The diameter of the symplectomorphism group is infinite

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1. Introduction

It is known (see [A] and [EM]) that the group of volume preserving diffeomorphisms of a compact oriented Riemannian manifold is the configuration space of incompressible ideal hydrodynamics. The group of symplectic diffeomorphisms plays a similar role in plasma physics (see [MW]). Both groups carry a structure of metric space, natural from the point of view of mechanics, which can be defined, starting with the L^2-norm on their Lie algebras, as the infimum of length of paths connecting two points (see §2 for exact definitions). Shnirelman in [Sh] has proved that the group of volume-preversing diffeomorphisms of the cube in $\mathbb{R}^3$ has finite diameter and has announced the result that this is false for the square.

   Despite the fact that Shnirelman formulated his theorem only for the 3-dimensional cube, his proof can be modified for the case of the group of volume preserving diffeomorphisms of any compact simply-connected Riemannian manifold of dimension >2. However, the diameter can become infinite if the fundamental group is not trivial (see Appendix).

   It turns out that the situation with the group of symplectic diffeomorphisms is completely different. We prove in this paper that the diameter of the symplectomorphism group of any compact exact symplectic manifold (necessarily with boundary) is infinite. We think that the result is true without the assumption of the exactness of the symplectic form.

   It is important to point out that, as a contrary to the volume-preserving case, infiniteness of the diameter of the group of symplectomorphisms has local nature and has nothing to do with the topology of the underlying manifold. The distinction between the cases comes from the fact that the fundamental group of the group of linear symplectic transformations is infinite while in the volume preserving case it is finite in dimension >2.

Although the structure of metric space on the group of symplectomorphisms $\mathscr{D}_\omega$ considered in this paper is natural from the point of view of continuum mechanics, there exist other metrics which are quite natural from the geometric point of view. In fact, we prove in this paper infiniteness of the diameter in the metric on $\mathscr{D}_\omega$ generated by the $L^1 -$ (rather than the $L^2 -$) norm on its Lie algebra. This implies, of course, the result for all L^p-norms, $p \geqq 1$. Moreover, the group $\mathscr{D}_\omega$ admits even more interesting metrics. One of them which is actually *biinvariant* and can be canonically defined in terms of the symplectic structure was recently discovered by H. Hofer (see [H]). Infiniteness of the diameter of $\mathscr{D}_\omega$ in Hofer's metric can also be seen within the philosophy of our paper. While the main ingredient in our proof is the Calabi invariant (which is a volume-like symplectic invariant), the proof for Hofer's metric requires more subtle symplectic invariants – Ekeland-Hofer-Zehnder's capacities or Gromov's width.

The paper is organized as follows. In Sect. 2 we define the metric on $\mathscr{D}_\omega$ and state the main theorem. Section 3 contains a reduction to the case of exact symplectomorphisms. Section 4 is devoted to a general inequality which we use in the Sect. 5 to prove the main results. In the Appendix we give a brief explanation of the role of the fundamental group in the volume preserving case.

Note that the proof can be greatly simplified for the case of the group of *exact* symplectomorphisms fixed at the boundary of the manifold. In this case the result follows (see [ER]) from three main ingredients: an inequality between the length of a path and the value of the Calabi invariant of the resulted symplectomorphism; the independence of the Calabi invariant from the choice of the path; the existence of symplectomorphisms with arbitrarily large values of the Calabi invariant.

The proof in the general case requires more work. But even in the special case above the approach of this paper has some advantages. It shows that not only $\mathscr{D}_\omega$ but all its non-trivial normal subgroups have infinite diameter (see Theorem 2.2 below).

2. Statement of the results

2.1 Metrics on the group of volume preserving diffeomorphisms

Let M be a compact manifold (possibly with boundary) with a volume form μ. Fix a Riemannian metric on M. Let $\mathscr{D}$ be the connected component of the identity in the group of volume preserving diffeomorphisms of (M, μ). A tangent vector to $\mathscr{D}$ at a point η is a map $X_\eta : M \to TM$ such that $X_\eta(x) \in T_{\eta(x)}(M), x \in M$ (see [EM]). Let us put $\|X_\eta\|_p = \left(\int\limits_M \|X_\eta(x)\|^p \mu \right)^{\frac{1}{p}}$, where the norm $\|X_\eta(x)\|$ is taken with respect to the chosen Riemannian on M. Note that $\|X_\eta\|_p \geqq c_p \|X_\eta\|_1$ for a positive constant c_p.

Given a path $\eta_t \in \mathscr{D}, t \in [0, 1]$, we define its p-length $l_p(\{\eta_t\})$ by the formula

$$l_p(\{\eta_t\}) = \int\limits_0^1 \left\| \frac{d\eta_t}{dt} \right\|_p dt .$$

The length function l_p defines a distance function d_p on $\mathscr{D}$ by $d_p(f, g) = \inf l_p(\{\eta_t\})$, where the infimum is taken over all paths $\eta_t, t \in [0, 1]$, joining $\eta_0 = f$ and $\eta_1 = g$.

