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Geometric representation theory for unitary groups of operator algebras

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Abstract

Geometric realizations for the restrictions of GNS representations to unitary groups of C^*-algebras are constructed. These geometric realizations use an appropriate concept of reproducing kernels on vector bundles. To build such realizations in spaces of holomorphic sections, a class of complex coadjoint orbits of the corresponding real Banach–Lie groups is described and some homogeneous holomorphic Hermitian vector bundles that are naturally associated with the coadjoint orbits are constructed.

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1. Introduction

The study of geometric properties of state spaces is a basic topic in the theory of operator algebras (see, e.g., [2] and [3]). The GNS construction produces representations of operator algebras out of states. From this point of view, we think it interesting to investigate the geometry behind these representations.

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One method to do this is to proceed as in the theory of geometric realizations of Lie group representations (see, e.g., [22,24,25,31]) and to try to build the representation spaces as spaces of sections of certain vector bundles. The basic ingredient in this construction is the reproducing kernel Hilbert space (see, e.g., [6,7,12,27,31,36,38]).

In the present paper we show that the aforementioned method can indeed be applied to the case of group representations obtained by restricting GNS representations to unitary groups of C^*-algebras. More precisely, for these representations, we construct one-to-one intertwining operators from the representation spaces onto reproducing kernel Hilbert spaces of sections of certain Hermitian vector bundles (Theorem 5.4). The construction of these vector bundles is based on a choice of a sub-C^*-algebra that is related in a suitable way to the state involved in the GNS construction (see Construction 3.1).

It turns out that, in the case of normal states of W^*-algebras, there is a natural choice of the subalgebra (namely the centralizer subalgebra), and the base of the corresponding vector bundle is just one of the symplectic leaves studied in our previous paper [11]. Since the corresponding symplectic leaves are just unitary orbits of states, the geometric representation theory initiated in the present paper provides, in particular, a geometric interpretation of the result in [26], namely the equivalence class of an irreducible GNS representation only depends on the unitary orbit of the corresponding pure state.

In [17] and references therein one can find several interesting results regarding the classification of unitary group representations of various operator algebras. The point of the present paper is to show that some of these representations (namely the ones obtained by restricting GNS representations to unitary groups) can be realized geometrically following the pattern of the classical Borel–Weil theorem for compact groups.

This raises the challenging problem of finding geometric realizations of more general representations of unitary groups of operator algebras. Similar results for other classes of infinite-dimensional groups have been already obtained: see [23,30,42] for direct limit groups, and [15,32,33] for groups related to operator ideals. The same problem of geometric realizations for representations of the restricted unitary group was raised at the end of [16].

The structure of the paper is as follows. Since the reproducing kernels we need in the present paper show up most naturally in a C^*-algebraic setting (Construction 5.1), we establish in Section 2 the appropriate versions of a number of results in [11]. Section 3 gives a general construction of homogeneous Hermitian vector bundles associated with GNS representations. Section 4 is devoted to the concept of reproducing kernel suitable for the applications we have in mind. In Section 5 we construct such reproducing kernels out of GNS representations and we prove our main theorems on geometric realizations of GNS representations (Theorems 5.4 and 5.8).

2. Coadjoint orbits and C^*-algebras with finite traces

In this section we extend to a C^*-algebraic framework a number of results that were proved in [11] for symplectic leaves in preduals of W^*-algebras. We begin by establishing some notation that will be used throughout the paper.

Notation 2.1. For a unital C^*-algebra A with the unit $\mathbf{1}$ we shall use the following notation:

$$\{a\}' = \{b \in A \mid ab = ba\} \quad \text{whenever } a \in A,$$

$$
\begin{aligned}
&A^{\varphi}=\big\{a\in A\mid (\forall b\in A)\ \varphi(ab)=\varphi(ba)\big\}\quad\text{whenever } \varphi\in A^*,\\
&\mathrm{G}_A=\{g\in A\mid g \text{ invertible}\},\\
&\mathrm{U}_A=\{u\in A\mid uu^*=u^*u=\mathbf{1}\}\subseteq \mathrm{G}_A,\\
&A^{\mathrm{sa}}=\{a\in A\mid a=a^*\},\\
&(A^*)^{\mathrm{sa}}=\big\{\varphi\in A^*\mid (\forall a\in A)\ \varphi(a^*)=\overline{\varphi(a)}\,\big\}.
\end{aligned}
$$

In the special case of a W^*-algebra M we will also use the notation

$$
\begin{aligned}
&M_*=\{\varphi\in M^*\mid \varphi \text{ is } w^*\text{-continuous}\},\\
&M_*^{\mathrm{sa}}=M_*\cap (M^*)^{\mathrm{sa}}.
\end{aligned}
$$

Proposition 2.2. *Let A be a unital C^*-algebra having a faithful tracial state $\tau: A\to\mathbb{C}$. Consider the mapping*

$$\Theta^\tau: A\to A^*,\quad a\mapsto \Theta_a^\tau,$$

where for each $a\in A$ we define

$$\Theta_a^\tau: A\to\mathbb{C},\quad \Theta_a^\tau(b):=\tau(ab).$$

The mapping Θ^τ has the following properties:

(a) $\operatorname{Ker}\Theta^\tau=\{0\}$.
(b) $A^{\Theta_a^\tau}=\{a\}'$ *for all $a\in A$.*
(c) *The mapping Θ^τ is G_A-equivariant with respect to the adjoint action of G_A on A and the coadjoint action of G_A on A^*. In particular, the mapping*

$$\Theta^\tau|_{A^{\mathrm{sa}}}: A^{\mathrm{sa}}\to (A^*)^{\mathrm{sa}}$$

is U_A-equivariant with respect to the adjoint action of U_A on A^{sa} and the coadjoint action of U_A on $(A^*)^{\mathrm{sa}}$.
(d) *For each $a\in A$ the mapping Θ^τ induces a bijection of the adjoint orbit $\mathrm{G}_A\cdot a$ onto the coadjoint orbit $\mathrm{G}_A\cdot\Theta_a^\tau$. In particular, if $a\in A^{\mathrm{sa}}$ and there exists a conditional expectation of A onto $\{a\}'$ then we have a commutative diagram of U_A-equivariant diffeomorphisms of Banach manifolds*

$$
\begin{tikzcd}
\mathrm{U}_A\cdot a \arrow[dr, "\Theta^\tau"] & \\
\mathrm{U}_A/\mathrm{U}_{\{a\}'} \arrow[u] \arrow[r] & \mathrm{U}_A\cdot\Theta_a^\tau.
\end{tikzcd}
$$

(e) *If, moreover, A is a W^*-algebra and the faithful tracial state τ is normal, then* $\operatorname{Ran}\Theta^\tau\subseteq A_*$ *and the hypothesis on conditional expectation from* (d) *holds for each $a\in A^{\mathrm{sa}}$.*

Proof. This is proved like Proposition 2.12 in [11]. □

Corollary 2.3. *Let A be a unital C^*-algebra having a faithful tracial state $\tau: A \to \mathbb{C}$ and let $a = a^* \in A$ be such that there exists a conditional expectation of A onto $\{a\}'$. Then the unitary orbit of a has a natural structure of U_A-homogeneous weakly symplectic manifold.*

Proof. Use Proposition 2.2 along with the reasoning that leads to Corollary 2.9 in [34]. □

Corollary 2.4. *In a W^*-algebra M that admits a faithful normal tracial state the unitary orbit of each self-adjoint element has a natural structure of U_M-homogeneous weakly symplectic manifold.*

Proof. Use Proposition 2.2 along with Corollary 2.9 in [11]. □

Proposition 2.5. *Let A be a unital C^*-algebra and $a = a^* \in A$ have the spectrum a finite set. Then the unitary orbit of a has a natural structure of U_A-homogeneous complex Banach manifold.*

