Singular reduction of implicit Hamiltonian systems

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Abstract

This paper develops the theory of singular reduction for implicit Hamiltonian systems admitting a symmetry Lie group. The reduction is performed at a singular value of the momentum map. This leads to a singular reduced topological space which is not a smooth manifold. A topological Dirac structure on this space is defined in terms of a generalized Poisson bracket and a vector space of derivations, both being defined on a set of smooth functions. A corresponding Hamiltonian formalism is described. It is shown that solutions of the original system descend to solutions of the reduced system. Finally, if the generalized Poisson bracket is nondegenerate, then the singular reduced space can be decomposed into a set of smooth manifolds called pieces. The singular reduced system restricts to a regular reduced implicit Hamiltonian system on each of these pieces. The results in this paper naturally extend the singular reduction theory as previously developed for symplectic or Poisson Hamiltonian systems.

keywords: implicit Hamiltonian systems, Dirac structures, symmetry, reduction

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1 Introduction

Consider a symplectic manifold (M, ω) admitting a symmetry Lie group G acting freely and properly on M. Denote by $\mathfrak{g}$ the Lie algebra of G and by $\mathfrak{g}^*$ its dual. Suppose this action admits an equivariant momentum map $P : M \to \mathfrak{g}^*$. In [24] it is shown that at a *regular* value $\mu \in \mathfrak{g}^*$ of the momentum map, the symplectic structure on M naturally reduces to a symplectic structure ω_μ on the reduced manifold $M_\mu = P^{-1}(\mu)/G_\mu$, where G_μ denotes the coadjoint isotropy subgroup of G at μ. Furthermore, the integral curves of a Hamiltonian vector field defined by a G-invariant Hamiltonian $H \in C^\infty(M)^G$ project to integral curves of the reduced Hamiltonian vector field on M_μ associated to the reduced Hamiltonian function $H_\mu \in C^\infty(M_\mu)$. This theory has been generalized in [22] to the case of Poisson manifolds. At a regular value μ of the momentum map, the Poisson bracket $\{\cdot,\cdot\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$ on M descends to a Poisson bracket $\{\cdot,\cdot\}_\mu$ on the reduced phase space M_μ. Again, the Hamiltonian flow defined by a G-invariant Hamiltonian $H \in C^\infty(M)^G$ reduces to a Hamiltonian flow on M_μ corresponding to $H_\mu \in C^\infty(M_\mu)$. We refer to [1, 11, 21, 23, 27, 36] for some excellent presentations of various aspects of this theory as well as several worked examples.

By Sard's Theorem (e.g. [2]), the set of regular values of the momentum map is dense in $\mathfrak{g}^*$. Hence regular reduction can be considered to be the "generic" case of reduction. On the other hand, certain interesting dynamics, such as bifurcation phenomena, may occur at *singular* values of the momentum map. The main difference with regular reduction is that at a singular value μ of the momentum map, the level set $P^{-1}(\mu)$ is, in general, not a manifold. Thus the reduced space M_μ will *not* be a smooth manifold either. A simple example is provided by a spherical pendulum moving with angular momentum zero, i.e. moving in a plane. Another example is the reduction by the gravitational S^1-symmetry of a Lagrange top obtained after *regular* reduction of its internal S^1-symmetry (corresponding to the top's homogeneous mass distribution). In [3, 4, 12, 13, 18, 29, 30, 34] the theory of singular reduction of symplectic and Poisson Hamiltonian systems has been developed. See also [11] for a nice overview and some worked examples (including the spherical pendulum and the Lagrange top). Since the reduced space M_μ is not a manifold, symplectic forms and Hamiltonian vector fields cannot be defined. As a consequence, the reduced

"Hamiltonian dynamics" cannot be written as a system of ordinary differential equations on M_μ as in the regular case. However, since M_μ is a topological space (relative to the natural quotient topology), a reduced Poisson bracket $\{\cdot,\cdot\}_\mu$ on the space of smooth functions $C^\infty(M_\mu)$ may still be defined. This bracket induces a Hamiltonian formalism that allows one to write the reduced Hamiltonian dynamics on the singular reduced space M_μ. The Hamiltonian flow corresponding to this Hamiltonian dynamics is exactly the projection to M_μ of the regular Hamiltonian flow on M. Finally, in [4, 11, 12, 13, 18, 30, 34] it has been shown that the singular reduced space M_μ resulting from a *symplectic* manifold (M, ω) may be stratified by symplectic manifolds, called pieces. The stratification is by orbit type decomposition. The Hamiltonian flow on M_μ leaves these pieces invariant and restricts to a regular Hamiltonian flow on each of them.

Regular reduction has been recently extended in [6, 7] to the more general case of implicit Hamiltonian systems, building on preliminary work in [10, 37]. The geometric structure underlying an implicit Hamiltonian system on a state space M is the *Dirac structure*, defined as a maximally isotropic smooth vector subbundle of $TM \oplus T^*M$. As opposed to the symplectic or Poisson formalism, the "Hamiltonian dynamics" associated to a function $H \in C^\infty(M)$ via a Dirac structure consists of a coupled set of differential and algebraic equations. Perhaps the most striking example of an implicit Hamiltonian system is the one defined by a Poisson bracket on M, restricted to a submanifold of M which is *not* the level set of a collection of Casimir functions. This example actually motivated the original definition of a Dirac structure in [10], later adopted in [15]. Implicit Hamiltonian systems were defined in [8, 25, 40, 41, 42] extending those introduced in [10, 15]. They were successfully employed in the context of network modeling of energy conserving physical systems. Examples are mechanical systems with nonholonomic kinematic constraints, electrical LC circuits, and electromechanical systems. We refer to [38] and references therein for more information; see also [6] for a detailed historical account.

The study of symmetries and reduction of implicit Hamiltonian systems evolved from preliminary results in [10, 15, 37] culminating to a regular reduction theory described in [6, 7]. It was shown that a Dirac structure on M admitting a symmetry Lie group G with corresponding equivariant momentum map P, reduces to a Dirac structure on the quotient manifold $M_\mu := P^{-1}(\mu)/G_\mu$, if μ is a *regular* value of P and G_μ acts regularly on the level set $P^{-1}(\mu)$. Furthermore, the projectable integral curves of the implicit system defined by a G-invariant function $H \in C^\infty(M)^G$, descend to integral curves of the reduced implicit Hamiltonian system corresponding to the reduced Hamiltonian H_μ. The theory is a generalization of the classical regular reduction theory for symplectic and Poisson Hamiltonian systems, as well as the recently developed reduction theories for constrained mechanical systems. Section 3 briefly recalls the main results of [6, 7] from a different perspective needed in this paper. For a discussion of the reduction theory for constrained mechanical systems we refer to [6, 7].

The goal of the present work is to develop a reduction theory for implicit Hamiltonian systems at *singular* values of the momentum map. We restrict our attention to a class of Dirac structures described by a generalized Poisson bracket and a ("characteristic") distribution of derivations (vector fields), both defined on a class of smooth functions. We consider the special subclass of symmetries that preserve both the generalized Poisson bracket and the distribution. Using these ingredients, we prove that one can define a so-called *topological Dirac structure* on the singular reduced space M_μ (where for ease of exposition we will take $\mu = 0$). This represents the reduced Dirac structure.

This topological Dirac structure, whose construction implicitly uses Sikorski differential spaces (see [13, 33]), defines a Hamiltonian formalism on the reduced space M_μ. The dynamics corresponding to an implicit Hamiltonian system with Hamiltonian $H \in C^\infty(M_\mu)$ are described. It is shown that the projectable integral curves of the implicit Hamiltonian system on M, corresponding to a G-invariant function $H \in C^\infty(M)^G$, induce "integral curves" of the Hamiltonian dynamics defined on the singular reduced space. In the last two sections of the paper we make the assumption that the generalized Poisson bracket corresponding to the Dirac structure is nondegenerate. Secondly, we assume that the infinitesimal generators of the Lie group action are Hamiltonian vector fields, whose Hamiltonian is defined by the contraction of the momentum map with the corresponding Lie algebra element. Furthermore, the differential of the momentum map is assumed to annihilate the characteristic distribution. Under these assumptions, the singular reduced space can be decomposed by orbit type into a disjoint set of smooth manifolds, called pieces. It is shown that the topological Dirac structure restricts to a regular reduced Dirac structure on each piece. The Hamiltonian flow leaves the pieces invariant and restricts to a regular Hamiltonian flow on each piece.

The paper is organized as follows. Section 2 gives a brief introduction to Dirac structures and implicit Hamiltonian systems. The basic results concerning symmetries and regular reduction of implicit Hamiltonian systems are recalled in Section 3. Section 4 describes the topological reduction of an implicit Hamiltonian system admitting a symmetry group. The result is a so-called topological Dirac structure on the singular reduced space (which is not a smooth manifold). It is shown that if the symmetry group acts regularly and the value of the momentum map is regular (hence the reduced space is a smooth manifold), then the singular reduced implicit Hamiltonian system equals the regular reduced implicit Hamiltonian system as described in Section 3. Section 5 discusses the dynamics of singular reduced implicit Hamiltonian systems. It is shown that the "projectable" solutions of the original system descend to solutions of the singular reduced system. Section 6 describes the decomposition of the singular reduced space into smooth manifolds called pieces. The singular reduced implicit Hamiltonian system restricts to a regular reduced implicit Hamiltonian system on each piece. In Section 7 the theory developed in this paper is illustrated by working out in detail the singular reduction of a spherical pendulum with angular momentum zero about the vertical axis. Some additional comments on the singular reduction of constrained mechanical systems are included in this section. Section 8 contains the conclusions. For ease of reference, a list of notation is included at the end of this paper.

2 Implicit Hamiltonian systems

This section gives a brief introduction to Dirac structures and implicit Hamiltonian systems. It includes the important example of mechanical systems with kinematic constraints. The next section gives an overview of the regular reduction theory developed for these systems. These two review sections, recalling previously published results (see e.g. [6, 7, 8, 10, 14, 15, 20, 25, 37, 38, 39, 40, 41, 42]), set up notations and conventions, give the necessary definitions and results, and present the general framework of Dirac geometry, in order to serve as reference to the singular reduction theory developed in this paper. The authors would like to stress, however, that some of the definitions and theorems, as well as their general presentation, slightly differ from previous treatments of the subject. In particular, the introduction of the Dirac structure in Definition 1 and the consistent use of local sections throughout the paper have some advantages over previous formulations facilitating the passage to singular reduction.

Let M be a smooth n-dimensional manifold and let $TM \oplus T^*M$ denote the vector bundle whose fiber at $x \in M$ is given by $T_xM \times T_x^*M$. Here TM denotes the tangent and T^*M the cotangent bundle of M. Throughout this paper all geometric objects are assumed to be smooth, so when manifolds, vector bundles, sections are mentioned, they are all smooth. A Dirac structure on M is defined as follows.

Definition 1. *A smooth vector subbundle $D \subset TM \oplus T^*M$ is called a* Dirac structure *if every fiber $D(x) \subset T_xM \times T_x^*M$, $x \in M$, satisfies $D(x) = D^{\perp}(x)$, where*

$$D^{\perp}(x) = \{(w, w^*) \in T_xM \times T_x^*M \mid \langle v^*, w\rangle + \langle w^*, v\rangle = 0,\ \forall (v, v^*) \in D(x)\}. \tag{2.1}$$

Here $\langle \cdot, \cdot \rangle$ denotes the duality pairing between TM and T^*M.

Notice that D being a vector subbundle of $TM \oplus T^*M$ implies, by definition, that its fibers all have the same dimension, i.e., $\dim D(x) = \dim D(x')$, $\forall x, x' \in M$. In particular, if D is a Dirac structure then $\dim D(x) = n$, $\forall x \in M$. Furthermore, $D(x) = D^{\perp}(x)$, $x \in M$, implies that

$$\langle v^*, v\rangle = 0,\ \forall (v, v^*) \in D(x). \tag{2.2}$$

Remark 1. In [14] a *constant Dirac structure* on a vector space $\mathcal{V}$ is defined as a vector subspace $D \subset \mathcal{V} \times \mathcal{V}^*$ such that $D = D^{\perp}$. An equivalent way of writing Definition 1 is therefore the following. A Dirac structure on a manifold M is a smooth vector subbundle $D \subset TM \oplus T^*M$ such that each fiber $D(x)$, $x \in M$, is a constant Dirac structure on T_xM. ♦

Remark 2. In [10] there is yet another slightly different definition of a Dirac structure. Denote by $\mathfrak{X}_{loc}(M)$, respectively $\mathfrak{X}(M)$, the space of local, respectively global, smooth sections of TM. That is, these are the spaces of smooth local, respectively global, vector fields on M. Similarly, $\Omega^k_{loc}(M)$ and $\Omega^k(M)$ denote the spaces of smooth local and global k-forms on M. The spaces of smooth local and global sections of the vector subbundle $D \subset TM \oplus T^*M$ are denoted by $\mathfrak{D}_{loc}$ and $\mathfrak{D}$ respectively. Throughout, let $X, Y \in \mathfrak{X}_{loc}(M)$ and $\alpha, \beta \in \Omega^1_{loc}(M)$. Define a pairing on smooth sections of $TM \oplus T^*M$ by

$$\langle\!\langle (X, \alpha),\ (Y, \beta) \rangle\!\rangle = \langle \alpha, Y\rangle + \langle \beta, X\rangle, \quad \text{for } (X, \alpha),\ (Y, \beta) \in \mathfrak{X}_{loc}(M) \oplus \Omega^1_{loc}(M). \tag{2.3}$$

According to [10], a Dirac structure on M is a smooth vector subbundle $D \subset TM \oplus T^*M$ such that

1. D is isotropic: For every two local sections (X, α), $(Y, \beta) \in \mathfrak{D}_{loc}$ we have $\langle\!\langle (X, \alpha),\ (Y, \beta) \rangle\!\rangle = 0$;
2. D is maximal: If (Y, β) is a local section of $TM \oplus T^*M$ such that $\langle\!\langle (X, \alpha),\ (Y, \beta) \rangle\!\rangle = 0$, $\forall (X, \alpha) \in \mathfrak{D}_{loc}$, then $(Y, \beta) \in \mathfrak{D}_{loc}$.

