Exact Solution of the Gauge Symmetric p-Spin Glass Model on a Complete Graph

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We consider a gauge symmetric version of the p-spin glass model on a complete graph. The gauge symmetry guarantees the absence of replica symmetry breaking and allows to fully use the interpolation scheme of Guerra [4] to rigorously compute the free energy. In the case of pairwise interactions ($p = 2$), where we have a gauge symmetric version of the Sherrington-Kirkpatrick model, we get the free energy and magnetization for all values of external parameters. Our analysis also works for even $p \geq 4$ except in a range of parameters surrounding the phase transition line, and for odd $p \geq 3$ in a more restricted region. We also obtain concentration estimates for the magnetization and overlap parameter that play a crucial role in the proofs for odd p and justify the absence of replica symmetry breaking. Our initial motivation for considering this model came from problems related to communication over a noisy channel, and is briefly explained.

1 Introduction and main results

1.1 Motivation

During the last decade substantial mathematical progress has been accomplished towards solutions of mean field spin-glass models (see [18] and references therein). These fall in two main categories: models on sparse graphs of Erdös-Renyi type and models on complete graphs. The general Hamiltonians on complete graphs (or more precisely hypergraphs) are of the form

$$H(\underline{s}) = -\sum_{1=i_1<\ldots<i_p}^{N} J_{i_1\ldots i_p} s_{i_1} s_{i_2} \ldots s_{i_p} - \sum_{i=1}^{N} h_i s_i \tag{1}$$

The standard p-spin model ($p \geq 2$) introduced by Derrida [1], Gross and Mézard [3] has random i.i.d. coupling constants $J_{i_1,\ldots,i_p} \sim \mathcal{N}(0, \frac{Jp!}{2N^{p-1}})$ and $h_i \sim \mathcal{N}(0, h)$. The variance is normalized by N^{p-1} to yield a non trivial free energy in the thermodynamic limit, while the $p!/2$ is important if one wants to take the $p \to +\infty$ limit where it reduces to the Random Energy Model [1]. The special case $p = 2$ is the Sherrington-Kirkpatrick model [16] for which the Parisi formula [14] for the free energy has been proven (for the whole parameter range) in a remarkable series of papers that developed the interpolation methods in various directions [4, 5, 6, 20]. These results have been extended also to even p, $p \geq 4$ [19].

Here we study a gauge symmetric version of model (1) where the random couplings

$$J_{i_1,\dots,i_p} \sim \mathcal{N}\left(\frac{Jp!}{2N^{p-1}}, \frac{Jp!}{2N^{p-1}}\right), \qquad h_i \sim \mathcal{N}(h,h). \tag{2}$$

have equal mean and variance. The average free energy is defined at inverse temperature $\beta = 1$

$$f_N = -\frac{1}{N}\mathbb{E}[\ln Z_N], \qquad Z_N = \sum_{\underline{s}} e^{-H(\underline{s})}$$

where $\mathbb{E}[-]$ is the expectation with respect to (2). In this setting the local transformations

$$s_i \to \tau_i s_i, \qquad h_i \to \tau_i h_i, \qquad J_{i_1 \dots i_p} \to \tau_{i_1} \dots \tau_{i_p} J_{i_1 \dots i_p} \tag{3}$$

where $\tau_i = \pm 1$, are a gauge transformation first studied by Nishimori [13]. This symmetry holds only for $\beta = 1$ which is referred to as the Nishimori line of the phase diagram (β, J, h). Along this line one does not expect any replica symmetry breaking to occur.

We show that, as a consequence of the gauge symmetry, the simplest version of the interpolation method [4], when suitably applied, suffices to compute rigorously the average free energy in the limit $N \to +\infty$. Our results confirm that the replica symmetric solution is indeed exact on the Nishimori line of the phase diagram (the full replica solution for $\beta \neq 1$ can be found in [13]). Our analysis applies to both even and odd p. The latter is more complicated and requires concentration results of the Edwards-Anderson overlap parameter, which seem to be new. Proofs of concentration of the free energy for the standard model [7] can be adapted to the present case and will therefore be omitted here. The appropriately defined limit $p \to \infty$ for the model results in a variant of the Random Energy Model and has been studied in [2] (for $h = 0$ but any β) and will therefore not be discussed further here.

Our initial motivation for studying the present model comes from problems in communication through noisy channels. Loosely speaking, Shannon's theorem assures that for transmission rates below the channel capacity there exist error correcting codes allowing error free communication. In fact as first shown by Sourlas [17] error correcting codes can be viewed as spin glass models where, the spins correspond to transmitted bits, the couplings are determined by the received values, and the geometry of the underlying graph is fixed by the error correcting code. The couplings are quenched random variables (due to channel noise), and the geometry of the underlying graph is defined by the random code drawn from an ensemble (following Shannon). Remarkably, it turns out that for a large class of relevant channels the spin glass models have a gauge symmetry[1] of the type (3). Because the Hamiltonian (1) is defined on a complete hypergraph it does not represent a sensible code in the thermodynamic limit, but does so only for N (large) finite, because

[1]In fact this depends on the decoder that is used. It is true for optimal bit-decoding, but not for optimal block-decoding which corresponds to $\beta = +\infty$.

the rate of transmission scales as $\frac{p!}{N^{p-1}}$ (this code is sometimes referred to as Sourlas code in the literature). However models of dilute spin glasses on random Erdös-Renyi type hypergraphs do represent sensible codes which have positive transmission rates even in the thermodynamic limit. These are the so-called Low Density Parity Check and/or Low Density Generator Matrix codes that have attracted a lot of attention in communication theory in recent times due to their excellent properties (see [15] for the state of the art, history and references). The analysis developed in the present work is useful in that (more complicated) context also where bounds on the capacity (and/or free energy) have been derived [12, 11, 8, 9] but a general solution is still missing.

A summary of the present results has appeared in [10].

1.2 Main results

The formal replica trick applied to the present model leads to the expression $\min_{m\in[0,1]} f_{RS}(m)$ for the infinite volume free energy, with a "replica symmetric" variational free energy

$$f_{RS}(m) = -\frac{J}{4}(1 - pm^{p-1} - (p-1)m^p) - \int_{-\infty}^{+\infty} Dz\, \ln\big(2\cosh(z\sqrt{v} + v)\big) \tag{4}$$

where

$$Dz = dz\,\frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}}, \qquad v = \frac{J}{2}pm^{p-1} + h.$$

Our first result is an upper bound on the free energy.

Theorem 1. *For Lebesgue almost every $h \geq 0$ we have*

$$\limsup_{N\to\infty} f_N \leq \min_{m\in[0,1]} f_{RS}(m).$$

For even p the inequality is true for all $h \geq 0$.

The proof (see Section 3) proceeds by an interpolation between the true and a mean field Hamiltonian which preserves the gauge symmetry. For odd p the interpolation argument is not quite sufficient and one has to combine it with a self-averaging result for the magnetization or the overlap parameter

$$m_1 = \frac{1}{N}\sum_{i=1}^{N} s_i, \qquad q_{12} = \frac{1}{N}\sum_{i=1}^{N} s_i^{(1)} s_i^{(2)}.$$

We can prove various forms of self averaging for these quantities, namely that $\mathbb{E}[\langle|A - \langle A\rangle|\rangle]$, $\mathbb{E}[|\langle A\rangle - \mathbb{E}[\langle A\rangle]|]$ and $\mathbb{E}[\langle|A - \mathbb{E}[\langle A\rangle]|\rangle]$ all tend to zero as $N \to +\infty$ where $A = m_1$ or q_{12} (of course if two of these quantities tend to zero then the third one also tends to zero). This issue is discussed in detail in Section 6.

The next result is a converse bound. Let $\widehat{m}$ be a minimizer of (4) and define the function

$$\widetilde{f}(m) = \frac{J}{2}(p-1)m^p - \int_{-\infty}^{+\infty} Dz\, \ln(2\cosh(z\sqrt{\widehat{v}} + v)) \tag{5}$$

where (note the difference between the integral terms in (5) and (4))

$$\widehat{v} = \frac{J}{2}p\widehat{m}^{p-1} + h.$$

Theorem 2. *For p an even integer and all $(J,h) \in \mathbb{R}_+^2$ we have*

$$\liminf_{N\to+\infty} f_N \geq -\frac{J}{4}(1 - p\widehat{m}^{p-1} + (p-1)\widehat{m}^p) + \min_{m\in[0,1]} \widetilde{f}(m).$$

For p odd this inequality is satisfied for all $(J,h) \in C_{+,p}$ where

$$C_{+,p} = \{(J,h) \in \mathbb{R}_+^2 \mid (p-1)\widehat{m}^p + p\widehat{m}^{p-1} - 1 \geq 0\}. \tag{6}$$

Note that for even p, $C_{+,p} = \mathbb{R}_+^2$. The proof (see Section 4) proceeds by a naive interpolation which does not preserve the gauge symmetry. Theorems 1 and 2 have an immediate corollary which forms our main result. Let $\widetilde{m}$ denote the minimizer of $\widetilde{f}(m)$ and set

$$C_p = \{(J,h) \in \mathbb{R}_+^2 \mid \widehat{m} = \widetilde{m}\} \cap C_{+,p}.$$

In Appendix A, we show that for pairwise interactions C_2 is equal to the whole two dimensional quadrant. However, for $p \geq 3$, C_p is not equal to the whole plane. For even $p \geq 4$ the region C_p does not include some parameter close to the phase transition. For odd $p \geq 3$ the region is even smaller due to the restriction (6). A graphical illustration for $p = 4$ is shown in Figure 1 in Appendix A.

