The universal covering and covered spaces of a symplectic Lie algebra action

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Dedicated to Alan Weinstein on the occasion of his 60th birthday.

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Abstract

We show that the category of Hamiltonian covering spaces of a given connected and paracompact symplectic manifold (M, ω) acted canonically upon by a Lie algebra admits a universal covering and covered space.

1 Introduction

Let (M, ω) be a connected symplectic manifold and $\mathfrak{g}$ be a Lie algebra acting symplectically on it. A Lie algebra action of $\mathfrak{g}$ on M is a Lie algebra antihomomorphism $\xi \in \mathfrak{g} \mapsto \xi_M \in \mathfrak{X}(M)$ such that the map $(m, \xi) \in M \times \mathfrak{g}^* \mapsto \xi_M(m) \in TM$ is smooth. The action is symplectic when $\pounds_{\xi_M}\omega = 0$, for any $\xi \in \mathfrak{g}$ and where $\pounds_{\xi_M}$ is the Lie derivative operator defined by the vector field ξ_M.

Definition 1.1 *Let (M, ω) be a connected symplectic manifold and $\mathfrak{g}$ be a Lie algebra acting symplectically on it. We say that the map $p_N : N \to M$ is a* ***Hamiltonian covering map*** *of (M, ω) when it satisfies the following conditions:*

(i) *p_N is a smooth covering map*

(ii) *(N, ω_N) is a connected symplectic manifold*

(iii) *p_N is a symplectic map*

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(iv) *$\mathfrak{g}$ acts symplectically on (N, ω_N) and admits a momentum map $\mathbf{K}_N : N \to \mathfrak{g}^*$*

(v) *p_N is $\mathfrak{g}$-equivariant, that is, $\xi_M(p_N(n)) = T_n p_N(\xi_N(n))$, for any $n \in N$ and any $\xi \in \mathfrak{g}$.*

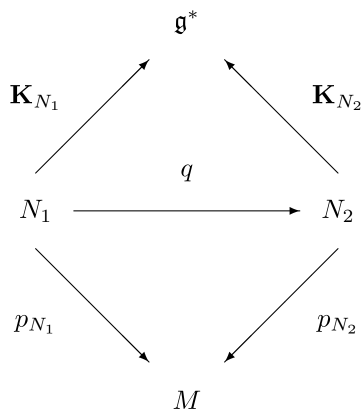
The connectedness hypothesis on N that we assumed in the previous definition implies that the momentum map $\mathbf{K}_N : N \to \mathfrak{g}^*$ is determined up to a constant element in $\mathfrak{g}^*$. We will denote by $[\mathbf{K}_N]$ the equivalence class consisting of all the maps $N \to \mathfrak{g}^*$ that differ from $\mathbf{K}_N$ by a constant map.

Definition 1.2 *Let (M, ω) be a connected symplectic manifold and $\mathfrak{g}$ be a Lie algebra acting symplectically on it. Let $\mathfrak{H}$ be the category whose objects $\mathrm{Ob}(\mathfrak{H})$ are the four-tuples $(p_N : N \to M, \omega_N, \mathfrak{g}, [\mathbf{K}_N])$ with $p_N : N \to M$ a Hamiltonian covering map of (M, ω) and whose morphisms $\mathrm{Mor}(\mathfrak{H})$ are the smooth maps $q : (N_1, \omega_1) \to (N_2, \omega_2)$ that satisfy the following properties:*

(i) *q is a symplectic covering map*

(ii) *q is $\mathfrak{g}$-equivariant*

(iii) *the diagram*



commutes for some $\mathbf{K}_{N_1} \in [\mathbf{K}_{N_1}]$ and $\mathbf{K}_{N_2} \in [\mathbf{K}_{N_2}]$.

We will refer to $\mathfrak{H}$ as the category of ***Hamiltonian covering maps****.*

The main goal of this paper is to show that the category $\mathfrak{H}$ admits universal covering and covered spaces. More explicitly, we will show that there exist two objects $(\widehat{p} : \widehat{M} \to M, \omega_{\widehat{M}}, \mathfrak{g}, [\widehat{\mathbf{K}}])$ (***universal Hamiltonian covering space***) and $(\widetilde{p} : \widetilde{M} \to M, \omega_{\widetilde{M}}, \mathfrak{g}, [\widetilde{\mathbf{K}}])$ (***universal Hamiltonian covered space***) in $\mathfrak{H}$ such that for any other object $(p_N : N \to M, \omega_N, \mathfrak{g}, [\mathbf{K}_N])$ in $\mathfrak{H}$, there exist morphisms (not necessarily unique) $\widehat{q} : \widehat{M} \to N$ and $\widetilde{q} : N \to \widetilde{M}$ in $\mathrm{Mor}(\mathfrak{H})$. Even though the objects that satisfy these properties are not necessarily unique, they are all isomorphic to $(\widehat{p} : \widehat{M} \to M, \omega_{\widehat{M}}, \mathfrak{g}, [\widehat{\mathbf{K}}])$ and $(\widetilde{p} : \widetilde{M} \to M, \omega_{\widetilde{M}}, \mathfrak{g}, [\widetilde{\mathbf{K}}])$ in $\mathfrak{H}$, respectively, which justifies the adjective "universal" when we will refer to the Hamiltonian covering and covered spaces.

The universal Hamiltonian covering space will be easily obtained in Section 2 from the standard simply connected universal covering manifold. The universal Hamiltonian covered space is constructed in Section 3 using a connection introduced by Condevaux, Dazord, and Molino [CDM88], Section I.3.1, and used by them in the definition of the so called "reduced momentum map". The universality of the Hamiltonian covered space is presented in Section 4.

