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The reduced spaces of a symplectic Lie group action

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Abstract There exist three main approaches to reduction associated with canonical Lie group actions on a symplectic manifold, namely, foliation reduction, introduced by Cartan, Marsden–Weinstein reduction, and optimal reduction, introduced by the authors. When the action is free, proper, and admits a momentum map these three approaches coincide. The goal of this paper is to study the general case of a symplectic action that does not admit a momentum map and one needs to use its natural generalization, a cylinder valued momentum map introduced by Condevaux et al. In this case, it will be shown that the three reduced spaces mentioned above do not coincide, in general. More specifically, the Marsden–Weinstein reduced spaces are not symplectic but Poisson and their symplectic leaves are given by the optimal reduced spaces. Foliation reduction produces a symplectic reduced space whose Poisson quotient by a certain Lie group associated to the group of symmetries of the problem equals the Marsden–Weinstein reduced space. We illustrate these constructions with concrete examples, special emphasis being given to the reduction of a magnetic cotangent bundle of a Lie group in the situation when the magnetic term ensures the non-existence of the momentum map for the lifted action. The precise relation of the cylinder valued momentum map with group valued momentum maps for Abelian Lie groups is also given.

Keywords Symplectic reduction · Marsden–Weintein reduction · Poisson reduction · Momentum map · Lie group action

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1. Introduction

Let (M, ω) be a connected paracompact symplectic manifold acted upon properly and canonically by a Lie group G. In this paper, it is assumed that the G-action is free; the non-free case is the subject of [24]. Let $\mathfrak{g}$ be the Lie algebra of G and $\mathfrak{g}^*$ its dual. Assume for the moment that the action admits a standard equivariant momentum map $\mathbf{J}\colon M \to \mathfrak{g}^*$. There are three main approaches to the symmetry reduction of (M, ω) by G that yield, up to connected components, the same spaces:

- *Foliation reduction* [5]: Consider the fiber $\mathbf{J}^{-1}(\mu)$ and the characteristic distribution $D = \ker T\mathbf{J} \cap (\ker T\mathbf{J})^{\omega}$ on it; the upper index ω on a vector subbundle of TM denotes the ω-orthogonal complement. The symplectic structure of (M, ω) drops naturally to the leaf space $\mathbf{J}^{-1}(\mu)/D$.
- *Marsden–Weinstein reduction* [15]: Let G_μ be the isotropy subgroup of the element $\mu \in \mathfrak{g}^*$ with respect to the coadjoint action of G on $\mathfrak{g}^*$. The orbit manifold $\mathbf{J}^{-1}(\mu)/G_\mu$ inherits from (M, ω) a natural symplectic form ω_μ uniquely characterized by the expression $i_\mu^*\omega = \pi_\mu^*\omega_\mu$, with $i_\mu\colon \mathbf{J}^{-1}(\mu) \hookrightarrow M$ the inclusion and $\pi_\mu\colon \mathbf{J}^{-1}(\mu) \to \mathbf{J}^{-1}(\mu)/G_\mu$ the projection.
- *Optimal reduction* [18, 20]: Let A'_G be the distribution on M defined by $A'_G := \{X_f \mid f \in C^\infty(M)^G\}$. The distribution A'_G is smooth and integrable in the sense of Stefan and Sussmann [29–31]. The optimal momentum map $\mathcal{J}\colon M \longrightarrow M/A'_G$ is defined as the canonical projection onto the leaf space of A'_G which is, in most cases, not even a Hausdorff topological space, let alone a manifold. For any $g \in G$, the map $\Psi_g(\rho) = \mathcal{J}(g \cdot m) \in M/A'_G$ defines a continuous G-action on M/A'_G with respect to which $\mathcal{J}$ is G-equivariant. The orbit space $M_\rho := \mathcal{J}^{-1}(\rho)/G_\rho$ is a smooth symplectic regular quotient manifold with symplectic form ω_ρ characterized by $\pi_\rho^*\omega_\rho = i_\rho^*\omega$, where $\pi_\rho\colon \mathcal{J}^{-1}(\rho) \to \mathcal{J}^{-1}(\rho)/G_\rho$ is the projection and $i_\rho\colon \mathcal{J}^{-1}(\rho) \hookrightarrow M$ the inclusion.

These reduction theorems are important for symmetric Hamiltonian dynamics since the flow associated to a G-invariant Hamiltonian function projects to a Hamiltonian flow on the symplectic reduced spaces.

Our goal in this paper is to carry out the regular reduction procedure for *any* symplectic action, even when a momentum map does not exist. As will be shown, the three approaches to reduction yield spaces that are, in general, distinct but non-trivially related to each other in very interesting ways. Our results are based on a key construction of Condevaux et al. [6] naturally generalizing the standard momentum map to a *cylinder valued momentum map* $\mathbf{K}\colon M \to \mathbb{R}^a \times \mathbb{T}^b$, $a, b \in \mathbb{N}$, that *always* exists for *any* symplectic Lie group action. The cylinder $\mathbb{R}^a \times \mathbb{T}^b$ is obtained as the quotient $\mathfrak{g}^*/\bar{\mathcal{H}}$, with $\mathcal{H}$ a zero-dimensional Lie subgroup of $(\mathfrak{g}^*, +)$ which is the holonomy of a flat connection on the trivial principal fiber bundle $\pi\colon M \times \mathfrak{g}^* \to M$ with $(\mathfrak{g}^*, +)$ as Abelian structure group. This flat connection is constructed using exclusively the canonical G-action and the symplectic form ω on M thereby justifying the name *Hamiltonian holonomy* for $\mathcal{H}$.

The main result. Let (M, ω) be a connected paracompact symplectic manifold and G a Lie group acting freely and properly on it by symplectic diffeomorphisms. Let $\mathbf{K}\colon M \to \mathfrak{g}^*/\bar{\mathcal{H}}$ be a cylinder valued momentum map for this action. Then $\mathfrak{g}^*/\bar{\mathcal{H}}$ carries a natural Poisson structure and there exists a smooth G-action on it with respect to which $\mathbf{K}$ is equivariant and Poisson. Moreover:

(i) The Marsden–Weinstein reduced space $M^{[\mu]} := \mathbf{K}^{-1}([\mu])/G_{[\mu]}$, $[\mu] \in \mathfrak{g}^*/\bar{\mathcal{H}}$, has a natural Poisson structure inherited from the symplectic structure (M, ω) that is, in general, degenerate. $M^{[\mu]}$ will be referred to as the *Poisson reduced space*.
(ii) The optimal reduced spaces can be naturally identified with the symplectic leaves of $M^{[\mu]}$.
(iii) The reduced spaces obtained by foliation reduction equal the orbit spaces $M_{[\mu]} := \mathbf{K}^{-1}([\mu])/N_{[\mu]}$, where N is a normal connected Lie subgroup of G whose Lie algebra is the annihilator $\mathfrak{n} := (\mathrm{Lie}(\bar{\mathcal{H}}))^\circ \subset \mathfrak{g}$ of $\mathrm{Lie}(\bar{\mathcal{H}}) \subset \mathfrak{g}^*$ in $\mathfrak{g}$. The manifolds $M_{[\mu]}$ will be referred to as the *symplectic reduced spaces*.
(iv) The quotient group $H_{[\mu]} := G_{[\mu]}/N_{[\mu]}$ acts canonically on $M_{[\mu]}$ and the quotient Poisson manifold $M_{[\mu]}/H_{[\mu]}$ is Poisson diffeomorphic to $M^{[\mu]}$.

As will be shown in the course of this paper, one of the reasons behind the existence of the three distinct reduced manifolds is the non-closedness of the zero-dimensional Hamiltonian holonomy $\mathcal{H}$ (as the holonomy group of a flat connection). In fact, $\bar{\mathcal{H}}$ measures in some sense the degree of degeneracy of the Poisson structure of the Marsden–Weinstein reduced space $M^{[\mu]}$. Moreover, when $\mathcal{H}$ is closed, the three reduction approaches yield (up to connected components) the same symplectic space.

The present paper deals only with free actions. In our forthcoming paper [24], we will study the situation in which this hypothesis has been dropped.

The contents of the paper are as follows. Section 2 introduces and presents in detail the properties of the cylinder valued momentum map. Section 3 studies the invariance properties of the Hamiltonian holonomy $\mathcal{H}$ and constructs a natural action on the target space of the cylinder valued momentum map with respect to which the cylinder valued momentum map is equivariant. This action is an essential ingredient for reduction. Section 4 defines a Poisson structure on the target space of the cylinder valued momentum map with respect to which this map is Poisson. It also provides a careful study of this Poisson structure and explicitly characterizes its symplectic leaves. This section also contains a general discussion on central extensions of Lie algebras and groups, their actions, and their role in the characterization of the symplectic leaves of affine and projected affine Lie–Poisson structures on duals of Lie algebras. Apart from its intrinsic interest, this information on central extensions will be heavily used in the example of Section 6. Section 5 contains a detailed statement and proof of the reduction results announced earlier. Section 6 contains an in-depth study of an example that illustrates some of the main results in the paper. The cotangent bundle of a Lie group is considered, but with a symplectic structure that is the sum of the canonical one and of an invariant magnetic term, whose value at the identity does not integrate to a group two-cocycle. This modification destroys, in general, the existence of a standard momentum map for the lift of left translations and forces the use of all the developments in the paper. This section contains an interesting generalization of the classical result, which states that the coadjoint orbits endowed with their canonical Kostant–Kirillov–Souriau symplectic structure are symplectic reduced spaces of the cotangent bundle of the corresponding Lie group. The paper concludes with an appendix that specifies the relation, in the context of Abelian Lie group actions, of the cylinder valued momentum map and the so called Lie group valued momentum maps.

Notations and general assumptions. Manifolds: In this paper, all manifolds are finite-dimensional. *Group actions:* The image of a point m in a manifold M under a group action $\Phi: G \times M \to M$ is denoted interchangeably by $\Phi(g, m) = \Phi_g(m) = g \cdot m$, for any $g \in G$. The symbol $L_g: G \to G$ (respectively $R_g: G \to G$) denotes left (respectively right) translation on G by the group element $g \in G$. The group orbit containing $m \in M$ is denoted by $G \cdot m$ and its tangent space by $T_m(G \cdot m)$ or $\mathfrak{g} \cdot m$. The Lie algebra of the group G is usually denoted by $\mathfrak{g}$. Given any $\xi \in \mathfrak{g}$, the symbol ξ_M denotes the infinitesimal generator vector field associated to ξ defined by $\xi_M(m) = \frac{\mathrm{d}}{\mathrm{d}t}|_{t=0} \exp t\xi \cdot m$, for any $m \in M$. A *right (left) Lie algebra action* of $\mathfrak{g}$ on M is a Lie algebra (anti)homomorphism $\xi \in \mathfrak{g} \longmapsto \xi_M \in \mathfrak{X}(M)$ such that the mapping $(m, \xi) \in M \times \mathfrak{g} \longmapsto \xi_M(m) \in TM$ is smooth. If $\mathfrak{g}$ acts on a symplectic manifold (M, ω) we say that the $\mathfrak{g}$-action is *canonical* when $\pounds_{\xi_M}\omega = 0$, for any $\xi \in \mathfrak{g}$. *The Chu map:* Given a symplectic manifold (M, ω) acted canonically upon by a Lie algebra $\mathfrak{g}$, the Chu map $\Psi\colon M \to Z^2(\mathfrak{g})$ is defined by the expression $\Psi(m)(\xi, \eta) := \omega(m)(\xi_M(m), \eta_M(m))$, for any $m \in M, \xi, \eta, \in \mathfrak{g}$.

2. The cylinder valued momentum map

In this section we define carefully the cylinder valued momentum map and study its elementary properties. This construction, first introduced by Condevaux et al. in [6] under the name of "reduced momentum map", is the key stone of the main results in this paper.

The following notations will be used throughout this work. If $\langle \cdot, \cdot \rangle\colon W^* \times W \to \mathbb{R}$ is a nondegenerate duality pairing and $V \subset W$, define the *annihilator* subspace $V^\circ := \{\alpha \in W^* \mid \langle \alpha, v \rangle = 0 \text{ for all } v \in V\} \subset W^*$ and similarly for a subset of W^*. If (S, ω) is a symplectic vector space and $U \subset S$, define the *ω-orthogonal subspace* $U^\omega := \{s \in S \mid \omega(s, u) = 0 \text{ for all } u \in U\}$.

Let (M, ω) be a connected and paracompact symplectic manifold and let $\mathfrak{g}$ be a Lie algebra that acts canonically on M. Take the Cartesian product $M \times \mathfrak{g}^*$ and let $\pi\colon M \times \mathfrak{g}^* \to M$ be the projection onto M. Consider π as the bundle map of the trivial principal fiber bundle $(M \times \mathfrak{g}^*, M, \pi, \mathfrak{g}^*)$ that has $(\mathfrak{g}^*, +)$ as Abelian structure group. The group $(\mathfrak{g}^*, +)$ acts on $M \times \mathfrak{g}^*$ by $\nu \cdot (m, \mu) := (m, \mu - \nu)$, with $m \in M$ and $\mu, \nu \in \mathfrak{g}^*$. Let $\alpha \in \Omega^1(M \times \mathfrak{g}^*; \mathfrak{g}^*)$ be the connection one-form defined by

$$\langle \alpha(m, \mu)(v_m, \nu), \xi \rangle := (\mathbf{i}_{\xi_M}\omega)(m)(v_m) - \langle \nu, \xi \rangle, \tag{1}$$

where $(m, \mu) \in M \times \mathfrak{g}^*$, $(v_m, \nu) \in T_mM \times \mathfrak{g}^*$, $\langle \cdot, \cdot \rangle$ denotes the natural pairing between $\mathfrak{g}^*$ and $\mathfrak{g}$, and ξ_M is the infinitesimal generator vector field associated to $\xi \in \mathfrak{g}$ defined by $\xi_M(m) = \frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0} \exp t\xi \cdot m$, $m \in M$. It is easy to check that α is a flat connection. For $(z, \mu) \in M \times \mathfrak{g}^*$, let $(M \times \mathfrak{g}^*)(z, \mu)$ be the holonomy bundle through (z, μ) and let $\mathcal{H}(z, \mu)$ be the holonomy group of α with reference point (z, μ) (which is an Abelian zero dimensional Lie subgroup of $\mathfrak{g}^*$ by the flatness of α). The Reduction Theorem [13, Theorem 7.1, page 83] guarantees that the principal bundle $((M \times \mathfrak{g}^*)(z, \mu), M, \pi|_{(M\times\mathfrak{g}^*)(z,\mu)}, \mathcal{H}(z, \mu))$ is a reduction of the principal bundle $(M \times \mathfrak{g}^*, M, \pi, \mathfrak{g}^*)$; it is here that we used the paracompactness of M since it is a technical hypothesis in the Reduction Theorem. To simplify notation, we will write $(\tilde{M}, M, \tilde{p}, \mathcal{H})$ instead of $((M \times \mathfrak{g}^*)(z, \mu), M, \pi|_{(M\times\mathfrak{g}^*)(z,\mu)}, \mathcal{H}(z, \mu))$. Let $\tilde{\mathbf{K}}\colon \tilde{M} \subset M \times \mathfrak{g}^* \to \mathfrak{g}^*$ be the projection into the $\mathfrak{g}^*$-factor.

Let $\bar{\mathcal{H}}$ be the closure of $\mathcal{H}$ in $\mathfrak{g}^*$. Since $\bar{\mathcal{H}}$ is a closed subgroup of $(\mathfrak{g}^*, +)$, the quotient $C := \mathfrak{g}^*/\bar{\mathcal{H}}$ is a cylinder (that is, it is isomorphic to the Abelian Lie group $\mathbb{R}^a \times \mathbb{T}^b$ for some $a, b \in \mathbb{N}$). Let $\pi_C\colon \mathfrak{g}^* \to \mathfrak{g}^*/\bar{\mathcal{H}} = C$ be the projection. Define $\mathbf{K}\colon M \to C$ to be the map that makes the following diagram commutative:

$$\begin{array}{ccc} \tilde{M} & \xrightarrow{\tilde{\mathbf{K}}} & \mathfrak{g}^* \\ {\scriptstyle \tilde{p}}\downarrow & & \downarrow{\scriptstyle \pi_C} \\ M & \xrightarrow{\mathbf{K}} & \mathfrak{g}^*/\bar{\mathcal{H}}. \end{array} \tag{2}$$

In other words, $\mathbf{K}$ is defined by $\mathbf{K}(m) = \pi_C(\nu)$, where $\nu \in \mathfrak{g}^*$ is any element such that $(m, \nu) \in \tilde{M}$. This is a good definition because if we have two points $(m, \nu), (m, \nu') \in \tilde{M}$, this implies that $(m, \nu), (m, \nu') \in \tilde{p}^{-1}(m)$ and, as $\mathcal{H}$ is the structure group of the principal fiber bundle $\tilde{p}\colon \tilde{M} \to M$, there exists an element $\rho \in \mathcal{H}$ such that $\nu' = \nu + \rho$. Consequently, $\pi_C(\nu) = \pi_C(\nu')$.

We will refer to $\mathbf{K}\colon M \to \mathfrak{g}^*/\bar{\mathcal{H}} := C$ as a *cylinder valued momentum map* associated to the canonical $\mathfrak{g}$-action on (M, ω). The cylinder valued momentum map is a strict generalization of the standard (Kostant–Souriau) momentum map since it is easy to prove (see for instance [22, Proposition 5.2.10]) that the G-action has a standard momentum map if and only if the holonomy group $\mathcal{H}$ is trivial. In such a case the cylinder valued momentum map *is* a standard momentum map.

Notice that we refer to *"a"* and not to *"the"* cylinder valued momentum map since each choice of the holonomy bundle of the connection (1) defines such a map. In order to see how the definition of $\mathbf{K}$ depends on the choice of the holonomy bundle $\tilde{M}$ take $\tilde{M}_1$ and $\tilde{M}_2$ two holonomy bundles of $(M \times \mathfrak{g}^*, M, \pi, \mathfrak{g}^*)$. We now notice three things. First, there exists $\tau \in \mathfrak{g}^*$ such that $\tilde{M}_2 = R_\tau(\tilde{M}_1)$, where $R_\tau(m, \mu) := (m, \mu + \tau)$, for any $(m, \mu) \in M \times \mathfrak{g}^*$. Second, since $(\mathfrak{g}^*, +)$ is Abelian all the holonomy groups based at any point are the same and hence the projection $\pi_C\colon \mathfrak{g}^* \to \mathfrak{g}^*/\bar{\mathcal{H}}$ in (2) does not depend on the choice of $\tilde{M}$; in view of this remark we will refer to $\mathcal{H}$ as the *Hamiltonian holonomy* of the G-action on (M, ω). Third, π_C is a group homomorphism. Let now $\tilde{p}_{\tilde{M}_i}\colon \tilde{M}_i \to M$, $\tilde{\mathbf{K}}_{\tilde{M}_i}\colon \tilde{M}_i \to \mathfrak{g}^*$, and $\mathbf{K}_{\tilde{M}_i}\colon M \to \mathfrak{g}^*$ be the maps in diagram (2) constructed using the holonomy bundles $\tilde{M}_i$, $i \in \{1, 2\}$. Let $m \in M$. By definition $\mathbf{K}_{\tilde{M}_2}(m) = \mathbf{K}_{\tilde{M}_2}(\tilde{p}_{\tilde{M}_2}(m, \nu))$, where $(m, \nu) \in \tilde{M}_2$. Since $\tilde{M}_2 = R_\tau(\tilde{M}_1)$ there exists an element $\nu' \in \mathfrak{g}^*$ such that $(m, \nu') \in \tilde{M}_1$ and $(m, \nu) = (m, \nu' + \tau)$. Hence,

$$\begin{aligned} \mathbf{K}_{\tilde{M}_2}(m) &= \mathbf{K}_{\tilde{M}_2}(\tilde{p}_{\tilde{M}_2}(m, \nu)) = \mathbf{K}_{\tilde{M}_2}(\tilde{p}_{\tilde{M}_2}(m, \nu' + \tau)) \\ &= \pi_C(\tilde{\mathbf{K}}_{\tilde{M}_2}(m, \nu' + \tau)) = \pi_C(\nu' + \tau) = \pi_C(\nu') + \pi_C(\tau) = \mathbf{K}_{\tilde{M}_1}(m) + \pi_C(\tau). \end{aligned}$$

Since in the previous chain of equalities the point $m \in M$ is arbitrary and $\tau \in \mathfrak{g}^*$ depends only on $\tilde{M}_1$ and $\tilde{M}_2$ we have that

$$\mathbf{K}_{\tilde{M}_2} = \mathbf{K}_{\tilde{M}_1} + \pi_C(\tau).$$

Remark 2.1. The definition of the cylinder valued momentum map that we just introduced can be reproduced if we assume that the form ω is only presymplectic.

Remark 2.2. The Hamiltonian holonomy $\mathcal{H}$ is the image of the *period homomorphism* P_ω: $\pi_1(M, z) \to \mathfrak{g}^*$ defined by

$$\langle P_\omega([\gamma]), \xi \rangle := \int_\gamma \mathbf{i}_{\xi_M} \omega, \qquad \text{for any } \xi \in \mathfrak{g}.$$

The following proposition summarizes the elementary properties of the cylinder valued momentum map.

Proposition 2.3. *Let (M, ω) be a connected and paracompact symplectic manifold and $\mathfrak{g}$ a Lie algebra acting canonically on it. Then any cylinder valued momentum map $\mathbf{K}: M \to C$ associated to this action has the following properties:*

(i) $\mathbf{K}$ *is a smooth map that satisfies Noether's Theorem, that is, for any $\mathfrak{g}$-invariant function $h \in C^\infty(M)^{\mathfrak{g}} := \{f \in C^\infty(M) \mid \mathbf{d}h(\xi_M) = 0 \text{ for all } \xi \in \mathfrak{g}\}$, the flow F_t of its associated Hamiltonian vector field X_h satisfies the identity*

$$\mathbf{K} \circ F_t = \mathbf{K}|_{\mathrm{Dom}(F_t)}.$$

(ii) *For any $v_m \in T_m M$, $m \in M$, we have the relation*

$$T_m\mathbf{K}(v_m) = T_\mu \pi_C(\nu),$$

where $\mu \in \mathfrak{g}^*$ is any element such that $\mathbf{K}(m) = \pi_C(\mu)$ and $\nu \in \mathfrak{g}^*$ is uniquely determined by:

$$\langle \nu, \xi \rangle = (\mathbf{i}_{\xi_M}\omega)(m)(v_m), \quad \textit{for any } \xi \in \mathfrak{g}. \tag{3}$$

(iii) $\ker(T_m\mathbf{K}) = ((\mathrm{Lie}(\bar{\mathcal{H}}))^\circ \cdot m)^\omega$.

(iv) *Bifurcation lemma:*

$$\mathrm{range}\,(T_m\mathbf{K}) = T_\mu \pi_C\left((\mathfrak{g}_m)^\circ\right),$$

where $\mu \in \mathfrak{g}^*$ is any element such that $\mathbf{K}(m) = \pi_C(\mu)$.

Remark 2.4. Later on in Theorem 5.4 we will show that the cylinder valued momentum map remains constant along the flow of functions that are less invariant than those in part (i) of the previous proposition.

Proof: Since $\mathfrak{g}^*/\bar{\mathcal{H}}$ is a homogeneous manifold, the canonical projection $\pi_C: \mathfrak{g}^* \to \mathfrak{g}^*/\bar{\mathcal{H}}$ is a surjective submersion. Moreover, by (2), $\mathbf{K} \circ \tilde{p} = \pi_C \circ \tilde{\mathbf{K}}$ is a smooth map. Thus, since $\tilde{p}$ is a surjective submersion, it follows that the map $\mathbf{K}$ is necessarily smooth.