Theorem 3. *For $(J,h) \in C_p$ the free energy is given by*

$$\lim_{N\to+\infty} f_N = \min_{m\in[0,1]} f_{RS}(m).$$

The minimizer $\widehat{m}$ equals 0 or is one of the fixed points of

$$\widehat{m} = \int_{-\infty}^{+\infty} Dz\ \tanh(z\sqrt{\widehat{v}} + \widehat{v}), \qquad \widehat{v} = \frac{p}{2}J\widehat{m}^{p-1} + h. \tag{7}$$

1.3 Notation and organization of the paper

The interpolating Hamiltonian that will be introduced in Section 3 depends on a parameter $t \in [0,1]$ and is denoted $H_t(\underline{s})$. The corresponding partition function and free energy are

$$Z_N(t) = \sum_{\underline{s}} e^{-H_t(\underline{s})}, \qquad f_N(t) = -\frac{1}{N}\mathbb{E}\big[\log Z_N(t)\big] \tag{8}$$

where $\mathbb{E}[-]$ is the expectation with respect to all quenched couplings involved in the interpolation. We will use the interpolating Gibbs brackets

$$\langle a(\underline{s})\rangle_t = \frac{1}{Z_N(t)} \sum_{\underline{s}} a(\underline{s}) e^{-H_t(\underline{x})}$$

and

$$\langle a(\underline{s}^{(1)}, \underline{s}^{(2)})\rangle_t = \frac{1}{Z_N(t)^2} \sum_{\underline{s}^{(1)},\underline{s}^{(2)}} a(\underline{s}^{(1)}, \underline{s}^{(2)}) e^{-(H_t(\underline{s}^{(1)}) + H_t(\underline{s}^{(2)}))}.$$

The replica indices will be omitted in Gibbs brackets $\langle - \rangle_t$ themselves, but always appear as a superscript, $\underline{s}^{(\alpha)}$, $\alpha = 1,2$, in the spin variables, so that there in no confusion (we will never need more than two replicas). In Section 4 we use another interpolating Hamiltonian $\widehat{H}_t(\underline{s})$ with the same corresponding definitions for $\widehat{Z}_N(t)$, $\widehat{f}_N(t)$ and $\langle - \rangle_t$. We define the polynomial

$$R_p(a,b) = (p-1)a^p - pa^{p-1}b + b^p = (b^p - a^p) - pa^{p-1}(b-a)$$

which plays an important role. We will make use of the following important property: convexity of x^p for even p implies $R_p(a,b) \geq 0$ if p is even. Also, convexity of x^p for all p if $x \geq 0$ implies $R_p(a,b) \geq 0$ for all p if a and b are non-negative.

The proof of the main theorem is given in the next section together with a few useful identities. We prove Theorem 1 and Theorem 2 in Section 3 and Section 4 respectively. In order to prove Theorem 2 we solve an intermediate model by saddle point calculations in Section 5. Section 6 is devoted to various extra results on the self-averaging of the magnetization and overlap parameters. The appendices contain more technical details.

2 Preliminary calculations

In this section we gather a few useful facts and in the process prove Theorem 3.

Minimizers of $f_{RS}(m)$ and $\widetilde{f}(m)$. Differentiating (4) with respect to m we obtain

$$p(p-1)\frac{J}{4}m^{p-2}\Big(1 + m - 2\int_{-\infty}^{+\infty} Dz\,\big(\frac{z}{2\sqrt{v}} + 1\big)\tanh(z\sqrt{v} + v)\Big).$$

Since $zDz = -\frac{\partial}{\partial z}Dz$, the term involving $\frac{z}{2\sqrt{v}}$ can be integrated by parts. One then finds

$$\frac{J}{4}p(p-1)m^{p-2}\Big(m + \int_{-\infty}^{+\infty} Dz\,\big(\tanh^2(z\sqrt{v} + v) - 2\tanh(z\sqrt{v} + v)\big)\Big).$$

One can prove the remarkable identity

$$\int_{-\infty}^{+\infty} Dz\,\tanh^2(z\sqrt{v} + v) = \int_{-\infty}^{+\infty} Dz\,\tanh(z\sqrt{v} + v). \tag{9}$$

Instead of giving a direct proof we give an indirect one below which shows that it is a special case of a larger set of Nishimori identities (11) related to gauge symmetry. Thus

$$\frac{\partial}{\partial m}f_{RS}(m) = \frac{J}{4}p(p-1)m^{p-2}\Big(m - \int_{-\infty}^{+\infty} Dz\,\tanh(z\sqrt{v} + v)\Big) \tag{10}$$

and therefore $\widehat{m} = 0$ or it satisfies (7) (for $p = 2$ the first possibility is excluded except possibly when $h = 0$). Differentiating $\widetilde{f}(m)$ we find

$$\frac{\partial}{\partial m}\widetilde{f}(m) = \frac{J}{2}p(p-1)m^{p-2}\Big(m - \int_{-\infty}^{+\infty} Dz\,\tanh(z\sqrt{\widehat{v}} + v)\Big).$$

We therefore conclude that, since $\widehat{m}$ is a minimizer of $f_{RS}(m)$ (by definition), it must necessarily be a critical point of $\widetilde{f}(m)$.

Nishimori identities. The gauge symmetry of the model implies a set of remarkable identities, called Nishimori identities. In this work we will use special cases of the formula (see Appendix B)

$$\mathbb{E}\Big[\prod_A \langle s_A\rangle\Big] = \mathbb{E}\Big[\langle \prod_A \tau_A\rangle \prod_A \langle s_A\rangle\Big] \tag{11}$$

where A denotes a set of spins and $s_A = \prod_{i\in A} s_i$. The simplest of these is $\mathbb{E}[\langle s_i\rangle] = \mathbb{E}[\langle s_i\rangle^2]$. Note that in the special case of non-interacting spins, $J = 0$, this becomes precisely (9). Below we make use of the four special cases

$$\begin{aligned}
&\mathbb{E}[\langle s_i s_j\rangle] = \mathbb{E}[\langle s_i s_j\rangle^2], \qquad (\text{choose } A = \{i,j\})\\
&\mathbb{E}[\langle s_i\rangle\langle s_j\rangle] = \mathbb{E}[\langle s_i s_j\rangle\langle s_i\rangle\langle s_j\rangle], \qquad (\text{choose } A = \{i\};\{j\})\\
&\mathbb{E}[\langle s_i s_j\rangle\langle s_j\rangle] = \mathbb{E}[\langle s_i s_j\rangle\langle s_i\rangle\langle s_j\rangle], \qquad (\text{choose } A = \{i,j\};\{j\})\\
&\mathbb{E}[\langle s_i\rangle\langle s_j\rangle^2] = \mathbb{E}[\langle s_i\rangle^2\langle s_j\rangle^2] \qquad (\text{choose } A = \{i\};\{j\};\{j\}).
\end{aligned}$$

Another consequence is

$$\mathbb{E}[\langle m_1^k\rangle] = \mathbb{E}[\langle q_{12}^k\rangle], \qquad k\in\mathbb{N}. \tag{12}$$

This is easily seen by expanding both moments and applying $\mathbb{E}[\langle s_A\rangle] = \mathbb{E}[\langle s_A\rangle^2] = \mathbb{E}[\langle s_A^{(1)} s_A^{(2)}\rangle]$. Thus the magnetization and the overlap parameter both have the same induced distribution under $\mathbb{E}[\langle -\rangle]$.

Magnetization and susceptibility. We show that the derivatives of the free energy with respect to h (which is the mean and variance of the random magnetic field) have simple expressions in terms of magnetization and correlation function.

$$\frac{\partial}{\partial h} f_N = -\frac{1}{2}(1+\mathbb{E}[\langle m_1\rangle]), \qquad \frac{\partial^2}{\partial h^2} f_N = -\frac{1}{2N}\sum_{i,j=1}^N \mathbb{E}[(\langle s_i s_j\rangle_t - \langle s_i\rangle\langle s_j\rangle)^2]. \tag{13}$$

To prove these formulas we use the identity

$$\frac{\partial}{\partial h}\frac{e^{-\frac{(h_i-h)^2}{2h}}}{\sqrt{2\pi h}} = \Big(-\frac{\partial}{\partial h_i} + \frac{1}{2}\frac{\partial^2}{\partial h_i^2}\Big)\frac{e^{-\frac{(h_i-h)^2}{2h}}}{\sqrt{2\pi h}}.$$

Then integrating by parts one finds contributions $\mathbb{E}[\langle s_i\rangle]$ coming from $\frac{\partial}{\partial h_i}$ and $\mathbb{E}[1-\langle s_i\rangle^2]$ from $\frac{\partial^2}{\partial h_i^2}$,

$$\frac{\partial}{\partial h} f_N = -\frac{1}{N}\sum_{i=1}^N\Big(\mathbb{E}[\langle s_i\rangle] + \frac{1}{2}\mathbb{E}[1-\langle s_i\rangle^2]\Big) = -\frac{1}{2N}\sum_{i=1}^N\Big(1+\mathbb{E}[\langle s_i\rangle]\Big). \tag{14}$$

The second equality is a consequence of the Nishimori identity (see after (11)). This proves the first equality in (13). Proceeding similarly one more time

$$\begin{aligned}\frac{\partial^2}{\partial h^2} f_N =& - \frac{1}{2N}\sum_{i,j=1}^N \mathbb{E}[(\frac{\partial}{\partial h_j} + \frac{1}{2}\frac{\partial}{\partial h_j^2})\langle s_i\rangle] \\ =& - \frac{1}{2N}\sum_{i,j=1}^N \mathbb{E}[\langle s_i s_j\rangle - \langle s_i\rangle\langle s_j\rangle - \langle s_i s_j\rangle\langle s_j\rangle + \langle s_i\rangle\langle s_j\rangle^2] \\ =& - \frac{1}{2N}\sum_{i,j=1}^N \mathbb{E}[(\langle s_i s_j\rangle - \langle s_i\rangle\langle s_j\rangle)^2]. \end{aligned} \tag{15}$$

To get the last equality we have used the four Nishimori identities stated after (11). This formula shows that f_N is concave as a function of h. We will show that for $(J, h) \in C_p$ the limit $N \to \infty$ exists and therefore it is concave and continuous as a function of h. Proceeding similarly (for m fixed)

$$\frac{\partial}{\partial h} f_{RS}(m) = -\frac{1}{2}\int Dz\big(1+\tanh(z\sqrt{v}+v)\big), \qquad \frac{\partial^2}{\partial h^2} f_{RS}(m) = -\frac{1}{2}\int Dz\big(\cosh(z\sqrt{v}+v)\big)^{-4}. \tag{16}$$

The proof is left to the reader. An important consequence of the second formula is that $\min_{m\in[0,1]} f_{RS}(m)$ is a concave and continuous function of h.