It is easily shown that this definition given in [10] and Definition 1 are equivalent. Indeed, since D is a smooth vector subbundle, every $(v, v^*) \in D(x)$ can be extended to a local section $(X, \alpha) \in \mathfrak{D}_{loc}$. Furthermore, D being a smooth vector subbundle and the duality pairing $\langle \cdot, \cdot \rangle$ between TM and T^*M being nondegenerate, implies that also $D^{\perp} \subset TM \oplus T^*M$, with fibers given by $D^{\perp}(x)$, is a smooth vector subbundle. Therefore, also every $(w, w^*) \in D^{\perp}(x)$ can be extended to a local section (Y, β) of $D^{\perp}$. Elementary linear algebra (see, e.g., [1], §5.3 for such an argument) shows that

- D is isotropic if and only if $D \subset D^\perp$ and that
- maximal isotropy is equivalent to $D = D^\perp$, or to the fact that $\dim D(x) = n$ for all $x \in M$.

Thus the two definitions are equivalent. ♦

A Dirac structure induces several (co-)distributions. Recall that a *distribution* Δ on a manifold M is an assignment of a vector subspace $\Delta(x) \subset T_xM$ to each $x \in M$. The distribution Δ is said to be *smooth* if for each $x_0 \in M$ there exist a neighborhood U of x_0 in M and smooth vector fields $X_1, \ldots, X_k \in \mathfrak{X}(U)$ such that $\Delta(x) = \operatorname{span}\{X_1(x), \ldots, X_k(x)\}$ for all $x \in U$. The distribution Δ is called *constant dimensional* if the dimension of the linear subspace $\Delta(x) \subset T_xM$ does not depend on the point $x \in M$. Notice that if Δ is a smooth constant dimensional distribution on M, then it defines a smooth vector subbundle (also denoted by Δ) of the tangent bundle TM, with fibers $\Delta(x)$, $x \in M$. Analogously, a *codistribution* Γ is an assignment of a vector subspace $\Gamma(x) \subset T_x^*M$ to each $x \in M$. Smoothness and constant dimensionality are defined in the same way as for distributions. A smooth constant dimensional codistribution defines a smooth vector subbundle of the cotangent bundle T^*M.

Any Dirac structure D naturally defines

- a distribution Δ whose fibers are given by[1]

$$\Delta(x) := \{X(x) \mid X \in \mathfrak{X}_{loc}(M),\ (X, 0) \in \mathfrak{D}_{loc}\} \tag{2.4}$$

- a codistribution Γ whose fibers are given by[2]

$$\Gamma(x) := \{\alpha(x) \mid \alpha \in \Omega^1_{loc}(M),\ \exists X \in \mathfrak{X}_{loc}(M) \text{ such that } (X, \alpha) \in \mathfrak{D}_{loc}\}. \tag{2.5}$$

Since D is isotropic it follows that $\Delta(x) \subset \Gamma^\circ(x)$. Here $\Gamma^\circ(x)$ denotes the annihilating vector subspace of $\Gamma(x)$ in T_xM, that is, $\Gamma^\circ(x) := \{v \in T_xM \mid \langle v^*, v\rangle = 0,\ \forall v^* \in \Gamma(x)\}$. Equivalently, $\Gamma(x) \subset \Delta^\circ(x)$, where $\Delta^\circ(x)$ denotes the annihilating vector subspace of $\Delta(x)$ in T_x^*M, that is, $\Delta^\circ(x) := \{v^* \in T_x^*M \mid \langle v^*, v\rangle = 0,\ \forall v \in \Delta(x)\}$. Furthermore, if Γ is constant dimensional, and hence defines a vector subbundle of T^*M, it follows, by maximal isotropy of D, that $\Delta(x) = \Gamma^\circ(x)$, or equivalently, $\Gamma(x) = \Delta^\circ(x)$. Notice that in this case, Δ is also constant dimensional and hence defines a vector subbundle of TM.

Remark 3. In order to obtain a smooth distribution, it is important to define Δ in terms of local sections, as in (2.4). In particular, it is *not* true that $v \in \Delta(x)$ if and only if $(v, 0) \in D(x)$. To see this, consider the following example. Let $M = \mathbb{R}^2$ with (global) coordinates $x = (x_1, x_2) \in M$. Consider the closed two-form $\omega = \|x\|^2 dx_1 \wedge dx_2$, and let D be given by

$$D(x) = \{(v, v^*) \in \mathbb{R}^2 \times \mathbb{R}^2 \mid v^* = \omega(x)(v, \cdot)\}. \tag{2.6}$$

Then D defines a Dirac structure on M. To see this, notice that D defines a smooth vector subbundle of $TM \oplus T^*M$. A basis for its fibers is given by the smooth tensor fields $\frac{\partial}{\partial x_1} + \|x\|^2 dx_2$

[1] In the literature on implicit Hamiltonian systems, this distribution is usually denoted by G_0 and sometimes called the *characteristic distribution*. However, in order to avoid confusion with notation defined later on in this paper we have decided to adopt a different notation here.

[2] This codistribution is usually denoted by P_1 in the literature.

and $\frac{\partial}{\partial x_2} - \|x\|^2 dx_1$, which immediately shows that $D^\perp(x) = D(x)$. Since ω is nondegenerate outside $x = (0,0)$ the smooth distribution Δ defined in (2.4) is given by $\Delta = \{0\}$ (the zero section of TM). However, $(v, 0) \in D(0)$ for *every* $v \in \mathbb{R}^2$.

This example illustrates also something else. Notice that Γ defined in (2.5) is the smooth codistribution whose basis is given by the one-forms $-\|x\|^2 dx_1$ and $\|x\|^2 dx_2$. In particular, $\Gamma^\circ(0) = (\{0\})^\circ = T_0 M = \mathbb{R}^2$, which does not equal $\Delta(0) = \{0\}$. Hence $\Gamma^\circ(x) \neq \Delta(x)$. The problem stems from the fact that Γ is not constant dimensional in this example. In general, *if* Γ is constant dimensional, *then* $\Gamma^\circ(x) = \Delta(x)$. ♦

Remark 4. The codistribution Γ defined in (2.5) can be equivalently defined pointwise by:

$$\Gamma(x) = \{v^* \in T^*_x M \mid \exists v \in T_x M \text{ such that } (v, v^*) \in D(x)\}. \tag{2.7}$$

To see this, recall that, by definition, D is a smooth vector subbundle of $TM \oplus T^*M$. Hence there exists a smooth local basis for its fibers. The canonical projection of this basis to T^*M yields a smooth local basis for Γ (around the point x). Therefore, definitions (2.5) and (2.7) are equivalent. (See e.g. the previous Remark for an example.) ♦

A Dirac structure can satisfy the following special property.

Definition 2. *A Dirac structure D is called* closed*, or* integrable*, if for all local sections* (X_1, α_1), (X_2, α_2), $(X_3, \alpha_3) \in \mathfrak{D}_{loc}$

$$\langle L_{X_1}\alpha_2, X_3\rangle + \langle L_{X_2}\alpha_3, X_1\rangle + \langle L_{X_3}\alpha_1, X_2\rangle = 0. \tag{2.8}$$

Equivalently (see [10, 14, 15]), D is closed if and only if for all (X_1, α_1), $(X_2, \alpha_2) \in \mathfrak{D}_{loc}$

$$([X_1, X_2], L_{X_1}\alpha_2 - L_{X_2}\alpha_1 + \mathrm{d}\langle \alpha_1, X_2\rangle) \in \mathfrak{D}_{loc}. \tag{2.9}$$

The notation L_X is reserved for the Lie derivative operator (acting on any type of tensor field) defined by the local vector field X on M.

It is easy to see that the graph of a symplectic form $\omega : TM \to T^*M$, or the graph of the skew-symmetric vector bundle map $J : T^*M \to TM$ induced by a Poisson bracket $\{\cdot,\cdot\}$ on M, defines a Dirac structure on M. As customary, we will call both the bundle map J and the two-tensor defined by $\{\cdot,\cdot\}$ the Poisson structure on M. The Dirac structure D being closed corresponds to the condition that ω is a closed two-form, respectively, that the Poisson bracket satisfies the Jacobi identity.

In this paper we will concentrate on a rather frequently occurring type of Dirac structure defined as follows. Let $\{\cdot,\cdot\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$ be a *generalized Poisson bracket* on M. That is, $\{\cdot,\cdot\}$ is skew-symmetric, bilinear, and satisfies the Leibniz property. Denote the corresponding vector bundle map by $J : T^*M \to TM$. By definition, $\langle dF, J(dH)\rangle = \{F, H\}$, for all $F, H \in C^\infty(M)$. Recall that J is skew-symmetric, that is, $J^* = -J$. Note that we do not require $\{\cdot,\cdot\}$ to satisfy the Jacobi identity, nor J to have constant rank. Moreover, given a subbundle Δ of TM (i.e., a smooth constant dimensional distribution Δ on M), it is easy to see that the vector subbundle $D \subset TM \oplus T^*M$ with fiber

$$D(x) = \{(v, v^*) \in T_x M \times T^*_x M \mid v - J(x)v^* \in \Delta(x),\ v^* \in \Delta^\circ(x)\} \tag{2.10}$$

defines a Dirac structure on M. In terms of its local sections this is expressed as

$$\mathfrak{D}_{loc} = \{(X, \alpha) \in \mathfrak{X}_{loc}(M) \oplus \Omega^1_{loc}(M) \mid X - J\alpha \text{ is a local section of } \Delta, \\ \alpha \text{ is local section of } \Delta^\circ\}. \tag{2.11}$$

Here Δ° denotes the vector subbundle of T^*M whose fiber at $x \in M$ equals $\Delta^\circ(x)$.

Remark 5. In [10, 14] it is shown that under a mild constant dimensionality assumption, *every* Dirac structure can be written in the form (2.10) or, equivalently, (2.11). Indeed, if D is an arbitrary Dirac structure on M, define the codistribution Γ as in (2.5), and *assume that* Γ *is constant dimensional* (hence defines a vector subbundle of T^*M). Then there exists a well defined (see [14]) skew-symmetric vector bundle map $J(x) : \Gamma(x) \subset T^*_xM \to (\Gamma(x))^* \subset T_xM,\ x \in M$, defined by

$$J(x)v^* = \bar{v} \in (\Gamma(x))^*, \quad v^* \in \Gamma(x), \tag{2.12}$$

where $\bar{v} \in (\Gamma(x))^*$ denotes the restriction of some $v \in T_xM$ to $\Gamma(x) \subset T^*_xM$ which satisfies the condition $(v, v^*) \in D(x)$. Notice that the kernel of $J(x)$ is given by the codistribution Γ_0 with fibers defined by[3]

$$\Gamma_0(x) := \{\alpha(x) \mid \alpha \in \Omega^1_{loc}(M), (0, \alpha) \in \mathfrak{D}_{loc}\}. \tag{2.13}$$

Then the Dirac structure D is of the form (2.10) or, equivalently, (2.11) with $\Delta = \Gamma^\circ$. The map J may be locally extended to a skew-symmetric vector bundle map $J(x) : T^*_xM \to T_xM$, defining a generalized Poisson structure on M. ⧫

Notice that, in general, the Dirac structure defined in (2.10) is not closed. Although it does not play an important role in the rest of the paper, we remark for completeness that (2.11) defines a closed Dirac structure if and only if (see [14])

1. Δ is involutive,
2. the bracket $\{\cdot, \cdot\}$ restricted to the set of *admissible functions* $\mathcal{A}_D := \{H \in C^\infty(M) \mid dH \text{ is a section of } \Delta^\circ\}$ defines a Poisson bracket on $\mathcal{A}_D$ (that is, the Jacobi identity holds).

Remark 6. Before leaving this brief introduction to Dirac structures and we proceed to the description of implicit Hamiltonian systems, we would like to mention the following generalization. In [20] a Dirac structure is defined as a maximal isotropic subbundle $D \subset \mathcal{A} \oplus \mathcal{A}^*$, where the pair $(\mathcal{A}, \mathcal{A}^*)$ is a *Lie bialgebroid* over a smooth manifold M. The isotropy condition is defined with respect to the natural pairing $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ defined analogously as in (2.3) by $\mathcal{A}$ and its dual $\mathcal{A}^*$. If we take the special case $\mathcal{A} = TM$ and dually $\mathcal{A}^* = T^*M$, then we recover Definition 1. For more information on this generalization we refer to [20] and references therein. We remark that [20] require the Dirac structure to be closed. In their terminology, *Dirac structures* are always closed, while Dirac structures *not* satisfying condition (2.9) are called *almost Dirac structures.* In this paper, however, we prefer to use the terminology as introduced above. That is, we shall call a maximally isotropic subbundle of $TM \oplus T^*M$ a Dirac structure and add the prefix *closed* if (and only if) the conditions of Definition 2 are satisfied. ⧫

[3] This codistribution is usually denoted by P_0 in the literature.

Now we turn to the definition of an implicit Hamiltonian system. Consider a Dirac structure D on M and a smooth function $H \in C^\infty(M)$, called the Hamiltonian or energy function. Then the three-tuple (M, D, H) defines an implicit Hamiltonian system in the following way.

Definition 3. *The* implicit Hamiltonian system (M, D, H) *is defined as a set of smooth time functions* $\{x(t) \mid x : \mathbb{R} \to M$ *of class* $C^\infty\}$*, called* solutions*, satisfying the condition*

$$(\dot{x}(t), dH(x(t))) \in D(x(t)), \ \forall t. \tag{2.14}$$

Equations (2.2) and (2.14) imply that implicit Hamiltonian systems are *energy conserving*, i.e.,

$$\frac{dH}{dt}(x(t)) = \langle dH(x(t)), \dot{x}(t) \rangle = 0, \ \forall t. \tag{2.15}$$

If D is the graph of a symplectic form ω or of a Poisson structure J, then Definition 3 yields a classical symplectic or Poisson Hamiltonian system. On the other hand, if D is defined by (2.10) then the system includes the *algebraic constraints*

$$dH(x(t)) \in \Delta^\circ(x(t)), \ \forall t. \tag{2.16}$$

Thus all solutions of the implicit Hamiltonian system necessarily lie in the *constraint manifold*

$$M_c := \{x \in M \mid dH(x) \in \Delta^\circ(x)\}. \tag{2.17}$$

Since implicit Hamiltonian systems consist of coupled differential and algebraic equations, there is no existence and uniqueness theorem as for classical Hamiltonian systems described by ordinary differential equations. In particular, not every point $x_0 \in M_c$ necessarily lies on the trajectory of some solution $x(t)$ of the system. Neither are the solutions through a point $x_0 \in M_c$, if they exist, necessarily unique. This happens, for instance, if the Lagrange multipliers corresponding to the algebraic constraints cannot be solved uniquely. In the sequel we will not investigate these problems. Instead, we will study the reduction of the underlying Dirac structure in the presence of symmetries (to be defined later on). We shall show that certain "projectable" solutions, whose existence will be postulated, will descend to solutions of an implicit Hamiltonian system on the reduced space.

