - f(\xi_0) - t\langle \lambda; \eta \rangle = 0, \ \forall t \in [0, 1].$$

We are therefore, choosing $t = 1$, in the framework of (ii) and we get $\left\langle \alpha^i; \eta^i \right\rangle = 0$, $i = 1, ..., m$, as wished. ■

Of course any strictly convex function is strictly convex in at least m directions, but the above condition is much weaker. For example in the scalar case, $m = 1$, it is enough that the function is not affine in a neighborhood of ξ_0, to guarantee the condition (see below).

We now have the following result established by Dacorogna-Marcellini in [6], although the concept of strict quasiconvexity does not appear there.

Proposition 22 *If a convex function $f : \mathbb{R}^{m\times n} \longrightarrow \mathbb{R}$ is strictly convex at $\xi_0 \in \mathbb{R}^{m\times n}$ in at least m directions, then it is strictly quasiconvex at ξ_0.*

Theorem 19, combined with the above proposition, gives immediately a sharp result for the scalar case, namely

Corollary 23 *Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be lower semicontinuous, $\xi_0 \in \mathbb{R}^n$ with $Cf(\xi_0) < f(\xi_0)$ and Cf not affine at ξ_0. Then (P) has no solution.*

Remark 24 *In the scalar case this result has been obtained by several authors, in particular Cellina [3], Friesecke [12] and Dacorogna-Marcellini [6]. It also gives, combined with the result of the preceding section, that, provided some appropriate boundedness is assumed, a necessary and sufficient condition for existence of minima for (P) is that f be affine on the connected component of $\{\xi : Cf(\xi) < f(\xi)\}$ that contains ξ_0.*

Before proceeding with the proof of Proposition 22 we need the following lemma whose proof is elementary (see Step 2 of Theorem 5.1 in [6]).

Lemma 25 *Let Ω be a bounded open set of $\mathbb{R}^n$ and $\varphi \in W_0^{1,\infty}(\Omega;\mathbb{R}^m)$ be such that*

$$\left\langle \alpha^i; D\varphi^i(x)\right\rangle = 0, \ \text{a.e. } x \in \Omega,\ i = 1, ..., m$$

for some $\alpha^i \neq 0$, $i = 1, ..., m$, then $\varphi \equiv 0$.

Proof. (Proposition 22). Assume that for a certain bounded domain $U \subset \mathbb{R}^n$ and for some $\varphi \in W_0^{1,\infty}(U;\mathbb{R}^m)$ we have

$$\int_U f(\xi_0 + D\varphi(x))\,dx = f(\xi_0)\ \text{meas}(U)$$

and let us show that $\varphi \equiv 0$.

Since f is convex and the above identity holds, we find

$$\begin{aligned} f(\xi_0)\ \text{meas}(U) &= \int_U \left[\frac{1}{2} f(\xi_0) + \frac{1}{2} f(\xi_0 + D\varphi(x))\right] dx \\ &\geq \int_U f\left(\xi_0 + \frac{1}{2} D\varphi(x)\right) dx \geq f(\xi_0)\ \text{meas}(U), \end{aligned}$$

which implies that

$$\int_U \left[\frac{1}{2} f(\xi_0) + \frac{1}{2} f(\xi_0 + D\varphi(x)) - f\left(\xi_0 + \frac{1}{2} D\varphi(x)\right)\right] dx = 0.$$

The convexity of f implies then that for almost every x in U, we have

$$\frac{1}{2} f(\xi_0) + \frac{1}{2} f(\xi_0 + D\varphi(x)) - f\left(\xi_0 + \frac{1}{2} D\varphi(x)\right) = 0.$$

The strict convexity in at least m directions leads to

$$\left\langle \alpha^{i}; D\varphi^{i}\left(x\right)\right\rangle =0,\ \text{a.e. } x\in U,\ i=1,...,m.$$

Lemma 25 gives the claim. ■

We will now generalize Proposition 22. Since the notations in the next result are involved, we will first write the proposition when $m=n=2$.

Proposition 26 *Let $f:\mathbb{R}^{m\times n}\longrightarrow\mathbb{R}$ be polyconvex, $\xi_0\in\mathbb{R}^{m\times n}$ and $\lambda=\lambda\left(\xi_0\right)\in\mathbb{R}^{\tau(m,n)}$ so that*

$$f\left(\xi_0+\eta\right)-f\left(\xi_0\right)-\left\langle \lambda; T\left(\xi_0+\eta\right)-T\left(\xi_0\right)\right\rangle \geq 0,\ \textit{for every } \eta\in\mathbb{R}^{m\times n}.$$

(i) Let $m=n=2$ and assume that there exist $\alpha^{1,1},\alpha^{1,2},\alpha^{2,2}\in\mathbb{R}^2$, $\alpha^{1,1}\neq 0,\alpha^{2,2}\neq 0$, $\beta\in\mathbb{R}$, so that if for some $\eta\in\mathbb{R}^{2\times 2}$ the following equality holds

$$f\left(\xi_0+\eta\right)-f\left(\xi_0\right)-\left\langle \lambda; T\left(\xi_0+\eta\right)-T\left(\xi_0\right)\right\rangle =0$$

then necessarily

$$\left\langle \alpha^{2,2};\eta^{2}\right\rangle =0\ \textit{and}\ \left\langle \alpha^{1,1};\eta^{1}\right\rangle +\left\langle \alpha^{1,2};\eta^{2}\right\rangle +\beta\det\eta=0.$$

Then f is strictly quasiconvex at ξ_0.