Proof of Theorem 3. Note that

$$-\frac{J}{4}(1 - p\widehat{m}^{p-1} + (p-1)\widehat{m}^p) + \widetilde{f}(\widehat{m}) = f_{RS}(\widehat{m}).$$

Therefore Theorem 1 and Theorem 2 immediately imply that for $(J, h) \in C_p$ and almost all h, $\lim_{N\to+\infty}$ exists and

$$\lim_{N\to+\infty} f_N = \min_{m\in[0,1]} f_{RS}(m)$$

Above we have shown that both members of this equality are continuous functions of h. Thus we can remove the restriction to Lebesgue almost every h.

3 Upper bound: Theorem 1

The integral term in the replica symmetric variational expression (4) suggests that we introduce the mean field random Hamiltonian

$$H_0(\underline{s}) = -\sum_i J_i s_i - \sum_i h_i s_i$$

where $J_i \sim \mathcal{N}(\frac{J}{2}pm^{p-1}, \frac{J}{2}pm^{p-1})$ and $0 \le m \le 1$ is a free parameter. Its free energy is

$$f_N(0) = -\int_{-\infty}^{+\infty} Dz\, \ln(2\cosh(z\sqrt{v} + v)) \tag{17}$$

and misses the term $-\frac{J}{4}(1-pm^{p-1}-(p-1)m^p)$. We choose a Hamiltonian that interpolates between $H_0(\underline{s})$ and $H(\underline{s})$ and also preserves the gauge symmetry

$$H_t(\underline{s}) = -\sum_{i_1<\ldots<i_p} J_{i_1,\ldots i_p} s_{i_1}\ldots s_{i_p} - \sum_i J_i s_i - \sum_i h_i s_i \tag{18}$$

where now

$$J_{i_1\ldots i_p} \sim \mathcal{N}\left(t\frac{Jp!}{2N^{p-1}}, t\frac{Jp!}{2N^{p-1}}\right), \qquad J_i \sim \mathcal{N}\left((1-t)\frac{J}{2}pm^{p-1}, (1-t)\frac{J}{2}pm^{p-1}\right)$$

and $h_i \sim \mathcal{N}(h,h)$ remains unchanged. Note that all Nishimori identities and formulas of Section 2, as well as their proofs, remain identical for the interpolated system. Of course the Hamiltonian of the original system is equal to $H_1(\underline{s})$. By the fundamental theorem of calculus the free energy can be computed as

$$f_N = f_N(1) = f_N(0) + \int_0^1 dt \frac{d}{dt} f_N(t). \tag{19}$$

In this equation the free energies are defined with the appropriate expectations on all the Gaussian random variables involved. In particular, the t- derivative has two contributions: one coming from $J_{i_1\ldots i_p}$ and one from J_i. It is best computed by using the identity

$$\frac{d}{dt}\frac{e^{-\frac{(Y-u(t))^2}{2u(t)}}}{\sqrt{2\pi u(t)}} = u'(t)\Big(-\frac{\partial}{\partial Y} + \frac{1}{2}\frac{\partial^2}{\partial Y^2}\Big)\frac{e^{-\frac{(Y-u(t))^2}{2u(t)}}}{\sqrt{2\pi u(t)}}.$$

For the contribution coming from $J_{i_1\ldots i_p}$, we have $u'(t) = \frac{Jp!}{2N^{p-1}}$ and $Y = J_{i_1\ldots i_p}$. For the one coming from J_i we have $u'(t) = -\frac{J}{2}pm^{p-1}$ and $Y = J_i$. Integration by parts with respect to the Y variable then leads to

$$\frac{d}{dt} f_N(t) = A + B$$

where A is produced by $\frac{\partial}{\partial Y}$,

$$\begin{aligned} A &= -\frac{Jp!}{2N^p}\sum_{i_1<\ldots<i_p=1}^{N} \mathbb{E}[\langle s_{i_1}\ldots s_{i_p}\rangle_t] + \frac{Jpm^{p-1}}{2N}\sum_{i=1}^{N}\mathbb{E}[\langle s_i\rangle_t] \\ &= -\frac{J}{2}\mathbb{E}[\langle m_1^p\rangle_t] + \frac{J}{2}pm^{p-1}\mathbb{E}[\langle m_1\rangle_t] + O\Big(\frac{1}{N}\Big) \end{aligned} \tag{20}$$

and B is produced by $\frac{\partial^2}{\partial Y^2}$,

$$\begin{aligned} B &= -\frac{Jp!}{4N^p}\sum_{i_1<\ldots<i_p=1}^{N}\mathbb{E}[1-\langle s_{i_1}\ldots s_{i_p}\rangle_t^2] + \frac{Jpm^{p-1}}{4N}\sum_{i=1}^{N}\mathbb{E}[1-\langle s_i\rangle_t^2] \\ &= -\frac{Jp!}{4N^p}\sum_{i_1<\ldots<i_p=1}^{N}\mathbb{E}[1-\langle s_{i_1}^{(1)}s_{i_1}^{(2)}\ldots s_{i_p}^{(1)}s_{i_p}^{(2)}\rangle_t] + \frac{Jpm^{p-1}}{4N}\sum_{i=1}^{N}\mathbb{E}[1-\langle s_i^{(1)}s_i^{(2)}\rangle_t] \\ &= -\frac{J}{4}\big(1-\mathbb{E}[\langle q_{12}^p\rangle_t]\big) + \frac{J}{4}pm^{p-1}\big(1-\mathbb{E}[\langle q_{12}\rangle_t]\big) + O\Big(\frac{1}{N}\Big). \end{aligned}$$

These results can be cast in the form

$$\frac{df_N(t)}{dt} = -\frac{J}{4}(1 - pm^{p-1} - (p-1)m^p) - \frac{J}{2}\mathbb{E}\left[\langle R_p(m, m_1)\rangle_t\right] + \frac{J}{4}\mathbb{E}\left[\langle R_p(m, q_{12})\rangle_t\right] + O\Big(\frac{1}{N}\Big). \tag{21}$$

From (12) for the interpolated system,

$$\mathbb{E}[\langle R_p(m, m_1)\rangle_t] = \mathbb{E}[\langle R_p(m, q_{12})\rangle_t].$$

Thus (17), (19) and (21), imply the simple sum rule

$$f_N = f_{RS}(m) - \frac{J}{4}\int_0^1 \mathbb{E}\left[\langle R_p(m, q_{12})\rangle_t\right] dt + O\left(\frac{1}{N}\right).$$

Now we derive the bound of Theorem 1 from this sum rule.

The case of even p**.** The positivity of $R_p(m, q_{12})$ immediately implies $f_N(1) \le f_{RS}(m) + O(\frac{1}{N})$. The theorem then follows by taking the $\limsup_{N\to+\infty}$ and optimizing over m.

The case of odd p**.** We cannot use the positivity of $R_p(m, q_{12})$ but we note that

$$\langle q_{12}\rangle_t = \frac{1}{N}\sum_{i=1}^N \langle s_i^{(1)} s_i^{(2)}\rangle_t = \frac{1}{N}\sum_{i=1}^N \langle s_i\rangle_t^2$$

is non-negative. Thus from the convexity of x^p for $x \ge 0$ we have

$$R_p(m, \langle q_{12}\rangle_t) \ge 0. \tag{22}$$

Therefore it is sufficient to prove that

Lemma 1. *For Lebesgue almost every* h *we have*

$$\lim_{N\to+\infty}\int_0^1 dt\,\Big(\mathbb{E}[\langle R_p(m, q_{12})\rangle_t] - \mathbb{E}[R_p(m, \langle q_{12}\rangle_t)]\Big) = 0$$

Thanks to this lemma we get

$$\limsup_{N\to+\infty} f_N(1) \le \min_{m\in[0,1]} f_{RS}(m)$$

for almost every h.

Proof of lemma 1. Using the identity $b^p - a^p = (b-a)\sum_{l=0}^{p-1} a^l b^{p-1-l}$ and $0 \le m \le 1$, $-1 \le q_{12} \le 1$,

$$\begin{aligned}|R_p(m, q_{12}) - R_p(m, \langle q_{12}\rangle_t)| =& \big(q_{12}^p - \langle q_{12}\rangle_t^p\big) - pm^{p-1}\big(q_{12} - \langle q_{12}\rangle_t\big)\\ \le& 2p|q_{12} - \langle q_{12}\rangle_t|\end{aligned}$$

Thus Schwarz inequality applied to $\int_0^1 dt\,\mathbb{E}[\langle - \rangle_t]$ yields

$$\begin{aligned}\int_0^1 dt\,\mathbb{E}[|\langle R_p(m, q_{12})\rangle_t - R_p(m, \langle q_{12}\rangle_t)|] \le& 2p\int_0^1 dt\,\mathbb{E}[\langle |q_{12} - \langle q_{12}\rangle_t|\rangle]\\ \le& 2p\Big(\int_0^1 dt\,\mathbb{E}[\langle q_{12}^2\rangle_t - \langle q_{12}\rangle_t^2]\Big)^{1/2}.\end{aligned} \tag{23}$$

From the definition of q_{12}, Schwarz and (15)

$$\begin{aligned}\mathbb{E}[\langle(q_{12}-\langle q_{12}\rangle_t)^2\rangle_t] &= \frac{1}{N^2}\sum_{i,j=1}^{N}\mathbb{E}[\langle s_i s_j\rangle_t^2 - \langle s_i\rangle_t^2\langle s_j\rangle_t^2] \\ &\le \frac{2}{N^2}\sum_{i,j=1}^{N}\mathbb{E}[|\langle s_i s_j\rangle_t - \langle s_i\rangle_t\langle s_j\rangle_t|] \\ &\le \left(\frac{1}{N^2}\sum_{i,j=1}^{N}\mathbb{E}[(\langle s_i s_j\rangle_t - \langle s_i\rangle_t\langle s_j\rangle_t)^2]\right)^{1/2} \\ &= \left(-\frac{2}{N}\frac{\partial^2}{\partial h^2}f_N(t)\right)^{1/2}.\end{aligned}$$

Let $\varphi(h)$ be a sufficiently smooth positive test function. We have

$$\begin{aligned}\int dh\varphi(h)\left\{\int_0^1 dt\mathbb{E}[\langle(q_{12}-\langle q_{12}\rangle_t)^2\rangle_t]\right\}^2 &\le \int dh\,\varphi(h)\int_0^1 dt\mathbb{E}\Big[\langle(q_{12}-\langle q_{12}\rangle)^2\rangle\Big]^2 \\ &\le -\int dh\,\varphi(h)\int_0^1 dt\frac{2}{N}\frac{\partial^2}{\partial h^2}f_N(t) \\ &= \frac{2}{N}\int dh\,\varphi'(h)\int_0^1 dt\frac{\partial}{\partial h}f_N(t) \\ &= \frac{1}{N}\int dh\,\varphi'(h)\int_0^1 dt(1+\mathbb{E}[\langle m_1\rangle_t]).\end{aligned}$$

The right hand side above is smaller than $\frac{2}{N}\int dh\,|\varphi'(h)|$. Dominated convergence then implies that for any convergent subsequence $N_k\to+\infty$,

$$\lim_{N_k\to+\infty}\int_0^1 dt\mathbb{E}[\langle(q_{12}-\langle q_{12}\rangle_t)^2\rangle_t]=0$$

for Lebesgue almost every h. Taking the intersection of the two measure one h-sets corresponding to the convergent subsequences attaining the $\liminf$ and $\limsup$ (which both vanish) implies that the $\lim_{N\to+\infty}$ exists and vanishes. Combining with (23) ends the proof of the lemma.