Proof. Let $a = a^* \in A$ such that there exist finitely many different numbers $\lambda_1, \ldots, \lambda_m \in \mathbb{R}$ with

$$a = \lambda_1 e_1 + \cdots + \lambda_m e_m,$$

where $e_1, \ldots, e_m \in A$ are orthogonal projections satisfying $e_i e_j = 0$ whenever $i \neq j$ and $e_1 + \cdots + e_m = \mathbf{1}$. It is clear that, denoting $p_j = e_1 + \cdots + e_j$ for $j = 1, \ldots, m-1$, we have

$$\begin{aligned} \{a\}' &= \{b \in A \mid be_j = e_j b \text{ for } 1 \leqslant j \leqslant m\} \\ &= \{b \in A \mid bp_j = p_j b \text{ for } 1 \leqslant j \leqslant m-1\}. \end{aligned}$$

Thus both the unitary orbit of a and the unitary orbit of $(p_1, \ldots, p_{m-1}) \in A \times \cdots \times A$ can be identified with $\mathrm{U}_A/\mathrm{U}_{\{a\}'}$, and now the desired conclusion follows by Corollary 16 in [9] or Example 6.20 in [10]. (See Proposition 2.7 below for the special case of W^*-algebras.) □

Theorem 2.6. *If a unital C^*-algebra A possesses a faithful tracial state, then the unitary orbit of each self-adjoint element with finite spectrum has a natural structure of an U_A-homogeneous weakly Kähler manifold.*

Proof. The proof is similar to the one of Proposition 4.8 in [11]. □

We conclude this section by a result that will be needed in the proof of Theorem 5.8. We say that a unital W^*-algebra M is *finite* if for every $u \in M$ with $u^*u = \mathbf{1}$ we have $uu^* = \mathbf{1}$ as well. In this case each projection $p \in M$ is finite, in the sense that, if $v \in M$, $v^*v = p$ and vv^* is a projection smaller than p, then $vv^* = p$. Clearly a unital W^*-algebra is finite if it has a faithful tracial state. The factorization used in the proof of the next statement resembles the one developed in the case of finite nests in [35].

Proposition 2.7. *Let M be a finite W^*-algebra and $e_1, \dots, e_n \in M$ orthogonal projections satisfying $e_i e_j = 0$ whenever $i \neq j$ and $e_1 + \cdots + e_n = \mathbf{1}$. Next consider the sub-$W^*$-algebra*

$$M_0 = \{a \in M \mid ae_j = e_j a \text{ for } j = 1, \dots, m\}$$

of M, the subgroup $P = \{g \in \mathrm{G}_M \mid e_k g e_j = 0 \text{ if } 1 \leqslant j < k \leqslant n\}$ of G_M, and define the mapping

$$\psi : \mathrm{U}_M / \mathrm{U}_{M_0} \to \mathrm{G}_M / P, \quad u\mathrm{U}_{M_0} \mapsto uP.$$

Then ψ is a real analytic U_M-equivariant diffeomorphism.

Proof. The mapping ψ is clearly real analytic. Next note that $\mathrm{U}_M \cap P = \mathrm{U}_{M_0}$, hence ψ is injective. The fact that ψ is surjective can be equivalently expressed by the following assertion: For all $g \in \mathrm{G}_M$ there exist $u \in \mathrm{U}_M$ and $q \in P$ such that $g = uq$. In order to prove this fact, denote $p_0 = 0$ and $p_j = e_1 + \cdots + e_j$ for $j = 1, \dots, n$, and note that

$$P = \{g \in \mathrm{G}_M \mid gp_j = p_j g p_j \text{ for } j = 1, \dots, p_n\}.$$

We have

$$p_1 \leqslant p_2 \leqslant \cdots \leqslant p_n \quad \text{and} \quad \mathrm{l}(gp_1) \leqslant \mathrm{l}(gp_2) \leqslant \cdots \leqslant \mathrm{l}(gp_n).$$

Now let us fix $j \in \{1, \dots, n\}$. Recall that for each $b \in M$ one denotes by $\mathrm{l}(b)$ the smallest projection $p \in M$ satisfying $pb = b$. By Lemma 3.2(iii) in [8] we have $\mathrm{l}(gp_j) \sim p_j$ and $\mathrm{l}(gp_{j-1}) \sim p_{j-1}$. Since p_j is a finite projection, it then follows that $\mathrm{l}(gp_j) - \mathrm{l}(gp_{j-1}) \sim p_j - p_{j-1} = e_j$ (see, for instance, Exercise 6.9.8 in [28]). In other words, there exists $v_j \in M$ such that $v_j v_j^* = e_j$ and $v_j^* v_j = \mathrm{l}(gp_j) - \mathrm{l}(gp_{j-1})$.

Now, denoting

$$u^* = v_1 + \cdots + v_n \in M$$

it is easy to check that $u \in \mathrm{U}_M$ and $ue_j u^* = \mathrm{l}(gp_j) - \mathrm{l}(gp_{j-1})$ for $j = 1, \dots, n$. By summing up these equalities for $j = 1, \dots, k$, we get $u^{-1}\mathrm{l}(gp_k) = p_k u^*$ for each $k \in \{1, \dots, n\}$.

It then follows that for every k we have $\mathrm{l}(u^{-1}\mathrm{l}(gp_k)) = \mathrm{l}(p_k u^*)$, so $\mathrm{l}(u^{-1}gp_k) = p_k$ by Lemma 3.2(i) in [8]. Consequently $q := u^{-1}g \in P$, and we are done. □

3. Homogeneous vector bundles

The present section is devoted to a general construction of homogeneous vector bundles in a C^*-algebraic setting. (A discussion similar in spirit appears also in the paper [18].)

Construction 3.1. Let A be a unital C^*-algebra, B a unital sub-C^*-algebra of A and $\varphi : A \to \mathbb{C}$ a state such that there exists a conditional expectation $E : A \to B$ with $\varphi \circ E = \varphi$. By *conditional expectation* we mean that E is a bounded linear mapping such that $\|E\| = 1$ and E is idempotent, that is, $E^2 = E$. It then follows by the theorem of Tomiyama (see [40] or [37]) that E has the additional properties:

$$E(a^*) = E(a)^*, \tag{3.1}$$

$$0 \leqslant E(a)^* E(a) \leqslant E(a^* a), \tag{3.2}$$

$$E(b_1 a b_2) = b_1 E(a) b_2, \tag{3.3}$$

for all $a \in A$ and $b_1, b_2 \in B$.

Now let

$$\rho : A \to \mathcal{B}(\mathcal{H}) \quad \text{and} \quad \rho_\varphi : B \to \mathcal{B}(\mathcal{H}_\varphi)$$

be the GNS unital $*$-representations of A and B corresponding to φ and $\varphi|_B$, respectively. We recall that, for instance, $\mathcal{H}$ is the Hilbert space obtained from A by factoring out the null-space of the nonnegative definite Hermitian sesquilinear form

$$A \times A \ni (a_1, a_2) \mapsto \varphi(a_2^* a_1) \in \mathbb{C}$$

and then taking the completion, and for each $a \in A$ the operator $\rho(a) : \mathcal{H} \to \mathcal{H}$ is the one obtained from the linear mapping $a' \mapsto aa'$ on A. The Hilbert space $\mathcal{H}_\varphi$ is obtained similarly, using the restriction of the aforementioned sesquilinear form to B. It is easy to see that $\mathcal{H}_\varphi$ is a closed subspace of $\mathcal{H}$ and for all $b \in B$ we have a commutative diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "E"'] & \mathcal{H} \arrow[r, "\rho(b)"] \arrow[d, "P_{\mathcal{H}_\varphi}"] & \mathcal{H} \arrow[d, "P_{\mathcal{H}_\varphi}"] \\ B \arrow[r] & \mathcal{H}_\varphi \arrow[r, "\rho_\varphi(b)"] & \mathcal{H}_\varphi \end{tikzcd} \tag{3.4}$$

where $P_{\mathcal{H}_\varphi} : \mathcal{H} \to \mathcal{H}_\varphi$ denotes the orthogonal projection, while $A \to \mathcal{H}$ and $B \to \mathcal{H}_\varphi$ are the natural maps.

The fact that $E : A \to B$ is continuous with respect to the GNS scalar products on A and B (whence its "extension" by continuity to $P_{\mathcal{H}_\varphi}$) follows since for all $a \in A$ we have by inequality (3.2) that $E(a)^* E(a) \leqslant E(a^* a)$, hence $\varphi(E(a)^* E(a)) \leqslant \varphi(E(a^* a)) = \varphi(a^* a)$.

