On a generalized Wirtinger inequality

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Abstract

Let

$$\alpha(p,q,r)=\inf\left\{\frac{\|u'\|_p}{\|u\|_q}:u\in W^{1,p}_{per}(-1,1)\setminus\{0\}\,,\ \int_{-1}^{1}|u|^{r-2}u=0\right\}.$$

We show that

$$\begin{aligned}\alpha(p,q,r)&=\alpha(p,q,q)\ \text{ if } q\le rp+r-1\\ \alpha(p,q,r)&<\alpha(p,q,q)\ \text{ if } q>(2r-1)p\end{aligned}$$

generalizing results of Dacorogna-Gangbo-Subía and others.

1 The main result

In the present article we discuss the following minimization problem

$$\alpha(p,q,r)=\inf\left\{\frac{\|u'\|_p}{\|u\|_q}:u\in W^{1,p}_{per}(-1,1)\setminus\{0\}\,,\ \int_{-1}^{1}|u|^{r-2}u=0\right\}$$

where $p>1$, $q\ge r-1\ge 1$ and

$$\|u\|_q=\left(\int_{-1}^{1}|u|^q\right)^{1/q}$$
$$W^{1,p}_{per}(-1,1)=\left\{u:u\in W^{1,p}(-1,1)\ \text{and}\ u(-1)=u(1)\right\}.$$

We will denote by p' the conjugate exponent of p (i.e. $\frac{1}{p}+\frac{1}{p'}=1$) and the Beta function

$$B(p,q)=\frac{\Gamma(p)\,\Gamma(q)}{\Gamma(p+q)}=\int_0^1 t^{p-1}(1-t)^{q-1}\,dt\,.$$

Our main result will be

Theorem 1 *Let $p > 1$, $q \geq r - 1 \geq 1$ then*

$$\alpha(p,q,r) = \alpha(p,q,q) \text{ if } q \leq rp + r - 1$$
$$\alpha(p,q,r) < \alpha(p,q,q) \text{ if } q > (2r-1)p.$$

Furthermore

$$\alpha(p,q,q) = 2\left(\frac{1}{p'}\right)^{\frac{1}{q}}\left(\frac{1}{q}\right)^{\frac{1}{p'}}\left(\frac{2}{p'+q}\right)^{\frac{1}{p}-\frac{1}{q}} B\left(\frac{1}{p'},\frac{1}{q}\right).$$

The above formula is also valid when $q = r > 1$, $q = 1$ ($p > 1$ and $r = 2$) and $p = \infty$ ($q \geq r - 1 \geq 1$).

Remark 2 *(i) If the domain of integration is (a,b) instead of $(-1,1)$ the best constant becomes*

$$\alpha_{a,b}(p,q,r) = \left(\frac{2}{b-a}\right)^{\frac{1}{p'}+\frac{1}{q}} \alpha(p,q,r).$$

(ii) The case $p = q = 2$ is the classical Wirtinger inequality and the constant is then

$$\alpha(2,2,2) = \pi.$$

(iii) The nonlinear case has first been investigated by Dacorogna-Gangbo-Subía [3] where the cases $r = q$ and $r = 2$ were considered. They computed the actual value of $\alpha(p,q,q)$ and proved that when $q \leq 2p$ then $\alpha(p,q,2) = \alpha(p,q,q)$, while for $q >> 2p$ then strict inequality holds, showing in particular that

$$\alpha(p,\infty,2) = 2^{1/p}(p'+1)^{1/p'}.$$

The problem with $r = 2$ was then improved by many authors. Belloni-Kawohl [1] and Kawohl [6] proved that the range where equality holds can be extended to $2p + 1$. Buslaev-Kondratiev-Nazarov [2], refining a result from Egorov [4], showed that strict inequality holds as soon as $q > 3p$.

(iv) The importance of these best constants is, when $r = q$, to generalize an isoperimetric inequality known as Wulff theorem, cf. [3] (see also Lindquist-Peetre [8]). The case $r = 2$ is important in many different contexts, see for example [5], [7], [9] or [10].

(v) In [3] the cases $r = q$ and $r = 2$ were treated separately. One of the aims of the present article is, by the introduction of the parameter r, to unify these treatments and, at the same time, to generalize the known results.

(vi) We would like to conclude this introduction by calling the attention to some problems that we were not able to resolve. It is believed, and supported by some numerical evidences, that the equality between $\alpha(p,q,r)$ and $\alpha(p,q,q)$ breaks down at exactly $(2r-1)p$ (B. Kawohl informed us that A.I. Nazarov has recently shown that when $r = 2$ the equality does indeed hold when $q \leq 3p$). A related question is to know the actual value of $\alpha(p,q,r)$ when the equality breaks

down. Another problem is to know for which $r \in [2, q+1]$ $\min_{2\le r\le q+1}\{\alpha(p,q,r)\}$ *is attained. By the theorem we know that*

$$\max_{2\le r\le q+1}\{\alpha(p,q,r)\} = \alpha(p,q,q).$$

2 Proof of the main result

We proceed first with three lemmas and then with the proof of the theorem.

Lemma 3 *Let* $p > 1$, $q \ge r - 1 \ge 1$. *Let* $F, K : (0,1] \to \mathbb{R}$ *be defined by*

$$K(m) = 2\left(\frac{p'}{q}\right)^{\frac{1}{p'}}\left[\frac{q(p-1)+p}{2p(1-r(m))}\right]^{\frac{p'+q}{p'q}}\int_{-m}^{1}\left[1 - r(m) + r(m)|z|^{r-2}z - |z|^q\right]^{\frac{1}{p'}}dz$$

$$F(m) = \int_{-m}^{1}\frac{|z|^{r-2}z}{\left[1 - r(m) + r(m)|z|^{r-2}z - |z|^q\right]^{1/p}}dz$$

where

$$r(m) = \frac{1-m^q}{1+m^{r-1}}.$$

The following then holds

$$\alpha(p,q,r) = \inf\{K(m) : m \in (0,1] \ \text{ and } F(m) = 0\}.$$

Proof. The proof is similar in spirit to the one of [3] but it differs in many technical aspects.