For any pathwise connected subset $A \subset \mathscr{D}$ we define its p-diameter by the formula

$$\operatorname{diam}_p(A) = \sup_{f,g \in A} d_p(f,g) .$$

For a pair $A \subset B$ in $\mathscr{D}$ we define a relative p-distance by the formula

$$d_p(f, g|B) = \inf l_p(\{\eta_t\})$$

where the infimum is taken over all paths $\eta_t \in B$, $t \in [0,1]$, joining $\eta_0 = f$, $\eta_1 = g \in A$. The relative p-diameter of A in B is now defined by the formula

$$\operatorname{diam}(A|B) = \sup_{f,g \in A} d_p(f, g|B) .$$

Note that

$$\operatorname{diam}_p(A|B) \geq c_p \operatorname{diam}_1(A|B) .$$

In this paper we are proving infiniteness of diameters of certain subgroups of $\mathscr{D}$. According to the inequality above the strongest results are with respect to diam_1. From now on we will consider only this case and we will drop the index "1" from our notations.

2.2 Groups of symplectomorphisms

Let M be a compact exact symplectic manifold (necessarily with boundary) with the symplectic form $\omega = d\lambda$. Then $\mu = \omega^n$ is a volume form on M. Let us denote by $\mathscr{D}_\omega, \mathscr{D}_\omega \subset \mathscr{D}$, the identity component of the group of symplectomorphisms of M, i.e. diffeomorphisms $f : M \to M$ with $f^* \omega = \omega$.

A symplectomorphism $f \in \mathscr{D}_\omega$ is called *exact* if the closed 1-form $\lambda_f = f^* \lambda - \lambda$ is exact. The exactness of λ_f is independent on the choice of λ with $d\lambda = \omega$. Let $\mathscr{D}_\omega^e$ be the subgroup of $\mathscr{D}_\omega$ which consists of exact symplectomorphisms. Note that $\mathscr{D}_\omega^e$ is path connected (see 3.5.2 below). By $\mathscr{D}_{\omega,\partial}^e$ we will denote the subgroup of $\mathscr{D}_\omega^e$ which consists of diffeomorphisms which are fixed near the boundary ∂M.

Calabi in [Ca] defined an epimorphism

$$\operatorname{Cal} : \mathscr{D}_{\omega,\partial}^e \to \mathbb{R} ;$$

see 5.1 below.

The kernel of this homomorphism which we will denote by $\mathscr{D}_{\omega,\partial}^{e,0}$ consists of symplectomorphisms with *zero Calabi invariant*. According to Banyaga (see [B]) $\mathscr{D}_{\omega,\partial}^{e,0}$ is the commutator $[\mathscr{D}_{\omega,\partial}^e, \mathscr{D}_{\omega,\partial}^e]$ and it is a simple group.

Theorem 2.2.1.

$$\operatorname{diam}(\mathscr{D}_\omega) = +\infty$$

Theorem 1 is a direct corollary of the following more precise result:

Theorem 2.2.2.

$$\operatorname{diam}(\mathscr{D}_{\omega,\partial}^{e,0}|\mathscr{D}_\omega) = +\infty$$

3. Reduction to the exact case

3.1 Exact symplectomorphisms

Let $(M, \omega = d\lambda)$ be an exact compact symplectic manifold. A vector field X on M is called *symplectic* if its dual 1-form $\theta = X \rfloor \omega$ is closed which is equivalent to $\mathscr{L}_X \omega = 0$. If θ is exact then X is called *Hamiltonian*. If $\eta_t, t \in [0, 1]$, is a path in $\mathscr{D}_\omega$ then for any $t \in [0, 1]$ the vector field $X_t(x) = \dfrac{d\eta_t(\eta_t^{-1}(x))}{dt}$ is symplectic. All paths in $\mathscr{D}_\omega$ which we will consider in this paper, will start at the identity and we will usually assume that without mentioning it. We will call the path $\eta_t, t \in [0, 1]$, *Hamiltonian* if the field X_t is Hamiltonian for all $t \in [0, 1]$. The function $H_t, t \in [0, 1]$, with $dH_t = \theta_t$ is called, in this case, the (time-dependent) Hamiltonian which generates $\eta_t, t \in [0, 1]$. The function H_t is determined up to adding a constant C_t.

The following standard fact relates the notions of exact symplectomorphisms and Hamiltonian paths.

Lemma 3.1.1. *The path $\eta_t, t \in [0, 1]$ is Hamiltonian if and only if it consists of exact symplectomorphisms.*
Indeed,

$$\eta_t^* \lambda - \lambda = \int_0^t \frac{d\eta_\tau^*(x)}{d\tau}\, d\tau = \int_0^t \eta_t^*(X_\tau \rfloor w)\, d\tau + d\int_0^t \eta_t^*(\lambda(x_\tau))\, d\tau .$$

Thus $\eta_t^* \lambda - \lambda$ is exact for all $t \in [0, 1]$ if and only if the form $\theta_t = X_t \rfloor w$ is exact for all $t \in [0, 1]$. $\square$

3.2 The characteristic line bundle

The restriction $\omega|_{\partial M}$ has 1-dimensional kernel $K \subset T(\partial M)$ which is called the *characteristic line bundle* of ∂M. A symplectic field X on M is tangent to ∂M if and only if the form $\theta = X \rfloor \omega$ vanishes on K. In this case X integrates to a flow $X^t : M \to M, t \in \mathbb{R}$, such that $X^t(\partial M) = \partial M, t \in \mathbb{R}$. Let us denote by $H_K^1(M; \mathbb{R})$ the subspace of $H^1(M; \mathbb{R})$ which consists of cohomology classes of closed 1-forms vanishing on K.