4 Lower Bound: Theorem 2

The lower bound will follow from an interpolation scheme which uses a Hamiltonian formally identical to (18)

$$\widehat{H}_t(\underline{s}) = -\sum_{i_1<i_2<\ldots<i_p=1}^{N}\widehat{J}_{i_1\ldots i_p}s_{i_1}\ldots s_{i_p} - \sum_{i=1}^{N}\widehat{J}_i s_i - \sum_{i=1}^{N}h_i s_i$$

but with

$$\widehat{J}_{i_1\ldots i_p}\sim\mathcal{N}\left(\frac{Jp!}{2N^{p-1}}, t\frac{Jp!}{2N^{p-1}}\right),\qquad \widehat{J}_i\sim\mathcal{N}\left(0,(1-t)\frac{J}{2}p\widehat{m}^{p-1}\right).$$

For $t = 1$ we find the initial gauge symmetric model, while for $t = 0$ we have an Ising model on a complete hypergraph with a random external magnetic field.

$$\widehat{H}_0(\underline{s}) = -\frac{J}{2}Nm_1^p - \sum_{i=1}^{N}\widehat{J}_i s_i - \sum_{i=1}^{N} h_i s_i + E(\underline{s}), \qquad \widehat{J}_i \sim \mathcal{N}\left(0, \frac{J}{2}p\widehat{m}^{p-1}\right).$$

Here $E(\underline{s})$ denotes an "error term" that does not contribute to the free energy because $\max_{\underline{s}} E(\underline{s}) = O(1)$. One can show that the free energy of this model is $\widehat{f}_N(0) = \min_{m\in[0,1]} \widetilde{f}(m)$ This follows from a saddle point calculation outlined in Section 5. There this calculation is made rigorous only for the lower bound

$$\widehat{f}_N(0) \geq \min_{m\in[0,1]} \widetilde{f}(m) \tag{24}$$

since this is all we really need.

At this point we wish to make a few remarks on the interpolation schemes that are use here. In the scheme of Section 3 in order to preserve the Nishimori symmetry we varied the mean and the variance: this lead to an upper bound on the free energy. Here we do not vary the mean but only the variance (hence the Nishimori symmetry is broken) and this leads to a lower bound. From this point of view, such an interpolation is identical to the Guerra's "first interpolation" [4] for the SK model. However one can also take the point of view that it is similar to the Guerra-Toninelli interpolation [6] (with coupled replica's) because of the identity

$$\widehat{H}_t(\underline{s}) = H_t(\underline{s})|_{m=\widehat{m}} - \frac{J}{2}(1-t)N(R_p(\widehat{m}, m_1) + (1-p)\widehat{m}^p).$$

We can in fact proceed as in [6] and prove that the fluctuations of the remainder vanish in the limit $N \to +\infty$.

Here however we proceed in a simpler way that is similar to Section 3. Let us calculate the derivative of $\widehat{f}_N(t)$ with respect to t. First we make the change of variable

$$\widehat{J}_{i_1\dots i_p} \to \sqrt{t}\sqrt{\frac{Jp!}{2N^{p-1}}}\widehat{J}_{i_1\dots i_p} + \frac{Jp!}{2N^{p-1}}, \qquad \widehat{J}_i \to \sqrt{1-t}\sqrt{\frac{Jp!}{2N^{p-1}}}\widehat{J}_i \tag{25}$$

where the new random couplings are distributed as

$$\widehat{J}_{i_1\dots i_p} \sim \mathcal{N}\left(0, \frac{Jp!}{2N^{p-1}}\right), \qquad \widehat{J}_i \sim \mathcal{N}\left(0, \frac{J}{2}p\widehat{m}^{p-1}\right).$$

For simplicity the Gibbs measure and the expectation with respect to the coupling constants pertaining to the transformed Hamiltonian are still denoted $\langle - \rangle_t$ and $\mathbb{E}$. We have

$$\begin{aligned}\frac{d}{dt}\widehat{f}_N(t) = &-\frac{1}{N}\frac{1}{2\sqrt{t}}\sqrt{\frac{Jp!}{2N^{p-1}}}\sum_{i_1<\dots<i_p}\mathbb{E}\left[\widehat{J}_{i_1\dots i_p}\langle s_{i_1}\dots s_{i_p}\rangle_t\right] \\ &+\frac{1}{N}\frac{1}{2\sqrt{1-t}}\sqrt{\frac{J}{2}p\widehat{m}^{p-1}}\sum_i \mathbb{E}\left[\widehat{J}_i\langle s_i\rangle_t\right].\end{aligned}$$

Integration by parts for standard Gaussian variables shows that we can make the replacements $\widehat{J}_{i_1\ldots i_p} \to \frac{\partial}{\partial \widehat{J}_{i_1\ldots i_p}}$ and $\widehat{J}_i \to \frac{\partial}{\partial \widehat{J}_i}$ in the last formula. Performing these derivatives yields

$$\frac{d}{dt}\widehat{f}_N(t) = -\frac{Jp!}{4N^p}\sum_{i_1<\ldots<i_p}\mathbb{E}\big[1-\langle s_{i_1}\ldots s_{i_p}\rangle_t^2\big] + \frac{Jp\widehat{m}^{p-1}}{4N}\sum_i \mathbb{E}\big[1-\langle s_i\rangle_t^2\big]. \tag{26}$$

At this point one can revert back to the original Gibbs measure and couplings by undoing the change of variables (25). Next we introduce replicas to write

$$\langle s_i\rangle_t^2 = \langle s_i^{(1)}s_i^{(2)}\rangle_t, \qquad \langle s_{i_1}\ldots s_{i_p}\rangle_t^2 = \langle s_{i_1}^{(1)}s_{i_1}^{(2)}\ldots s_{i_p}^{(1)}s_{i_p}^{(2)}\rangle_t.$$

Replacing in (26) we find

$$\begin{aligned}\frac{d}{dt}\widehat{f}_N(t) &= \frac{J}{4}(p\widehat{m}^{p-1}-1) + \frac{J}{4}\big(\langle q_{12}^p\rangle_t - p\widehat{m}^{p-1}\langle q_{12}\rangle_t\big)\\ &= \frac{J}{4}(p\widehat{m}^{p-1}-1-(p-1)\widehat{m}^p) + \frac{J}{4}\big(\langle q_{12}^p\rangle_t - p\widehat{m}^{p-1}\langle q_{12}\rangle_t + (p-1)\widehat{m}^p\big).\end{aligned}$$

Applying the fundamental theorem of calculus we find

$$\widehat{f}_N(1) = -\frac{J}{4}(1-p\widehat{m}^{p-1}+(p-1)\widehat{m}^p) + \widehat{f}_N(0) + \frac{J}{4}\int_0^1 dt\langle R_p(\widehat{m},q_{12})\rangle_t.$$

As long as the integral term is non-negative, using (24) we find

$$f(J,h) = \lim_{N\to\infty}\widehat{f}_N(1) \geq -\frac{J}{4}(1-p\widehat{m}^{p-1}+(p-1)\widehat{m}^p) + \min_{m\in[0,1]}\widetilde{f}(m).$$

Clearly the integral term is positive for even p. For odd p we have to show that $R_p(\widehat{m},q_{12}) \geq 0$ as long as $(p-1)\widehat{m}^p + p\widehat{m}^{p-1} - 1 \geq 0$ (see condition (6)). This is easily done by studying the graph of the polynomial $x^p - p\widehat{m}^{p-1}x + (p-1)\widehat{m}^p$ for $-1 \leq x \leq +1$. This concludes the proof of Theorem 2.

5 Complete graph Ising model in a random field

The goal of this section is to prove (24). We have to study a model of the form

$$\mathcal{H}(\underline{s}) = -\sum_{i=1}^N J_i s_i - \frac{J}{2}Nm_1^p, \qquad m_1 = \frac{1}{N}\sum_{i=1}^N s_i \tag{27}$$

where $J_i \sim \mathcal{N}(\mu,\sigma^2)$. Saddle point calculations lead to the following expression for the free energy

$$\mathcal{F}(\mu,\sigma) = \min_{m\in[0,1]}\left[\frac{J}{2}(p-1)m^p - \int_{-\infty}^{+\infty}Dz\ln 2\cosh(\sigma z + \frac{J}{2}pm^{p-1}+\mu)\right]. \tag{28}$$

In our application we take $J_i = \widehat{J}_i + h_i$ so that $\mu = h$ and $\sigma^2 = \frac{J}{2}p\widehat{m}^{p-1} + h$.

The rigorous proof of (28) is complicated by two facts: first we have $p \geq 2$ so the problem is not easily "linearized" (for $p > 2$) and second the magnetic field is random so one has to control fluctuations of the saddle point. Here we prove the following.