By restriction we get a norm-continuous unitary representation of the unitary group of B,

$$\rho_\varphi|_{\mathrm{U}_B} : \mathrm{U}_B \to \mathrm{U}_{\mathcal{B}(\mathcal{H}_\varphi)}.$$

Since U_B is a Lie group with the topology inherited from U_A and the self-adjoint mapping E gives a continuous projection of the Lie algebra of U_A onto the Lie algebra of U_B, it then follows by Proposition 11 in Chapter III, §1.6 in [14] that we have a principal U_B-bundle

$$\pi_B : \mathrm{U}_A \to \mathrm{U}_A/\mathrm{U}_B, \quad u \mapsto u \cdot \mathrm{U}_B.$$

Now we can construct the U_A-homogeneous vector bundle associated with π_B and $\rho|_{\mathrm{U}_B}$, which we denote by

$$\Pi_{\varphi,B} : D_{\varphi,B} \to \mathrm{U}_A/\mathrm{U}_B$$

(see, for instance, 6.5.1 in [13] for this construction). We recall that $D_{\varphi,B} = \mathrm{U}_A \times_{\mathrm{U}_B} \mathcal{H}_\varphi$, in the sense that $D_{\varphi,B}$ is the quotient of $\mathrm{U}_A \times \mathcal{H}_\varphi$ by the equivalence relation $\sim$ defined by

$$(u_1, f_1) \sim (u_2, f_2) \quad \iff \quad (\exists v \in \mathrm{U}_B) \quad u_1 = u_2 v \quad \text{and} \quad f_1 = \rho_\varphi\big(v^{-1}\big) f_2,$$

while the mapping $\Pi_{\varphi,B}$ takes the equivalence class $[(u, f)]$ of any pair (u, f) to $u \cdot \mathrm{U}_B$.

Definition 3.2. With the notations of Construction 3.1, the U_A-homogeneous vector bundle $\Pi_{\varphi,B} : D_{\varphi,B} \to \mathrm{U}_A/\mathrm{U}_B$ is called the *homogeneous bundle associated with φ and B*.

Remark 3.3. In the setting of Construction 3.1, there exist additional structures on the vector bundle $\Pi_{\varphi,B} : D_{\varphi,B} \to \mathrm{U}_A/\mathrm{U}_B$, which we discuss below.

(i) Denote by $\alpha_B : \mathrm{U}_A \times \mathrm{U}_A/\mathrm{U}_B \to \mathrm{U}_A/\mathrm{U}_B$ the natural action of U_A on $\mathrm{U}_A/\mathrm{U}_B$. Define the mapping

$$\beta_{\varphi,B} : \mathrm{U}_A \times D_{\varphi,B} \to D_{\varphi,B}$$

that takes the pair consisting of $u' \in \mathrm{U}_A$ and the equivalence class of $(u, f) \in \mathrm{U}_A \times \mathcal{H}_\varphi$ to the equivalence class of $(u'u, f)$. Then $\beta_{\varphi,B}$ is a real analytic action of U_A on $D_{\varphi,B}$ and the diagram

$$\begin{array}{ccc} \mathrm{U}_A \times D_{\varphi,B} & \xrightarrow{\beta_{\varphi,B}} & D_{\varphi,B} \\ {\scriptstyle \mathrm{id}_{\mathrm{U}_A} \times \Pi_{\varphi,B}} \downarrow & & \downarrow {\scriptstyle \Pi_{\varphi,B}} \\ \mathrm{U}_A \times \mathrm{U}_A/\mathrm{U}_B & \xrightarrow{\alpha_B} & \mathrm{U}_A/\mathrm{U}_B \end{array}$$

is commutative.

(ii) The bundle $\Pi_{\varphi,B}$ is actually a Hermitian vector bundle, in the sense that its fibers come equipped with structures of complex Hilbert spaces, and moreover for each $u \in \mathrm{U}_A$ the mapping $\beta_{\varphi,B}(u, \cdot) : D_{\varphi,B} \to D_{\varphi,B}$ is (bounded linear and) fiberwise unitary.

Example 3.4. Let A be a unital C^*-algebra and $\varphi : A \to \mathbb{C}$ a state with the corresponding GNS representation $\rho : A \to \mathcal{B}(\mathcal{H})$. We describe below the extreme situations of the unital sub-C^*-algebra B of A, when the hypothesis of Construction 3.1 is satisfied.

(i) For $B = A$ we can take $E = \mathrm{id}_A$. In this case the homogeneous vector bundle associated with φ and B is just the vector bundle with the base reduced to a single point and with the fiber equal to $\mathcal{H}$.

(ii) The other extreme situation is for $B = \mathbb{C}\mathbf{1}$, when we can take $E(\cdot) = \varphi(\cdot)\mathbf{1}$. Then, denoting $\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$, it follows that the homogeneous vector bundle associated with φ and B is a line bundle whose base is the Banach–Lie group $\mathrm{U}_A/\mathbb{T}\mathbf{1}$.

Example 3.5. If M is a W^*-algebra and $0 \leqslant \varphi \in M_*$, then there always exists a conditional expectation

$$E_\varphi : M \to M^\varphi \quad \text{with } \varphi \circ E_\varphi = \varphi.$$

We recall that the existence of E_φ follows by Theorem 4.2 in Chapter IX of [39] in the case when φ is faithful. In the general case, denote by $p = p^* = p^2 \in M$ the support of φ. Since in a W^*-algebra every element is a linear combination of unitary elements, it follows by Lemma 2.7 in [11] that, denoting $\varphi_p := \varphi|_{(pMp)} \in (pMp)_*$, we have

$$M^\varphi = \left\{a \in M \mid ap = pa \text{ and } pap \in (pMp)^{\varphi_p}\right\} = (pMp)^{\varphi_p} \oplus (\mathbf{1} - p)M(\mathbf{1} - p).$$

Now, since φ_p is faithful on pMp, it follows by the aforementioned theorem from [39] that there exists a conditional expectation $E_{\varphi_p} : pMp \to (pMp)^{\varphi_p}$ with $\varphi_p \circ E_{\varphi_p} = E_{\varphi_p}$. Then

$$E_\varphi : M \to M^\varphi, \quad a \mapsto E_{\varphi_p}(pap) + (\mathbf{1} - p)a(\mathbf{1} - p)$$

is a conditional expectation with $\varphi \circ E_\varphi = \varphi$ (see also [11, Remark 2.5]).

Example 3.6. Let A be a unital C^*-algebra, B a unital sub-C^*-algebra of A and $E : A \to B$ a conditional expectation as in Construction 3.1. If $\varphi_0 : B \to \mathbb{C}$ is a *pure* state of B, then it is easy to see that $\varphi := \varphi_0 \circ E : A \to \mathbb{C}$ is in turn a pure state of A provided it is the unique extension of φ_0 to A. Since pure states lead to irreducible representations, this easy remark can be viewed as a version of Theorem 2.5 in [12] asserting that (roughly speaking) irreducibility of the isotropy representation implies global irreducibility. (Compare also Theorem 5.4 below.)

We refer to the papers [4] and [5] for situations when, converse to the above remark, the existence of a unique conditional expectation from A onto B follows from the assumption that each pure state of B extends uniquely to a pure state of A.

4. Reproducing kernels

In what follows, by *continuous vector bundle* we mean a continuous mapping $\Pi : D \to T$, where D and T are topological spaces, the fibers $D_t := \Pi^{-1}(t)$ $(t \in T)$ are complex Banach spaces, and Π is locally trivial with fiberwise bounded linear trivializations. We say that Π is *Hermitian* if it is equipped with a continuous Hermitian structure

$$(\cdot \mid \cdot) : D \oplus D \to \mathbb{C}$$

that makes each fiber D_t into a complex Hilbert space. We define in a similar manner the *smooth* vector bundles and *holomorphic* vector bundles. For instance, for a Hermitian holomorphic vector bundle we require that both D and T are complex Banach manifolds, Π is holomorphic and the Hermitian structure is smooth. We denote by $\mathcal{C}(T, D)$, $\mathcal{C}^\infty(T, D)$ and $\mathcal{O}(T, D)$ the spaces of continuous, $\mathcal{C}^\infty$ and holomorphic sections of the bundle $\Pi : D \to T$, respectively, whenever the corresponding smoothness condition makes sense. We refer to [29] and [1] for basic facts on vector bundles.