3.3 Special flows

For a closed 1-form α we denote by $[\alpha]$ its cohomology class.

Let us fix closed 1-forms $\theta_1, \ldots, \theta_N$ vanishing on K and such that $[\theta_1], \ldots, [\theta_N]$ form a basis of $H_K^1(M; \mathbb{R})$. Let ∂_k be the symplectic field dual to the forms θ_k, $k = 1, \ldots, N$, namely, $\theta_k = \partial_k \rfloor \omega$.

Take continuous functions $A_1, \ldots, A_N : [0, 1] \to \mathbb{R}$ with $A_k(0) = 0, k = 1, \ldots, N$ and define a vector field $A(t), t \in [0, 1]$, by the formula

$$A(t) = \sum_{k=1}^N A_k(t) \partial_k .$$

Let $A^u(t), u \in \mathbb{R}$, be the flow generated by $A(t)$, for a fixed $t \in [0, 1]$. Thus

$$\frac{dA^u(t)}{du} = A(t) \circ A^u(t) .$$

Lemma 3.3.1.

$$[(A^u(t))^* \lambda - \lambda] = u \sum_{k=1}^{N} A_k(t)[\theta_k].$$

Proof. We have (comp. 3.1.1):

$$[(A^u(t))^* \lambda - \lambda] = \int_0^u [(A^u(\tau))^* (A(\tau)\rfloor \omega)] d\tau$$

$$= \int_0^u [A(\tau)\rfloor \omega] d\tau = u[A(\tau)\rfloor \omega]$$

$$= u \sum_{k=1}^{N} A_k(t)[\theta_k]. \quad \square$$

The following lemma 3.3.2 immediately follows from compactness arguments.

Lemma 3.3.2. *Let* $D = \max(|A_k(t)|), k = 1, \ldots, N, t \in [0,1]$. *Then there exists a* $C = C(D)$ *such that* $\|dA^u(t)(x)\| \leq C(D)$ *for any* $t \in [0,1], x \in M, |u| \leq 1$.

Lemma 3.3.3. *Suppose that* $\int_0^1 \left| \frac{dA_k(t)}{dt} \right| dt \leq D'$. *Then there exists a constant* $C' = C'(D')$ *such that* $l(\{A^u(t)\}) \leq C'(D')$ *for* $|u| \leq 1$.

Proof. Let us embed the compact manifold with boundary M isometrically in an Euclidean space. From

$$A^u(t)(x) - x = \int_0^u \frac{d}{ds} (A^s(t)(x)) ds$$

$$= \int_0^u A(t)(A^s(t)(x)) ds$$

we conclude that

$$\frac{dA^u(t)(x)}{dt} = \int_0^u A'(t)(A^s(t)(x)) ds$$

$$+ \int_0^u dA(t)(A^s(t)(x)) \left(\frac{dA^s(t)(x)}{dt} \right) ds$$

and hence

$$\int_0^1 \left\| \frac{dA^u(t)(x)}{dt} \right\| dt \leq \int_0^1 \int_0^u \|A'(t)(A^s(t)(x)\| ds dt$$

$$+ \int_0^1 \int_0^u \|dA(t)(A^s(t)(x)\| \left\| \frac{dA^s(t)(x)}{dt} \right\| ds dt.$$

However, since $A(t) = \sum_{k=1}^{N} A_k(t) \partial_k$, it follows that $\|A'(t)(y)\| \leq C_1 \max_{k=1,\ldots,N} |A_k'(t)|$, for any $y \in M$, where the constant C_1 depends only on M and the vector fields ∂_k, $k = 1, \ldots, N$. Thus the first term in the previous inequality is bounded above by

$$C_1 \max_{k=1,\ldots,N} \int_0^1 |A_k'(t)| dt \leq C_1 D'.$$

By compactness of M, $\|dA(t)(y)\| \leq C_2 \max\limits_{k=1,\dots,N} |A_k(t)|$, for all $y \in M$, where the constant C_2 depends only on the vector fields $\partial_k, k = 1, \dots, N$. But

$$|A_k(t)| \leq \int_0^1 |A_k'(s)|ds \leq D', \quad \text{i.e.}$$

for all $t \in [0, 1]$ and all $k = 1, \dots, N$. Therefore,

$$\|dA(t)(y)\| \leq C_2 D'.$$

Thus we have

$$\int_0^1 \left\| \frac{dA^u(t)(x)}{dt} \right\| dt \leq C_1 D' + C_2 D' \int_0^u \left[\int_0^1 \left\| \frac{dA^s(t)(x)}{dt} \right\| dt \right] ds.$$