2 The standard universal covering as a Hamiltonian covering

Let (M,ω) be a connected symplectic manifold and $\mathfrak{g}$ be a Lie algebra acting symplectically on it. Let $\widehat{p}:\widehat{M}\to M$ be a simply connected universal covering of M. This can be made into a Hamiltonian covering map in a straightforward manner. First, since $\widehat{p}$ is a local diffeomorphism, the two–form $\omega_{\widehat{M}}:=\widehat{p}^*\omega$ is a symplectic form on $\widehat{M}$. Thus, properties (**i**), (**ii**), and (**iii**) of Definition 1.1 hold. Second, we can use the $\mathfrak{g}$-action on M to define a symplectic $\mathfrak{g}$-action on $(\widehat{M},\omega_{\widehat{M}})$ by

$$\xi_{\widehat{M}}(z):=(T_{(z)}\widehat{p})^{-1}\xi_M(\widehat{p}(z)),\quad \text{for any}\quad \xi\in\mathfrak{g}\quad\text{and}\quad z\in\widehat{M}. \tag{2.1}$$

This is a good definition since $\widehat{p}$ is a covering map and hence a local diffeomorphism. Moreover, the map $(z,\xi)\in\widehat{M}\times\mathfrak{g}\mapsto\xi_{\widehat{M}}(z)\in T\widehat{M}$ is clearly smooth. Note that by definition, the vector fields $\xi_{\widehat{M}}$ and ξ_M are $\widehat{p}$–related for all $\xi\in\mathfrak{g}$. This immediately shows that $[\xi,\eta]_{\widehat{M}}=-[\xi_{\widehat{M}},\eta_{\widehat{M}}]$ for any $\xi,\eta\in\mathfrak{g}$ and that $\pounds_{\xi_{\widehat{M}}}\omega_{\widehat{M}}=\pounds_{\xi_{\widehat{M}}}\widehat{p}^*\omega=\widehat{p}^*\pounds_{\xi_M}\omega=0$, for any $\xi\in\mathfrak{g}$. Thus, expression (2.1) defines a symplectic action of $\mathfrak{g}$ on $(\widehat{M},\omega_{\widehat{M}})$ relative to which $\widehat{p}$ is equivariant by construction. Finally, the $\mathfrak{g}$-action on $\widehat{M}$ admits a momentum map $\widehat{\mathbf{K}}:\widehat{M}\to\mathfrak{g}^*$ because $\widehat{M}$ is simply connected and therefore $H^1(\widehat{M},\mathbb{R})=0$; we recall from [We77] that the canonical action of a Lie algebra $\mathfrak{h}$ on the symplectic manifold (S,ω) admits an associated momentum map if and only if the linear map $[\xi]\in\mathfrak{h}/[\mathfrak{h},\mathfrak{h}]\mapsto[\mathbf{i}_{\xi_S}\omega]\in H^1(S,\mathbb{R})$ is identically zero. Thus, conditions (**iv**) and (**v**) also hold, which makes $(\widehat{p}:\widehat{M}\to M,\omega_{\widehat{M}},\mathfrak{g},[\widehat{\mathbf{K}}])$ into an object of $\mathfrak{H}$.

Proposition 2.1 *Let (M,ω) be a connected symplectic manifold and $\mathfrak{g}$ be a Lie algebra acting symplectically on it. Let $(\widehat{p}:\widehat{M}\to M,\omega_{\widehat{M}},\mathfrak{g},[\widehat{\mathbf{K}}])$ be the object in $\mathfrak{H}$ constructed above using a simply connected universal covering of M. Then for any other object $(p_N:N\to M,\omega_N,\mathfrak{g},[\mathbf{K}_N])$ of $\mathfrak{H}$, there exists a morphism $q:\widehat{M}\to N$ in $\mathrm{Mor}(\mathfrak{H})$. Any other object in $\mathfrak{H}$ that satisfies the same universality property is isomorphic to $(\widehat{p}:\widehat{M}\to M,\omega_{\widehat{M}},\mathfrak{g},[\widehat{\mathbf{K}}])$.*

Proof. Since $\widehat{M}$ is the universal covering space of M, there exists a smooth covering map $q:\widehat{M}\to M$ (in general not unique) such that $p_N\circ q=\widehat{p}$. We shall prove that this is a morphism in $\mathfrak{H}$. Indeed, since p_N and $\widehat{p}$ are symplectic maps we have

$$\omega_{\widehat{M}}=\widehat{p}^*\omega=(p_N\circ q)^*\omega=q^*p_N^*\omega=q^*\omega_N,$$

so condition **(i)** in Definition 1.2 is satisfied. Additionally, since $\widehat{p}$ and p_N are $\mathfrak{g}$-equivariant we have, for any $z\in\widehat{M}$ and $\xi\in\mathfrak{g}$

$$T_{q(z)}p_N\left(T_zq\left(\xi_{\widehat{M}}(z)\right)\right)=T_z\widehat{p}\left(\xi_{\widehat{M}}(z)\right)=\xi_M(\widehat{p}(z))=\xi_M(p_N(q(z)))=T_{q(z)}p_N\left(\xi_N(q(z))\right).$$

Since $T_{q(z)}p_N$ is an isomorphism it follows that $T_z q\left(\xi_{\widehat{M}}(z)\right) = \xi_N(q(z))$ and so **(ii)** is satisfied. To verify **(iii)** it suffices to note that $\mathbf{K}_N \circ q$ is a momentum map for the $\mathfrak{g}$-action on $\widehat{M}$.