Step 1 (Existence of minima). The minimum is easily seen to be attained. Moreover there exists a minimum u that satisfies also

$$u = u(x) = u(p,q,r,x) \in W_0^{1,p}(-1,1) \setminus \{0\}$$

(which means, in particular, that we can assume that $u(-1) = u(1) = 0$) so that

$$\alpha(p,q,r) = \frac{\|u'\|_p}{\|u\|_q} \text{ and } \int_{-1}^{1}|u|^{r-2}u = 0.$$

In this step we only need $p, r > 1$, $q \ge 1$.

Step 2 (Euler-Lagrange equation). The function u found in the preceding step satisfies

$$u,\ |u'|^{p-2}u' \in C^1([-1,1])$$

and there exists $\mu \in \mathbb{R}$ so that

$$p\left(|u'|^{p-2}u'\right)' + p\,\alpha^p\,\|u\|_q^{p-q}\,|u|^{q-2}u - \mu(r-1)|u|^{r-2} = 0. \qquad (1)$$

Before briefly explaining how this equation can be derived we want to point out that it can be shown, with simple arguments, that the Lagrange multiplier $\mu = 0$ when $q = r$. It is this fact that makes the whole analysis easier when $q = r$ and that allows also to treat the case $1 < q = r < 2$; however we do not discuss this case in details and we refer to [3].
Let u be a minimum and let $\varphi,\ \theta \in C_0^\infty(-1,1)$ with $(r-1)\int_{-1}^1 |u|^{r-2}\theta = 1$ and let $|\varepsilon|,\ |t| < 1$. Define then

$$\Phi(\varepsilon,t) = \int_{-1}^1 |u' + \varepsilon\varphi' + t\theta'|^p - \alpha^p \left[\int_{-1}^1 |u + \varepsilon\varphi + t\theta|^q\right]^{\frac{p}{q}},$$

$$\Psi(\varepsilon,t) = \int_{-1}^1 |u + \varepsilon\varphi + t\theta|^{r-2}[u + \varepsilon\varphi + t\theta].$$

It is easily seen that Φ is differentiable as well as Ψ if $r \geq 2$ and that $\Psi_t(0,0) = 1 \neq 0$ for any choice of θ as above. Therefore, applying the implicit function theorem to Ψ, we find that there exist $\varepsilon_0 << 1$ and a function $\tau \in C^1(-\varepsilon_0,\varepsilon_0)$, with $\tau(0) = 0$ such that $\Psi(\varepsilon,\tau(\varepsilon)) = 0$, $\forall\varepsilon \in (-\varepsilon_0,\varepsilon_0)$; in particular we deduce that $\tau'(0) = (1-r)\int_{-1}^1 |u|^{r-2}\varphi$. Since $\Phi(\varepsilon,\tau(\varepsilon))$ is minimum at $\varepsilon = 0$ we deduce that $\Phi_\varepsilon(0,0) + \Phi_t(0,0)\tau'(0) = 0$. This leads to the Euler-Lagrange equation in the weak form which holds for every $\varphi \in C_0^\infty(-1,1)$, namely

$$p\int_{-1}^1 |u'|^{p-2}u'\,\varphi' - \alpha^p\,p\,\|u\|_q^{p-q}\int_{-1}^1 |u|^{q-2}u\,\varphi + \mu(r-1)\int_{-1}^1 |u|^{r-2}\,\varphi = 0$$

where $\mu = \mu(\alpha,\theta,u) = -\Phi_t(0,0) \in \mathbb{R}$ is a constant. We then deduce that $|u'|^{p-2}u' \in C^1$ and that (1) holds. Moreover since the function $g(t) = |t|^{p-2}t$ has a continuous inverse we have that $u' = g^{-1}\left(|u'|^{p-2}u'\right)$ is continuous and hence $u \in C^1$.
Note, for further reference, that we also have

$$\left(|u'|^p\right)' = p'\left(|u'|^{p-2}u'\right)'u'. \tag{2}$$

This is obviously true if $u \in C^2$. In our context this follows from the fact that the functions $f(v) = |v|^{p'}$ and $v = |u'|^{p-2}u'$ are both C^1 which hence implies the claim, namely

$$\left(|u'|^p\right)' = \left(f\left(|u'|^{p-2}u'\right)\right)' = f'\left(|u'|^{p-2}u'\right)\left(|u'|^{p-2}u'\right)' = p'u'\left(|u'|^{p-2}u'\right)'.$$

Step 3 (Integrated Euler-Lagrange equation). Multiplying the preceding equation by u', using (2) and integrating we get (c being a constant)

$$(p-1)|u'(x)|^p + \frac{p}{q}\alpha^p\|u\|_q^{p-q}|u(x)|^q - \mu|u(x)|^{r-2}u(x) = c. \tag{3}$$

It is this version of the Euler-Lagrange equation that we will almost always use (but not exclusively).