(ii) Let $m,n\geq 2$ and assume that there exist, for every $\nu=1,...,m$,

$$\alpha^{\nu,\nu},\alpha^{\nu,\nu+1},...,\alpha^{\nu,m}\in\mathbb{R}^{n},\alpha^{\nu,\nu}\neq 0,\beta^{\nu,s}\in\mathbb{R}^{\binom{n}{s}},2\leq s\leq n\wedge\left(m-\nu+1\right)$$

so that if for some $\eta\in\mathbb{R}^{m\times n}$ the following equality holds

$$f\left(\xi_0+\eta\right)-f\left(\xi_0\right)-\left\langle \lambda; T\left(\xi_0+\eta\right)-T\left(\xi_0\right)\right\rangle =0$$

then necessarily

$$\sum_{s=\nu}^{m}\left\langle \alpha^{\nu,s};\eta^{s}\right\rangle +\sum_{s=2}^{n\wedge(m-\nu+1)}\left\langle \beta^{\nu,s}; adj_{s}\left(\eta^{\nu},...,\eta^{m}\right)\right\rangle =0,\ \nu=1,...,m.$$

Then f is strictly quasiconvex at ξ_0.

Remark 27 *(i) The existence of a λ as in the hypotheses of the proposition is automatically guaranteed by the polyconvexity of f (see (1) in Section 2, it corresponds in the case of a convex function to an element of $\partial f\left(\xi_0\right)$).*

(ii) We have adopted the convention that if $l>k>0$ are integers, then $\sum_{l}^{k}=0.$

Example 28 *Let $m=n=2$ and consider the function*

$$f\left(\eta\right)=\left(\eta_2^2\right)^2+\left(\eta_1^1+\det\eta\right)^2.$$

This function is trivially polyconvex and according to the proposition it is also strictly quasiconvex at $\xi_0=0$ (choose $\lambda=0\in\mathbb{R}^5$, $\alpha^{2,2}=(0,1)$, $\alpha^{1,2}=(0,0)$, $\alpha^{1,1}=(1,0)$, $\beta=1$).

Proof. We will prove the proposition only in the case $m = n = 2$, the general case being handled similarly.

Assume that for a certain bounded domain $U \subset \mathbb{R}^2$ and for some $\varphi \in W_0^{1,\infty}\left(U;\mathbb{R}^2\right)$ we have

$$\int_U f\left(\xi_0 + D\varphi\left(x\right)\right) dx = f\left(\xi_0\right) \text{meas}(U)$$

and let us prove that $\varphi \equiv 0$. This is equivalent, for every $\mu \in \mathbb{R}^{\tau(2,2)}$, to

$$\left[\int_U f\left(\xi_0 + D\varphi\left(x\right)\right) - f\left(\xi_0\right) - \left\langle \mu; T\left(\xi_0 + D\varphi\left(x\right)\right) - T\left(\xi_0\right)\right\rangle\right] dx = 0.$$

Choosing $\mu = \lambda$ (λ as in the statement of the proposition) in the previous equation and using the polyconvexity of the function f, we get

$$f\left(\xi_0 + D\varphi\left(x\right)\right) - f\left(\xi_0\right) - \left\langle \lambda; T\left(\xi_0 + D\varphi\left(x\right)\right) - T\left(\xi_0\right)\right\rangle = 0, \text{ a.e. } x \in U.$$

We hence infer that, for almost every $x \in U$, we have

$$\left\langle \alpha^{2,2}; D\varphi^2\right\rangle = 0 \text{ and } \left\langle \alpha^{1,1}; D\varphi^1\right\rangle + \left\langle \alpha^{1,2}; D\varphi^2\right\rangle + \beta \det D\varphi = 0.$$

Lemma 25, applied to the first equation, implies that $\varphi^2 \equiv 0$. Using this result in the second equation we get

$$\left\langle \alpha^{1,1}; D\varphi^1\right\rangle = 0$$

and hence, appealing once more to the lemma, we have the claim, namely $\varphi^1 \equiv 0$. ■

Summarizing the results of Theorem 19, Proposition 22 and Proposition 26, we get

Corollary 29 *Let $f : \mathbb{R}^{m\times n} \longrightarrow \mathbb{R}$ be lower semicontinuous, $\xi_0 \in \mathbb{R}^{m\times n}$ with*

$$Qf\left(\xi_0\right) < f\left(\xi_0\right).$$

If either one of the two following conditions hold

(i) $Qf\left(\xi_0\right) = Cf\left(\xi_0\right)$ and Cf is strictly convex at ξ_0 in at least m directions;

(ii) $Qf\left(\xi_0\right) = Pf\left(\xi_0\right)$ and Pf is strictly polyconvex at ξ_0 (in the sense of Proposition 26);

then (QP) has a unique solution, namely u_{ξ_0}, while (P) has no solution.