Definition 4.1. Let $\Pi : D \to T$ be a continuous Hermitian vector bundle, $p_1, p_2 : T \times T \to T$ the natural projections, and

$$p_j^*\Pi : p_j^* D \to T \times T$$

the pull-back of Π by p_j, $j = 1, 2$, which are in turn Hermitian vector bundles. Also consider the continuous vector bundle $\mathrm{Hom}(p_2^*\Pi, p_1^*\Pi)$ over $T \times T$, whose fiber over $(s, t) \in T \times T$ is the Banach space $\mathcal{B}(D_t, D_s)$.

A *positive definite reproducing kernel* on Π is a continuous section

$$K \in \mathcal{C}\big(T \times T, \mathrm{Hom}(p_2^*\Pi, p_1^*\Pi)\big)$$

having the property that for every integer $n \geqslant 1$ and all choices of $t_1, \ldots, t_n \in T$ and $\xi_1 \in D_{t_1}, \ldots, \xi_n \in D_{t_n}$, we have

$$\sum_{j,l=1}^{n} \big(K(t_l, t_j)\xi_j \mid \xi_l\big)_{D_{t_l}} \geqslant 0.$$

If Π is a holomorphic vector bundle, we say that the reproducing kernel K is *holomorphic* if for each $t \in T$ and every $\xi \in D_t$ we have $K(\cdot\,, t)\xi \in \mathcal{O}(T, D)$.

The following properties of a reproducing kernel K are immediate consequences of the above definition:

(i) For all $t \in T$ we have $K(t, t) \geqslant 0$ in $\mathcal{B}(D_t)$.
(ii) For all $t, s \in T$ we have $K(t, s) \in \mathcal{B}(D_s, D_t)$, $K(s, t) \in \mathcal{B}(D_t, D_s)$ and $K(t, s)^* = K(s, t)$.

The first part of the next statement is a version of Theorem 1.4 in [12]. The proof consists in adapting to the present setting a number of basic ideas from the classical theory of reproducing kernel Hilbert spaces (see, for instance, [27] and [21]).

Theorem 4.2. *Let $\Pi : D \to T$ be a continuous Hermitian vector bundle, denote by $p_1, p_2 : T \times T \to T$ the projections and consider*

$$K \in \mathcal{C}\big(T \times T, \mathrm{Hom}(p_2^*\Pi, p_1^*\Pi)\big).$$

Then K is a reproducing kernel on Π if and only if there exists a linear mapping $\iota : \mathcal{H} \to \mathcal{C}(T, D)$, where $\mathcal{H}$ is a complex Hilbert space and $\mathrm{ev}_t^\iota := (\iota(\cdot))(t) : \mathcal{H} \to D_t$ is a bounded linear operator for all $t \in T$, and

$$K(s, t) = \mathrm{ev}_s^\iota \big(\mathrm{ev}_t^\iota\big)^* : D_t \to D_s, \tag{4.1}$$

for all $s, t \in T$. If this is the case, then the mapping ι can be chosen to be injective. If, moreover, Π is a holomorphic vector bundle then the reproducing kernel K is holomorphic if and only if any mapping ι as above has the range contained in $\mathcal{O}(T, D)$.

Proof. Given a mapping ι as described in the statement, it is clear that K defined by (4.1) is a reproducing kernel.

Conversely, given the reproducing kernel K, consider the complex linear subspace $\mathcal{H}_0^K \subset \mathcal{C}(T, D)$ spanned by $\{K_\xi := K(\cdot, \Pi(\xi))\xi \mid \xi \in D\}$ and denote by $\mathcal{H}^K$ its completion with respect to the scalar product defined by

$$(\Theta \mid \Delta)_{\mathcal{H}^K} = \sum_{j,l=1}^{n} \big(K\big(\Pi(\xi_l), \Pi(\eta_j)\big)\eta_j \bigm| \xi_l\big)_{D_{\Pi(\xi_l)}},$$

where $\Theta = \sum_{j=1}^n K(\cdot, \Pi(\eta_j))\eta_j$, $\Delta = \sum_{l=1}^n K(\cdot, \Pi(\xi_l))\xi_l \in \mathcal{H}_0^K$. In order to see that this formula is independent on the choices used to define Θ and Δ, let us first note the reproducing kernel property

$$(\Theta \mid K_\xi)_{\mathcal{H}^K} = \big(\Theta\big(\Pi(\xi)\big) \bigm| \xi\big)_{D_{\Pi(\xi)}} \tag{4.2}$$

which holds whenever $\xi \in D$ and $\Theta \in \mathcal{H}_0^K$. Now, for Θ and Δ as above we get

$$(\Theta \mid \Delta)_{\mathcal{H}^K} = \sum_{l=1}^{n} \big(\Theta\big(\Pi(\xi_l)\big) \bigm| \xi_l\big)_{D_{\Pi(\xi_l)}}.$$

Hence if $\Theta = 0$, then $(\Theta \mid \Delta)_{\mathcal{H}^K} = 0$.

Moreover,

$$(\Theta \mid \Theta)_{\mathcal{H}^K} = \sum_{j,l=1}^{n} \big(K\big(\Pi(\eta_l), \Pi(\eta_j)\big)\eta_j \bigm| \eta_l\big)_{D_{\Pi(\eta_l)}} \geqslant 0,$$

which shows that $(\cdot \mid \cdot)_{\mathcal{H}^K}$ is a nonnegative definite Hermitian sesquilinear form on $\mathcal{H}_0^K$. Therefore, it satisfies the Cauchy–Schwarz inequality and, in particular, the reproducing kernel property (4.2) implies that

$$\big|\big(\Theta\big(\Pi(\xi)\big) \bigm| \xi\big)_{D_{\Pi(\xi)}}\big| \leqslant (\Theta \mid \Theta)_{\mathcal{H}^K}^{1/2} (K_\xi \mid K_\xi)_{\mathcal{H}^K}^{1/2} \quad \text{for all } \xi \in D.$$

Thus, if $(\Theta \mid \Theta)_{\mathcal{H}^K} = 0$, then $\Theta = 0$. On the other hand, for each $\xi \in D$ we have

$$(K_\xi \mid K_\xi)_{\mathcal{H}^K} = \big(K\big(\Pi(\xi), \Pi(\xi)\big)\xi \bigm| \xi\big)_{D_{\Pi(\xi)}} \leqslant \big\|K\big(\Pi(\xi), \Pi(\xi)\big)\big\| \|\xi\|_{D_{\Pi(\xi)}}^2.$$

Now it follows by this inequality along with the previous one that for all $\Theta \in \mathcal{H}_0^K$ and $t \in T$ we have

$$\big\|\Theta(t)\big\|_{D_t} \leqslant \big\|K(t,t)\big\|^{1/2} (\Theta \mid \Theta)_{\mathcal{H}^K}^{1/2} = \big\|K(t,t)\big\|^{1/2} \|\Theta\|_{\mathcal{H}^K}.$$

Consequently we can uniquely define for each $t \in T$ the bounded linear mapping $\mathrm{ev}_t^\iota : \mathcal{H}^K \to D_t$ satisfying $\mathrm{ev}_t^\iota(\Theta) = \Theta(t)$ for all $\Theta \in \mathcal{H}_0^K$. Then we define $\iota : \mathcal{H}^K \to \mathcal{C}(T, D)$ by

$$\big(\iota(\cdot)\big)(t) := \mathrm{ev}_t^\iota(\cdot) : \mathcal{H}^K \to D_t \quad \text{for all } t \in T.$$

Now note that the reproducing kernel property (4.2) extends by continuity to arbitrary $\Theta \in \mathcal{H}^K$ under the following form:

$$(\Theta \mid K_\xi)_{\mathcal{H}^K} = \big(\big(\iota(\Theta)\big)\big(\Pi(\xi)\big) \bigm| \xi\big)_{D_{\Pi(\xi)}} \quad \text{for all } \Theta \in \mathcal{H}^K \text{ and } \xi \in D. \tag{4.3}$$

In particular, (4.3) shows that $\iota : \mathcal{H}^K \to \mathcal{C}(T, D)$ is injective: If $\iota(\Theta) = 0$, then $\Theta \perp K_\xi$ in $\mathcal{H}^K$ for all $\xi \in D$, hence $\Theta = 0$ since the linear span of $\{K_\xi \mid \xi \in D\}$ is dense in $\mathcal{H}^K$. The proof of the first part of the statement is finished.