By Gronwall's inequality we conclude

$$\int_0^1 \left\| \frac{dA^u(t)(x)}{dt} \right\| dt \leq C_1 D' e^{C_2 D'} = \tilde{C}(D').$$

Therefore

$$l(\{A^u(t)\}) = \int_0^1 \int_M \left\| \frac{dA^u(t)(x)}{dt} \right\| \mu(x) dt \leq C'(D'). \quad \square$$

Remark 3.3.4. The functions $C(D)$ and $C'(D)$ in 3.3.2 and 3.3.3 can obviously be chosen monotonically increasing with D.

3.4. Estimates on the length of a non-Hamiltonian path

Let $\eta_t, t \in [0, 1]$, be a path in $\mathscr{D}_\omega$ (starting at $\eta_0 = id$, see 3.1). Let $X_t(x) = \dfrac{d\eta_t}{dt}(\eta_t^{-1}(x))$ and α_t be the closed 1-form ω-dual to $X_t, t \in [0, 1]$.

Lemma 3.4.1. *The cohomology classes of the 1-forms $\eta_t^* \lambda - \lambda$ and $\int_0^t \alpha_s ds$ coincide. In particular, $[\eta_t^* \lambda - \lambda] \in H_K^1(M)$.*

Proof.

$$[\eta_t^* \lambda - \lambda] = \int_0^t \eta_s^* [\alpha_s + d(\lambda(X_s))] ds = \int_0^t [\alpha_s] ds.$$

However, α_t annihilates the characteristic line bundle K since X_t is necessarily tangent to ∂M. Therefore

$$\int_0^t [\alpha_s] ds \in H_K^1(M). \quad \square$$

Let η_t, X_t and α_t be as above. Let $[\theta_1], \dots [\theta_N]$ be the basis of $H_K^1(M)$ considered in 3.3. Then $[\eta_t^* \lambda - \lambda] = \sum_1^N A_k(t)[\theta_k]$.

Lemma 3.4.2.

$$l(\{\eta_t\}) \geq Q \max_k \int_0^1 |A_k'(t)|dt.$$

The following lemma 3.4.3 is a key ingredient in the proof of 3.4.2.

Lemma 3.4.3. *Let $S \subset M$ be an embedded circle and Γ, $S \subset \Gamma \subset M$, be its tubular neighborhood. Then for any closed 1-form θ on M we have*

$$C \left| \int_C \theta \right| \leqq \int_\Gamma \|\theta\| \mu$$

for a constant C independent on θ.

Proof. Let $\chi : S \times \mathscr{D}^{2n-1} \to \Gamma$ be a decomposition of the tubular neighborhood Γ and $\dfrac{\partial}{\partial t}$ be a unit tangent vector field along S^1. Then

$$\int_\Gamma \|\theta\| \mu \geqq C_1 \int_\Gamma \left| \theta \left(d_\chi \left(\frac{\partial}{\partial t} \right) \right) \right| \mu \geqq C_1' \int_{\mathscr{D}^{2n-1} \times S^1} \left| \chi^*(\theta) \left(\frac{\partial}{\partial t} \right) \right| d^{2n-t} x \, dt$$

$$\geqq C_1' \int_{\mathscr{D}^{2n-1}} \left| \int_{x \times S^1} \chi^*(\theta) \left(\frac{\partial}{\partial t} \right) dt \right| d^{2n-1} x$$

$$= C_1' \int_{\mathscr{D}^{2n-1}} \left| \int_S \theta \right| d^{2n-1} x = C \left| \int_S \theta \right|. \quad \square$$

Proof of 3.4.2. Let us choose disjoint embedded circles $S_1, \ldots, S_l \subset M$ such that their homology classes $[S_1], \ldots [S_l]$ form a basis of $H_1(M)$. Choose disjoint tubular neighborhoods $\Gamma_i \supset S_i, i = 1, \ldots, l$. Let us complete the basis $[\theta_1], \ldots [\theta_N]$ of $H_K^1(M) \subset H^1(M)$ to a basis $[\theta_1], \ldots, [\theta_N], \ldots, [\theta_l]$ of $H^1(M)$. Let $B = (b_{ij})$ be a $(l \times l)$-matrix $b_{ij} = \int_{S_j} \theta_i$. Then B is nondegenerate and, therefore, $\|By\| \geqq b \|y\|$ for $y \in \mathbb{R}^l, b > 0$. We have

$$l(\{\eta_t\}) = \int_0^1 \int_M \|X_t(\eta_t(x))\| \mu \, dt$$

$$\geqq \int_0^1 \sum_{k=1}^l \int_{\Gamma_k} \|\alpha_t(x)\| \mu \, dt .$$

Applying 3.4.3 and 3.4.1 we get

$$l(\{\eta_t\}) \geqq C_1 \int_0^1 \sum_{k=1}^l \left| \int_{S_k} \alpha_t \right| dt$$

$$= C_1 \int_0^1 \sum_{k=1}^l \left| \sum_{j=1}^N A_j'(t) \int_{S_k} \theta_j \right| dt$$

$$= C_1 \int_0^1 \sum_{k=1}^l \left| \sum_{j=1}^N b_{jk} A_j'(t) \right| dt \geqq C_2 \int_0^1 \|By\| dt$$

where $y = (A_1'(t), \ldots, A_N'(t), 0, \ldots, 0) \in \mathbb{R}^l$. Therefore

$$l(\{\eta_t\}) \geqq C_2 b \int_0^1 \|y\| dt \geqq Q \max_k \int_0^1 |A_k'(t)| dt . \quad \square$$