The problem of existence and uniqueness of solutions to implicit systems is an important and active area of research and will not be touched upon here. We only would like to mention the special case of so-called *index 1* systems. Consider the implicit Hamiltonian system defined by a Dirac structure of type (2.10) and the Hamiltonian function $H \in C^\infty(M)$. Let the vector subbundle Δ be locally expressed as the span of the independent vector fields $g_1, \dots, g_m$. Then the constraint manifold can be written as

$$M_c = \{x \in M \mid L_{g_j} H(x) = 0, \ j = 1, \dots, m\}. \tag{2.18}$$

Now *assume* that the constraints are of index 1, that is, the matrix

$$\left[L_{g_i} L_{g_j} H(x)\right]_{i,j=1,\dots,m} \tag{2.19}$$

is nonsingular for all $x \in M_c$. In that case, the restriction of the implicit Hamiltonian system (M, D, H) to M_c yields an *explicit* Hamiltonian system on M_c. This system is defined by a (possibly non-integrable) Poisson bracket on M_c (see [6, 41]). Its corresponding dynamics is thus given by a set of ordinary differential equations on M_c. Standard existence and uniqueness results now yield the usual conclusion: through every point $x_0 \in M_c$ there exists a unique (local) solution of the implicit Hamiltonian system restricted to M_c.

Example 1. As an important example of implicit Hamiltonian systems, and in anticipation to Section 7, we mention the class of mechanical systems with kinematic constraints. These systems are described by implicit Hamiltonian systems (M, D, H) with D of the form (2.10). The phase space $M = T^*Q$ is the cotangent bundle of the configuration space Q. Local coordinates are denoted, as usual, by $(q, p) \in T^*Q$. The function $H \in C^\infty(M)$ denotes the total energy of the system. The Poisson bracket $\{\cdot, \cdot\}$ is the usual one associated to the canonical symplectic form $\omega = dq \wedge dp$ on T^*Q. It induces a Poisson structure J. We assume that the kinematic constraints are linear in the velocities, and that they are independent. Then there exists a set of k independent one-forms $\alpha_1, \ldots, \alpha_k$ on Q such that the constraints are given by

$$\alpha_i(q)\dot{q} = 0, \quad i = 1, \ldots, k. \tag{2.20}$$

Define the matrix A^T, whose i-th row expresses the one-form α_i. Then the kinematic constraints can be equivalently written in the familiar form

$$A^T(q)\dot{q} = 0. \tag{2.21}$$

The matrix $A^T(q)$ is a $k \times n$ matrix, $n = \dim Q$, with full row rank k at every point $q \in Q$. The distribution $\Lambda = \ker A^T(q)$ is called the constraint distribution. It describes the allowed infinitesimal motions of the system. Recall that the constraints are called *holonomic* if they can be integrated to a set of configuration constraints $\{f_1(q) = 0, \ldots, f_k(q) = 0\}$. If this is not possible, then the constraints are called *nonholonomic*. A necessary and sufficient condition for the constraints to be holonomic is that the constraint distribution is involutive. (This is Frobenius' Theorem.)

By d'Alembert's principle, the constraints (2.21) generate constraint forces of the form $F_c = A(q)\lambda$, where $\lambda \in \mathbb{R}^k$ are called the Lagrange multipliers. Hence, the equations of motion of the system are given by the implicit Hamiltonian system

$$\begin{pmatrix} \dot{q} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} 0_n & I_n \\ -I_n & 0_n \end{pmatrix} \begin{pmatrix} \frac{\partial H}{\partial q}(q,p) \\ \frac{\partial H}{\partial p}(q,p) \end{pmatrix} + \begin{pmatrix} 0_{n\times k} \\ A(q) \end{pmatrix} \lambda, \tag{2.22a}$$

$$0 = \begin{pmatrix} 0_{k\times n} & A^T(q) \end{pmatrix} \begin{pmatrix} \frac{\partial H}{\partial q}(q,p) \\ \frac{\partial H}{\partial p}(q,p) \end{pmatrix}. \tag{2.22b}$$

From (2.22a) it follows that the distribution Δ is locally spanned by the columns of the matrix $\begin{pmatrix} 0 & A^T(q) \end{pmatrix}^T$, where each column is understood as a vector field on T^*Q. With this in mind, we write (with some abuse of notation)

$$\Delta(q, p) = \operatorname{Im} \begin{bmatrix} 0_{n\times k} \\ A(q) \end{bmatrix}, \quad (q, p) \in T^*Q. \tag{2.23}$$

More intrinsically, Δ can be defined in the following way: Define the bundle projection $\pi_Q : T^*Q \to Q$. Lift the one-forms $\alpha_1, \ldots, \alpha_k$ on Q by means of a vertical lift to the one-forms

$\pi_Q^*\alpha_1, \ldots, \pi_Q^*\alpha_k$ on T^*Q. The span of these lifted one-forms defines a codistribution on T^*Q, which in fact equals Γ_0 as defined in (2.13). Now, recall that the symplectic form ω is nondegenerate and hence defines two isomorphisms: $\omega^\flat : T(T^*Q) \to T^*(T^*Q)$ and its inverse $\omega^\sharp : T^*(T^*Q) \to T(T^*Q)$. These are sometimes called the *musical isomorphisms*. Define the vector fields $X_i = -\omega^\sharp(\pi_Q^*\alpha_i)$, with $i = 1, \ldots, k$, on T^*Q. The span of these vector fields defines a distribution on T^*Q, which is exactly Δ. Since the vector fields are independent, the distribution is constant dimensional (with dimension k) and defines a vector subbundle of $T(T^*Q)$.

Finally, if the kinetic energy is defined by a positive definite metric on Q, then the constraints are of index 1. In that case, the Lagrange multipliers λ can be solved uniquely. Hence the constrained mechanical system on T^*Q can be written as an *unconstrained* generalized Hamiltonian system on M_c. In [39] it is shown that the corresponding generalized Poisson bracket on M_c satisfies the Jacobi identity if and only if the kinematic constraints are *holonomic*. $\Diamond$

3 Symmetries and regular reduction

In this section we recall some of the results in [6, 7] concerning symmetries and regular reduction of implicit Hamiltonian systems. They will serve as reference for the rest of the paper. Specifically, they are needed in Section 6 where we shall prove that the singular reduced Dirac structure restricts to regular reduced Dirac structures on the pieces defined by the orbit type decomposition of the singular reduced space M_0. We refer to [6, 7] for all technical details and proofs. We stress that, unless specifically stated otherwise, the results in this section are valid for arbitrary Dirac structures, not necessarily of the form (2.10).

Definition 4. *A smooth vector field Y on M is called a* symmetry *of the Dirac structure D if for every local section (X, α) of D, the pair $(L_Y X, L_Y \alpha)$ is also a local section of D. The vector field Y is called a* symmetry *of the implicit Hamiltonian system (M, D, H) if Y is a symmetry of D and a symmetry of H, i.e., $L_Y H = 0$.*[4]

Definition 4 generalizes the classical notion of symmetry for symplectic or Poisson Hamiltonian systems. To see this, let $J : T^*M \to TM$ be the vector bundle map over the identity defined by the Poisson structure: $\{F_1, F_2\} = \langle dF_1, J(dF_2)\rangle$ for any smooth locally defined functions $F_1, F_2 : M \to \mathbb{R}$. Denote by the same letter the map induced on local sections, that is, $(J\alpha)(x) = J(\alpha(x))$, for any $\alpha \in \Omega^1_{loc}(M)$ and any x in the domain of definition of α. Thus we can think of J also as a $C^\infty(M)$-linear map $J : \Omega^1_{loc}(M) \to \mathfrak{X}_{loc}(M)$. It is easy to check that the vector field Y is a symmetry of D if and only if $L_Y \circ J = J \circ L_Y$. This is equivalent to the condition that Y is a derivation of the Poisson bracket, that is, $L_Y\{F_1, F_2\} = \{L_Y F_1, F_2\} + \{F_1, L_Y F_2\}$ for any $F_1, F_2 \in C^\infty(M)$. In turn, this is equivalent to the statement that the flow of Y consists of Poisson diffeomorphisms. If $J = \omega^{-1}$, where ω is a symplectic form on M, then this condition is clearly equivalent to $L_Y\omega = 0$.

In this paper we will consider a special subclass of symmetries, defined in the following Proposition.

Proposition 1. *Consider a Dirac structure D of the type defined in* (2.10)*. Assume that the vector field Y on M is a derivation of the generalized Poisson bracket (equivalently, $L_Y \circ J = J \circ L_Y$).*

[4] This is called a *strong* symmetry in [6, 7].

Furthermore, assume that $L_Y Z$ is a local section of Δ whenever Z is a local section of Δ. Then Y is a symmetry of D.

In particular, this means that we restrict our attention to the case where Y is a symmetry of the generalized Poisson bracket, as well as a symmetry of the vector subbundle $\Delta \subset TM$. These kinds of symmetries often arise in constrained mechanical systems, as will be seen in Section 7.

More specifically, we will consider *Lie algebra symmetries*, defined as follows. Recall that a left Lie algebra action on a manifold M is a Lie algebra anti-homomorphism $\xi \in \mathfrak{g} \mapsto \xi_M \in \mathfrak{X}(M)$ such that the map $(x, \xi) \in M \times \mathfrak{g} \mapsto \xi_M(x) \in T_xM$ is smooth.

Definition 5. *The Lie algebra $\mathfrak{g}$ is called a* symmetry Lie algebra *of D if ξ_M is a symmetry of D for every $\xi \in \mathfrak{g}$. It is called a* symmetry Lie algebra *of the implicit Hamiltonian system (M, D, H) if ξ_M is a symmetry of D and also a symmetry of H, i.e. $L_{\xi_M} H = 0$, for every $\xi \in \mathfrak{g}$.*

In particular, if the Dirac structure is of the type (2.10), then the criterion (sufficient condition) in Proposition 1 applies.

Lie algebra symmetries are often induced by Lie group actions. Let G be a Lie group with Lie algebra $\mathfrak{g}$ and $\phi : G \times M \to M$ a smooth left action of G on the manifold M. The *infinitesimal generator* of the action associated to $\xi \in \mathfrak{g}$ is defined by

$$\xi_M(x) = \left.\frac{d}{dt}\right|_{t=0} \phi(\exp(t\xi), x) \in T_xM, \quad x \in M. \tag{3.1}$$

The assignment $\xi \in \mathfrak{g} \mapsto \xi_M \in \mathfrak{X}(M)$ is a left Lie algebra action of $\mathfrak{g}$ on M.

Definition 6. *The Lie group G is said to be a* symmetry Lie group *of D if for every $(X, \alpha) \in \mathfrak{D}_{loc}$ and every $g \in G$ it follows that $\left(\phi_g^* X, \phi_g^* \alpha\right) \in \mathfrak{D}_{loc}$. It is said to be a* symmetry Lie group *of the implicit Hamiltonian system (M, D, H) if, in addition, H is G-invariant, that is, $H \circ \phi_g = H$ for all $g \in G$.*

Setting $g = \exp(t\xi)$ for $\xi \in \mathfrak{g}$ and taking the derivative of the defining relations in Definition 6, it follows that the conditions in Definitions 4 and 5 hold. Thus a G-symmetry of the Dirac structure D (respectively of the implicit Hamiltonian system (M, D, H)) induces a similar $\mathfrak{g}$-symmetry of D (respectively of (M, D, H)).

We turn now to the analysis of the regular reduction process of Dirac structures and implicit Hamiltonian systems. We start by explaining how an implicit Hamiltonian system on M can be restricted to an implicit Hamiltonian system on a submanifold N of M. Let D be a Dirac structure on M and let $N \subset M$ be a submanifold of M. Following [10], define for each $x \in N$ the map $\sigma(x) : T_xN \times T_x^*M \to T_xN \times T_x^*N$, $x \in N$, by $\sigma(x)(v, v^*) = (v, v^*|_{T_xN})$. Here $v^*|_{T_xN}$ denotes the restriction of the covector $v^* \in T_x^*M$ to the subspace $T_xN \subset T_xM$. Define a vector subspace of $T_xN \times T_x^*N$ by

$$D_N(x) = \sigma(x) \left(D(x) \cap \left(T_xN \times T_x^*M\right)\right), \quad x \in N. \tag{3.2}$$

It is clear that $D_N(x) \subset D_N^\perp(x)$, $\forall x \in N$. To prove the reverse inclusion, suppose that $(w, w^*) \in D_N^\perp(x) \subset T_xN \times T_x^*N$. That is, $\langle v^*, w\rangle + \langle w^*, v\rangle = 0$, $\forall (v, v^*) \in D_N(x)$. Then there exists a $\bar{v}^* \in T_x^*M$ such that $(v, v^*) = \sigma(x)(v, \bar{v}^*)$, i.e. $\bar{v}^*|_{T_xN} = v^*$, and $(v, \bar{v}^*) \in D(x)$. Since $v, w \in T_xN$, one gets

$$0 = \langle v^*, w\rangle + \langle w^*, v\rangle = \langle \bar{v}^*, w\rangle + \langle \bar{w}^*, v\rangle,$$

where $\bar{w}^*$ is an arbitrary extension of w^* to T_xM. Since this relation holds for all $(v, \bar{v}^*) \in D(x)$ with $v \in T_xN$, this implies that

$$\begin{aligned}(w, \bar{w}^*) \in [D(x) \cap (T_xN \times T_x^*M)]^\perp &= D^\perp(x) + (T_xN \times T_x^*M)^\perp \\ &= D(x) + (\{0\} \times T_xN^\circ).\end{aligned} \tag{3.3}$$

Hence there exists a $\bar{u}^* \in T_xN^\circ \subset T_x^*M$ such that $(w, \bar{w}^* + \bar{u}^*) \in D(x)$. However, since $w \in T_xN$ and $\sigma(x)(w, \bar{w}^* + \bar{u}^*) = (w, (\bar{w}^* + \bar{u}^*)|_{T_xN}) = (w, w^*)$, it follows that $(w, w^*) \in D_N(x)$. This shows that $D_N^\perp(x) \subset D_N(x)$.

Now assume that the dimension of $D(x) \cap (T_xN \times T_x^*M)$ is independent of $x \in N$, i.e. that $D \cap (TN \times T^*M)$ is a vector subbundle of $TN \times T^*M$. Then it follows that σ is a vector bundle map and hence that D_N is a vector subbundle of $TN \times T^*N$. So we have proved the following result (which is a slightly rewritten version of the result in [10]).