We propose, although this is not necessary for the future developments, to derive directly (3) without using (1). The advantage of this direct derivation is that it is also valid if $1 < r < 2$; however it is not clear how to infer the required regularity of u from (3). We now sketch the proof of this fact. Consider the functional

$$G(v) = \|v'\|_p^p - \alpha^p \|v\|_q^p$$

where $v \in \mathcal{W}_r = \left\{ v \in W_0^{1,p}(-1,1) \text{ and } \int_{-1}^1 |v|^{r-2}\, v = 0 \right\}$. We know that it has a minimum at u. Consider for any $\varphi \in W_0^{1,\infty}(-1,1)$ such that $\int_{-1}^1 |u|^{r-2} u\, \varphi' = 0$ and for $|\varepsilon| \leq 1$ the function

$$w_\varepsilon(x) = x + \varepsilon \frac{\varphi(x)}{2\,\|\varphi'\|_\infty}.$$

Observe that $w_\varepsilon : [-1,1] \to [-1,1]$ is a homeomorphism. It is easy to see that if $v_\varepsilon\,(x) = u(w_\varepsilon^{-1}(x))$, then $v_\varepsilon \in \mathcal{W}_r$ and therefore $G(u) = 0 \leq G(v_\varepsilon)$, which in turn implies that

$$\left. \frac{d}{d\varepsilon} G(v_\varepsilon) \right|_{\varepsilon=0} = 0.$$

We then deduce that

$$(1-p) \int_{-1}^1 \frac{|u'(t)|^p \varphi'(t)}{2\,\|\varphi'\|_\infty} dt = \alpha^p \frac{p}{q} \|u\|_q^{p-q} \int_{-1}^1 \frac{|u(t)|^q \varphi'(t)}{2\,\|\varphi'\|_\infty} dt\,. \tag{4}$$

In order to have a more classical weak form of the integrated Euler-Lagrange equation, we need to remove the hypothesis that $\int_{-1}^1 |u|^{r-2} u\, \varphi' = 0$. To do this we let $\psi \in W_0^{1,\infty}(-1,1)$ be arbitrary and we choose

$$\varphi\,(x) = \psi(x) - \left[\int_{-1}^1 |u(t)|^{r-2} u(t) \psi'(t) dt \right] f(x),$$

where

$$f(x) = \frac{\int_{-1}^x |u(s)|^{r-2} u(s) ds}{\int_{-1}^1 |u(a)|^{2r-2} da}.$$

With this choice we obtain from (4) that

$$(1-p) \int_{-1}^1 |u'|^p \psi' + \sigma \int_{-1}^1 |u|^{r-2} u \psi' - \alpha^p \frac{p}{q} \|u\|_q^{p-q} \int_{-1}^1 |u|^q \psi' = 0$$

for an appropriate $\sigma = \sigma\,(\alpha, f, u) \in \mathbb{R}$ (which turns out to be identical to the μ in (3)) and for any $\psi \in W_0^{1,\infty}(-1,1)$. The integrated form (3) follows then immediately.

Step 4 (Value of μ, c and $\|u\|_q$ in terms of m). First observe that by rescaling u, we can assume that

$$\max_{x \in [-1,1]} \{u\,(x)\} = 1$$
$$\min_{x \in [-1,1]} \{u\,(x)\} = -m,\ m \in (0,1]\,.$$

Writing

$$r\,(m) = \frac{1 - m^q}{1 + m^{r-1}}, \tag{5}$$

we claim that

$$\mu = \frac{p}{q}\alpha^p \left\| u \right\|_q^{p-q} r\,(m)\,, \tag{6}$$

$$c = \alpha^p \frac{p}{q} \left\| u \right\|_q^{p-q} (1 - r(m)), \tag{7}$$

$$\left\| u \right\|_q = \left[\frac{2p(1 - r(m))}{q(p-1) + p} \right]^{\frac{1}{q}}. \tag{8}$$

We start by establishing (6) and (7). Writing the integrated Euler-Lagrange equation (3) for the point of maximum x_0, i.e. $u(x_0) = 1$ (and $u'(x_0) = 0$), and for the point of minimum x_1, i.e. $u(x_1) = -m$ (and $u'(x_1) = 0$), we get

$$\mu m^{r-1} + \alpha^p \frac{p}{q} \left\| u \right\|_q^{p-q} m^q = -\mu + \alpha^p \frac{p}{q} \left\| u \right\|_q^{p-q} = c. \tag{9}$$

The identities (6) and (7) follow then immediately. The integrated Euler-Lagrange equation becomes then

$$\begin{aligned} (p-1)|u'|^p + \alpha^p \tfrac{p}{q} \|u\|_q^{p-q} |u|^q - \alpha^p \tfrac{p}{q} \|u\|_q^{p-q} r(m) |u|^{r-2} u \\ = \alpha^p \tfrac{p}{q} \|u\|_q^{p-q} [1 - r(m)]. \end{aligned} \tag{10}$$

or equivalently

$$|u'| = \left[\frac{p'}{q} \alpha^p \left\| u \right\|_q^{p-q} \right]^{\frac{1}{p}} [1 - r(m) + r(m)|u|^{r-2} u - |u|^q]^{\frac{1}{p}}. \tag{11}$$

Integrating (10) over $(-1, 1)$ and recalling that $\left\| u' \right\|_p = \alpha \left\| u \right\|_q$ we get (8).

Step 5 (Qualitative properties of the solution). We now show that

$$u'\,(x) = 0 \Longleftrightarrow u\,(x) = 1 \text{ or } u\,(x) = -m. \tag{12}$$

Indeed the implication ($\Leftarrow$) is trivial. We next discuss the counter implication. Let

$$g\,(X) = 1 - r(m) + r(m)|X|^{r-2} X - |X|^q,\ X \in [-m, 1]$$

so that

$$|u'| = \left[\frac{p'}{q} \alpha^p \left\| u \right\|_q^{p-q} \right]^{\frac{1}{p}} [g\,(u)]^{\frac{1}{p}}.$$

Observe that $g\,(-m) = g\,(1) = 0$. Using the hypothesis $q \geq r - 1$ one easily shows that at points $\overline{X}$ where $g'\left(\overline{X}\right) = 0$ then $g\left(\overline{X}\right) > 0$. This shows that the function g never vanishes in $(-m, 1)$. This implies that $u'\,(x) \neq 0$ if $u\,(x) \neq 1$ and $u\,(x) \neq -m$, as claimed.