We start by proving (ii). Let $m \in M$ and $(m, \mu) \in \tilde{p}^{-1}(m)$. If $v_m = T_{(m,\mu)}\tilde{p}(v_m, \nu)$ then (2) gives

$$T_m\mathbf{K}(v_m) = T_m\mathbf{K}(T_{(m,\mu)}\tilde{p}(v_m, \nu)) = T_\mu \pi_C\left(T_{(m,\mu)}\tilde{\mathbf{K}}(v_m, \nu)\right) = T_\mu \pi_C(\nu).$$

(i) We now check that $\mathbf{K}$ satisfies Noether's condition. Let $h \in C^\infty(M)^{\mathfrak{g}}$ and let F_t be the flow of the associated Hamiltonian vector field X_h. Using the expression for the derivative $T_m\mathbf{K}$ in (ii) it follows that $T_m\mathbf{K}(X_h(m)) = T_\mu\pi_C(\nu)$, where $\mu \in \mathfrak{g}^*$ is any element such that $\mathbf{K}(m) = \pi_C(\mu)$ and $\nu \in \mathfrak{g}^*$ is uniquely determined by

$$\langle \nu, \xi \rangle = \left(\mathbf{i}_{\xi_M}\omega\right)(m)(X_h(m)) = -\mathbf{d}h(m)(\xi_M(m)) = \xi_M[h](m) = 0,$$

for all $\xi \in \mathfrak{g}$, which proves that $\nu = 0$ and, consequently, $T_m\mathbf{K}(X_h(m)) = 0$, for all $m \in M$. Finally, as

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{K} \circ F_t)(m) = T_{F_t(m)}\mathbf{K}\,(X_h(F_t(m))) = 0,$$

we have $\mathbf{K} \circ F_t = \mathbf{K}|_{\mathrm{Dom}(F_t)}$, as required.
(iii) Due to the expression in (ii), a vector $v_m \in \ker T_m\mathbf{K}$ if and only if the unique element $\nu \in \mathfrak{g}^*$ determined by (3) satisfies $T_\nu\pi_C(\nu) = 0$, that is, $\nu \in \mathrm{Lie}(\bar{\mathcal{H}})$. Equivalently, we have that $\langle \nu, \xi \rangle = 0$, for any $\xi \in (\mathrm{Lie}(\bar{\mathcal{H}}))^\circ \subset (\mathfrak{g}^*)^* = \mathfrak{g}$ which, in terms of v_m, yields that $(\mathbf{i}_{\xi_M}\omega)(m)(v_m) = 0$ for any $\xi \in (\mathrm{Lie}(\bar{\mathcal{H}}))^\circ$. This can obviously be rewritten by saying that $v_m \in ((\mathrm{Lie}(\bar{\mathcal{H}}))^\circ \cdot m)^\omega$.
(iv) We start by checking that range$\,(T_m\mathbf{K}) \subset T_\mu\pi_C\,((\mathfrak{g}_m)^\circ)$. Let $T_m\mathbf{K}(v_m) \in$ range$\,(T_m\mathbf{K})$. Let $\nu \in \mathfrak{g}^*$ be the element determined by (3) which hence satisfies $T_m\mathbf{K}(v_m) = T_\mu\pi_C(\nu)$. Now, notice that for any $\xi \in \mathfrak{g}_m$ we have that

$$\langle \nu, \xi \rangle = \omega(m)(\xi_M(m), v_m) = 0$$

which implies that $\nu \in (\mathfrak{g}_m)^\circ$. This proves the inclusion range$\,(T_m\mathbf{K}) \subset T_\mu\pi_C((\mathfrak{g}_m)^\circ)$. Hence, the equality will be proven if we show that

$$\mathrm{rank}\,(T_m\mathbf{K}) = \dim(T_\mu\pi_C\,((\mathfrak{g}_m)^\circ)). \tag{4}$$

On one hand we can use the equality in (iii) to obtain

$$\begin{aligned}\mathrm{rank}\,(T_m\mathbf{K}) &= \dim M - \dim(\ker T_m\mathbf{K}) = \dim M - \dim M + \dim((\mathrm{Lie}(\bar{\mathcal{H}}))^\circ \cdot m)\\ &= \dim((\mathrm{Lie}(\bar{\mathcal{H}}))^\circ) - \dim(\mathfrak{g}_m \cap (\mathrm{Lie}(\bar{\mathcal{H}}))^\circ).\end{aligned} \tag{5}$$

On the other hand,

$$\begin{aligned}&\dim(T_\mu\pi_C((\mathfrak{g}_m)^\circ))\\ &\quad= \dim(\mathfrak{g}_m)^\circ - \dim(\ker T_\mu\pi_C|_{(\mathfrak{g}_m)^\circ})\\ &\quad= \dim\mathfrak{g} - \dim\mathfrak{g}_m - \dim(\ker T_\mu\pi_C \cap (\mathfrak{g}_m)^\circ)\\ &\quad= \dim\mathfrak{g} - \dim\mathfrak{g}_m - \dim(\mathrm{Lie}(\bar{\mathcal{H}}) \cap (\mathfrak{g}_m)^\circ)\\ &\quad= \dim\mathfrak{g} - \dim\mathfrak{g}_m - \dim(\mathrm{Lie}(\bar{\mathcal{H}})) - \dim(\mathfrak{g}_m)^\circ + \dim(\mathrm{Lie}(\bar{\mathcal{H}}) + (\mathfrak{g}_m)^\circ)\\ &\quad= -\dim(\mathrm{Lie}(\bar{\mathcal{H}})) + \dim\mathfrak{g} - \dim([\mathrm{Lie}(\bar{\mathcal{H}}) + (\mathfrak{g}_m)^\circ]^\circ)\\ &\quad= \dim(\mathrm{Lie}(\bar{\mathcal{H}}))^\circ) - \dim(\mathfrak{g}_m \cap (\mathrm{Lie}(\bar{\mathcal{H}}))^\circ),\end{aligned}$$

which coincides with (5), thereby establishing (4). □

Proposition 2.5 (The cylinder valued momentum map and restricted actions). *Let (M, ω) be a connected and paracompact symplectic manifold, $\mathfrak{g}$ a Lie algebra acting symplectically on it, and $\mathbf{K}_\mathfrak{g}: M \to \mathfrak{g}^*/\bar{\mathcal{H}}_\mathfrak{g}$ an associated cylinder valued momentum map. Let $\mathfrak{h}$ be a Lie subalgebra of $\mathfrak{g}$ and let $\mathcal{H}_\mathfrak{h}$ be the Hamiltonian holonomy of the $\mathfrak{h}$-action. Then*

(i) *$i^*(\mathcal{H}_{\mathfrak{g}}) \subset \mathcal{H}_{\mathfrak{h}}$, where $i^*\colon \mathfrak{g}^* \to \mathfrak{h}^*$ is the dual of the inclusion $i\colon \mathfrak{h} \hookrightarrow \mathfrak{g}$. Hence, there is a unique Lie group epimorphism $\bar{i}^*\colon \mathfrak{g}^*/\bar{\mathcal{H}}_{\mathfrak{g}} \to \mathfrak{h}^*/\bar{\mathcal{H}}_{\mathfrak{h}}$ such that $\bar{i}^* \circ \pi_{\mathfrak{g}^*} = \pi_{\mathfrak{h}^*} \circ i^*$, with $\pi_{\mathfrak{h}^*}\colon \mathfrak{h}^* \to \mathfrak{h}^*/\bar{\mathcal{H}}_{\mathfrak{h}}$ and $\pi_{\mathfrak{g}^*}\colon \mathfrak{g}^* \to \mathfrak{g}^*/\bar{\mathcal{H}}_{\mathfrak{g}}$ the natural projections.*

(ii) *Let $\hat{i}^*\colon M \times \mathfrak{g}^* \to M \times \mathfrak{h}^*$ be the map given by $\hat{i}^*(m, \mu) := (m, i^*(\mu))$, $(m, \mu) \in M \times \mathfrak{g}^*$, and $\tilde{M}_{\mathfrak{g}}$ the holonomy bundle used in the construction of $\mathbf{K}_{\mathfrak{g}}$ and that contains the point (m_0, μ_0). Let $\tilde{M}_{\mathfrak{h}}$ be the holonomy bundle for the H-action containing the point $(m_0, i^*(\mu_0))$. Then*

$$\hat{i}^*(\tilde{M}_{\mathfrak{g}}) \subset \tilde{M}_{\mathfrak{h}}. \tag{6}$$

(iii) *Let $\mathbf{K}_{\mathfrak{h}}\colon M \to \mathfrak{h}^*/\bar{\mathcal{H}}_{\mathfrak{h}}$ be the $\mathfrak{h}$-cylinder valued momentum map constructed using $\tilde{M}_{\mathfrak{h}}$. Then*

$$\mathbf{K}_{\mathfrak{h}} = \bar{i}^* \circ \mathbf{K}_{\mathfrak{g}}. \tag{7}$$

Proof:

(i) Let $\mu \in \mathcal{H}_{\mathfrak{g}}$ and $c(t) \subset M$ be a loop in M such that $c(0) = c(1) = m_0$ whose horizontal lift $(c(t), \mu(t))$ is such that $\mu(0) = \mu_0$ and $\mu(1) - \mu_0 = \mu$. The horizontality of $(c(t), \mu(t))$ means that for any $\xi \in \mathfrak{g}$ the equality $\langle \mu'(t), \xi \rangle = \omega(c(t))(\xi_M(c(t)), c'(t))$ holds. Consequently $i^*\mu(1) - i^*\mu_0 = i^*\mu$ and $\langle i^*\mu'(t), \eta \rangle = \omega(c(t))(\eta_M(c(t)), c'(t))$, for any $\eta \in \mathfrak{h}$ which proves that $(c(t), i^*\mu(t))$ is the $\mathfrak{h}$-horizontal lift of $c(t)$ passing through $(m_0, i^*\mu_0)$ and hence that $i^*\mu \in \mathcal{H}_{\mathfrak{h}}$. The rest of the statement is a straightforward verification.

(ii) Let $(z, \nu) \in \tilde{M}_{\mathfrak{g}}$. By definition there exists a piecewise smooth $\mathfrak{g}$-horizontal curve $(m(t), \mu(t))$ such that $(m(0), \mu(0)) = (m_0, \mu_0)$ and $(m(1), \mu(1)) = (z, \nu)$. An argument similar to the one that we just used in the proof of (i) shows that the $\mathfrak{g}$-horizontality of $(m(t), \mu(t))$ implies the $\mathfrak{h}$-horizontality of $(m(t), i^*\mu(t)) = \hat{i}^*((m(t), \mu(t)))$ and hence $(m(1), i^*\mu(1)) = \hat{i}^*(z, \nu) \in \tilde{M}_{\mathfrak{h}}$.

(iii) Let $m \in M$ arbitrary. For some $\mu \in \mathfrak{g}^*$ such that $(m, \mu) \in \tilde{M}_{\mathfrak{g}}$ we have that

$$(\bar{i}^* \circ \mathbf{K}_{\mathfrak{g}})(m) = (\bar{i}^* \circ \pi_{\mathfrak{g}^*})(\mu) = (\pi_{\mathfrak{h}^*} \circ i^*)(\mu).$$

On the other hand $\mathbf{K}_{\mathfrak{h}}(m) = \pi_{\mathfrak{h}^*}(\nu)$, for some $\nu \in \mathfrak{h}^*$ such that $(m, \nu) \in \tilde{M}_{\mathfrak{h}}$. Since by (6) $(m, i^*(\mu)) \in \tilde{M}_{\mathfrak{h}}$ we have that $\mathbf{K}_{\mathfrak{h}}(m) = \pi_{\mathfrak{h}}(i^*(\mu)) = (\bar{i}^* \circ \mathbf{K}_{\mathfrak{g}})(m)$, as required. □

3. The equivariance properties of the cylinder valued map

Suppose now that the $\mathfrak{g}$-Lie algebra action on (M, ω) considered in the previous section is obtained from a canonical action of the Lie group G on (M, ω) by taking the infinitesimal generators of all elements in $\mathfrak{g}$. The main goal of this section is the construction of a G-action on the target space of the cylinder valued momentum map $\mathbf{K}\colon M \to \mathfrak{g}^*/\bar{\mathcal{H}}$ with respect to which it is G-equivariant. The following paragraphs generalize to the context of the cylinder valued momentum map the strategy followed by Souriau [27] for the standard momentum map. We start with an important fact about the Hamiltonian holonomy of a symplectic action.

Proposition 3.1. *Let (M, ω) be a connected and paracompact symplectic manifold and $\Phi: G \times M \to M$ a symplectic Lie group action. Then the Hamiltonian holonomy $\mathcal{H}$ is invariant under the coadjoint action, that is,*

$$\mathrm{Ad}^*_{g^{-1}}\mathcal{H} \subset \mathcal{H}, \quad \text{for any } g \in G. \tag{8}$$

Moreover, if G is connected then $\mathcal{H}$ is pointwise fixed by the coadjoint action, that is,

$$\mathrm{Ad}^*_{g^{-1}}\nu = \nu, \quad \text{for any } g \in G \text{ and } \nu \in \mathcal{H}. \tag{9}$$

Proof: We first show that $\mathcal{H}$ is fixed by the coadjoint action as a set, that is, $\mathrm{Ad}^*_{g^{-1}}(\mathcal{H}) \subset \mathcal{H}$, for any $g \in G$. Let $\nu \in \mathcal{H}$ arbitrary. By definition of the holonomy group there exists a loop $c: [0, 1] \to M$ at a point $m \in M$, that is, $c(0) = c(1) = m$, whose horizontal lift $\tilde{c}(t) = (c(t), \mu(t))$ satisfies the relations $\tilde{c}(0) = (m, \mu)$ and $\tilde{c}(1) = (m, \mu + \nu)$, for some $\mu \in \mathfrak{g}^*$. We now show that $\mathrm{Ad}^*_{g^{-1}}\nu \in \mathcal{H}$, for any $g \in G$. Take the loop $d: [0, 1] \to M$ at the point $g \cdot m$ defined by $d(t) := \Phi_g(c(t))$. We will prove the claim by showing that the horizontal lift $\tilde{d}$ of d is given by

$$\tilde{d}(t) = (\Phi_g(c(t)), \mathrm{Ad}^*_{g^{-1}}\mu(t)). \tag{10}$$

If this is the case then $\mathrm{Ad}^*_{g^{-1}}\nu \in \mathcal{H}$ necessarily since $\tilde{d}(0) = (g \cdot m, \mathrm{Ad}^*_{g^{-1}}\mu)$ and $\tilde{d}(1) = (g \cdot m, \mathrm{Ad}^*_{g^{-1}}\mu + \mathrm{Ad}^*_{g^{-1}}\nu)$. In order to establish (10) it suffices to check that $\tilde{d}(t)$ is horizontal. Notice first that

$$\tilde{d}'(t) = (T_{c(t)}\Phi_g(c'(t)), \mathrm{Ad}^*_{g^{-1}}\mu'(t)).$$

Then, for any $\xi \in \mathfrak{g}$, we have

$$\begin{aligned}
&\mathbf{i}_{\xi_M}\omega(\Phi_g(c(t)))(T_{c(t)}\Phi_g(c'(t))) + \langle \mathrm{Ad}^*_{g^{-1}}\mu'(t), \xi\rangle \\
&\quad = \omega(\Phi_g(c(t)))(T_{c(t)}\Phi_g((\mathrm{Ad}_{g^{-1}}\xi)_M(c(t))), T_{c(t)}\Phi_g(c'(t))) + \langle \mu'(t), \mathrm{Ad}_{g^{-1}}\xi\rangle \\
&\quad = \omega(c(t))((\mathrm{Ad}_{g^{-1}}\xi)_M(c(t)), c'(t)) + \langle \mu'(t), \mathrm{Ad}_{g^{-1}}\xi\rangle = 0
\end{aligned}$$

because of the symplectic character of the action and the fact that the curve $\tilde{c}(t)$ is horizontal.

Suppose now that G is connected. In this situation, the inclusion (8) implies that $\mathcal{H}$ is a collection of connected coadjoint orbits. Since $\mathcal{H}$ is zero-dimensional so are these coadjoint orbits, whose connectedness ensures that the points of $\mathcal{H}$ are fixed by the coadjoint action. □

Remark 3.2. The connected component of the identity $\bar{\mathcal{H}}_0$ of $\bar{\mathcal{H}}$ is a vector space. Consequently,

$$\mathrm{Lie}(\bar{\mathcal{H}}) = \bar{\mathcal{H}}_0 \subset \bar{\mathcal{H}}.$$

In order to prove this recall that any Abelian connected Lie group, like $\bar{\mathcal{H}}_0$, is isomorphic to $\mathbb{T}^a \times \mathbb{R}^b$, for some $a, b \in \mathbb{N}$. Since $\bar{\mathcal{H}}_0$ is a closed Lie subgroup of $(\mathfrak{g}^*, +)$ it cannot contain any compact subgroup and hence $a = 0$ necessarily.

Corollary 3.3. *In the hypotheses of Proposition 3.1 the following statements hold*

(i) $\mathrm{Ad}^*_{g^{-1}}(\bar{\mathcal{H}}) \subset \bar{\mathcal{H}}$, *for any* $g \in G$.
(ii) $\mathrm{Ad}^*_{g^{-1}}(\mathrm{Lie}(\bar{\mathcal{H}})) \subset \mathrm{Lie}(\bar{\mathcal{H}})$, *for any* $g \in G$.
(iii) *There is a unique group action* $\mathcal{A}d^* \colon G \times \mathfrak{g}^*/\bar{\mathcal{H}} \to \mathfrak{g}^*/\bar{\mathcal{H}}$ *such that for any* $g \in G$

$$\mathcal{A}d^*_{g^{-1}} \circ \pi_C = \pi_C \circ \mathrm{Ad}^*_{g^{-1}}. \tag{11}$$

The map $\pi_C \colon \mathfrak{g}^* \to \mathfrak{g}^*/\bar{\mathcal{H}}$ *is the projection. We will refer to* $\mathcal{A}d^*$ *as the projected coadjoint action of* G *on* $\mathfrak{g}^*/\bar{\mathcal{H}}$.

Proof:

(i) By (8) and the continuity of the coadjoint action we have $\mathrm{Ad}^*_{g^{-1}}(\bar{\mathcal{H}}) \subset \overline{\mathrm{Ad}^*_{g^{-1}}(\mathcal{H})} \subset \bar{\mathcal{H}}$.
(ii) The inclusion (8) guarantees that the restricted map $\mathrm{Ad}^*_{g^{-1}} \colon \bar{\mathcal{H}} \to \bar{\mathcal{H}}$ is a Lie group homomorphism so it induces a Lie algebra homomorphism (which is itself) $\mathrm{Ad}^*_{g^{-1}} \colon \mathrm{Lie}(\bar{\mathcal{H}}) \to \mathrm{Lie}(\bar{\mathcal{H}})$. In particular this implies the statement.
(iii) The map $\mathcal{A}d^*_{g^{-1}}$ given by $\mathcal{A}d^*_{g^{-1}}(\mu + \bar{\mathcal{H}}) := \mathrm{Ad}^*_{g^{-1}}\mu + \bar{\mathcal{H}}$ is well defined by part (i) and satisfies (11). A straightforward verification shows that the map $\mathcal{A}d^*$ defines an action.

□

As we will see in the following paragraphs, the results that we just proved allow us to reproduce in the context of the cylinder valued momentum map the techniques introduced by Souriau [27] to study the equivariance properties of the standard momentum map.

Proposition 3.4. *Let* (M, ω) *be a connected and paracompact symplectic manifold and* $\Phi \colon G \times M \to M$ *a symplectic Lie group action. Let* $\mathbf{K} \colon M \to \mathfrak{g}^*/\bar{\mathcal{H}}$ *be a cylinder valued momentum map for this action. Define* $\bar{\sigma} \colon G \times M \to \mathfrak{g}^*/\bar{\mathcal{H}}$ *by*

$$\bar{\sigma}(g, m) := \mathbf{K}(\Phi_g(m)) - \mathcal{A}d^*_{g^{-1}}\mathbf{K}(m). \tag{12}$$

Then:

(i) *The map* $\bar{\sigma}$ *does not depend on the points* $m \in M$ *and hence it defines a map* $\sigma \colon G \to \mathfrak{g}^*/\bar{\mathcal{H}}$.
(ii) *The map* $\sigma \colon G \to \mathfrak{g}^*/\bar{\mathcal{H}}$ *is a group valued one-cocycle, that is, for any* $g, h \in G$, *it satisfies the equality*

$$\sigma(gh) = \sigma(g) + \mathcal{A}d^*_{g^{-1}}\sigma(h).$$

(iii) *The map*

$$\begin{aligned} \Theta \colon G \times \mathfrak{g}^*/\bar{\mathcal{H}} &\longrightarrow \mathfrak{g}^*/\bar{\mathcal{H}} \\ (g, \pi_C(\mu)) &\longmapsto \mathcal{A}d^*_{g^{-1}}(\pi_C(\mu)) + \sigma(g), \end{aligned}$$

defines a G*-action on* $\mathfrak{g}^*/\bar{\mathcal{H}}$ *with respect to which the cylinder valued momentum map* $\mathbf{K}$ *is* G*-equivariant, that is, for any* $g \in G$, $m \in M$,

$$\mathbf{K}(\Phi_g(m)) = \Theta_g(\mathbf{K}(m)).$$

(iv) *The infinitesimal generators of the affine G-action on $\mathfrak{g}^*/\bar{\mathcal{H}}$ are given by the expression*

$$\xi_{\mathfrak{g}^*/\bar{\mathcal{H}}}(\pi_C(\mu)) = -T_\mu \pi_C\,(\Psi(m)(\xi, \cdot)), \tag{13}$$

for any $\xi \in \mathfrak{g}$, $(m, \mu) \in \tilde{M}$, and where $\Psi\colon M \to Z^2(\mathfrak{g})$ is the Chu map defined by $\Psi(\xi, \eta) := \omega\,(\xi_M, \eta_M)$, for any $\xi, \eta \in \mathfrak{g}$.

We will refer to $\sigma\colon G \to \mathfrak{g}^*/\bar{\mathcal{H}}$ as the non-equivariance one-cocycle of the cylinder valued momentum map $\mathbf{K}\colon M \to \mathfrak{g}^*/\bar{\mathcal{H}}$ and to Θ as the affine G-action on $\mathfrak{g}^*/\bar{\mathcal{H}}$ induced by σ.

Proof:

(i) For any $g \in G$ define the map $\tau_g\colon M \to \mathfrak{g}^*/\bar{\mathcal{H}}$ by $\tau_g(m) := \bar{\sigma}(g, m)$. We will prove the claim by showing that τ_g is a constant map. Indeed, for any point $z \in M$ and any vector $v_z \in T_z M$

$$T_z\tau_g(v_z) = T_{g\cdot z}\mathbf{K}(T_z\Phi_g(v_z)) - T_{\mathbf{K}(z)}\mathcal{A}d^*_{g^{-1}}\,(T_z\mathbf{K}(v_z))\,. \tag{14}$$

Recall now that by part (ii) of Proposition 2.3, $T_z\mathbf{K}(v_z) = T_\mu\pi_C(\nu)$, where $\mu \in \mathfrak{g}^*$ is any element such that $\mathbf{K}(z) = \pi_C(\mu)$ and $\nu \in \mathfrak{g}^*$ is uniquely determined by the equality $\langle \nu, \xi\rangle = (\mathbf{i}_{\xi_M}\omega)(z)(v_z)$, for any $\xi \in \mathfrak{g}$. Equivalently, the relation between v_z and ν can be expressed by saying that the pair $(v_z, \nu) \in T_{(z,\mu)}\tilde{M}$, where $\tilde{M}$ is the holonomy bundle of the connection α in (1) used in the definition of the cylinder valued momentum map $\mathbf{K}$. Let now $\mu' \in \mathfrak{g}^*$ be such that $(g \cdot z, \mu') \in \tilde{M}$. We now show that $(v_z, \nu) \in T_{(z,\mu)}\tilde{M}$ implies that $(T_z\Phi_g(v_z), \mathrm{Ad}^*_{g^{-1}}\nu) \in T_{(g\cdot z,\mu')}\tilde{M}$ due to the symplectic character of the G-action. Indeed, for any $\xi \in \mathfrak{g}$ we have

$$\begin{aligned}
\mathbf{i}_{\xi_M}\omega(g\cdot z)(T_z\Phi_g(v_z)) &= \omega(g\cdot z)(T_z\Phi_g(\mathrm{Ad}_{g^{-1}}\xi)_M(z), T_z\Phi_g(v_z))\\
&= \omega(z)((\mathrm{Ad}_{g^{-1}}\xi)_M(z), v_z)\\
&= \langle \nu, \mathrm{Ad}_{g^{-1}}\xi\rangle = \langle \mathrm{Ad}^*_{g^{-1}}\nu, \xi\rangle,
\end{aligned}$$

which proves that $(T_z\Phi_g(v_z), \mathrm{Ad}^*_{g^{-1}}\nu) \in T_{(g\cdot z,\mu')}\tilde{M}$ and hence that $T_{g\cdot z}\mathbf{K}(T_z\Phi_g(v_z)) = T_{\mu'}\pi_C(\mathrm{Ad}^*_{g^{-1}}\nu)$. If we use this fact to compute the derivatives on the right-hand side of (14) we obtain

$$\begin{aligned}
T_z\tau_g(v_z) &= T_{\mu'}\pi_C(\mathrm{Ad}^*_{g^{-1}}\nu) - T_{\mathbf{K}(z)}\mathcal{A}d^*_{g^{-1}}(T_\mu\pi_C(\nu))\\
&= T_{\mu'}\pi_C(\mathrm{Ad}^*_{g^{-1}}\nu) - T_\mu(\pi_C \circ \mathrm{Ad}^*_{g^{-1}})(\nu)\\
&= T_{\mu'}\pi_C(\mathrm{Ad}^*_{g^{-1}}\nu) - T_\mu\pi_C(\mathrm{Ad}^*_{g^{-1}}\nu) = 0.
\end{aligned} \tag{15}$$

The last equality follows from the fact that π_C is an Abelian Lie group homomorphism. Indeed, for any $\rho, \mu \in \mathfrak{g}^*$

$$T_\mu\pi_C(\rho) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}\pi_C(\mu + t\rho) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}(\pi_C(\mu) + \pi_C(t\rho)) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}\pi_C(t\rho) = T_0\pi_C(\rho),$$

and hence $T_\mu \pi_C = T_0 \pi_C$, for any $\mu \in \mathfrak{g}^*$. Finally, the expression (15) shows that $T\tau_g = 0$, for any $g \in G$. As M is, by hypothesis, connected this guarantees that $\bar{\sigma}(g, m) = \bar{\sigma}(g, m')$ for any $m, m' \in M$, which proves the claim.

(ii) Using the definition (12) at the point $h \cdot m$ we obtain $\sigma(g) = \mathbf{K}(gh \cdot m) - \mathcal{A}d^*_{g^{-1}}\mathbf{K}(h \cdot m)$. If we now use the point $m \in M$ we can write $\sigma(h) = \mathbf{K}(h \cdot m) - \mathcal{A}d^*_{h^{-1}}\mathbf{K}(m)$. Consequently,

$$\sigma(g) + \mathcal{A}d^*_{g^{-1}}\sigma(h) = \mathbf{K}(gh \cdot m) - \mathcal{A}d^*_{g^{-1}}\mathbf{K}(h \cdot m) + \mathcal{A}d^*_{g^{-1}}\mathbf{K}(h \cdot m) - \mathcal{A}d^*_{g^{-1}}\mathcal{A}d^*_{h^{-1}}\mathbf{K}(m)$$
$$= \mathbf{K}(gh \cdot m) - \mathcal{A}d^*_{(gh)^{-1}}\mathbf{K}(m) = \sigma(gh),$$

as required.

(iii) It is a straightforward consequence of the definition.

(iv) By definition

$$\xi_{\mathfrak{g}^*/\bar{\mathcal{H}}}(\pi_C(\mu)) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \Theta_{\exp t\xi}(\pi_C(\mu))$$
$$= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \left(\pi_C\left(\mathrm{Ad}^*_{\exp(-t\xi)}\mu\right) + \sigma(\exp t\xi)\right) = -T_\mu\pi_C(\mathrm{ad}^*_\xi\mu) + T_e\sigma(\xi). \quad (16)$$

Additionally, if $\mathbf{K}(m) = \pi_C(\mu)$ then

$$T_e\sigma(\xi) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \left(\mathbf{K}(\exp t\xi \cdot m) - \mathcal{A}d^*_{\exp(-t\xi)}\mathbf{K}(m)\right) = T_m\mathbf{K}\left(\xi_M(m)\right) + T_\mu\pi_C(\mathrm{ad}^*_\xi\mu).$$

By (3), $T_m\mathbf{K}\left(\xi_M(m)\right) = T_\mu\pi_C(\nu)$ with $\nu \in \mathfrak{g}^*_m$ uniquely determined by the expression $\langle \nu, \eta\rangle = (\mathbf{i}_{\eta_M}\omega)(m)\left(\xi_M(m)\right) = \Psi(m)(\eta, \xi)$, for any $\eta \in \mathfrak{g}$. Hence (16) yields (13). □

4. Poisson structures on $\mathfrak{g}^*/\bar{\mathcal{H}}$

In the following theorem, we present a Poisson structure on the target space of the cylinder valued momentum map $\mathbf{K}: M \to \mathfrak{g}^*/\bar{\mathcal{H}}$ with respect to which this mapping becomes a Poisson map. We also see that the symplectic leaves of this Poisson structure can be described as the orbits of the affine action introduced in the previous section with respect to a subgroup of G whose definition is related to the non-closedness of the Hamiltonian holonomy $\mathcal{H}$ as a subspace of $\mathfrak{g}^*$.

Theorem 4.1. *Let (M, ω) be a connected paracompact symplectic manifold acted symplectically upon by the Lie group G. Let $\mathbf{K}: M \to \mathfrak{g}^*/\bar{\mathcal{H}}$ be a cylinder valued momentum map for this action with non-equivariance cocycle $\sigma: M \to \mathfrak{g}^*/\bar{\mathcal{H}}$ and defined using the holonomy bundle $\tilde{M} \subset M \times \mathfrak{g}^*$. The bracket $\{\cdot,\cdot\}_{\mathfrak{g}^*/\bar{\mathcal{H}}}: C^\infty(\mathfrak{g}^*/\bar{\mathcal{H}}) \times C^\infty(\mathfrak{g}^*/\bar{\mathcal{H}}) \to \mathbb{R}$ defined by*

$$\{f, g\}_{\mathfrak{g}^*/\bar{\mathcal{H}}}(\pi_C(\mu)) = \Psi(m)\left(\frac{\delta(f\circ\pi_C)}{\delta\mu}, \frac{\delta(g\circ\pi_C)}{\delta\mu}\right), \quad (17)$$

where $f, g \in C^\infty(\mathfrak{g}^*/\bar{\mathcal{H}})$, $(m,\mu) \in \tilde{M}$, $\pi_C : \mathfrak{g}^* \to \mathfrak{g}^*/\bar{\mathcal{H}}$ *is the projection, and* $\Psi : M \to Z^2(\mathfrak{g})$ *is the Chu map, defines a Poisson structure on* $\mathfrak{g}^*/\bar{\mathcal{H}}$ *such that*

(i) $\mathbf{K}: M \to \mathfrak{g}^*/\bar{\mathcal{H}}$ *is a Poisson map.*

(ii) *The annihilator* $(\mathrm{Lie}(\bar{\mathcal{H}}))^\circ \subset \mathfrak{g}$ *of* $\mathrm{Lie}(\bar{\mathcal{H}})$ *in* $\mathfrak{g}^*$ *is an ideal in* $\mathfrak{g}$. *Let* $N \subset G$ *be a connected normal Lie subgroup of* G *whose Lie algebra is* $\mathfrak{n} := (\mathrm{Lie}(\bar{\mathcal{H}}))^\circ$. *The symplectic leaves of* $(\mathfrak{g}^*/\bar{\mathcal{H}}, \{\cdot,\cdot\}_{\mathfrak{g}^*/\bar{\mathcal{H}}})$ *are the orbits of the affine* N*-action on* $\mathfrak{g}^*/\bar{\mathcal{H}}$ *induced by* $\sigma : G \to \mathfrak{g}^*/\bar{\mathcal{H}}$.

