Some remarks on a certain class of axisymmetric fluids of differential type

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Abstract

We prove global existence and uniqueness for axisymmetric solutions without swirl for the three-dimensional second grade fluid and the α-Euler equations. The domain considered is either a bounded domain (invariant with respect to rotations about some axis) or the full space $\mathbb{R}^3$. For a certain class of stationary solutions of the α-Euler equations Lyapunov stability is proved.
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1. Introduction

The constitutive laws of non-Newtonian fluids have been introduced in order to express some features that cannot be found in the behavior of a standard Newtonian fluid. Some of these anomalous features are the ability to shear, to thin or thicken, to creep, to relax stresses, and the presence of yield stress. Different equations can be used depending on what unusual property must be modeled.

We consider in this paper the following second grade fluid equations

$$\partial_t v - \nu \Delta u + u \cdot \nabla v + \sum_j v_j \nabla u_j = -\nabla p, \quad v = u - \alpha \Delta u, \qquad \text{div}\, u = 0, \quad u = 0 \text{ on } \partial\Omega, \tag{1}$$

where $\nu \geq 0$ is the viscosity, $\alpha > 0$ is a material coefficient, Ω is an open set of $\mathbb{R}^3$, and (u, p) represent the velocity and the pressure of the fluid, the unknowns of the system.

This fluid model belongs to the particular class of non-Newtonian fluids given by the fluids of grade n. The constitutive laws of these fluids have been introduced by Rivlin and Ericksen [29] in 1955. Three of these models are well-known, corresponding to the cases $n = 1, 2$ or 3. If $n = 1$ then we obtain the (Newtonian) Navier–Stokes equations; we study in this paper the case $n = 2$.

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In the analysis of second grade fluids due to Dunn and Fosdick [9] (see also [13]), the following constitutive law is considered:

$$T = -pI + \nu A + \alpha(\dot{A} + AL + L^T A - A^2), \quad A = L + L^T, \;\; L = \nabla u,$$

where the dot denotes the material derivative. As a consequence, it is shown that the velocity field must obey system (1).

Concerning the original physical meaning of the second grade fluid equation, we mention that the coefficient α represents the elastic response of the fluid. These fluids can also be interpreted as having short memory represented by α. It is also interesting to note that at least two other completely different physical interpretations have been found for this equation. Fokas and Fuchssteiner [12], and independently Camassa and Holm [5], proposed a shallow water model obeying the following equation:

$$u_t - u_{xxt} + 2ku_x = -3uu_x + 2u_x u_{xx} + uu_{xxx}. \tag{2}$$

On the other hand, Camassa and Holm [5] and Kouranbaeva [21] showed that the case $k = 0$ of (2) has the interesting geometrical interpretation of being the spatial representation of the geodesic spray on the diffeomorphism group endowed with the right invariant H^1 metric. Misiolek [26] treated the case $k \neq 0$ and showed that (2) is the spatial representation of the geodesic spray of the right invariant H^1 metric on the Bott–Virasoro group. This geometric point of view was then used by Holm et al. [17,18] to generalize these equations to higher dimensions for incompressible flows. The resulting equation is now known as α-Euler (or $LAE - \alpha$) since the same authors show that a special averaging procedure in the standard Euler equations yields the same equation. What is more astonishing is that these equations correspond to the vanishing viscosity case of the second grade fluid equations (1).

In addition, Oliver and Shkoller [27] proved that the α-Euler model corresponds exactly to a regularization of the point vortex algorithm for ideal bidimensional hydrodynamics, the Chorin vortex blob method, with a particular choice of the cut-off function.

We conclude these comments on the physical importance of second grade fluids by noting that these equations are connected to turbulence theory. A discussion on this subject can already be found in a paper by Rivlin [28]. Recently, starting from the existing α-models (α-Euler and second grade fluids), Foias et al. [10] and Holm [16], proposed a new viscous α-model, called α-Navier–Stokes, where the viscosity term $-\nu \Delta u$ in (1) is replaced by $-\nu \Delta(u - \alpha \Delta u)$. As pointed out by these authors, the α-Navier–Stokes equations are related to large eddy simulation turbulence modeling. From the mathematical point of view, this change in the viscosity term implies a stronger dissipation compared to the second grade fluids and allows to prove global existence in 3D without any smallness assumption.

Axisymmetric flows are physically relevant particular cases of three-dimensional flows for which the fluid velocity is assumed to have cylindrical symmetry and is therefore of the form

$$u(x) = u^r(r, x_3)e_r + u^\theta(r, x_3)e_\theta + u^3(r, x_3)e_3,$$

where e_r, e_θ, and e_3 denote the standard orthonormal cylindrical coordinate system

$$e_r = \left(\frac{x_1}{r}, \frac{x_2}{r}, 0\right), \qquad e_\theta = \left(\frac{x_2}{r}, -\frac{x_1}{r}, 0\right), \qquad e_3 = (0, 0, 1), \qquad r = (x_1^2 + x_2^2)^{1/2},$$

and we assumed that the axis of rotation is $\mathbb{R}(0, 0, 1)$. If the swirl velocity u^θ vanishes, then the flow is said to be axisymmetric without swirl.

Several authors have considered this type of flow for the Euler and Navier–Stokes equations. Since the velocity depends only on two variables and both the Euler and Navier–Stokes equations are globally well-posed in

dimension two, it is natural to expect global well-posedness for axisymmetric flows. However, for the Euler equations, global well-posedness is known only for the axisymmetric case *without swirl*, see [24,23,32,30,31]. For the Navier–Stokes equations, the smoothing effect of the viscosity allows to estimate the swirl velocity, which implies global well-posedness even in the case with swirl, see [8,14,15,20].

Our aim is to prove a similar global well-posedness result for axisymmetric second grade fluids. As far as well-posedness results are concerned, these fluids bear a certain similarity with the Navier–Stokes equations: global well-posedness holds in 2D for large data and in 3D for small data, see [3,6,7,22,25]. Nevertheless, we would like to point out that, unlike the Navier–Stokes case, the viscosity term is not regularizing. Indeed, this term has the good sign but is of the same order as the time-derivative term. Even though this allows to prove global existence for small data, it does not seem to be very useful in the case of large data. Therefore, as far as axisymmetric large solutions are concerned, the second grade fluid equations resemble more the Euler equations than the Navier–Stokes equations. This paper shows global well-posedness only for axisymmetric solutions *without swirl*. The authors have tried to consider the case with swirl, but ran into the same type of problem as encountered in the standard Euler equations. Even though the swirl velocity verifies an equation from which the pressure is missing, the estimates on this equation are not good enough to couple with the estimate on the remaining part of the velocity; one would need an estimate for the uniform norm of the gradient of the swirl velocity and this could not be achieved from the swirl velocity's equation.

We state two different theorems for the cases Ω bounded and $\Omega = \mathbb{R}^3$ as the hypothesis is different. The domain Ω is assumed to be invariant with respect to rotations about the $\mathbb{R}(0, 0, 1)$-axis. We prove the following theorem.