Proof. The proof is almost identical under both hypotheses and so we will establish the corollary only in the first case. The result will follow from Theorem 19 if we can show that Qf is strictly convex at ξ_0. So assume that

$$\int_\Omega Qf\left(\xi_0 + D\varphi\left(x\right)\right)\, dx = Qf\left(\xi_0\right) \text{meas}\,\Omega$$

for some $\varphi \in W_0^{1,\infty}(\Omega;\mathbb{R}^m)$ and let us prove that $\varphi \equiv 0$. Using Jensen inequality combined with the hypothesis $Qf(\xi_0) = Cf(\xi_0)$ and the fact that $Qf \geq Cf$, we find that the above identity implies

$$\int_\Omega Cf(\xi_0 + D\varphi(x))\, dx = Cf(\xi_0) \operatorname{meas}\Omega.$$

The hypotheses on Cf and Proposition 22 imply that $\varphi \equiv 0$, as wished. ■

5 Examples

We now consider two examples of the form studied in the previous sections, namely

$$(P) \qquad \inf\left\{ I(u) = \int_\Omega f(Du(x))\, dx : \; u \in u_{\xi_0} + W_0^{1,\infty}(\Omega;\mathbb{R}^m) \right\}$$

where Ω is a bounded open set of $\mathbb{R}^n$, u_{ξ_0} is affine, i.e. $Du_{\xi_0} = \xi_0$ and $f : \mathbb{R}^{m\times n} \longrightarrow \mathbb{R}$ is a lower semicontinuous function.

1) We consider in Subsection 5.1 the case where $m = n$ and

$$f(\xi) = g(\lambda_2(\xi), ..., \lambda_{n-1}(\xi), \det \xi)$$

where $0 \leq \lambda_1(\xi) \leq \cdots \leq \lambda_n(\xi)$ are the singular values of $\xi \in \mathbb{R}^{n\times n}$. Functions of the above type are simplified versions of stored energy functions that appear in nonlinear elasticity.

2) In Subsection 5.2 we deal with the minimal surface case, namely when $m = n+1$ and $f(\xi) = g(adj_n\xi)$. One should note that if $u : \mathbb{R}^n \longrightarrow \mathbb{R}^{n+1}$ is a surface in parametric form, then $adj_n Du$ is the normal to this surface. In this sense the minimisation problem is of minimal surface type.

The same analysis could be applied to the following examples that have been treated by other authors.

– Integrands of the form

$$f(\xi) = g(\Phi(\xi))$$

where $\Phi : \mathbb{R}^{m\times n} \longrightarrow \mathbb{R}$ is quasiaffine. This problem has first been considered by Mascolo-Schianchi [17] and later by Dacorogna-Marcellini [6] for the case of the determinant ($m = n$, $\Phi(\xi) = \det \xi$). The general case was studied by Cellina-Zagatti [4] and later by Dacorogna-Ribeiro [10]. We remark that the result of Subsection 5.1 includes the case $f(\xi) = g(\det \xi)$.

– The Saint Venant-Kirchhoff energy functional (here $m = n$ and $\nu \in (0, 1/2)$ is a parameter):

$$f(\xi) = \left|\xi\xi^t - I\right|^2 + \frac{\nu}{1-2\nu}\left(|\xi|^2 - n\right)^2$$

or in terms of the singular values $0 \leq \lambda_1(\xi) \leq \cdots \leq \lambda_n(\xi)$ of $\xi \in \mathbb{R}^{n \times n}$

$$f(\xi) = \sum_{i=1}^{n} \left(\lambda_i^2(\xi) - 1\right)^2 + \frac{\nu}{1 - 2\nu} \left(\sum_{i=1}^{n} \lambda_i^2(\xi) - n\right)^2.$$

This case has been studied by Dacorogna-Marcellini [6] with the help of a representation formula due to Le Dret-Raoult. With the present theory it is possible to establish more general results, but we do not discuss the details here.

– Optimal design problem (with $m = n = 2$):

$$f(\xi) = \begin{cases} 1 + |\xi|^2 & \text{if } \xi \neq 0 \\ 0 & \text{if } \xi = 0. \end{cases}$$

This problem was introduced by Kohn-Strang [15]. The existence of minimizers was then established by Dacorogna-Marcellini in [6] and [7] and in a different manner by Dacorogna-Tanteri [11].

5.1 The case of singular values

In this section we let $m = n$ and we denote by $\lambda_1(\xi), ..., \lambda_n(\xi)$ the singular values of $\xi \in \mathbb{R}^{n \times n}$ with $0 \leq \lambda_1(\xi) \leq \cdots \leq \lambda_n(\xi)$ and by Q the set

$$Q = \{x = (x_2, ..., x_{n-1}) \in \mathbb{R}^{n-2} : \ 0 \leq x_2 \leq \cdots \leq x_{n-1}\}$$

which is the natural set where to consider $(\lambda_2(\xi), ..., \lambda_{n-1}(\xi))$ for $\xi \in \mathbb{R}^{n \times n}$.

The functions under consideration are functions depending not only on some singular values, but also on the determinant.

Theorem 30 *Let $g : Q \times \mathbb{R} \longrightarrow \mathbb{R}$ be a function such that $g(\cdot, s)$ is continuous and bounded from below for all $s \in \mathbb{R}$. Let $f : \mathbb{R}^{n \times n} \longrightarrow \mathbb{R}$ be defined by*

$$f(\xi) = g(\lambda_2(\xi), ..., \lambda_{n-1}(\xi), \det \xi)$$

then

$$Pf(\xi) = Qf(\xi) = Rf(\xi) = Ch(\det \xi),$$

where $h : \mathbb{R} \longrightarrow \mathbb{R}$ is given by $h(s) = \inf_{x \in Q} g(x, s)$.