(iii) *For any* $[\mu] := \pi_C(\mu) \in \mathfrak{g}^*/\bar{\mathcal{H}}$, *the symplectic form* $\omega^+_{N\cdot[\mu]}$ *on the affine orbit* $N \cdot [\mu]$ *induced by the Poisson structure (17) is given by*

$$\begin{aligned}\omega^+_{N\cdot[\mu]}([\mu])(\xi_{\mathfrak{g}^*/\bar{\mathcal{H}}}([\mu]), \eta_{\mathfrak{g}^*/\bar{\mathcal{H}}}([\mu])) &= \omega^+_{N\cdot[\mu]}([\mu])(-T_\mu\pi_C(\Psi(m)(\xi,\cdot)),\\ &\quad - T_\mu\pi_C(\Psi(m)(\eta,\cdot)))\\ &= \Psi(m)(\xi,\eta),\end{aligned}$$

for any $\xi, \eta \in \mathfrak{n}$, $(m,\mu) \in \tilde{M}$.

The proof of this theorem requires several preliminary considerations.

Lemma 4.2. *Let* $\mathcal{H}$ *be the Hamiltonian holonomy in the statement of the previous theorem. Then*

$$\bar{\mathcal{H}} \subset [\mathfrak{g}, \mathfrak{g}]^\circ. \tag{18}$$

Moreover

$$[\mathfrak{g}, \mathfrak{g}] \subset (\mathrm{Lie}(\bar{\mathcal{H}}))^\circ \tag{19}$$

and hence $(\mathrm{Lie}(\bar{\mathcal{H}}))^\circ$ *is an ideal of* $\mathfrak{g}$.

Proof: Let $\mu \in \mathcal{H}$ be arbitrary. By definition, there exists a loop $c(t)$ in M such that $c(0) = c(1) = m$ and a horizontal lift $\gamma(t) := (c(t), \mu(t)) \in \tilde{M} \subset M \times \mathfrak{g}^*$ such that $\mu(1) - \mu(0) = \mu$. Since $\gamma(t)$ is horizontal we have $\mathbf{i}_{\xi_M}\omega(c'(t)) = \langle \mu'(t), \xi\rangle$, for any $\xi \in \mathfrak{g}$. Since the G-action is symplectic, the infinitesimal generator vector fields ξ_M and η_M are locally Hamiltonian, for any $\xi, \eta \in \mathfrak{g}$, and hence $[\xi,\eta]_M = -[\xi_M, \eta_M]$ is globally Hamiltonian. Let $f \in C^\infty(M)$ be such that $X_f = [\xi,\eta]_M = -[\xi_M,\eta_M]$. The relation $\mu = \mu(1) - \mu(0) = \int_0^1 \mu'(t)\mathrm{d}t$ implies thus

$$\begin{aligned}\langle \mu, [\xi,\eta]\rangle &= \int_0^1 \langle \mu'(t), [\xi,\eta]\rangle \mathrm{d}t = \int_0^1 \mathbf{i}_{[\xi,\eta]_M}\omega(c'(t))\mathrm{d}t\\ &= \int_0^1 \omega(c(t))([\xi,\eta]_M(c(t)), c'(t))\mathrm{d}t\\ &= \int_0^1 \omega(c(t))(X_f(c(t)), c'(t))\mathrm{d}t = \int_0^1 \mathbf{d}f(c(t))(c'(t))\mathrm{d}t\\ &= f(c(1)) - f(c(0)) = f(m) - f(m) = 0.\end{aligned}$$

This shows that $\mathcal{H} \subset [\mathfrak{g}, \mathfrak{g}]^\circ$ and hence that $\bar{\mathcal{H}} \subset \overline{[\mathfrak{g}, \mathfrak{g}]^\circ} = [\mathfrak{g}, \mathfrak{g}]^\circ$. The inclusion (19) is a consequence of (18) and Remark 3.2. □

4.1. Projected Poisson structures

The Poisson structure in Theorem 4.1 will be obtained as the projection onto the quotient $\mathfrak{g}^*/\bar{\mathcal{H}}$ of an affine Lie–Poisson structure on $\mathfrak{g}^*$. The next proposition proves the existence of this projected Poisson structure.

Proposition 4.3. *Let $\mathcal{H}$ be the Hamiltonian holonomy in the statement of Theorem 4.1. Let $\Sigma\colon \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ be an arbitrary Lie algebra two-cocycle on $\mathfrak{g}$ and $\{\cdot, \cdot\}^\Sigma_\pm$ the associated affine Lie–Poisson bracket on $\mathfrak{g}^*$ defined by*

$$\{f, g\}^\Sigma_\pm(\mu) := \pm\left\langle \mu, \left[\frac{\delta f}{\delta \mu}, \frac{\delta g}{\delta \mu}\right]\right\rangle \mp \Sigma\left(\frac{\delta f}{\delta \mu}, \frac{\delta g}{\delta \mu}\right), \qquad f, g \in C^\infty(\mathfrak{g}^*). \tag{20}$$

The action $\phi\colon \bar{\mathcal{H}} \times \mathfrak{g}^* \to \mathfrak{g}^*$ given by $\phi_\nu(\mu) := \mu + \nu$ on $(\mathfrak{g}^*, \{\cdot, \cdot\}^\Sigma_\pm)$ is free, proper, and Poisson and hence it induces a unique Poisson structure $\{\cdot, \cdot\}^{\pm\Sigma}_{\mathfrak{g}^*/\bar{\mathcal{H}}}$ on $\mathfrak{g}^*/\bar{\mathcal{H}}$ such that

$$\{F, H\}^{\pm\Sigma}_{\mathfrak{g}^*/\bar{\mathcal{H}}} \circ \pi_C = \{F \circ \pi_C, H \circ \pi_C\}^\Sigma_\pm, \qquad F, H \in C^\infty(\mathfrak{g}^*/\bar{\mathcal{H}}). \tag{21}$$

where $\pi_C\colon \mathfrak{g}^* \to \mathfrak{g}^*/\bar{\mathcal{H}}$ is the projection.

Proof: We first note that if $\mu, \nu \in \mathfrak{g}^*$ and $f \in C^\infty(\mathfrak{g}^*)$ then

$$\frac{\delta(f \circ \phi_\nu)}{\delta \mu} = \frac{\delta f}{\delta(\mu + \nu)}.$$

With this equality in mind we can write, for any $f, g \in C^\infty(\mathfrak{g}^*)$, $\mu \in \mathfrak{g}^*$, and $\nu \in \bar{\mathcal{H}}$

$$\begin{aligned}
\{\phi^*_\nu f, \phi^*_\nu g\}^\Sigma_\pm(\mu) &= \pm\left\langle \mu, \left[\frac{\delta(f \circ \phi_\nu)}{\delta \mu}, \frac{\delta(g \circ \phi_\nu)}{\delta \mu}\right]\right\rangle \mp \Sigma\left(\frac{\delta(f \circ \phi_\nu)}{\delta \mu}, \frac{\delta(g \circ \phi_\nu)}{\delta \mu}\right) \\
&= \pm\left\langle \mu, \left[\frac{\delta f}{\delta(\mu + \nu)}, \frac{\delta g}{\delta(\mu + \nu)}\right]\right\rangle \mp \Sigma\left(\frac{\delta f}{\delta(\mu + \nu)}, \frac{\delta g}{\delta(\mu + \nu)}\right) \\
&= \{f, g\}^\Sigma_\pm(\phi_\nu(\mu)) - \left\langle \nu, \left[\frac{\delta f}{\delta(\mu + \nu)}, \frac{\delta g}{\delta(\mu + \nu)}\right]\right\rangle = \{f, g\}^\Sigma_\pm(\phi_\nu(\mu))
\end{aligned}$$

since $\langle \nu, [\frac{\delta f}{\delta(\mu+\nu)}, \frac{\delta g}{\delta(\mu+\nu)}]\rangle = 0$ by Lemma 4.2. This shows that the action ϕ is Poisson. It is clearly free and proper. □

4.2. Central extensions and their actions

The description of the symplectic leaves of the projected Poisson structures $(\mathfrak{g}^*/\bar{\mathcal{H}}, \{\cdot, \cdot\}^{\pm\Sigma}_{\mathfrak{g}^*/\bar{\mathcal{H}}})$ requires the use of certain facts on central extensions that we present in the following paragraphs.

Let $\mathfrak{g}_\Sigma := \mathfrak{g} \oplus \mathbb{R}$ be the one-dimensional central extension of the Lie algebra $\mathfrak{g}$ determined by the cocycle Σ whose bracket is given by the expression

$$[(\xi, s), (\eta, t)] := ([\xi, \eta], -\Sigma(\xi, \eta)) .$$

Let G_Σ be the simply connected Lie group whose Lie algebra is $\mathfrak{g}_\Sigma$. Let $\pi_{\mathfrak{g}} \colon \mathfrak{g}_\Sigma \to \mathfrak{g}$ be the projection and let $\pi_G \colon G_\Sigma \to G$ be the unique Lie group homomorphism whose derivative is $\pi_{\mathfrak{g}}$ and makes the diagram

$$\begin{array}{ccccccccc}
\{0\} & \longrightarrow & \mathbb{R} & \longrightarrow & \mathfrak{g}_\Sigma := \mathfrak{g} \oplus \mathbb{R} & \xrightarrow{\pi_{\mathfrak{g}} = T_e \pi_G} & \mathfrak{g} & \longrightarrow & \{0\} \\
 & & \Big\downarrow \exp_{\mathfrak{g}_\Sigma}\big|_{\mathbb{R}} & & \Big\downarrow \exp_{\mathfrak{g}_\Sigma} & & \Big\downarrow \exp_{\mathfrak{g}} & & \\
\{e\} & \longrightarrow & \ker \pi_G & \longrightarrow & G_\Sigma & \xrightarrow{\pi_G} & G & \longrightarrow & \{e\}
\end{array}$$

commutative. We notice that the connected component of the identity $(\ker \pi_G)_0$ of $\ker \pi_G$ equals

$$(\ker \pi_G)_0 = \{\exp_{\mathfrak{g}_\Sigma}(0, a) \mid a \in \mathbb{R}\}. \tag{22}$$

Indeed, for any $a \in \mathbb{R}$, $\pi_G(\exp_{\mathfrak{g}_\Sigma}(0, a)) = \exp_{\mathfrak{g}}(0) = e$. This shows that the one-dimensional connected Lie subgroup $\{\exp_{\mathfrak{g}_\Sigma}(0, a) \mid a \in \mathbb{R}\}$ of G_Σ is included in $(\ker \pi_G)_0$. Since $\dim(\ker \pi_G) = 1$ it follows that $\{\exp_{\mathfrak{g}_\Sigma}(0, a) \mid a \in \mathbb{R}\}$ is open and hence closed in $(\ker \pi_G)_0$ thus they are equal. Additionally,

$$(\ker \pi_G)_0 \subset (Z(G_\Sigma))_0, \tag{23}$$

where $(Z(G_\Sigma))_0$ denotes the connected component of the identity of the center $Z(G_\Sigma)$ of G_Σ. To see this, note that any element $g \in G_\Sigma$ can be written as $g = \exp_{\mathfrak{g}_\Sigma}(\xi_1, a_1) \cdots \exp_{\mathfrak{g}_\Sigma}(\xi_1, a_1)$, $\xi_1, \ldots, \xi_n \in \mathfrak{g}$, $a_1, \ldots, a_n \in \mathbb{R}$ and hence, by the Baker–Campbell–Hausdorff formula, the elements of the form $\exp_{\mathfrak{g}_\Sigma}(0, a)$ commute with every factor in the decomposition of g.

Furthermore, if $Z(G) = \{e\}$ then (23) is an equality, that is,

$$(\ker \pi_G)_0 = (Z(G_\Sigma))_0. \tag{24}$$

In order to prove this fact recall that, by general theory, the map $\exp_{\mathfrak{g}_\Sigma}$ is an isomorphism between the center of $\mathfrak{g}_\Sigma$ and $Z(G_\Sigma)_0$. Since $\mathfrak{g}$ has no center the dimension of $Z(G_\Sigma)$ equals one. Additionally, since by (22) and (23), $(\ker \pi_G)_0$ is a one-dimensional connected subgroup of $(Z(G_\Sigma))_0$ the equality (24) follows.

Proposition 4.4. *Let G be a Lie group with Lie algebra $\mathfrak{g}$, Σ a Lie algebra two-cocycle on $\mathfrak{g}$, and $\mathfrak{g}_\Sigma$ the one-dimensional central extension of $\mathfrak{g}$ determined by Σ. Let G_Σ be the connected and simply connected Lie group whose Lie algebra is $\mathfrak{g}_\Sigma$. There exists a smooth group one-cocycle $\mu_\Sigma \colon G_\Sigma \longrightarrow \mathfrak{g}^*$ such that for any $g, h \in G_\Sigma$, $(\xi, s) \in \mathfrak{g}_\Sigma$, $(\nu, a) \in \mathfrak{g}_\Sigma^*$ we have*

(i) $\mathbf{Ad}_g(\xi, s) = (\mathbf{Ad}_{\pi_G(g)}\xi, s + \langle \mu_\Sigma(g), \xi\rangle)$.
(ii) $\mathrm{Ad}^*_{g^{-1}}(\nu, a) = (\mathrm{Ad}^*_{\pi_G(g)^{-1}}\nu + a\mu_\Sigma(g^{-1}), a)$.
(iii) $\mu_\Sigma(gh) = \mu_\Sigma(h) + \mathrm{Ad}^*_{\pi_G(h)}\mu_\Sigma(g)$.
(iv) $\mu_\Sigma(e) = 0$.
(v) $\mu_\Sigma(g^{-1}) = -\mathrm{Ad}^*_{\pi_G(g)^{-1}}\mu_\Sigma(g)$.
(vi) $T_g\mu_\Sigma(T_eL_g(\xi, s)) = -\Sigma(\xi, \cdot) + \mathrm{ad}^*_\xi\mu_\Sigma(g)$.

In this statement we have identified $\mathfrak{g}^*_\Sigma$ with $\mathfrak{g}^* \oplus \mathbb{R}$ by using the pairing $\langle(\nu, a), (\xi, s)\rangle := \langle \nu, \xi\rangle + as$. We will refer to μ_Σ as the $\mathfrak{g}^*$-valued one-cocycle associated to Σ.

Proof: We first notice that

$$\pi_{\mathfrak{g}}(\mathrm{Ad}_g(\xi, s)) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \pi_G(g \exp_{\mathfrak{g}_\Sigma} t(\xi, s)g^{-1}) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \pi_G(g)\pi_G(\exp_{\mathfrak{g}_\Sigma} t(\xi, s))\pi_G(g^{-1})$$
$$= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \pi_G(g)\exp_{\mathfrak{g}}(t\xi)\pi_G(g^{-1}) = \mathrm{Ad}_{\pi_G(g)}\xi.$$

Consequently, there exists a smooth function $f\colon G_\Sigma \times \mathfrak{g} \times \mathbb{R} \longrightarrow \mathbb{R}$ such that

$$\mathrm{Ad}_g(\xi, s) = (\mathrm{Ad}_{\pi_G(g)}\xi, f(g, \xi, s)).$$

Note that for g fixed the map $f(g, \cdot, \cdot)$ is linear by the linearity of Ad_g, hence there exist smooth maps $\mu_\Sigma\colon G_\Sigma \to \mathfrak{g}^*$ and $s_\Sigma\colon G_\Sigma \to \mathbb{R}$ such that

$$f(g, \xi, s) = \langle \mu_\Sigma(g), \xi\rangle + s_\Sigma(g)s. \tag{25}$$

Additionally, by (22) and (23), the elements of the form $\exp_{\mathfrak{g}_\Sigma}(0, s)$ belong to the center $Z(G_\Sigma)$ of G_Σ and hence we have

$$\mathrm{Ad}_g(0, s) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} g\exp_{\mathfrak{g}_\Sigma} t(0, s)g^{-1} = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \exp_{\mathfrak{g}_\Sigma} t(0, s) = (0, s).$$

Hence, combining this with (25), we get $\mathrm{Ad}_g(0, s) = (0, s_\Sigma(g)s) = (0, s)$, which implies that $s_\Sigma(g) = 1$, for any $g \in G_\Sigma$. This proves point (i).

(ii) For $(\xi, s) \in \mathfrak{g}_\Sigma$ we have by (i),

$$\langle \mathrm{Ad}^*_{g^{-1}}(\nu, a), (\xi, s)\rangle = \left\langle (\nu, a), \mathrm{Ad}_{g^{-1}}(\xi, s)\right\rangle = \left\langle (\nu, a), \left(\mathrm{Ad}_{\pi_G(g)^{-1}}\xi, s + \langle \mu_\Sigma(g^{-1}), \xi\rangle\right)\right\rangle$$
$$= \left\langle \nu, \mathrm{Ad}_{\pi_G(g)^{-1}}\xi\right\rangle + as + a\langle \mu_\Sigma(g^{-1}), \xi\rangle$$
$$= \left\langle \mathrm{Ad}^*_{\pi_G(g)^{-1}}\nu + a\mu_\Sigma(g^{-1}), \xi\right\rangle + as,$$

which proves (ii).

(iii) For any $g, h \in G_\Sigma$ and any $(\xi, s) \in \mathfrak{g}_\Sigma$ we have by (i),

$$\begin{aligned}\left(\mathrm{Ad}_{\pi_G(gh)}\xi, s + \langle \mu_\Sigma(gh), \xi\rangle\right) &= \mathrm{Ad}_{gh}(\xi, s) = \mathrm{Ad}_g(\mathrm{Ad}_h(\xi, s))\\ &= \mathrm{Ad}_g(\mathrm{Ad}_{\pi_G(h)}\xi, s + \langle \mu_\Sigma(h), \xi\rangle)\\ &= \left(\mathrm{Ad}_{\pi_G(g)\pi_G(h)}\xi, s + \langle \mu_\Sigma(h), \xi\rangle + \langle \mu_\Sigma(g), \mathrm{Ad}_{\pi_G(h)}\xi\rangle\right)\\ &= \left(\mathrm{Ad}_{\pi_G(g)\pi_G(h)}\xi, s + \langle \mu_\Sigma(h), \xi\rangle + \langle \mathrm{Ad}^*_{\pi_G(h)}\mu_\Sigma(g), \xi\rangle\right)\end{aligned}$$

which implies (iii).

(iv) By part (iii) we can write $\mu_\Sigma(e) = \mu_\Sigma(ee) = \mu_\Sigma(e) + \mu_\Sigma(e)$, and hence $\mu_\Sigma(e) = 0$.

(v) By parts (iii) and (iv) we have $0 = \mu_\Sigma(e) = \mu_\Sigma(gg^{-1}) = \mu_\Sigma(g^{-1}) + \mathrm{Ad}^*_{\pi_G(g)^{-1}}\mu_\Sigma(g)$.

(vi) For any $(\xi, s), (\eta, t) \in \mathfrak{g}_\Sigma$,

$$\begin{aligned}([\xi, \eta], -\Sigma(\xi, \eta)) = [(\xi, s), (\eta, t)] = \mathrm{ad}_{(\xi,s)}(\eta, t) &= \left.\frac{\mathrm{d}}{\mathrm{d}\epsilon}\right|_{\epsilon=0} \mathrm{Ad}_{\exp_{\mathfrak{g}_\Sigma} \epsilon(\xi,s)}(\eta, t)\\ &= \left.\frac{\mathrm{d}}{\mathrm{d}\epsilon}\right|_{\epsilon=0} \left(\mathrm{Ad}_{\exp_{\mathfrak{g}} \epsilon\xi}\eta, t + \langle \mu_\Sigma(\exp_{\mathfrak{g}_\Sigma} \epsilon(\xi, s)), \eta\rangle\right)\\ &= ([\xi, \eta], \langle T_e\mu_\Sigma(\xi, s), \eta\rangle),\end{aligned}$$

which proves that

$$T_e\mu_\Sigma(\xi, s) = -\Sigma(\xi, \cdot).$$

If we now use this equality together with (iii) we obtain

$$\begin{aligned}T_g\mu_\Sigma(T_eL_g(\xi, s)) &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \mu_\Sigma\left(g \exp_{\mathfrak{g}_\Sigma} t(\xi, s)\right)\\ &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \left(\mu_\Sigma\left(\exp_{\mathfrak{g}_\Sigma} t(\xi, s)\right) + \mathrm{Ad}^*_{\exp_{\mathfrak{g}} t\xi}\mu_\Sigma(g)\right)\\ &= T_e\mu_\Sigma(\xi, s) + \mathrm{ad}^*_\xi\mu_\Sigma(g) = -\Sigma(\xi, \cdot) + \mathrm{ad}^*_\xi\mu_\Sigma(g).\end{aligned}$$ □

Remark 4.5. It is worth mentioning that μ_Σ induces a Lie algebra two-coboundary $\bar{\Sigma}$ on $\mathfrak{g}_\Sigma$ and hence the one-dimensional central extension of $\mathfrak{g}_\Sigma$ defined by it is the product Lie algebra $\mathfrak{g}_\Sigma \oplus \mathbb{R}$. Indeed, using part (vi) in the previous proposition $\bar{\Sigma}((\xi, s), (\eta, t)) := \langle T_e\mu_\Sigma(\xi, s), \eta\rangle = -\Sigma(\xi, \eta)$, which is clearly a coboundary.

Corollary 4.6. *Let G be a Lie group with Lie algebra $\mathfrak{g}$, Σ a Lie algebra two-cocycle on $\mathfrak{g}$, and $\mathfrak{g}_\Sigma$ the one-dimensional central extension of $\mathfrak{g}$ determined by Σ. Let G_Σ be the connected and simply connected Lie group whose Lie algebra is $\mathfrak{g}_\Sigma$ and $\mu_\Sigma\colon G_\Sigma \longrightarrow \mathfrak{g}^*$ the one-cocycle associated to Σ. The map $\bar{\Xi}\colon G_\Sigma \times \mathfrak{g}^* \longrightarrow \mathfrak{g}^*$ defined by*

$$\bar{\Xi}(g, \mu) := \mathrm{Ad}^*_{\pi_G(g)^{-1}}\mu + \mu_\Sigma(g^{-1}), \qquad g \in G_\Sigma, \mu \in \mathfrak{g}^*,$$

is an action of the Lie group G_Σ on $\mathfrak{g}^*$. We will refer to $\bar{\Xi}$ as the "extended affine action" of G_Σ on $\mathfrak{g}^*$. This action projects to a G_Σ-action on the quotient $\mathfrak{g}^*/\bar{\mathcal{H}}$ via the map Ξ: $G_\Sigma \times \mathfrak{g}^*/\bar{\mathcal{H}} \longrightarrow \mathfrak{g}^*/\bar{\mathcal{H}}$ defined by

$$\Xi(g, \mu + \bar{\mathcal{H}}) := \mathrm{Ad}^*_{\pi_G(g)^{-1}}\mu + \mu_\Sigma(g^{-1}) + \bar{\mathcal{H}}, \qquad g \in G_\Sigma, \mu \in \mathfrak{g}^*,$$

with respect to which π_C is G_Σ-equivariant. We will call Ξ the "projected extended affine action" of G_Σ on $\mathfrak{g}^*/\bar{\mathcal{H}}$.

Proof: The proof of the first statement is a straightforward verification. The projected extended affine action is well-defined by Proposition 3.1. □

Remark 4.7. For any $\xi \in \mathfrak{g}_\Sigma$, the associated infinitesimal generators $\xi_{\mathfrak{g}^*}$ and $\xi_{\mathfrak{g}^*/\bar{\mathcal{H}}}$ of the extended and the projected extended affine action of G_Σ on $\mathfrak{g}^*$ and $\mathfrak{g}^*/\bar{\mathcal{H}}$, respectively, are given by the expressions,

$$\xi_{\mathfrak{g}^*}(\mu) = -\mathrm{ad}^*_{\pi_{\mathfrak{g}}(\xi)}\mu + \Sigma(\pi_{\mathfrak{g}}(\xi), \cdot) =: (\pi_{\mathfrak{g}}(\xi))_{\mathfrak{g}^*}(\mu), \tag{26}$$

$$\xi_{\mathfrak{g}^*/\bar{\mathcal{H}}}(\pi_C(\mu)) = T_\mu\pi_C\big(-\mathrm{ad}^*_{\pi_{\mathfrak{g}}(\xi)}\mu + \Sigma(\pi_{\mathfrak{g}}(\xi), \cdot)\big) = T_\mu\pi_C((\pi_{\mathfrak{g}}(\xi))_{\mathfrak{g}^*}(\mu)), \tag{27}$$

for any $\mu \in \mathfrak{g}^*$. The second equality in (26) emphasizes that for any $\zeta \in \mathfrak{g}$, the expression $\zeta_{\mathfrak{g}^*}(\mu) := -\mathrm{ad}^*_\zeta\mu + \Sigma(\zeta, \cdot)$ defines a Lie algebra action of $\mathfrak{g}$ on $\mathfrak{g}^*$. Unlike the $\mathfrak{g}_\Sigma$-action defined by the first equality in (26), the $\mathfrak{g}$-action cannot, in general, be integrated to a group action.