Theorem 1 (Ω bounded). *Let Ω be a bounded smooth axisymmetric domain of $\mathbb{R}^3$. Suppose that the initial velocity u_0 is divergence free and axisymmetric without swirl, belongs to $H^3(\Omega)$, vanishes on the boundary, and $\operatorname{curl}(v_0)/r \in L^2(\Omega)$. Then there exists a unique global H^3 solution of system* (1).

Theorem 2 ($\Omega = \mathbb{R}^3$). *Consider* Eq. (1) *in $\mathbb{R}^3$. Suppose that the initial velocity is divergence free and axisymmetric without swirl, belongs to H^3, that $\operatorname{curl}(v_0)/r \in L^2(\mathbb{R}^3)$, and that $\operatorname{curl}(v_0) \in L^p(\mathbb{R}^3)$ for some $p \in [1, 2)$. Then there exists a unique global H^3 solution of system* (1) *in $\mathbb{R}^3$.*

We complete these global well-posedness results by showing that certain stationary solutions of this system for vanishing viscosity are Lyapunov stable. This is done in the same spirit as [1,2], see also [4,23]. We deliberately toned down the geometric aspect of this method (as presented in [19]) and formulated everything directly in order to match the analytic character of the previous sections. The precise statement of this stability result requires some additional notations; for this reason we prefer to formulate it later in Theorem 3 of Section 5.

The plan of the article is the following. We first give some calculations that will be used in the sequel. The next two sections treat the case Ω bounded and $\Omega = \mathbb{R}^3$, respectively. The next section proves the Lyapunov stability. We conclude the paper with an Appendix A that contains a regularity theorem whose direct application improves some existing results in the literature.

2. Some calculations

It is an easy computation to show that system (1) is rotation invariant, i.e. for any rotation matrix Q and solution (u, p), the couple $(\tilde{u}, \tilde{p})$, where $\tilde{u}(t, x) = Q^T u(t, Qx)$ and $\tilde{p}(t, x) = p(t, Qx)$ is again a solution. By uniqueness of solutions we deduce that a solution with axisymmetric initial data stays axisymmetric, i.e. the solution u must be of the form

$$u = u^r(r, x_3, t)e_r + u^\theta(r, x_3, t)e_\theta + u^3(r, x_3, t)e_3.$$

We therefore obtain a problem which is almost bidimensional with the usual known difficulties that appear in the vicinity of the axis or when r is large.

We will consider in the following the axisymmetric case *without swirl*, i.e. we will assume that the swirl velocity u^θ vanishes. Note that if the initial velocity is without swirl, it will stay like that; this follows from the fact that the equation for the swirl velocity has no pressure term.

To simplify the writing, we redefine

$$u(x, t) = f(r, x_3, t)e_r + g(r, x_3, t)e_3. \tag{3}$$

We first consider the case $\nu = 0$ and indicate afterwards how to adjust the proofs to the case $\nu > 0$.

In the following, all vector fields are regarded as three-dimensional vector fields and the derivatives $\partial_i = \partial/\partial x_i$, ∇, and Δ refer to the spatial variables (x_1, x_2, x_3). All norms are considered to be taken with respect to these variables unless otherwise specified.

A repeated use of the formula $\partial_i f(r, x_3) = (x_i/r)\partial_r f(r, x_3), i = 1, 2$, shows that the vorticity of u can be expressed in the form

$$\omega(u) = (\partial_2 u_3 - \partial_3 u_2, \partial_3 u_1 - \partial_1 u_3, \partial_1 u_2 - \partial_2 u_1) = \tilde{\omega}(r, x_3)(x_2, -x_1, 0), \tag{4}$$

where

$$\tilde{\omega}(r, x_3) = \frac{\partial_r g - \partial_3 f}{r}.$$

More important is the curl of v

$$\omega(v) = \omega(u) - \alpha\Delta\omega(u) = (\tilde{\omega} - \alpha\Delta\tilde{\omega} - \frac{2\alpha}{r}\partial_r\tilde{\omega})(x_2, -x_1, 0) = \check{\omega}(r, x_3)(x_2, -x_1, 0), \tag{5}$$

where

$$\check{\omega} = \tilde{\omega} - \alpha\Delta\tilde{\omega} - \frac{2\alpha}{r}\partial_r\tilde{\omega}.$$

We now show that $\check{\omega}$ verifies a transport equation. We start with the equation for $\omega(v)$ which is well-known

$$\partial_t\omega(v) = \omega(v) \cdot \nabla u - u \cdot \nabla\omega(v). \tag{6}$$

According to (5), the left-hand side is equal to

$$\partial_t\omega(v) = \partial_t\check{\omega}(x_2, -x_1, 0) = r\partial_t\check{\omega}e_\theta,$$

a multiple of e_θ. It is therefore sufficient to determine the coefficient of e_θ on the right-hand side of (6). First, the term $\omega(v) \cdot \nabla u$ can be written as

$$\omega(v) \cdot \nabla u = (\text{multiples of } e_r \text{ and } e_3) + \check{\omega}fe_\theta. \tag{7}$$

Some simple calculations also show that the last term of (6) can expressed as

$$u \cdot \nabla\omega(v) = (ru \cdot \nabla\check{\omega} + \check{\omega}f)e_\theta. \tag{8}$$

We now deduce from (6)–(8) that $\check{\omega}$ is transported by the velocity u:

$$\partial_t\check{\omega} + u \cdot \nabla\check{\omega} = 0. \tag{9}$$

Remark 1. In the previous equation, the derivatives are with respect to the variables x_1, x_2, and x_3. It is possible to obtain a transport equation in the variables r and x_3:

$$\partial_t \check{\omega} + f \partial_r \check{\omega} + g \partial_3 \check{\omega} = 0,$$

but this equation is less interesting as the vector field (f, g) is not divergence free in the variables (r, x_3), i.e. $\partial_r f + \partial_3 g \neq 0$.

Remark 2. As u is divergence free, the transport Eq. (9) implies the conservation of all L^p norms of $\check{\omega}$ (and therefore of $\omega(v)/r$), $p \in [1, \infty]$.

Remark 3. The hypothesis made on u in the standard well-posedness theory of second grade fluids is $u \in H^3(\Omega)$. However, this may not be sufficient in our case, depending on the position of the domain with respect to the axis. More precisely:

- if $r \to 0$ (i.e. the domain intersects the axis), the sole hypothesis $u_0 \in H^3(\Omega)$ does not imply that $\omega(v_0)/r \in L^p$ for some p, so Remark 2 is difficult to use;
- if $r \to \infty$ (i.e. the domain is not bounded in the radial direction) we have that $\omega(v_0)/r \in L^2$ which implies that $\omega(v)/r \in L^2$, but we need instead the control of $\|\omega(v)\|_{L^2}$.

Some additional hypotheses will be required in order to prove the global existence of H^3 solutions. The cases Ω bounded and $\Omega = \mathbb{R}^3$ will be treated separately.