In order to prove the last sentence in the statement let $(\widehat{p}' : \widehat{M}' \to M, \omega_{\widehat{M}'}, \mathfrak{g}, [\widehat{\mathbf{K}}'])$ be another object in $\mathfrak{H}$ satisfying the same universality property as $(\widehat{p} : \widehat{M} \to M, \omega_{\widehat{M}}, \mathfrak{g}, [\widehat{\mathbf{K}}])$. Let $q : \widehat{M} \to \widehat{M}'$ and $q' : \widehat{M}' \to \widehat{M}$ be the corresponding morphisms. Since both q and q' are symplectic covering maps their composition $q' \circ q : \widehat{M} \to \widehat{M}$ is also a symplectic covering map (see Theorems 3, 5, 6 in Section 2.2 and Theorem 10 in Section 2.4 of [Sp66]). Thus $q' \circ q$ is a local diffeomorphism. Since $\widehat{M}$ is simply connected, this map is also injective [Sp66, Theorem 9, page 73]. Consequently, $\varphi := q' \circ q$ is a bijective local diffeomorphism, hence a diffeomorphism. Finally, this proves that both q and q' are isomorphisms in $\mathfrak{H}$ with inverses $\varphi^{-1} \circ q'$ and $q \circ \varphi^{-1}$, respectively. ■

Remark 2.2 It should be noticed that the universality property for $(\widehat{p} : \widehat{M} \to M, \omega_{\widehat{M}}, \mathfrak{g}, [\widehat{\mathbf{K}}])$ stated in the previous proposition does not imply that this is an initial object in $\mathfrak{H}$ due to the non-uniqueness of the morphism q. This is in agreement with the situation encountered for general manifolds.

3 The universal Hamiltonian covered space in $\mathfrak{H}$

In this section we will construct an object in the category $\mathfrak{H}$ defined in the introduction using a principal connection introduced by Condevaux, Dazord, and Molino in Section I.3.1 of [CDM88]. In the next section we will prove that this object has the universality property stated in the introduction to define the universal Hamiltonian covered space. The setup is identical to the one in the introduction, but from now on we will assume that M is also paracompact.

The connection α. Let $\pi : M \times \mathfrak{g}^* \to M$ be the projection onto M. Consider π as the bundle map of the trivial principal fiber bundle $(M \times \mathfrak{g}^*, M, \pi, \mathfrak{g}^*)$ that has $(\mathfrak{g}^*, +)$ as Abelian structure group. The group $(\mathfrak{g}^*, +)$ acts on $M \times \mathfrak{g}^*$ by $\nu \cdot (m, \mu) := (m, \mu - \nu)$, with $m \in M$ and $\mu, \nu \in \mathfrak{g}^*$. Let $\alpha \in \Omega^1(M \times \mathfrak{g}^*, \mathfrak{g}^*)$ be the connection one-form defined by

$$\langle \alpha(m, \mu)(v_m, \nu), \xi \rangle := (\mathbf{i}_{\xi_M}\omega)(m)(v_m) - \langle \nu, \xi \rangle, \tag{3.1}$$

where $(m, \mu) \in M \times \mathfrak{g}^*$, $(v_m, \nu) \in T_m M \times \mathfrak{g}^*$, $\xi \in \mathfrak{g}$, and $\langle \cdot, \cdot \rangle$ denotes the natural pairing between $\mathfrak{g}^*$ and $\mathfrak{g}$.

We briefly check that α is indeed a connection one-form on $M \times \mathfrak{g}^*$. Notice that the infinitesimal generator $\nu_{M \times \mathfrak{g}^*}$ associated to an element $\nu \in \mathfrak{g}^*$ is given by $\nu_{M \times \mathfrak{g}^*}(m, \mu) = (0, -\nu)$. Consequently, for any $\xi \in \mathfrak{g}$, $\langle \alpha(m, \mu)(\nu_{M \times \mathfrak{g}^*}(m, \mu)), \xi \rangle = \langle \alpha(m, \mu)(0, -\nu), \xi \rangle = \langle \nu, \xi \rangle$, that is, $\alpha(m, \mu)(\nu_{M \times \mathfrak{g}^*}(m, \mu)) = \nu$. Also, it is obvious that for any $\rho \in \mathfrak{g}^*$ we have $\langle \alpha(m, \mu - \rho)(v_m, \nu), \xi \rangle = (\mathbf{i}_{\xi_M}\omega)(m)(v_m) - \langle \nu, \xi \rangle = \langle \alpha(m, \mu)(v_m, \nu), \xi \rangle$, hence α is a well defined connection one–form on $M \times \mathfrak{g}^*$.

The horizontal and vertical bundles of α. By definition, the horizontal subspace $H(m, \mu)$ at the point (m, μ) determined by α is given by

$$H(m, \mu) = \{(v_m, \nu) \in T_{(m,\mu)}(M \times \mathfrak{g}^*) \mid (\mathbf{i}_{\xi_M}\omega)(m)(v_m) - \langle \nu, \xi \rangle = 0, \forall \xi \in \mathfrak{g}\}. \tag{3.2}$$

Consequently, given any vector $(v_m, \nu) \in T_{(m,\mu)}(M \times \mathfrak{g}^*)$, its horizontal $(v_m, \nu)^H$ and vertical $(v_m, \nu)^V$ parts are such that

$$(v_m, \nu)^H = (v_m, \rho) \quad \text{and} \quad (v_m, \nu)^V = (0, \rho'),$$

where $\rho, \rho' \in \mathfrak{g}^*$ are uniquely determined by the relations

$$\langle \rho, \xi \rangle = (\mathbf{i}_{\xi_M} \omega)(m)(v_m) \quad \text{and} \quad \rho' = \nu - \rho, \quad \text{for any} \quad \xi \in \mathfrak{g}.$$