Remark 31 *We remark that if some dependence on λ_1 or λ_n is allowed, then no simple and general expression for the envelopes is known; see Proposition 32 below, when there is dependence on λ_1, and Theorem 3.5 in Buttazzo-Dacorogna-Gangbo [2], when there is dependence on λ_n.*

Proof. Firstly we remark that, using Theorem 3.1 of Dacorogna-Ribeiro [10], we can easily obtain the following assertion. Let

$$E = \{\xi \in \mathbb{R}^{n \times n} : \ \lambda_i(\xi) = m_i, \ i = 2, ..., n-1, \ \det \xi = c\}$$

for some $0 < m_2 \leq \cdots \leq m_{n-1}$, $c \in \mathbb{R}$, then

$$\operatorname{Rco} E = \{\xi \in \mathbb{R}^{n\times n} : \ \det \xi = c\}.$$

We now prove the result. Since we always have

$$Ch(\det \xi) \leq Pf(\xi) \leq Qf(\xi) \leq Rf(\xi),$$

we only need to prove that $Rf(\xi) \leq Ch(\det \xi)$ which follows if $Rf(\xi) \leq h(\det \xi)$. In fact, if we get $Rf(\xi) \leq h(\det \xi)$ then the rank-one convex envelope of each member preserves the inequality and since the rank-one convex envelope of $h(\det \xi)$ is $Ch(\det \xi)$ we get $Rf(\xi) \leq Ch(\det \xi)$. Let $\xi \in \mathbb{R}^{n\times n}$ and let $x^k = (x_2^k, ..., x_{n-1}^k) \in Q$ be a sequence such that

$$h(\det \xi) = \inf g(\cdot, \det \xi) = \lim g(x^k, \det \xi).$$

For each $k \in \mathbb{N}$ we define

$$G_k(\eta) = Rf(\eta) - g(x^k, \det \xi)$$

which is a rank one convex function. We will prove that $G_k(\xi) \leq 0$ for all $k \in \mathbb{N}$ and the result follows by passing to the limit.

There are two different cases to consider. For fixed k let $s \in \{1, ..., n-1\}$ be such that $x_2^k = \cdots = x_s^k = 0$ and $x_{s+1}^k > 0$.

Case 1: $s = 1$. Consider the set

$$E_k = \{\eta \in \mathbb{R}^{n\times n} : \lambda_i(\eta) = x_i^k, \ i = 2, ..., n-1, \ \det \eta = \det \xi\}.$$

Since in E_k, $G_k(\eta) = Rf(\eta) - f(\eta)$ then G_k is non positive in this set. Besides, the rank one convexity of G_k implies that G_k is also non positive in $Rco\, E_k$. By the remark made above we obtain $G_k(\xi) \leq 0$ since $\xi \in \operatorname{Rco} E_k$.

Case 2: $s > 1$. Let, for $t > 0$,

$$H_k^t(\eta) = Rf(\eta) - g\left(\frac{x_{s+1}^k}{t}, ..., \frac{x_{s+1}^k}{t}, x_{s+1}^k, ..., x_{n-1}^k, \det \xi\right)$$

(write $x_{s+1}^k = 1$ in the case $s = n-1$) and let

$$E_k^t = \left\{ \eta \in \mathbb{R}^{n\times n} : \begin{array}{c} \lambda_i(\eta) = \dfrac{x_{s+1}^k}{t}, \ i = 2, ..., s, \\ \lambda_j(\eta) = x_j^k, \ j = s+1, ..., n-1, \\ \det \eta = \det \xi \end{array} \right\}.$$

As before, H_k^t is non positive in this set. Since $\xi \in \operatorname{Rco} E_k^t$ and H_k^t is rank one convex we obtain that $H_k^t(\xi) \leq 0$:

$$Rf(\xi) \leq g\left(\frac{x_{s+1}^k}{t}, ..., \frac{x_{s+1}^k}{t}, x_{s+1}^k, ..., x_{n-1}^k, \det \xi\right).$$

Passing to the limit as $t \to +\infty$, we get $G_k(\xi) \leq 0$ as wished. ■

We next see that the previous result is not true for functions depending also on λ_1.

Proposition 32 *Let $f:\mathbb{R}^{2\times 2}\longrightarrow\mathbb{R}$ be defined by $f(\xi)=|\lambda_1(\xi)-1|+|\det\xi|$. Then $Pf(\xi)\neq|\det\xi|$.*

Proof. Let us suppose for the sake of contradiction that $Pf(\xi)=|\det\xi|$. Then, for ξ such that $\lambda_1(\xi)=0$, $Pf(\xi)=|\det\xi|=0$. From the representation formula for the polyconvex envelope (see Theorem 5.1.1 in Dacorogna [5]), we therefore get that there exist $A_i^n\in\mathbb{R}^{2\times 2}$, $t_i^n\in[0,1]$ and $\sum_{i=1}^6 t_i^n=1$ such that

$$\lim_{n\to\infty}\sum_{i=1}^{6}t_i^n f(A_i^n)=0 \text{ with } \sum_{i=1}^{6}t_i^n(A_i^n,\det A_i^n)=(\xi,\det\xi).$$