3. The case of a bounded domain

We prove in this section Theorem 1. Since $u_0 \in H^3(\Omega)$, we know by standard results (see, for instance, [6]) that a local H^3 solution exists such that if T^* is the maximal time existence and $T^* < \infty$ then

$$\lim_{t \to T^*} \|u(t)\|_{H^3(\Omega)} = +\infty.$$

We prove that the H^3 norm of u cannot blow up in finite time. This will imply the global existence of the solution.

Let us start with the case of vanishing viscosity.

3.1. Case $\nu = 0$

Since we know, by hypothesis, that $|\check{\omega}(0)| = |\omega(v_0)|/r \in L^2(\Omega)$, we deduce from the transport Eq. (9) that $\|\check{\omega}(t)\|_{L^2(\Omega)} = \|\check{\omega}(0)\|_{L^2(\Omega)}$ for all t. But $\|\omega(v)\|_{L^2} = \|r\check{\omega}\|_{L^2} \leq M\|\check{\omega}\|_{L^2}$, where $M = \sup_{x \in \Omega} \sqrt{x_1^2 + x_2^2} < \infty$. Since the H^1 norm of the velocity is bounded in time, using Corollary A.1 and the fact that $\omega(v)$ is bounded in $L^2(\Omega)$ we get that u stays bounded in $H^3(\Omega)$. This completes the proof in the case $\nu = 0$.

3.2. Case $\nu > 0$

If $\nu \neq 0$, then Eq. (6) for $\omega(v)$ is no longer valid since it contains the additional viscosity term

$$-\nu \Delta \omega(u) = \frac{\nu}{\alpha}(\omega(v) - \omega(u)) = \frac{\nu}{\alpha}(\check{\omega} - \tilde{\omega})(x_2, -x_1, 0)$$

on the right-hand side (relations (4) and (5) were used). The transport equation for $\check{\omega}$ is modified accordingly and becomes

$$\partial_t \check{\omega} + \frac{\nu}{\alpha}(\check{\omega} - \tilde{\omega}) + u \cdot \nabla_x \check{\omega} = 0.$$

Taking the L^2 scalar product of this equation with $\check{\omega}$ gives

$$0 = \frac{1}{2}\partial_t \|\check{\omega}\|_{L^2}^2 + \frac{\nu}{\alpha} \int_\Omega (\check{\omega} - \tilde{\omega})\check{\omega} \, dx = \frac{1}{2}\partial_t \|\check{\omega}\|_{L^2}^2 + \frac{\nu}{2\alpha} \int_\Omega (\check{\omega}^2 + (\tilde{\omega} - \check{\omega})^2 - \tilde{\omega}^2) \, dx.$$

We deduce that

$$\partial_t \|\check{\omega}\|_{L^2}^2 + \frac{\nu}{\alpha}\|\check{\omega}\|_{L^2}^2 \leq \frac{\nu}{\alpha}\|\tilde{\omega}\|_{L^2}^2. \tag{10}$$

We know by standard H^1 estimates that the H^1 norm of u is bounded in time, i.e. the L^2 norm of $\omega(u)$ is bounded in time. Unfortunately, the right-hand side of (10) is not equal to $\|\omega(u)\|_{L^2}$ but to $\|\tilde{\omega}\|_{L^2}^2 = \|\omega(u)/r\|_{L^2}^2$. Nevertheless, it is possible to eliminate this annoying r by adding a derivative that can be still controlled in terms of the H^1 norm and the left-hand side of (10). More precisely, a straightforward calculation shows that

$$\partial_1 \omega(u) = \frac{x_1}{r}\partial_r \tilde{\omega}[x_2, -x_1, 0] + \tilde{\omega}[0, -1, 0], \qquad \partial_2 \omega(u) = \frac{x_2}{r}\partial_r \tilde{\omega}[x_2, -x_1, 0] + \tilde{\omega}[1, 0, 0],$$

so that

$$x_2 \partial_1 \omega(u) - x_1 \partial_2 \omega(u) = \tilde{\omega}[-x_1, -x_2, 0].$$

We further deduce that

$$|\tilde{\omega}| = \left| \frac{x_2}{r}\partial_1 \omega(u) - \frac{x_1}{r}\partial_2 \omega(u) \right|.$$

This immediately shows that $|\tilde{\omega}| \leq |\partial_1 \omega(u)| + |\partial_2 \omega(u)|$ so $\|\tilde{\omega}\|_{L^2} \leq C\|\nabla \omega(u)\|_{L^2} \leq C\|u\|_{H^2}$. Relation (10) now becomes

$$\partial_t \|\check{\omega}\|_{L^2}^2 + \frac{\nu}{\alpha}\|\check{\omega}\|_{L^2}^2 \leq C\|u\|_{H^2}^2.$$

Gronwall's lemma implies

$$\|\check{\omega}(t)\|_{L^2}^2 \leq \|\check{\omega}(0)\|_{L^2}^2 + C \sup_{[0,t]} \|u\|_{H^2}^2,$$

so

$$\sup_{[0,t]} \|\check{\omega}(t)\|_{L^2}^2 \leq C + C \sup_{[0,t]} \|u\|_{H^2}^2.$$

Since the domain is bounded, we can bound $\|\omega(v)\|_{L^2} = \|r\check{\omega}\|_{L^2} \leq C\|\check{\omega}\|_{L^2}$. Combining this with Corollary A.1 we deduce that

$$\|u\|_{H^3}^2 \leq C\|u\|_{H^1}^2 + C\|\check{\omega}\|_{L^2}^2.$$

As $\|u\|_{H^1}$ is bounded in time, we get

$$\sup_{[0,t]} \|u\|_{H^3}^2 \leq C + C \sup_{[0,t]} \|u\|_{H^2}^2,$$

which implies by interpolation that

$$\sup_{[0,t]} \|u\|_{H^3}^2 \le C + C \sup_{[0,t]} \|u\|_{H^1} \|u\|_{H^3} \le C + \tfrac{1}{2} \sup_{[0,t]} \|u\|_{H^3}^2 + C \sup_{[0,t]} \|u\|_{H^1}^2.$$

Using again that the H^1 norm of u is bounded in time we finally get that the H^3 norm of u is bounded. This completes the proof of Theorem 1.

4. The case of $\mathbb{R}^3$

The aim of this section is to prove Theorem 2.

As in the case of a bounded domain, we first consider the vanishing viscosity case.

4.1. Case $v = 0$

We can assume without loss of generality that $p > 3/2$. Indeed, from the hypothesis $u_0 \in H^3$ we obtain that $\omega(v_0) \in L^2 \cap L^p$, so, by interpolation, $\omega(v_0)$ belongs to any intermediate space L^r, $r \in [p, 2]$.