Now assume that Π is a holomorphic vector bundle and let ι be a corresponding injective mapping. Note that the expression (4.1) of K in terms of the mappings ev_s^ι implies that for each $t \in T$ and $\xi \in D_t$ we have

$$K(s,t)\xi = \mathrm{ev}_s^\iota\big(\big(\mathrm{ev}_t^\iota\big)^*\xi\big) = \iota\big(\big(\mathrm{ev}_t^\iota\big)^*\xi\big)(s) \quad \text{for all } s \in T. \tag{4.4}$$

Thus, if $\operatorname{Ran}\iota \subseteq \mathcal{O}(T, D)$ then K is a holomorphic kernel. Conversely, assume that ι is injective and the kernel K is holomorphic. Then (4.4) shows that $\iota(h) \in \mathcal{O}(T, D)$ for all $h \in \mathcal{H}_0 := \bigcup_{\xi \in D} \operatorname{Ran}(\mathrm{ev}_{\Pi(\xi)}^\iota)^*$ $(\subseteq \mathcal{H})$. Note that in $\mathcal{H}$ we have

$$\mathcal{H}_0^\perp = \bigcap_{\xi \in D} \big(\operatorname{Ran}\big(\mathrm{ev}_{\Pi(\xi)}^\iota\big)^*\big)^\perp = \bigcap_{\xi \in D} \operatorname{Ker}\big(\mathrm{ev}_{\Pi(\xi)}^\iota\big) = \{0\},$$

where the latter equality follows since ι is injective. Consequently we know that $\iota(h) \in \mathcal{O}(D, T)$ when h runs through a dense linear subspace of $\mathcal{H}$. Now note that, as above, we can show that

$$\big\|\big(\iota(h)\big)(t)\big\|_{D_t} \leqslant \big\|K(t,t)\big\| \|h\|_{\mathcal{H}} \quad \text{for all } h \in \mathcal{H},$$

hence $\iota : \mathcal{H} \to \mathcal{C}(D, T)$ is continuous when $\mathcal{C}(D, T)$ is equipped with the topology of uniform convergence on the compact subsets of D. On the other hand, $\mathcal{O}(D, T)$ is a closed subspace of $\mathcal{C}(D, T)$ with respect to that topology (see, e.g., [41, Corollary 1.14]). Since we have already seen that $\iota(\mathcal{H}_0) \subseteq \mathcal{O}(D, T)$ and $\mathcal{H}_0$ is dense in $\mathcal{H}$, it then follows that $\operatorname{Ran}\iota \subseteq \mathcal{O}(D, T)$, as desired. □

5. Reproducing kernels and GNS representations

In this section we obtain our main results concerning geometric realizations of GNS representations in spaces of sections of Hermitian vector bundles (see Theorems 5.4 and 5.8). Since the corresponding spaces of sections will be reproducing kernel Hilbert spaces, we begin by constructing the reproducing kernels. In the extreme case $B = \mathbb{C}\mathbf{1}$ (see Example 3.4(ii)) the kernel $K_{\varphi,B}$ from the following construction is related to a certain Hermitian kernel studied in [20] (more precisely, see formula (6.2) in [20]).

Construction 5.1. Let A be a unital C^*-algebra, B a unital sub-C^*-algebra of A and $\varphi : A \to \mathbb{C}$ a state such that there exists a conditional expectation $E : A \to B$ with $\varphi \circ E = \varphi$. Recall from Definition 3.2 the homogeneous vector bundle

$$\Pi_{\varphi,B} : D_{\varphi,B} = \mathrm{U}_A \times_{\mathrm{U}_B} \mathcal{H}_\varphi \to \mathrm{U}_A/\mathrm{U}_B$$

associated with φ and B. Then we define

$$\iota_{\varphi,B} : \mathcal{H} \to \mathcal{C}(\mathrm{U}_A/\mathrm{U}_B, D_{\varphi,B})$$

by

$$\iota_{\varphi,B}(h)(u\mathrm{U}_B) = \big[\big(u, P_{\mathcal{H}_\varphi}\big(\rho(u)^{-1}h\big)\big)\big]$$

for all $h \in \mathcal{H}$ and $u \in \mathrm{U}_A$. We also define

$$K_{\varphi,B} \in \mathcal{C}\big(\mathrm{U}_A/\mathrm{U}_B \times \mathrm{U}_A/\mathrm{U}_B, \mathrm{Hom}\big(p_2^*(\Pi_{\varphi,B}), p_1^*(\Pi_{\varphi,B})\big)\big)$$

by

$$K_{\varphi,B}(u_1\mathrm{U}_B, u_2\mathrm{U}_B)\big[(u_2, f)\big] = \big[\big(u_1, P_{\mathcal{H}_\varphi}\big(\rho\big(u_1^{-1}u_2\big)f\big)\big)\big],$$

where $u_1, u_2 \in \mathrm{U}_A$, $f \in \mathcal{H}_\varphi$, and $p_1, p_2 : \mathrm{U}_A/\mathrm{U}_B \times \mathrm{U}_A/\mathrm{U}_B \to \mathrm{U}_A/\mathrm{U}_B$ are the projections.

We note that both $\iota_{\varphi,B}$ and $K_{\varphi,B}$ are well defined by the commutativity of the diagram (3.4).

Definition 5.2. In the setting of Construction 5.1, the operator $\iota_{\varphi,B} : \mathcal{H} \to \mathcal{C}(\mathrm{U}_A/\mathrm{U}_B, D_{\varphi,B})$ is called the *realization operator* associated with φ and B and the map $K_{\varphi,B} \in \mathcal{C}(\mathrm{U}_A/\mathrm{U}_B \times \mathrm{U}_A/\mathrm{U}_B, \mathrm{Hom}(p_2^*(\Pi_{\varphi,B}), p_1^*(\Pi_{\varphi,B})))$ is called the *reproducing kernel* associated with φ and B.

Remark 5.3.

(i) It will follow from the proof of Theorem 5.4 below that the reproducing kernel $K_{\varphi,B}$ associated with a state φ and B is indeed a reproducing kernel in the sense of Definition 4.1.

(ii) Recall the action $\beta_{\varphi,B} : \mathrm{U}_A \times D_{\varphi,B} \to D_{\varphi,B}$ from Remark 3.3(i) and, for all $u, u' \in \mathrm{U}_A$ and $f \in \mathcal{H}_\varphi$, denote $u' \cdot [(u, f)] := \beta_{\varphi,B}(u', [(u, f)]) = [(u'u, f)]$ for the sake of simplicity. Then, in the notation of Definition 5.2, for all $u, u_1, u_2 \in \mathrm{U}_A$ and $f \in \mathcal{H}$ we have

$$\begin{aligned}
K_{\varphi,B}(u \cdot u_1\mathrm{U}_B, u_2\mathrm{U}_B)\big[(u_2, f)\big] &= \big[\big(uu_1, P_{\mathcal{H}_\varphi}\big(\rho(u_1^*u^*u_2)f\big)\big)\big] \\
&= u \cdot \big[\big(u_1, P_{\mathcal{H}_\varphi}\big(\rho(u_1^*u^*u_2)f\big)\big)\big] \\
&= u \cdot K_{\varphi,B}(u_1\mathrm{U}_B, u^*u_2\mathrm{U}_B)\big[(u^*u_2, f)\big] \\
&= u \cdot K_{\varphi,B}(u_1\mathrm{U}_B, u^*u_2\mathrm{U}_B)u^* \cdot \big[(u_2, f)\big].
\end{aligned}$$

Thus

$$u^* \cdot K_{\varphi,B}(u \cdot t_1, t_2)\xi = K_{\varphi,B}(t_1, u^* \cdot t_2)(u^* \cdot \xi)$$

for $u \in \mathrm{U}_A$, $t_1, t_2 \in \mathrm{U}_A/\mathrm{U}_B$ and $\xi \in (D_{\varphi,B})_{t_2}$.