α is a flat connection. We compute the curvature form Ω associated to α. Let $(m, \mu) \in M \times \mathfrak{g}^*$, $v_m, u_m \in T_mM$, $\xi \in \mathfrak{g}$, and $\nu, \rho \in \mathfrak{g}^*$ arbitrary. By definition,

$$\langle \Omega(m, \mu)((v_m, \nu), (u_m, \rho)), \xi \rangle = \langle \mathbf{d}\alpha(m, \mu)((v_m, \nu)^H, (u_m, \rho)^H), \xi \rangle. \tag{3.3}$$

Let now (X_1, Y_1) and (X_2, Y_2) be vector fields on $M \times \mathfrak{g}^*$ such that $(X_1(m), Y_1(\mu)) = (v_m, \nu)$ and $(X_2(m), Y_2(\mu)) = (u_m, \rho)$. Using these vector fields, the right hand side of (3.3) can be rewritten as

$$\langle (X_1, Y_1)[\alpha(X_2, Y_2)](m, \mu), \xi \rangle - \langle (X_2, Y_2)[\alpha(X_1, Y_1)](m, \mu), \xi \rangle - \langle \alpha([X_1, X_2], 0)(m, \mu), \xi \rangle. \tag{3.4}$$

Let (m_t^1, μ_t^1) and (m_t^2, μ_t^2) be the flows of (X_1, Y_1) and (X_2, Y_2), respectively. We choose Y_1 and Y_2 such that their flows are given by $\mu_t^1(\mu) = \mu + t\nu$ and $\mu_t^2(\mu) = \mu + t\rho$. We can use these flows to compute

$$\begin{aligned}
\langle (X_1, Y_1)[\alpha(X_2, Y_2)](m, \mu), \xi \rangle &= \frac{d}{dt}\Big|_{t=0} \left\langle \alpha(m_t^1(m), \mu_t^1(\mu)) \left(X_2(m_t^1(m)), Y_2(\mu_t^1(\mu)) \right), \xi \right\rangle \\
&= \frac{d}{dt}\Big|_{t=0} \left((\mathbf{i}_{\xi_M} \omega)(m_t^1) \left(X_2(m_t^1(m) \right) - \langle Y_2(\mu_t^1(\mu)), \xi \rangle \right) \\
&= X_1[\mathbf{i}_{\xi_M} \omega(X_2)](m) - \frac{d}{dt}\Big|_{t=0} \frac{d}{ds}\Big|_{s=0} \langle \mu + t\nu + s\rho, \xi \rangle \\
&= X_1[\mathbf{i}_{\xi_M} \omega(X_2)](m).
\end{aligned}$$

Analogously, we have

$$\langle (X_2, Y_2)[\alpha(X_1, Y_1)](m, \mu), \xi \rangle = X_2[\mathbf{i}_{\xi_M} \omega(X_1)](m).$$

Consequently, the expression (3.4) equals

$$\begin{aligned}
X_1[\mathbf{i}_{\xi_M} \omega(X_2)](m) - X_2[\mathbf{i}_{\xi_M} \omega(X_1)](m) - \mathbf{i}_{\xi_M} \omega(m)([X_1, X_2](m)) &= \mathbf{d}(\mathbf{i}_{\xi_M} \omega)(m)(X_1(m), X_2(m)) \\
= (\pounds_{\xi_M} \omega)(m)(X_1(m), X_2(m)) - (\mathbf{i}_{\xi_M} \mathbf{d}\omega)(m)(X_1(m), X_2(m)) &= 0,
\end{aligned}$$

which guarantees the flatness of α.

The holonomy bundles of α. The flatness of α implies that the associated horizontal distribution is integrable and that its maximal integral leaves coincide with the holonomy bundles $(\widetilde{p} : \widetilde{M} \to M, \mathcal{H})$ of α, where $\mathcal{H}$ is the holonomy group of α based at any point of $\widetilde{M}$ (the paracompactness of M is used at this point in the proof [KN63, Theorem 7.1, page 83] that $(\widetilde{p} : \widetilde{M} \to M, \mathcal{H})$ is a reduction of the bundle $(\pi : M \times \mathfrak{g}^* \to M, \mathfrak{g}^*)$). Notice that since $(\mathfrak{g}^*, +)$ is Abelian, any two holonomy bundles $\widetilde{M}_1$ and $\widetilde{M}_2$ are isomorphic as principal bundles with the same structure group $\mathcal{H}$, via the map $R_\tau : \widetilde{M}_2 \to \widetilde{M}_1$ defined by $R_\tau(m,\mu) := (m, \mu+\tau)$, for some fixed $\tau \in \mathfrak{g}^*$ and for any $(m,\mu) \in \widetilde{M}_2$. The group $\mathcal{H}$ will be referred to as the ***holonomy group of the*** $\mathfrak{g}$***-action***. It is easy to prove that the $\mathfrak{g}$-action on (M,ω) admits a standard momentum map if and only if the holonomy group of the action $\mathcal{H}$ is trivial.