The same argument as in the bounded domain case shows that $\|\check{\omega}(t)\|_{L^2} = \|\check{\omega}(0)\|_{L^2}$, so that $\|\omega(v)/r\|_{L^2}$ is bounded. Next, we estimate the L^p norm of $\omega(v)$, i.e. the norm of $r\check{\omega}$. Multiplying the transport Eq. (9) for $\check{\omega}$ by $r^p \check{\omega} |\check{\omega}|^{p-2}$ and integrating gives

$$\int \partial_t \check{\omega} \check{\omega} |\check{\omega}|^{p-2} r^p \, \mathrm{d}x + \int u \cdot \nabla \check{\omega} \check{\omega} |\check{\omega}|^{p-2} r^p \, \mathrm{d}x = 0,$$

that is

$$\partial_t \int |\check{\omega}|^p r^p \, \mathrm{d}x = -\int u \cdot \nabla(|\check{\omega}|^p) r^p \, \mathrm{d}x = \int u \cdot \nabla(r^p) |\check{\omega}|^p \, \mathrm{d}x = p \int u \cdot e_r r^{p-1} |\check{\omega}|^p \, \mathrm{d}x$$

$$= p \int f r^{p-1} |\check{\omega}|^p \, \mathrm{d}x, \tag{11}$$

where we used that $\nabla(r^p) = p r^{p-1} e_r$ and $u \cdot e_r = f$ (according to (3)). We infer that

$$\partial_t \int |\omega(v)|^p \, \mathrm{d}x = p \int \frac{f}{r} |\omega(v)|^p \, \mathrm{d}x. \tag{12}$$

If f/r would be bounded then we would immediately get bounds on $\|\omega(v)\|_{L^p}$. We do not know that f/r is bounded, but we know that f is bounded in a weighted H^1 space and depends on two variables only. By Sobolev embeddings we can therefore obtain estimates on weighted L^q norms of f for all $q < \infty$. More precisely, we prove the following lemma.

Lemma 1. *For all $q \in [2, +\infty)$ there exists a constant $C(q)$ such that*

$$\|f r^{1/2 - 1/q}\|_{L^q(\mathbb{R}^3)} \le C(q), \quad \forall t \ge 0.$$

Proof of Lemma 1. Since $u = f e_r + g e_3$ we deduce that $|u| \ge |f|$ so

$$\|u\|_{L^2}^2 = \int |u|^2 \, \mathrm{d}x \ge \int |f(r, x_3)|^2 \, \mathrm{d}x = 2\pi \int r |f(r, x_3)|^2 \, \mathrm{d}r \, \mathrm{d}x_3,$$

that is,

$$\int r|f(r, x_3)|^2 \, \mathrm{d}r \, \mathrm{d}x_3 \le C. \tag{13}$$

On the other hand, we have by explicit computation that

$$\partial_1 u = \frac{x_1}{r} \partial_r f e_r + f \frac{x_2}{r^2} e_\theta + \frac{x_1}{r} \partial_r g e_3, \qquad \partial_2 u = \frac{x_2}{r} \partial_r f e_r - f \frac{x_1}{r^2} e_\theta + \frac{x_2}{r} \partial_r g e_3, \qquad \partial_3 u = \partial_3 f e_r + \partial_3 g e_3.$$

As $\{e_r, e_\theta, e_3\}$ form an orthonormal basis, we infer that

$$|\nabla u|^2 = |\partial_r f|^2 + \frac{|f|^2}{r^2} + |\partial_3 f|^2 + |\partial_r g|^2 + |\partial_3 g|^2, \tag{14}$$

so that

$$\int |\nabla u|^2 \, \mathrm{d}x \ge \int \left(|\nabla_{r,x_3} f|^2 + \frac{|f|^2}{r^2} \right) \, \mathrm{d}x = 2\pi \int \left(r|\nabla_{r,x_3} f|^2 + \frac{|f|^2}{r} \right) \, \mathrm{d}r \, \mathrm{d}x_3.$$

Using that the H^1 norm of u is bounded in time as well as relation (13), we deduce that

$$\int \left(r|\nabla_{r,x_3} f|^2 + r|f|^2 + \frac{|f|^2}{r} \right) \, \mathrm{d}r \, \mathrm{d}x_3 \le C. \tag{15}$$

Since $\nabla_{r,x_3}(f\sqrt{r}) = \sqrt{r}\nabla_{r,x_3} f + f[1/2\sqrt{r}, 0]$, relation (15) implies that

$$\|f\sqrt{r}\|_{H^1(\mathbb{R}^2)} \le C.$$

By the (bidimensional) Sobolev embedding $H^1 \hookrightarrow L^q$ for all $q \in [2, \infty)$, we deduce that $\|f\sqrt{r}\|_{L^q(\mathbb{R}^2)} \le C(q)$ for all $q \in [2, \infty)$. Going back to x coordinates implies the conclusion of Lemma 1:

$$\|fr^{1/2-1/q}\|_{L^q(\mathbb{R}^3)}^q = \int |f|^q r^{q/2-1} \, \mathrm{d}x = 2\pi \int |f|^q r^{q/2} \, \mathrm{d}r \, \mathrm{d}x_3 = \|f\sqrt{r}\|_{L^q(\mathbb{R}^2)}^q \le C(q).$$

$$\square$$

We now go back to relation (12), we fix $q > 2$, we write

$$\frac{f}{r}|\omega(v)|^p = fr^{1/2-1/q} \left(\frac{|\omega(v)|}{r} \right)^{3/2-1/q} |\omega(v)|^{p+1/q-3/2},$$

and we use Hölder's estimate with the triple $(q, 4q/(3q-2), 4q/(q-2))$ to deduce that

$$\partial_t \|\omega(v)\|_{L^p}^p \le C\|fr^{1/2-1/q}\|_{L^q} \left\| \frac{\omega(v)}{r} \right\|_{L^2}^{3/2-1/q} \|\omega(v)\|_{L^{(4pq+4-6q)/(q-2)}}^{p+1/q-3/2}.$$

On the right-hand side, the first term is bounded according to Lemma 1. The second term is also bounded according to the observations at the beginning of the proof. As for the last term, we simply choose q such that $(4pq+4-6q)/(q-2) = p$, i.e. $q = 2(2+p)/3(2-p)$. With this choice we get

$$\partial_t \|\omega(v)\|_{L^p}^p \le C\|\omega(v)\|_{L^p}^{p+1/q-3/2}. \tag{16}$$

After integration

$$\|\omega(v(t))\|_{L^p}^{3/2-1/q} \le \|\omega(v_0)\|_{L^p}^{3/2-1/q} + Ct.$$

Therefore, $\|\omega(v)\|_{L^p}$ is controlled.

Since $p < 3$ we can apply the Hardy–Littlewood–Sobolev inequality:

$$\|v\|_{L^{p^*}} \leq C\|\omega(v)\|_{L^p},$$

where $p^* = 3p/(3-p)$. This implies that $\|v\|_{L^{p^*}}$ is also controlled. Standard elliptic regularity results for the operator $1 - \alpha\Delta$ implies that $\|u\|_{W^{2,p^*}} \leq C\|v\|_{L^{p^*}}$ is also controlled. Finally, from the relation $p^* > 3$ (that follows from the assumption $p > 3/2$) we have the embedding $W^{1,p^*} \hookrightarrow L^\infty$ so we obtain that $\|\nabla u\|_{L^\infty}$ is controlled. Recalling relation (14) we further infer that $\|f/r\|_{L^\infty}$ is controlled.