Remark 4.8. Suppose that there exists a right group one-cocycle $\sigma: G \to \mathfrak{g}^*$ that integrates Σ, that is,

$$T_e\sigma(\xi) = -\Sigma(\xi, \cdot), \tag{28}$$

for any $\xi \in \mathfrak{g}$. This happens to be the case when, for instance, the manifold underlying the group G_Σ is diffeomorphic to $G \times \mathbb{R}$ (see Chapter 2 of [10]). Then the extended affine G_Σ-action on $\mathfrak{g}^*$ drops to the affine G-action $\bar{\Theta}$: $G \times \mathfrak{g}^* \to \mathfrak{g}^*$ on $\mathfrak{g}^*$ determined by σ via the expression $\bar{\Theta}(g, \mu) = \mathrm{Ad}^*_{g^{-1}}\mu + \sigma(g^{-1})$, $(g, \mu) \in G \times \mathfrak{g}^*$. More specifically,

$$\bar{\Xi}(\hat{g}, \mu) = \bar{\Theta}(\pi_G(\hat{g}), \mu), \tag{29}$$

for any $\hat{g} \in G_\Sigma$ and $\mu \in \mathfrak{g}^*$. This equality implies that

$$\sigma \circ \pi_G = \mu_\Sigma.$$

Analogously, the affine G-action Θ: $G \times \mathfrak{g}^*/\bar{\mathcal{H}} \to \mathfrak{g}^*/\bar{\mathcal{H}}$ on $\mathfrak{g}^*/\bar{\mathcal{H}}$ induced by σ via the expression $\Theta(g, \mu + \bar{\mathcal{H}}) = \mathrm{Ad}^*_{g^{-1}}\mu + \sigma(g^{-1}) + \bar{\mathcal{H}}$, $(g, \mu) \in G \times \mathfrak{g}^*$, is such that

$$\Xi(\hat{g}, \mu + \bar{\mathcal{H}}) = \Theta(\pi_G(\hat{g}), \mu + \bar{\mathcal{H}}),$$

for any $\hat{g} \in G_\Sigma$ and $\mu \in \mathfrak{g}^*$. In order to prove (29) notice that (26) and (28) guarantee that

$$\xi_{\mathfrak{g}^*}(\mu) = (\pi_{\mathfrak{g}}(\xi))_{\mathfrak{g}^*}(\mu), \tag{30}$$

for any $\xi \in \mathfrak{g}_\Sigma$, $\mu \in \mathfrak{g}^*$. The infinitesimal generator on the left-hand side of this expression corresponds to the extended affine G_Σ-action $\bar{\Xi}$ on $\mathfrak{g}^*$ while the one on the right-hand side is constructed using the affine G-action $\bar{\Theta}$ induced by σ. The equality (30) implies (29). Indeed, since G_Σ is connected, any element $\hat{g} \in G_\Sigma$ can be written as $\hat{g} = \exp_{\mathfrak{g}_\Sigma} \xi_1 \cdots \exp_{\mathfrak{g}_\Sigma} \xi_n$, $\xi_1, \ldots, \xi_n \in \mathfrak{g}_\Sigma$. Hence, $\bar{\Xi}(\hat{g}, \mu) = \bar{\Xi}(\exp_{\mathfrak{g}_\Sigma} \xi_1 \cdots \exp_{\mathfrak{g}_\Sigma} \xi_n, \mu) = F_1^{\xi_1} \circ \cdots \circ F_1^{\xi_n}(\mu)$, where $F_t^{\xi_i}$ is the flow of the infinitesimal generator vector field $(\xi_i)_{\mathfrak{g}^*}$. Since by (30) $(\xi_i)_{\mathfrak{g}^*} = (\pi_{\mathfrak{g}}(\xi_i))_{\mathfrak{g}^*}$, the flow $F_t^{\xi_i}$ coincides with the flow $F_t^{\pi_{\mathfrak{g}}(\xi_i)}$ of $(\pi_{\mathfrak{g}}(\xi_i))_{\mathfrak{g}^*}$ and hence

$$\begin{aligned}\bar{\Xi}(\hat{g}, \mu) &= F_1^{\pi_{\mathfrak{g}}(\xi_1)} \circ \cdots \circ F_1^{\pi_{\mathfrak{g}}(\xi_n)}(\mu) = \bar{\Theta}(\exp_{\mathfrak{g}}(\pi_{\mathfrak{g}}(\xi_1)) \cdots \exp_{\mathfrak{g}}(\pi_{\mathfrak{g}}(\xi_n)), \mu) \\ &= \bar{\Theta}\big(\pi_G\big(\exp_{\mathfrak{g}_\Sigma} \xi_1\big) \cdots \pi_G(\exp_{\mathfrak{g}_\Sigma} \xi_n), \mu\big) = \bar{\Theta}(\pi_G(\hat{g}), \mu),\end{aligned}$$

which proves (29).

4.3. The characteristic distributions of $(\mathfrak{g}^*, \{\cdot,\cdot\}_\pm^\Sigma)$ and $(\mathfrak{g}^*/\bar{\mathcal{H}}, \{\cdot,\cdot\}_{\mathfrak{g}^*/\bar{\mathcal{H}}}^{\pm\Sigma})$

The characteristic distribution $\bar{E}$ of $(\mathfrak{g}^*, \{\cdot,\cdot\}_\pm^\Sigma)$ is given by (see for instance Section 4.5.27 in [22])

$$\bar{E}(\mu) = \{\mathrm{ad}_\xi^* \mu - \Sigma(\xi, \cdot) \mid \xi \in \mathfrak{g}\}. \tag{31}$$

We now show that the characteristic distribution E of the Poisson manifold $(\mathfrak{g}^*/\bar{\mathcal{H}}, \{\cdot,\cdot\}_{\mathfrak{g}^*/\bar{\mathcal{H}}}^{\pm\Sigma})$ is given by

$$E(\pi_C(\mu)) = \{T_\mu \pi_C(\mathrm{ad}_\xi^* \mu - \Sigma(\xi, \cdot)) \mid \xi \in (\mathrm{Lie}(\bar{\mathcal{H}}))^\circ\}. \tag{32}$$

Indeed, since the projection $\pi_C \colon \mathfrak{g}^* \to \mathfrak{g}^*/\bar{\mathcal{H}}$ is a Poisson map, for any $f \in C^\infty(\mathfrak{g}^*/\bar{\mathcal{H}})$ and $\mu \in \mathfrak{g}^*$ we have that $X_f(\pi_C(\mu)) = T_\mu \pi_C(X_{f\circ\pi_C}(\mu))$. Since $X_{f\circ\pi_C}$ is known (see for instance (4.5.20) in [22]) we have

$$X_f(\pi_C(\mu)) = T_\mu \pi_C \left(\mp \mathrm{ad}^*_{\frac{\delta(f\circ\pi_C)}{\delta\mu}} \mu \pm \Sigma\left(\frac{\delta(f \circ \pi_C)}{\delta\mu}, \cdot \right)\right). \tag{33}$$

We now recall a result in [17] (see [22, Theorem 2.5.10]) that characterizes the span of the differentials of the invariant functions $f \in C^\infty(M)^G$ with respect to a proper G-action on a manifold M by means of the equality

$$\{\mathbf{d} f(m) \mid f \in C^\infty(M)^G\} = ((\mathfrak{g} \cdot m)^\circ)^{G_m}. \tag{34}$$

If we apply this result to the free and proper action of $\bar{\mathcal{H}}$ on $\mathfrak{g}^*$ we obtain that

$$\{\mathbf{d} h(\mu) \mid h \in C^\infty(\mathfrak{g}^*)^{\bar{\mathcal{H}}}\} = ((\mathrm{Lie}(\bar{\mathcal{H}})) \cdot \mu)^\circ. \tag{35}$$

However, since for any element $\nu \in \mathrm{Lie}(\bar{\mathcal{H}})$ its associated infinitesimal generator satisfies $\nu_{\mathfrak{g}^*}(\mu) = \nu$, for any $\mu \in \mathfrak{g}^*$, we can conclude that $\mathrm{Lie}(\bar{\mathcal{H}}) \cdot \mu = \mathrm{Lie}(\bar{\mathcal{H}})$ and hence (35) can be rewritten as

$$\{\mathbf{d} h(\mu) \mid h \in C^\infty(\mathfrak{g}^*)^{\bar{\mathcal{H}}}\} = (\mathrm{Lie}(\bar{\mathcal{H}}))^\circ. \tag{36}$$

Now, since for any $\rho \in \mathfrak{g}^*$, we have $\langle \rho, \frac{\delta(f\circ\pi_C)}{\delta\mu}\rangle = \mathbf{d}(f \circ \pi_C)(\mu)(\rho)$ and $f \circ \pi_C \in C^\infty(\mathfrak{g}^*)^{\bar{\mathcal{H}}}$, expressions (33) and (36) guarantee the validity of (32).

4.4. The symplectic leaves of $(\mathfrak{g}^*, \{\cdot,\cdot\}^{\Sigma}_{\pm})$ and $(\mathfrak{g}^*/\bar{\mathcal{H}}, \{\cdot,\cdot\}^{\pm\Sigma}_{\mathfrak{g}^*/\bar{\mathcal{H}}})$

As we saw in Lemma 4.2 the subspace $\mathfrak{n} := \mathrm{Lie}(\bar{\mathcal{H}})^\circ$ is an ideal of $\mathfrak{g}$. This implies that $\mathfrak{n}_\Sigma := \mathfrak{n} \oplus \mathbb{R}$ is an ideal of $\mathfrak{g}_\Sigma$. Let N_Σ be the connected and simply connected normal Lie subgroup of G_Σ whose Lie algebra is $\mathfrak{n}_\Sigma$.

Proposition 4.9. *Let $(\mathfrak{g}^*, \{\cdot,\cdot\}^{\Sigma}_{\pm})$ be an affine Lie–Poisson structure and $(\mathfrak{g}^*/\bar{\mathcal{H}}, \{\cdot,\cdot\}^{\pm\Sigma}_{\mathfrak{g}^*/\bar{\mathcal{H}}})$ the corresponding projected Lie–Poisson structure introduced in Proposition 4.3. Let N_Σ be the normal subgroup of G_Σ introduced in the paragraph above. Then for any $\mu \in \mathfrak{g}^*$, the symplectic leaves $\mathcal{L}_\mu$ and $\mathcal{L}_{\pi_C(\mu)}$ of $(\mathfrak{g}^*, \{\cdot,\cdot\}^{\Sigma}_{\pm})$ and $(\mathfrak{g}^*/\bar{\mathcal{H}}, \{\cdot,\cdot\}^{\pm\Sigma}_{\mathfrak{g}^*/\bar{\mathcal{H}}})$ that contain μ and $\pi_C(\mu)$, respectively, equal*

$$\mathcal{L}_\mu = \bar{\Xi}(G_\Sigma, \mu) \qquad \textit{and} \qquad \mathcal{L}_{\pi_C(\mu)} = \Xi(N_\Sigma, \pi_C(\mu)), \tag{37}$$

where $\bar{\Xi}$ and Ξ are actions introduced in Corollary 4.6.

Proof: If we compare (26) and (27) with (31) and (32), it is clear that the tangent spaces to the orbits of the G_Σ and N_Σ-actions produce distributions in $\mathfrak{g}^*$ and $\mathfrak{g}^*/\bar{\mathcal{H}}$ equal to the characteristic distributions of $(\mathfrak{g}^*, \{\cdot,\cdot\}^{\Sigma}_{\pm})$ and $(\mathfrak{g}^*/\bar{\mathcal{H}}, \{\cdot,\cdot\}^{\pm\Sigma}_{\mathfrak{g}^*/\bar{\mathcal{H}}})$, respectively, and hence they have the same maximal integral leaves. Since G_Σ and N_Σ are connected these are the G_Σ- and N_Σ-orbits, respectively. □

Proof of Theorem 4.1. We will prove Theorem 4.1 by showing that the Poisson structure in (17) is one of the projected Poisson structures presented in Proposition 4.3 when we use a Lie algebra two-cocycle $\Sigma\colon \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ that comes naturally with the construction that lead us to the cylinder valued momentum map. We start by recalling a proposition whose proof can be found in [23].

Proposition 4.10. *Let (M, ω) be a connected and paracompact symplectic manifold acted symplectically upon by the Lie group G. Let $\mathbf{K}\colon M \to \mathfrak{g}^*/\bar{\mathcal{H}}$ be a cylinder valued momentum map for this action defined using the holonomy bundle $\tilde{M} \subset M \times \mathfrak{g}^*$. Let $p\colon \tilde{M} \to M$ be the projection. Then the pair $(\tilde{M}, \omega_{\tilde{M}} := \tilde{p}^*\omega)$ is a symplectic manifold where the Lie algebra $\mathfrak{g}$ acts canonically via the map*

$$\xi_{\tilde{M}}(m, \mu) := (\xi_M(m), -\Psi(m)(\xi, \cdot)),$$

for any $\xi \in \mathfrak{g}$, $(m, \mu) \in \tilde{M}$, and where $\Psi\colon M \to Z^2(\mathfrak{g})$ is the Chu map. The projection $\tilde{\mathbf{K}}\colon \tilde{M} \subset M \times \mathfrak{g}^* \to \mathfrak{g}^*$ is a standard momentum map for this action with infinitesimal non-equivariance cocycle $\tilde{\Sigma}\colon \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ given by

$$\tilde{\Sigma}(\xi, \eta) := \langle \mu, [\xi, \eta]\rangle - \Psi(m)(\xi, \eta), \tag{38}$$

for any $(m, \mu) \in \tilde{M}$. The value of $\tilde{\Sigma}$ does not depend on the point $(m, \mu) \in \tilde{M}$ used to define it.

The Poisson structure in the statement of Theorem 4.1 is one of the projected Poisson structures (21) when we take in (20) the two-cocycle $\tilde{\Sigma}$ introduced in the statement of the previous proposition. More specifically, for any $f, g \in C^\infty(\mathfrak{g}^*)$ we have

$$\{f, g\}_{\mathfrak{g}^*/\bar{\mathcal{H}}}(\pi_C(\mu)) = \{f \circ \pi_C, g \circ \pi_C\}_+^{\tilde{\Sigma}}(\mu). \tag{39}$$

Indeed, using the independence of $\tilde{\Sigma}$ on the point (m, μ), we can write

$$\begin{aligned}\{f \circ \pi_C, g \circ \pi_C\}_+^{\tilde{\Sigma}}(\mu) &= \left\langle \mu, \left[\frac{\delta(f \circ \pi_C)}{\delta\mu}, \frac{\delta(g \circ \pi_C)}{\delta\mu}\right]\right\rangle - \left\langle \mu, \left[\frac{\delta(f \circ \pi_C)}{\delta\mu}, \frac{\delta(g \circ \pi_C)}{\delta\mu}\right]\right\rangle \\ &\quad + \Psi(m)\left(\frac{\delta(f \circ \pi_C)}{\delta\mu}, \frac{\delta(g \circ \pi_C)}{\delta\mu}\right) = \Psi(m)\left(\frac{\delta(f \circ \pi_C)}{\delta\mu}, \frac{\delta(g \circ \pi_C)}{\delta\mu}\right).\end{aligned}$$

Proof of point (i). Let $f, g \in C^\infty(\mathfrak{g}^*/\bar{\mathcal{H}})$ be arbitrary functions. We will show that for any $m \in M$

$$\{f \circ \mathbf{K}, g \circ \mathbf{K}\}(m) = \{f, g\}_{\mathfrak{g}^*/\bar{\mathcal{H}}}(\mathbf{K}(m)).$$

Indeed, by definition

$$\begin{aligned}\{f, g\}_{\mathfrak{g}^*/\bar{\mathcal{H}}}(\mathbf{K}(m)) &= \Psi(m)\left(\frac{\delta(f \circ \pi_C)}{\delta\mu}, \frac{\delta(g \circ \pi_C)}{\delta\mu}\right) \\ &= \omega(m)\left(\left(\frac{\delta(f \circ \pi_C)}{\delta\mu}\right)_M(m), \left(\frac{\delta(g \circ \pi_C)}{\delta\mu}\right)_M(m)\right),\end{aligned}$$

where $\mathbf{K}(m) = \pi_C(\mu)$. Since $\{f \circ \mathbf{K}, g \circ \mathbf{K}\}(m) = \omega(m)(X_{f\circ\mathbf{K}}(m), X_{g\circ\mathbf{K}}(m))$, it suffices to show that for any $f \in C^\infty(\mathfrak{g}^*/\bar{\mathcal{H}})$

$$X_{f\circ\mathbf{K}}(m) = \left(\frac{\delta(f \circ \pi_C)}{\delta\mu}\right)_M(m). \tag{40}$$

This equality holds since by (3)

$$\begin{aligned}\omega(m)(X_{f\circ\mathbf{K}}(m), v_m) &= \mathbf{d}(f \circ \mathbf{K})(m)(v_m) = \mathbf{d}f(\mathbf{K}(m))(T_m\mathbf{K}(v_m)) \\ &= \mathbf{d}f(\mathbf{K}(m))(T_\mu\pi_C(\nu)) = \mathbf{d}(f \circ \pi_C)(\mu)(\nu) = \left\langle \nu, \frac{\delta(f \circ \pi_C)}{\delta\mu}\right\rangle \\ &= \omega(m)\left(\left(\frac{\delta(f \circ \pi_C)}{\delta\mu}\right)_M(m), v_m\right)\end{aligned}$$

for any $v_m \in T_mM$, which proves (40).

Proof of point (ii). $(\mathrm{Lie}(\bar{\mathcal{H}}))^\circ$ is an ideal in $\mathfrak{g}$ by Lemma 4.2. In order to prove the second statement, note that since N is connected so are its orbits and hence it suffices to show that the distribution given by the tangent spaces to the N-orbits coincides with the characteristic distribution of the Poisson bracket (17). By (32) and (39) the characteristic distribution is given by

$$E(\pi_C(\mu)) = \{T_\mu\pi_C\,(\Psi(m)(\xi, \cdot)) \mid \xi \in (\mathrm{Lie}(\bar{\mathcal{H}}))^\circ\}, \tag{41}$$

where $\mathbf{K}(m) = \pi_C(\mu)$. Since, by Proposition 3.4, the infinitesimal generators of the affine G-action on $\mathfrak{g}^*/\bar{\mathcal{H}}$ are given by the expression $\xi_{\mathfrak{g}^*/\bar{\mathcal{H}}}(\pi_C(\mu)) = -T_\mu \pi_C\,(\Psi(m)(\xi, \cdot))$, for any $\xi \in \mathfrak{g}$, $(m, \mu) \in \tilde{M}$, the statement follows.

Point (iii) is a straightforward consequence of (ii). □

5. The reduction theorems

The symplectic reduction procedure introduced by Marsden and Weinstein in [15] consists of two steps. First, one restricts to the level sets of the momentum map and second, one projects it to the space of orbits of the group that leaves that level set invariant. The elements of the group that leave a given level set of the momentum map invariant form a closed subgroup of the original symmetry group. Indeed, if the manifold is connected, one can always find an action on the dual of the Lie algebra with respect to which the momentum map is equivariant [27] and hence this subgroup is the isotropy subgroup of the momentum value defining the level set.

In the preceding sections, we have introduced all the necessary ingredients to reproduce this construction in the context of the cylinder valued momentum map. Nevertheless, if we blindly reproduce the Marsden–Weinstein construction it turns out that we do not obtain symplectic reduced spaces but orbit spaces that are endowed with a naturally defined Poisson structure that is, in general, degenerate. As we will see, the reason behind this surprising phenomenon is the eventual non-closedness of the Hamiltonian holonomy in the dual of the Lie algebra. Indeed, when this happens to be the case one may consider two different reduced spaces: first, one based on the Marsden–Weinstein construction that suggests a reduction by the G-action and that, as we already said, yields a Poisson manifold; second, the foliation reduction theorem of Cartan [5] (see Section 6.1.5 in [22] for a self-contained presentation) imposes the reduction by the group N introduced in Theorem 4.1 that integrates $(\mathrm{Lie}(\bar{\mathcal{H}}))^\circ$. The resulting reduced space is symplectic. The two reduced spaces are related via a Poisson reduction procedure that will be described in detail and they coincide when the Hamiltonian holonomy is closed in the dual of the Lie algebra.

Throughout this section all group actions are free and proper. This ensures the smoothness of all the orbit spaces that we will encounter. The generalization of these results to the context of non-free actions is the subject of another paper [24].

Theorem 5.1 (Symplectic reduction). *Let (M, ω) be a connected and paracompact symplectic manifold and G a Lie group acting freely, properly, and symplectically on it. Let $\mathbf{K}\colon M \to \mathfrak{g}^*/\bar{\mathcal{H}}$ be a cylinder valued momentum map for this action with associated non-equivariance one-cocycle $\sigma\colon M \to \mathfrak{g}^*/\bar{\mathcal{H}}$. Let N be a normal connected Lie subgroup of G that has $(\mathrm{Lie}(\bar{\mathcal{H}}))^\circ$ as Lie algebra. Then for any $[\mu] \in \mathfrak{g}^*/\bar{\mathcal{H}}$ the orbit spaces $M_{[\mu]} := \mathbf{K}^{-1}([\mu])/N_{[\mu]}$ are regular quotient manifolds that are endowed with a natural symplectic structure $\omega_{[\mu]}$ uniquely determined by the equality*

$$i_{[\mu]}^*\omega = \pi_{[\mu]}^*\omega_{[\mu]}, \tag{42}$$

where $N_{[\mu]}$ denotes the isotropy subgroup of $[\mu]$ with respect to the N-action on $\mathfrak{g}^*/\bar{\mathcal{H}}$ obtained by restriction of the affine G-action constructed using the non-equivariance cocycle σ of $\mathbf{K}$, $i_{[\mu]}\colon \mathbf{K}^{-1}([\mu]) \hookrightarrow M$ is the inclusion, and $\pi_{[\mu]}\colon \mathbf{K}^{-1}([\mu]) \to \mathbf{K}^{-1}([\mu])/N_{[\mu]}$ is the projection. We will refer to the spaces $M_{[\mu]}$ as the "symplectic reduced spaces".

The proof of this theorem requires an intermediate result that generalizes in the context of the cylinder valued momentum map the so called *reduction lemma.*

Proposition 5.2 (Reduction Lemma). *Let (M, ω) be a connected and paracompact symplectic manifold and G a Lie group acting symplectically on it. Let $\mathbf{K}\colon M \to \mathfrak{g}^*/\bar{\mathcal{H}}$ be a cylinder valued momentum map for this action with associated non-equivariance one-cocycle $\sigma\colon M \to \mathfrak{g}^*/\bar{\mathcal{H}}$ and N a normal connected Lie subgroup of G that has $\mathfrak{n} := (\mathrm{Lie}(\bar{\mathcal{H}}))^\circ$ as Lie algebra. Then for any $m \in M$ such that $\mathbf{K}(m) = \pi_C(\mu) =: [\mu]$ we have*

(i) $\mathfrak{g}_{[\mu]} \cdot m = \ker T_m\mathbf{K} \cap \mathfrak{g} \cdot m$.
(ii) $\mathfrak{n}_{[\mu]} \cdot m = \ker T_m\mathbf{K} \cap (\ker T_m\mathbf{K})^\omega = (\mathfrak{n} \cdot m)^\omega \cap \mathfrak{n} \cdot m$.
(iii) *If the Hamiltonian holonomy $\mathcal{H}$ is closed in $\mathfrak{g}^*$ then $\mathfrak{g}_{[\mu]} \cdot m = (\mathfrak{g} \cdot m)^\omega \cap \mathfrak{g} \cdot m$.*

Proof: Let $\xi \in \mathfrak{g}$. The equivariance of the cylinder valued momentum map $\mathbf{K}$ with respect to the affine action implies that

$$
\begin{aligned}
T_m\mathbf{K}\,(\xi_M(m)) &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \mathbf{K}(\exp t\xi \cdot m) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \Theta_{\exp t\xi}\mathbf{K}(m) \\
&= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \Theta_{\exp t\xi}[\mu] = \xi_{\mathfrak{g}^*/\bar{\mathcal{H}}}([\mu]).
\end{aligned} \tag{43}
$$

This chain of equalities shows that $\xi_M(m) \in \ker T_m\mathbf{K} \cap \mathfrak{g} \cdot m$ if and only if $\xi \in \mathfrak{g}_{[\mu]}$ which proves (i). As to (ii), the second equality is a consequence of part (iii) in Proposition 2.3. This equality and (i) imply the first one. Part (iii) follows from part (i) by noticing that when $\mathcal{H}$ is closed in $\mathfrak{g}^*$ then $\ker T_m\mathbf{K} = (\mathfrak{g} \cdot m)^\omega$ by part (iii) in Proposition 2.3. □

Remark 5.3. Notice that the reduction lemma shows how the reduced space that we consider in Theorem 5.1 is the one that is hinted at in the foliation reduction theorem of Cartan. This theorem studies the leaf space of the characteristic distribution $\ker T\mathbf{K} \cap (\ker T\mathbf{K})^\omega$ in the level set $\mathbf{K}^{-1}([\mu])$ that, by part (ii) of the previous proposition, coincides with the distribution spanned by the tangent spaces to the $N_{[\mu]}$-orbits.

Proof of Theorem 5.1. The freeness of the action implies, via part (iv) of Proposition 2.3, that $\mathbf{K}$ is a submersion and hence the fiber $\mathbf{K}^{-1}([\mu])$ is a closed embedded submanifold of M. Additionally, the properness condition guarantees that the orbit space $M_{[\mu]} := \mathbf{K}^{-1}([\mu])/N_{[\mu]}$ is a regular quotient manifold.

We now show that (42) defines a two-form on $M_{[\mu]}$. Let $m, m' \in M$, $v_m, w_m \in T_m\mathbf{K}^{-1}([\mu])$, $v_{m'}, w_{m'} \in T_{m'}\mathbf{K}^{-1}([\mu])$ be such that

$$\pi_{[\mu]}(m) = \pi_{[\mu]}(m'), \tag{44}$$

$$T_m\pi_{[\mu]}(v_m) = T_{m'}\pi_{[\mu]}(v_{m'}) \quad \text{and} \quad T_m\pi_{[\mu]}(w_m) = T_{m'}\pi_{[\mu]}(w_{m'}). \tag{45}$$

The equality (44) implies that there exists an element $n \in N_{[\mu]}$ such that $m' = n \cdot m$ and hence $T_m\pi_{[\mu]} = T_{m'}\pi_{[\mu]} \circ T_m\Phi|_{\ker T_m\mathbf{K}}$, which, substituted in (45), yields $(T_{m'}\pi_{[\mu]} \circ T_m\Phi)(v_m) = T_{m'}\pi_{[\mu]}(v_{m'})$ and $(T_{m'}\pi_{[\mu]} \circ T_m\Phi)(w_m) = T_{m'}\pi_{[\mu]}(w_{m'})$. This in turn implies the existence of two elements $\xi, \eta \in \mathfrak{n}_{[\mu]}$ such that $T_m\Phi(v_m) - v_{m'} = \xi_M(m')$, $T_m\Phi(w_m) - w_{m'} = \eta_M(m')$. Consequently

$$\omega(m')(v_{m'}, w_{m'}) = \omega(\Phi_n(m))(T_m\Phi(v_m) - \xi_M(m'), T_m\Phi(w_m) - \eta_M(m')) = \omega(m)(v_m, w_m).$$

In the previous chain of equalities we have used three things. First, by the canonical character of the action, $\omega(\Phi_n(m))(T_m\Phi_n(v_m), T_m\Phi_n(w_m)) = \omega(m)(v_m, w_m)$. Second, the canonical character of the action and the reduction lemma imply that

$$\begin{aligned}\omega(\Phi_n(m))(T_m\Phi_n(v_m), \eta_M(m')) &= \omega(\Phi_n(m))(T_m\Phi_n(v_m), T_m\Phi_n\,(\mathrm{Ad}_{n^{-1}}\eta)_M\,(m))\\ &= \omega(m)(v_m, (\mathrm{Ad}_{n^{-1}}\eta)_M(m)) = 0.\end{aligned}$$

Finally, for the same reasons, $\omega(\Phi_n(m))(T_m\Phi_n(w_m), \xi_M(m')) = \omega(\Phi_n(m))(\eta_M(m'), \xi_M(m')) = 0$ which shows that (42) defines a two-form $\omega_{[\mu]}$ on $M_{[\mu]}$. Since $\pi_{[\mu]}$ is a surjective submersion and ω is closed, so is $\omega_{[\mu]}$. In order to show that $\omega_{[\mu]}$ is non-degenerate let $v_m \in \ker T_m\mathbf{K}$ be such that $\omega_{[\mu]}(\pi_{[\mu]}(m))(T_m\pi_{[\mu]}(v_m), T_m\pi_{[\mu]}(w_m)) = 0$ for any $w_m \in \ker T_m\mathbf{K}$. By (42), $\omega(m)(v_m, w_m) = 0$ for any $w_m \in \ker T_m\mathbf{K}$. Consequently, by the reduction lemma, $v_m \in (\ker T_m\mathbf{K})^\omega \cap \ker T_m\mathbf{K} = \mathfrak{n}_{[\mu]} \cdot m$ and hence $T_m\pi_{[\mu]}(v_m) = 0$, as required.