A fact that will be important later on is that the holonomy bundles $\widetilde{M}$ are initial submanifolds of $M \times \mathfrak{g}^*$, that is, they satisfy the following universality property: the inclusion $i : \widetilde{M} \hookrightarrow M \times \mathfrak{g}^*$ is a smooth immersion such that for any manifold Z, a given mapping $f : Z \to \widetilde{M}$ is smooth if and only if $i \circ f : Z \to M \times \mathfrak{g}^*$ is smooth. The initial submanifold property is satisfied by the maximal integral leaves of any smooth integrable distribution, such as the horizontal distribution in our case.

The holonomy bundles of α are Hamiltonian coverings of $(M,\omega,\mathfrak{g})$. We now prove the following proposition.

Proposition 3.1 *Let (M,ω) be a connected paracompact symplectic manifold and let $\mathfrak{g}$ be a Lie algebra acting symplectically on it. Let α be the connection on the trivial bundle $(\pi : M\times\mathfrak{g}^* \to M, \mathfrak{g}^*)$ introduced in* (3.1) *and $(\widetilde{p} : \widetilde{M} \to M, \mathcal{H})$ be one of its holonomy bundles. If we define $\omega_{\widetilde{M}} := \widetilde{p}^*\omega$, then the pair $(\widetilde{M}, \omega_{\widetilde{M}})$ is a symplectic manifold acted symplectically upon by the Lie algebra $\mathfrak{g}$ via the expression*

$$\xi_{\widetilde{M}}(m,\mu) := (\xi_M(m), -\Psi(m)(\xi,\cdot)), \quad \textit{for any} \quad \xi \in \mathfrak{g} \quad \textit{and} \quad (m,\mu) \in \widetilde{M}. \tag{3.5}$$

The symbol Ψ denotes the Chu map [Ch75] *$\Psi : M \to Z^2(\mathfrak{g})$, defined by $\Psi(m)(\xi,\eta) := \omega(m)(\xi_M(m), \eta_M(m))$, for any $\xi,\eta \in \mathfrak{g}$. Finally, the projection $\widetilde{\mathbf{K}} : \widetilde{M} \to \mathfrak{g}^*$ of $\widetilde{M}$ onto $\mathfrak{g}^*$ is a momentum map for this action. Moreover, the four tuple $(\widetilde{p} : \widetilde{M} \to M, \omega_{\widetilde{M}}, \mathfrak{g}, [\widetilde{\mathbf{K}}])$ is an object in the category $\mathfrak{H}$ introduced in Definition 1.2.*

Proof. We start by noticing that the projection $\widetilde{p} : \widetilde{M} \to M$ is a smooth covering projection as a consequence of the flatness of α. Indeed, since the connection is flat, the Ambrose-Singer Theorem [AS53] implies that the Lie algebra Lie$(\mathcal{H})$ of the holonomy group is trivial and hence $\mathcal{H}$ is a discrete (possibly not closed) subgroup of $(\mathfrak{g}^*, +)$. As $(\widetilde{p} : \widetilde{M} \to M, \mathcal{H})$ is a locally trivial bundle, any point $m \in M$ has an open neighborhood U such that $\widetilde{p}^{-1}(U)$ is diffeomorphic to $U \times \mathcal{H}$. Since $\mathcal{H}$ is discrete, each subset $U \times \{\mu\}$, $\mu \in \mathcal{H}$, is an open subset diffeomorphic to U. Hence, $\widetilde{p}$ is a covering map.

Now, as $\widetilde{p}$ is a local diffeomorphism, the equality $\omega_{\widetilde{M}} := \widetilde{p}^*\omega$ defines a symplectic form on $\widetilde{M}$ with respect to which $\widetilde{p}$ is a symplectomorphism. We have hence shown that $\widetilde{p} : \widetilde{M} \to M$ satisfies properties **(i)**, **(ii)**, and **(iii)** in Definition 1.1.

We now define a $\mathfrak{g}$-action on $\widetilde{M}$ by

$$\xi_{\widetilde{M}}(m,\mu) := (T_{(m,\mu)}\widetilde{p})^{-1}\xi_M(m), \quad \text{for any} \quad \xi \in \mathfrak{g} \quad \text{and} \quad (m,\mu) \in \widetilde{M}. \tag{3.6}$$

This is a good definition since $\widetilde{p}$ is a covering map and hence a local diffeomorphism. Moreover, the map $((m,\mu),\xi) \in \widetilde{M} \times \mathfrak{g} \mapsto \xi_{\widetilde{M}}(m,\mu) \in T\widetilde{M}$ is clearly smooth. Note that, by definition, the vector fields $\xi_{\widetilde{M}}$ and ξ_M are $\widetilde{p}$-related for all $\xi \in \mathfrak{g}$. This immediately shows that $[\xi,\eta]_{\widetilde{M}} = -[\xi_{\widetilde{M}}, \eta_{\widetilde{M}}]$ for any $\xi,\eta \in \mathfrak{g}$ and that $\pounds_{\xi_{\widetilde{M}}}\omega_{\widetilde{M}} = \pounds_{\xi_{\widetilde{M}}}\widetilde{p}^*\omega = \widetilde{p}^*\pounds_{\xi_M}\omega = 0$, for any $\xi \in \mathfrak{g}$. Thus, expression (3.6) defines a symplectic action of $\mathfrak{g}$ on $(\widetilde{M}, \omega_{\widetilde{M}})$. We now show that (3.6) can be rewritten as (3.5), that is,

$$\xi_{\widetilde{M}}(m,\mu) := (\xi_M(m), -\Psi(m)(\xi,\cdot)), \quad \text{for any} \quad \xi \in \mathfrak{g} \quad \text{and} \quad (m,\mu) \in \widetilde{M}. \tag{3.7}$$