In the statement of the next theorem we refer to the correspondence established in Theorem 4.2. Representations of this type were also considered from another point of view in [32].

Theorem 5.4. *Let A be a unital C^*-algebra, B a unital sub-C^*-algebra of A and $\varphi: A \to \mathbb{C}$ a state such that there exists a conditional expectation $E: A \to B$ with $\varphi \circ E = \varphi$. Consider the GNS representation $\rho: A \to \mathcal{B}(\mathcal{H})$ associated with φ. Then the realization operator $\iota_{\varphi,B}: \mathcal{H} \to \mathcal{C}(\mathrm{U}_A/\mathrm{U}_B, D_{\varphi,B})$ associated with φ and B is injective, has the property that $(\iota_{\varphi,B}(\cdot))(s): \mathcal{H} \to (D_{\varphi,B})_s$ is bounded linear for all $s \in \mathrm{U}_A/\mathrm{U}_B$, and the corresponding reproducing kernel is precisely the reproducing kernel $K_{\varphi,B}$ associated with φ and B. Moreover, $\iota_{\varphi,B}$ intertwines the unitary representation $\rho|_{\mathrm{U}_A}$ of U_A on $\mathcal{H}$ and the natural representation of U_A by linear mappings on $\mathcal{C}(\mathrm{U}_A/\mathrm{U}_B, D_{\varphi,B})$.*

Proof. According to Theorem 4.2, the asserted relationship between $\iota_{\varphi,B}$ and $K_{\varphi,B}$ will follow as soon as we prove that for all $s, t \in \mathrm{U}_A/\mathrm{U}_B$ the equality

$$K_{\varphi,B}(s,t) = \mathrm{ev}_s^{\iota_{\varphi,B}} \big(\mathrm{ev}_t^{\iota_{\varphi,B}}\big)^* : (D_{\varphi,B})_t \to (D_{\varphi,B})_s$$

holds, where

$$\mathrm{ev}_s^{\iota_{\varphi,B}} = \big(\iota_{\varphi,B}(\cdot)\big)(s): \mathcal{H} \to (D_{\varphi,B})_s$$

and similarly for $\mathrm{ev}_t^{\iota_{\varphi,B}}$.

To this end, let $u_1, u_2 \in \mathrm{U}_A$ such that $s = u_1\mathrm{U}_B$ and $t = u_2\mathrm{U}_B$. Then

$$(D_{\varphi,B})_s = \big\{\big[(u_1, f)\big] \bigm| f \in \mathcal{H}_\varphi\big\}$$

and

$$\mathrm{ev}_s^{\iota_{\varphi,B}}(h) := \big[\big(u_1, P_{\mathcal{H}_\varphi}\big(\rho\big(u_1^{-1}\big)h\big)\big)\big] = \big[\big(u_1, \big(P_{\mathcal{H}_\varphi} \circ \rho(u_1)^*\big)h\big)\big] \quad \text{for all } h \in \mathcal{H},$$

and a similar formula holds with s replaced by t and u_1 replaced by u_2.

Now, since $(P_{\mathcal{H}_\varphi})^*$ is just the inclusion mapping $\mathcal{H}_\varphi \hookrightarrow \mathcal{H}$, it follows that for an arbitrary element $[(u_2, f)] \in (D_{\varphi,B})_t$ (where $f \in \mathcal{H}_\varphi$) we have

$$\big(\mathrm{ev}_t^{\iota_{\varphi,B}}\big)^*\big[(u_2, f)\big] = \rho(u_2)f;$$

hence

$$\mathrm{ev}_s^{\iota_{\varphi,B}}\big(\mathrm{ev}_t^{\iota_{\varphi,B}}\big)^*\big[(u_2, f)\big] = \big[\big(u_1, \big(P_{\mathcal{H}_\varphi} \circ \rho(u_1)^*\big)\rho(u_2)f\big)\big] = K_{\varphi,B}(s,t)\big[(u_2, f)\big],$$

as desired.

In order to prove that $\iota_{\varphi,B}$ is injective, let $h \in \mathcal{H}$ such that $\iota_{\varphi,B}(h) = 0$. By the definition of $\iota_{\varphi,B}$ (see Construction 5.1), it follows that for all $u \in \mathrm{U}_A$ we have $P_{\mathcal{H}_\varphi}(\rho(u^{-1})h) = 0$. On the other hand, A is the linear span of U_A. (For instance, if $a = a^* \in A$ and $\|a\| \leqslant 1$, then $a = (u + u^*)/2$, where $u = a + i(\mathbf{1} - a^2)^{1/2} \in \mathrm{U}_A$.) Consequently $P_{\mathcal{H}_\varphi}(\rho(a)h) = 0$ for all $a \in A$.

In particular, denoting by $h_0 \in \mathcal{H}_\varphi \subseteq \mathcal{H}$ the image of $\mathbf{1} \in A$ in $\mathcal{H}_\varphi$, we get $0 = (\rho(a)h \mid h_0) = (h \mid \rho(a^*)h_0)$ for all $a \in A$. Since h_0 is a cyclic vector for the representation $\rho: A \to \mathcal{B}(\mathcal{H})$, according to the GNS construction (see, e.g., [2, Proposition 1.100 and Definition 1.101]), it follows that $h \perp \mathcal{H}$, whence $h = 0$. Thus $\iota_{\varphi,B}$ is injective.

Since it is clear that $(\iota_{\varphi,B}(\cdot))(u\mathrm{U}_B):\mathcal{H}\to(D_{\varphi,B})_{u\mathrm{U}_B}$ is bounded linear for all $u\in\mathrm{U}_A$, it follows that the realization operator $\iota_{\varphi,B}:\mathcal{H}\to\mathcal{C}(\mathrm{U}_A/\mathrm{U}_B,D_{\varphi,B})$ is of the type occurring in Theorem 4.2.

It remains to prove the intertwining property of $\iota_{\varphi,B}$. So consider, as in the proof of Theorem 4.2, the linear subspace $\mathcal{H}_0^{K_{\varphi,B}}$ spanned by $\{(K_{\varphi,B})_\xi := K_{\varphi,B}(\cdot\,,\Pi_{\varphi,B}(\xi))\xi \mid \xi\in D_{\varphi,B}\}$ in $\mathcal{C}(\mathrm{U}_A/\mathrm{U}_B,D_{\varphi,B})$. Then it follows by Remark 5.3(ii) that for all $u\in\mathrm{U}_A$ we have $u\cdot(K_{\varphi,B})_\xi=(K_{\varphi,B})_{u\cdot\xi}$, where the left-hand side denotes the action of u on $(K_{\varphi,B})_\xi\in\mathcal{C}(\mathrm{U}_A/\mathrm{U}_B,D_{\varphi,B})$.