In particular, $t_i^n\,|\lambda_1(A_i^n)-1|\to 0$ and $t_i^n\,|\det A_i^n|\to 0$, $i=1,...,6$. Up to a subsequence, $t_i^n\to t_i\in[0,1]$ with $\sum_{i=1}^6 t_i=1$. So, there is some j such that $t_j\neq 0$ and thus

$$\left|\lambda_1(A_j^n)-1\right|=\frac{1}{t_j^n}t_j^n\left|\lambda_1(A_j^n)-1\right|\to 0$$

and

$$\left|\det A_j^n\right|=\frac{1}{t_j^n}t_j^n\left|\det A_j^n\right|\to 0.$$

The first condition implies that $\lambda_1(A_j^n)\to 1$, which contradicts the second one, since then we would have $\left|\det A_j^n\right|\geq(\lambda_1(A_j^n))^2\to 1$. ■

We next apply the theory of Section 3 to get the following existence result.

Theorem 33 *Let*

$$f(\xi)=g(\lambda_2(\xi),...,\lambda_{n-1}(\xi))+h(\det\xi)$$

where g and h are lower semicontinuous functions such that $g:Q\longrightarrow\mathbb{R}$ verifies

$$\inf g=g(m_2,...,m_{n-1}),\ \text{with } 0<m_2\leq\cdots\leq m_{n-1}$$

and $h:\mathbb{R}\longrightarrow\mathbb{R}$ verifies

$$\lim_{|t|\to+\infty}\frac{h(t)}{|t|}=+\infty. \tag{10}$$

Then (P) has a solution.

Proof. We note that, by Theorem 30, $Qf(\xi)=\inf g+Ch(\det\xi)$. Letting

$$K=\left\{\xi\in\mathbb{R}^{n\times n}:Qf\left(\xi\right)<f\left(\xi\right)\right\}$$

we see that

$$K=L_1\cup L_2$$

where

$$L_1=\left\{\xi\in\mathbb{R}^{n\times n}:\ Ch(\det\xi)<h(\det\xi)\right\}$$

$$L_2=\left\{\xi\in\mathbb{R}^{n\times n}:\ Ch(\det\xi)=h(\det\xi),\ \inf g<g(\lambda_2(\xi),...,\lambda_{n-1}(\xi))\right\}.$$

We first observe that hypothesis (10) allows us to write

$$S = \{t \in \mathbb{R} : Ch(t) < h(t)\} = \bigcup_{j \in \mathbb{N}} (\alpha_j, \beta_j),$$

Ch being affine in each interval (α_j, β_j); thus

$$L_1 = \left\{ \xi \in \mathbb{R}^{n \times n} : \ \det \xi \in \bigcup_{j \in \mathbb{N}} (\alpha_j, \beta_j) \right\}.$$

Note that Qf is quasiaffine on K.

We now prove the result. Clearly, if $\xi_0 \notin K$ then u_{ξ_0} is a solution of (P). Let us suppose that $\xi_0 \in K$. There are three different cases to consider.

Case 1: $\xi_0 \in L_1$. Let (α_j, β_j) be an interval as above such that $\det \xi_0 \in (\alpha_j, \beta_j)$. We get the result applying Theorem 12 with

$$K_0 = \left\{ \xi \in \mathbb{R}^{n \times n} : \ \det \xi \in (\alpha_j, \beta_j) \, , \ \prod_{i=\nu}^{n} \lambda_i(\xi) < \prod_{i=\nu}^{n} m_i, \ \nu = 2, ..., n \right\},$$

where m_n is chosen sufficiently large so that

$$m_{n-1} \leq m_n, \tag{11}$$

$$\prod_{i=\nu}^{n} \lambda_i(\xi_0) < \prod_{i=\nu}^{n} m_i, \ \nu = 2, ..., n, \tag{12}$$

$$\max \{|\alpha_j| \, , |\beta_j|\} < m_2 \prod_{i=2}^{n} m_i. \tag{13}$$

Clearly $K_0 \subset L_1 \subset K$, moreover (12) ensures that $\xi_0 \in K_0$ and (13) ensures the relaxation property of K_0 with respect to

$$E = \left\{ \xi \in \mathbb{R}^{n \times n} : \ \det \xi \in \{\alpha_j, \beta_j\} \, , \ \lambda_\nu(\xi) = m_\nu, \ \nu = 2, ..., n \right\} \subset \overline{K}_0 \cap \partial K$$

through Theorem 10 and the family of sets

$$E_\delta = \left\{ \xi \in \mathbb{R}^{n \times n} : \ \det \xi \in \{\alpha_j + \delta, \beta_j - \delta\} \, , \ \lambda_i(\xi) = m_i - \delta, \ i = 2, ..., n \right\},$$

(cf. the proof of Theorem 1.1 of Dacorogna-Ribeiro [10] for details). Consequently K_0 has the relaxation property with respect to $\overline{K}_0 \cap \partial K$.