It can then be proved, exactly as in [3] and we omit the details, that u has only one zero $\alpha \in (-1, 1)$ (± 1 being, by Step 1, also zeroes of u) and u' has only two zeroes $\eta_1 = (\alpha - 1)/2$ and $\eta_2 = (\alpha + 1)/2$. Furthermore the function is symmetric in the following sense

$$u(x) = \begin{cases} u(2\eta_1 - x) & \text{if } x \in [-1, \alpha] \\ \\ u(2\eta_2 - x) & \text{if } x \in [\alpha, 1]. \end{cases}$$

As a consequence we obtain that, for every continuous function $f : \mathbb{R}^2 \to \mathbb{R}$,

$$\int_{-1}^{1} f(u(x), |u'(x)|)\, dx = 2\int_{\eta_1}^{\eta_2} f(u(x), |u'(x)|)\, dx. \tag{13}$$

Step 6 (The functions K and F). Let m be the minimal value of the solution u. We will then establish that

$$\alpha = K(m) \equiv$$
$$2\left(\frac{p'}{q}\right)^{\frac{1}{p'}} \left[\frac{q(p-1)+p}{2p(1-r(m))}\right]^{\frac{p'+q}{p'q}} \int_{-m}^{1} \left[1 - r(m) + r(m)|z|^{r-2} z - |z|^q\right]^{\frac{1}{p'}} dz \tag{14}$$

$$F(m) \equiv \int_{-m}^{1} \frac{|z|^{r-2} z}{\left[1 - r(m) + r(m)|z|^{r-2} z - |z|^q\right]^{1/p}} dz = 0. \tag{15}$$

We now briefly explain how to derive these identities. We start with (14). (Note that the derivation here is done in a slightly different manner than in [3]. There a function L was derived instead of the present function K below; they coincide at the minimal value). Using (10) we obtain

$$|u'|^p = \left[\frac{p'}{q}\alpha^p \|u\|_q^{p-q}\right]^{\frac{1}{p'}} [1 - r(m) + r(m)|u|^{r-2}u - |u|^q]^{\frac{1}{p'}} |u'|.$$

Let η_1, η_2 be the zeroes of u'. Recalling (13), integrating the above equation over (η_1, η_2) and performing the change of variable $z = u(x)$ in the right hand side of the equation we obtain

$$\|u'\|_p^p = 2\left(\frac{p'}{q}\alpha^p \|u\|_q^{p-q}\right)^{\frac{1}{p'}} \int_{-m}^{1} [1 - r(m) + r(m)|z|^{r-2}z - |z|^q]^{\frac{1}{p'}} dz$$

which combined with (8) and with $\|u'\|_p = \alpha \|u\|_q$ implies (14).
To obtain (15) we rewrite the condition $\int_{-1}^{1} |u|^{r-2} u = 0$ in the following manner. We first observe that (η_1, η_2 being the zeroes of u'), we have, appealing to (10) and (13),

$$\begin{aligned} 0 &= \int_{-1}^{1} |u|^{r-2}u = 2\int_{\eta_1}^{\eta_2} |u|^{r-2}u = 2\int_{\eta_1}^{\eta_2} \frac{|u|^{r-2}uu'}{u'} \\ &= 2\left[\frac{p'}{q}\alpha^p \|u\|_q^{p-q}\right]^{\frac{-1}{p}} \int_{\eta_1}^{\eta_2} \frac{|u|^{r-2}uu'}{\left[1 - r(m) + r(m)|u|^{r-2} u - |u|^q\right]^{1/p}}. \end{aligned}$$

Performing the change of variable $z = u(x)$ we get (15).

Step 7 (Equivalence of minima). Denote by

$$\beta = \inf \{K(m) : m \in (0,1] \text{ and } F(m) = 0\}.$$

The aim of this step is to show that $\alpha = \beta$ concluding thus the proof of the lemma. From the previous steps we know that $\beta \le \alpha$. We now wish to show the reverse inequality. Let $m \in (0,1]$ be such that $\beta = K(m)$ and $F(m) = 0$ (such an m exists by continuity of the functions F and K and by the fact that $F(0) \neq 0$). To conclude to the inequality $\alpha \le \beta$ it is enough to show that we can find $u \in W^{1,p}_{per}(-1,1)$ with $\int_{-1}^{1} |u|^{r-2} u = 0$ such that

$$K(m) = \frac{\|u'\|_p}{\|u\|_q}\ . \tag{16}$$

This u will be first constructed on $(-1,0)$ as follows. We claim that we can find $u \in W^{1,p}(-1,0)$ a solution of the problem

$$(E_m) \begin{cases} u' = \gamma h(u),\ x \in (-1,0) \\ u(-1) = -m,\ u(0) = 1 \\ \max u(x) = \max |u(x)| = 1 \end{cases}$$

where

$$h(s) = [1 - r(m) + r(m) |s|^{r-2} s - |s|^q]^{\frac{1}{p}}\ ,\ \gamma = \int_{-m}^{1} \frac{ds}{h(s)}.$$

Note, for further reference, that since $h(-m) = h(1) = 0$ then

$$u'(-1) = u'(0) = 0.$$

A solution of (E_m) is constructed as follows. Let $H : [-m,1] \to [-\gamma,0]$ be defined by $H(y) = \int_1^y \frac{dx}{h(x)}$. The solution of (E_m) is then given by

$$u(x) = H^{-1}(\gamma x).$$

Using the fact that $F(m) = 0$ we obtain that

$$\int_{-1}^{0} |u|^{r-2} u = \int_{-1}^{0} \frac{|u|^{r-2} u u'}{u'} = \frac{1}{\gamma} \int_{-1}^{0} \frac{|u|^{r-2} u u'}{h(u)}$$
$$= \frac{1}{\gamma} \int_{-m}^{1} \frac{|z|^{r-2} z}{h(z)} dz = \frac{1}{\gamma} F(m) = 0.$$

We then extend u to $(0,1)$ so as to be even. It is then clear that $u \in W^{1,p}_{per}(-1,1)$ and that $\int_{-1}^{1} |u|^{r-2} u = 0$.