Theorem 4. *Let $\mathcal{Z}$ be the partition function of the model* (27)*. Then*

$$\lim_{N\to\infty} -\frac{1}{N}\mathbb{E}\big[\ln \mathcal{Z}\big] \geq \mathcal{F}(\mu,\sigma). \tag{29}$$

The derivation of a converse bound is more difficult except for p even for which we can use a simple trick. Although the converse bound is not needed in the present work we briefly explain its derivation for p even at the end of the section.

In the following let $f(N) \doteq (\dot{\leq}) g(N)$ denote

$$\lim_{N\to\infty} \frac{1}{N} \log f(N) = (\leq) \lim_{N\to\infty} \frac{1}{N} \log g(N).$$

The first step is to reduce oneself to a Gaussian model. This is accomplished by the following lemma which is proven in Appendix C. Of course for $p=2$ we can proceed more simply by a standard direct linearization of the Gaussian term. The present treatment unifies the cases $p=2$ and $p\geq 3$.

Lemma 2. *Fix $0<\alpha<1$. The following equality holds,*

$$\lim_{N\to\infty} \frac{1}{N}\mathbb{E}[\ln \mathcal{Z}] = \lim_{N\to\infty} \frac{1}{N}\mathbb{E}\big[\ln \widetilde{\mathcal{Z}}\big] \tag{30}$$

where

$$\widetilde{\mathcal{Z}} = \sum_{\underline{s}} \int_{-1}^{1} du e^{-N^{1+\alpha}(u-m_1)^2 + \frac{J}{2}Nu^p + \sum_{i=1}^N J_i s_i}.$$

Using the Gaussian identity $\int_{-\infty}^{\infty} e^{-(y+c)^2} dy = \sqrt{\pi}$ for any $c\in\mathbb{C}$, we get

$$e^{-N^{1+\alpha}(u-m_1)^2} = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} dy e^{-y^2 - 2iN^{\frac{1+\alpha}{2}}(u-m_1)y}.$$

Now we can perform the sum over $\underline{s}$ and obtain (integrals over u and y are exchangeable by Fubini's theorem and the statistical sum is finite)

$$\widetilde{\mathcal{Z}} \doteq \int_{-1}^{1} du\, e^{N\frac{J}{2}u^p} \int_{-\infty}^{\infty} dy\, \Pi(y) e^{-N^{1-\alpha}y^2 - 2iNuy} \tag{31}$$

with

$$\Pi(y) = \prod_{j=1}^{N} 2\cosh(J_j + 2iy)$$

where we have made the substitution $\frac{y}{N^{\frac{1-\alpha}{2}}} \to y$. We evaluate both integrals by two controlled saddle point calculations. Let us first deal with the y integral. Set

$$I_R(u) = \int_{-R}^{+R} dy\, \Pi(y) e^{-y^2 N^{1-\alpha} - 2iNuy}$$

and let y^* be a solution of the formal stationary phase equation

$$u = \frac{1}{N}\sum_{j=1}^{N} \tanh(J_j + 2iy^*) + \frac{iy^*}{N^\alpha}. \tag{32}$$

It is easy to see that we must have $y^* = iy_0(u)$ purely imaginary with $|y_0(u)| < 2N^\alpha$. We deform the y integral over $[-R, +R]$ to the contour $-R \to -R + iy_0(u)$ (along a vertical line), $-R + iy_0(u) \to +R + iy_0(u)$ (along an horizontal line), and $+R + iy_0(u) \to +R$ (along a vertical line). It is easily seen that the two contributions along the vertical parts of the contour tend to zero as $R \to \infty$, thus by Cauchy's theorem

$$\lim_{R \to +\infty} I_R(u) = \int_{-\infty}^{+\infty} dt \, \Pi(t + iy_0(u)) e^{-(t+iy_0(u))^2 N^{1-\alpha} - 2iNu(t+iy_0(u))}.$$

Using $|\cosh(J_j + 2it - 2y_0(u))| \leq \cosh(J_j - 2y_0(u))$ we find

$$\lim_{R \to +\infty} I_R(u) \leq \sqrt{\frac{\pi}{N^{1-\alpha}}} e^{y_0(u)^2 N^{1-\alpha} + 2Nuy_0(u)} \prod_{j=1}^{N} 2\cosh(J_j - 2y_0(u))$$

and replacing in (31) we have

$$\widetilde{\mathcal{Z}} \dot{\leq} \int_{-1}^{+1} du \, \exp(NL(u, \{J_j\}))$$

where

$$L(u, \{J_j\}) = \frac{J}{2} u^p + N^{-\alpha} y_0(u)^2 + 2uy_0(u) + \frac{1}{N} \sum_{j=1}^{N} \ln 2\cosh(J_j - 2y_0(u)). \tag{33}$$

It remains to evaluate the u integral on the right hand side. In Appendix D we prove that for almost every realization of $\{J_j\}$ the maximum of $L(u, \{J_j\})$ cannot be attained at the boundary points ± 1. Therefore in what follows we do not take this possibility in to account. Let us first find the stationary points of $L(u, \{J_j\})$. Differentiating with respect to u we find that they must satisfy

$$y_0{}'(u) \Big(\frac{y_0(u)}{N^\alpha} - \frac{1}{N} \sum_{j=1}^{N} \tanh(J_j - 2y_0(u)) + u \Big) + y_0(u) = -\frac{J}{4} p u^{p-1} \tag{34}$$

which, using (32), implies $y_0(u) = -\frac{J}{4} p u^{p-1}$. Hence the stationary points of (33) are solutions of the equation

$$u_* = \frac{1}{N} \sum_{j=1}^{N} \tanh(J_j + \frac{J}{2} p u_*{}^{p-1}) + \frac{J}{4N^\alpha} p u_*{}^{p-1} \tag{35}$$

For these points we have $L(u_*, \{J_j\}) = G(u_*, \{J_j\})$ where

$$G(u, \{J_j\}) = \frac{J}{2} u^p (1-p) + N^{-\alpha} \left(\frac{J}{4} p u^{p-1} \right)^2 + \frac{1}{N} \sum_{j=1}^{N} \ln 2\cosh\Big(J_j + \frac{J}{2} p u^{p-1} \Big).$$

At this point, note for further use that we necessarily have $\max_{u \in (-1,+1)} L(u, \{J_j\}) \leq \max_{u \in (-1,+1)} G(u, \{J_j\})$. Consider now the equation (the thermodynamic limit of (35))

$$u = \int Dz \tanh(\sigma z + \frac{J}{2} p u^{p-1} + \mu), \qquad u \in [-1, +1] \tag{36}$$

and let $\mathcal{S} = \{u_s\}$ be the set of its solutions for $u \in (-1, +1)$. Let $\mathcal{E}_N$ be the event that the maximum of $L(u, \{J_j\})$ over $u \in [-1, +1]$ is attained in the set $\mathcal{G}_N \equiv \cup_s (u_s - CN^{-d}, u_s + CN^{-d})$ (C a numerical constant independent of N). In particular, this set does not contain the points ± 1 for N large enough. In Appendix D we prove the following.

Lemma 3. *There exists $\epsilon > 0$ (small) such that for N large enough we have $\mathbb{P}(\mathcal{E}_N^c) \leq e^{-N^\epsilon}$.*

We have

$$\begin{aligned}\frac{1}{N}\mathbb{E}[\ln \widetilde{\mathcal{Z}}] &\leq \mathbb{E}[\max_{u\in(-1,+1)} L(u, \{J_j\})] \\ &\leq \mathbb{E}[\max_{u\in\mathcal{G}_N} L(u, \{J_j\}) \mid \mathcal{E}_N]\mathbb{P}(\mathcal{E}_N) + \mathbb{E}[\max_{u\in[-1,+1]} L(u, \{J_j\}) \mid \mathcal{E}_N^c]\mathbb{P}(\mathcal{E}_N^c) \\ &\leq \mathbb{E}[\max_{u\in\mathcal{G}_N} G(u, \{J_j\})] + \left(\frac{J}{2} + N^{-\alpha}\frac{J^2p^2}{16} + \frac{1}{N}\sum_{j=1}^N \mathbb{E}[|J_j| \mid \mathcal{E}_N^c]\right)\mathbb{P}(\mathcal{E}_N^c).\end{aligned}$$

The second term on the right hand side of the last inequality can easily be shown to vanish in the limit $N \to +\infty$ thanks to Lemma 3. Thus from (30) we conclude

$$\lim_{N\to+\infty} \frac{1}{N}\mathbb{E}[\ln \mathcal{Z}] = \lim_{N\to+\infty} \frac{1}{N}\mathbb{E}[\ln \widetilde{\mathcal{Z}}] \leq \lim_{N\to+\infty} \mathbb{E}[\max_{u\in\mathcal{G}_N} G(u, \{J_j\})].$$

For $u \in (u_s - CN^{-d}, u_s + CN^{-d})$, the variation in the value of $G(u, \{J_j\})$ from $G(u_s, \{J_j\})$ can be bounded by,

$$2CN^{-d} \max_{u\in(-1,1)} |G'(u, \{J_j\})| = O(N^{-d})$$

uniformly in J_j because $\tanh(J_j + \frac{J}{2}pu^{p-1}) \leq 1$. Therefore,

$$\lim_{N\to+\infty} \frac{1}{N}\mathbb{E}[\ln \mathcal{Z}] \leq \lim_{N\to\infty} \mathbb{E}[\max_{u\in\mathcal{G}_N} G(u, \{J_j\})] = \lim_{N\to\infty} \mathbb{E}[\max_{u_s\in\mathcal{S}} G(u_s, \{J_j\})].$$

The set $\mathcal{S}$ is not random, so the maximum on the right hand side is taken for some u_s independent of $\{J_j\}$, say u_{max}. Thus

$$\begin{aligned}\lim_{N\to+\infty} \frac{1}{N}\mathbb{E}[\ln \mathcal{Z}] &\leq \lim_{N\to\infty} \mathbb{E}[G(u_{max}, \{J_j\})] \\ &= \frac{J}{2}u_{max}^p(1-p) + \frac{1}{N}\sum_{j=1}^N \mathbb{E}[\ln 2\cosh\left(J_j + \frac{J}{2}pu_{max}^{p-1}\right) \\ &= \frac{J}{2}(1-p)u_{max}^p + \int Dz \ln 2\cosh(\sigma z + \frac{J}{2}pu_{max}^{p-1} + \mu) \\ &\leq \max_{m\in[-1,+1]}\left[\frac{J}{2}(1-p)m^p + \int Dz \ln 2\cosh(\sigma m + \frac{J}{2}pm^{p-1} + \mu)\right].\end{aligned}$$

We conclude that

$$-\lim_{N\to\infty} \frac{1}{N}\mathbb{E}[\ln \mathcal{Z}] \geq \mathcal{F}(\mu, \sigma)$$

which proves Theorem 4.