Proposition 2. *Consider a Dirac structure D on M and let N be a submanifold of M. Assume that $D(x) \cap (T_xN \times T_x^*M)$, $x \in N$, has constant dimension on N. Then the bundle D_N with fibers defined by* (3.2) *is a Dirac structure on N. This is called the* restriction *of D to N.*

In order to do computations it is convenient to describe the restricted Dirac structure D_N in terms of its local sections. This gives the following proposition (an improved version of [6, 7]). Let $\iota : N \hookrightarrow M$ denote the inclusion map.

Proposition 3. *Consider a Dirac structure D on M and let N be a submanifold of M. Assume that $D(x) \cap (T_xN \times T_x^*M)$, $x \in N$, has constant dimension on N and let D_N denote the restriction of D to N. Then $(\bar{X}, \bar{\alpha})$ is a local section of D_N if and only if there exists a local section (X, α) of D such that $\bar{X} \sim_\iota X$ and $\bar{\alpha} = \iota^*\alpha$. Otherwise stated, in terms of local sections,*

$$(\mathfrak{D}_N)_{loc} = \{(\bar{X}, \bar{\alpha}) \in \mathfrak{X}_{loc}(N) \oplus \Omega^1_{loc}(N) \mid \exists (X, \alpha) \in \mathfrak{D}_{loc} \text{ such that } \bar{X} \sim_\iota X \text{ and } \bar{\alpha} = \iota^*\alpha\}. \tag{3.4}$$

Here ι^* denotes the pull back by ι and $\bar{X} \sim_\iota X$ signifies that $\bar{X}$ and X are ι-related, that is, $T\iota \circ \bar{X} = X \circ \iota$. In particular, if $\bar{X} \sim_\iota X$, this means that X is tangent to N and its restriction to N is exactly $\bar{X}$.

If D is closed, then also D_N is closed. This fact follows immediately from Definition 2 by observing that for $(\bar{X}_1, \bar{\alpha}_1)$, $(\bar{X}_2, \bar{\alpha}_2)$, $(\bar{X}_3, \bar{\alpha}_3) \in (\mathfrak{D}_N)_{loc}$, we have

$$\langle L_{\bar{X}_1}\bar{\alpha}_2, \bar{X}_3\rangle + \langle L_{\bar{X}_2}\bar{\alpha}_3, \bar{X}_1\rangle + \langle L_{\bar{X}_3}\bar{\alpha}_1, \bar{X}_2\rangle = \big(\langle L_{X_1}\alpha_2, X_3\rangle + \langle L_{X_2}\alpha_3, X_1\rangle + \langle L_{X_3}\alpha_1, X_2\rangle\big) \circ \iota. \tag{3.5}$$

Now let (M, D, H) be an implicit Hamiltonian system on M and let N be a submanifold of M such that the constant dimensionality condition of Proposition 2 is satisfied. Assume that (the flow corresponding to) the solutions of (M, D, H) leave the submanifold N invariant. Restrict the Hamiltonian H to a smooth function H_N on N by $H_N = H \circ \iota$ and define the implicit Hamiltonian system (N, D_N, H_N) on N. Then we have the following result.

Proposition 4. *Every solution $x(t)$ of (M, D, H) which leaves N invariant (i.e. which is contained in N) is a solution of (N, D_N, H_N).*

We remark that, in general, there is not a one-to-one correspondence between the solutions generated by the original system (M, D, H) and those generated by the restricted system (N, D_N, H_N).

(For comparison, one can consider the restriction of a symplectic form ω on M to an arbitrary submanifold N, leading to a nontrivial kernel and hence a presymplectic form ω_N.) An exceptional case is obtained when N is the level set of a *Casimir* function.

Definition 7. *A* Casimir *function of a Dirac structure D on M is a smooth function $C \in C^\infty(M)$ such that $(0, dC) \in \mathfrak{D}_{loc}$.*

If D is of the form (2.10) then a *sufficient* condition for C to be a Casimir function of D is that C is a Casimir of the generalized Poisson structure and dC is a section of Δ°. It can easily be seen that if C is a Casimir function of a Dirac structure D (not necessarily of the form (2.10)), then it is a *first integral* (or, conserved quantity) of the implicit Hamiltonian system (M, D, H), for arbitrary $H \in C^\infty(M)$.

Now, in case the submanifold N in Proposition 4 happens to be the level set of a Casimir function of the Dirac structure D then there *does* exist a one-to-one correspondence between the solutions of the original system and those of the restricted system. See [6], Proposition 4.1.7, for a proof.

Next, we explain how an implicit Hamiltonian system on M, admitting a symmetry Lie group G, can be projected to an implicit Hamiltonian system on the orbit space M/G. Consider a Dirac structure D on M and let G be a symmetry Lie group of D, acting *regularly* on M. That is, the orbit space M/G is a smooth manifold and the canonical projection map $\pi : M \to M/G$ is a surjective submersion. Let $V = \ker T\pi$ denote the vertical subbundle of TM, with fiber $V(x) = \text{span}\,\{\xi_M(x) \mid \xi \in \mathfrak{g}\}$ for every $x \in M$. We assume that $V + \Delta$ is a smooth vector subbundle of TM, i.e., its fibers all have the same dimension. Furthermore, define the smooth vector subbundle $E \subset TM \oplus T^*M$ in terms of its local sections by

$$\Gamma_{loc}(E) = \{(X, \alpha) \in \mathfrak{X}_{loc}(M) \oplus \Omega^1_{loc}(M) \mid \alpha = \pi^*\hat{\alpha} \text{ for some } \hat{\alpha} \in \Omega^1_{loc}(M/G)\}; \tag{3.6}$$

$\Gamma_{loc}(E)$ denotes the space of local sections of the subbundle E. Assume that $D \cap E$ is a smooth vector subbundle of $TM \oplus T^*M$, i.e., its fibers all have the same dimension. Then we have the following result.

Proposition 5. *[6, 7, 37] Consider a Dirac structure D on M admitting a symmetry Lie group G acting regularly on M. Assume that $V + \Delta$ is a smooth vector subbundle of TM and that $D \cap E$ is a smooth vector subbundle of $TM \oplus T^*M$. Then D projects to a Dirac structure $\hat{D}$ on $\hat{M} := M/G$, described in terms of its local sections by*

$$\hat{\mathfrak{D}}_{loc} = \{(\hat{X}, \hat{\alpha}) \in \mathfrak{X}_{loc}(\hat{M}) \times \Omega^1_{loc}(\hat{M}) \mid \exists (X, \alpha) \in \mathfrak{D}_{loc} \textit{ such that } X \sim_\pi \hat{X} \textit{ and } \alpha = \pi^*\hat{\alpha}\}. \tag{3.7}$$

This is called the *projection* of D to M/G. Just as in the case of restriction, D being closed implies that $\hat{D}$ is closed. Let (M, D, H) be an implicit Hamiltonian system admitting a symmetry Lie group acting regularly on M and such that the conditions in Proposition 5 are satisfied. The G-invariant function H defines a function $\hat{H} \in C^\infty(M/G)$ by $H = \hat{H} \circ \pi$. Consider the implicit Hamiltonian system $(M/G, \hat{D}, \hat{H})$. A *G-projectable* solution $x(t)$ of (M, D, H) is defined as a solution $x(t)$ of (M, D, H) for which there exists a projectable vector field $X \in \mathfrak{X}_{loc}(M)$ having $x(t)$ as an integral curve. This condition means that $X \sim_\pi \hat{X}$, for some $\hat{X} \in \mathfrak{X}_{loc}(M/G)$, and the solution $x(t)$ of (M, D, H) satisfies $\dot{x}(t) = X(x(t))$. The following proposition was obtained in [6, 7].

Proposition 6. *If $x(t)$ is a G-projectable solution of (M, D, H) then $\hat{x}(t) := \pi(x(t))$ is a solution of $(M/G, \hat{D}, \hat{H})$. Conversely, every solution $\hat{x}(t)$ of $(M/G, \hat{D}, \hat{H})$ is locally the projection under π of a G-projectable solution $x(t)$ of (M, D, H).*

In [6] a simple example is given showing that not every solution of an implicit Hamiltonian system (M, D, H) admitting a symmetry Lie group G is necessarily G-projectable. However, if the constraints are of index 1, then it can be shown that *every* solution is G-projectable and hence descends to a solution of the reduced implicit Hamiltonian system on M/G.

Finally, let us briefly recall some results in [6, 7] concerning the reduction of implicit Hamiltonian systems admitting a symmetry Lie group having an associated momentum map. Consider an implicit Hamiltonian system (M, D, H) admitting a symmetry Lie group G with Lie algebra $\mathfrak{g}$. Assume that there exists an Ad^*-equivariant map $P : M \to \mathfrak{g}^*$ (Ad^* denoting the coadjoint action), called *momentum map*, such that

$$(\xi_M, dP_\xi) \in \mathfrak{D}, \ \forall \xi \in \mathfrak{g}, \tag{3.8}$$

where $P_\xi \in C^\infty(M)$ is defined by $P_\xi(x) = \langle P(x), \xi \rangle, \ x \in M$. Notice that if D is the graph of a symplectic form ω, or a Poisson structure J, then this corresponds to the classical definition of a momentum map. Assuming that $\mu \in \mathfrak{g}^*$ is a *regular* value of P, it follows that the level set $P^{-1}(\mu)$ is a closed submanifold of M. Since the Hamiltonian is G-invariant, the solutions of (M, D, H) leave the level set $P^{-1}(\mu)$ invariant. Thus by (2.14), (3.8), and the identity $D(x) = D^\perp(x)$, it follows that

$$\frac{dP_\xi}{dt}(x(t)) = \langle dP_\xi(x(t)), \dot{x}(t) \rangle = -\langle dH, \xi_M \rangle(x(t)) = 0, \ \forall t, \ \forall \xi \in \mathfrak{g}. \tag{3.9}$$

In other words, P is a *first integral* of the implicit Hamiltonian system (M, D, H). Assuming that the conditions in Proposition 2 hold, we can restrict the implicit Hamiltonian system (M, D, H) to an implicit Hamiltonian system (N, D_N, H_N) on $N := P^{-1}(\mu)$. The system (N, D_N, H_N) admits the symmetry Lie group $G_\mu := \{g \in G \mid Ad_g^* \mu = \mu\}$. Assume that G_μ acts *regularly* on N, that is, N/G_μ is a smooth manifold with the canonical projection a surjective submersion, and assume that the conditions in Proposition 5 are satisfied. Then we can project the implicit Hamiltonian system (N, D_N, H_N) to an implicit Hamiltonian system (M_μ, D_μ, H_μ). Here $M_\mu := N/G_\mu = P^{-1}(\mu)/G_\mu$ is the regular reduced space and $H_\mu \in C^\infty(M_\mu)$, defined by $H_\mu \circ \pi = H_N$, is the reduced Hamiltonian. The implicit Hamiltonian system (M_μ, D_μ, H_μ) is called the *reduced* implicit Hamiltonian system corresponding to (M, D, H). Moreover, D_μ is called the reduced Dirac structure. If D happens to be the graph of a symplectic form ω, then D_μ is precisely the graph of the Marsden-Weinstein [24] reduced symplectic form ω_μ. Likewise, if D is the graph of a Poisson structure J on M, then D_μ is the graph of the reduced Poisson structure J_μ [22]. Notice however, that contrary to these classical reduction results, the Dirac structure is *not* required to be closed for the reduction scheme to work. This observation is important since it allows the reduction method to be applied to the class of mechanical systems with (possibly nonholonomic) kinematic constraints. See [6, 7] for further information and a discussion of other recent results in this area. Finally, notice that if D happens to be closed then also the reduced Dirac structure D_μ will be closed.

As a last remark we would like to mention that in [6, 7] it is shown that the reduction scheme can also be applied the other way round. One starts with factorizing by the symmetry group G, thus obtaining M/G, and afterwards restricting to a level set $\tilde{M}_\mu$ of the remaining first integrals (which actually turn out to be Casimir functions). The resulting implicit Hamiltonian system on $\tilde{M}_\mu$ is *isomorphic* to the system (M_μ, D_μ, H_μ). Notice that $\tilde{M}_\mu$ equals the *orbit reduced space* $P^{-1}(\mathcal{O}_\mu)/G$, where $\mathcal{O}_\mu$ denotes the coadjoint orbit in $\mathfrak{g}^*$ through μ. See also [19, 21, 27] for classical orbit reduction if $\mathcal{O}_\mu$ is embedded in $\mathfrak{g}^*$ and [30] for the general regular and singular cases.

Intrinsic reductions. In this paragraph we compare the reduction results described in this section with what we call *intrinsic reductions.* The latter are independent of any symmetry properties of the Dirac structure and, in fact, can be performed on *any closed Dirac structure.* They are analogous in spirit to the Cartan reduction of a symplectic form to the leaf space of the characteristic distribution of its pull back to a given submanifold (see e.g. §4.3 in [1]) or the reduction of a Poisson structure relative to a subbundle compatible with the bracket (see [22]). These kind of reductions have been described in the literature by various authors [6, 7, 10, 15, 20].

Consider a closed Dirac structure D on M. By (2.9) it follows that the characteristic distribution Δ is involutive. Assume, in addition, that Δ has constant rank, that is, $\Delta \subset TM$ is a vector subbundle. Then, by Frobenius' Theorem, Δ defines a regular foliation Φ_Δ partitioning M into integral submanifolds of Δ. On the other hand, the distribution defined by[5]

$$\Theta(x) := \{X(x) \mid X \in \mathfrak{X}_{loc}(M), \exists \alpha \in \Omega^1_{loc}(M) \text{ such that } (X, \alpha) \in \mathfrak{D}_{loc}\} \tag{3.10}$$

is clearly also involutive. Assuming, as before, that $\Theta \subset TM$ is a subbundle, it follows that Θ defines a regular foliation Φ_Θ partitioning M into integral submanifolds of Θ.

There a two logical ways to "reduce" the Dirac structure on M to a lower dimensional manifold. The first is to project the Dirac structure to the quotient manifold M/Φ_Δ, i.e., by factoring out the characteristic distribution. This was done in [10] where it was shown that the Dirac structure D on M induces a well defined Poisson bracket on the quotient manifold M/Φ_Δ ([10], Corollary 2.6.3). This remarkable result was generalized in [20] to Dirac structures on Lie bialgebroids as described in Remark 6, where it was referred to as *Poisson reduction.* In [6] it was observed that this reduction can be interpreted as a special case of *symmetry reduction* if one notices that the distribution Δ is a symmetry distribution of D, which means that every section Y of Δ is a symmetry of D as in Definition 4. The Dirac structure D can be projected to a Dirac structure $\hat{D}$ on M/Φ_Δ using Proposition 5. It turns out that $\hat{D}$ is exactly the graph of the Poisson structure corresponding to the Poisson bracket defined by Courant [10]. We refer to [6], Example 4.2.4, p. 73, for more details.