It therefore remains to prove (16). From (E_m), the fact that $\int_{-1}^{1} |u|^{r-2} u = 0$ and the evenness of u we deduce that

$$\|u'\|_p^p = \gamma^p [2(1 - r(m)) - \|u\|_q^q]. \tag{17}$$

In a similar way we have from (E_m) that

$$(u')^p = \gamma^{p-1} \left[h(u)\right]^{p-1} u' \, ;$$

using the evenness of u, and after a change of variables we deduce that

$$\|u'\|_p^p = 2\gamma^{p-1} \int_{-m}^{1} [h(s)]^{p-1} \, .$$

Recalling the definition of $K(m)$ we obtain

$$\|u'\|_p^p = \gamma^{p-1} \left(\frac{q}{p'}\right)^{\frac{1}{p'}} \left[\frac{2p\,(1-r\,(m))}{q\,(p-1)+p}\right]^{\frac{p'+q}{p'q}} K\,(m) \, . \tag{18}$$

From (E_m) we also have

$$|u'|^p = \gamma^p [1 - r(m) + r\,(m)\,|u|^{r-2} u - |u|^q].$$

Differentiating this equation, using (2), we get after a simplification by u' that

$$p' \left(|u'|^{p-2} \, u'\right)' = \gamma^p \left[(r-1)\, r(m)|u|^{r-2} - q|u|^{q-2} u\right] .$$

Multiplying this equation by u, integrating, bearing in mind that $\int_{-1}^{1} |u|^{r-2} u = 0$, that u is even and that $u'(-1) = u'(1) = 0$, and using (17) we get

$$\|u'\|_p^p = \frac{q}{p'} \gamma^p \, \|u\|_q^q \text{ and } \|u\|_q^q = \frac{2p'\,(1-r\,(m))}{q+p'} = \frac{2p\,(1-r\,(m))}{q\,(p-1)+p}. \tag{19}$$

Combining (18) and (19) we find the claimed result

$$K(m) = \frac{\|u'\|_p}{\|u\|_q}.$$

■

We now study the functions F (cf. Lemma 4) and K (cf. Lemma 5).

Lemma 4 *Let $F : (0,1] \to \mathbb{R}$ be the function defined in the preceding lemma. The following properties then hold.*
(i) $F(1) = 0$, for every $p > 1$ and $q \geq r - 1 \geq 1$.
(ii) If $q \leq rp + r - 1$ then $F(m) \neq 0$ for every $m \in (0,1)$.
(iii) If $q > (2r-1)\,p$, then there exists $m \in (0,1)$ (i.e. $m \neq 1$) such that $F(m) = 0$. Moreover $F < 0$ for m close to 1 ($m < 1$).

Proof. The function $F \in C^1((0,1])$ and we can rewrite F in the following way

$$F(m) = \int_0^1 g_m(t)\, dt,$$

where

$$g_m(t) = \frac{t^{r-1}}{[1 - r(m) + r(m)t^{r-1} - t^q]^{\frac{1}{p}}} - \frac{t^{r-1}m^r}{[1 - r(m) - r(m)t^{r-1}m^{r-1} - m^q t^q]^{\frac{1}{p}}}.$$

Step 1. Note that since $r(m) = 0$, recalling that $r(m) = \frac{1-m^q}{1+m^{r-1}}$, whenever $m = 1$, we deduce that $g_1(t) \equiv 0$ and thus $F(1) = 0$.

Step 2. We will now prove that, when $q \le rp + r - 1$, then

$$g_m(t) \ge 0, \ \forall t \in [0, 1]$$

leading to the claim. Observe that $g_m(t) \ge 0$ if and only if

$$\begin{aligned} h_m(t) &\equiv 1 - r(m) - r(m)t^{r-1}m^{r-1} - m^q t^q - m^{rp}[1 - r(m) + r(m)t^{r-1} - t^q] \\ &= (1 - m^{rp})(1 - r(m)) - \left(m^{r-1} + m^{rp}\right) r(m)\, t^{r-1} - (m^q - m^{rp})\, t^q \ge 0\,. \end{aligned} \tag{20}$$

Note that

$$h'_m(t) = -(r-1)\left(m^{r-1} + m^{rp}\right) r(m)\, t^{r-2} - q\,(m^q - m^{rp})\, t^{q-1}. \tag{21}$$

To establish (20) we divide the proof into two cases.

Case 1: $q \le rp$. Observe that, in this case, since $0 < m \le 1$, then trivially $h'_m(t) \le 0$. On the other hand $h_m(0) \ge 0 = h_m(1)$, therefore (20) is proved.