For even p we can prove a converse bound by the following trick. Consider the interpolating Hamiltonian

$$\mathcal{H}_t(\underline{x}) = -\sum J_i x_i - t\frac{J}{2}Nm_1^p - (1-t)\frac{J}{2}Npm^{p-1}m_1$$

The fundamental theorem of calculus applied to the corresponding free energies reads

$$\begin{aligned}
-\frac{1}{N}\mathbb{E}[\ln \mathcal{Z}_{t=1}] &= -\frac{1}{N}\mathbb{E}[\ln \mathcal{Z}_{t=0}] - \frac{J}{2}\int_0^1 \mathbb{E}[\langle m_1^p - pm^{p-1}m_1\rangle_t]dt \\
&= -\frac{J}{2}(1-p)m^p - \int Dz \ln 2\cosh(\sigma z + \frac{J}{2}pm^{p-1} + \mu) \\
&- \frac{J}{2}\int_0^1 \mathbb{E}[\langle R_p(m, m_1)\rangle_t]dt.
\end{aligned}$$

Since the remainder is positive for even p we get

$$-\lim_{N\to\infty}\frac{1}{N}\mathbb{E}[\ln \mathcal{Z}_{t=1}] \leq \min_{m\in[0,1]}\left[-\frac{J}{2}(1-p)m^p - \int Dz \ln 2\cosh(\sigma z + \frac{J}{2}pm^{p-1} + \mu)\right]$$

which is exactly the converse of Theorem 4.

6 Concentration of Magnetization

In this section we show the various forms of concentration of the magnetization. The main statement of this section is the following theorem.

Theorem 5. *For any two constants $0 < a \leq b < \infty$, we have the three identities*

$$\lim_{N\to\infty}\int_a^b dh\mathbb{E}\langle|m_1 - \mathbb{E}\langle m_1\rangle|\rangle = 0, \tag{37}$$

$$\lim_{N\to\infty}\int_a^b dh\mathbb{E}\langle|m_1 - \langle m_1\rangle|\rangle = 0, \tag{38}$$

$$\lim_{N\to\infty}\int_a^b dh\mathbb{E}|\langle m_1\rangle - \mathbb{E}\langle m_1\rangle| = 0. \tag{39}$$

Similar identities are true for q_{12}.

The identity (38) is proved by using similar arguments as in Lemma 1 and we do not reproduce them here. Identity (39) follows from (37),(38) using the triangle inequality. It remains to prove (37). It is sufficient to prove this identity for the case of m_1 because gauge symmetry implies m_1 and q_{12} are identically distributed under $\mathbb{E}[\langle - \rangle]$.

The proof of (37) is based on the idea used in [9] which involves proving Ghirlanda-Guerra type identities for our model. For a brief review of these identities for the SK model and their applications please refer to [19, Section 2.12].

Consider the following Hamiltonian

$$H'(\underline{s}) = H(\underline{s}) + \sum_{i=1}^N |h_i - h|. \tag{40}$$

The additional term is independent of the configuration $\underline{s}$. Therefore, Gibbs average with respect to $H'(\underline{s})$ is same as that of $H(\underline{s})$. Let $Z_N(h), f_N(h)$ denote the partition function and free energy with respect to this new Hamiltonian.

The proof is organized in a succession of lemmas. By using similar interpolation method as in [7] we can prove the following concentration of the free energy.

Lemma 4. *There exists a strictly positive constant* α *(which remains positive for all* h*) such that*

$$\mathbb{P}[|f_N(h) - \mathbb{E}[f_N(h)]| \geq \epsilon] = O(e^{-\alpha\epsilon^2 N})$$

The perturbation term (40) has been chosen carefully so that the following holds,

Lemma 5. *When considered as a function of* h, $-f_N(h)$ *is convex in* h.

Proof. First write the Hamiltonian (40) as

$$H'(\underline{s}) = -\sum_{1=i_1<\cdots<i_p}^{N} J_{i_1\ldots i_p} s_{i_1}\ldots s_{i_p} - \sqrt{h}\sum_{i=1}^{N} h_i s_i - h\sum_{i=1}^{N} s_i + \sqrt{h}\sum_{i=1}^{N}|h_i|$$

where $h_i \sim \mathcal{N}(0,1)$. We simply evaluate the second derivative and show it is positive.

$$-\frac{df_N(h)}{dh} = \langle L(\underline{s})\rangle - \frac{1}{N2\sqrt{h}}\sum_k |h_k| \tag{41}$$

where we have defined

$$L(\underline{s}) = \frac{1}{N}\frac{1}{2\sqrt{h}}\sum_k h_k s_k + \frac{1}{N}\sum_k s_k.$$

Differentiating again,

$$\begin{aligned} -\frac{d^2 f_N(h)}{dh^2} &= \frac{1}{N}\Big\langle \frac{-1}{4h^{3/2}}\sum_k h_k s_k\Big\rangle + \frac{1}{4h^{3/2}N}\sum_k |h_k| \\ &\quad + N(\langle L(\underline{s})^2\rangle - \langle L(\underline{s})\rangle^2) \geq 0. \end{aligned} \tag{42}$$

□

The quantity $L(\underline{s})$ turns out to be very useful and satisfies the following concentration properties.

Lemma 6. *For any two constants* $0 < a \leq b < \infty$,

$$\int_a^b dh \mathbb{E}\Big\langle \Big| L(\underline{s}) - \langle L(\underline{s})\rangle \Big| \Big\rangle = O\Big(\frac{1}{\sqrt{N}}\Big).$$

Proof. From equation (42), we have

$$\int_a^b dh\mathbb{E}\Big\langle\Big(L(\underline{s}) - \langle L(\underline{s})\rangle\Big)^2\Big\rangle \leq -\int_a^b dh\frac{1}{N}\frac{d^2}{dh^2}\mathbb{E}[f_N(h)]$$
$$\leq \frac{1}{N}\Big(\frac{d}{dh}\mathbb{E}[f_N(h)]\Big|_a - \frac{d}{dh}\mathbb{E}[f_N(h)]\Big|_b\Big) = O\Big(\frac{1}{N}\Big).$$

The very last equality follows from the boundedness of the first derivative of $\mathbb{E}[f_N(h)]$ for $h \geq a > 0$ (see (41)). Using Cauchy-Schwarz inequality for $\int \mathbb{E}\langle - \rangle$ we obtain the lemma. □

Lemma 7. *For any two constants* $0 < a \leq b < \infty$*,*

$$\int_a^b dh\mathbb{E}\Big|\langle L(\underline{s})\rangle - \mathbb{E}\langle L(\underline{s})\rangle_h\Big| = O\Big(\frac{1}{\sqrt[4]{N}}\Big).$$

Proof. From convexity of $-f_N(h)$ with respect to h (Lemma 5) we have for any $\delta > 0$,

$$\frac{d}{dh}\mathbb{E}[f_N(h)] - \frac{d}{dh}f_N(h) \leq \frac{f_N(h) - f_N(h+\delta)}{\delta} + \frac{d}{dh}\mathbb{E}[f_N(h)]$$
$$\leq \frac{f_N(h) - \mathbb{E}[f_N(h)]}{\delta} - \frac{f_N(h+\delta) - \mathbb{E}[f_N(h+\delta)]}{\delta}$$
$$+ \frac{d}{dh}\mathbb{E}[f_N(h)] - \frac{d}{dh}\mathbb{E}[f_N(h+\delta)].$$

A similar lower bound holds with δ replaced by $-\delta$. Now from Lemma 4 we know that the fluctuations of the first two terms are $O(N^{-\frac{1}{2}})$. Thus from the formula for the first derivative (41) and the fact that the fluctuations of $\frac{1}{N}\sum_{k=1}^N |h_k|$ are $O(N^{-\frac{1}{2}})$ we get

$$\mathbb{E}\Big|\langle L(\underline{s})\rangle - \mathbb{E}\langle L(\underline{s})\rangle\Big| \leq \frac{1}{\delta}O\Big(\frac{1}{\sqrt{N}}\Big) + \frac{1}{\delta}O\Big(\frac{1}{\sqrt{N}}\Big)$$
$$+\frac{d}{dh}\mathbb{E}[f_N(h)] - \frac{d}{dh}\mathbb{E}[f_N(h+\delta)].$$

We will choose $\delta = N^{-\frac{1}{4}}$. Note that we cannot assume that the difference of the two derivatives is small because the first derivative of the free energy is not uniformly continuous in N (as $N \to \infty$ it may develop jumps at the phase transition points). The free energy itself is uniformly continuous. Using $|h_i x_i + hx_i + |h_i|| \leq 2|h_i| + h$, we get

$$|\mathbb{E}[f_N(h)] - \mathbb{E}[f_N(0)]| \leq 2\sqrt{h}\mathbb{E}[|h_k|] + h.$$

Therefore, if we integrate with respect to h, we get

$$\int_a^b dh\mathbb{E}\Big|\langle L(\underline{s})\rangle - \mathbb{E}\langle L(\underline{s})\rangle\Big| \leq O\Big(\frac{1}{\sqrt[4]{N}}\Big).$$

□

Proof of Theorem 5: Combining Lemma 6 and Lemma 7 we get

$$\int_a^b dh \mathbb{E}\langle |L(\underline{s}) - \mathbb{E}\langle L(\underline{s})\rangle|\rangle \le O\Big(\frac{1}{\sqrt[4]{N}}\Big).$$