We start by checking that the right hand side of this expression is a horizontal vector with respect to α and thereby tangent to $\widetilde{M}$, which means that $\left\langle \alpha(m,\mu)\left(\xi_{\widetilde{M}}(m,\mu)\right),\eta\right\rangle = 0$, for any $\eta \in \mathfrak{g}$. By the definition of α we have that

$$\begin{aligned}\left\langle \alpha(m,\mu)\left(\xi_{\widetilde{M}}(m,\mu)\right),\eta\right\rangle &= (\mathbf{i}_{\eta_M}\omega)(m)\left(\xi_M(m)\right) + \langle \Psi(m)(\xi,\cdot),\eta\rangle \\ &= \omega(m)(\eta_M(m),\xi_M(m)) + \omega(m)(\xi_M(m),\eta_M(m)) = 0.\end{aligned}$$

Consequently $(\xi_M(m), -\Psi(m)(\xi,\cdot))$ is horizontal and therefore it suffices to notice that $\widetilde{p}$ is the projection onto M to prove the equivalence between (3.6) and (3.7). The same remark proves the $\mathfrak{g}$-equivariance of $\widetilde{p}:\widetilde{M}\to M$.

We conclude by showing that the projection $\widetilde{\mathbf{K}}:\widetilde{M}\to\mathfrak{g}^*$ of $\widetilde{M}$ onto $\mathfrak{g}^*$ is a momentum map for the $\mathfrak{g}$-action on $\widetilde{M}$ defined in (3.5). Let $\xi\in\mathfrak{g}$ be arbitrary and $\widetilde{\mathbf{K}}^\xi := \langle\widetilde{\mathbf{K}},\xi\rangle$. On one hand we have that $\mathbf{d}\widetilde{\mathbf{K}}^\xi(m,\mu)(v_m,\nu) = \langle\nu,\xi\rangle$, for any $(m,\mu)\in\widetilde{M}$ and any $(v_m,\nu)\in T_{(m,\mu)}\widetilde{M} = H(m,\mu)$. On the other hand, $\mathbf{i}_{\xi_{\widetilde{M}}}\omega_{\widetilde{M}}(m,\mu)(v_m,\nu) = \mathbf{i}_{\xi_{\widetilde{M}}}(\widetilde{p}^*\omega)(m,\mu)(v_m,\nu) = (\widetilde{p}^*\omega)(m,\mu)(\xi_{\widetilde{M}}(m,\mu),(v_m,\nu)) = (\widetilde{p}^*\omega)(m,\mu)((\xi_M(m),-\Psi(m)(\xi,\cdot)),(v_m,\nu)) = \omega(m)(\xi_M(m),v_m) = \langle\nu,\xi\rangle$, which proves the claim. ■

Remark 3.2 The momentum map $\widetilde{\mathbf{K}}$ is not equivariant in general. Indeed, its infinitesimal non-equivariance cocycle is given by

$$\begin{aligned}\Sigma(\xi,\eta) := \widetilde{\mathbf{K}}^{[\xi,\eta]}(m,\mu) - \{\widetilde{\mathbf{K}}^\xi,\widetilde{\mathbf{K}}^\eta\}(m,\mu) &= \langle\mu,[\xi,\eta]\rangle - (\widetilde{p}^*\omega)(m,\mu)(\xi_{\widetilde{M}}(m,\mu),\eta_{\widetilde{M}}(m,\mu)) \\ &= \langle\mu,[\xi,\eta]\rangle - \omega(m)(\xi_M(m),\eta_M(m)) = \langle\mu,[\xi,\eta]\rangle - \Psi(m)(\xi,\eta), \quad (3.8)\end{aligned}$$

for any $\xi,\eta\in\mathfrak{g}$. The value of Σ does not depend on the point $(m,\mu)\in\widetilde{M}$ used to define it because for any $(v_m,\nu)\in T_{(m,\mu)}\widetilde{M}$ the function $f(m,\mu):=\langle\mu,[\xi,\eta]\rangle-\Psi(m)(\xi,\eta)$ is such that

$$\mathbf{d}f(m,\mu)(v_m,\mu) = \langle\nu,[\xi,\eta]\rangle - T_m\Psi(v_m)(\xi,\eta) = \langle\nu,[\xi,\eta]\rangle - \omega(m)([\xi,\eta]_M(m),v_m) = 0,$$

where we used the horizontality of (v_m,ν) in the last equality. The connectedness of $\widetilde{M}$ concludes the argument.

4 The universality theorem

In this section we state and prove the main result of the paper.

Theorem 4.1 *Let (M,ω) be a connected paracompact symplectic manifold and $\mathfrak{g}$ a Lie algebra acting symplectically on it. The Hamiltonian covering $(\widetilde{p}:\widetilde{M}\to M,\omega_{\widetilde{M}},\mathfrak{g},[\widetilde{\mathbf{K}}])$ constructed in Proposition 3.1 is a universal Hamiltonian covered space in the category $\mathfrak{H}$ of Hamiltonian covering maps, that is, given any other object $(p_N:N\to M,\omega_N,\mathfrak{g},[\mathbf{K}_N])$ in $\mathfrak{H}$, there exists a (not necessarily unique) morphism $q:N\to\widetilde{M}$ in $\mathrm{Mor}(\mathfrak{H})$. Any other object of $\mathfrak{H}$ that satisfies this universality property is isomorphic to $(\widetilde{p}:\widetilde{M}\to M,\omega_{\widetilde{M}},\mathfrak{g},[\widetilde{\mathbf{K}}])$.*