Case 2: $\xi_0 \in L_2$ and $\det \xi_0 \neq 0$. We consider in this case the set

$$K_1 = \left\{ \xi \in \mathbb{R}^{n \times n} : \ \det \xi = \det \xi_0, \ \prod_{i=\nu}^{n} \lambda_i(\xi) < \prod_{i=\nu}^{n} m_i, \ \nu = 2, ..., n \right\}$$

where m_n satisfies the conditions (11) and (12) of the first case (with strict inequality for the first one: $m_n > m_{n-1}$). It was shown by Dacorogna-Tanteri [11] that K_1 has the relaxation property with respect to

$$E = \left\{ \xi \in \mathbb{R}^{n \times n} : \ \det \xi = \det \xi_0, \ \lambda_\nu(\xi) = m_\nu, \ \nu = 2, ..., n \right\}$$

and the existence of $u \in u_{\xi_0} + W_0^{1,\infty}(\Omega, \mathbb{R}^n)$ such that $Du \in E$, *a.e.* in Ω. Since $Qf = f$ in E and $Qf(\xi_0) = Qf(Du)$, we can apply Lemma 11 and get the result.

Case 3: $\xi_0 \in L_2$ and $\det \xi_0 = 0$. Since any matrix $\xi \in \mathbb{R}^{n \times n}$ can be decomposed in the form RDQ, where $R, Q \in O(n)$ and $D = diag(\lambda_1(\xi), ..., \lambda_n(\xi))$ (cf. [13]) we can reduce ourselves to the case of $\xi_0 = diag(\lambda_1(\xi_0), ..., \lambda_n(\xi_0))$. In particular, as $\det \xi_0 = 0$, we have $\lambda_1(\xi_0) = 0$ and thus the first line of ξ_0 equal to zero. Let $m_n \geq m_{n-1}$ and define

$$\mathcal{K} = \left\{ \xi \in \mathbb{R}^{(n-1) \times n} : \ \prod_{i=\nu}^{n-1} \lambda_i(\xi) < \prod_{i=\nu}^{n-1} m_{i+1}, \ \nu = 2, ..., n-1 \right\},$$

$$\mathcal{E} = \left\{ \xi \in \mathbb{R}^{(n-1) \times n} : \ \lambda_i(\xi) = m_{i+1}, \ i = 1, ..., n-1 \right\}.$$

It is then easy to show, by use of the approximation property, that $\mathcal{K}$ has the relaxation property with respect to $\mathcal{E}$ (cf. Dacorogna-Marcellini in [7, Theorem 7.28] for more details).

Using the above, if we define

$$K_1 = \left\{ \xi \in \mathbb{R}^{n \times n} : \ \xi^1 = 0, \ \prod_{i=\nu}^{n} \lambda_i(\xi) < \prod_{i=\nu}^{n} m_i, \ \nu = 2, ..., n \right\}$$

$$E = \left\{ \xi \in \mathbb{R}^{n \times n} : \ \xi^1 = 0, \ \lambda_i(\xi) = m_i, \ i = 2, ..., n \right\}$$

we get that K_1 has the relaxation property with respect to E. If we chose m_n sufficiently large such that $\xi_0 \in K_1$ we can apply Theorem 7 to get the existence of $u \in u_{\xi_0} + W_0^{1,\infty}(\Omega, \mathbb{R}^n)$ such that $Du \in E$. Finally, as $Qf = f$ in E and $Qf(\xi_0) = Qf(Du)$, applying Lemma 11, we conclude the result. ■

5.2 The minimal surface case

We now deal with the case where $m = n + 1$ and

$$f(\xi) = g(adj_n \xi).$$

The minimization problem is then

$$(P) \qquad \inf \left\{ \int_\Omega g(adj_n(Du(x)))\, dx : \ u \in u_{\xi_0} + W_0^{1,\infty}(\Omega; \mathbb{R}^{n+1}) \right\}$$

where Ω is a bounded open set of $\mathbb{R}^n$, $Du_{\xi_0} = \xi_0$ and $g : \mathbb{R}^{n+1} \longrightarrow \mathbb{R}$ is a lower semicontinuous non convex function.

It was proved by Dacorogna (see [5]) that

$$Qf(\xi) = Cg(adj_n\xi).$$

We next set

$$S = \{y \in \mathbb{R}^{n+1} : \ Cg(y) < g(y)\}$$

and assume, in order to avoid the trivial situation, that $adj_n\xi_0 \in S$. We also assume that S is connected, otherwise we replace it by its connected component that contains $adj_n\xi_0$.

Observe that

$$K = \{\xi \in \mathbb{R}^{(n+1)\times n} : \ Qf(\xi) < f(\xi)\} = \left\{\xi \in \mathbb{R}^{(n+1)\times n} : adj_n\xi \in S\right\}.$$

Theorem 34 *If S is bounded, Cg is affine in S and* rank $\xi_0 \geq n-1$, *then (P) has a solution.*

Remark 35 *(i) The fact that Cg be affine in S is not a necessary condition for existence of minima, as seen in Proposition 36.*

(ii) We will apply Corollary 16 to obtain the above result. If instead we apply Theorem 12 we could also obtain the existence of solution to (P) with no restriction on the rank of ξ_0.