For any function $g(\underline{s})$ such that $|g(\underline{s})| \le 1$, we have

$$\int_a^b dh |\mathbb{E}\langle L(\underline{s}) g(\underline{s})\rangle - \mathbb{E}\langle L(\underline{s})\rangle \mathbb{E}\langle g(\underline{s})\rangle|\rangle \le \int_a^b dh \mathbb{E}\langle |L(\underline{s}) - \mathbb{E}\langle L(\underline{s})\rangle|\rangle.$$

More generally the same inequality holds if one takes a function depending on many replicas such as $g(\underline{s}^{(1)}, \underline{s}^{(2)}) = q_{12}$. Using integration by parts formula with respect to h_k,

$$\begin{aligned}
&\mathbb{E}\langle L(\underline{s}) q_{12}\rangle = \mathbb{E}\Big\langle \frac{1}{2N\sqrt{h}} \sum_k h_k s_k q_{12} \Big\rangle + \mathbb{E}\langle m_1 q_{12}\rangle \\
&= \frac{1}{2}\mathbb{E}\langle (1+q_{12}) q_{12}\rangle - \frac{1}{2}\mathbb{E}\langle (q_{13}+q_{14}) q_{12}\rangle + \mathbb{E}\langle m_1 q_{12}\rangle \\
&= \frac{1}{2}\mathbb{E}\langle (1+q_{12}) q_{12}\rangle \\
&= \frac{1}{2}\mathbb{E}\langle m_1 + m_1^2\rangle. \qquad (43)
\end{aligned}$$

We used a Gaussian integration by parts formula for the second equality, gauge transformation for the third and Nishimori identities for the fourth equality. Moreover using similar tricks we get,

$$\begin{aligned}
\mathbb{E}\langle L(\underline{s})\rangle \mathbb{E}\langle q_{12}\rangle &= \frac{1}{2}\mathbb{E}\langle 1 - q_{12} + 2m_1\rangle \mathbb{E}\langle q_{12}\rangle \\
&= \frac{1}{2}(\mathbb{E}\langle m_1\rangle + (\mathbb{E}\langle m_1\rangle)^2). \qquad (44)
\end{aligned}$$

From equations (43) and (44), we get

$$\int_a^b dh |\mathbb{E}\langle m_1^2\rangle - (\mathbb{E}\langle m_1\rangle)^2| \le O\Big(\frac{1}{\sqrt[4]{N}}\Big).$$

By Cauchy-Schwarz this implies (37). □

A Region C_p

A.1 $p = 2$

The minima of (4) and (5) are attained at one of their stationary points. For $p = 2$ these points are given by the solutions of the following fixed point equations respectively.

$$m = \int Dz \tanh(z\sqrt{Jm+h} + Jm + h) \qquad (45)$$

$$m = \int Dz \tanh(z\sqrt{J\widehat{m}+h} + Jm + h) \qquad (46)$$

Here we show $C_2 = \mathbb{R}_+^2$ by arguing that $\widehat{m} = \widetilde{m}$ for all $(J, h) \in \mathbb{R}_+^2$.
Case $h > 0$ and any J. Both (45) and (46) have a unique positive solution which is the minimizer of both (4) and (5). Hence $\widehat{m} = \widetilde{m}$.
Case $h = 0$ and $J \leq 1$. Both (45) and (46) have a unique solution $\widehat{m} = \widetilde{m} = 0$ which is the minimizer of both (4) and (5). Hence $\widehat{m} = \widetilde{m}$.
Case $h = 0$ and $J \geq 1$. Both (45) and (46) have two solutions $\{0, \widehat{m}\}$, and $\widehat{m}$ is the minimizer of both (4) and (5). Hence $\widehat{m} = \widetilde{m}$.

A.2 $p \geq 3$

For $p \geq 3$ the fixed point equations

$$m = \int Dz \tanh(z\sqrt{\frac{J}{2}pm^{p-1} + h} + \frac{J}{2}pm^{p-1} + h)$$
$$m = \int Dz \tanh(z\sqrt{\frac{J}{2}p\widehat{m}^{p-1} + h} + \frac{J}{2}pm^{p-1} + h)$$

have 3 solutions with 2 of them being local minima. The minimizers $\widehat{m}$ and $\widetilde{m}$ are not always equal and this results in $C_p \subset \mathbb{R}_+^2$. For even p the equality for the free energy is not valid in some region close to the phase transition line (jump in magnetization). The region C_4 is shown in Figure 1.

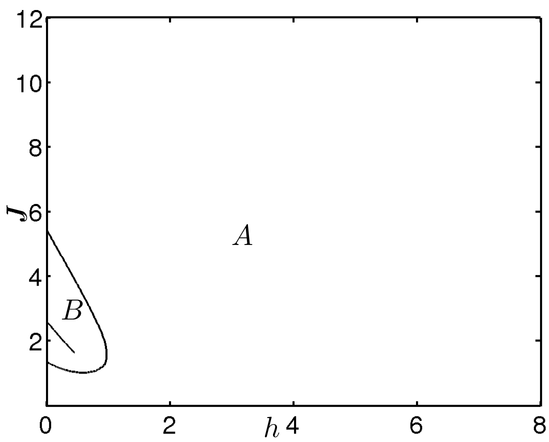

Figure 1: C_4 is equal to A. The line in region B is the phase transition line. As mentioned before, close to this phase transition line (region B), we cannot show the equality for the free energy.

B Nishimori identities

The gauge symmetry leads to remarkable identities first discussed by Nishimori. For the ease of the reader we give a brief streamlined proof of the necessary facts that are used in the present work. The following arguments are also valid for the interpolating Hamiltonian of Section 3 for any t.

Let $s_A = \prod_{i\in A} s_i$, $A \subset \{1, ..., N\}$. Under a gauge transformation the Hamiltonian remains invariant, thus $\langle s_A\rangle \to \tau_A\langle s_A\rangle$. On the other hand the Gaussian distribution of the couplings transforms as $\mathbb{E}[(-)] \to \mathbb{E}[(-)e^{-H(\underline{\tau})+H(\underline{1})}]$ where $H(\underline{1})$ is the Hamiltonian evaluated for $\tau_i = 1$, all i. Therefore

$$\mathbb{E}\Big[\prod_A \langle s_A\rangle\Big] = \mathbb{E}\Big[\prod_A \tau_A \prod_A \langle s_A\rangle e^{-H(\underline{\tau})+H(\underline{1})}\Big].$$

Summing both sides over $\underline{\tau}$,

$$\begin{aligned} 2^N\mathbb{E}\Big[\prod_A \langle s_A\rangle\Big] &= \mathbb{E}\Big[Z\langle \prod_A \tau_A\rangle \langle\prod_A \langle s_A\rangle e^{H(\underline{1})}\Big] \\ &= \sum_{\underline{\rho}} \mathbb{E}\Big[\langle \prod_A \tau_A\rangle \prod_A \langle s_A\rangle e^{-H(\underline{\rho})+H(\underline{1})}\Big]. \end{aligned} \tag{47}$$

The last step is to perform an extra gauge transformation for each term in the ρ sum: $s_i \to \rho_i s_i$, $\tau_i \to \rho_i\tau_i$, $J_i \to \rho_i J_i$, $J_{i_1\ldots i_p} \to \rho_{i_1}\ldots\rho_{i_p} J_{i_1\ldots i_p}$. The terms in the last exponent of the right hand side transform as $H(\rho) \to H(\underline{1})$, $H(\underline{1}) \to H(\rho)$. Then each term of the right hand side becomes

$$\mathbb{E}\Big[\prod_A \rho_A^2 \langle\prod_A \tau_A\rangle \prod_A \langle s_A\rangle e^{-H(\underline{1})+H(\underline{\rho})} e^{-H(\underline{\rho})+H(\underline{1})}\Big]$$

which is independent of ρ. Thus (47) implies the general identity

$$\mathbb{E}\Big[\prod_A \langle s_A\rangle\Big] = \mathbb{E}\Big[\langle\prod_A \tau_A\rangle \prod_A \langle s_A\rangle\Big].$$

C Proof of Lemma 2

We have to study the integral

$$\mathcal{I} = \int_{-1}^{1} du\, e^{-N^{1+\alpha}F(u)}$$

with $F(u) = (u - m_1)^2 - \frac{J}{2}N^{-\alpha}u^p$ and $p \geq 2$. For $|u - m_1| \leq N^{-\frac{3\alpha}{4}}$ we have $|F(u) + \frac{J}{2}N^{-\alpha}m_1^p| \leq CN^{-\frac{3\alpha}{2}}$, C a positive constant depending only upon p and J. The following simple lower bound will suffice,

$$\begin{aligned} \mathcal{I} &= e^{\frac{J}{2}Nm_1^p}\int_{-1}^{+1} du\, e^{-N^{1+\alpha}\left(F(u)+\frac{J}{2}N^{-\alpha}m_1^p\right)} \\ &\geq e^{\frac{J}{2}Nm_1^p}\int_{m_1-N^{-\frac{3\alpha}{4}}}^{m_1+N^{-\frac{3\alpha}{4}}} du\, e^{-N^{1+\alpha}\left(F(u)+\frac{J}{2}N^{-\alpha}m_1^p\right)} \\ &\geq 2N^{-\frac{3\alpha}{4}} e^{-CN^{1-\frac{\alpha}{2}}} e^{\frac{J}{2}Nm_1^p}. \end{aligned} \tag{48}$$

For an upper bound we separate the integral over $[-1,+1]$ into two contributions $\mathcal{I}_1$ over $\Delta_1 = \{u : |u-m_1| \leq N^{-\frac{3\alpha}{4}}\}$ and $\mathcal{I}_2$ over $\Delta_2 = \{u : |u-m_1| \geq N^{-\frac{3\alpha}{4}}\}$. We have

$$\mathcal{I}_1 = e^{\frac{J}{2}Nm_1^p} \int_{\Delta_1} du\, e^{-N^{1+\alpha}\left(F(u)+\frac{J}{2}N^{-\alpha}m_1^p\right)} \leq 2N^{-\frac{3\alpha}{4}} e^{CN^{1-\frac{\alpha}{2}}} e^{\frac{J}{2}Nm_1^p}.$$

To estimate $\mathcal{I}_2$ we first note

$$\frac{d}{du}\left(F(u) + \frac{J}{2}N^{-\alpha}m_1^p\right) = 2(u-m_1) - \frac{J}{2}N^{-\alpha}pu^{p-1}.$$

Now for $u - m_1 \geq N^{-\frac{3\alpha}{4}}$ and N large enough this derivative is necessarily positive, therefore $F(u) + \frac{J}{2}N^{-\alpha}m_1^p$ takes its minimal value at $u = m_1 + N^{-\frac{3\alpha}{4}}$. Similarly for $u - m_1 \leq -N^{-\frac{3\alpha}{4}}$ the same function takes its minimal value at $u = m_1 - N^{-\frac{3\alpha}{4}}$. Thus

$$\mathcal{I}_2 = e^{\frac{J}{2}Nm_1^p} \int_{\Delta_2} du\, e^{-N^{1+\alpha}\left(F(u)+\frac{J}{2}N^{-\alpha}m_1^p\right)} \leq 2e^{O(N^{1-\frac{\alpha}{2}})} e^{\frac{J}{2}Nm_1^p}.$$

Finally for N large enough,

$$\mathcal{I} = \mathcal{I}_1 + \mathcal{I}_2 \leq 3e^{O(N^{1-\frac{\alpha}{2}})} e^{\frac{J}{2}Nm_1^p}. \tag{49}$$

The two bounds (48) and (49) immediately imply

$$\left| \frac{1}{N}\mathbb{E}[\ln \widetilde{\mathcal{Z}}] - \frac{1}{N}\mathbb{E}[\ln \mathcal{Z}] \right| = O(N^{-\frac{\alpha}{2}})$$

and this completes the proof.