3.5 Reduction to the exact case

In this section we continue to use the notations which were introduced above. In particular, η_t, $t \in [0,1]$, is a path in $\mathcal{D}_\omega$, X_t is the corresponding symplectic vector field, α_t is ω-dual to X_t, $t \in [0,1]$. We denote by $[\theta_1]...[\theta_N]$ a basis in $H^1_K(M)$ and by $\partial_1,...,\partial_N$ symplectic fields dual to $\theta_1,...,\theta_N$. We define functions $A_1,...,A_N : [0,1] \to \mathbb{R}$ by the formula $[\eta_t^* \lambda - \lambda] = \sum_{k=1}^{N} A_k(t)[\theta_k]$. We keep the notation $A(t)$ for the vector field $\sum_{k=1}^{N} A_k(t)\partial_k$, $t \in [0,1]$, and denote by $A^u(t)$, $u \in \mathbb{R}$, the flow generated by $A(t)$.

Lemma 3.5.1. *Let* $D = \max_k \int_0^1 |A_k'(t)|dt$. *Suppose that* η_1 *is an exact symplectomorphism. Let* $\psi_t = A^{-1}(t) \circ \eta_t$, $t \in [0,1]$. *Then the path* $\{\psi_t\}$ *is Hamiltonian,* $\psi_1 = \eta_1$ *and*

$$l(\{\psi_t\}) \leqq C(D)l(\{\eta_t\}) + C'(D)$$

for increasing functions $C(D)$ *and* $C'(D)$.

Proof. First note that $[\psi_t^* \lambda - \lambda] = [(A^{-1}(t))^* \lambda - \lambda] + [\eta_t^* \lambda - \lambda] = 0$ according to 3.3.1. Therefore, $\{\psi_t\}$ is Hamiltonian by 3.1.1. Now we have

$$l(\{\psi_t\}) = \int_0^1 \int_M \left\| \frac{d\psi_t(x)}{dt} \right\| \mu dt$$

$$\leqq \int_0^1 \int_M \left(\left\| \frac{dA^{-1}(t)}{dt}(\eta_t(x)) \right\| + \| dA^{-1}(t)(X_t(\eta_t(x))) \| \right) \mu dt$$

$$= l(\{A^{-1}(t)\}) + \int_0^1 \int_M \| dA^{-1}(t)(X_t(x)) \| \mu dt.$$

Applying 3.3.2 and 3.3.3, we get

$$l(\{\psi_t\}) \leqq C'(D) + C(D)l(\{\eta_t\}).$$

Finally note that $A(1) = 0$ because η_1 is exact. Therefore $\psi_1 = \eta_1$. $\square$

The following lemma 3.5.2 is the main result of this section.

Lemma 3.5.2. *Let* $\mathcal{D}_\omega^e$ *be the subgroup of* $\mathcal{D}_\omega$ *which consists of exact xmplectomorphisms* $M \to M$. *Then there exists an increasing function* $\varphi : \mathbb{R}_+ \to \mathbb{R}$ *such that* $d(id, f|\mathcal{D}_\omega^e) \leqq \varphi(d(id, f|\mathcal{D}_\omega))$ *for any* $f \in \mathcal{D}_\omega$.

Proof. Take a diffeomorphism $f = \eta_1 \in \mathcal{D}_\omega^e$. Let $L = d(id, f|\mathcal{D}_\omega)$. Then η_1 can be joined with the identity by a path $\eta_t \in \mathcal{D}_\omega$ with $l(\{\eta_t\}) < 2L$. Applying the construction from 3.5.1 we can find a Hamiltonian path ψ_t, $t \in [0,1]$, with $\psi_1 = \eta_1$ such that

$$l(\{\psi_t\}) \leqq C(D)l\{\eta_t\} + C'(D) \leqq 2C(D)L + C'(D).$$

But Lemma 3.4.2 implies that

$$D \leqq \frac{1}{Q} l\{\eta_t\} \leqq \frac{2L}{Q}.$$

Therefore,

$$l(\{\psi_t\}) \leqq 2C\left(\frac{2L}{Q}\right)L + C'\left(\frac{2L}{Q}\right) = \varphi(L)$$

or

$$d(id, f|\mathcal{D}_\omega^e) \leqq \varphi(d(id, f|\mathcal{D}_\omega^e)). \quad \square$$

Remark 3.5.3. Lemma 3.5.1 implies, in particular, that the group $\mathcal{D}_\omega^e$ is path-connected (see Section 2).