Now, relation (12) holds also for $p = 2$. Therefore

$$\partial_t \|\omega(v)\|_{L^2}^2 \leq 2 \left\| \frac{f}{r} \right\|_{L^\infty} \|\omega(v)\|_{L^2}^2 \leq C(t)\|\omega(v)\|_{L^2}^2,$$

where the dependence of $C(t)$ on t can be made explicit. Gronwall's lemma now gives

$$\|\omega(v(t))\|_{L^2}^2 \leq \|\omega(v_0)\|_{L^2}^2 \exp^{\int_0^t C(\tau)\,\mathrm{d}\tau}.$$

This is the control of $\|\omega(v)\|_{L^2}$ and suffices to complete the proof of Theorem 2 in the case $\nu = 0$.

4.2. Case $\nu \neq 0$

We first show that $\|\omega(v)/r\|_{L^2}$ is still bounded by proving that the viscosity term can be ignored. Recall the equation for $\check{\omega}$:

$$\partial_t \check{\omega} + \frac{\nu}{\alpha}(\check{\omega} - \tilde{\omega}) + u \cdot \nabla_x \check{\omega} = 0, \tag{17}$$

where

$$\check{\omega} = \tilde{\omega} - \alpha\Delta_x\tilde{\omega} - \frac{2\alpha}{r}\partial_r\tilde{\omega}.$$

Multiplying (17) by $\check{\omega}$, the viscosity term yields

$$\int \frac{\nu}{\alpha}(\check{\omega} - \tilde{\omega})\check{\omega}\,\mathrm{d}x = \frac{\nu}{\alpha}\int (\check{\omega} - \tilde{\omega})^2\,\mathrm{d}x + \frac{\nu}{\alpha}\int (\check{\omega} - \tilde{\omega})\tilde{\omega}\,\mathrm{d}x = \frac{\nu}{\alpha}\|\check{\omega} - \tilde{\omega}\|_{L^2}^2 - \nu \int \tilde{\omega}\left(\Delta_x\tilde{\omega} + \frac{2}{r}\partial_r\tilde{\omega}\right)\mathrm{d}x$$

$$= \frac{\nu}{\alpha}\|\check{\omega} - \tilde{\omega}\|_{L^2}^2 - \nu \int \tilde{\omega}\Delta_x\tilde{\omega}\,\mathrm{d}x - 2\nu \int \frac{\tilde{\omega}\partial_r\tilde{\omega}}{r}\,\mathrm{d}x.$$

A trivial integration by parts shows that

$$-\nu \int \tilde{\omega}\Delta_x\tilde{\omega}\,\mathrm{d}x = \nu\|\nabla\tilde{\omega}\|_{L^2}^2 \geq 0.$$

Also

$$\int \frac{\tilde{\omega}\partial_r\tilde{\omega}}{r}\,\mathrm{d}x = 2\pi \int \tilde{\omega}\partial_r\tilde{\omega}\,\mathrm{d}r\,\mathrm{d}x_3 = \pi \int \partial_r(\tilde{\omega})^2\,\mathrm{d}r\,\mathrm{d}x_3 = 0,$$

so

$$\int \frac{\nu}{\alpha}(\check{\omega} - \tilde{\omega})\tilde{\omega}\,\mathrm{d}x = \frac{\nu}{\alpha}\|\check{\omega} - \tilde{\omega}\|_{L^2}^2 + \nu\|\nabla\tilde{\omega}\|_{L^2}^2 \geq 0.$$

This means that the viscosity term may be ignored and shows that $\check{\omega}$ is bounded in L^2.

It remains to prove that we still have L^p estimates on $\omega(v)$ (i.e. the analogue of relation (16)). Relation (11) now contains the additional term

$$I = \frac{\nu}{\alpha} \int (\check{\omega} - \tilde{\omega}) r^p \check{\omega} |\check{\omega}|^{p-2} \, \mathrm{d}x.$$

We estimate I in the following way:

$$|I| \leq \frac{\nu}{\alpha} \int (|\check{\omega}| + |\tilde{\omega}|) r^p |\check{\omega}|^{p-1} \, \mathrm{d}x = \frac{\nu}{\alpha} \int (|\omega(u)| + |\omega(v)|) |\omega(v)|^{p-1} \, \mathrm{d}x$$

$$= \frac{\nu}{\alpha} \left(\|\omega(v)\|_{L^p}^p + \int |\omega(u)| |\omega(v)|^{p-1} \right) \mathrm{d}x.$$

Next, we use Hölder's inequality

$$|I| \leq \frac{\nu}{\alpha} \|\omega(v)\|_{L^p}^p + \frac{\nu}{\alpha} \|\omega(u)\|_{L^p} \|\omega(v)\|_{L^p}^{p-1},$$

the trivial estimate $\|\omega(u)\|_{L^p} \leq \|\omega(u)\|_{W^{2,p}}$, and the L^p regularity result for the operator $1 - \alpha \Delta$ that says that $\|\omega(u)\|_{W^{2,p}} \leq C \|\omega(v)\|_{L^p}$, to finally deduce that

$$|I| \leq C \|\omega(v)\|_{L^p}^p.$$

With this additional term, the analogue of relation (16) is now

$$\partial_t \|\omega(v)\|_{L^p}^p \leq C(\|\omega(v)\|_{L^p}^p + \|\omega(v)\|_{L^p}^{p+1/q-3/2}).$$

Since one can easily check that $p + 1/q - 3/2 \in (0, 1)$, an application of Young's inequality yields

$$\partial_t \|\omega(v)\|_{L^p}^p \leq C + C \|\omega(v)\|_{L^p}^p,$$

so, by Gronwall's lemma

$$\|\omega(v(t))\|_{L^p}^p \leq (\|\omega(v_0)\|_{L^p} + Ct) \exp(Ct).$$

This is the control of $\|\omega(v)\|_{L^p}^p$ and starting from here the proof is similar to the vanishing viscosity case.

5. Lyapunov stability of some stationary solutions

We are concerned in this section with the vanishing viscosity case of the second grade fluids, also known as the α-Euler equation:

$$\partial_t v + u \cdot \nabla v + \sum_j v_j \nabla u_j = -\nabla p, \quad v = u - \alpha \Delta u, \ \ \mathrm{div}\, u = 0, \ \ u = 0 \ \ \mathrm{on} \ \ \partial\Omega. \tag{18}$$

We assume in this section that the axisymmetric domain Ω is obtained by rotation from a *simply connected* domain Ω' in the (r, x_3) variables. The domain Ω' is also supposed to be bounded and not intersecting the axis $\{r = 0\}$.