Case 2: $rp < q \le rp + r - 1$. We will show that if there exists $\bar{t} \in [0, 1]$ with $h'_m(\bar{t}) = 0$ then necessarily $h_m(\bar{t}) \ge 0$. This fact coupled with the observation that $h_m(0) \ge 0 = h_m(1)$ shows (20). Note that $h'_m(\bar{t}) = 0$ if and only if

$$\bar{t}^{q-r+1} = \frac{r(m)(r-1)(m^{rp} + m^{r-1})}{q\,(m^{rp} - m^q)}.$$

We therefore have (assuming that $\bar{t} \le 1$, otherwise nothing is to be proved)

$$\begin{aligned} h_m(\bar{t}) &= [1 - r(m)](1 - m^{rp}) - \bar{t}^{r-1} r(m)(m^{r-1} + m^{rp})\frac{q-r+1}{q} \\ &\ge [1 - r(m)](1 - m^{rp}) - r(m)(m^{r-1} + m^{rp})\frac{q-r+1}{q} \\ &\ge \frac{1}{q}\frac{m^{r-1}}{1+m^{r-1}}\{q(m^{q-r+1} + 1)(1 - m^{rp}) - (q - r + 1)(1 - m^q)(1 + m^{rp+1-r})\}. \end{aligned}$$

To obtain the claim it is thus sufficient to show that, for every $m \in [0, 1]$,

$$G(m) \equiv q(m^{q-r+1} + 1)(1 - m^{rp}) - (q - r + 1)(1 - m^q)(1 + m^{rp+1-r}) \ge 0.$$

Observe first that $G(0) \ge G(1) = 0$. Define, for $\alpha \ge 0$,

$$H(\alpha, m) = q(m^{\alpha-r+1} + 1)(1 - m^{rp}) - (q - r + 1)(1 - m^\alpha)(1 + m^{rp+1-r}).$$

Note that $H(q, m) = G(m)$. Moreover if $\alpha \ge \beta \ge 0$, then $H(\beta, m) \ge H(\alpha, m)$. If we can show that $H(rp + r - 1, m) \ge 0$ we would obtain

$$G(m) = H(q, m) \ge H(rp + r - 1, m) \ge 0$$

as claimed. It therefore remains to show that, for every $m \in [0,1]$,

$$\begin{aligned}\widetilde{H}\,(m) &\equiv H(rp+r-1,m)\\ &= (r-1)\,(1-m^{2rp}) + (q-r+1)(m^{rp+r-1}-m^{rp+1-r}) \geq 0.\end{aligned}$$

This is proved by observing that $\widetilde{H}\,(0) = r-1 \geq \widetilde{H}\,(1) = 0$ and that $\widetilde{H}'\,(m) \leq 0$. To prove this last inequality we first observe that

$$\widetilde{H}'\,(m) = -m^{rp+r-2}\varphi\,(m)$$

where

$$\varphi\,(m) = 2rp\,(r-1)\,m^{rp-r+1} + (q-r+1)\,(rp-r+1)\,m^{-2r+2} - (q-r+1)\,(rp+r-1)\,.$$

We next see that $\varphi\,(0) = +\infty$ and from the hypothesis of Case 2

$$\varphi\,(1) = 2\,(r-1)\,(rp+r-1-q) \geq 0.$$

To conclude to $\varphi\,(m) \geq 0$ for every $m \in [0,1]$, we observe that at a point $\overline{m}$ where $\varphi'\,(\overline{m}) = 0$, one has $\varphi\,(\overline{m}) \geq 0$ and this concludes the proof.

Step 3. We will now prove that $F'(1) > 0$ if and only if $q > (2r-1)p$. Since $F(1) = 0$ and $F(0) > 0$ this will show, as wished, that there exists $m_0 \in (0,1)$ such that $F(m_0) = 0$ and that $F < 0$ for m close to 1 ($m < 1$). A direct computation shows that

$$F'(1) = \frac{q}{p}\int_0^1 \frac{t^{2r-2} - t^{r-1+q}}{(1-t^q)^{1+\frac{1}{p}}}dt - r\int_0^1 \frac{t^{r-1}}{(1-t^q)^{\frac{1}{p}}}dt\,.$$

Changing the variable $s = t^q$ we get

$$\begin{aligned}F'(1) &= \frac{1}{p}\int_0^1 \frac{s^{\frac{2r-1}{q}-1} - s^{\frac{r}{q}}}{(1-s)^{1+\frac{1}{p}}}ds - \frac{r}{q}\int_0^1 \frac{s^{\frac{r}{q}-1}}{(1-s)^{\frac{1}{p}}}ds\\ &= \frac{1}{p}\int_0^1 \frac{s^{\frac{2r-1}{q}-1} - s^{\frac{2r-1}{q}}}{(1-s)^{1+\frac{1}{p}}}ds + \frac{1}{p}\int_0^1 \frac{s^{\frac{2r-1}{q}} - s^{\frac{r}{q}}}{(1-s)^{1+\frac{1}{p}}}ds - \frac{r}{q}\int_0^1 \frac{s^{\frac{r}{q}-1}}{(1-s)^{\frac{1}{p}}}ds\,.\end{aligned}$$

Note that the first expression is readily given as

$$\frac{1}{p}\int_0^1 \frac{s^{\frac{2r-1}{q}-1} - s^{\frac{2r-1}{q}}}{(1-s)^{1+\frac{1}{p}}}ds = \frac{1}{p}\int_0^1 \frac{s^{\frac{2r-1}{q}-1}}{(1-s)^{\frac{1}{p}}}ds = \frac{1}{p}B\left(\frac{2r-1}{q},\frac{1}{p'}\right).$$

Integrating by parts the second term in $F'\,(1)$ and applying L'Hôpital's rule we obtain

$$\begin{aligned}\frac{1}{p}\int_0^1 \frac{s^{\frac{2r-1}{q}} - s^{\frac{r}{q}}}{(1-s)^{1+\frac{1}{p}}}ds &= \left[\frac{s^{\frac{2r-1}{q}} - s^{\frac{r}{q}}}{(1-s)^{\frac{1}{p}}}\right]\Bigg|_0^1 - \int_0^1 \frac{\frac{2r-1}{q}s^{\frac{2r-1}{q}-1} - \frac{r}{q}s^{\frac{r}{q}-1}}{(1-s)^{\frac{1}{p}}}ds\\ &= -\frac{2r-1}{q}\int_0^1 \frac{s^{\frac{2r-1}{q}-1}}{(1-s)^{\frac{1}{p}}}ds + \frac{r}{q}\int_0^1 \frac{s^{\frac{r}{q}-1}}{(1-s)^{\frac{1}{p}}}ds\\ &= -\frac{2r-1}{q}B\left(\frac{2r-1}{q},\frac{1}{p'}\right) + \frac{r}{q}\int_0^1 \frac{s^{\frac{r}{q}-1}}{(1-s)^{\frac{1}{p}}}ds\,.\end{aligned}$$