4. An inequality

The goal of this section is the following inequality 4.1.

Proposition 4.1. *Let μ be a compact Riemannian manifold (with or without boundary) with volume form μ, $H : M \to \mathbb{R}$ be a smooth function. Then*

$$\int_M \int_M |H(x) - H(y)|\mu_x \mu_y \leqq C \int_M \|\nabla H\|\mu.$$

Proof. Let us start with the case when M is a ball B of radius 1 in $\mathbb{R}^n$. Fix any nonzero vector $z \in \mathbb{R}^n$. Let us denote by $\Pi_z : \mathbb{R}^n \to \mathbb{R}^n$ a parallel translation $\Pi_z(x) = x - z$. Let $B_z = B \cap \Pi_z(B)$ and D_z be the image of B_z under the orthogonal projection $\pi : B \to L$ onto the hyperplane $L \subset \mathbb{R}^n$ which is perpendicular to z. For a point $u \in D_z$ we denote by l_u a fiber $\pi^{-1}(u) \subset B$ over the point $u \in D_z$. Let $l'_u = l_u \cap B_z$. So a point $y \in B_z \subset B$ has coordinates $u \in D_z, t \in l_u$. Then we have

$$\int_{B_z} |H(y+z) - H(y)|d^n y = \int_{D_z} \int_{l'_u} |H(y+z) - H(y)|dt\, d^{n-1}u$$

$$\leqq \int_{D_z} \int_{l'_u} \int_{l_u} \left|\frac{dH}{d\tau}(\tau, u)\right| d\tau dt\, d^{n-1}u$$

$$\leqq \int_{D_z} \int_{l'_u} \int_{l_u} \|\nabla H(\tau, u)\| d\tau dt\, d^{n-1}u$$

$$\leqq 2 \int_{D_z} \int_{l_u} \|\nabla H(\tau, u)\| d\tau\, d^{n-1}u$$

$$= 2 \int_B \|\nabla H(y)\| d^n y$$

Therefore,

$$\int_B \int_B |H(x) - H(y)|d^n x\, d^n y$$

$$\int_B \int_{B_z} |H(y+z) - H(y)|d^n y\, d^n z$$

$$\leqq \int_B \int_{B_z} \|\nabla H(y)\| d^n y\, d^n z \leqq 2\,\mathrm{vol}\,(B) \int_B \|\nabla H(y)\| d^n y$$

Now consider the general case. Take embeddings $f_1, \ldots, f_N : B \to M$ such that $M \times M = \bigcup_1^N f_i(B) \times f_i(B)$. Then

$$\int_M \int_M |H(x) - H(y)| \mu_x \mu_y \leqq \sum_1^N \int_{f_i(B)} \int_{f_i(B)} |H(x) - H(y)| \mu_x \mu_y$$

$$\leqq C_1 \sum_1^N C \int_B \int_B |H(f_i(u)) - H(f_i(v))| d^n u \, d^n v$$

$$\leqq C_2 \sum_1^N \int_B \|\nabla(H \circ f_i)(u)\| d^n u$$

$$\leqq C_3 \sum_1^N \int_{f_i(B)} \|\nabla H\| \mu \leqq C_4 \sum_1^N \int_M \|\nabla H\| \mu$$

$$= C \int_M \|\nabla H\| \mu . \quad \square$$

5. Exact case

Let now $(M, \omega = d\lambda)$ be a compact exact symplectic manifold.

5.1 Calabi invariant

Let $\eta \in \mathscr{D}^e_{\omega,\partial}$ be an exact symplectomorphism which is fixed near ∂M. Then the form $\lambda_\eta = \eta^* \lambda - \lambda$ is exact and is 0 near ∂M. Let $h : M \to \mathbb{R}$ be a function with $h|_{\partial M} = 0$, $dh = \lambda_\eta$. Then the number $\int_M h \mu$ does not depend on the choice of the primitive λ of ω (see [C] and [B]) and is called the *Calabi invariant* of η;

$$\mathrm{Cal}(\eta) = \int_M h \mu .$$

The function Cal defines an epimorphism of the group $\mathscr{D}^e_{\omega,\partial}$ onto $\mathbb{R}$.

Let $\mathrm{Fix}(\eta) \subset M$ be the set of fixed points of η and let $U \subset \mathrm{Fix}(\eta)$ be the component of $\mathrm{Fix}(\eta)$ which contains the boundary ∂M. Fix a point $x \in U$ and for any $y \in M$ let us choose a path γ_{xy} which starts at x and ends at y. Then

$$h(y) = \int_{\eta(\gamma_{xy})} \lambda - \int_{\gamma_{xy}} \lambda$$

and, therefore,

$$\mathrm{Cal}(\eta) = \int_M \left(\int_{\eta(\gamma_{xy})} \lambda - \int_{\gamma_{xy}} \lambda \right) \mu .$$

5.2 An identity

Let $\eta_t, t \in [0,1]$ be a Hamiltonian path in $\mathscr{D}_\omega$ with $\eta_0 = id, \eta_1 = \eta$ and let $H_t : M \to \mathbb{R}, t \in [0,1]$ be its generating Hamiltonian, $dH_t = X_t \rfloor \omega$.