Theorem 5.4. *Let (M, ω) be a connected and paracompact symplectic manifold and G a Lie group acting freely, properly, and symplectically on it. Let $\mathbf{K}\colon M \to \mathfrak{g}^*/\bar{\mathcal{H}}$ be a cylinder valued momentum map for this action with associated non-equivariance one-cocycle $\sigma\colon M \to \mathfrak{g}^*/\bar{\mathcal{H}}$. Let N be a normal connected Lie subgroup of G that has $(\mathrm{Lie}(\bar{\mathcal{H}}))^\circ$ as Lie algebra.*

(i) *Let $F \in C^\infty(M)^N$ be an N-invariant function on M and let F_t be the flow of the associated Hamiltonian vector field X_F. Then*

$$\mathbf{K} \circ F_t = \mathbf{K}|_{\mathrm{Dom}(F_t)}.$$

(ii) *Let $F \in C^\infty(M)^N$ and let $M_{[\mu]}$ be the symplectic reduced space introduced in Theorem 5.1, for some $[\mu] \in \mathfrak{g}^*/\bar{\mathcal{H}}$. Let $f \in C^\infty(M_{[\mu]})$ be the function uniquely determined by $f \circ \pi_{[\mu]} = F \circ i_{[\mu]}$. Then*

$$T\pi_{[\mu]} \circ X_F \circ i_{[\mu]} = X_f \circ \pi_{[\mu]}.$$

(iii) *The bracket $\{\cdot, \cdot\}_{M_{[\mu]}}$ induced by $\omega_{[\mu]}$ on $M_{[\mu]}$ can be expressed as*

$$\{f, h\}_{M_{[\mu]}}(\pi_{[\mu]}(m)) = \{F, H\}(m),$$

with $F, H \in C^\infty(M)^N$ two local N-invariant extensions at the point m of $f \circ \pi_{[\mu]}, h \circ \pi_{[\mu]} \in C^\infty(\mathbf{K}^{-1}([\mu]))^{N_{[\mu]}}$, respectively (see the remark below for an explanation of this terminology).

Remark 5.5. Let M be a smooth manifold and S an embedded submanifold of M. Let $D \subset TM|_S$ be a subbundle of the tangent bundle of M restricted to S such that $D_S := D \cap TS$ is a smooth, integrable, regular distribution on S. Let $\pi_{D_S}\colon S \to S/D_S$ be the projection onto the leaf space of the D_S-distribution in S. It can be proved (see Lemma 10.4.14

in [22]) that for any open set $V \subset S/D_S$, any function $f \in C^\infty(V)$, and any $z \in V$ there exist a point $m \in \pi_{D_S}^{-1}(V)$, an open neighborhood U_m of m in M, and $F \in C^\infty_M(U_m)$ such that

$$f \circ \pi_{D_S}|_{\pi_{D_S}^{-1}(V)\cap U_m} = F|_{\pi_{D_S}^{-1}(V)\cap U_m},$$

and

$$\mathbf{d}F(n)|_{D(n)} = 0, \quad \text{for any} \quad n \in \pi_{D_S}^{-1}(V) \cap U_m.$$

We say that F is a *local D-invariant extension* of $f \circ \pi_{D_S}$ at the point $m \in \pi_{D_S}^{-1}(V)$.

The existence of the local N-invariant extensions in part (iii) of the statement of Theorem 5.4 is a consequence of the result that we just quoted by taking $S = \mathbf{K}^{-1}([\mu])$ and $D(m) = \mathfrak{n} \cdot m$, $m \in \mathbf{K}^{-1}([\mu])$. In this setup, the reduction lemma implies that $D_S(m) = \mathfrak{n}_{[\mu]} \cdot m, m \in \mathbf{K}^{-1}([\mu])$.

Proof of Theorem 5.4. To prove (i) it suffices to show that $T\mathbf{K}(X_F) = 0$. To see this, note that for any $m \in M$ such that $\mathbf{K}(m) = \pi_C(\mu)$ we have that $T_m\mathbf{K}(X_F(m)) = T_\mu\pi_C(\nu)$ where

$$\langle \nu, \xi \rangle = \omega(m)\,(\xi_M(m), X_F(m)) = -\mathbf{d}F(m)\,(\xi_M(m)),$$

for any $\xi \in \mathfrak{g}$. This equality and the N-invariance of F obviously imply that $\nu \in \mathfrak{n}^\circ = ((\mathrm{Lie}(\bar{\mathcal{H}}))^\circ)^\circ = \mathrm{Lie}(\bar{\mathcal{H}})$. Hence, $T_\mu\pi_C(\nu) = 0$, as required. Part (ii) is a straightforward verification. To show (iii) recall from Remark 5.5 that the existence of the local N-invariant extensions is guaranteed by Lemma 10.4.14 in [22]. Moreover, by part (ii),

$$\begin{aligned}
\{f, h\}_{M_{[\mu]}}(\pi_{[\mu]}(m)) &= \omega_{[\mu]}\big(\pi_{[\mu]}(m)\big)\big(X_f\big(\pi_{[\mu]}(m)\big), X_h\big(\pi_{[\mu]}(m)\big)\big)\\
&= \omega_{[\mu]}\big(\pi_{[\mu]}(m)\big)\big(T_m\pi_{[\mu]}\big(X_F(m)\big), T_m\pi_{[\mu]}\big(X_H(m)\big)\big)\\
&= \big(\pi_{[\mu]}^*\omega_{[\mu]}\big)(m)(X_F(m), X_H(m))\\
&= \omega(m)(X_F(m), X_H(m)) = \{F, H\}(m).
\end{aligned}$$

□

The optimal momentum map and optimal reduction: A quick overview. As stated in the introduction to this section, the analog of the Marsden–Weinstein reduction scheme in the cylinder valued momentum map setup yields a Poisson manifold that is related to the symplectic reduced space in Theorem 5.1 via Poisson reduction. The description of the symplectic leaves of this Poisson reduced space will require the use of the *optimal momentum map* [20] and of the reduction procedure that can be carried out with it [18] that we quickly review in the following paragraphs. We refer to these papers or to [22] for the proofs of the statements quoted below.

Let $(M, \{\cdot, \cdot\})$ be a Poisson manifold and G a Lie group that acts properly on M by Poisson diffeomorphisms via the left action $\Phi: G \times M \to M$. The group of canonical transformations associated to this action will be denoted by $A_G := \{\Phi_g : M \to M \mid g \in G\}$. Let A'_G be the *G-characteristic* or the *polar distribution* on M associated to A_G [19] defined for any $m \in M$ by $A'_G(m) := \{X_f(m) \mid f \in C^\infty(M)^G\}$. The distribution A'_G is a smooth integrable generalized distribution in the sense of Stefan and Sussman [29–31]. The *optimal momentum map* $\mathcal{J}$ is defined as the canonical projection onto the leaf space of A'_G, that is, $\mathcal{J}\colon M \longrightarrow M/A'_G$. By its very definition, the levels sets of $\mathcal{J}$ are preserved by the Hamiltonian flows associated to G-invariant Hamiltonian functions and $\mathcal{J}$ is *universal* with respect to this property, that is, any other map whose level sets are preserved by G-equivariant Hamiltonian dynamics factors necessarily through $\mathcal{J}$. By construction, the fibers of $\mathcal{J}$ are the leaves of an integrable generalized distribution and thereby *initial immersed submanifolds* of M [8]. Recall that N is an *initial submanifold* of M when the injection $i\colon N \to M$ is a smooth immersion that satisfies the following property: for any manifold Z, a mapping $f\colon Z \to N$ is smooth if and only if $i \circ f\colon Z \to M$ is smooth.

The leaf space M/A'_G is called the *momentum space* of $\mathcal{J}$. We will consider it as a topological space with the quotient topology. If $m \in M$ let $\rho := \mathcal{J}(m) \in M/A'_G$. Then, for any $g \in G$, the map $\Psi_g(\rho) = \mathcal{J}(g \cdot m) \in M/A'_G$ defines a continuous G-action on M/A'_G with respect to which $\mathcal{J}$ is G-equivariant. Notice that since this action is not smooth and M/A'_G is not Hausdorff in general, there is no guarantee that the isotropy subgroups G_ρ are closed, and therefore embedded, subgroups of G. However, there is a unique smooth structure on G_ρ for which this subgroup becomes an initial Lie subgroup of G with Lie algebra $\mathfrak{g}_\rho$ given by

$$\mathfrak{g}_\rho = \{\xi \in \mathfrak{g} \mid \xi_M(m) \in T_m\mathcal{J}^{-1}(\rho), \text{ for all } m \in \mathcal{J}^{-1}(\rho)\}.$$

With this smooth structure for G_ρ, the left action $\Phi^\rho\colon G_\rho \times \mathcal{J}^{-1}(\rho) \to \mathcal{J}^{-1}(\rho)$ defined by $\Phi^\rho(g, z) := \Phi(g, z)$ is smooth.

Theorem 5.6 ([18]). *Let $(M, \{\cdot, \cdot\})$ be a smooth Poisson manifold and G a Lie group acting canonically and properly on M. Let $\mathcal{J}\colon M \to M/A'_G$ be the optimal momentum map associated to this action. Then, for any $\rho \in M/A'_G$ whose isotropy subgroup G_ρ acts properly on $\mathcal{J}^{-1}(\rho)$, the orbit space $M_\rho := \mathcal{J}^{-1}(\rho)/G_\rho$ is a smooth symplectic regular quotient manifold with symplectic form ω_ρ defined by:*

$$\begin{aligned} &\pi_\rho^*\omega_\rho(m)(X_f(m), X_h(m)) = \{f, h\}(m),\\ &\textit{for any } m \in \mathcal{J}^{-1}(\rho) \textit{ and any } f, h \in C^\infty(M)^G. \end{aligned} \tag{46}$$

The map $\pi_\rho : \mathcal{J}^{-1}(\rho) \to \mathcal{J}^{-1}(\rho)/G_\rho$ is the projection.

Suppose now that the G-action is free and proper. It is well known that the orbit space M/G is a Poisson manifold with the Poisson bracket $\{\cdot, \cdot\}_{M/G}$, uniquely characterized by the relation

$$\{f,\ g\}_{M/G}(\pi(m)) = \{f \circ \pi,\ g \circ \pi\}(m), \tag{47}$$

for any $m \in M$ and where $f, g\colon M/G \to \mathbb{R}$ are two arbitrary smooth functions. A fact that we will use in the sequel is that the symplectic leaves of $(M/G, \{\cdot, \cdot\}_{M/G})$ are given by the optimal orbit reduced spaces $(\mathcal{J}^{-1}(\mathcal{O}_\rho)/G, \omega_{\mathcal{O}_\rho})$, $\mathcal{O}_\rho := G \cdot \rho$, $\rho \in M/A'_G$, that are symplectically diffeomorphic to the optimal point reduced spaces introduced in Theorem 5.6 via the map $\pi_\rho(m) \longmapsto \pi_{\mathcal{O}_\rho}(m)$, with $m \in \mathcal{J}^{-1}(\rho)$ and $\pi_{\mathcal{O}_\rho}\colon \mathcal{J}^{-1}(\mathcal{O}_\rho) \to \mathcal{J}^{-1}(\mathcal{O}_\rho)/G$ the projection.

Theorem 5.7. *Let (M, ω) be a connected and paracompact symplectic manifold and G a Lie group acting freely, properly, and symplectically on it. Let $\mathbf{K}\colon M \to \mathfrak{g}^*/\bar{\mathcal{H}}$ be a cylinder valued momentum map for this action with associated non-equivariance one-cocycle $\sigma\colon M \to \mathfrak{g}^*/\bar{\mathcal{H}}$. Let N be a normal connected Lie subgroup of G that has $(\mathrm{Lie}(\bar{\mathcal{H}}))^\circ$ as Lie algebra. Then for any $[\mu] \in \mathfrak{g}^*/\bar{\mathcal{H}}$:*

- **(i)** *The orbit space $M^{[\mu]} := \mathbf{K}^{-1}([\mu])/G_{[\mu]}$ is a regular quotient manifold endowed with a natural Poisson structure induced by the bracket $\{\cdot,\cdot\}_{M^{[\mu]}}$ determined by the expression*

$$\{f, h\}_{M^{[\mu]}}\left(\pi^{[\mu]}(m)\right) = \{F, H\}(m), \tag{48}$$

where $G_{[\mu]}$ denotes the isotropy subgroup of $[\mu]$ with respect to the affine G-action on $\mathfrak{g}^*/\bar{\mathcal{H}}$ constructed using the non-equivariance cocycle σ of $\mathbf{K}$, $\pi^{[\mu]}\colon \mathbf{K}^{-1}([\mu]) \to \mathbf{K}^{-1}([\mu])/G_{[\mu]}$ is the projection, and $F, H \in C^\infty(M)^G$ are local G-invariant extensions of $f \circ \pi^{[\mu]}$ and $h \circ \pi^{[\mu]}$ around the point m, respectively. We will refer to the spaces $M^{[\mu]}$ as the "Poisson reduced spaces".
- **(ii)** *The Lie group $H_{[\mu]} := G_{[\mu]}/N_{[\mu]}$ acts canonically, freely, and properly on $(M_{[\mu]}, \omega_{[\mu]})$. The reduced Poisson manifold $(M_{[\mu]}/H_{[\mu]}, \{\cdot,\cdot\}_{H_{[\mu]}})$ is Poisson isomorphic to $(M^{[\mu]}, \{\cdot,\cdot\}_{M^{[\mu]}})$ via the map*

$$\begin{array}{rccc} \Psi\colon & M_{[\mu]}/H_{[\mu]} & \longrightarrow & M^{[\mu]} \\ & \pi_{H_{[\mu]}}\left(\pi_{[\mu]}(m)\right) & \longmapsto & \pi^{[\mu]}(m), \end{array}$$

where $\pi_{H_{[\mu]}}\colon M_{[\mu]} \to M_{[\mu]}/H_{[\mu]}$ is the projection.
- **(iii)** *Let $\mathcal{J}_{H_{[\mu]}}\colon M_{[\mu]} \to M_{[\mu]}/A'_{H_{[\mu]}}$ be the optimal momentum map associated to the $H_{[\mu]}$-action on $M_{[\mu]}$. Suppose that for any $\rho \in M_{[\mu]}/A'_{H_{[\mu]}}$, the isotropy subgroup $(H_{[\mu]})_\rho$ acts properly on the level set $\mathcal{J}^{-1}_{H_{[\mu]}}(\rho)$. Then the symplectic leaves of $(M^{[\mu]}, \{\cdot,\cdot\}_{M^{[\mu]}})$ are given by the manifolds $\Psi(\mathcal{J}^{-1}_{H_{[\mu]}}(\mathcal{O}_\rho)/H_{[\mu]})$, for any $\rho \in M_{[\mu]}/A'_{H_{[\mu]}}$.*
- **(iv)** *Let $\mathcal{J}\colon M \to M/A'_G$ be the optimal momentum map associated to the G-action on M and let $m \in M$ be such that $\mathbf{K}(m) = [\mu]$. Let $\rho = \mathcal{J}(m)$. Then $\mathcal{J}^{-1}(\rho) \subset \mathbf{K}^{-1}([\mu])$ and $G_\rho \subset G_{[\mu]}$. If the isotropy subgroup G_ρ acts properly on $\mathcal{J}^{-1}(\rho)$ then $(M_\rho := \mathcal{J}^{-1}(\rho)/G_\rho, \omega_\rho)$ is a smooth regular quotient symplectic manifold and the map*

$$\begin{array}{rccc} L\colon & \mathcal{J}^{-1}(\rho)/G_\rho & \longrightarrow & \mathbf{K}^{-1}([\mu])/G_{[\mu]} \\ & \pi_\rho(z) & \longmapsto & \pi^{[\mu]}(z) \end{array}$$

is well defined, smooth, injective, and its image is the symplectic leaf $\mathcal{L}_{\pi^{[\mu]}(m)}$ of $(M^{[\mu]}, \{\cdot,\cdot\}_{M^{[\mu]}})$ that contains $\pi^{[\mu]}(m)$. If, additionally, $\mathcal{J}^{-1}(\rho)$ is a closed subset of M then L is a Poisson map and the Poisson bracket $\{\cdot,\cdot\}_{M_\rho}$ induced by the symplectic form ω_ρ is determined by the expression

$$\{f, h\}_{M_\rho}(\pi_\rho(z)) = \{F, H\}(z), \tag{49}$$

where $F, H \in C^\infty(M)^G$ are local G-invariant extensions of $f \circ \pi_\rho$ and $h \circ \pi_\rho$ around the point $z \in \mathcal{J}^{-1}(\rho)$, respectively.

Remark 5.8. This theorem links all the three reduction schemes induced by a free, proper, and canonical action on a symplectic manifold. On one hand, we have the two possible reductions using the cylinder valued momentum map: $M_{[\mu]}$ the symplectic and $M^{[\mu]}$ the

Poisson reduced spaces that are related to each other via Poisson reduction by the quotient group $G_{[\mu]}/N_{[\mu]}$. On the other hand, we can carry out optimal reduction; point (iv) in the previous statement shows that the optimal reduced spaces are the symplectic leaves of the Poisson reduced space $M^{[\mu]}$.

All these three reduced spaces are, in general, different (see the example below). Nevertheless, we note that if $\mathcal{H}$ is closed in $\mathfrak{g}^*$ then $(\mathrm{Lie}(\bar{\mathcal{H}}))^\circ = (\mathrm{Lie}(\mathcal{H}))^\circ = \mathfrak{g}$ and hence $G_{[\mu]} = N_{[\mu]}$, $H_{[\mu]} = \{e\}$, $M_{[\mu]} = M^{[\mu]}$. Moreover, in this situation, the closedness hypothesis needed in point (iv) of the statement of the theorem always holds and the optimal reduced spaces are the connected components of $M_{[\mu]} = M^{[\mu]}$. This is so because whenever $\mathcal{H}$ is closed in $\mathfrak{g}^*$ then, by Proposition 2.3, $\ker T_z\mathbf{K} = (\mathfrak{g}\cdot z)^\omega$, $z \in M$, and hence $T_z\mathcal{J}^{-1}(\rho) = \ker T_z\mathbf{K}$, $z \in \mathcal{J}^{-1}(\rho)$, which shows that $\mathcal{J}^{-1}(\rho)$ is one of the connected components of $\mathbf{K}^{-1}([\mu])$ and hence is closed in M.

We emphasize that the closedness of $\mathcal{H}$ in $\mathfrak{g}^*$ is a sufficient, but in general not necessary, condition for the three reduced spaces to coincide (see Example 6.11).

Example 5.9. The following elementary example shows that the three reduced spaces in the statement of Theorems 5.1 and 5.7, that is, the Poisson, the symplectic, and the optimal reduced spaces, are in general distinct. Let $M := \mathbb{T}^2 \times \mathbb{T}^2$ be the product of two tori whose elements will be denoted by the four-tuples $(e^{i\theta_1}, e^{i\theta_2}, e^{i\psi_1}, e^{i\psi_2})$. Endow M with the symplectic structure ω defined by $\omega := \mathbf{d}\theta_1 \wedge \mathbf{d}\theta_2 + \sqrt{2}\,\mathbf{d}\psi_1 \wedge \mathbf{d}\psi_2$. Consider the canonical circle action given by $e^{i\phi}\cdot(e^{i\theta_1}, e^{i\theta_2}, e^{i\psi_1}, e^{i\psi_2}) := (e^{i(\theta_1+\phi)}, e^{i\theta_2}, e^{i(\psi_1+\phi)}, e^{i\psi_2})$ and the trivial principal bundle $(M\times\mathbb{R}) \to M$ with $(\mathbb{R}, +)$ as structure group. It is easy to see that the horizontal vectors in $T(M\times\mathbb{R})$ with respect to the connection α defined in (1) are of the form $((a_1, a_2, b_1, b_2), -a_2 - \sqrt{2}b_2)$, with $a_1, a_2, b_1, b_2 \in \mathbb{R}$. The surfaces $\tilde{M}_\tau \subset M\times\mathbb{R}$ of the form $\tilde{M}_\tau := \{((e^{i\theta_1}, e^{i\theta_2}, e^{i\psi_1}, e^{i\psi_2}), \tau - \theta_2 - \sqrt{2}\psi_2) \in M\times\mathbb{R} \mid \theta_1, \theta_2, \psi_1, \psi_2 \in \mathbb{R}\}$ integrate the horizontal distribution spanned by these vectors.

Take now $\tilde{M} := \tilde{M}_0$ and consider the projection $\tilde{p}\colon \tilde{M} \to M$. It is clear that $\tilde{p}^{-1}(e^{i\theta_1}, e^{i\theta_2}, e^{i\psi_1}, e^{i\psi_2}) = \{(e^{i\theta_1}, e^{i\theta_2}, e^{i\psi_1}, e^{i\psi_2}, -(\theta_2 + 2n\pi) - \sqrt{2}(\psi_2 + 2m\pi)) \mid m, n \in \mathbb{Z}\}$. Since the Hamiltonian holonomy $\mathcal{H}$ coincides with the structure group of the fibration $\tilde{p}\colon \tilde{M} \to M$, it follows that $\mathcal{H} = \mathbb{Z} + \sqrt{2}\mathbb{Z} \subset \mathbb{R}$. In this case $\mathcal{H}$ is indeed not closed; moreover $\mathcal{H}$ is dense in $\mathbb{R}$, that is, $\bar{\mathcal{H}} = \mathbb{R}$. Therefore, in this case, the cylinder valued momentum map is a constant map since its range is just a point and the group $N = \{e\}$. Hence, the symplectic reduced space $M_{[\mu]}$ equals the entire symplectic manifold $\mathbb{T}^4$ and the Poisson reduced space $M^{[\mu]}$ equals the orbit space $\mathbb{T}^4/S^1$.

We now compute the optimal reduced spaces. In this case, $C^\infty(M)^{S^1}$ consists of all the functions f of the form $f(e^{i\theta_1}, e^{i\theta_2}, e^{i\psi_1}, e^{i\psi_2}) \equiv g(e^{i\theta_2}, e^{i\psi_2}, e^{i(\theta_1-\psi_1)})$, for some function $g \in C^\infty(\mathbb{T}^3)$. An inspection of the Hamiltonian flows associated to such functions readily shows that the leaves of A'_{S^1}, that is, the level sets $\mathcal{J}^{-1}(\rho)$ of the optimal momentum map $\mathcal{J}$, are the product of a two-torus with a leaf of an irrational foliation (Kronecker submanifold) of another two-torus. The isotropy subgroups S^1_ρ coincide with the circle S^1, whose compactness guarantees that its action on $\mathcal{J}^{-1}(\rho)$ is proper. Theorem 5.6 automatically guarantees that the quotients

$$M_\rho := \mathcal{J}^{-1}(\rho)/S^1_\rho \simeq (\mathbb{T}^2 \times \{\text{Kronecker submanifold of } \mathbb{T}^2\})/S^1.$$

are symplectic and, by Theorem 5.7, they are the symplectic leaves of the quotient Poisson manifold $\mathbb{T}^4/S^1$.

Proof of Theorem 5.7.

(i) The smooth structure of $M^{[\mu]}$ as a regular quotient manifold is a consequence of the freeness and properness of the G-action, using the same arguments as in the proof of Theorem 5.1. Let $S := \mathbf{K}^{-1}([\mu])$ and $D \subset TM|_S$ be defined by $D(m) := \mathfrak{g} \cdot m, m \in M$. The reduction lemma guarantees that the distribution $D_S := D \cap TS$ in S coincides with the tangent spaces to the $G_{[\mu]}$-orbits and hence Lemma 10.4.14 in [22] guarantees that the local G-invariant extensions used in the expression (48) do exist. Let $B^\sharp\colon T^*M \to TM$ be vector bundle isomorphism induced by the symplectic form ω on M. Since for any $m \in \mathbf{K}^{-1}([\mu])$

$$B^\sharp(m)((D(m))^\circ) = (D(m))^\omega = (\mathfrak{g} \cdot m)^\omega \subset (\mathfrak{n} \cdot m)^\omega = \ker T_m\mathbf{K},$$

it is clear that $B^\sharp(D^\circ) \subset TS + D$ and hence the Marsden–Ratiu Theorem on Poisson reduction [14] guarantees that (48) is a well-defined bracket.

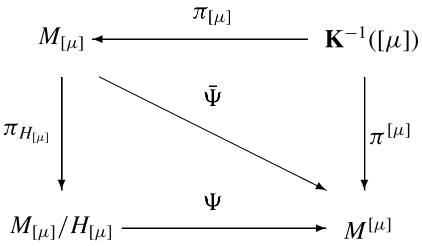
(ii) The $H_{[\mu]}$-action on $M_{[\mu]}$ is given by

$$[g]_\mu \cdot \pi_{[\mu]}(m) := \pi_{[\mu]}(g \cdot m), \quad [g]_{[\mu]} \in G_{[\mu]}/H_{[\mu]}.$$

This action is obviously canonical, free, and proper, and hence $M_{[\mu]}/H_{[\mu]}$ is a smooth Poisson manifold such that the projection $\pi_{H_{[\mu]}}$ is a Poisson surjective submersion. It is easily verified that Ψ is a well-defined smooth bijective map with smooth inverse given by

$$\pi^{[\mu]}(m) \longmapsto \pi_{H_{[\mu]}}(\pi_{[\mu]}(m)), \quad m \in \mathbf{K}^{-1}([\mu]).$$

Consequently, Ψ is a diffeomorphism. In order to show that Ψ is Poisson let $\bar{\Psi} := \Psi \circ \pi_{H_{[\mu]}}\colon M_{[\mu]} \to M^{[\mu]}$. Notice that $\bar{\Psi}$ makes the diagram



commutative. Since $\pi_{[\mu]}$ and $\pi_{H_{[\mu]}}$ are surjective submersions it follows that Ψ is Poisson if and only if

$$\{f, h\}_{M^{[\mu]}} \circ \Psi \circ \pi_{H_{[\mu]}} \circ \pi_{[\mu]} = \{f \circ \Psi \circ \pi_{H_{[\mu]}}, h \circ \Psi \circ \pi_{H_{[\mu]}}\}_{M_{[\mu]}} \circ \pi_{[\mu]},$$

which is equivalent to

$$\{f, h\}_{M^{[\mu]}}\big(\pi^{[\mu]}(m)\big) = \{f \circ \bar{\Psi}, h \circ \bar{\Psi}\}_{M_{[\mu]}}(\pi_{[\mu]}(m)), \tag{50}$$

for any $f, h \in C^\infty(M^{[\mu]})$ and $m \in \mathbf{K}^{-1}([\mu])$. By part (i), the left-hand side of (50) equals $\{F, H\}$, with $F, H \in C^\infty(M)^G$ local G-invariant extensions at m of $f \circ \pi^{[\mu]}$ and $g \circ \pi^{[\mu]}$, respectively. Additionally, since the bracket $\{\cdot, \cdot\}_{M_{[\mu]}}$ is induced by the symplectic form $\omega_{[\mu]}$ it equals, by part (iii) in Theorem 5.4, $\{\bar{F}, \bar{H}\}(m)$ with $\bar{F}, \bar{H} \in C^\infty(M)^N$ local N-invariant extensions at m of $f \circ \bar{\Psi} \circ \pi_{[\mu]}$ and $h \circ \bar{\Psi} \circ \pi_{[\mu]}$, respectively. Since $\bar{\Psi} \circ \pi_{[\mu]} = \pi^{[\mu]}$, $\bar{F}$ and $\bar{H}$ can be taken to be F and H, respectively, which proves (50).