The second possibility to obtain a Dirac structure on a lower dimensional manifold is to restrict the Dirac structure on M to each of the integral submanifolds of Θ. This can be done using Proposition 2 and results in a Dirac structure on each of the integral submanifolds of Θ. In [6, 7] it is shown that each of the reduced Dirac structures represents a presymplectic structure on the corresponding leaf of the foliation. See [6], Example 4.1.8, p. 69, and [7], Example 9, p. 79. This corresponds to Theorem 2.3.6 in Courant [10] and Theorem 2.2 in Dorfman [15], stating that a closed Dirac structure has a foliation by presymplectic leaves.

Once more we want to stress that the reductions described above are "intrinsic" and have nothing to do with the existence of any symmetry groups of the implicit Hamiltonian system. (Although as explained above, the first reduction can be interpreted in terms of symmetries of the Dirac structure.) They can be performed on any *closed* Dirac structure. We will not deal with these intrinsic

[5] This distribution is usually denoted by G_1 in the literature.

reductions in the paper and instead will investigate *symmetry Lie groups* of implicit Hamiltonian systems, together with their (singular) reductions. Doing so, we do *not* assume that the Dirac structure is closed and, in fact, all our results will be presented for the general case.

4 Singular reduction

Contrary to the regular reduction reviewed in the previous section, we now describe a *topological* method of reducing a Dirac structure on M to one on the reduced space M_μ, even if M_μ is not a manifold. This occurs when μ is a singular value of the momentum map P. In such a case, vector fields and differential one-forms on M_μ are not defined. Therefore, the results described in the previous section cannot be used. Describing the dynamics corresponding to such a topologically reduced Dirac structure on M_μ will be done in section 5. For ease of exposition we will take $\mu = 0$ throughout the rest of this paper.

From this point on we specifically consider Dirac structures of the form (2.10), admitting symmetries as described in Proposition 1. Let us specify the precise setting of the problem. Given is a vector subbundle $\Delta \subset TM$ and a generalized Poisson structure $J : T^*M \to TM$ (not necessarily satisfying the Jacobi identity). Define the Dirac structure by

$$D(x) = \{(v, v^*) \in T_xM \times T_x^*M \mid v - J(x)v^* \in \Delta(x),\ v^* \in \Delta^\circ(x)\}. \tag{4.1}$$

The vector field $Y \in \mathfrak{X}(M)$ is a symmetry of D if

$$L_Y \circ J = J \circ L_Y, \text{ and } L_Y Z \text{ is a local section of } \Delta \text{ whenever } Z \text{ is a local section of } \Delta. \tag{4.2}$$

A symmetry Lie group of a Dirac structure of the type (4.1) is defined as a smooth left action $\phi : G \times M \to M$ satisfying for every $g \in G$

$$\phi_g^* \circ J = J \circ \phi_g^*, \text{ and } \phi_g^* Z \text{ is a local section of } \Delta \text{ whenever } Z \text{ is a local section of } \Delta. \tag{4.3}$$

It immediately follows that if G is a symmetry group in the sense defined above then for every $(X, \alpha) \in \mathfrak{D}_{loc}$ and every $g \in G$ it follows that $\left(\phi_g^* X, \phi_g^* \alpha\right) \in \mathfrak{D}_{loc}$. Thus G is a symmetry Lie group in the sense of Definition 6. Also, if (4.3) holds, then every infinitesimal generator ξ_M, $\xi \in \mathfrak{g}$, is a symmetry of D as defined in (4.2). Assume, in addition, that the action ϕ admits an Ad^*-equivariant momentum map $P : M \to \mathfrak{g}^*$. Recall from (3.8) that this means $(\xi_M, dP_\xi) \in \mathfrak{D}$ for all $\xi \in \mathfrak{g}$, where $P_\xi(x) := \langle P(x), \xi \rangle$, $x \in M$. It is not assumed that G acts regularly on M.

Let $\mu = 0 \in \mathfrak{g}^*$ be a singular value of P and consider $N := P^{-1}(0)$. The level set N is, in general, not a smooth submanifold of M. However, N is a closed subset of M and it is a topological space relative to the induced subspace topology. The level set N is G-invariant, since the coadjoint isotropy subgroup G_0 at zero equals G. Hence one can endow the orbit space $M_0 := N/G = P^{-1}(0)/G$ with the quotient topology. Denote by $\pi : N \to M_0$ the canonical projection map, that is, π maps $x \in N$ onto its orbit $G \cdot x \in M_0$.

Define the set of smooth functions on M_0 as follows (see [11] or, for the original source, [32]).

Definition 8. *A continuous function f_0 on M_0 is called* smooth*, denoted by $f_0 \in C^\infty(M_0)$, if there exists a smooth G-invariant function $f \in C^\infty(M)^G$ such that $f_0 \circ \pi = f|_{P^{-1}(0)}$.*

Given the singular reduced space M_0 together with its topology and a set of smooth functions $C^\infty(M_0)$ on M_0, our goal is to define a reduced Dirac structure on M_0.

We begin by constructing the generalized Poisson structure. By (4.3), G is a symmetry Lie group of the generalized Poisson bracket $\{\cdot,\cdot\}: C^\infty(M)\times C^\infty(M) \to C^\infty(M)$ corresponding to the bundle map J. Therefore we can use the theory in [3, 11, 28, 29, 30] to define a generalized Poisson bracket $\{\cdot,\cdot\}_0 : C^\infty(M_0) \times C^\infty(M_0) \to C^\infty(M_0)$ on the singular reduced space M_0. This goes as follows. Let $f_0, h_0 \in C^\infty(M_0)$ and let $f, h \in C^\infty(M)^G$ be such that $f_0 \circ \pi = f|_{P^{-1}(0)}$ and $h_0 \circ \pi = h|_{P^{-1}(0)}$. Define the bracket

$$\{f_0, h_0\}_0 \circ \pi = \{f, h\}|_{P^{-1}(0)}. \tag{4.4}$$

This yields a well defined generalized Poisson bracket on M_0. In particular, (4.4) does not depend on the choice of the G-invariant extensions f and h (whose existence is assumed, by definition).

Remark 7. The reduction theory in [3, 11] is only developed for the singular reduction of *symplectic* manifolds under a symmetry Lie group action. That is, the Poisson bracket $\{\cdot,\cdot\}$ is assumed to be nondegenerate and to satisfy the Jacobi identity. In particular, [3, 11] show that (under the assumption that G acts properly) nondegeneracy of $\{\cdot,\cdot\}$ implies that of $\{\cdot,\cdot\}_0$. In [29, 30] the reduction theory is carried out for *Poisson brackets* in the singular case and for *presheaves of Poisson algebras*. This extends easily to the case of generalized Poisson brackets, as described above in (4.4). Note that by (4.4), the bracket $\{\cdot,\cdot\}_0$ satisfies the Jacobi identity if $\{\cdot,\cdot\}$ does. The general theory of reduction for Leibniz brackets (one drops also the skew-symmetry condition and is left only with a bilinear operation $\{\cdot,\cdot\}: C^\infty(M) \times C^\infty(M) \to C^\infty(M)$ satisfying the derivation property in every factor) and various applications thereof can be found in [28].

Once again for clarity: In this paper we neither assume that the generalized Poisson bracket $\{\cdot,\cdot\}$ is nondegenerate, nor that it satisfies the Jacobi identity. Furthermore, properness of the group action is not needed until Section 6. ♦

Next, we construct the analogue of the subbundle $\Delta \subset TM$ on M_0. Since M_0 is not a manifold, it is hopeless to search for a subbundle of the inexistent tangent bundle of M_0, so instead one seeks a vector space of derivations $\hat{\Delta}$ on $C^\infty(M_0)$ naturally induced by Δ. To do this, denote by $\Gamma_{loc}(\Delta)$ the local sections of the subbundle $\Delta \subset TM$. We shall show that every vector field $X \in \Gamma_{loc}(\Delta)$ is "tangent" to $N = P^{-1}(0)$. For regular values μ, when $N = P^{-1}(\mu)$ is a smooth submanifold of M, this means that X restricts to a well defined vector field $\bar{X}$ on N. However, if $\mu = 0$ is a singular value of the momentum map, then N is not a smooth manifold and hence we have to define what "tangent" means. Recall that a vector field $X \in \mathfrak{X}(M)$ is in one-to-one correspondence with a derivation on $C^\infty(M)$, denoted by the same letter $X : C^\infty(M) \to C^\infty(M)$. This correspondence is given by the Lie derivative formula

$$L_X f := X[f] := \langle df, X\rangle, \quad \forall f \in C^\infty(M). \tag{4.5}$$

A derivation X on $C^\infty(M)$ is said to be *tangent* to the subset $N \subset M$ if it restricts to a well defined derivation $\bar{X}$ on the set of Whitney smooth functions on N. A continuous function $\bar{f}$ on N is said to be a *Whitney smooth function* if there exists a smooth function f on M such that $\bar{f} = f|_N$; the set of Whitney smooth functions on N is denoted by $W^\infty(N)$. Otherwise stated, X *is tangent to* N *if there exists a derivation* $\bar{X}$ *on* $W^\infty(N)$ *such that* $X[f](x) = \bar{X}[f|_N](x)$ *for all* $f \in C^\infty(M)$

and all $x \in N$. A necessary and sufficient condition for X to be tangent to N is that

$$X[f](x) = X[h](x), \quad \forall x \in N, \tag{4.6}$$

for all $f, h \in C^\infty(M)$ such that $f|_N = h|_N$. Notice that if N is a smooth *closed submanifold* of M and M is *paracompact*, then the set $W^\infty(N)$ of Whitney smooth functions on N equals the set $C^\infty(N)$ of all smooth functions on N (as defined by the differential structure on the submanifold N)[6]. In this case, the previous definition has the usual meaning of a vector field X being tangent to the submanifold N. Consequently, its restriction $\bar{X}$ to N yields a vector field on N.

Consider a vector field (or equivalently, a derivation) X on M. Define $\gamma(t)$ to be an *integral curve* of X through $x_0 \in M$ if[7]

$$\frac{d}{dt} f(\gamma(t)) = X[f](\gamma(t)), \quad \forall t, \; \forall f \in C^\infty(M), \; \gamma(0) = x_0. \tag{4.7}$$

Now, consider an arbitrary vector field $X \in \Gamma_{loc}(\Delta)$ and let $\gamma(t)$ be an integral curve of X through $x_0 \in P^{-1}(0)$. In particular,

$$\frac{d}{dt} P_\xi(\gamma(t)) = X[P_\xi](\gamma(t)) = 0, \; \forall t, \; \forall \xi \in \mathfrak{g}. \tag{4.8}$$

This follows from the fact that, by (4.1) and (3.8), we have $dP_\xi(x) \in \Delta^\circ(x)$, $\forall x \in M$. Thus the integral curve of $X \in \Gamma_{loc}(\Delta)$ through every $x_0 \in P^{-1}(0)$ is contained in $P^{-1}(0)$. By the equivalence of derivations and velocity vectors it then follows that

$$X[f](x_0) = \left.\frac{d}{dt}\right|_{t=0} f(\gamma(t)) = \left.\frac{d}{dt}\right|_{t=0} h(\gamma(t)) = X[h](x_0), \tag{4.9}$$

for all $f, h \in C^\infty(M)$ satisfying $f|_N = h|_N$. (Remember that M is a smooth manifold.) Hence (4.6) holds which shows that every vector field $X \in \Gamma_{loc}(\Delta)$ is tangent to $N = P^{-1}(0)$. Consequently, every vector field $X \in \Gamma_{loc}(\Delta)$ restricts to a well defined derivation $\bar{X}$ on $W^\infty(N)$. In conclusion, the constant dimensional distribution Δ on M restricts to a vector space $\bar{\Delta}$ of derivations on $W^\infty(N)$. If Δ is locally spanned by the independent vector fields $X_1, \ldots, X_m$, then $\bar{\Delta}$ is locally spanned by the independent derivations $\bar{X}_1, \ldots, \bar{X}_m$.

Using the results mentioned above, we will now show that the distribution Δ on M projects to a well defined vector space $\hat{\Delta}$ of derivations on the smooth functions $C^\infty(M_0)$. A vector field X on M is said to *project* to M_0 if there exists a derivation $\hat{X}$ on $C^\infty(M_0)$ such that for every $f \in C^\infty(M)^G$, $X[f](x) = \hat{X}[f_0](\pi(x))$, $\forall x \in N$, where f_0 is defined by $f_0 \circ \pi = f|_N$. It is clear that X restricts to a well defined derivation $\hat{X}$ on $C^\infty(M_0)$ if and only if

1. $X[f](x)$ does not depend on the extension of $f_0 \circ \pi$ off N to M, and

2. $X[f](x) = X[f](y)$ for all $x, y \in N$ such that $\pi(x) = \pi(y)$.

[6]The definition of Whitney smooth functions can be relaxed as in [30]. Proposition 1.1.23 in [30] guarantees this equality if N is an embedded submanifold of M. With the more common defintion given here, that coincides with the one in [11], even if N is open, the inclusion $W^\infty(N) \subset C^\infty(N)$ is strict, in general. However, with our definition, if M is paracompact, and hence admits partitions of unity, and N is closed, then $W^\infty(N) = C^\infty(N)$.

[7]Take the coordinate functions $f = x^i$ to obtain the usual definition, $\dot{\gamma}^i(t) = X^i(\gamma(t))$.

Now, let X be a local section of Δ. Since X is tangent to N it follows that $X[f](x) = \bar{X}[f|_N](x) = \bar{X}[f_0 \circ \pi](x)$, $\forall x \in N$. Therefore its value does not depend on the extension of $f_0 \circ \pi$ off N to M. It remains to show that

$$X[f](x) = X[f](y), \quad \forall x, y \in N \text{ such that } \pi(x) = \pi(y). \tag{4.10}$$

In general, this condition will not be satisfied by every local section X of Δ. To see this, assume that Y is a local section of Δ for which condition (4.10) is satisfied. Clearly, $X = hY$ is also a local section of Δ, for any $h \in C^\infty(M)$. However, X will satisfy condition (4.10) if *and only if* h is G-invariant, i.e. $h \in C^\infty(M)^G$.