Recall that

$$u = f e_r + g e_3, \quad \mathrm{curl}\, u = \tilde{\omega}(x_2, -x_1, 0), \quad \tilde{\omega} = \frac{\partial_r g - \partial_3 f}{r},$$

$$v = \check{f} e_r + \check{g} e_3, \quad \mathrm{curl}\, v = \check{\omega}(x_2, -x_1, 0), \quad \check{\omega} = \frac{\partial_r \check{g} - \partial_3 \check{f}}{r},$$

and that $\check{\omega}$ is transported by u in the (r, x_3) variables:

$$\partial_t \check{\omega} + f \partial_r \check{\omega} + g \partial_3 \check{\omega} = 0. \tag{19}$$

It is also easy to check that the vector field (rf, rg) is divergence free in the (r, x_3) coordinates. Since we assumed Ω' simply connected, there exists a stream function $\psi(r, x_3)$ vanishing on the boundary such that $(rf, rg) = \nabla^\perp_{r,x_3} \psi$.

From (19) we see that u_e is a stationary solution if and only if $\nabla_{r,x_3} \psi_e$ and $\nabla_{r,x_3} \check{\omega}_e$ are proportional. In fact, a stationary solution verifies

$$\nabla_{r,x_3} \psi_e = -h(\check{\omega}_e) \nabla_{r,x_3} \check{\omega}_e \tag{20}$$

for some function h. In the following theorem we study the stability of such stationary solutions.

Theorem 3 (Lyapunov stability). *Let u_e be a stationary solution of (18) such that there exists a (continuous) function h and positive constants $0 < C_1 < C_2$ such that $C_1 \leq h \leq C_2$ and (20) is valid. Then the solution u_e is Lyapunov stable in the H^3 norm with respect to axisymmetric perturbations.*

Proof. Let ϕ be such that $\phi'' = h$. Note that we have some freedom in the choice of ϕ; namely ϕ' is defined up to a constant. For two axisymmetric (swirl free) vector fields u^1 and u^2 we use the corresponding superscripts to denote the related quantities and we introduce the bilinear functional

$$H(u^1, u^2) = \frac{1}{2} \int_\Omega u^1 \cdot v^2 \, dx.$$

We will also use the functional

$$C(u) = \int_\Omega \phi(\check{\omega}) \, dx.$$

The properties of these functionals are summarized in the following lemma.

Lemma 2. *Let u^1 and u^2 be two divergence free axisymmetric (swirl free) vector fields vanishing on the boundary. The following holds*

$$H(u^1, u^2) = H(u^2, u^1) = \frac{1}{2} \int_\Omega \psi^1 \check{\omega}^2 \, dx \quad and \quad H(u^1, u^1) = \frac{1}{2}(\|u^1\|^2_{L^2} + \alpha \|\nabla u^1\|^2_{L^2}).$$

Moreover, if u is an axisymmetric solution of (18) then $H(u) \overset{\text{def}}{=} H(u, u)$ and $C(u)$ are constant in time.

Proof of Lemma 2. The relations $H(u^1, u^2) = H(u^2, u^1)$ and $H(u^1, u^1) = (1/2)(\|u^1\|^2_{L^2} + \alpha \|\nabla u^1\|^2_{L^2})$ follow from the definition after applying a trivial integration by parts that we will not detail. Next

$$\int_\Omega u^1 \cdot v^2 \, dx = \int_\Omega (f^1, g^1) \cdot (\check{f}^2, \check{g}^2) \, dx = 2\pi \int_{\Omega'} (rf^1, rg^1) \cdot (\check{f}^2, \check{g}^2) \, dr \, dx_3$$

$$= 2\pi \int_{\Omega'} (\nabla^\perp_{r,x_3} \psi^1) \cdot (\check{f}^2, \check{g}^2) \, dr \, dx_3 = 2\pi \int_{\Omega'} \psi^1 \, \text{curl}_{r,x_3}(\check{f}^2, \check{g}^2) \, dr \, dx_3$$

$$= 2\pi \int_{\Omega'} \psi^1 r \check{\omega}^2 \, dr \, dx_3 = \int_\Omega \psi^1 \check{\omega}^2 \, dx.$$

The fact that, for a solution u, $H(u)$ is constant in time simply follows from the H^1 estimates for Eq. (18): the norm $\|u\|^2_{L^2} + \alpha \|\nabla u\|^2_{L^2}$ is constant in time. It remains to prove that $C(u)$ is also constant in time. This follows from the

following sequence of calculations:

$$\partial_t C(u) = \int_\Omega \phi'(\check\omega)\partial_t\check\omega\,\mathrm{d}x = -\int_\Omega \phi'(\check\omega)u\cdot\nabla\check\omega\,\mathrm{d}x = -\int_\Omega u\cdot\nabla\phi(\check\omega)\,\mathrm{d}x = \int_\Omega \phi(\check\omega)\,\mathrm{div}\,u\,\mathrm{d}x = 0. \qquad \square$$

Let now u_e be a stationary solution as in Theorem 3 and let u an arbitrary axisymmetric solution. Consider the following quantity

$$E(t) = (H+C)(u) - (H+C)(u_e) - \int_\Omega [\psi_e + \phi'(\check\omega_e)](\check\omega - \check\omega_e)\,\mathrm{d}x.$$

We claim that E is in fact constant in time. Indeed, $(H+C)(u)$ is constant in time as a consequence of Lemma 2. Also, from (20) we get that $\nabla(\psi_e + \phi'(\check\omega_e)) = 0$ which implies that $\psi_e + \phi'(\check\omega_e) = \text{Const}$. This constant can be made equal to 0 by adding a suitable constant to ϕ'. This means that the last term in $E(t)$ vanishes and this now implies that E is constant in time.

On the other hand, we can also write E under the form

$$E = \underbrace{C(u) - C(u_e) - \int_\Omega \phi'(\check\omega_e)(\check\omega - \check\omega_e)\,\mathrm{d}x}_{E_1} + \underbrace{H(u) - H(u_e) - \int_\Omega \psi_e(\check\omega - \check\omega_e)\,\mathrm{d}x}_{E_2}.$$

From the definition of the functional C we see that

$$E_1 = \int_\Omega \phi(\check\omega) - \phi(\check\omega_e) - \phi'(\check\omega_e)(\check\omega - \check\omega_e)\,\mathrm{d}x = \frac{1}{2}\int_\Omega \phi''(\xi)|\check\omega - \check\omega_e|^2\,\mathrm{d}x$$

for some ξ between $\check\omega$ and $\check\omega_e$. From the hypothesis we know that $\phi'' = h$ is of the order of a constant, so we deduce that $E_1 \simeq \|\check\omega - \check\omega_e\|_{L^2}^2 = \|(1/r)(\omega(v) - \omega(v_e))\|_{L^2}^2$. The assumptions made on the domain Ω imply that r is bounded from below and above by two positive constants. Therefore $E_1 \simeq \|\omega(v) - \omega(v_e)\|_{L^2}^2$.

As for E_2, we use the definition, the symmetry and the bilinearity of H to write

$$E_2 = H(u,u) - H(u_e,u_e) - 2H(u_e, u-u_e) = H(u-u_e, u-u_e) \simeq \|u-u_e\|_{H^1}^2.$$

Using Corollary A.1 this finally implies that

$$E = E_1 + E_2 \simeq \|u-u_e\|_{H^1}^2 + \|\omega(v) - \omega(v_e)\|_{L^2}^2 \simeq \|u-u_e\|_{H^3}^2.$$

Since we know that E is constant in time, we obtain that $\|u(t) - u_e\|_{H^3}^2 \simeq \|u(0) - u_e\|_{H^3}^2$. This completes the proof of Theorem 3.