In view of the construction of $\mathcal{H}^K$ in the proof of Theorem 4.2, it follows that the natural action of u on $\mathcal{C}(\mathrm{U}_A/\mathrm{U}_B,D_{\varphi,B})$ leaves $\mathcal{H}^{K_{\varphi,B}}$ invariant and actually induces a unitary operator on it. In fact, for $t_1,\dots,t_n\in\mathrm{U}_A/\mathrm{U}_B$, $\xi_1\in(D_{\varphi,B})_{t_1},\dots,\xi_n\in(D_{\varphi,B})_{t_n}$ and $\Theta=\sum_{j=1}^n(K_{\varphi,B})_{\xi_j}=\sum_{j=1}^n K_{\varphi,B}(\cdot\,,t_j)\xi_j$, we have $\|\Theta\|^2_{\mathcal{H}^{K_{\varphi,B}}}=\sum_{j,l=1}^n(K(t_l,t_j)\xi_j\mid\xi_l)_{(D_{\varphi,B})_{t_l}}$ and

$$\begin{aligned}\|u\cdot\Theta\|^2_{\mathcal{H}^{K_{\varphi,B}}}&=\sum_{j,l=1}^n\bigl(K(u^*\cdot t_l,u^*\cdot t_j)(u^*\cdot\xi_j)\mid u^*\cdot\xi_l\bigr)_{(D_{\varphi,B})_{u^*\cdot t_l}}\\&=\sum_{j,l=1}^n\bigl(u\cdot K(u^*\cdot t_l,u^*\cdot t_j)(u^*\cdot\xi_j)\mid\xi_l\bigr)_{(D_{\varphi,B})_{t_l}}\\&=\sum_{j,l=1}^n\bigl(K(t_l,t_j)\xi_j\mid\xi_l\bigr)_{(D_{\varphi,B})_{t_l}}\\&=\|\Theta\|^2_{\mathcal{H}^{K_{\varphi,B}}},\end{aligned}$$

where the next-to-last equality follows by Remark 5.3(ii). The intertwining property of $\iota_{\varphi,B}$ is straightforward: for all $u,v\in\mathrm{U}_A$ and $h\in\mathcal{H}$ we have

$$\begin{aligned}\iota_{\varphi,B}\bigl(\rho(v)h\bigr)(u\mathrm{U}_B)&=\bigl[\bigl(u,P_{\mathcal{H}_\varphi}\bigl(\rho\bigl(u^{-1}\bigr)\rho(v)h\bigr)\bigr)\bigr]=\bigl[\bigl(u,P_{\mathcal{H}_\varphi}\bigl(\rho\bigl(\bigl(v^{-1}u\bigr)^{-1}\bigr)h\bigr)\bigr)\bigr]\\&=v\cdot\bigl[\bigl(v^{-1}u,P_{\mathcal{H}_\varphi}\bigl(\rho\bigl(\bigl(v^{-1}u\bigr)^{-1}\bigr)h\bigr)\bigr)\bigr]=v\cdot\iota_{\varphi,B}(h)\bigl(v^{-1}u\mathrm{U}_B\bigr)\end{aligned}$$

as desired. □

Remark 5.5. As a by-product of the proof of Theorem 5.4 it follows that for each $f\in\mathcal{H}_\varphi$ we have $\iota_{\varphi,B}(f)=(K_{\varphi,B})(\cdot\,,\mathbf{1}\mathrm{U}_B)[(\mathbf{1},f)]$, whence we get $\|\iota_{\varphi,B}(f)\|_{\mathcal{H}^{K_{\varphi,B}}}=\|f\|_{\mathcal{H}_\varphi}$.

Remark 5.6. In the setting of Theorem 5.4, if it happens that φ is a pure state then the fact that the realization operator $\iota_{\varphi,B}:\mathcal{H}\to\mathcal{C}(\mathrm{U}_A/\mathrm{U}_B,D_{\varphi,B})$ is injective can be proved in an alternative way as follows. Let $h\in H$ with $\iota_{\varphi,B}(h)=0$. Then for all $u\in\mathrm{U}_A$ we have $0=(\iota_{\varphi,B}(h))(u\mathrm{U}_B)=[(u,P_{\mathcal{H}_\varphi}(\rho(u^{-1})h))]$. Since every element of A is a linear combination of unitary elements, we get $P_{\mathcal{H}_\varphi}(\rho(A)h)=0$, hence the closure $\mathcal{H}_0$ of $\rho(A)h$ is contained in $\operatorname{Ker}P_{\mathcal{H}_\varphi}$. On the other hand, we cannot have $P_{\mathcal{H}_\varphi}=0$, since, denoting by h_0 the image of $\mathbf{1}\in B$ in $\mathcal{H}_\varphi$, we have $P_{\mathcal{H}_\varphi}h_0=h_0\neq 0$. Hence $\mathcal{H}_0\neq\mathcal{H}$. Now note that $\mathcal{H}_0$ is an invariant subspace for the GNS representation $\rho:A\to\mathcal{B}(\mathcal{H})$ and this representation is irreducible since φ is pure. Consequently we must have $\mathcal{H}_0=\{0\}$, whence $h=0$, and thus $\iota_{\varphi,B}$ is injective.

In connection with Remark 5.6, we note that pure normal states can exist only in the case of W^*-algebras of type I, for instance direct sums of algebras of the form $\mathcal{B}(\mathcal{H})$ with $\mathcal{H}$ a Hilbert space. We discuss below the case of finite-dimensional $\mathcal{H}$.

A few details on the infinite-dimensional case can be found in [19].

Example 5.7. Let $n \geqslant 1$ be an integer, $M = M_n(\mathbb{C})$ with its structure of W^*-algebra, and think of $\mathbb{C}^n$ as a Hilbert space with the usual scalar product $(\cdot \mid \cdot)$. Next, denote

$$h = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad \text{and} \quad p = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} \in M.$$

Now define

$$\varphi : M \to \mathbb{C}, \quad \varphi(a) = (ah \mid h) = \operatorname{Tr}(ap) = a_{11} \quad \text{for } a = (a_{ij})_{1 \leqslant i,j \leqslant n} \in M.$$

It is clear that φ is a pure normal state of M. We want to construct the GNS representation of M with respect to φ and to see what Theorem 5.4 says in this special case.

We have

$$M_0 := \{a \in M \mid \varphi(a^*a) = 0\} = \{a \in M \mid ah = 0\} = \left\{ \begin{pmatrix} 0 & * & \cdots & * \\ 0 & * & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & * & \cdots & * \end{pmatrix} \right\}$$

and

$$\varphi(a^*a) = |a_{11}|^2 + |a_{21}|^2 + \cdots + |a_{n1}|^2 \quad \text{for } a = \begin{pmatrix} a_{11} & 0 & \cdots & 0 \\ a_{21} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & 0 & \cdots & 0 \end{pmatrix}.$$

Hence the completion $\mathcal{H}$ of M/M_0 with respect to the scalar product induced by $(a, b) \mapsto \varphi(b^*a)$ is just the Hilbert space $\mathbb{C}^n$ with the usual scalar product, viewed as the set of column vectors, and the natural mapping $M \to \mathcal{H}$ is

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \mapsto \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{pmatrix}.$$

Moreover, according to the way matrices multiply, it follows that for each $a \in M$ the operator $\rho(a): \mathcal{H} \to \mathcal{H}$ that makes the diagram

$$\begin{array}{ccc} M & \xrightarrow{b \mapsto ab} & M \\ \Big\downarrow & & \Big\downarrow \\ \mathcal{H} & \xrightarrow{\rho(a)} & \mathcal{H} \end{array}$$

commutative is just the natural action of $a \in M = M_n(\mathbb{C})$ on $\mathbb{C}^n$. Thus the GNS representation of M associated to φ is just the natural representation of $M = M_n(\mathbb{C})$ on $\mathbb{C}^n$.