What we will show, however, is the following: *There exists a basis of local sections* $X_1, \ldots, X_m \in \Gamma_{loc}(\Delta)$, *spanning* Δ, *which satisfy* (4.10). To do this, denote by $\mathfrak{V}_{loc}$ the space of local sections of the vertical distribution V. Recall that V is defined by $V(x) = \ker T_x\pi = \text{span}\,\{\xi_M(x) \mid \xi \in \mathfrak{g}\}$. Since by (4.2), $L_{\xi_M}\Gamma_{loc}(\Delta) \subset \Gamma_{loc}(\Delta)$ for every $\xi \in \mathfrak{g}$, it follows that $[\Gamma_{loc}(\Delta), \mathfrak{V}_{loc}] \subset \mathfrak{V}_{loc} + \Gamma_{loc}(\Delta)$. To see this, take an arbitrary local section of V of the form $Y = \sum_i h_i \xi^i_M$, with $h_i \in C^\infty(M)$, for $i = 1, \ldots, r = \dim \mathfrak{g}$. Here $\xi^1_M, \ldots, \xi^r_M$ denotes a local basis of V. If $X \in \Gamma_{loc}(\Delta)$, then

$$[X, Y] = \left[X, \sum_{i=1}^r h_i \xi^i_M\right] = \sum_{i=1}^r h_i \left[X, \xi^i_M\right] + (L_X h_i)\xi^i_M \in \Gamma_{loc}(\Delta) + \mathfrak{V}_{loc}. \tag{4.11}$$

This proves the inclusion $[\Gamma_{loc}(\Delta), \mathfrak{V}_{loc}] \subset \mathfrak{V}_{loc} + \Gamma_{loc}(\Delta)$.

Now assume that the distribution $V + \Delta$ has constant dimension on M. Then the above mentioned inclusion implies that there exists a basis of local sections $X_1, \ldots, X_m \in \Gamma_{loc}(\Delta)$, spanning Δ, such that $[X_i, \mathfrak{V}_{loc}] \subset \mathfrak{V}_{loc}$, $i = 1, \ldots, m$; see e.g. [26], Theorem 7.5, p. 214, for the proof of this statement. (In the notation used in that theorem: the involutive distribution D is V, the distribution G is Δ, and one takes $f = 0$.) In particular, $[X_i, \xi_M] \in \mathfrak{V}_{loc}$, which implies that for all $f \in C^\infty(M)^G$

$$0 = [X_i, \xi_M]\,[f] = X_i\,[L_{\xi_M} f] - L_{\xi_M}(X_i[f]) = -L_{\xi_M}(X_i[f]),\ \forall \xi \in \mathfrak{g}. \tag{4.12}$$

This means that the function $X_i[f]$ is G-invariant and therefore satisfies (4.10). In conclusion, we have shown that there exists a basis of local sections $X_1, \ldots, X_m \in \Gamma_{loc}(\Delta)$, spanning Δ, such that each X_i projects to a well defined derivation $\hat{X}_i$ on $C^\infty(M_0)$. The derivations $\hat{X}_1, \ldots, \hat{X}_m$ locally span (in other words, form a basis of) a vector space of derivations on $C^\infty(M_0)$, which we will denote by $\hat{\Delta}$.

Remark 8. Notice that the constructed basis $\hat{X}_1, \ldots, \hat{X}_m$ of $\hat{\Delta}$ depends on the choice of a "projectable" basis $X_1, \ldots, X_m$ of Δ (which, as we have shown, exists). Of course, any other choice $Y_1, \ldots, Y_m$ of a projectable basis of Δ will lead to another constructed basis $\hat{Y}_1, \ldots, \hat{Y}_m$. It is clear from the construction of these bases that this second basis spans the *same* vector space $\hat{\Delta}$ of derivations on $C^\infty(M_0)$. Hence, the definition of $\hat{\Delta}$ is independent on the choice of a projectable basis for Δ. This means that $\hat{\Delta}$ is an *intrinsically* defined vector space, only determined by Δ and the symmetry group action. ♦

Remark 9. In the regular case, i.e., when $\mu = 0$ is a regular value of the momentum map and G acts freely and properly on M, the reduced space M_0 is a smooth manifold. Furthermore, under the assumption that M is paracompact, the set of smooth functions $C^\infty(M_0)$ equals the set of smooth functions as defined by the differential structure on M_0. To see this, notice that since $N = P^{-1}(0)$ is closed in M, every smooth G-invariant function on N can be smoothly extended to a G-invariant function on M (see [3]). In this case, the notion of a "projecting derivation" as employed above has the usual meaning of the projection of a vector field on M to a vector field on the reduced space M_0. In particular, there exists a basis $X_1, \ldots, X_m$ of local sections of Δ, tangent to N, such that their restrictions $\bar{X}_1, \ldots, \bar{X}_m$ project to M_0. That is, each $\bar{X}_i$ is π-related to a vector field $\hat{X}_i$ on M_0. The projected vector fields $\hat{X}_1, \ldots, \hat{X}_m$ form a basis of the local sections of $\hat{\Delta}$. ♦

So far we have defined the following three objects on the singular reduced space M_0:

1. a set of smooth functions $C^\infty(M_0)$,
2. a generalized Poisson bracket $\{\cdot,\cdot\}_0$ on $C^\infty(M_0)$, and
3. a vector space $\hat{\Delta}$ of derivations on $C^\infty(M_0)$.

Recall that the original Dirac structure D on the manifold M (of the type (4.1)) was completely determined by the generalized Poisson bracket $\{\cdot,\cdot\}$, corresponding to J and the distribution Δ. Therefore it makes sense to define a reduced Dirac structure on M_0 as follows.

Definition 9. *Consider the singular reduced space M_0 together with the set of smooth functions $C^\infty(M_0)$. The* singular reduced Dirac structure D_0 *is defined as the pair $(\{\cdot,\cdot\}_0, \hat{\Delta})$.*

We call D_0 a *topological Dirac structure*. It will be shown in the next section that the singular reduced Dirac structure D_0 defines a Hamiltonian formalism on the singular reduced space M_0.

In order to get a better understanding of the topological Dirac structure introduced in Definition 9, let us see how things look like in case of regular reduction. We now show that, in the case of *regular* reduction, the topological Dirac structure D_0 equals the regular reduced Dirac structure on M_0 as described in Section 3.

Regular reduction. Suppose that $\mu = 0$ is a regular value of the momentum map and that G acts regularly on M. That is, M/G is a smooth manifold and the projection $M \to M/G$ is a surjective submersion. For example, if G acts freely and properly, then these conditions are satisfied. According to the results described in Section 3, the Dirac structure D on M is reduced to a Dirac structure $\hat{D}$ on the manifold M_0 in two steps. Firstly, D is restricted to a Dirac structure D_N on $N = P^{-1}(0)$ defined by (3.4). Secondly, D_N is projected to a Dirac structure $\hat{D}$ on M_0 defined by (3.7) (where $\mathfrak{D}_{loc}$ should be replaced by $(\mathfrak{D}_N)_{loc}$). Otherwise stated, in terms of its local sections we have

$$\begin{aligned}\hat{\mathfrak{D}}_{loc} = \{(\hat{X}, \hat{\alpha}) \in \mathfrak{X}_{loc}(M_0) \oplus \Omega^1_{loc}(M_0) \mid \exists (X, \alpha) \in \mathfrak{D}_{loc} \text{ such that } X \text{ is tangent to } N \\ \text{and } X|_N \sim_\pi \hat{X},\ \iota^*\alpha = \pi^*\hat{\alpha}\}.\end{aligned} \tag{4.13}$$

Here $\iota : N \hookrightarrow M$ denotes the inclusion map and $X|_N$ denotes the restriction of X to N, which exists, since N is a closed submanifold of M and X is tangent to N by hypothesis.

Consider the topological Dirac structure D_0 introduced in Definition 9. Since M_0 is a manifold, the generalized Poisson bracket $\{\cdot,\cdot\}_0$ on the set of smooth functions $C^\infty(M_0)$ (see also Remark 9) defines a skew-symmetric vector bundle map $J_0 : T^*M_0 \to TM_0$. This map is completely defined by the condition that $\langle df_0, J_0(dh_0)\rangle = \{f_0, h_0\}_0$, $\forall f_0, h_0 \in C^\infty(M_0)$. The vector space $\hat{\Delta}$ of derivations on $C^\infty(M_0)$ defines a constant dimensional distribution of vector fields on M_0 (in other words, a vector subbundle of TM_0), also denoted by $\hat{\Delta}$. It now follows that the topological Dirac structure D_0 defines a usual Dirac structure on the manifold M_0. Indeed, this Dirac structure, also denoted by D_0, is given in terms of its local sections by

$$(\mathfrak{D}_0)_{loc} = \{(\hat{X}, \hat{\alpha}) \in \mathfrak{X}_{loc}(M_0) \oplus \Omega^1_{loc}(M_0) \mid \hat{X} - J_0\hat{\alpha} \in \Gamma_{loc}(\hat{\Delta}),\ \hat{\alpha} \in \Gamma_{loc}(\hat{\Delta}^\circ)\}. \tag{4.14}$$

This makes it obvious that D_0 is a Dirac structure on M_0 according to Definition 1, since it has the form given in (2.11). Now we show that $\hat{D} = D_0$. Since both are Dirac structures, and therefore their fibers are of the same dimension (i.e., equal to dim M_0), it is enough to show that $\hat{D} \subset D_0$.

Let $(\hat{X}, \hat{\alpha})$ be a local section of $\hat{D}$. Then, by (4.13), there exists a local section (X, α) of D such that $X|_N \sim_\pi \hat{X}$ and $\iota^*\alpha = \pi^*\hat{\alpha}$. Since (X, α) is a local section of D (being of the form (2.11)) we conclude that

$$Z := X - J\alpha \text{ is a local section of } \Delta, \text{ and } \alpha \text{ is a local section of } \Delta^\circ. \tag{4.15}$$

Consider the vector field $J\alpha \in \mathfrak{X}_{loc}(M)$. Since $(J\alpha, \alpha) \in \mathfrak{D}_{loc}$, it follows from (3.8) and $D = D^\perp$ that

$$(J\alpha)[P_\xi](x) = \langle dP_\xi, J\alpha\rangle(x) = -\langle \alpha, \xi_M\rangle(x) = -\langle \hat{\alpha}, 0\rangle(\pi(x)) = 0,\ \forall x \in N,\ \forall \xi \in \mathfrak{g}. \tag{4.16}$$

This implies that the vector field $J\alpha$ is tangent to N. Furthermore, by construction of the reduced generalized bracket (4.4), it follows that $(J\alpha)|_N \sim_\pi J_0\hat{\alpha}$. Since also $X|_N \sim_\pi \hat{X}$, equation (4.15) implies that there exists a vector field $\hat{Z} \in \mathfrak{X}_{loc}(M_0)$ such that $Z|_N \sim_\pi \hat{Z}$. It follows that $\hat{Z} \in \Gamma_{loc}(\hat{\Delta})$, by construction of $\hat{\Delta}$. This yields

$$\hat{X} - J_0\hat{\alpha} = \hat{Z} \in \Gamma_{loc}(\hat{\Delta}). \tag{4.17}$$

By construction, the distribution $\hat{\Delta}$ is spanned by vector fields $\hat{Z}_1, \ldots, \hat{Z}_m$ for which there exists a basis of vector fields $Z_1, \ldots, Z_m \in \Gamma_{loc}(\Delta)$ such that $Z_j|_N \sim_\pi \hat{Z}_j$, for $j = 1, \ldots, m$. Since $\iota^*\alpha = \pi^*\hat{\alpha}$ and $\alpha \in \Gamma_{loc}(\Delta^\circ)$, it follows immediately that

$$\langle \hat{\alpha}, \hat{Z}_j\rangle \circ \pi = \langle \alpha, Z_j\rangle \circ \iota = 0, \quad j = 1, \ldots, m. \tag{4.18}$$

Therefore $\hat{\alpha} \in \Gamma_{loc}(\hat{\Delta}^\circ)$. By (4.14) it follows that $(\hat{X}, \hat{\alpha})$ is a local section of D_0. Hence we have shown that $\hat{D} \subset D_0$. Since both are Dirac structures on M_0 this implies that $\hat{D} = D_0$.

We conclude that in the case of regular reduction, the topological Dirac structure D_0 equals the regular reduced Dirac structure on M_0.

5 Singular dynamics

This section introduces a Hamiltonian formalism for the singular reduced Dirac structure D_0 in Definition 9. It defines the dynamics corresponding to an implicit Hamiltonian system (M_0, D_0, H_0) on the singular reduced space M_0. We shall show that if (M_0, D_0, H_0) is the reduction of the implicit

Hamiltonian system (M, D, H), then the G-projectable solutions of (M, D, H) project to solutions of the reduced system (M_0, D_0, H_0).

First let us define a Hamiltonian formalism on a topological space in the spirit of Sikorski differential spaces (see [13, 33]). Consider a topological space M_0 together with a subalgebra $C^\infty(M_0)$ of the algebra of continuous functions on M_0, called the set of *smooth functions* on M_0. A continuous curve $\gamma(t)$ on M_0 is said to be *smooth* (see [33]) if $f_0 \circ \gamma$ is smooth, as a function from (a subinterval of) $\mathbb{R}$ to $\mathbb{R}$, for every $f_0 \in C^\infty(M_0)$. Let $\hat{X}$ denote a derivation on $C^\infty(M_0)$. An *integral curve* of $\hat{X}$ through some point $x_0 \in M_0$ is defined (see [33]) as a smooth curve $\gamma(t)$ for which, cf. (4.7),

$$\frac{d}{dt} f_0(\gamma(t)) = \hat{X}[f_0](\gamma(t)), \quad \forall t,\ \forall f_0 \in C^\infty(M_0),\ \gamma(0) = x_0. \tag{5.1}$$

Let D_0 be a topological Dirac structure on M_0 as given in Definition 9. It consists of a generalized Poisson bracket $\{\cdot, \cdot\}_0$ on $C^\infty(M_0)$ (not necessarily satisfying the Jacobi identity) and a vector space $\hat{\Delta}$ of derivations on $C^\infty(M_0)$. Furthermore, let $H_0 \in C^\infty(M_0)$ be a smooth function on M_0, called the Hamiltonian function. Notice that $\{\cdot, H_0\}_0 : C^\infty(M_0) \to C^\infty(M_0)$ defines a derivation on $C^\infty(M_0)$ by $\{\cdot, H_0\}_0[f_0] := \{f_0, H_0\}_0,\ f_0 \in C^\infty(M_0)$. If $\hat{X}$ is a derivation on $C^\infty(M_0)$ and $x \in M_0$, then $\hat{X}(x) : C^\infty(M_0) \to \mathbb{R}$ is defined by $(\hat{X}(x))[f_0] := \hat{X}[f_0](x),\ f_0 \in C^\infty(M_0)$. The three-tuple (M_0, D_0, H_0) defines an implicit Hamiltonian system in the following way.