Proof. Let $(p_N : N \to M, \omega_N, \mathfrak{g}, [\mathbf{K}_N]) \in \mathfrak{H}$ and $n_0 \in N$. Define $\widetilde{m}_0 := (p_N(n_0), \mathbf{K}_N(n_0)) \in M \times \mathfrak{g}^*$. Since $M \times \mathfrak{g}^*$ is foliated by the holonomy bundles of the connection α in (3.1), the point $\widetilde{m}_0$ lies in one of them, say $\widetilde{M}'$. Let $\tau \in \mathfrak{g}^*$ be such that $\widetilde{M}' = R_\tau(\widetilde{M})$ and define $\overline{\mathbf{K}}_N := \mathbf{K}_N - \tau$. The map $\overline{\mathbf{K}}_N : N \to \mathfrak{g}^*$ is also a momentum map for the $\mathfrak{g}$-action on N, $[\overline{\mathbf{K}}_N] = [\mathbf{K}_N]$, and moreover $(p_N(n_0), \overline{\mathbf{K}}_N(n_0)) \in \widetilde{M}$. Hence, we can assume without loss of generality that $(p_N : N \to M, \omega_N, \mathfrak{g}, [\mathbf{K}_N])$ is such that $(p_N(n_0), \mathbf{K}_N(n_0)) \in \widetilde{M}$. Using this choice we define the map $g : N \to M \times \mathfrak{g}^*$ by $n \longmapsto (p_N(n), \mathbf{K}_N(n))$, $n \in N$.

We will now show that $g(N) \subset \widetilde{M}$. We start by proving that $T_n g(v_n) \in H(p_N(n), \mathbf{K}_N(n))$ for all $n \in N$ and $v_n \in T_n N$. Indeed, since $T_n g(v_n) = (T_n p_N(v_n), T_n \mathbf{K}_N(v_n))$ we have for any $\xi \in \mathfrak{g}$

$$\begin{aligned} \langle \alpha(g(n))\left(T_n g(v_n)\right), \xi\rangle &= \omega(p_N(n))(\xi_M(p_N(n)), T_n p_N(v_n)) - \langle T_n \mathbf{K}_N(v_n), \xi\rangle \\ &= \omega(p_N(n))(T_n p_N(\xi_N(n)), T_n p_N(v_n)) - \mathbf{d}\mathbf{K}_N^\xi(n)(v_n) \\ &= \omega_N(n)(\xi_N(n), v_n) - \mathbf{d}\mathbf{K}_N^\xi(n)(v_n) = 0, \end{aligned}$$

where we used the $\mathfrak{g}$-equivariance and the symplectic character of p_N. Let now $n \in N$ be arbitrary. As N is connected, there exists a smooth curve $c : [0,1] \to N$ such that $c(0) = n_0$ and $c(1) = n$. Since the derivative Tg of g maps into the horizontal bundle of α, the chain rule implies that $g(c(t))$ is a horizontal curve starting at $g(c(0)) = g(n_0) \in \widetilde{M}$. Hence, by the definition of the holonomy bundle, $g(c(1)) = g(n) \in \widetilde{M}$. This argument and the arbitrary character of $n \in N$ show that $g(N) \subset \widetilde{M}$.

Let $q : N \to \widetilde{M}$ be the map obtained from g by restriction of the range. We will show that q is the morphism needed to prove the statement of the theorem. First, the map q is smooth since g is smooth and $\widetilde{M}$ is an initial submanifold of $M \times \mathfrak{g}^*$. Second, we verify that q satisfies the three conditions in Definition 1.2 that characterize an element in $\mathrm{Mor}(\mathfrak{H})$.

(i) q is a symplectic covering projection: Since $p_N : N \to M$ and $\widetilde{p} : \widetilde{M} \to M$ are covering projections and $\widetilde{p} \circ q = p_N$ it follows that $q : N \to \widetilde{M}$ is a covering projection [Sp66, Lemma 1, page 79]. Since p_N and $\widetilde{p}$ are symplectic so is q.

(ii) q is $\mathfrak{g}$-equivariant: Let $\xi \in \mathfrak{g}$, $n \in N$ be arbitrary. On one hand

$$\xi_{\widetilde{M}}(q(n)) = \xi_{\widetilde{M}}(p_N(n), \mathbf{K}_N(n)) = (\xi_M(p_N(n)), -\Psi(p_N(n))(\xi, \cdot)).$$

On the other hand

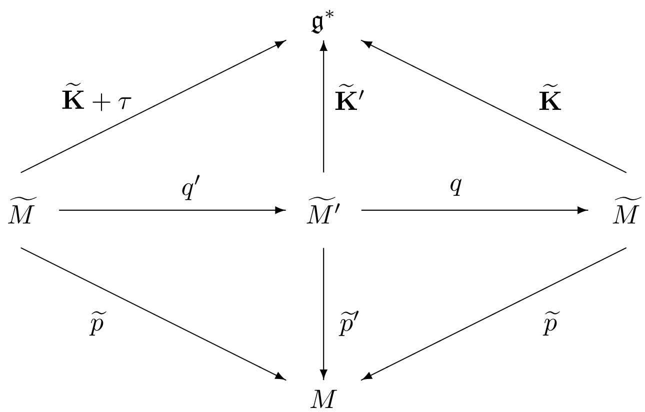
$$T_n q(\xi_N(n)) = (T_n p_N(\xi_N(n)), T_n \mathbf{K}_N(\xi_N(n))) = (\xi_M(p_N(n)), T_n \mathbf{K}_N(\xi_N(n))).$$