(iii) This follows from part (iv) in Theorem 10.1.1 of [22].

(iv) By Proposition 2.3 we can think of the connected components of the level sets of $\mathbf{K}$ as the maximal integral manifolds of the distribution D in M given by $D(z) = (\mathfrak{n} \cdot z)^\omega, z \in M$. Since $(\mathfrak{n} \cdot z)^\omega \supset (\mathfrak{g} \cdot z)^\omega = A'_G(z)$ and the level sets of the optimal momentum map $\mathcal{J}$ are the maximal integral manifolds of A'_G, we obviously have that $\mathcal{J}^{-1}(\rho) \subset \mathbf{K}^{-1}([\mu])$. Let now $g \in G_\rho$ and $m' := \mathfrak{g} \cdot m$. Since $\mathcal{J}$ is G-equivariant we have $\mathcal{J}(m') = \mathcal{J}(g \cdot m) = g \cdot \rho = \rho$ and hence $m' \in \mathbf{K}^{-1}([\mu])$ which in turn implies that $[\mu] = \mathbf{K}(m') = \mathbf{K}(g \cdot m) = g \cdot \mathbf{K}(m) = g \cdot [\mu]$ and hence guarantees that $g \in G_{[\mu]}$.

If G_ρ acts freely and properly on $\mathcal{J}^{-1}(\rho)$ then the hypotheses of the optimal reduction theorem 5.6 are satisfied. The map L is just the projection of the $(G_\rho, G_{[\mu]})$-equivariant inclusion $\mathcal{J}^{-1}(\rho) \hookrightarrow \mathbf{K}^{-1}([\mu])$ and is hence well defined and smooth. Injectivity is obvious. We now show that $L(M_\rho) = \mathcal{L}_{\pi^{[\mu]}(m)}$. Let $\pi_\rho(z) \in M_\rho$. By the definition of the optimal momentum map there exists a finite composition of Hamiltonian flows $F_{t_1}, \ldots, F_{t_n}$ corresponding to G-invariant Hamiltonian functions such that $z = (F^1_{t_1} \circ \cdots \circ F^n_{t_n})(m)$. For simplicity in the exposition take $n = 1$ and let $F \in C^\infty(M)^G$ be the function whose Hamiltonian flow is F_t. Let $f \in C^\infty(M^{[\mu]})$ be the function defined by $F \circ i^{[\mu]} = f \circ \pi^{[\mu]}$. It is easy to see that the Hamiltonian flow F^f_t of X_f in $M^{[\mu]}$ is such that $F^f_t \circ \pi^{[\mu]} = \pi^{[\mu]} \circ F_t \circ i^{[\mu]}$. Consequently

$$L(\pi_\rho(z)) = L(\pi_\rho(F_t(m))) = \pi^{[\mu]}(F_t(m)) = F^f_t\left(\pi^{[\mu]}(m)\right) \in \mathcal{L}_{\pi^{[\mu]}(m)}.$$

This argument can be reversed by using local G-invariant extensions of the compositions of the functions in $M^{[\mu]}$ with $\pi^{[\mu]}$ (that do exist by part (i)) and hence proving the converse inclusion $\mathcal{L}_{\pi^{[\mu]}(m)} \subset L(M_\rho)$.

The closedness hypothesis on $\mathcal{J}^{-1}(\rho)$ implies that this set is an embedded submanifold of M (recall that closed integral leaves of constant rank integrable distributions are always embedded [4]). Additionally, it can be proved (see Proposition 4.5 in [20]) that under this hypothesis the isotropy subgroup G_ρ is closed in G and that

$$\mathfrak{g}_\rho \cdot z = T_z\mathcal{J}^{-1}(\rho) \cap \mathfrak{g} \cdot z, \quad z \in \mathcal{J}^{-1}(\rho). \tag{51}$$

Consequently, the form of the bracket (49) is obtained by mimicking the proof of part (iii) of Theorem 5.4. Notice that the local G-invariant extensions needed in (49) always exist by Lemma 10.4.14 in [22], which can be applied due to the fact that $\mathcal{J}^{-1}(\rho)$ is an embedded submanifold of M and (51) holds. The Poisson character of L is a straightforward consequence of (48) and (49). □

6. Example: Magnetic cotangent bundles of Lie groups

Let G be a finite-dimensional Lie group and T^*G its cotangent bundle endowed with the magnetic symplectic structure $\overline{\omega_\Sigma} := \omega_{\text{can}} - \pi^* B_\Sigma$, where ω_{can} is the canonical symplectic form on T^*G, $\pi: T^*G \to G$ is the projection onto the base, and $B_\Sigma \in \Omega^2(G)^G$ is a left invariant two-form on G whose value at the identity is the Lie algebra two-cocycle Σ:

$\mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$. Since Σ is a cocycle it follows that B_Σ is closed and hence $\overline{\omega_\Sigma}$ is a symplectic form.

The cotangent lift of the action of G on itself by left translations produces, due to the invariance of B_Σ, a canonical G-action on $(T^*G, \overline{\omega_\Sigma})$. In the absence of magnetic terms this action has an associated coadjoint equivariant momentum map $\mathbf{J}\colon T^*G \to \mathfrak{g}^*$ given by $\langle \mathbf{J}(\alpha_g), \xi \rangle := \langle \alpha_g, T_eR_g(\xi) \rangle$, $\alpha_g \in T^*G$, $\xi \in \mathfrak{g}$. It is well known that the Marsden–Weinstein reduced spaces associated to this action are naturally symplectomorphic to the coadjoint orbits of the G-action on $\mathfrak{g}^*$, endowed with their canonical Kostant–Kirillov–Souriau orbit symplectic form. The magnetic term destroys this picture in most cases. As we will see later on in this section, $(T^*G, \overline{\omega_\Sigma})$ has, in general, a non-zero Hamiltonian holonomy $\mathcal{H}$ and hence the lift of left translation does not admit anymore a standard momentum map. This forces us, when carrying out reduction, to work in the degree of generality of the preceding section. We will hence compute in this setup the cylinder valued and the optimal momentum maps and will characterize the three reduced spaces introduced in Theorems 5.1 and 5.7. One of the conclusions of our discussion will be the fact that the resulting reduced spaces are related not to the coadjoint orbits but to the orbits of the extended affine action of G_Σ on $\mathfrak{g}^*$, with G_Σ the connected and simply connected Lie group that integrates $\mathfrak{g}_\Sigma$, the one-dimensional central extension of $\mathfrak{g}$ constructed using the cocycle Σ. More specifically, the optimal reduced spaces are naturally symplectomorphic to these orbits which shows that, in this context, optimal reduction is the natural generalization of the picture that allows one to see in the standard setup the Kostant–Kirillov–Souriau coadjoint orbits as symplectic reduced spaces.

In order to make the problem more tractable we introduce the left trivialization $\lambda\colon T^*G \to G \times \mathfrak{g}^*$ of T^*G given by $\lambda(\alpha_g) := (g, T_e^*L_g(\alpha_g))$, for any $\alpha_g \in T_g^*G$. Its inverse $\lambda^{-1}\colon G \times \mathfrak{g}^* \to T^*G$ is $\lambda^{-1}(g, \mu) = T_g^*L_{g^{-1}}(\mu)$, $(g, \mu) \in G \times \mathfrak{g}^*$. The cotangent lift of left translations on G to T^*G is given in this trivialization by $h \cdot (g, \mu) = (hg, \mu)$, $h, g \in G$, $\mu \in \mathfrak{g}^*$, and hence the infinitesimal generators take the form

$$\xi_{G\times\mathfrak{g}^*}(g, \mu) = (T_eR_g(\xi), 0) = (T_eL_g(\mathrm{Ad}_{g^{-1}}\xi), 0), \quad \xi \in \mathfrak{g}. \tag{52}$$

Additionally, the magnetic symplectic form $\omega_\Sigma := (\lambda^{-1})^*\overline{\omega_\Sigma}$ on $G \times \mathfrak{g}^*$ has the expression

$$\omega_\Sigma(g, \nu)((T_eL_g(\xi), \rho), (T_eL_g(\eta), \sigma)) = \langle \sigma, \xi \rangle - \langle \rho, \eta \rangle + \langle \nu, [\xi, \eta] \rangle - \Sigma(\xi, \eta). \tag{53}$$

In order to compute the cylinder valued momentum map of the canonical G-action on $(G \times \mathfrak{g}^*, \omega_\Sigma)$ we note that the horizontal distribution associated to the connection α on the bundle $(G \times \mathfrak{g}^*) \times \mathfrak{g}^* \to G \times \mathfrak{g}^*$ defined in (1) equals, by (52) and (53),

$$H((g, \mu), \nu) = \{(T_eL_g(\xi), \rho, \mathrm{Ad}^*_{g^{-1}}(\rho - \mathrm{ad}^*_\xi\mu + \Sigma(\xi, \cdot))) \mid \xi \in \mathfrak{g}, \rho \in \mathfrak{g}^*\}. \tag{54}$$

Proposition 6.1. *Let $G \times \mathfrak{g}^* \to G$ be the trivial principal fiber bundle with structure group $(\mathfrak{g}^*, +)$ and where the action is given by $R_\nu(g, \mu) := (g, \mu - \nu)$, $g \in G$, $\mu, \nu \in \mathfrak{g}^*$.*

(i) *The distribution H_G on $G \times \mathfrak{g}^*$ given by*

$$H_G(g, \mu) = \{(T_eL_g(\xi), \mathrm{Ad}^*_{g^{-1}}(\Sigma(\xi, \cdot))) \mid \xi \in \mathfrak{g}\}$$

defines a flat connection α_G on $G \times \mathfrak{g}^* \to G$.

(ii) *The holonomy bundle $(\widetilde{G \times \mathfrak{g}^*})_{(g,\mu,\nu)}$ of the connection α defined in (1) that contains the point $(g, \mu, \nu) \in (G \times \mathfrak{g}^*) \times \mathfrak{g}^*$ is given by*

$$(\widetilde{G \times \mathfrak{g}^*})_{(g,\mu,\nu)} = \left\{(h, \rho, \mathrm{Ad}^*_{h^{-1}}\rho + \tau) \mid \rho \in \mathfrak{g}^*, (h, \tau) \in \tilde{G}_{(g,\nu - \mathrm{Ad}^*_{g^{-1}}\mu)}\right\},$$

where $\tilde{G}_{(g,\nu-\mathrm{Ad}^*_{g^{-1}}\mu)}$ is the holonomy bundle of the connection α_G containing the point $(g, \nu - \mathrm{Ad}^*_{g^{-1}}\mu)$.

(iii) *The holonomy group $\mathcal{H}_G$ of α_G and the Hamiltonian holonomy $\mathcal{H}$ of the G-action on $(G \times \mathfrak{g}^*, \omega_\Sigma)$ coincide, that is, $\mathcal{H} = \mathcal{H}_G$.*

Proof:

(i) The vertical bundle V_G of $G \times \mathfrak{g}^* \to G$ is given by $V_G(g, \mu) = \{(0, \rho) \mid \rho \in \mathfrak{g}^*\}$. It easily follows that $T_{(g,\mu)}(G \times \mathfrak{g}^*) = H_G(g, \mu) \oplus V_G(g, \mu)$ and that $T_{(g,\mu)}R_\nu(H_G(g, \mu)) = H_G(g, \mu - \nu)$, for any $g \in G$, $\mu, \nu \in \mathfrak{g}^*$, which proves that H_G is the horizontal bundle of a connection on $G \times \mathfrak{g}^* \to G$ whose associated one-form $\alpha_G \in \Omega^1(G \times \mathfrak{g}^*; \mathfrak{g}^*)$ is

$$\alpha_G(g, \mu)(T_e L_g(\xi), \rho) = \mathrm{Ad}^*_{g^{-1}}(\Sigma(\xi, \cdot)) - \rho.$$

In the computation of this expression we used that the horizontal $v^H_{(g,\mu)}$ and vertical $v^V_{(g,\mu)}$ components of any vector $v_{(g,\mu)} = (T_e L_g(\xi), \rho)$ are given by $v^H_{(g,\mu)} = (T_e L_g(\xi), \mathrm{Ad}^*_{g^{-1}}(\Sigma(\xi, \cdot)))$ and $v^V_{(g,\mu)} = (0, \rho - \mathrm{Ad}^*_{g^{-1}}(\Sigma(\xi, \cdot)))$. The flatness of α_G will be obtained as a consequence of point (iii) and of the zero-dimensional character of $\mathcal{H}$ (see below).

(ii) Let $(h, \tau) \in \tilde{G}_{(g,\nu-\mathrm{Ad}^*_{g^{-1}}\mu)}$. By definition, there exists a piecewise smooth horizontal curve $(g(t), \tau(t))$ such that $g(0) = g$, $g(1) = h$, $\tau(0) = \nu - \mathrm{Ad}^*_{g^{-1}}\mu$, $\tau(1) = \tau$, and $\tau'(t) = \mathrm{Ad}^*_{g(t)^{-1}}(\Sigma(T_{g(t)}L_{g(t)^{-1}}(g'(t)), \cdot))$. The piecewise smooth curve $\gamma(t) := (g(t), \mu(t), \mathrm{Ad}^*_{g(t)^{-1}}\mu(t) + \tau(t))$, with $\mu(t)$ an arbitrary smooth curve in $\mathfrak{g}^*$ such that $\mu(0) = \mu$ and $\mu(1) = \rho$, satisfies $\gamma(0) = (g, \mu, \nu)$, $\gamma(1) = (h, \rho, \mathrm{Ad}^*_{h^{-1}}\rho + \tau)$, and is horizontal. Indeed, let $\xi(t) := T_{g(t)}L_{g(t)^{-1}}(g'(t)) \in \mathfrak{g}$. Then

$$\begin{aligned}\gamma'(t) &= \left(T_e L_{g(t)}(\xi(t)), \mu'(t), \mathrm{Ad}^*_{g(t)^{-1}}\mu'(t) - \mathrm{Ad}^*_{g(t)^{-1}}(\mathrm{ad}^*_{\xi(t)}\mu(t)) + \tau'(t)\right)\\ &= \left(T_e L_{g(t)}(\xi(t)), \mu'(t), \mathrm{Ad}^*_{g(t)^{-1}}(\mu'(t) - \mathrm{ad}^*_{\xi(t)}\mu(t) + \Sigma(\xi(t), \cdot))\right),\end{aligned}$$

which belongs to $H(\gamma(t))$ by (54). This proves that $(h, \rho, \mathrm{Ad}^*_{h^{-1}}\rho + \tau) \in (\widetilde{G \times \mathfrak{g}^*})_{(g,\mu,\nu)}$.

Conversely, let $(h, \rho, \sigma) \in (\widetilde{G \times \mathfrak{g}^*})_{(g,\mu,\nu)}$. By definition, there exists a piecewise smooth horizontal curve $\gamma(t) := (g(t), \mu(t), \nu(t))$ such that $\gamma(0) = (g, \mu, \nu)$, $\gamma(1) = (h, \rho, \sigma)$, and

$$\nu'(t) = \mathrm{Ad}^*_{g(t)^{-1}}\left(\mu'(t) - \mathrm{ad}^*_{\xi(t)}\mu(t) + \Sigma(\xi(t), \cdot)\right), \tag{55}$$

where $\xi(t) = T_{g(t)}L_{g(t)^{-1}}(g'(t))$. We now show that $(h, \rho, \sigma) = (h, \rho, \mathrm{Ad}^*_{h^{-1}}\rho + \tau)$ where $(h, \tau) \in \tilde{G}_{(g,\nu-\mathrm{Ad}^*_{g^{-1}}\mu)}$. Let $\tau(t) := \nu(t) - \mathrm{Ad}^*_{g(t)^{-1}}\mu(t)$. Notice that $\tau(0) = \nu - \mathrm{Ad}^*_{g^{-1}}\mu$, $\tau(1) = \sigma - \mathrm{Ad}^*_{h^{-1}}\rho$, and that, by (55), $\tau'(t) = \mathrm{Ad}^*_{g(t)^{-1}}(\Sigma(\xi(t), \cdot))$, which shows that $(g(t), \tau(t))$ is horizontal. Consequently $(h, \rho, \sigma) = (g(1), \rho, \mathrm{Ad}^*_{h^{-1}}\rho + \tau(1))$ and hence the claim follows.

(iii) Let $\mu \in \mathcal{H}$ be arbitrary. Then there exists a loop $(g(t), \rho(t))$ in $G \times \mathfrak{g}^*$ whose horizontal lift $(g(t), \rho(t), \mu(t))$ satisfies that $\mu = \mu(1) - \mu(0)$. Consequently, using horizontality

$$\mu = \int_0^1 \mu'(t)\mathrm{d}t = \int_0^1 \mathrm{Ad}^*_{g(t)^{-1}}(\rho'(t) - \mathrm{ad}^*_{\xi(t)}\rho(t) + \Sigma(\xi(t), \cdot))\,\mathrm{d}t$$
$$= \int_0^1 \left(\frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{Ad}^*_{g(t)^{-1}}\rho(t)) + \mathrm{Ad}^*_{g(t)^{-1}}\Sigma(\xi(t), \cdot)\right)\mathrm{d}t = \int_0^1 \mathrm{Ad}^*_{g(t)^{-1}}\Sigma(\xi(t), \cdot)\,\mathrm{d}t, \quad (56)$$

where $\xi(t) = T_{g(t)}L_{g(t)^{-1}}(g'(t))$. Therefore, the loop $g(t)$ in G has a horizontal lift $(g(t), \nu(t))$ such that

$$\nu(1) - \nu(0) = \int_0^1 \nu'(t)\mathrm{d}t = \int_0^1 \mathrm{Ad}^*_{g(t)^{-1}}\Sigma(\xi(t), \cdot)\,\mathrm{d}t = \mu,$$

which proves that $\mu \in \mathcal{H}_G$. The converse inclusion is obtained by reading backwards the previous argument.
Finally, the equality $\mathcal{H} = \mathcal{H}_G$ implies that the connection α_G is flat since the tangent spaces to the holonomy bundles equal the horizontal distribution because Lie $(\mathcal{H}) = \{0\}$. Indeed, as the distribution associated to the holonomy bundles is integrable by general theory this shows that the horizontal distribution is integrable and hence the associated connection is flat. □

Remark 6.2. If the Lie algebra two-cocycle Σ can be integrated to a smooth $\mathfrak{g}^*$-valued group one-cocycle $\sigma: G \to \mathfrak{g}^*$, that is

$$\Sigma(\xi, \cdot) = T_e\sigma(\xi), \quad \text{for any } \xi \in \mathfrak{g},$$

then the holonomy bundles $\tilde{G}_{(g,\mu)}$ are the graphs of σ. More specifically

$$\tilde{G}_{(g,\mu)} = \{(h, \sigma(h) + \mu - \sigma(g)) \mid h \in G\}.$$

In this particular case $\mathcal{H}_G = \{0\}$ and hence, by Proposition 6.1, $\mathcal{H} = \{0\}$.

We are now going to use the central extensions and their actions introduced in Section 4 to better characterize the holonomy bundles of α and α_G. This will allow us to give an explicit expression for the cylinder valued momentum map of the G-action on the magnetic cotangent bundle $(G \times \mathfrak{g}^*, \omega_\Sigma)$.

Proposition 6.3. *Let $\mathfrak{g}_\Sigma$ be the one-dimensional central extension of the Lie algebra $\mathfrak{g}$ determined by the co-cycle Σ, G_Σ the connected and simply connected Lie group that integrates it, and $\mu_\Sigma: G_\Sigma \to \mathfrak{g}^*$ the $\mathfrak{g}^*$-valued one-cocycle introduced in Proposition 4.4. The holonomy bundle $\tilde{G}_{(h,\nu)}$ of the connection α_G that contains the point $(h, \nu) \in G \times \mathfrak{g}^*$ equals*

$$\tilde{G}_{(h,\nu)} = \{(\pi_G(\hat{g}), \mu_\Sigma(\hat{g}^{-1}) - \mu_\Sigma(\hat{h}^{-1}) + \nu) \mid \hat{g}, \hat{h} \in G_\Sigma \text{ such that } \pi_G(\hat{h}) = h\}, \quad (57)$$

with $\pi_G: G_\Sigma \to G$ the projection.

Proof: Since $\tilde{G}_{(h,\nu)} = R_{-\nu}(\tilde{G}_{(h,0)})$ it suffices to show that

$$\tilde{G}_{(h,0)} = \{(\pi_G(\hat{g}), \mu_\Sigma(\hat{g}^{-1}) - \mu_\Sigma(\hat{h}^{-1})) \mid \hat{g}, \hat{h} \in G_\Sigma \text{ such that } \pi_G(\hat{h}) = h\}. \tag{58}$$

We begin with the inclusion $\supset$. Let $\hat{g} \in G_\Sigma$ and $\hat{g}(t)$ a piecewise smooth curve in G_Σ such that $\hat{g}(0) = \hat{h}$, $\hat{g}(1) = \hat{g}$, and where $\pi_G(\hat{h}) = h$. We will show that the element $(\pi_G(\hat{g}), \mu_\Sigma(\hat{g}^{-1}) - \mu_\Sigma(\hat{h}^{-1}))$ belongs to $\tilde{G}_{(h,0)}$ by proving that the curve $\gamma(t) := (\pi_G(\hat{g}(t)), \mu_\Sigma(\hat{g}(t)^{-1}) - \mu_\Sigma(\hat{h}^{-1}))$ is horizontal and connects $(h, 0)$ with $(\pi_G(\hat{g}), \mu_\Sigma(\hat{g}^{-1}) - \mu_\Sigma(\hat{h}^{-1}))$. Indeed, $\gamma(0) = (h, 0)$, $\gamma(1) = (\pi_G(\hat{g}), \mu_\Sigma(\hat{g}^{-1}) - \mu_\Sigma(\hat{h}^{-1}))$, and $\gamma(t)$ is horizontal because if we write

$$\frac{\mathrm{d}}{\mathrm{d}t}\pi_G(\hat{g}(t)) = T_e L_{g(t)}(\xi(t)),$$

where $g(t) := \pi_G(\hat{g}(t))$ and $\xi(t) := T_{g(t)} L_{g(t)^{-1}}(g'(t))$, then by Proposition 4.4

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mu_\Sigma(\hat{g}(t)^{-1}) &= -\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{Ad}^*_{g(t)^{-1}}\mu_\Sigma(\hat{g}(t))\right)\\ &= -\mathrm{Ad}^*_{g(t)^{-1}}\big(-\mathrm{ad}^*_{\xi(t)}\mu_\Sigma(\hat{g}(t)) + T_{\hat{g}(t)}\mu_\Sigma(\hat{g}'(t))\big).\end{aligned} \tag{59}$$

Now let $(\hat{\xi}(t), s(t)) := T_{\hat{g}(t)} L_{\hat{g}(t)^{-1}}(\hat{g}'(t))$. It turns out that $\hat{\xi}(t) = \xi(t)$ because

$$\begin{aligned}\hat{\xi}(t) = \pi_{\mathfrak{g}}(\hat{\xi}(t), s(t)) = T_e\pi_G(\hat{\xi}(t), s(t)) &= \left.\frac{\mathrm{d}}{\mathrm{d}\epsilon}\right|_{\epsilon=0} \pi_G(\hat{g}(t)^{-1}\hat{g}(t+\epsilon))\\ &= \left.\frac{\mathrm{d}}{\mathrm{d}\epsilon}\right|_{\epsilon=0} \pi_G(\hat{g}(t)^{-1})\pi_G(\hat{g}(t+\epsilon))\\ &= \left.\frac{\mathrm{d}}{\mathrm{d}\epsilon}\right|_{\epsilon=0} g(t)^{-1}g(t+\epsilon) = T_{g(t)}L_{g(t)^{-1}}(g'(t)) = \xi(t),\end{aligned}$$

and hence (59) equals, by Proposition 4.4,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mu_\Sigma(\hat{g}(t)^{-1}) &= -\mathrm{Ad}^*_{g(t)^{-1}}\left(-\mathrm{ad}^*_{\xi(t)}\mu_\Sigma(\hat{g}(t)) + \mathrm{ad}^*_{\xi(t)}\mu_\Sigma(\hat{g}(t)) - \Sigma(\xi(t), \cdot)\right)\\ &= \mathrm{Ad}^*_{g(t)^{-1}}\left(\Sigma(\xi(t), \cdot)\right),\end{aligned}$$

which shows that $\gamma(t)$ is horizontal, as required.

We prove next the reverse inclusion

$$\tilde{G}_{(h,0)} \subset \{(\pi_G(\hat{g}), \mu_\Sigma(\hat{g}^{-1}) - \mu_\Sigma(\hat{h}^{-1})) \mid \hat{g}, \hat{h} \in G_\Sigma \text{ such that } \pi_G(\hat{h}) = h\}$$

by showing that any piecewise smooth horizontal curve $\gamma(t) \subset \tilde{G}_{(h,0)}$ such that $\gamma(0) = (h, 0)$ satisfies $\gamma(t) \in \{(\pi_G(\hat{g}), \mu_\Sigma(\hat{g}^{-1}) - \mu_\Sigma(\hat{h}^{-1})) \mid \hat{g}, \hat{h} \in G_\Sigma \text{ such that } \pi_G(\hat{h}) = h\}$ for all $t \in I$, where I is the time interval on which γ is defined. Let $g(t)$ and $\mu(t)$, $t \in I$, be two curves in G and $\mathfrak{g}^*$, respectively, such that $\gamma(t) = (g(t), \mu(t))$. The horizontality of $\gamma(t)$ implies that $\mu'(t) = \mathrm{Ad}^*_{g(t)^{-1}}(\Sigma(\xi(t), \cdot))$, with $\xi(t) = T_{g(t)}L_{g(t)^{-1}}(g'(t))$. Since the map $\pi_G \colon G_\Sigma \to G$ is a surjective submersion it admits local sections. In particular, there exists an open neighborhood U of h in G and a map $\sigma \colon U \subset G \to G_\Sigma$ such that $\pi_G \circ \sigma = \mathrm{id}|_U$ and $\sigma(h) = \hat{h}$.