Definition 10. *A smooth curve $\gamma(t)$ on M_0 is called an* integral curve *(or,* solution*) of (M_0, D_0, H_0) if there exists a derivation $\hat{X}$ on $C^\infty(M_0)$ such that $\gamma(t)$ is an integral curve of $\hat{X}$, and*

$$\hat{X}(\gamma(t)) - \{\cdot, H_0\}_0(\gamma(t)) \in \hat{\Delta}(\gamma(t)), \quad \forall t, \tag{5.2}$$

$$\hat{Z}[H_0](\gamma(t)) = 0, \quad \forall t,\ \forall \hat{Z} \in \hat{\Delta}. \tag{5.3}$$

The implicit Hamiltonian system (M_0, D_0, H_0) is defined as the collection of all integral curves $\gamma(t)$ of (M_0, D_0, H_0).

Note that if M_0 would be a smooth manifold, then Definition 10 of an implicit Hamiltonian system equals Definition 3 given in Section 2 (with D_0 defined by (4.14)). However, since M_0 is not a smooth manifold but only a topological space, the implicit Hamiltonian system (M_0, D_0, H_0) *cannot* be written as a set of differential and algebraic equations. As for implicit Hamiltonian systems defined on manifolds, the implicit Hamiltonian system (M_0, D_0, H_0) is energy conserving, cf. (2.15),

$$\frac{dH_0}{dt}(\gamma(t)) = \hat{X}[H_0](\gamma(t)) = \{H_0, H_0\}_0(\gamma(t)) = 0,\ \forall t. \tag{5.4}$$

Remark 10. Equation (5.2) implies that

$$\frac{d}{dt} f_0(\gamma(t)) = \{f_0, H_0\}_0(\gamma(t)), \quad \forall t,\ \forall f_0 \in \mathcal{A}_{D_0}, \tag{5.5}$$

where $\mathcal{A}_{D_0} = \{f_0 \in C^\infty(M_0) \mid \hat{Z}[f_0] = 0,\ \forall \hat{Z} \in \hat{\Delta}\}$. However, (5.5) does not imply (5.2). Even in the regular case it is not true that $\hat{Z}$ being a local section of $\hat{\Delta}$ is equivalent to $\hat{Z}[f_0] = 0,\ \forall f_0 \in \mathcal{A}_{D_0}$. A counterexample can easily be constructed by considering a suitable *noninvolutive* distribution $\hat{\Delta}$. ♦

Remark 11. If $\hat{\Delta} = 0$, then $\mathcal{A}_{D_0} = C^\infty(M_0)$ and (5.2, 5.3) are equivalent to (5.5). This happens when $\Delta = 0$, i.e. when the original system (M, D, H) does not include constraints. In this case, the Hamiltonian dynamics defined by (5.5) is exactly the singular reduced Hamiltonian dynamics as defined in [11, 12, 29, 30, 34]. ♦

Recall that implicit Hamiltonian systems defined on manifolds define a set of differential and algebraic equations. As a consequence, the standard results on existence and uniqueness of solutions for ordinary differential equations do not apply. As explained in Section 2, in general one cannot expect existence or uniqueness of solutions for these systems. Therefore one can neither expect existence nor uniqueness of solutions for implicit Hamiltonian systems on topological spaces, such as given in Definition 10. In particular, all solutions necessarily lie on the *constraint space*

$$M_0^c = \{x \in M_0 \mid \hat{Z}[H_0](x) = 0,\ \forall \hat{Z} \in \hat{\Delta}\}. \tag{5.6}$$

(This is a topological space, with topology induced from M_0.) What we can show however, is the following. Suppose (M_0, D_0, H_0) is the singular reduction of an implicit Hamiltonian system (M, D, H). Then, as we will show next, every G-projectable solution of (M, D, H), *if it exists*, will project to a solution of (M_0, D_0, H_0).

So let $x(t)$ be a solution of (M, D, H) with $x(0) \in N = P^{-1}(0)$. Then, by (3.9), the curve $x(t)$ is contained in N. Now assume that $x(t)$ is a *G-projectable* solution. Recall that this means that there exists a projectable vector field X on M, with integral curve $x(t)$, such that X projects to a well defined derivation $\hat{X}$ on $C^\infty(M_0)$. By (2.10) and (2.14) it follows that

$$X(x(t)) - \{\cdot, H\}(x(t)) =: Z(x(t)) \in \Delta(x(t)), \quad \forall t, \tag{5.7}$$

$$Y[H](x(t)) = 0, \quad \forall t,\ \forall Y \in \Gamma_{loc}(\Delta). \tag{5.8}$$

Let M_0 be the singular reduced space and D_0 the singular reduced Dirac structure on M_0. Since H is assumed to be G-invariant, its restriction to N projects to a well defined function $H_0 \in C^\infty(M_0)$ defined by $H_0 \circ \pi = H|_N$. (For the definition of $C^\infty(M_0)$, see Definition 8 in Section 4.) Define the singular reduced implicit Hamiltonian system (M_0, D_0, H_0) as in Definition 10. Project the curve $x(t)$ to M_0 to obtain the smooth curve $\gamma(t) = \pi(x(t))$ on M_0. (Recall that $\pi : N \to M_0$ denotes the projection map.) Then $\gamma(t)$ is an integral curve of the derivation $\hat{X}$. To see this, take an arbitrary $f_0 \in C^\infty(M_0)$ and let $f \in C^\infty(M)^G$ be such that $f_0 \circ \pi = f|_N$. Then

$$\frac{d}{dt} f_0(\gamma(t)) = \frac{d}{dt} f(x(t)) = X[f](x(t)) = \hat{X}[f_0](\gamma(t)), \quad \forall t, \tag{5.9}$$

where we used the fact that $x(t)$ is an integral curve of X, cf. (4.7), and that X projects to a derivation $\hat{X}$ on $C^\infty(M_0)$. According to (5.1) it follows that $\gamma(t)$ is an integral curve of $\hat{X}$. Furthermore, suppose that $Y_1, \ldots, Y_m \in \Gamma_{loc}(\Delta)$ is a basis of projectable local sections spanning Δ, hence projecting to a basis $\hat{Y}_1, \ldots, \hat{Y}_m$ of $\hat{\Delta}$. Then it follows that

$$0 = Y_j[H](x(t)) = \hat{Y}_j[H_0](\gamma(t)), \quad \forall t,\ j = 1, \ldots, m. \tag{5.10}$$

This yields equation (5.3). It remains to be proven that condition (5.2) is satisfied. Notice that by (4.4) the derivation $\{\cdot, H\}$ projects to a well defined derivation $\{\cdot, H_0\}_0$ on $C^\infty(M_0)$. Since also X

projects to a derivation $\hat{X}$ on $C^\infty(M_0)$, it follows from (5.7) that the derivation Z projects to a well defined derivation on $C^\infty(M_0)$, denoted by $\hat{Z}$. Again, suppose that $Y_1, \dots, Y_m \in \Gamma_{loc}(\Delta)$ is a basis of projectable local sections spanning Δ. Then it follows from (5.7) that at each point x_0 on the curve $x(t)$, one has

$$Z(x_0) = c_1 Y_1(x_0) + \cdots + c_m Y_m(x_0), \tag{5.11}$$

for some constants $c_1, \dots, c_m \in \mathbb{R}$. We claim that the reduced derivation $\hat{Z}$ can be expressed as

$$\hat{Z}(\gamma_0) = c_1 \hat{Y}_1(\gamma_0) + \cdots + c_m \hat{Y}(\gamma_0), \quad \text{where } \gamma_0 = \pi(x_0). \tag{5.12}$$

To see this, take an arbitrary $f_0 \in C^\infty(M_0)$ and let $f \in C^\infty(M)^G$ be such that $f_0 \circ \pi = f|_N$. Then

$$\begin{aligned}(\hat{Z}(\gamma_0))[f_0] = \hat{Z}[f_0](\gamma_0) = Z[f](x_0) \\ &= (c_1 Y_1[f] + \cdots + c_m Y_m[f])\,(x_0) \\ &= \Big(c_1 \hat{Y}_1[f_0] + \cdots + c_m \hat{Y}_m[f_0]\Big)\,(\gamma_0) \\ &= \Big(c_1 \hat{Y}_1(\gamma_0) + \cdots + c_m \hat{Y}_m(\gamma_0)\Big)\,[f_0], \end{aligned} \tag{5.13}$$

which proves (5.12). Since $\hat{Y}_1, \dots, \hat{Y}_m$ forms a basis of $\hat{\Delta}$ it follows that $\hat{Z}(\gamma(t)) \in \hat{\Delta}(\gamma(t))$, $\forall t$. Therefore condition (5.2) is satisfied. This implies that $\gamma(t)$ is an integral curve of the reduced implicit Hamiltonian system (M_0, D_0, H_0). We have obtained the following result.

Proposition 7. *Every G-projectable solution $x(t)$ of (M, D, H), with $x(0) \in P^{-1}(0)$, projects to a solution $\gamma(t) = \pi(x(t))$ of the singular reduced implicit Hamiltonian system (M_0, D_0, H_0).*

Remark 12. Suppose that the implicit Hamiltonian system (M, D, H) is of index 1. As remarked in Section 2, the system can be restricted to an explicit Hamiltonian system on the constraint manifold M_c. This restricted system is defined by a generalized Poisson bracket, denoted by $\{\cdot,\cdot\}_c$, on $C^\infty(M_c)$. Since this system is explicit, it is described by a set of ordinary differential equations on M_c. Hence we can conclude the local existence and uniqueness of solutions of this system on M_c. The G-action leaves the manifold M_c invariant. Hence G is a symmetry Lie group of the explicit Hamiltonian system on M_c. That is, $L_{\xi_{M_c}}\{f, g\}_c = \{L_{\xi_{M_c}} f, g\}_c + \{f, L_{\xi_{M_c}} g\}_c$, $\forall f, g \in C^\infty(M_c)$, and $L_{\xi_{M_c}} H_c = 0$ (where $H_c = H|_{M_c}$), for all $\xi \in \mathfrak{g}$. The corresponding equivariant momentum map is given by the restriction of P to M_c. Every solution of the (restricted) system is G-projectable. See [6, 7, 37] for details on all this. Then we can use the singular reduction theory developed in [11, 12, 29, 30, 34] to reduce the system to a Hamiltonian system on the singular reduced space M_0. Alternatively, we can use the theory developed in this paper, considering $\Delta = 0$. The reduced generalized Poisson bracket $(\{\cdot,\cdot\}_c)_0$ is defined in the same way as in (4.4). The reduced dynamics is given by equation (5.5), where $\{\cdot,\cdot\}_0$ is replaced by $(\{\cdot,\cdot\}_c)_0$, and $\mathcal{A}_{D_0} = C^\infty((M_c)_0)$. Local existence of solutions now follows from the local existence of solutions on M_c and Proposition 7. Furthermore, if the G-action is *proper* then also uniqueness of solutions of the singular reduced system can be proved [11, 34]. ♦

6 Orbit type decomposition

Consider a *symplectic* manifold (M, ω) admitting a symmetry Lie group with a corresponding equivariant momentum map. Let M_0 denote the singular reduced space, and $\{\cdot,\cdot\}_0$ the singular reduced Poisson bracket. The singular reduced Hamiltonian dynamics is defined by equation (5.5), cf. Remark 11. In [4, 11, 12, 13, 30, 34] it is shown that the space M_0 may be decomposed into a family of symplectic manifolds, called pieces. The decomposition is by orbit type and defines a stratification of the singular reduced space M_0. Furthermore, the Hamiltonian flow corresponding to (5.5) leaves the pieces invariant and restricts to a regular Hamiltonian flow on each of the pieces. In this section we show that, to some extent, these results can be generalized to singular reduced implicit Hamiltonian systems. We specifically consider Dirac structures of the type (4.1) and assume that the generalized Poisson structure is nondegenerate. Furthermore, we assume that the infinitesimal generators corresponding to the Lie group action are Hamiltonian vector fields. As will become clear in the sequel, we need these assumptions to prove that the pieces corresponding to the orbit type decomposition are smooth manifolds. We refer to Remark 15 for a brief discussion of these assumptions.

Consider an implicit Hamiltonian system (M, D, H) with a Dirac structure D as defined in (4.1). The generalized Poisson structure defined by J is assumed to be nondegenerate. That is, the Poisson bracket is defined by a nondegenerate two-form ω on M. In fact, we have that $\{f, h\} = \langle df, J(dh)\rangle = \omega(X_f, X_h)$, $\forall f, h \in C^\infty(M)$, where X_f is defined by $df = \omega(X_f, \cdot)$. Notice that we do *not* assume that ω is a closed two-form. This means that we do not assume that $\{\cdot, \cdot\}$ satisfies the Jacobi identity.

Let G be a symmetry Lie group of (M, D, H), that is, it satisfies conditions (4.3). From now on we will assume that the action of G is *proper*. Furthermore, we assume that there exists an Ad^*-equivariant momentum map $P : M \to \mathfrak{g}^*$ such that

$$dP_\xi = \omega(\xi_M, \cdot) \quad \text{(equivalently: } \xi_M = JdP_\xi\text{)}, \quad \text{and } dP_\xi \text{ is a section of } \Delta^\circ, \quad \forall \xi \in \mathfrak{g}. \tag{6.1}$$

In other words, the infinitesimal generators of the Lie group action are Hamiltonian vector fields with Hamiltonian P_ξ and the differential of P_ξ annihilates Δ. Notice that this is a special case of (3.8).

In the sequel we will show that, under the assumptions given above, the singular reduced space M_0 can be decomposed into a set of smooth manifolds (pieces). The Dirac structure D on M reduces to a Dirac structure on each of the pieces. Hence, the implicit Hamiltonian system (M, D, H) can be reduced to an implicit Hamiltonian system on each of the pieces. This will be the result of Theorem 1. If, additionally, some constant dimensionality conditions are satisfied, then the reduced system on each of the pieces can be obtained by regular reduction; see Theorem 2 and Corollary 1. In the last part of this section we will show that the regular reduced implicit Hamiltonian system on each of the pieces equals the restriction of the singular reduced implicit Hamiltonian system (M_0, D_0, H_0) to the relevant piece. In order to enhance the readability of this section, we have subdivided it into a number of subsections.

The orbit type decomposition of M_0. The manifold M can be decomposed into submanifolds as follows [11, 30, 34]. Let K be a compact subgroup of G and define $M_{(K)}$ to be the set of points in M whose stabilizer group $G_x = \{g \in G \mid \phi(g, x) = x\}$ is conjugate to K, that is,

$$M_{(K)} = \{x \in M \mid \exists g \in G \text{ such that } gG_xg^{-1} = K\}. \tag{6.2}$$

Notice that since the G-action is assumed to be proper, every stabilizer group G_x, $x \in M$, is a compact subgroup of G. The set $M_{(K)}$ is a submanifold of M called the manifold of *orbit type* (K).