Acknowledgements

Appendix A

The aim of this appendix is to prove the following regularity result.

Proposition A.1. *Let Ω be a bounded smooth domain (not necessarily simply connected). If u is a $H^2(\Omega)$ divergence free vector field vanishing on the boundary such that $\mathrm{curl}(u - \alpha \Delta u) \in L^2(\Omega)$, then $u \in H^3(\Omega)$.*

Proof. In order to show the ideas, we first consider the case of a domain with flat boundary, $\Omega = \{x_3 > 0\}$. The Dirichlet boundary conditions implies that $\partial_1 u|_{\partial\Omega} = \partial_2 u|_{\partial\Omega} = 0$, and the divergence free condition shows that $\partial_3 u_3|_{\partial\Omega} = -(\partial_1 u_1 + \partial_2 u_2)|_{\partial\Omega} = 0$, so $\nabla u_3|_{\partial\Omega} = 0$, i.e. $u_3 \in H_0^2(\Omega)$.

Set $v = u - \alpha \Delta u$. Since v is divergence free, we have that $\Delta v = -\mathrm{curl}\,\mathrm{curl}\,v \in H^{-1}(\Omega)$; in particular $\Delta v_3 \in H^{-1}(\Omega)$. We infer that $\Delta^2 u_3 = (1/\alpha)(\Delta u_3 - \Delta v_3) \in H^{-1}(\Omega)$. But we saw that u_3 as well as its first order derivatives vanish on the boundary, so it satisfies boundary conditions compatible with the bi-Laplacian. The standard regularity theory for the bi-Laplacian now implies that $u_3 \in H^3(\Omega)$. This means that $v_3 \in H^1(\Omega)$ and by trace theorems $v_3|_{\partial\Omega} \in H^{1/2}(\partial\Omega)$, that is, $v \cdot n|_{\partial\Omega} \in H^{1/2}(\partial\Omega)$. Furthermore, we also have that $\mathrm{div}\,v = 0$ and $\mathrm{curl}\,v \in L^2(\Omega)$. Standard regularity results for the curl operator (see, for instance, [11]) imply that $v \in H^1(\Omega)$. Finally, the regularity theory for the Laplacian implies that $u \in H^3(\Omega)$ which is the desired conclusion in this particular case.

We now indicate how to modify this proof in the general case. Let n be the normal vector to the boundary and $\{\tau_1, \tau_2\}$ be an orthonormal basis of the tangent space. We extend these vectors smoothly inside Ω. As above, we have that $\Delta v \in H^{-1}(\Omega)$ and therefore $\Delta^2(u \cdot n) \in H^{-1}(\Omega)$. Now, if we denote by $\mathrm{Col}[\tau_1, \tau_2, n]$, respectively $\mathrm{Row}[\tau_1, \tau_2, n]$, the matrices with columns, respectively rows, equal (in this order) to τ_1, τ_2, n, then we obviously have that

$$\begin{bmatrix} \partial_{\tau_1} \\ \partial_{\tau_2} \\ \partial_n \end{bmatrix} = \mathrm{Row}[\tau_1, \tau_2, n] \begin{bmatrix} \partial_1 \\ \partial_2 \\ \partial_3 \end{bmatrix}.$$

Since the basis $\{\tau_1, \tau_2, n\}$ is orthonormal, the inverse of $\mathrm{Row}[\tau_1, \tau_2, n]$ is $\mathrm{Col}[\tau_1, \tau_2, n]$. Therefore, at the boundary

$$\begin{bmatrix} \partial_1 \\ \partial_2 \\ \partial_3 \end{bmatrix} = \mathrm{Col}[\tau_1, \tau_2, n] \begin{bmatrix} \partial_{\tau_1} \\ \partial_{\tau_2} \\ \partial_n \end{bmatrix}.$$

We deduce that the divergence free condition may be expressed at the boundary in the basis $\{n, \tau_1, \tau_2\}$ under the form

$$\mathrm{Col}[\tau_1, \tau_2, n] \begin{bmatrix} \partial_{\tau_1} \\ \partial_{\tau_2} \\ \partial_n \end{bmatrix} \cdot \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = 0.$$

Expanding the above expression and taking into account that the tangential derivatives of u vanish at the boundary, we end up with the following relation: $n \cdot \partial_n u = 0$. This can also be expressed under the form $\partial_n(u \cdot n) = u \cdot \partial_n n$ at the boundary. As $u \in H^2(\Omega)$, we deduce that $\partial_n(u \cdot n) \in H^{3/2}(\partial\Omega)$. But we also have that $\partial_{\tau_1}(u \cdot n)|_{\partial\Omega} = 0$ and $\partial_{\tau_2}(u \cdot n)|_{\partial\Omega} = 0$ so $\nabla(u \cdot n) \in H^{3/2}(\partial\Omega)$. This last relation plus the facts that $u \cdot n|_{\partial\Omega} = 0$ and $\Delta^2(u \cdot n) \in H^{-1}(\Omega)$, implies by the regularity theory for the bi-Laplacian that $u \cdot n \in H^3(\Omega)$. Taking the Laplacian of this quantity and noting that $\Delta(u \cdot n) - \Delta u \cdot n$ can be expressed in terms of u and first order derivatives of u, we deduce that $\Delta u \cdot n \in H^1(\Omega)$. This means that $v \cdot n \in H^1(\Omega)$ so $v \cdot n|_{\partial\Omega} \in H^{1/2}(\partial\Omega)$. With this information we can conclude, as in the flat boundary case, that $u \in H^3(\Omega)$. □

We get the following corollary.

Corollary A.1. *Let Ω be a bounded smooth domain* (*not necessarily simply connected*) *and u a $H^3(\Omega)$ divergence free vector field vanishing on the boundary. The following quantities are equivalent*:

$$\|u\|_{H^3} \simeq \|u\|_{H^1} + \|\mathrm{curl}(u - \alpha \triangle u)\|_{L^2}.$$

Proof. The bound

$$\|u\|_{H^1} + \|\mathrm{curl}(u - \alpha \triangle u)\|_{L^2} \leq \mathrm{Const.}\|u\|_{H^3}$$

is obvious. In order to prove the converse inequality, first note that the proof of Proposition A.1 actually shows that

$$\|u\|_{H^3} \leq \mathrm{Const.}\|u\|_{H^2} + \mathrm{Const.}\|\mathrm{curl}(u - \alpha \triangle u)\|_{L^2}. \tag{21}$$

We now use the standard interpolation inequality $\|u\|_{H^2} \leq \mathrm{Const.}\|u\|_{H^1}^{1/2}\|u\|_{H^3}^{1/2}$ and estimate in the usual manner $\|u\|_{H^1}^{1/2}\|u\|_{H^3}^{1/2} \leq \varepsilon\|u\|_{H^3} + C(\varepsilon)\|u\|_{H^1}$. Choosing ε small enough and plugging this estimate into relation (21) we get

$$\|u\|_{H^3} \leq \tfrac{1}{2}\|u\|_{H^3} + \mathrm{Const.}\|u\|_{H^1} + \mathrm{Const.}\|\mathrm{curl}(u - \alpha \triangle u)\|_{L^2}.$$

This completes the proof. $\qquad\square$

Corollary A.1 allows to extend the well-posedness results of [6,7] to domains that are not necessarily simply connected. Indeed, the hypothesis of simple connectedness was used in those articles to prove the equivalence of the norms

$$\|u\|_{H^3} \simeq \|\mathrm{curl}(u - \alpha \triangle u)\|_{L^2}.$$

If the domain is not simply connected, we get the same relation but with an additional term $\|u\|_{H^1}$. However, this term is not a problem as it is trivially bounded from the H^1 estimates that are always true. We also refer to [4] where this kind of estimate was used to get well-posedness for second grade fluids equations for not necessarily simply connected domains but with other boundary conditions.