The smoothness of the curve $g(t)$ implies that there exists $0 < t_0 \in I$ such that $g(t) \in U$ for any $t \in [0, t_0]$. Let $\hat{g}(t) := \sigma(g(t))$ and $\nu(t) := \mu_\Sigma(\hat{g}(t)^{-1}) - \mu_\Sigma(\hat{h}^{-1})$. We will show that the curve $\gamma_\sigma(t) := (g(t), \nu(t)) = (\pi_G(\hat{g}(t)), \mu_\Sigma(\hat{g}(t)^{-1}) - \mu_\Sigma(\hat{h}^{-1})) \subset \{(\pi_G(\hat{g}), \mu_\Sigma(\hat{g}^{-1}) - \mu_\Sigma(\hat{h}^{-1})) \mid \hat{g}, \hat{h} \in G_\Sigma \text{ such that } \pi_G(\hat{h}) = h\}$ is such that $\gamma_\sigma(t) = \gamma(t)$, for any $t \in [0, t_0]$. Notice that since $\gamma_\sigma(0) = (g(0), \nu(0)) = (h, \mu_\Sigma(\sigma(g(0))^{-1}) - \mu_\Sigma(\hat{h}^{-1})) = (h, \mu_\Sigma(\hat{h}^{-1}) - \mu_\Sigma(\hat{h}^{-1})) = (h, 0)$, the uniqueness of horizontal lifts guarantees that it suffices to check that $\gamma_\sigma(t)$ is horizontal. Given that $\pi_G(\hat{g}(t)) = g(t)$ for any $t \in [0, t_0]$, an argument similar to the one in the first part of the proof shows that

$$\hat{g}'(t) = T_{\hat{g}(t)} L_{\hat{g}(t)^{-1}} (\xi(t), s(t)), \tag{60}$$

where $\xi(t) = T_{g(t)} L_{g(t)^{-1}}(g'(t))$ and $s(t)$ is some piecewise smooth curve in $\mathbb{R}$. The use of (60) and of Proposition 4.4 in a straightforward computation show that $\nu'(t) = \mathrm{Ad}^*_{g(t)^{-1}}(\Sigma(\xi(t), \cdot))$, which proves that $\gamma_\sigma(t)$ is horizontal and hence $\gamma_\sigma(t) = \gamma(t) \subset \{(\pi_G(\hat{g}), \mu_\Sigma(\hat{g}^{-1}) - \mu_\Sigma(\hat{h}^{-1})) \mid \hat{g}, \hat{h} \in G_\Sigma, \pi_G(\hat{h}) = h\}$, for any $t \in [0, t_0]$, as required. We can repeat what we just did by taking a local section σ_1 of π_G around $g(t_0)$ which will allow us to construct a curve $\gamma_{\sigma_1}(t) \subset \{(\pi_G(\hat{g}), \mu_\Sigma(\hat{g}^{-1}) - \mu_\Sigma(\hat{h}^{-1})) \mid \hat{g}, \hat{h} \in G_\Sigma, \pi_G(\hat{h}) = h\}$, $t \in [t_0, t_1]$, $t_1 > t_0$, such that $\gamma_{\sigma_1}(t) = \gamma(t)$, for any $t \in [t_0, t_1]$. The compactness of I guarantees that by repeating this procedure a finite number of times we can write γ as a broken path made of finite smooth curves included in $\{(\pi_G(\hat{g}), \mu_\Sigma(\hat{g}^{-1}) - \mu_\Sigma(\hat{h}^{-1})) \mid \hat{g}, \hat{h} \in G_\Sigma, \pi_G(\hat{h}) = h\}$ which proves that γ itself is included in $\{(\pi_G(\hat{g}), \mu_\Sigma(\hat{g}^{-1}) - \mu_\Sigma(\hat{h}^{-1})) \mid \hat{g}, \hat{h} \in G_\Sigma, \pi_G(\hat{h}) = h\}$, hence proving the desired inclusion. □

Corollary 6.4. *In the setup of Proposition 6.3 the following inclusion holds*

$$\mathcal{H} \subset \mu_\Sigma(G_\Sigma).$$

Proof: Since $\mathcal{H} = \mathcal{H}_G$ by Proposition 6.1 it suffices to show that $\mathcal{H}_G \subset \mu_\Sigma(G_\Sigma)$. Let $\nu \in \mathcal{H}_G$. By definition, there exists a loop $g(t)$ in G such that $g(0) = g(1) = e$ with a horizontal lift $\gamma(t) = (g(t), \mu(t))$ that satisfies $\gamma(0) = (e, 0)$ and $\gamma(1) = (e, \nu)$. The compactness of the interval $[0, 1]$ implies that we can take local sections $\sigma_1, \ldots, \sigma_n$ of the projection $\pi_G \colon G_\Sigma \to G$ and that we can split the interval $[0, 1]$ into n intervals of the form $[t_{i-1}, t_i]$ with $0 = t_0 < t_1 < \cdots < t_n = 1$ such that for any $i \in \{1, \ldots, n\}$ we can define $\hat{g}_i(t) := \sigma_i(g(t))$, $t \in [t_{i-1}, t_i]$. The sections $\sigma_1, \ldots, \sigma_n$ are chosen in such a way that $\hat{g}(0) = e$ and $\hat{g}_i(t_i) = \hat{g}_{i+1}(t_i)$. Moreover, by construction, $g(t) = \pi_G(\hat{g}_i(t))$ for any $t \in [t_{i-1}, t_i]$ and hence a strategy similar to the one in the first part of the proof of Proposition 6.3 shows that $\gamma_1(t) := (\pi_G(\hat{g}_i(t)), \mu_\Sigma(\hat{g}_i(t)^{-1}))$ is a piecewise smooth horizontal curve for α_G such that $\gamma_1(0) = (e, 0)$ and hence $\gamma_1(t) = \gamma(t)$, for any $t \in [0, 1]$. Consequently, $\nu = \mu_\Sigma(\hat{g}_n(1)) \in \mu_\Sigma(G_\Sigma)$, as required. □

Theorem 6.5. *Let $(G \times \mathfrak{g}^*, \omega_\Sigma)$ be a magnetic cotangent bundle of the Lie group G. The map $\mathbf{K} \colon G \times \mathfrak{g}^* \to \mathfrak{g}^*/\bar{\mathcal{H}}$ given by the expression*

$$\mathbf{K}(g, \eta) = \pi_C(\mathrm{Ad}^*_{g^{-1}}\eta + \mu_\Sigma(\hat{g}^{-1}) + \nu_0) = \Xi(\hat{g}, \eta + \bar{\mathcal{H}}) + \pi_C(\nu_0) \tag{61}$$

is a cylinder valued momentum map for the canonical G-action on $(G \times \mathfrak{g}^*, \omega_\Sigma)$. The element $\nu_0 \in \mathfrak{g}^*$ is an arbitrary constant, $(g, \eta) \in G \times \mathfrak{g}^*$ is arbitrary, and $\hat{g} \in G_\Sigma$ is any element such that $\pi_G(\hat{g}) = g$. The map $\mu_\Sigma \colon G_\Sigma \to \mathfrak{g}^*$ is the $\mathfrak{g}^*$-valued one-cocycle associated to Σ and $\Xi \colon G_\Sigma \times \mathfrak{g}^*/\bar{\mathcal{H}} \to \mathfrak{g}^*/\bar{\mathcal{H}}$ the associated G_Σ-action on $\mathfrak{g}^*/\bar{\mathcal{H}}$.

The non-equivariance cocycle $\sigma: G \to \mathfrak{g}^*/\bar{\mathcal{H}}$ *of* $\mathbf{K}$ *is given by*

$$\sigma(g) = \pi_C(\mu_\Sigma(\hat{g}^{-1}) + \nu_0 - \mathrm{Ad}^*_{g^{-1}}(\nu_0)), \tag{62}$$

with $g \in G$ *and* $\hat{g} \in G_\Sigma$ *such that* $\pi_G(\hat{g}) = g$. *Finally*

$$\mu_\Sigma(\ker \pi_G) \subset \bar{\mathcal{H}}. \tag{63}$$

Proof: If we put together the conclusions of Propositions 6.1 and 6.3 we can conclude that for any $(h, \mu, \nu) \in G \times \mathfrak{g}^* \times \mathfrak{g}^*$

$$\widetilde{(G \times \mathfrak{g}^*)}_{(h,\mu,\nu)} = \Big\{\big(\pi_G(\hat{g}), \eta, \mathrm{Ad}^*_{\pi_G(\hat{g}^{-1})}\eta + \mu_\Sigma(\hat{g}^{-1}) - \mu_\Sigma(\hat{h}^{-1}) + \nu - \mathrm{Ad}^*_{h^{-1}}\mu\big) \mid \hat{g} \in G_\Sigma, \eta \in \mathfrak{g}^*\Big\}.$$

Hence setting $\nu_0 := \nu - \mathrm{Ad}^*_{h^{-1}}\mu - \mu_\Sigma(\hat{h}^{-1})$ and using the definition of the cylinder valued momentum map with this holonomy bundle we obtain that

$$\mathbf{K}(g, \eta) = \pi_C(\mathrm{Ad}^*_{g^{-1}}\eta + \mu_\Sigma(\hat{g}^{-1}) + \nu_0) = \Xi(\hat{g}, \eta + \bar{\mathcal{H}}) + \pi_C(\nu_0).$$

In order to prove (62) recall that by Proposition 3.4

$$\begin{aligned}\sigma(g) &= \mathbf{K}(g \cdot (e, 0)) - \mathcal{A}d^*_{g^{-1}}\mathbf{K}(e, 0) = \pi_C(\mu_\Sigma(\hat{g}^{-1}) + \nu_0) - \pi_C(\mathrm{Ad}^*_{g^{-1}}\nu_0)\\ &= \pi_C(\mu_\Sigma(\hat{g}^{-1}) + \nu_0 - \mathrm{Ad}^*_{g^{-1}}\nu_0).\end{aligned}$$

Finally, let $\hat{h} \in \ker \pi_G$. By (62), $0 = \sigma(e) = \pi_C(\mu_\Sigma(\hat{h}^{-1}))$. Consequently $\mu_\Sigma(\hat{h}^{-1}) = -\mathrm{Ad}^*_{h^{-1}}\mu_\Sigma(\hat{h}) \in \bar{\mathcal{H}}$ and hence, by (8), $\mu_\Sigma(\hat{h}) \in \bar{\mathcal{H}}$. □

Remark 6.6. If there exists a group one-cocycle that integrates Σ, that is, the equality (28) holds then we can use the affine G-action $\bar{\Theta}: G \times \mathfrak{g}^* \to \mathfrak{g}^*$ introduced in Remark 4.7 in order to express the holonomy bundle $\widetilde{(G \times \mathfrak{g}^*)}_{(h,\mu,\nu)}$ as the graph

$$\widetilde{(G \times \mathfrak{g}^*)}_{(h,\mu,\nu)} = \{(g, \eta, \bar{\Theta}(g, \eta) - \bar{\Theta}(h, \mu) + \nu) \mid g \in G, \eta \in \mathfrak{g}^*\}.$$

In this case, the Hamiltonian holonomy $\mathcal{H}$ is obviously trivial and the cylinder valued momentum map is the standard momentum map given by

$$\mathbf{K}(g, \eta) = \bar{\Theta}(g, \eta) + \nu_0 = \Theta(g, \eta) + \nu_0, \qquad (g, \eta) \in G \times \mathfrak{g}^*.$$

Once we have computed the cylinder valued momentum map for the symplectic G-action on the magnetic cotangent bundle $(G \times \mathfrak{g}^*, \omega_\Sigma)$ we will carry out reduction in this context. According to Theorem 5.7, the optimal reduced spaces provide the symplectic leaves of the Poisson reduced spaces. In the next two theorems we will describe these two reduced spaces in the setup of this section. As we will see, the optimal reduced spaces can be seen as the G_Σ-orbits in $\mathfrak{g}^*$ of the extended affine action, while the Poisson reduced spaces are, roughly speaking, the $\bar{\mathcal{H}}$-saturation of these orbits.

Theorem 6.7 (Optimal reduction of magnetic cotangent bundles). *The optimal reduced spaces of the canonical G-action on the magnetic cotangent bundle $(G \times \mathfrak{g}^*, \omega_\Sigma)$ are symplectically diffeomorphic to the orbits corresponding to the extended affine action $\bar{\Xi}$: $G_\Sigma \times \mathfrak{g}^* \to \mathfrak{g}^*$ of G_Σ on $\mathfrak{g}^*$ endowed with the symplectic structure that makes them the symplectic leaves of $(\mathfrak{g}^*, \{\cdot,\cdot\}_-^\Sigma)$ (see Proposition 4.9).*

Remark 6.8. When the magnetic term is set to zero then $G_\Sigma = G$ (suppose G is connected) and the extended affine orbits become the G-coadjoint orbits. Consequently, this theorem shows that optimal reduction, and not the other reduction schemes presented in this paper, generalizes to the magnetic setup the well-known result that says that the Marsden–Weinstein reduced spaces of the lifted action of a connected Lie group G on its cotangent bundle (endowed with the canonical symplectic form) are symplectomorphic to the G-coadjoint orbits in $\mathfrak{g}^*$.

Proof: In order to compute the polar distribution A'_G of the G-action on $(G \times \mathfrak{g}^*, \omega_\Sigma)$ notice that for any $f \in C^\infty(G \times \mathfrak{g}^*)^G \simeq C^\infty(\mathfrak{g}^*)$, the corresponding Hamiltonian vector field X_f is

$$X_f(g,\nu) = \left(T_eL_g\left(\frac{\delta f}{\delta \nu}\right), \mathrm{ad}^*_{\delta f/\delta\nu}\nu - \Sigma\left(\frac{\delta f}{\delta\nu},\cdot\right)\right)$$

and hence by (26) we can write

$$A'_G(g,\nu) = \{(T_eL_g(\xi), \mathrm{ad}^*_\xi\nu - \Sigma(\xi,\cdot)) \mid \xi \in \mathfrak{g}\} = \{(T_eL_g(\pi_{\mathfrak{g}}(\eta)), -\eta_{\mathfrak{g}^*}(\nu)) \mid \eta \in \mathfrak{g}_\Sigma\}.$$

Consequently, the leaves of A'_G are given by the orbits of the right G_Σ-action $\Upsilon\colon G_\Sigma \times (G \times \mathfrak{g}^*) \to G \times \mathfrak{g}^*$ defined by $\Upsilon(g,(h,\nu)) := (h\pi_G(g), \bar{\Xi}(g^{-1},\nu))$, $(h,\nu) \in G \times \mathfrak{g}^*$, $g \in G_\Sigma$. The momentum space $(G \times \mathfrak{g}^*)/A'_G$ can be identified with the orbit space $(G \times \mathfrak{g}^*)/G_\Sigma$ and hence for any $\rho \in (G \times \mathfrak{g}^*)/A'_G$ there exists an element $\mu \in \mathfrak{g}^*$ such that $\mathcal{J}^{-1}(\rho) = G_\Sigma \cdot (e,\mu)$. Moreover

$$\begin{aligned} G_\rho &= \{g \in G \mid g \cdot (e,\mu) = \Upsilon(h,(e,\mu)) \text{ for some } h \in G_\Sigma\} \\ &= \{g \in G \mid (g,\mu) = (\pi_G(h), \bar{\Xi}(h^{-1},\mu)) \text{ for some } h \in G_\Sigma\}, \end{aligned}$$

which guarantees that

$$G_\rho = \pi_G((G_\Sigma)_\mu). \tag{64}$$

Consider now the smooth surjective map

$$\begin{aligned} \bar{\phi}\colon \mathcal{J}^{-1}(\rho) = G_\Sigma \cdot (e,\mu) &\longrightarrow \quad G_\Sigma \cdot \mu \\ (\pi_G(g), \bar{\Xi}(g^{-1},\mu)) &\longmapsto \bar{\Xi}(g^{-1},\mu). \end{aligned}$$

The map $\bar{\phi}$ is clearly G_ρ-invariant and hence it drops to a smooth surjective map

$$\phi : \mathcal{J}^{-1}(\rho)/G_\rho \to G_\Sigma \cdot \mu.$$

The map ϕ is injective because if $\phi(\pi_\rho(\pi_G(g), \bar{\Xi}(g^{-1},\mu))) = \phi(\pi_\rho(\pi_G(h), \bar{\Xi}(h^{-1},\mu)))$ then $\bar{\Xi}(hg^{-1},\mu) = \mu$ and hence $\pi_G(h)\pi_G(g^{-1}) \in G_\rho$ by (64). Therefore,

$\pi_G(h)\pi_G(g^{-1}) \cdot (\pi_G(g), \bar{\Xi}(g^{-1}, \mu)) = (\pi_G(h), \bar{\Xi}(g^{-1}, \mu)) = (\pi_G(h), \bar{\Xi}(h^{-1}, \mu))$ and hence $\pi_\rho(\pi_G(g), \bar{\Xi}(g^{-1}, \mu)) = \pi_\rho(\pi_G(h), \bar{\Xi}(h^{-1}, \mu))$.

The map ϕ is a diffeomorphism because for any $\bar{\Xi}(g^{-1}, \mu) \in G_\Sigma \cdot \mu$ there is a smooth local section $\sigma\colon U_{\bar{\Xi}(g^{-1},\mu)} \subset G_\Sigma \cdot \mu \simeq G_\Sigma/(G_\Sigma)_\mu \to G_\Sigma$ such that the map

$$\begin{array}{rccc} \phi^{-1}_{U_{\bar{\Xi}(g^{-1},\mu)}} : & U_{\bar{\Xi}(g^{-1},\mu)} & \longrightarrow & \mathcal{J}^{-1}(\rho) \\ & \bar{\Xi}(h^{-1}, \mu) & \longmapsto & (\pi_G(\sigma(\bar{\Xi}(h^{-1}, \mu))^{-1}), \bar{\Xi}(h^{-1}, \mu)) \end{array}$$

is a local smooth inverse of ϕ. Finally, the symplectic character of ϕ is proved via a straightforward diagram chasing exercise. □

We are now going to provide a model for the Poisson reduced space $M^{[\mu]} := \mathbf{K}^{-1}([\mu])/G_{[\mu]}$ similar to the one provided in the previous theorem for the optimal reduced space and in which it will be very easy to see how, as we proved in general in Theorem 5.7, the optimal reduced spaces M_ρ are the symplectic leaves of the Poisson reduced spaces $M^{[\mu]}$.

For the sake of simplicity, we will take in our computations the cylinder valued momentum map in Theorem 6.5 for which $\nu_0 = 0$, that is,

$$\mathbf{K}(g, \eta) = \pi_C(\mathrm{Ad}^*_{g^{-1}}\eta + \mu_\Sigma(\hat{g}^{-1})) = \Xi(\hat{g}, \eta + \bar{\mathcal{H}}), \quad (g, \eta) \in G \times \mathfrak{g}^*, \ \pi_G(\hat{g}) = g. \tag{65}$$

In this situation, the non-equivariance cocycle (62) induces an affine G-action on $\mathfrak{g}^*/\bar{\mathcal{H}}$ (see Proposition 3.4) given by

$$\Theta(g, \pi_C(\mu)) = \pi_C(\mathrm{Ad}^*_{g^{-1}}\mu + \mu_\Sigma(\hat{g}^{-1})) = \Xi(\hat{g}, \pi_C(\mu)), \tag{66}$$

where $\hat{g} \in G_\Sigma$ is such that $\pi_G(\hat{g}) = g$ and hence $\mathbf{K}(g, \eta) = \Theta(g, \pi_C(\eta))$, $(g, \eta) \in G \times \mathfrak{g}^*$.

It is easy to show that in this specific situation, the Lie algebra two-cocycle $\tilde{\Sigma}$ in (38) coincides with Σ and hence the Poisson structure on $\mathfrak{g}^*/\bar{\mathcal{H}}$ introduced in Theorem 4.1 with respect to which $\mathbf{K}\colon G \times \mathfrak{g}^* \to \mathfrak{g}^*/\bar{\mathcal{H}}$ is a Poisson map is $(\mathfrak{g}^*/\bar{\mathcal{H}}, \{\cdot, \cdot\}^{+\Sigma}_{\mathfrak{g}^*/\bar{\mathcal{H}}})$.

In the next theorem, we will show that the Poisson reduced space $M^{[\mu]}$ is Poisson diffeomorphic to $G_\Sigma \cdot \mu + \bar{\mathcal{H}}$ with the smooth and Poisson structures that we now discuss.

The smooth structure of $G_\Sigma \cdot \mu + \bar{\mathcal{H}}$. Consider the orbit $G_\Sigma \cdot [\mu] := \Xi(G_\Sigma, [\mu])$. By general theory, $G_\Sigma \cdot [\mu]$ is an initial submanifold of $\mathfrak{g}^*/\bar{\mathcal{H}}$. Since $\pi_C\colon \mathfrak{g}^* \to \mathfrak{g}^*/\bar{\mathcal{H}}$ is a surjective submersion, the inverse image $\pi_C^{-1}(G_\Sigma \cdot [\mu]) = G_\Sigma \cdot \mu + \bar{\mathcal{H}}$ is an initial submanifold of $\mathfrak{g}^*$ by the transversality theorem for initial submanifolds. This is the smooth structure for $G_\Sigma \cdot \mu + \bar{\mathcal{H}}$ that we will consider in what follows.

The Poisson structure of $G_\Sigma \cdot \mu + \bar{\mathcal{H}}$. Consider the Poisson manifold $(\mathfrak{g}^*, \{\cdot, \cdot\}^\Sigma_-)$. By Proposition 4.9, the symplectic leaves of this Poisson manifold are the G_Σ-orbits of the $\bar{\Xi}$-action on $\mathfrak{g}^*$. Consequently $G_\Sigma \cdot \mu + \bar{\mathcal{H}}$ is automatically a quasi-Poisson submanifold of $(\mathfrak{g}^*, \{\cdot, \cdot\}^\Sigma_-)$ (see Section 4.1.21 of [22]) and hence a Poisson manifold on its own (see Proposition 4.1.23 of [22]) with the bracket $\{\cdot, \cdot\}^{G_\Sigma \cdot \mu + \bar{\mathcal{H}}}$ given by

$$\{f, g\}^{G_\Sigma \cdot \mu + \bar{\mathcal{H}}}(\nu) = \{F, G\}^\Sigma_-(\nu),$$

where $f, g \in C^\infty(G_\Sigma \cdot \mu)$ and $F, G \in C^\infty(\mathfrak{g}^*)$ are local extensions of f and g around $\nu \in G_\Sigma \cdot \mu + \bar{\mathcal{H}}$, respectively.

Theorem 6.9 (Poisson reduction of magnetic cotangent bundles). *Let $(G \times \mathfrak{g}^*, \omega_\Sigma)$ be the magnetic cotangent bundle of the Lie group G associated to the Lie algebra two-cocycle Σ. Let $\mathbf{K}: G \times \mathfrak{g}^* \to \mathfrak{g}^*/\bar{\mathcal{H}}$ be the cylinder valued momentum map in* (65) *for the canonical lifted action of G on $(G \times \mathfrak{g}^*, \omega_\Sigma)$. Then, for any $\mu \in \mathfrak{g}^*$, the associated Poisson reduced space $M^{[\mu]} := \mathbf{K}^{-1}([\mu])/G_{[\mu]}$ is naturally Poisson diffeomorphic to $(G_\Sigma \cdot \mu + \bar{\mathcal{H}}, \{\cdot, \cdot\}^{G_\Sigma \cdot \mu + \bar{\mathcal{H}}})$.*

Remark 6.10. In view of Corollary 6.4, notice that if G is Abelian and $\mu_\Sigma(G_\Sigma)$ is closed in $\mathfrak{g}^*$ then μ_Σ is a group homomorphism by Proposition 4.4 (iii) and hence

$$G_\Sigma \cdot \mu + \bar{\mathcal{H}} = \mu + \mu_\Sigma(G_\Sigma) + \bar{\mathcal{H}} = \mu + \mu_\Sigma(G_\Sigma) = G_\Sigma \cdot \mu.$$

Consequently, by the Theorems 6.7 and 6.9, the optimal and Poisson reduced spaces coincide in this case.

Proof: Since $\mathbf{K}(g, \eta) = \pi_C(\mathrm{Ad}^*_{g^{-1}}\eta + \mu_\Sigma(\hat{g}^{-1})) = \Xi(\hat{g}, \eta + \bar{\mathcal{H}})$, for any $(g, \eta) \in G \times \mathfrak{g}^*$ and $\pi_G(\hat{g}) = g$, we have that

$$\begin{aligned}\mathbf{K}^{-1}([\mu]) &= \{(g, \eta) \in G \times \mathfrak{g}^* \mid \Xi(\hat{g}, \pi_C(\eta)) = \pi_C(\mu) \text{ for any } \hat{g} \in G_\Sigma \text{ such that } \pi_G(\hat{g}) = g\}\\ &= \{(\pi_G(\hat{g}), \bar{\Xi}(\hat{g}^{-1}, \mu) + \nu) \mid \hat{g} \in G_\Sigma, \nu \in \bar{\mathcal{H}}\}.\end{aligned}$$

Consider the smooth surjective map

$$\begin{aligned}\bar{\phi} : \mathbf{K}^{-1}([\mu]) &\longrightarrow G_\Sigma \cdot \mu + \bar{\mathcal{H}}\\ (\pi_G(\hat{g}), \bar{\Xi}(\hat{g}^{-1}, \mu) + \nu) &\longmapsto \bar{\Xi}(\hat{g}^{-1}, \mu) + \nu.\end{aligned}$$

The map $\bar{\phi}$ is clearly $G_{[\mu]}$-invariant and hence it drops to a smooth surjective map

$$\phi : \mathbf{K}^{-1}([\mu])/G_{[\mu]} \longrightarrow G_\Sigma \cdot \mu + \bar{\mathcal{H}}.$$

We now show that ϕ is injective. Let $\pi^{[\mu]}(\pi_G(\hat{g}), \bar{\Xi}(\hat{g}^{-1}, \mu) + \nu)$ and $\pi^{[\mu]}(\pi_G(\hat{h}), \bar{\Xi}(\hat{h}^{-1}, \mu) + \nu')$ be two points in $M^{[\mu]}$ such that

$$\bar{\Xi}(\hat{g}^{-1}, \mu) + \nu = \bar{\Xi}(\hat{h}^{-1}, \mu) + \nu'. \tag{67}$$

Applying π_C to both sides of this equality we obtain $\Xi(\hat{g}^{-1}, \pi_C(\mu)) = \Xi(\hat{h}^{-1}, \pi_C(\mu))$. This implies that $\hat{k} := \hat{h}\hat{g}^{-1} \in (G_\Sigma)_{\pi_C(\mu)}$ and hence by (66)

$$\Theta(\pi_G(\hat{k}), \pi_C(\mu)) = \Xi(\hat{k}, \pi_C(\mu)) = \pi_C(\mu)$$

which guarantees that $\pi_G(\hat{h}\hat{g}^{-1}) \in G_{\pi_C(\mu)}$. Therefore, by (67), $\pi_G(\hat{h}\hat{g}^{-1}) \cdot (\pi_G(\hat{g}), \bar{\Xi}(\hat{g}^{-1}, \mu) + \nu) = (\pi_G(\hat{h}), \bar{\Xi}(\hat{h}^{-1}, \mu) + \nu')$ and hence $\pi^{[\mu]}(\pi_G(\hat{g}), \bar{\Xi}(\hat{g}^{-1}, \mu) + \nu) = \pi^{[\mu]}(\pi_G(\hat{h}), \bar{\Xi}(\hat{h}^{-1}, \mu) + \nu')$, which shows that ϕ is injective. A standard argument using local sections of π_C and of $G_\Sigma \to G_\Sigma/(G_\Sigma)_{[\mu]}$ (see proof of Theorem 6.7) shows that ϕ

has smooth local inverses and it is hence a diffeomorphism. The Poisson character of ϕ is a straightforward verification. □

Example 6.11. In order to illustrate how to explicitly implement the constructions introduced in this section we will carry them out for the cotangent bundle of a four-torus whose canonical symplectic structure has been modified with the invariant magnetic term induced by the Lie algebra two-cocycle $\Sigma\colon \mathbb{R}^4 \times \mathbb{R}^4 \to \mathbb{R}$ given by the matrix

$$\Sigma = \begin{pmatrix} 0 & 0 & -1 & -\sqrt{2} \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 & 0 \end{pmatrix}.$$

The entries in this matrix have been chosen in such a way that we obtain a Hamiltonian holonomy group that is not closed in the dual of the Lie algebra. We will compute a cylinder valued momentum map for the canonical $\mathbb{T}^4$-action on $(T^*\mathbb{T}^4, \omega_\Sigma)$ as well as the associated reduced spaces that, as we will see, are all identical despite the non-closedness of the Hamiltonian holonomy in $\mathbb{R}^4$.