Proof. The result follows if we choose a convenient rank-one direction $\lambda = \alpha \otimes \beta \in \mathbb{R}^{(n+1)\times n}$ satisfying the hypothesis of Corollary 16. We remark that, since we suppose Cg affine in S, Qf is quasiaffine in $L_K(\xi_0 + \alpha \otimes B_\epsilon, \lambda)$ (cf. Definition 14) independently of the choice of λ. So we only have to prove that K is stably bounded at ξ_0 in a direction $\lambda = \alpha \otimes \beta$.

Firstly we observe that we can find (cf. Theorem 3.1.1 in [13]) $P \in O\,(n+1)$, $Q \in SO\,(n)$ and $0 \leq \lambda_1 \leq ... \leq \lambda_n$, so that

$$\xi_0 = PLQ, \text{ where } L = \left(\lambda_j^i\right)_{1\leq j\leq n}^{1\leq i\leq n+1} \text{ with } \lambda_j^i = \lambda_j \delta_{ij},\ 1 \leq i \leq n+1,\ 1 \leq j \leq n;$$

in particular when $n = 2$ we have

$$L = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \\ 0 & 0 \end{pmatrix}.$$

Since rank $\xi_0 \geq n-1$ we have that $\lambda_2 > 0$. We also note that

$$adj_n\xi_0 = adj_nP \,.\, adj_nL \ \text{ and } adj_nL = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ (-1)^n \lambda_1 ... \lambda_n \end{pmatrix}.$$

Without loss of generality we assume $\xi_0 = L$. We then choose $\lambda = \alpha \otimes \beta$ where $\alpha = (1, 0, ..., 0) \in \mathbb{R}^{n+1}$ and $\beta = (1, 0, ..., 0) \in \mathbb{R}^n$. We will see that

$L_K(\xi_0 + \alpha \otimes B_\epsilon, \lambda)$ is bounded for some $\epsilon > 0$. Let $\eta \in L_K(\xi_0 + \alpha \otimes B_\epsilon, \lambda)$ then we can write $\eta = \xi_0 + \alpha \otimes \gamma_\epsilon + t\lambda$ for some $\gamma_\epsilon \in B_\epsilon$ and $t \in \mathbb{R}$. By definition of $L_K(\xi_0 + \alpha \otimes B_\epsilon, \lambda)$ we have $adj_n \eta \in \overline{S}$. Since S is bounded and

$$|adj_n \eta| = \left|\lambda_1 + \gamma_\epsilon^1 + t\right| \lambda_2 ... \lambda_n$$

it follows, using the fact that rank $\xi_0 \geq n-1$, that $|t|$ is bounded by a constant depending on S, ξ_0 and ϵ. Consequently $|\eta| \leq |\xi_0| + |\alpha \otimes \gamma_\epsilon| + |t| \ |\lambda|$ is bounded for any fixed positive ϵ and we get the result. ■

As already alluded in Section 4, we obtain now a result of non existence although the integrand of the relaxed problem is not strictly quasiconvex. We will consider the case where $m = 3$, $n = 2$ and $f : \mathbb{R}^{3\times 2} \to \mathbb{R}$ is given by

$$f(\xi) = g(adj_2 \xi)$$

where $g : \mathbb{R}^3 \to \mathbb{R}$ is defined by

$$g(\nu) = \left(\nu_1^2 - 4\right)^2 + \nu_2^2 + \nu_3^2 \,.$$

We therefore get $Qf(\xi) = Cg(adj_2 \xi)$ and

$$Cg(\nu) = \left[\nu_1^2 - 4\right]_+^2 + \nu_2^2 + \nu_3^2$$

where

$$[x]_+ = \begin{cases} x & \text{if } x \geq 0 \\ 0 & \text{if } x < 0. \end{cases}$$

We will choose the boundary datum as follows

$$u_{\xi_0}(x) = \begin{pmatrix} u_{\xi_0}^1(x) = \alpha_1 x_1 + \alpha_2 x_2 \\ \\ u_{\xi_0}^2(x) = 0 \\ \\ u_{\xi_0}^3(x) = 0 \end{pmatrix}$$

and hence

$$Du_{\xi_0}(x) = \xi_0 = \begin{pmatrix} \alpha_1 & \alpha_2 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \ adj_2 Du_{\xi_0}(x) = adj_2 \xi_0 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

The problem is then

$$(P) \qquad \inf \left\{ I(u) = \int_\Omega f(Du(x))\, dx : \ u \in u_{\xi_0} + W_0^{1,\infty}(\Omega; \mathbb{R}^3) \right\}.$$

Note also that $Qf(\xi_0) = 0 < f(\xi_0) = 16$.

In terms of the preceding notations we have

$$S = \{y \in \mathbb{R}^3 : \ Cg(y) < g(y)\} = \{y = (y_1, y_2, y_3) \in \mathbb{R}^3 : \ |y_1| < 2\}$$
$$K = \{\xi \in \mathbb{R}^{3\times 2} : \ Qf(\xi) < f(\xi)\} = \left\{\xi \in \mathbb{R}^{3\times 2} : adj_2 \xi \in S\right\}$$

and we observe that Cg is not affine on S, which in turn implies that Qf is not quasiaffine on K.

The following result shows that the hypothesis of strict quasiconvexity of Qf is not necessary for *non existence*.