Combining these results we have

$$F'(1)=\left(\frac{1}{p}-\frac{2r-1}{q}\right)B\left(\frac{2r-1}{q},\frac{1}{p'}\right)$$

which leads to the assertion. ■

Lemma 5 *Let $q>(2r-1)\,p$, then there exists $m_0\in(0,1)$ (i.e. $m_0<1$) such that* $\inf\{K(m):m\in(0,1]$ *and* $F(m)=0\}=K(m_0)<K(1)$*.*

Proof. *Step 1.* As already mentioned it is easy to see that the minimum is attained and we therefore wish to show that $m_0<1$ and $K(m_0)<K(1)$. To this aim we first observe that $K\in C^1((0,1])$ and that $\lim\limits_{m\to 0}K(m)=+\infty$. We will then prove, in the next step, that there exists a constant $c(p,q)>0$ such that

$$K'(m)=\frac{c(p,q)}{p'}r'(m)\,(1-r(m))^{\frac{1}{p}-\frac{1}{q}-2}\left(1-\frac{r-1}{q}r(m)\right)F(m)\;.$$

Recall that $r'(m)<0$ for every $m\in(0,1]$. Since $F<0$ for m close to 1 ($m<1$) (by Lemma 4), we deduce that $m=1$ is a local maximum of K in $(0,1]$. Therefore the global minimum of K in $(0,1]$ is at a point $m_0\in(0,1)$ where $F(m_0)=0$. This is the claimed result.

Step 2. We now compute $K'(m)$. Recall that

$$\begin{aligned}K(m)&=c(p,q)\,(1-r(m))^{-\frac{1}{p'}-\frac{1}{q}}\int_{-m}^{1}\left[1-r(m)+r(m)\,|z|^{r-2}z-|z|^q\right]^{\frac{1}{p'}}dz\\&=c(p,q)\,(1-r(m))^{-\frac{1}{q}}\\&\int_{-m}^{1}\left[1+r(m)\,(1-r(m))^{-1}|z|^{r-2}z-(1-r(m))^{-1}|z|^q\right]^{\frac{1}{p'}}dz\end{aligned}$$

where

$$c(p,q)=2\left(\frac{p'}{q}\right)^{\frac{1}{p'}}\left[\frac{q(p-1)+p}{2p}\right]^{\frac{p'+q}{p'q}}\;.$$

Writing $z=(1-r(m))^{\frac{1}{q}}t$, $\alpha(m)=-m\,(1-r(m))^{-\frac{1}{q}}$, $\beta(m)=(1-r(m))^{-\frac{1}{q}}$ and $\gamma(m)=r(m)\,(1-r(m))^{-1+\frac{r-1}{q}}$ we obtain

$$K(m)=c(p,q)\int_{\alpha(m)}^{\beta(m)}\left[1+\gamma(m)\,|t|^{r-2}t-|t|^q\right]^{\frac{1}{p'}}dt\,.$$

Noting that

$$1+\gamma(m)\,|\beta(m)|^{r-1}-|\beta(m)|^q=1-\gamma(m)\,|\alpha(m)|^{r-1}-|\alpha(m)|^q=0$$

we obtain

$$K'(m)=\frac{c(p,q)}{p'}\gamma'(m)\int_{\alpha(m)}^{\beta(m)}\frac{|t|^{r-2}t}{\left[1+\gamma(m)\,|t|^{r-2}t-|t|^q\right]^{\frac{1}{p}}}dt\,.$$

Performing backward the change of variable $t = (1 - r(m))^{-\frac{1}{q}} z$ we get

$$\begin{aligned}
K'(m) &= \frac{c(p,q)}{p'} \gamma'(m) (1 - r(m))^{-\frac{r}{q}} \\
&\int_{-m}^{1} \frac{|z|^{r-2} z \, dz}{\left[1 + r(m)(1 - r(m))^{-1} |z|^{r-2} z - (1 - r(m))^{-1} |z|^q\right]^{\frac{1}{p}}} \\
&= \frac{c(p,q)}{p'} \gamma'(m) (1 - r(m))^{\frac{1}{p} - \frac{r}{q}} F(m) \\
&= \frac{c(p,q)}{p'} r'(m) (1 - r(m))^{\frac{1}{p} - \frac{1}{q} - 2} \left(1 - \frac{r-1}{q} r(m)\right) F(m)
\end{aligned}$$

as wished. ■

We are now in a position to conclude the proof of the main theorem.

Proof. (Theorem 1). *Step 1.* If $q \leq rp + r - 1$, then, since $F(m) = 0$ if and only if $m = 1$, we deduce that

$$\alpha(p,q,r) = \alpha(p,q,q) = K(1) = 2 \left(\frac{p'}{q}\right)^{\frac{1}{p'}} \left[\frac{q(p-1) + p}{2p}\right]^{\frac{p'+q}{p'q}} \int_{-1}^{1} [1 - |z|^q]^{\frac{1}{p'}} dz$$

which easily leads to the value given in the theorem.
If $q > (2r - 1)p$, we have, as claimed, that

$$\alpha(p,q,q) = K(1) > \inf\{K(m) : \ m \in (0,1], \ F(m) = 0\} = \alpha(p,q,r).$$

It remains to discuss the limit cases.

Step 2. The case $q = r \geq 2$ is part of the previous analysis. If, however $1 < q = r < 2$, then the result still holds and we refer to [3] for more details.