We write $G := \mathbb{T}^4$ and denote by $\mathfrak{g} = \mathbb{R}^4$ its Lie algebra. We start by noting that the one-dimensional central extension $\mathfrak{g}_\Sigma = \mathbb{R}^4 \oplus \mathbb{R}$ of $\mathfrak{g}$ is integrated by the Heisenberg group $G_\Sigma = \mathbb{R}^4 \oplus \mathbb{R}$ with multiplication given by

$$\begin{aligned}(\mathbf{u}, a) \cdot (\mathbf{v}, b) :&= \left(\mathbf{u} + \mathbf{v}, a + b - \frac{1}{2}\Sigma(\mathbf{u}, \mathbf{v})\right) \\ &= (\mathbf{u} + \mathbf{v}, a + b - u_1 v_3 - \sqrt{2} u_1 v_4 + u_3 v_1 + \sqrt{2} u_4 v_2),\end{aligned}$$

with $(\mathbf{u}, a), (\mathbf{v}, b) \in G_\Sigma$. An easy calculation shows that for any $(\mathbf{u}, a) \in G_\Sigma$ and $(\xi, s) \in \mathfrak{g}_\Sigma$,

$$\mathrm{Ad}_{(u,a)}(\xi, s) = (\xi, s - \Sigma(u, \xi)).$$

Consequently, in view of Proposition 4.4, the $\mathfrak{g}^*$-valued one-cocycle $\mu_\Sigma\colon G_\Sigma \to \mathfrak{g}^*$ associated to Σ is given by $\mu_\Sigma(u, a) = \Sigma(\cdot, u)$, $(\mathbf{u}, a) \in G_\Sigma$. Using Proposition 6.1 (iii) and Proposition 6.3 it is easy to see that the Hamiltonian holonomy $\mathcal{H}$ of our setup is given by

$$\mathcal{H} = (\mathbb{Z} + \sqrt{2}\mathbb{Z}) \times \{0\} \times \mathbb{Z}(1, \sqrt{2}),$$

which is clearly not closed in $\mathbb{R}^4$ since $\bar{\mathcal{H}} = \mathbb{R} \times \{0\} \times \mathbb{Z}(1, \sqrt{2})$. In order to write down a cylinder valued momentum map for our example we start by noting that the map $\mathbb{R}^4/\bar{\mathcal{H}} \to \mathbb{R}^2 \times S^1$ given by $[a, b, c, d] \longmapsto (b, \frac{1}{3}(d - \sqrt{2}c), e^{2\pi i/3(c+\sqrt{2}d)})$ is a group isomorphism and hence we can write $\pi_C(a, b, c, d) = (b, \frac{1}{3}(d - \sqrt{2}c), e^{2\pi i/3(c+\sqrt{2}d)})$. Consequently, since the map $\pi_G\colon G_\Sigma \to G$ is given by $(\mathbf{u}, a) \mapsto (e^{2\pi i u_1}, \ldots, e^{2\pi i u_4})$, we have by Theorem 6.5 that the map $\mathbf{K}\colon \mathbb{T}^4 \times \mathbb{R}^4 \to \mathbb{R}^2 \times S^1$ defined by $\mathbf{K}(g, \eta) := \pi_C(\eta + \mu_\Sigma(-u)) = \pi_C(\eta - \Sigma(\cdot, u))$ is a cylinder valued momentum map for the $\mathbb{T}^4$-action on $(T^*\mathbb{T}^4, \omega_\Sigma)$; in this definition, the elements $g \in \mathbb{T}^4$ and $u \in \mathbb{R}^4$ are related by the equality $g = (e^{2\pi i u_1}, \ldots, e^{2\pi i u_4})$. We can be even more specific by writing

$$\mathbf{K}((e^{2\pi i u_1}, \ldots, e^{2\pi i u_4}), \eta) = \left(\eta_2, \frac{1}{3}(\eta_4 - \sqrt{2}u_1 - \sqrt{2}(\eta_3 - u_1)), e^{(2\pi i/3)(\eta_3 - u_1 + \sqrt{2}(\eta_4 - \sqrt{2}u_1))}\right).$$

We now compute the reduced spaces. We start by noticing that in this particular case the Lie algebra $\mathfrak{n} := (\mathrm{Lie}(\bar{\mathcal{H}}))^\circ = \{0\} \times \mathbb{R}^3$. The subgroup $N = \{e\} \times \mathbb{T}^3 \subset \mathbb{T}^4$ clearly has $\mathfrak{n}$ as Lie algebra. Let now $[\mu] \in \mathbb{R}^4/\bar{\mathcal{H}} \simeq \mathbb{R}^2 \times S^1$ be arbitrary. We have

$$N_{[\mu]} = \{n \in N \mid \Theta(n, [\mu]) = [\mu]\} = \{n \in N \mid \pi_C(\mu_\Sigma(\hat{n})) = 0, \pi_G(\hat{n}) = n\} = N.$$

A similar computation shows that $G_{[\mu]} = N = N_{[\mu]}$. Consequently, the Poisson $M^{[\mu]}$ and symplectic $M_{[\mu]}$ reduced spaces coincide. Moreover, by Theorem 6.9 they are naturally Poisson diffeomorphic to $G_\Sigma \cdot \mu + \bar{\mathcal{H}} = \mu + \mu_\Sigma(G_\Sigma) + \bar{\mathcal{H}}$. It can be checked that, in this particular case, $\bar{\mathcal{H}} \subset \mu_\Sigma(G_\Sigma)$ and hence, by Theorem 6.7,

$$\mu + \mu_\Sigma(G_\Sigma) + \bar{\mathcal{H}} = \mu + \mu_\Sigma(G_\Sigma) = G_\Sigma \cdot \mu \simeq M_\rho.$$

Consequently,

$$M_{[\mu]} = M^{[\mu]} \simeq M_\rho \simeq \mu + \mu_\Sigma(G_\Sigma) = \mu + (\mathbb{R} \times \{0\} \times \mathbb{R}(1, \sqrt{2})).$$

Appendix: The relation between Lie group and cylinder valued momentum maps

The cylinder valued momentum maps are closely related to the *Lie group valued momentum maps* introduced in [1, 9, 11, 12, 16]. We give the definition of these objects only for Abelian symmetry groups because in the non-Abelian case these momentum maps are defined on spaces that are neither symplectic nor Poisson (they are referred to as *quasi-Hamiltonian spaces* [1]).

Definition A.1. Let G be an Abelian Lie group whose Lie algebra $\mathfrak{g}$ acts canonically on the symplectic manifold (M, ω). Let $(\cdot, \cdot)$ be some bilinear symmetric nondegenerate form on the Lie algebra $\mathfrak{g}$. The map $\mathbf{J}\colon M \to G$ is called a *G-valued momentum map* for the $\mathfrak{g}$-action on M whenever

$$\mathbf{i}_{\xi_M}\omega(m)(v_m) = (T_m(L_{\mathbf{J}(m)^{-1}} \circ \mathbf{J})(v_m), \xi), \tag{A.1}$$

for any $\xi \in \mathfrak{g}$, $m \in M$, and $v_m \in T_m M$.

Proposition A.2. *Let G be an Abelian Lie group whose Lie algebra $\mathfrak{g}$ acts canonically on the symplectic manifold (M, ω). Let $\mathbf{J}\colon M \to G$ be a G-valued momentum map for this action.*

- **(i)** *The fibers of $\mathbf{J}\colon M \to G$ are invariant under the Hamiltonian flows corresponding to $\mathfrak{g}$-invariant Hamiltonian functions.*
- **(ii)** $\ker T_m\mathbf{J} = (\mathfrak{g} \cdot m)^\omega$ *for any $m \in M$.*

Proof:

(i) Let F_t be the flow of the Hamiltonian vector field X_h associated to a $\mathfrak{g}$-invariant function $h \in C^\infty(M)^{\mathfrak{g}}$. By the defining relation (A.1) of the Lie group valued momentum maps we have for any $m \in M$ and any $\xi \in \mathfrak{g}$

$$\left(\left(T_{\mathbf{J}(F_t(m))}L_{\mathbf{J}(F_t(m))^{-1}} \circ T_{F_t(m)}\mathbf{J}\right)(X_h(F_t(m))), \xi\right) = \left(T_{F_t(m)}\left(L_{\mathbf{J}(F_t(m))^{-1}} \circ \mathbf{J}\right)(X_h(F_t(m))\right), \xi\right)$$
$$= \mathbf{i}_{\xi_M}\omega(F_t(m)))(X_h(F_t(m))) = -\mathbf{d}h(F_t(m)))(\xi_M(F_t(m))) = 0.$$

Consequently,

$$\left(T_{\mathbf{J}(F_t(m))}L_{\mathbf{J}(F_t(m))^{-1}} \circ T_{F_t(m)}\mathbf{J}\right)(X_h(F_t(m))) = 0$$

and hence $T_{F_t(m)}\mathbf{J}(X_h(F_t(m))) = 0$, which can be rewritten as

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{J} \circ F_t)(m) = 0.$$

The arbitrary character of t and m implies that $\mathbf{J} \circ F_t = \mathbf{J}|_{\mathrm{Dom}(F_t)}$, as required.

(ii) A vector $v_m \in \ker T_m\mathbf{J}$ if and only if $T_m\mathbf{J}(v_m) = 0$. This identity is equivalent to $((T_{\mathbf{J}(m)}L_{\mathbf{J}(m)^{-1}} \circ T_m\mathbf{J})(v_m), \xi) = 0$, for any $\xi \in \mathfrak{g}$ and, by (A.1), to $\mathbf{i}_{\xi_M}\omega(m)(v_m) = 0$, for all $\xi \in \mathfrak{g}$, which in turn amounts to $v_m \in (\mathfrak{g} \cdot m)^\omega$. □

Lie group and cylinder valued momentum maps. We start with a proposition that states that any cylinder valued momentum map associated to an Abelian Lie algebra action whose corresponding holonomy group is closed can be understood as a Lie group valued momentum map.

Proposition A.3. *Let (M, ω) be a connected paracompact symplectic manifold and $\mathfrak{g}$ an Abelian Lie algebra acting canonically on it. Let $\mathcal{H} \subset \mathfrak{g}^*$ be the holonomy group associated to the connection α in (1) and $(\cdot, \cdot)\colon \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ some bilinear symmetric nondegenerate form on $\mathfrak{g}$. Let $f\colon \mathfrak{g} \to \mathfrak{g}^*$ be the isomorphism given by $\xi \longmapsto (\xi, \cdot)$, $\xi \in \mathfrak{g}$ and $\mathcal{T} := f^{-1}(\mathcal{H})$. The map f induces an Abelian group isomorphism $\bar{f}\colon \mathfrak{g}/\mathcal{T} \to \mathfrak{g}^*/\mathcal{H}$ by $\bar{f}(\xi + \mathcal{T}) := (\xi, \cdot) + \mathcal{H}$. Suppose that $\mathcal{H}$ is closed in $\mathfrak{g}^*$ and define $\mathbf{J} := \bar{f}^{-1} \circ \mathbf{K}\colon M \to \mathfrak{g}/\mathcal{T}$, where $\mathbf{K}\colon M \to \mathfrak{g}^*/\mathcal{H}$ is a cylinder valued momentum map for the $\mathfrak{g}$-action on (M, ω). Then*

$$\mathbf{i}_{\xi_M}\omega(m)(v_m) = \left(T_m\left(L_{\mathbf{J}(m)^{-1}} \circ \mathbf{J}\right)(v_m), \xi\right), \tag{A.2}$$

for any $\xi \in \mathfrak{g}$ and $v_m \in T_mM$. Consequently, the map $\mathbf{J}\colon M \to \mathfrak{g}/\mathcal{T}$ constitutes a $\mathfrak{g}/\mathcal{T}$-valued momentum map for the canonical action of the Lie algebra $\mathfrak{g}$ of $(\mathfrak{g}/\mathcal{T}, +)$ on (M, ω).

Proof: We start by noticing that the right-hand side of (A.2) makes sense due to the closedness hypothesis on $\mathcal{H}$. Indeed, this condition and the fact that $\mathcal{H}$ is zero-dimensional due to the flatness of α imply that $\mathfrak{g}^*/\mathcal{H}$, and therefore $\mathfrak{g}/\mathcal{T}$, are Abelian Lie groups whose Lie algebras can be naturally identified with $\mathfrak{g}^*$ and $\mathfrak{g}$, respectively. This identification is used in (A.2), where we think of $T_m(L_{\mathbf{J}(m)^{-1}} \circ \mathbf{J})(v_m) \in \mathrm{Lie}(\mathfrak{g}/\mathcal{T})$ as an element of $\mathfrak{g}$.

In what follows we will use the following notation: if $\mu \in \mathfrak{g}^*$ arbitrary, denote by $\xi_\mu \in \mathfrak{g}$ the unique element such that $\mu = (\xi_\mu, \cdot)$.

We now compute $T_m\mathbf{J}(v_m)$. Let $\mu + \mathcal{H} := \mathbf{K}(m)$ and hence $\mathbf{J}(m) = \xi_\mu + \mathcal{T}$. Now, by Theorem 2.3 (ii) we have

$$T_m\mathbf{J}(v_m) = T_m(\bar{f}^{-1} \circ \mathbf{K})(v_m) = T_{\mu+\mathcal{H}}\bar{f}^{-1}(T_m\mathbf{K}(v_m)) = T_{\mu+\mathcal{H}}\bar{f}^{-1}(T_\mu\pi_C(\nu)),$$

where the element $\nu \in \mathfrak{g}^*$ is given by

$$\langle \nu, \eta \rangle = \mathbf{i}_{\eta_M}\omega(m)(v_m), \quad \text{for all } \eta \in \mathfrak{g}. \tag{A.3}$$

Since $(\bar{f}^{-1} \circ \pi_C)(\rho) = \xi_\rho + \mathcal{T}$ for any $\rho \in \mathfrak{g}^*$, we can write

$$\begin{aligned} T_{\mu+\mathcal{H}}\bar{f}^{-1}(T_\mu\pi_C(\nu)) = T_\mu(\bar{f}^{-1} \circ \pi_C)(\nu) &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (\bar{f}^{-1} \circ \pi_C)(\mu + t\nu) \\ &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (\xi_\mu + t\xi_\nu + \mathcal{T}). \end{aligned}$$

Hence,

$$T_m\mathbf{J}(v_m) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (\xi_\mu + t\xi_\nu + \mathcal{T}) \in T_{\xi_\mu+\mathcal{T}}(\mathfrak{g}/\mathcal{T}).$$

Now,

$$\begin{aligned} \left(T_m(L_{\mathbf{J}(m)^{-1}} \circ \mathbf{J})(v_m), \xi\right) &= \left(T_{\mathbf{J}(m)}L_{\mathbf{J}(m)^{-1}}(T_m\mathbf{J}(v_m)), \xi\right) \\ &= \left(\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (-\xi_\mu + \mathcal{T}) + (\xi_\mu + t\xi_\nu + \mathcal{T}), \xi\right) \\ &= (\xi_\nu, \xi) = \langle \nu, \xi \rangle = \mathbf{i}_{\xi_M}\omega(m)(v_m), \end{aligned}$$

where the last equality is a consequence of (A.3). □

Lie group valued momentum maps produce closed Hamiltonian holonomies. So far we have investigated how cylinder valued momentum maps can be viewed as Lie group valued momentum maps. Now we shall focus on the converse relation, that is, we shall isolate hypotheses that guarantee that a Lie group valued momentum map naturally induces a cylinder valued momentum map.

Theorem A.4. *Let (M, ω) be a connected paracompact symplectic manifold and $\mathfrak{g}$ an Abelian Lie algebra acting canonically on it. Let $\mathcal{H} \subset \mathfrak{g}^*$ be the Hamiltonian holonomy group associated to the connection α in* (1) *associated to the $\mathfrak{g}$-action and let $(\cdot,\cdot)\colon \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ be a bilinear symmetric nondegenerate form on $\mathfrak{g}$. Let $f\colon \mathfrak{g} \to \mathfrak{g}^*$, $\bar{f}\colon \mathfrak{g}/\mathcal{T} \to \mathfrak{g}^*/\mathcal{H}$, and let $\mathcal{T} := f^{-1}(\mathcal{H})$ be as in the statement of Proposition* A.3. *Let G be a connected Abelian Lie group whose Lie algebra is $\mathfrak{g}$ and suppose that there exists a G-valued momentum map $\mathbf{A}\colon M \to G$ associated to the $\mathfrak{g}$-action whose definition uses the form $(\cdot,\cdot)$.*

(i) *If* $\exp\colon \mathfrak{g} \to G$ *is the exponential map, then*

$$\mathcal{H} \subset f(\ker \exp). \tag{A.4}$$

(ii) *$\mathcal{H}$ is closed in $\mathfrak{g}^*$.*

Let $\mathbf{J} := \bar{f}^{-1} \circ \mathbf{K}$*:* $M \to \mathfrak{g}/\mathcal{T}$*, where* $\mathbf{K}$*:* $M \to \mathfrak{g}^*/\mathcal{H}$ *is a cylinder valued momentum map for the* $\mathfrak{g}$*-action on* (M, ω)*. If* $f(\ker \exp) \subset \mathcal{H}$*, then* $\mathbf{J}$*:* $M \to \mathfrak{g}/\mathcal{T} = \mathfrak{g}/\ker \exp \simeq G$ *is a* G*-valued momentum map that differs from* $\mathbf{A}$ *by a constant in* G*.*

Conversely, if $\mathcal{H} = f(\ker \exp)$*, then* $\mathbf{J}$*:* $M \to \mathfrak{g}/\ker \exp \simeq G$ *is a* G*-valued momentum map.*

Remark A.5. The presence of a Lie group valued momentum map associated to a canonical Lie algebra action does not imply the reverse inclusion in (A.4). A simple example that illustrates this statement is the canonical action of a two torus $\mathbb{T}^2$ on itself via

$$(e^{i\phi_1}, e^{i\phi_2}) \cdot (e^{i\theta_1}, e^{i\theta_2}) := \left(e^{i(\theta_1+\phi_1)}, e^{i\theta_2}\right),$$

where we consider the torus as a symplectic manifold with the area form. A straightforward computation shows that the surface

$$\tilde{\mathbb{T}}^2 := \{((e^{i\theta_1}, e^{i\theta_2}), (\theta_2, 0)) \in \mathbb{T}^2 \times \mathbb{R}^2 \mid \theta_1, \theta_2 \in \mathbb{R}\},$$

is the holonomy bundle containing the point $((e, e), (0, 0)) \in \mathbb{T}^2 \times \mathbb{R}^2$ associated to the connection that defines the corresponding cylinder valued momentum map. This immediately shows that $\mathcal{H} = \mathbb{Z} \times \{0\}$ while $f(\ker \exp) = \mathbb{Z} \times \mathbb{Z}$ which is clearly not contained in $\mathcal{H}$.

Proof: Here we give proof of the theorem. We start by assuming that the $\mathfrak{g}$-action on (M, ω) has an associated G-valued momentum map $\mathbf{A}$: $M \to G$ and we will show that $\mathcal{H} \subset f(\ker \exp)$.

Let $\mu \in \mathcal{H}$. The definition of the holonomy group $\mathcal{H}$ implies the existence of a piecewise smooth loop m: $[0, 1] \to M$ at the point m, that is, $m(0) = m(1) = m \in M$, whose horizontal lift $\tilde{m}(t) = (m(t), \mu(t))$ starting at the point $(m, 0)$ satisfies $\mu = \mu(1)$. The horizontality of $\tilde{m}(t)$ implies that

$$\langle \dot{\mu}(t), \xi \rangle = \mathbf{i}_{\xi_M} \omega(m(t)) (\dot{m}(t)) = \left(T_{m(t)}(L_{\mathbf{A}(m(t))^{-1}} \circ \mathbf{A}\right) (\dot{m}(t)), \xi),$$

for any $\xi \in \mathfrak{g}$ or, equivalently,

$$\dot{\mu}(t) = f\left(\left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} \mathbf{A}(m(t))^{-1}\mathbf{A}(m(s))\right). \tag{A.5}$$

Fix $t_0 \in [0, 1]$. Since the exponential map exp: $\mathfrak{g} \to G$ is a local diffeomorphism, there exists a smooth curve ξ: $I_{t_0} := (t_0 - \epsilon, t_0 + \epsilon) \to \mathfrak{g}$, for $\epsilon > 0$ sufficiently small, such that for any $s \in (-\epsilon, \epsilon)$

$$\mathbf{A}(m(t_0 + s)) = \exp \xi(t_0 + s)\mathbf{A}(m(t_0)). \tag{A.6}$$

We now reformulate locally the expression (A.5) using the function $\xi\colon I_{t_0} \to \mathfrak{g}$. Let $\tau, \sigma \in (-\epsilon, \epsilon)$ be such that $t = t_0 + \tau$ and $s = t_0 + \sigma$. Expression (A.5) can be rewritten in I_{t_0} as

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\tau}\mu(t_0+\tau) &= f\left(\left.\frac{\mathrm{d}}{\mathrm{d}\sigma}\right|_{\sigma=\tau} \mathbf{A}(m(t_0+\tau))^{-1}\mathbf{A}(m(t_0+\sigma))\right) \\
&= f\left(\left.\frac{\mathrm{d}}{\mathrm{d}\sigma}\right|_{\sigma=\tau} \exp(-\xi(t_0+\tau))\exp\xi(t_0+\sigma)\mathbf{A}(m(t_0))^{-1}\mathbf{A}(m(t_0))\right) \\
&= f\left(\left.\frac{\mathrm{d}}{\mathrm{d}\sigma}\right|_{\sigma=\tau} \exp\left(\xi(t_0+\sigma)-\xi(t_0+\tau)\right)\right) \\
&= f\left(T_0\exp\left(\left.\frac{\mathrm{d}}{\mathrm{d}\sigma}\right|_{\sigma=\tau} (\xi(t_0+\sigma)-\xi(t_0+\tau))\right)\right) \\
&= f\left(\left.\frac{\mathrm{d}}{\mathrm{d}\sigma}\right|_{\sigma=\tau} (\xi(t_0+\sigma)-\xi(t_0+\tau))\right) \\
&= f\left(\left.\frac{\mathrm{d}}{\mathrm{d}\sigma}\right|_{\sigma=\tau} \xi(t_0+\sigma)\right) = f\left(\frac{\mathrm{d}}{\mathrm{d}\tau}\xi(t_0+\tau)\right),
\end{aligned}$$

which shows that for any $t \in I_{t_0}$

$$\dot\mu(t) = f(\dot\xi(t)). \tag{A.7}$$

We now cover the interval [0, 1] with a finite number of intervals $I_1, \ldots, I_n$ such that in each of them we define a function $\xi_i\colon I_i \to \mathfrak{g}$ that satisfies (A.6) and (A.7). We now write $I_i = [a_i, a_{i+1}]$, with $i \in \{1, \ldots, n\}$, $a_1 = 0$, and $a_{n+1} = 1$. Using these intervals, since $\mu(0) = 0$, we can write

$$\begin{aligned}
\mu = \mu(1) &= \int_0^1 \dot\mu(t)\,\mathrm{d}t = \int_{I_1} \dot\mu(t)\,\mathrm{d}t + \cdots + \int_{I_n} \dot\mu(t)\,\mathrm{d}t \\
&= f\left(\int_{I_1} \dot\xi_1(t)\,\mathrm{d}t + \cdots + \int_{I_n} \dot\xi_n(t)\,\mathrm{d}t\right) \\
&= f\left(\xi_1(a_2) - \xi_1(a_1) + \cdots + \xi_n(a_{n+1}) - \xi_n(a_n)\right).
\end{aligned} \tag{A.8}$$

The construction of the intervals I_i, $i \in \{1, \ldots, n\}$ implies that $\mathbf{A}(m(a_i)) = \exp\xi_i(a_i) \times \mathbf{A}(m(a_i))$. Hence

$$\exp\xi_i(a_i) = e \tag{A.9}$$

and hence $\xi_i(a_i) \in \ker\exp$ for all $i \in \{1, \ldots, n\}$. We also have that

$$\begin{aligned}
\mathbf{A}(m(1)) = \mathbf{A}(m(a_{n+1})) &= \exp\xi_n(a_{n+1})\mathbf{A}(m(a_n)) = \exp\xi_n(a_{n+1})\exp\xi_{n-1}(a_n)\mathbf{A}(m(a_{n-1})) \\
&= \exp\xi_n(a_{n+1})\exp\xi_{n-1}(a_n)\cdots\exp\xi_1(a_2)\mathbf{A}(m(a_1)) \\
&= \exp(\xi_1(a_2) + \cdots + \xi_n(a_{n+1}))\mathbf{A}(m(0)).
\end{aligned}$$

Since $m(0) = m(1) = m$ we have $\mathbf{A}(m(0)) = \mathbf{A}(m(1))$ and therefore $\exp(\xi_1(a_2) + \cdots + \xi_n(a_{n+1})) = e$, which implies that $\xi_1(a_2) + \cdots + \xi_n(a_{n+1}) \in \ker \exp$. If we substitute this relation and (A.9) in (A.8) we obtain that $\mu \in f(\ker \exp)$, which proves the inclusion $\mathcal{H} \subset f(\ker \exp)$.

We now show that $\mathcal{H}$ is closed in $\mathfrak{g}^*$. The inclusion $\mathcal{H} \subset f(\ker \exp)$, the closedness of $\ker \exp$ in $\mathfrak{g}$, and the fact that f is an isomorphism of Lie groups imply that

$$\bar{\mathcal{H}} \subset \overline{f(\ker \exp)} = f(\ker \exp).$$

Because G is Abelian, $\ker \exp$ is a zero-dimensional Lie subgroup of $(\mathfrak{g}, +)$ and hence $\bar{\mathcal{H}}$ is a zero-dimensional Lie subgroup of $\mathfrak{g}^*$. This implies that $\bar{\mathcal{H}} \subset \mathcal{H}$. Indeed, for any element $\mu \in \bar{\mathcal{H}}$ there exists an open neighborhood $U_\mu \subset \mathfrak{g}^*$ such that $U_\mu \cap \bar{\mathcal{H}} = \{\mu\}$ ($\bar{\mathcal{H}}$ is zero-dimensional). As $\mu \in \bar{\mathcal{H}}$ we have that $\varnothing \neq U_\mu \cap \mathcal{H} \subset U_\mu \cap \bar{\mathcal{H}} = \{\mu\}$, which implies that $\mu \in \mathcal{H}$. This shows that $\mathcal{H} = \bar{\mathcal{H}}$ and therefore that $\mathcal{H}$ is closed in $\mathfrak{g}^*$.

Assume now that $f(\ker \exp) \subset \mathcal{H}$. The hypothesis on the existence of a Lie group valued momentum map implies, via the inclusion (A.4) that we just proved, that $f(\ker \exp) = \mathcal{H}$ and that $\mathcal{H}$ is closed in $\mathfrak{g}^*$. Proposition A.3 implies that $\mathbf{J}\colon M \to \mathfrak{g}/\ker \exp \simeq G$ is a G-valued momentum map for the $\mathfrak{g}$-action on (M, ω). We now show that $\mathbf{A}$ and $\mathbf{J}$ differ by a constant in G. The expression (A.1) for $\mathbf{A}$ and (A.2) for $\mathbf{J}$ imply that for any $\xi \in \mathfrak{g}$ and any $v_m \in T_m M$ we have

$$\left(T_m\left(L_{\mathbf{A}(m)^{-1}} \circ \mathbf{A}\right)(v_m), \xi\right) = \mathbf{i}_{\xi_M}\omega(m)(v_m) = \left(T_m\left(L_{\mathbf{J}(m)^{-1}} \circ \mathbf{J}\right)(v_m), \xi\right),$$

which implies that $T\mathbf{J} = T\mathbf{A}$. Since the manifold M is connected we have that $\mathbf{A}$ and $\mathbf{J}$ coincide up to a constant element in G.

The last claim in the theorem is a straightforward corollary of Proposition A.3. □

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