Consequently, the map q is $\mathfrak{g}$-equivariant if and only if $T_n \mathbf{K}(\xi_N(n)) = -\Psi(p_N(n))(\xi, \cdot)$. This identity holds because for any $\eta \in \mathfrak{g}$ we have

$$\begin{aligned} \langle T_n \mathbf{K}_N(\xi_N(n)), \eta\rangle &= \mathbf{d}\mathbf{K}_N^\eta(n)(\xi_N(n)) = \omega_N(n)(\eta_N(n), \xi_N(n)) \\ &= (p_N^*\omega)(n)(\eta_N(n), \xi_N(n)) = \omega(p_N(n))(T_n p_N(\eta_N(n)), T_n p_N(\xi_N(n))) \\ &= \omega(p_N(n))(\eta_M(p_N(n)), \xi_M(p_N(n))) = -\Psi(p_N(n))(\xi, \eta). \end{aligned}$$

(iii) The diagram in Definition 1.2 commutes since $\widetilde{p} \circ q = p_N$ and $\widetilde{\mathbf{K}} \circ q = \mathbf{K}_N$ by the definition of q.

We conclude by showing that any other object $(\widetilde{p}' : \widetilde{M}' \to M, \omega_{\widetilde{M}'}, \mathfrak{g}, [\widetilde{\mathbf{K}}'])$ that satisfies the just proved universality property of $(\widetilde{p} : \widetilde{M} \to M, \omega_{\widetilde{M}}, \mathfrak{g}, [\widetilde{\mathbf{K}}])$ is necessarily isomorphic to it. Indeed, the universality property satisfied by $(\widetilde{p} : \widetilde{M} \to M, \omega_{\widetilde{M}}, \mathfrak{g}, [\widetilde{\mathbf{K}}])$ and $(\widetilde{p}' : \widetilde{M}' \to M, \omega_{\widetilde{M}'}, \mathfrak{g}, [\widetilde{\mathbf{K}}'])$ implies the existence of two morphisms $q : \widetilde{M}' \to \widetilde{M}$ and $q' : \widetilde{M} \to \widetilde{M}'$ in $\mathrm{Mor}(\mathfrak{H})$ and of an element $\tau \in \mathfrak{g}^*$ such that the following diagram commutes



Since q and q' are covering maps so is the composition $q \circ q' : \widetilde{M} \to \widetilde{M}$ (see Theorems 3, 5, 6 in Section 2.2 and Theorem 10 in Section 2.4 of [Sp66]). Thus $q \circ q'$ is a local surjective diffeomorphism. We now show that it is also injective. If $(m, \mu) \in \widetilde{M}$, the definition of q and the commutativity of the diagram above yield

$$(q \circ q')(m, \mu) = (\widetilde{p}'(q'(m, \mu)), \widetilde{\mathbf{K}}'(q'(m, \mu))) = (\widetilde{p}(m, \mu), \widetilde{\mathbf{K}}(m, \mu) + \tau) = (m, \mu + \tau). \tag{4.1}$$

Hence if $(m, \mu), (m', \mu') \in \widetilde{M}$ satisfy $(q \circ q')(m, \mu) = (q \circ q')(m', \mu')$, then (4.1) implies that $(m, \mu) = (m', \mu')$. Consequently, $\varphi := q \circ q'$ is a bijective local diffeomorphism and hence a diffeomorphism. This proves that both q and q' are isomorphisms is $\mathfrak{H}$. ■

Remark 4.2 It should be noticed that the universality property for $(\widetilde{p} : \widetilde{M} \to M, \omega_{\widetilde{M}}, \mathfrak{g}, [\widetilde{\mathbf{K}}])$ stated in the theorem does not imply that it is a final object in $\mathfrak{H}$ due to the non-uniqueness of the morphism q.

Remark 4.3 One could consider larger categories than $\mathfrak{H}$ in which case the universality result in Theorem 4.1 would be weaker. For example, if we drop the condition that $p_N : N \to M$ is a covering map in the definition of the objects of $\mathfrak{H}$ then the morphism $q : N \to \widetilde{M}$ is not necessarily a covering map.

Example 4.4 We shall illustrate the difference between the universal Hamiltonian covering and covered spaces by considering the following elementary example. Let $\mathbb{T}^2 = \{(e^{i\theta_1}, e^{i\theta_2})\}$ be the two-torus considered as a symplectic manifold with its area form $\omega := \mathbf{d}\theta_1 \wedge \mathbf{d}\theta_2$ and the circle $S^1 = \{e^{i\phi}\}$ acting canonically on it by $e^{i\phi}\cdot(e^{i\theta_1}, e^{i\theta_2}) := (e^{i(\theta_1+\phi)}, e^{i\theta_2})$. Proposition 2.1 guarantees that the universal covering space $\mathbb{R}^2$ of $\mathbb{T}^2$ can be endowed with the necessary structure to make it the universal Hamiltonian covering space of $(\mathbb{T}^2, \omega)$.

On the other hand, a straightforward computation shows that, in this case, the horizontal vectors in $T(\mathbb{T}^2\times\mathbb{R})$ with respect to the connection α defined in (3.1) are of the form $((a, b), b)$, with $a, b \in \mathbb{R}$. Since any surface $\widetilde{\mathbb{T}^2}_\tau := \{((e^{i\theta_1}, e^{i\theta_2}), \tau + \theta_2) \in \mathbb{T}^2 \times \mathbb{R} \mid \theta_1, \theta_2 \in \mathbb{R}\}$ integrates the horizontal distribution, it is immediately clear that the universal Hamiltonian covered space is given in this example by any of the cylinders $\widetilde{\mathbb{T}^2}_\tau$.

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