Proposition 36 *(P) has a solution if and only if $u_{\xi_0} \equiv 0$. Moreover Qf is not strictly quasiconvex at any $\xi_0 \in \mathbb{R}^{3\times 2}$ of the form*

$$\xi_0 = \begin{pmatrix} \alpha_1 & \alpha_2 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Proof. *Step 1.* We first show that if (P) has a solution then $u_{\xi_0} \equiv 0$. If $u \in u_{\xi_0} + W_0^{1,\infty}(\Omega; \mathbb{R}^3)$ is a solution of (P) we necessarily have, denoting by $\nu(\xi) = adj_2 \xi$,

$$|\nu_1(Du)| = 2\ , \nu_2(Du) = \nu_3(Du) = 0\ ,$$

since

$$Qf(Du_{\xi_0}) = Cg(adj_2 Du_{\xi_0}) = Cg(0) = 0.$$

The three equations read as

$$\begin{cases} \left|u_{x_1}^2 u_{x_2}^3 - u_{x_2}^2 u_{x_1}^3\right| = 2 \\ u_{x_1}^1 u_{x_2}^3 - u_{x_2}^1 u_{x_1}^3 = 0 \\ u_{x_1}^1 u_{x_2}^2 - u_{x_2}^1 u_{x_1}^2 = 0. \end{cases} \tag{14}$$

Multiplying the second equation of (14) first by $u_{x_1}^2$, then by $u_{x_2}^2$, using the third equation of (14), we get

$$0 = u_{x_1}^2 u_{x_1}^1 u_{x_2}^3 - u_{x_1}^2 u_{x_2}^1 u_{x_1}^3 = u_{x_1}^2 u_{x_1}^1 u_{x_2}^3 - u_{x_1}^1 u_{x_2}^2 u_{x_1}^3 = u_{x_1}^1 \left(u_{x_1}^2 u_{x_2}^3 - u_{x_2}^2 u_{x_1}^3\right)$$
$$0 = u_{x_2}^2 u_{x_1}^1 u_{x_2}^3 - u_{x_2}^2 u_{x_2}^1 u_{x_1}^3 = u_{x_1}^2 u_{x_2}^1 u_{x_2}^3 - u_{x_2}^2 u_{x_2}^1 u_{x_1}^3 = u_{x_2}^1 \left(u_{x_1}^2 u_{x_2}^3 - u_{x_2}^2 u_{x_1}^3\right).$$

Combining these last equations with the first one of (14), we find

$$u_{x_1}^1 = u_{x_2}^1 = 0, \text{ a.e.}$$

We therefore find that any solution of (P) should have $Du^1 = 0$ a.e. and hence $u^1 \equiv$constant on each connected component of Ω. Since u^1 agrees with $u_{\xi_0}^1$ on the boundary of Ω, we deduce that $u_{\xi_0}^1 \equiv 0$ and thus $u_{\xi_0} \equiv 0$, as claimed.

Step 2. We next show that if $u_{\xi_0} \equiv 0$, then (P) has a solution. It suffices to choose $u^1 \equiv 0$ and to solve

$$\begin{cases} \left|u_{x_1}^2 u_{x_2}^3 - u_{x_2}^2 u_{x_1}^3\right| = 2 & \text{a.e. in } \Omega \\ u^2 = u^3 = 0 & \text{on } \partial\Omega. \end{cases}$$

This is possible by virtue of, for example, Corollary 7.30 in [7].

Step 3. We finally prove that Qf is not strictly quasiconvex at any $\xi_0 \in \mathbb{R}^{3\times 2}$ of the form given in the statement of the proposition. Indeed let $0 < R_1 < R_2 < R$ and denote by B_R the ball centered at 0 and of radius R. Choose $\lambda, \mu \in C^\infty(B_R)$ such that

1) $\lambda = 0$ on ∂B_R and $\lambda \equiv 1$ on B_{R_2}.

2) $\mu \equiv 0$ on $B_R \diagdown \overline{B}_{R_2}$, $\mu \equiv 1$ on B_{R_1} and

$$\left|\mu^2 + \mu\left(x_1 \mu_{x_1} + x_2 \mu_{x_2}\right)\right| < 2 \text{ for every } x \in B_R \, .$$

This last condition (which is a restriction only in $B_{R_2} \diagdown \overline{B}_{R_1}$) is easily ensured by choosing appropriately R_1, R_2 and R.

We then choose $u(x) = u_{\xi_0}(x) + \varphi(x)$ where

$$\varphi^1(x) = -\lambda(x)\, u^1_{\xi_0}(x)\,,\ \varphi^2(x) = \mu(x)\, x_1 \text{ and } \varphi^3(x) = \mu(x)\, x_2\, .$$

We therefore have that $\varphi \in W_0^{1,\infty}(B_R;\mathbb{R}^3)$, $adj_2 Du \equiv 0$ on $B_R \diagdown \overline{B}_{R_2}$, while on B_{R_2} we have

$$adj_2 Du = \left(\mu^2 + \mu\left(x_1\mu_{x_1} + x_2\mu_{x_2}\right), 0, 0\right).$$

We have thus obtained that $Cg\left(adj_2 Du\right) \equiv 0$ and hence

$$Qf\left(\xi_0 + D\varphi\right) \equiv Qf\left(\xi_0\right) = 0.$$

This implies that (QP) has infinitely many solutions. However since φ does not vanish identically, we deduce that Qf is not strictly quasiconvex at any ξ_0 of the given form. ■

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