References

[1] V.I. Arnold, Conditions for nonlinear stability of the stationary plane curvilinear flows of an ideal fluid, Dokl. Akad. Nauk. 162 (5) (1965) 773–777.
[2] V.I. Arnold, An a priori estimate in the theory of hydrodynamic stability [English translation], Am. Math. Soc. Trans. 19 (1969) 267–269.
[3] A.V. Busuioc, On second grade fluids with vanishing viscosity, Port. Math. (N.S.) 59 (1) (2002) 47–65.
[4] A.V. Busuioc, T.S. Ratiu, The second grade fluid and averaged Euler equations with Navier-slip boundary conditions, Nonlinearity 16 (3) (2003) 1119–1149.
[5] R. Camassa, D.D. Holm, An integrable shallow water equation with peaked solitons, Phys. Rev. Lett. 71 (11) (1993) 1661–1664.
[6] D. Cioranescu, V. Girault, Weak and classical solutions of a family of second grade fluids, Internat. J. Non-linear Mech. 32 (2) (1997) 317–335.
[7] D. Cioranescu, E.H. Ouazar, Existence and uniqueness for fluids of second grade, in: Nonlinear Partial Differential Equations and Their Applications, Collège de France Seminar, vol. VI, Paris, 1982/1983, Pitman, Boston, MA, 1984, pp. 178–197.
[8] D. Chae, J. Lee, On the regularity of the axisymmetric solutions of the Navier–Stokes equations, Math. Z. 239 (4) (2002) 645–671.
[9] J.E. Dunn, R.L. Fosdick, Thermodynamics, stability, and boundedness of fluids of complexity 2 and fluids of second grade, Arch. Ration. Mech. Anal. 56 (1974) 191–252.
[10] C. Foias, D.D. Holm, E.S. Titi, The Navier–Stokes-alpha model of fluid turbulence. Advances in nonlinear mathematics and science, Physica D 152–153 (2001) 505–519.
[11] C. Foias, R. Temam, Remarques sur les équations de Navier–Stokes stationnaires et les phénomènes successifs de bifurcation, Ann. Sci. Norm. Pisa V (1978) 29–63.

[12] A.S. Fokas, B. Fuchssteiner, Symplectic structures, their Bäcklund transformation and hereditary symmetries, Physica D 4 (1981) 47–66.

[13] R.L. Fosdick, K.R. Rajagopal, Anomalous features in the model of second order fluids, Arch. Ration. Mech. Anal. 70 (2) (1979) 145–152.

[14] I. Gallagher, S. Ibrahim, M. Majdoub, Existence et unicité de solutions pour le système de Navier–Stokes axisymétrique, Commun. Partial Diff. Equ. 26 (5–6) (2001) 883–907.

[15] I. Gallagher, Stability and weak–strong uniqueness for axisymmetric solutions of the Navier–Stokes equations, Diff. Integral Equ. 16 (5) (2003) 557–572.

[16] D.D. Holm, Averaged Lagrangians and the mean effects of fluctuations in ideal fluid dynamics, Physica D 170 (3–4) (2002) 253–286.

[17] D.D. Holm, J.E. Marsden, T.S. Ratiu, Euler–Poincaré models of ideal fluids with nonlinear dispersion, Phys. Rev. Lett. 349 (1998) 4173–4177.

[18] D.D. Holm, J.E. Marsden, T.S. Ratiu, The Euler–Poincaré equations and semidirect products with applications to continuum theories, Adv. Math. 137 (1) (1998) 1–81.

[19] D.D. Holm, J.E. Marsden, T.S. Ratiu, A.D. Weinstein, Nonlinear stability of fluid and plasma equilibria, Phys. Rep. 123 (1985) 1–116.

[20] N. Kim, Remarks for the axisymmetric Navier–Stokes equations, J. Diff. Equ. 187 (2) (2003) 226–239.

[21] S. Kouranbaeva, The Camassa–Holm equation as a geodesic flow on the diffeomorphism group, J. Math. Phys. 40 (2) (1999) 857–868.

[22] S. Kouranbaeva, M. Oliver, Global well-posedness for the averaged Euler equations in two dimensions, Physica D 138 (3–4) (2000) 197–209.

[23] C. Maffei, C. Marchioro, A result of global existence and stability for axisymmetric Euler flows, Bollettino UMI 7 (1994) 101–111.

[24] A. Majda, A. Bertozzi, Vorticity and Incompressible Flow, Cambridge University Press, Cambridge, 2002.

[25] J.E. Marsden, T.S. Ratiu, S. Shkoller, The geometry and analysis of the averaged Euler equations and a new diffeomorphism group, Geom. Func. Anal. 10 (2000) 582–599.

[26] G. Misiolek, A shallow water equation as a geodesic flow on the Bott–Virasoro group, J. Geom. Phys. 24 (3) (1998) 203–208.

[27] M. Oliver, S. Shkoller, The vortex blob method as a second-grade non-Newtonian fluid, Commun. Partial Diff. Equ. 26 (2001) 295–314.

[28] R.S. Rivlin, The relation between the flow of non-Newtonian fluids and turbulent Newtonian fluids, Q. Appl. Math. 15 (1957) 212–215.

[29] R.S. Rivlin, J.L. Ericksen, Stress-deformation relations for isotropic materials, J. Ration. Mech. Anal. 4 (1955) 323–425.

[30] X. Saint-Raymond, Remarks on axisymmetric solutions of the incompressible Euler system, Commun. Partial Diff. Equ. 19 (1–2) (1994) 321–334.

[31] T. Shirota, T. Yanagisawa, Note on global existence for axially symmetric solutions of the Euler system, Proc. Jpn. Acad. Ser. A: Math. Sci. 70 (10) (1994) 299–304.

[32] M.R. Ukhovskii, V.I. Yudovich, Axially symmetric flows of ideal and viscous fluids filling the whole space, Prikl. Mat. Meh. 32 (1968) 59–69 (Russian) [Translated as J. Appl. Math. Mech. 32 (1968) 52–61].