Next, Proposition 2.2(b) shows that

$$\begin{aligned} M^\varphi = \{p\}' &= \left\{ \begin{pmatrix} * & 0 & \cdots & 0 \\ 0 & * & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & * & \cdots & * \end{pmatrix} \right\} = \left\{ \begin{pmatrix} z & 0 \\ 0 & W \end{pmatrix} \;\middle|\; z \in \mathbb{C},\ W \in M_{n-1}(\mathbb{C}) \right\} \\ &\simeq M_1(\mathbb{C}) \times M_{n-1}(\mathbb{C}), \end{aligned}$$

hence $M^\varphi \cap M_0 = (\mathbf{1} - p)M(\mathbf{1} - p) \simeq M_{n-1}(\mathbb{C})$ and thus the space of the GNS representation $\rho_\varphi : M^\varphi \to \mathcal{B}(\mathcal{H}_\varphi)$ of M^φ corresponding to the state $\varphi|_{M^\varphi} : M^\varphi \to \mathbb{C}$ is one-dimensional, i.e., $\mathcal{H}_\varphi = \mathbb{C}$. Moreover $\rho_\varphi\left(\begin{smallmatrix} z & 0 \\ 0 & W \end{smallmatrix}\right)$ is the multiplication-by-z operator on $\mathcal{H}_\varphi$ for all $z \in \mathbb{C}$ and $W \in M_{n-1}(\mathbb{C})$. Now

$$\mathrm{U}_M / \mathrm{U}_{M^\varphi} = \mathrm{U}(n) \big/ \big(\mathrm{U}(1) \times \mathrm{U}(n-1)\big) = \mathbb{P}^{n-1}(\mathbb{C}),$$

and it follows at once that the homogeneous vector bundle $\Pi_{\varphi, M^\varphi} : D_{\varphi, M^\varphi} \to \mathrm{U}_M / \mathrm{U}_{M^\varphi}$ associated with φ is dual to the tautological line bundle over the complex projective space $\mathbb{P}^{n-1}(\mathbb{C})$. Thus, in this special case, Theorem 5.4 says that the natural representation of $\mathrm{U}(n)$ on $\mathbb{C}^n$ can be geometrically realized as a representation in the finite-dimensional vector space of global holomorphic sections of the dual to the tautological line bundle over the $\mathrm{U}(n)$-homogeneous compact Kähler manifold $\mathbb{P}^{n-1}(\mathbb{C})$, which is a special case of the Borel–Weil theorem (see, e.g., [25] and [24]).

We now describe a situation when the homogeneous vector bundle occurring in Theorem 5.4 is holomorphic and the reproducing kernel Hilbert space consists only of holomorphic sections.

Theorem 5.8. *Let M be a W^*-algebra with a faithful normal tracial state $\tau : M \to \mathbb{C}$. Pick $a \in M$ such that $0 \leqslant a$, $\tau(a) = 1$ and the spectrum of a is a finite set, and define*

$$\varphi : M \to \mathbb{C}, \quad \varphi(b) = \tau(ab).$$

Then the homogeneous vector bundle $\Pi_{\varphi, M^\varphi} : D_{\varphi, M^\varphi} \to \mathrm{U}_M / \mathrm{U}_{M^\varphi}$ associated with φ and M^φ is holomorphic and the reproducing kernel K_{φ, M^φ} is holomorphic as well. Also, if

$$\rho : M \to \mathcal{B}(\mathcal{H})$$

is the GNS representation corresponding to the normal state φ, then the image of the realization operator $\iota_{\varphi,M^\varphi} : \mathcal{H} \to \mathcal{C}(\mathrm{U}_M/\mathrm{U}_{M^\varphi}, D_{\varphi,M^\varphi})$ consists of holomorphic sections.

Proof. As in the proof of Proposition 2.5, write $a = \lambda_1 e_1 + \cdots + \lambda_n e_n$, where $e_1, \ldots, e_n \in M$ are orthogonal projections such that $e_i e_j = 0$ whenever $i \neq j$ and $e_1 + \cdots + e_n = \mathbf{1}$, and observe that, by Proposition 2.2(b), we have

$$M^\varphi = \{a\}' = \{b \in M \mid be_j = e_j b \text{ for } j = 1, \ldots, n\}.$$

Next denote by

$$\rho_\varphi : M_0 \to \mathcal{B}(\mathcal{H}_\varphi)$$

the GNS representation corresponding to $\varphi|_{M^\varphi}$.

Then the representation $\rho_\varphi|_{\mathrm{G}_{M^\varphi}}$ of the group G_{M^φ} can be extended to a representation of the group

$$P := \{g \in \mathrm{G}_M \mid e_k g e_j = 0 \text{ if } 1 \leqslant j < k \leqslant n\}$$

by defining

$$\tilde{\rho}_\varphi : P \to \mathcal{B}(\mathcal{H}_\varphi), \quad \tilde{\rho}_\varphi(g) := \rho_\varphi(e_1 g e_1 + \cdots + e_n g e_n).$$

Using this representation, it makes sense to consider the space $\widetilde{D}_\varphi = \mathrm{G}_M \times_P \mathcal{H}_\varphi$, that is, the quotient of $\mathrm{G}_M \times \mathcal{H}_\varphi$ by the equivalence relation given by $(gc, f) \sim (g, \tilde{\rho}_\varphi(c) f)$ whenever $c \in P$.

For an arbitrary pair $(g, f) \in \mathrm{G}_M \times \mathcal{H}_\varphi$ we denote by $[(g, f)]^\sim$ its equivalence class in $\widetilde{D}_\varphi$, in order to distinguish it from the equivalence class $[(g, f)] \in D_{\varphi,M^\varphi} = \mathrm{U}_M \times_{\mathrm{U}_{M^\varphi}} \mathcal{H}_\varphi$ when it happens that $g \in \mathrm{U}_M$. Also we define

$$\widetilde{\Pi}_\varphi\big([(g, f)]^\sim\big) = gP \in \mathrm{G}_M/P$$

for all $[(g, f)]^\sim \in \widetilde{D}_\varphi$, and $\Psi : D_\varphi \to \widetilde{D}_\varphi$ by $[(u, f)] \mapsto [(u, f)]^\sim$. Then we get a commutative diagram

$$\begin{tikzcd} D_\varphi \arrow[r, "\Psi"] \arrow[d, "\Pi_{\varphi,M^\varphi}"'] & \widetilde{D}_\varphi \arrow[d, "\widetilde{\Pi}_\varphi"] \\ \mathrm{U}_M/\mathrm{U}_{M^\varphi} \arrow[r, "\psi"] & \mathrm{G}_M/P \end{tikzcd}$$

where $\psi : \mathrm{U}_M/\mathrm{U}_{M^\varphi} \to \mathrm{G}_M/P$ is the U_M-equivariant diffeomorphism given by Proposition 2.7. In this diagram, $\widetilde{\Pi}_\varphi$ is a holomorphic vector bundle (according to its construction) and Ψ is an U_M-equivariant real analytic mapping that is fiberwise bounded linear. We will prove shortly that Ψ is bijective and then, since ψ is a real analytic diffeomorphism by Proposition 2.7, it will follow that the pair (Ψ, ψ) gives an isomorphism of vector bundles. Thus Π_{φ,M^φ} is a holomorphic vector bundle and, moreover, for each $h \in \mathcal{H}$ we have $(\Psi \circ \iota_{\varphi,M^\varphi}(h) \circ \psi^{-1})(gP) =$

$[(g, P_{\mathcal{H}_\varphi}(\rho(g)^{-1}h))]^{\sim}$ for all $g \in G$, hence clearly $\Psi \circ \iota_{\varphi,M^\varphi}(h) \circ \psi^{-1} \in \mathcal{O}(\mathrm{G}_M/P, \widetilde{D}_\varphi)$, whence $\iota_{\varphi,M^\varphi}(h) \in \mathcal{O}(\mathrm{U}_M/\mathrm{U}_{M^\varphi}, D_\varphi)$.

Thus it only remains to show that the mapping Ψ is bijective. To see that it is injective, let $u_1, u_2 \in \mathrm{U}_M$ and $f_1, f_2 \in \mathcal{H}_\varphi$ such that $[(u_1, f_1)]^{\sim} = [(u_2, f_2)]^{\sim}$. Then there exists $c \in P$ such that $u_1 = u_2 c$ and $f_1 = \tilde{\rho}_\varphi(c^{-1}) f_2$. In particular, $c = u_2^{-1} u_1 \in \mathrm{U}_M$, whence $c \in \mathrm{U}_M \cap P = \mathrm{U}_{M^\varphi}$, and then $[(u_1, f_1)] = [(u_2, f_2)]$. Thus Ψ is injective. To prove that it is surjective, let $g \in \mathrm{G}_M$ and $f \in \mathcal{H}_\varphi$ arbitrary. According to the proof of Proposition 2.7, there exist $u \in \mathrm{U}_M$ and $c \in P$ such that $g = uc$, whence $[(g, f)]^{\sim} = [(uc, f)]^{\sim} = [(u, \tilde{\rho}_\varphi(c) f)]^{\sim} = \Psi([(u, \tilde{\rho}_\varphi(c) f)])$. $\square$

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