Lemma 5.2.1. *Let γ be a path in M joining two points $x, y \in M$. Then*

$$\int_{\eta(\gamma)} \lambda - \int_\gamma \lambda = \int_0^1 \lambda(X_t(\eta_t(y))) dt - \int_0^1 \lambda(X_t(\eta_t(x))) dt + \int_0^1 (H_t(\eta_t(y)) - H_t(\eta_t(x))) dt .$$

Proof. Let $K = [0, 1] \times [0, 1] \subset \mathbb{R}^2$. Consider a map $S : K \to M$ defined by the formula $S(u, t) = \eta_t(\gamma(u))$, then

$$\int_K S^*w = \int_{\partial K} S^*\lambda = \int_\gamma \lambda + \int_{\{\eta_t(y)\}} \lambda - \int_{\eta(\gamma)} \lambda - \int_{\{\eta_t(x)\}} \lambda,$$

but

$$\int_K S^*w = \int_K w\left(d\eta_t\left(\frac{d\gamma(u)}{du}\right), X_t(\eta_t(\gamma(u)))\right) du\,dt$$

$$= -\int_K d(H_t \circ \eta_t)(\gamma(u))\left(\frac{d\gamma(u)}{du}\right) du\,dt$$

$$= -\int_0^1 (H_t(\eta_t(y))) - H_t(\eta_t(x))\,dt.$$

Therefore,

$$\int_{\eta(\gamma)} \lambda - \int_\gamma \lambda = \int_0^1 \lambda(X_t(\eta_t(y))\,dt - \int_0^1 \lambda(X_t(\eta_t(X))\,dt + \int_0^1 (H_t(\eta_t(y)) - H_t(\eta_t(x)))\,dt.$$

$\square$

5.3 Exact symplectomorphisms and Calabi invariant

Let us fix an embedded ball $B \subset \text{Int } M$ and a smaller ball $B_0 \subset \text{Int } B$. Let $A = B\backslash\text{Int } B_0$. We will call a symplectomorphism $f : M \to M$ *special* if $f(B) = B$ and $f|_A = id$. For a diffeomorphism $h : B \to B$ which is fixed near ∂B we denote as in 2.2 and 5.1 by $\text{Cal}\,(h)$ its Calabi invariant.

Proposition 5.3.1. *Let η_t, $t \in [0, 1]$, be a Hamiltonian path in $\mathcal{D}_\omega$ such that $\eta = \eta_1$ is special. Then $l(\{\eta_t\}) > K\,\text{Cal}\,(\eta|_B)$ where the constant K is independent on the choice of path $\{\eta_t\}$.*

Proof. Let us fix a point $x \in A$ and for any $x \in B$ let us choose a path γ_{xy} joining the points $x, y \in B$ we have

$$\text{Cal}\,(\eta|_B) = \int_B \left(\int_{\eta(\gamma_{xy})} \lambda - \int_{\gamma_{xy}} \lambda\right) \mu_y.$$

Applying now 5.2 we get

$$\text{Cal}\,(\eta|_B) = \int_B \left[\int_0^1 \lambda(X_t(\eta_t(y)))\,dt - \int_0^1 \lambda(X_t(\eta_t(x)))\,dt \right.$$

$$\left. + \int_0^1 (H_t(\eta_t(y)) - H_t(\eta_t(x))\,dt\right] \mu_y.$$

Integrating both parts of the inequality over A we get

$$\text{Vol}\,(A)\,\text{Cal}\,(\eta|_B) = \text{Vol}\,(A) \int_B \int_0^1 \lambda(X_t(\eta_t(y))\,dt\,\mu_x$$

$$- \text{Vol}\,(B) \int_A \int_0^1 \lambda(X_t(\eta_t(x))\,dt\,\mu_x$$

$$+ \int_0^1 \int_A \int_B (H_t(\eta_t(y) - H_t(\eta_t(x))\,dt\,\mu_y\mu_x.$$

Thus

$$\text{Vol}(A)|\text{Cal}(\eta|_B)| \leqq C\left[\text{Vol}(A)\int_0^1\int_M \|X_t\|dt\mu + \text{Vol}(B)\int_0^1\int_M \|X_t\|dt\mu\right]$$

$$+\int_0^1\int_M\int_M |H_t(\eta_t(x)) - H_t(\eta_t(y)|\mu_x\mu_y$$

$$= C(\text{Vol}(A) + \text{Vol}(B))l(\{\eta_t\})$$

$$+\int_0^1\int_M\int_M |H_t(x) - H_t(y)|\mu_x\mu_y.$$

Now we apply the inequality 4.1 and get

$$|\text{Cal}(\eta|_B)| \leqq C_1 l(\{\eta_t\}) + C_2\int_0^1\int_M \|\nabla H_t\|\mu dt = (C_1 + C_2)l(\{\eta_t\}). \quad \square$$

6. Proof of Theorem 2.2.2

We start with the following simple lemma.