Step 3. We now discuss the value of $\alpha(p,1,2)$. Let

$$\mathcal{W}_2 = \left\{ v \in W^{1,p}_{per}(-1,1), \ \int_{-1}^{1} v = 0 \right\}.$$

We have just seen that for every $q > 1$ then, using also Hölder inequality,

$$\|v'\|_p \geq \alpha(p,q,2) \|v\|_q \geq \alpha(p,q,2) 2^{-\frac{1}{q'}} \|v\|_1 \ , \ \forall v \in \mathcal{W}_2$$

and hence denoting by

$$\bar{\alpha} = \lim_{q \to 1} \alpha(p,q,2) = 2^{\frac{1}{p}} \frac{(p'+1)^{\frac{1}{p'}}}{p'} B\left(\frac{1}{p'}, 1\right) = 2^{\frac{1}{p}} (p'+1)^{\frac{1}{p'}}$$

we get

$$\|v'\|_p \geq \bar{\alpha} \|v\|_1 \ , \ \forall v \in \mathcal{W}_2.$$

We therefore have just obtained that $\alpha(p,1,2) \geq \overline{\alpha}$. We now prove the reverse inequality. Let for $q > 1$, $u_q \in \mathcal{W}_2$ be a minimizer, i.e.

$$\frac{\|u_q'\|_p}{\|u_q\|_q} = \alpha(p,q,2).$$

Recall that with our conventions $-1 \leq -m \leq u_q(x) \leq 1$ and hence $\|u_q\|_\infty \leq 1$. From the integrated Euler-Lagrange equation (11) we then get $\|u_q'\|_\infty \leq C(p)$. Therefore there exists $\bar{u}$ and a subsequence, still denoted by u_q, such that $u_q \overset{*}{\rightharpoonup} \bar{u}$ in $W^{1,\infty}$ and $u_q \to \bar{u}$ in L^∞. This implies by weak lower semicontinuity

$$\|\overline{u}'\|_p \leq \liminf_{q\to 1} \|u_q'\|_p .$$

Moreover

$$\begin{aligned}\left| \|u_q\|_q - \|\overline{u}\|_1 \right| &\leq \|u_q - \overline{u}\|_q + \left| \|\overline{u}\|_q - \|\overline{u}\|_1 \right| \\ &\leq 2^{\frac{1}{q}} \|u_q - \overline{u}\|_\infty + \left| \|\overline{u}\|_q - \|\overline{u}\|_1 \right|\end{aligned}$$

and, since $\lim_{q\to 1} \|\overline{u}\|_q = \|\overline{u}\|_1$, we get

$$\lim_{q\to 1} \|u_q\|_q = \|\overline{u}\|_1 .$$

Combining these facts we have the claim, namely

$$\alpha(p,1,2) \leq \frac{\|\bar{u}'\|_p}{\|\bar{u}\|_1} \leq \liminf_{q\to 1} \frac{\|u_q'\|_p}{\|u_q\|_q} = \lim_{q\to 1} \alpha(p,q,2) = \bar{\alpha}.$$

Step 4. We now compute $\alpha(\infty,q,r)$. We let

$$\mathcal{W}_r = \left\{ v \in W^{1,p}_{per}(-1,1),\ \int_{-1}^{1} |v|^{r-2} v = 0 \right\}.$$

As above we have

$$2^{\frac{1}{p}} \|v'\|_\infty \geq \|v'\|_p \geq \alpha(p,q,r) \|v\|_q \ , \ \forall v \in \mathcal{W}_r$$

and hence, if we denote by

$$\bar{\alpha} = \lim_{p\to\infty} \alpha(p,q,r) = \lim_{p\to\infty} \alpha(p,q,q) = \frac{2}{q} \left(\frac{q+1}{2} \right)^{\frac{1}{q}} B\left(1, \frac{1}{q}\right) = 2^{\frac{1}{q'}} (q+1)^{\frac{1}{q}} ,$$

we obtain

$$\|v'\|_\infty \geq \bar{\alpha} \|v\|_q \ , \ \forall v \in \mathcal{W}_r.$$

We therefore proved that $\alpha(\infty,q,r) \geq \overline{\alpha}$. Let us now show the reverse inequality. For $p > 1$, let $u_p \in \mathcal{W}_r$ be a minimizer, i.e.

$$\frac{\|u_p'\|_p}{\|u_p\|_q} = \alpha(p,q,r) .$$

Recall that since we assumed $-1 \leq -m \leq u_p(x) \leq 1$ we have $\|u_p\|_\infty \leq 1$. Choosing p sufficiently large so that $rp + r - 1 \geq q$, we can rewrite (11) as (recalling that we are then in the case where $m = 1$ and thus $r(m) = 0$)

$$|u_p'| = \alpha(p,q,r) \|u_p\|_q \left(\frac{p'}{q} \right)^{\frac{1}{p}} \|u_p\|_q^{-\frac{q}{p}} [1 - |u_p|^q]^{\frac{1}{p}},$$

which implies

$$\frac{\|u_p'\|_\infty}{\|u_p\|_q} = \alpha(p,q,r) \left(\frac{p'}{q}\right)^{\frac{1}{p}} \|u_p\|_q^{-\frac{q}{p}} \ .$$

Note that by (8) we have

$$\|u_p\|_q \underset{p\to\infty}{\longrightarrow} \left(\frac{2}{q+1}\right)^{\frac{1}{q}} \ .$$

By definition of $\alpha(\infty,q,r)$ we therefore get

$$\alpha(\infty,q,r) \le \alpha(p,q,r) \left(\frac{p'}{q}\right)^{\frac{1}{p}} \|u_p\|_q^{-\frac{q}{p}} \underset{p\to\infty}{\longrightarrow} \bar{\alpha}$$

which is the desired inequality. ■

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