D Bound on $P(\mathcal{E}_N^c)$

In the following lemma we show that the largest stationary point less than 1 is a local maximum almost surely over $\{J_j\}$. Similarly we can show that the smallest stationary point larger than -1 is also a local maximum. This implies that the maximum of $L(u, \{J_j\})$ is not attained at the boundary points ± 1.

Lemma 8. *Let $\overline{u}$ be the largest stationary point of $L(u, \{J_j\})$. Then $L(1, \{J_j\}) \leq L(\overline{u}, \{J_j\})$ for almost all realizations of $\{J_j\}$.*

Proof. The above statement is true if $\overline{u}$ is a local maximum, i.e, $L''(\bar{u}, \{J_j\}) < 0$. Let $t(u, y)$ denote

$$t(u, y) = \frac{1}{N}\sum_{j=1}^{N} \tanh(J_j - 2y) - \frac{y}{N^\alpha} - u$$

and recall that $y_0(u)$ is defined as $t(u, y_0(u)) = 0$.

From (33), we can write

$$L'(u, \{J_j\}) = -2y_0'(u)t(u, y_0(u)) + \frac{J}{2}pu^{p-1} + 2y_0(u)$$

Differentiating once more

$$L''(u,\{J_j\}) = \frac{J}{2}p(p-1)u^{p-2} + 2y_0'(u)$$

where $y_0'(u)$ can be computed from the equation $t(u, y_0(u)) = 0$ as

$$y_0'(u) = -\Big(\frac{1}{N^\alpha} + \frac{2}{N}\sum_{j=1}^{N}\text{sech}^2(J_j - 2y_0(u))\Big)^{-1}.$$

Now at $u = \bar{u}$ we have $y_0(\bar{u}) = -\frac{J}{4}p\bar{u}^{p-1}$. So, the condition that $\bar{u}$ is a local maximum is

$$\frac{J}{4}p(p-1)\bar{u}^{p-2}\Big(\frac{1}{N^\alpha} + \frac{2}{N}\sum_{j=1}^{N}\text{sech}^2\big(J_j + \frac{J}{2}p\bar{u}^{p-1}\big)\Big) - 1 < 0 \tag{50}$$

To prove this let us define

$$q(u) = t\Big(u, -\frac{J}{4}pu^{p-1}\Big).$$

From (36), we have $t(\bar{u}) = 0$ and note that for N large enough, we have $q(1) < 0$. Since $\bar{u}$ is the largest solution of $L'(u,\{J_j\}) = 0$ we must have

$$q(u) < 0 \text{ for } \bar{u} < u \leq 1.$$

Therefore $\frac{d}{du}q(u)|_{u=\bar{u}} \leq 0$. Computing the derivative gives

$$\frac{J}{4}p(p-1)\bar{u}^{p-2}\Big(\frac{1}{N^\alpha} + \frac{2}{N}\sum_{j=1}^{N}\text{sech}^2\big(J_j + \frac{J}{2}p\bar{u}^{p-1}\big)\Big) - 1 \leq 0.$$

We see that we have obtained (50) except for the possible equality. However, the $\{J_j\}$s have to satisfy both the equalities in (50) and (36), which happens with only an exponentially small probability. Therefore (50) is a strict inequality for almost all $\{J_j\}$. □

To prove Lemma 3, we need the following result on the concentration of bounded monotonic functions.

Lemma 9. *Let $f : \mathbb{R} \to \mathbb{R}$ be a bounded monotonic function and $\{X_1, X_2 ...\}$ be a sequence of i.i.d. real random variables. Then for any $0 \leq d < 1/2$ there exist constants A and B such that*

$$\mathbb{P}\Big(\sup_{u\in\mathbb{R}}\Big|\frac{1}{N}\sum_{i=1}^{N} f(X_i + u) - \mathbb{E}[f(X + u)]\Big| \leq N^{-d}\Big) \geq 1 - AN^d e^{-BN^{1-2d}}.$$

Proof. w.l.o.g. assume that $|f(x)| < 1$ and $f(x)$ is an increasing function. Since, $f(x)$ is bounded, its expectation exists. Using the concentration inequality for random variables with bounded difference, we get for any u,

$$\mathbb{P}\Big(\Big|\frac{1}{N}\sum_{i=1}^{N} f(X_i + u) - \mathbb{E}\big[f(X + u)\big]\Big| \geq \delta\Big) \leq 2e^{-N\delta^2/2}.$$

Consider any $\epsilon > 0$ and let $M = 1/\epsilon$. Let us define a sequence of numbers $\{u_{-M}, ...u_0, ..., u_M\}$ such that $|\mathbb{E}[f(X+u_k)] - \mathbb{E}[f(X+u_{k+1})]| = \epsilon$. From union bound we get,

$$\mathbb{P}\Big(\forall j \in \{-M, \dots, M\} : \Big|\frac{1}{N}\sum_{i=1}^{N} f(X_i + u_j) - \mathbb{E}\big[f(X+u_j)\big]\Big| \le \delta\Big) \ge 1 - (2M+1)2e^{-N\delta^2/2}. \tag{51}$$

Now consider any $v \notin \{u_{-M}, ...u_M\}$. Let $u_k < v < u_{k+1}$, for all the realizations of $\{X_i\}$ which satisfy (51), if $\frac{1}{N}\sum_{i=1}^{N} f(X_i+v) - \mathbb{E}\big[f(X+v)\big] \ge 0$

$$\begin{aligned}\Big|\frac{1}{N}\sum_{i=1}^{N} f(X_i + v) - \mathbb{E}\big[f(X+v)\big]\Big| &\le \Big|\frac{1}{N}\sum_{i=1}^{N} f(X_i + u_{k+1}) - \mathbb{E}\big[f(X+u_k)\big]\Big| \\ \le \Big|\frac{1}{N}\sum_{i=1}^{N} f(X_i + u_{k+1}) - \mathbb{E}\big[f(X+u_{k+1})\big]\Big| &+ \Big|\mathbb{E}f(X+u_k) - \mathbb{E}\big[f(X+u_{k+1})\big]\Big| \le \delta + \epsilon\end{aligned} \tag{52}$$

and if $\frac{1}{N}\sum_{i=1}^{N} f(X_i+v) - \mathbb{E}\big[f(X+v)\big] \le 0$, we have similarly

$$\begin{aligned}\Big|\frac{1}{N}\sum_{i=1}^{N} f(X_i + v) - \mathbb{E}\big[f(X+v)\big]\Big| &\le \Big|\frac{1}{N}\sum_{i=1}^{N} f(X_i + u_k) - \mathbb{E}\big[f(X+u_{k+1})\big]\Big| \\ \le \Big|\frac{1}{N}\sum_{i=1}^{N} f(X_i + u_k) - \mathbb{E}\big[f(X+u_k)\big]\Big| &+ \Big|\mathbb{E}f(X+u_k) - \mathbb{E}\big[f(X+u_{k+1})\big]\Big| \le \delta + \epsilon.\end{aligned}$$

Then using (51) and (52), we get

$$\mathbb{P}\Big(\sup_{u\in\mathbb{R}}\Big|\frac{1}{N}\sum_{i=1}^{N} f(X_i+u) - \mathbb{E}\big[f(X+u)\big]\Big| \le \delta + \epsilon\Big) \ge 1 - (2M+1)2e^{-N\delta^2/2}.$$

Taking $\delta = \epsilon = 1/(2N^d)$,

$$\mathbb{P}\Big(\sup_{u\in\mathbb{R}}\Big|\frac{1}{N}\sum_{i=1}^{N} f(X_i+u) - \mathbb{E}f\big[(X+u)\big]\Big| \le N^{-d}\Big) \ge 1 - (8N^d+2)e^{-N^{1-2d}/8}.$$

$\square$

Proof of Lemma 3: Applying the above lemma to $f(x) = \tanh(x)$, we get

$$\begin{aligned}\mathbb{P}\Big(\sup_{u\in\mathbb{R}}\Big|\frac{1}{N}\sum_{j=1}^{N}\tanh\Big(J_j + \frac{J}{2}pu^{p-1}\Big) - \mathbb{E}\tanh\Big(J_1 + \frac{J}{2}pu^{p-1}\Big)\Big| \le N^{-d}\Big) \\ \ge 1 - 9N^d e^{-N^{1-2d}/8}.\end{aligned}$$

Therefore, with probability at least $1 - 9N^d e^{-N^{1-2d}/8}$, the solutions of (35) belong to $(u_i - K_1N^{-d}, u_i + K_1N^{-d})$ where u_i denote the solutions of (36). Therefore, $L(u, \{J_j\})$ attains its maximum in $\cup_i(u_i - K_1N^{-d}, u_i + K_1N^{-d})$. $\square$

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