Lemma 6.1. *Let $B \subset M$ be an embedded ball as in 5.2. Let $h : B \to B$ be a symplecto-morphism fixed near the boundary ∂B and isotopic to the identity. Then h can be extended to a symplectomorphism $g : M \to M$ fixed near ∂M, isotopic to the identity and such that.*

$$\text{Cal}(g) = 0.$$

Proof. Let us choose a ball $B_1 \subset \text{Int}(M\backslash B)$. Take a symplectomorphism $h_1 : B_1 \to B_1$ fixed near ∂B_1 which is isotopic to the identity and such that $\text{Cal}(h_1) = -\text{Cal}(h)$. The required symplectomorphism $g : M \to M$ is equal to h on B, h_1 on B_1 and is the identity on $M\backslash(B_1 \cup B)$. $\quad \square$

Recall that $\mathcal{D}_\omega^e$ denotes the group of exact symplectomorphisms.

Proposition 6.2.
$$\text{diam}(\mathcal{D}_{\omega,\partial}^{e,0}|\mathcal{D}_\omega^e) = +\infty.$$

Proof. The Calabi homomorphism is surjective. Therefore, for any $N > 0$ there exists a symplectomorphism $h_N : B \to B$ with $\text{Cal}(h_N) > N$ which is isotopic to the identity. (The required h_N can be, for example, constructed as the time-one map of the Hamiltonian flow generated by a Hamiltonian $H : B \to \mathbb{R}$ with $H = 0$ near ∂B and $\int_B H\mu > N$.) Extend h_N using 6.1 to a symplectomorphism $g_N \in \mathcal{D}_{\omega,\partial}$. Take a Hamiltonian path $\eta_t \in \mathcal{D}_\omega^e, t \in [0,1]$ with $\eta_0 = id, \eta_1 = g_N$. According to 5.3.1 $l(\{\eta_t\}) > KN$. Hence $d(id, g_N) = \inf_{\{\eta_t\}}l(\{\eta_t\}) \geqq KN$. Therefore, $\text{diam}(\mathcal{D}_{\omega,\partial}^{e,0}|\mathcal{D}_\omega^e) = +\infty$. $\quad \square$

6.3 Proof of Theorem 2.2

Apply 6.2 and 3.5.2. $\quad \square$

Appendix

As we mentioned in the Introduction, the diameter of the group of the volume preserving diffeomorphisms of a manifold is finite when the manifold is simply connected (see [Sh]). We will explain in this section that in the non-simply-connected case the diameter can be infinite.

Theorem A.1. *Let M be a manifold with a volume form μ and $\mathscr{D}$ be (the identity component of) the group of volume preserving diffeomorphisms of M. Suppose that $H^1(M;\mathbb{R}) \neq 0$ and $\pi_1(M)$ has trivial center. Then*

$$\operatorname{diam}(\mathscr{D}) = \infty .$$

Proof. Let a 1-form a represent a nontrivial cohomology class from $H^1(M;\mathbb{R})$. Let g be the Lie algebra of divergent free vector fields. The formula $\mathscr{A}(X) = \int_M a(X)\mu$, $X \in g$, defines $\mathscr{A}$ as a 1-form on $\mathscr{D}$. A straightforward computation shows that this is a closed 1-form (comp. [Fa]) i.e., for any contractible loop η in $\mathscr{D}$ we have $\int_\eta \mathscr{A} = 0$. In fact this form is exact if the evaluation homomorphism $\pi_1(\mathscr{D}) \to H_1(M;\mathbb{R})$ is trivial (see [McD]). For example, this is so if the center of $\pi_1(M)$ is trivial. Then the formula

$$\mathscr{F}(x) = \int_{\eta_x} \mathscr{A} ,$$

where $x \in \mathscr{D}$ and η_x is a path connecting id with x defines a functional on $\mathscr{D}$ which is a primitive of $\mathscr{A}$. This is the so-called *mass flow* homomorphism (see [Fa]). It is easy to see that $\mathscr{F}$ is not trivial and, therefore, takes arbitrarily large values.

Now note that

$$|\mathscr{F}(x)| = \left| \int_{\eta_x} \mathscr{A} \right| = \left| \int_0^1 \left(\int_M a\left(\frac{d\eta_x(t)}{dt} \right) \mu \right) dt \right|$$

$$\leq C \int_0^1 \int_M \left\| \frac{d\eta_x(t)}{dt} \right\| \mu \, dt \leq \tilde{C} l(\eta_x)$$

where $l(\eta_x)$ is the length of the path η_x for the metric we consider in this paper (see (2-2)).

Hence,

$$d(id, x) \geq \tilde{C} |\mathscr{F}(x)| ,$$

Therefore, $d(id, x)$ takes arbitrarily large values together with $\mathscr{F}(x)$ which means that $\operatorname{diam}(\mathscr{D}) = \infty$. $\quad\square$

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