On the set of compact subgroups of G we can define an equivalence relation by saying that $\tilde{K} \sim K$ if and only if $\tilde{K}$ is conjugate to K. The equivalence class of K under this relation is denoted by (K). As (K) runs over the set of equivalence classes, the manifolds $M_{(K)}$ partition M. Since the G-action is proper it follows that this partition is locally finite. This partition is called the *orbit type decomposition* of M. We refer to [11, 16, 30, 34] for more details and proofs.

Next, we show that the image of the tangent of the momentum map at a point $x \in M$, is equal to the annihilator in $\mathfrak{g}^*$ of the Lie algebra of the stabilizer group G_x, that is,

$$\text{Im } T_x P = \mathfrak{g}_x^\circ, \ \forall x \in M. \tag{6.3}$$

See also [1, 11, 19, 23, 30]. This can be seen from the following set of equivalent expressions.

$$\xi \in \mathfrak{g}_x \Longleftrightarrow \xi_M(x) = 0 \Longleftrightarrow dP_\xi(x) = 0 \Longleftrightarrow (T_x P(v))\xi = 0, \ \forall v \in T_x M \Longleftrightarrow \xi \in (\text{Im } T_x P)^\circ \tag{6.4}$$

Here we used (6.1) and the fact that ω is nondegenerate. This yields $\mathfrak{g}_x = (\text{Im } T_x P)^\circ$. Taking the annihilator of both sides yields (6.3).

Equation (6.3) implies that the tangent of the restriction of the momentum map P to the submanifold $M_{(K)}$ has constant rank, equal to the codimension of K in G. It follows that the intersection $P^{-1}(0) \cap M_{(K)}$ is a smooth submanifold of M. The level set $P^{-1}(0)$ is invariant under the action of G. Clearly $M_{(K)}$ is also G-invariant. Hence, the manifold $P^{-1}(0) \cap M_{(K)}$ is G-invariant. It turns out that the quotient $(M_0)_{(K)} := (P^{-1}(0) \cap M_{(K)})/G = \pi(P^{-1}(0) \cap M_{(K)})$ is a smooth manifold [11, 30, 34]. Consequently, the singular reduced space M_0 is decomposed into a disjoint set of manifolds, called pieces,

$$M_0 = \coprod_{(K)} (M_0)_{(K)}. \tag{6.5}$$

Here (K) runs over the set of conjugacy classes of compact subgroups of G. Since the orbit type decomposition of M is locally finite, the decomposition of M_0 is also locally finite.

A generalized Poisson bracket on the pieces. Next, let us define a generalized Poisson bracket on each of the pieces $(M_0)_{(K)}$. For clarity of exposition, consider the commuting diagram in Figure 1. In this diagram ι, $\iota_{(K)}$, $\widetilde{\iota}_{(K)}$, and $\iota^0_{(K)}$ are inclusions and $\pi_{(K)}$ is the restriction of π to $P^{-1}(0) \cap M_{(K)}$. The commutativity of the diagram is obvious.

First, we define a set of smooth functions on $(M_0)_{(K)}$ called Whitney smooth functions. A continuous function $\bar{f}_0$ on $(M_0)_{(K)}$ is said to be a *Whitney smooth function* if there exists a smooth G-invariant function $f \in C^\infty(M)^G$ such that $\bar{f}_0 \circ \pi_{(K)} = f|_{P^{-1}(0) \cap M_{(K)}}$. The set of Whitney smooth functions on $(M_0)_{(K)}$ is denoted by $W^\infty((M_0)_{(K)})$.

Remark 13. In fact, $W^\infty((M_0)_{(K)})$ is equal to the set of functions obtained by restricting the functions in $C^\infty(M_0)$ to $(M_0)_{(K)}$. This is the reason why we have called it the set of Whitney smooth functions. To see this, notice that any G-invariant function f on M descends to a smooth function f_0 on M_0, whose restriction to $(M_0)_{(K)}$ is precisely $\bar{f}_0$. Indeed, use the commutativity of the diagram to obtain the following:

$$f_0 \circ \iota^0_{(K)} \circ \pi_{(K)} = f_0 \circ \pi \circ \tilde{\iota}_{(K)} = f \circ \iota \circ \tilde{\iota}_{(K)} = f \circ \iota_{(K)} = \bar{f}_0 \circ \pi_{(K)} \tag{6.6}$$

Since $\pi_{(K)}$ is surjective the result follows. ♦

$$
\begin{tikzcd}
 & M & \\
P^{-1}(0) \arrow[ur, "\iota"] \arrow[d, "\pi"] & & P^{-1}(0)\cap M_{(K)} \arrow[ul, "\iota_{(K)}"'] \arrow[ll, "\widetilde{\iota}_{(K)}"'] \arrow[d, "\pi_{(K)}"] \\
M_0 & & (M_0)_{(K)} \arrow[ll, "\iota^0_{(K)}"']
\end{tikzcd}
$$

Figure 1: Orbit type decomposition.

Just as in (4.4), we can define a generalized Poisson bracket $\{\cdot,\cdot\}_{(K)}$ on $W^\infty((M_0)_{(K)})$ by

$$\{\bar f_0, \bar h_0\}_{(K)} \circ \pi_{(K)} = \{f,h\}|_{P^{-1}(0)\cap M_{(K)}}. \tag{6.7}$$

We need to show that this bracket is well defined and does not depend on the choice of G-invariant extensions f and h. If we can prove that for every $f \in C^\infty(M)^G$ the flow of the Hamiltonian vector field $X_f = \{\cdot, f\}$ preserves the submanifold $P^{-1}(0)\cap M_{(K)}$, then $\mathcal{I} = \{f \in C^\infty(M)^G \mid f|_{P^{-1}(0)\cap M_{(K)}} = 0\}$ is a Poisson ideal of $C^\infty(M)^G$. In that case it follows that the bracket (6.7) is well defined (see e.g. [11], Appendix B, Claims (5.2) and (5.3)). Now consider an arbitrary $f \in C^\infty(M)^G$. The function f being G-invariant means that $\phi_g^* f = f$, for all $g \in G$. Here $\phi_g : M \to M$ denotes the diffeomorphism induced by the action of the element $g \in G$ on M. Since the action leaves the two-form ω invariant, it follows that $\phi_g^* X_f = X_{\phi_g^* f} = X_f$, for all $g \in G$. This implies that the flow ψ_t^f of X_f commutes with the action ϕ_g, for every $g \in G$. What we need to show is that ψ_t^f preserves the submanifold $M_{(K)}$. Notice that $M_{(K)} = G\cdot M_K$, where $M_K = \{x \in M \mid G_x = K\}$ (which again is a submanifold of M; see e.g. [11, 16, 30, 34]). Since the flow of X_f and the G-action commute, it is enough to show that ψ_t^f preserves M_K. In order to do that, proceed as follows. Observe that for every $g \in K$ and $x \in M_K$ we have

$$\phi_g(\psi_t^f(x)) = \psi_t^f(\phi_g(x)) = \psi_t^f(x), \tag{6.8}$$

since $\phi_g(x) = x$. Therefore, $K \subset G_{\psi_t^f(x)}$ (the stabilizer group of $\psi_t^f(x)$). Suppose $g \in G_{\psi_t^f(x)}$, that is, $\phi_g(\psi_t^f(x)) = \psi_t^f(x)$. Since the flow and the G-action commute, this implies that $\psi_t^f(\phi_g(x)) = \psi_t^f(x)$. Applying the backwards flow ψ_{-t}^f to this relation yields $\phi_g(x) = x$, so $g \in K$. Hence $G_{\psi_t^f(x)} \subset K$. It follows that $G_{\psi_t^f(x)} = K$ and therefore $\psi_t^f(x) \in M_K$. Hence we have shown that ψ_t^f preserves the submanifold M_K and therefore also the submanifold $M_{(K)}$. Since the flow ψ_t^f of any G-invariant function f preserves the level set $P^{-1}(0)$, we can conclude the following:

$$\text{The flow of any } G\text{-invariant function } f \text{ preserves the submanifold } P^{-1}(0)\cap M_{(K)}. \tag{6.9}$$

As indicated above, this implies that $\mathcal{I}$ is a Poisson ideal of $C^\infty(M)^G$ and hence the bracket (6.7) is well defined (see [11]).

The projection of Δ to the pieces. Next we show that the distribution Δ projects to a vector space $\hat{\Delta}_{(K)}$ of derivations on $W^\infty((M_0)_{(K)})$. Recall the following from Section 4 (cf. the text above formula (4.12)). There exists a basis $X_1, \dots, X_m \in \Gamma_{loc}(\Delta)$ of projectable local sections spanning Δ, such that $[X_j, \xi_M] \in \mathfrak{V}_{loc}$, for all $\xi \in \mathfrak{g}$, and $j = 1, \dots, m$. By condition (4.2) it follows that

$$[X_j, \xi_M] \in \Gamma_{loc}(\Delta \cap V), \quad \forall \xi \in \mathfrak{g},\ j = 1, \dots, m. \tag{6.10}$$

Now, if we make the additional assumption that $\Delta \cap V = 0$, then it follows that $[X_j, \xi_M] = 0$ and hence the flow of X_j commutes with the G-action. Just as in the previous subsection, it then follows that the flow of X_j preserves the submanifold $P^{-1}(0) \cap M_{(K)}$. This implies that X_j is tangent to the manifold $P^{-1}(0) \cap M_{(K)}$, for all $j = 1, \dots, m$. As in Section 4, the basis $X_1, \dots, X_m$ projects to a set of independent derivations $\hat{X}_1', \dots, \hat{X}_m'$ on $W^\infty((M_0)_{(K)})$, which define $\hat{\Delta}_{(K)}$.

Remark 14. If the system is of index 1, then it is always true that $\Delta(x) \cap V(x) = 0,\ \forall x \in M_c$. See [6, 7]. ♦

Summary and main results. So far in this section we have defined the following objects:

1. a decomposition of the singular reduced space M_0 into a set of smooth manifolds $(M_0)_{(K)}$, called pieces;
2. a generalized Poisson bracket $\{\cdot, \cdot\}_{(K)}$ on $W^\infty((M_0)_{(K)})$, for each of the pieces $(M_0)_{(K)}$;
3. a vector space of derivations $\hat{\Delta}_{(K)}$ on $W^\infty((M_0)_{(K)})$.

At this point, and throughout the rest of this Section, we will make the assumption that M is *paracompact*. Together with the properness of the G-action this implies that M admits G-partitions of unity. This, in turn, implies that the set of Whitney smooth functions $W^\infty((M_0)_{(K)})$ is dense in the set $C^\infty((M_0)_{(K)})$ of smooth functions as defined by the differential structure on $(M_0)_{(K)}$. To see this, notice that the pull back to $P^{-1}(0) \cap M_{(K)}$ of a smooth function $\bar{f}_0 \in C^\infty((M_0)_{(K)})$, compactly supported on $(M_0)_{(K)}$, can be extended to a smooth G-invariant function f on M.

Hence the bracket $\{\cdot, \cdot\}_{(K)}$ in (6.7) yields a well defined generalized Poisson bracket $\{\cdot, \cdot\}_{(K)}$ on $C^\infty((M_0)_{(K)})$. The associated generalized Poisson structure is denoted by $J_{(K)}$. It is equal to the inverse of the nondegenerate two-form $\omega_{(K)}$, which is defined by the following condition: the pull back of $\omega_{(K)}$ to $P^{-1}(0) \cap M_{(K)}$ equals the restriction of ω to $P^{-1}(0) \cap M_{(K)}$. See also [11, 30, 34].

Furthermore, the vector space $\hat{\Delta}_{(K)}$ of derivations on $W^\infty((M_0)_{(K)})$ defines a smooth distribution of vector fields on $(M_0)_{(K)}$.

Using these objects we can define a Dirac structure $D_{(K)}$ on each of the pieces $(M_0)_{(K)}$. In terms of its local sections, this Dirac structure is defined by

$$\begin{aligned}\left(\mathfrak{D}_{(K)}\right)_{loc} = \Big\{ (\hat{X}, \hat{\alpha}) \in \mathfrak{X}_{loc}\left((M_0)_{(K)}\right) \oplus \Omega^1_{loc}\left((M_0)_{(K)}\right) \Big| \\ \hat{X} - J_{(K)}\hat{\alpha} \in \Gamma_{loc}\left(\hat{\Delta}_{(K)}\right),\ \hat{\alpha} \in \Gamma_{loc}\left(\left(\hat{\Delta}_{(K)}\right)^\circ\right) \Big\}.\end{aligned} \tag{6.11}$$

We summarize the results we have obtained so far in this section in the following theorem.

Theorem 1. *Consider an implicit Hamiltonian system (M, D, H) with a Dirac structure D of type* (4.1)*, and assume M is paracompact. Suppose that the generalized Poisson structure defined by J*

is nondegenerate. Assume the system admits a symmetry Lie group G acting properly on M and satisfying the conditions (4.3)*. Assume also that this action admits an Ad^*-equivariant momentum map P satisfying* (6.1)*. Then the singular reduced space M_0 can be decomposed into a disjoint set of manifolds $(M_0)_{(K)}$, cf.* (6.5)*, called pieces.*

The generalized Poisson structure J induces a generalized Poisson structure $J_{(K)}$ on each of the pieces $(M_0)_{(K)}$, cf. (6.7)*. Assume furthermore that $\Delta \cap V = 0$. Then the distribution Δ projects to a distribution $\hat{\Delta}_{(K)}$ on $(M_0)_{(K)}$. These two objects define a Dirac structure $D_{(K)}$ on $(M_0)_{(K)}$, given by* (6.11)*. Define the reduced Hamiltonian $H_{(K)} \in C^\infty((M_0)_{(K)})$ by $H_{(K)} \circ \pi_{(K)} = H|_{P^{-1}(0)\cap M_{(K)}}$. Then the triple $((M_0)_{(K)}, D_{(K)}, H_{(K)})$ defines an implicit Hamiltonian system on the piece $(M_0)_{(K)}$.*

Remark 15. Suppose now that the generalized Poisson bracket is degenerate, or that the momentum map does not satisfy condition (6.1). It is not clear at the moment if in this case one can still prove that the sets $P^{-1}(0) \cap M_{(K)}$ and $(M_0)_{(K)}$ are smooth manifolds. On the other hand, a topological Dirac structure $D_{(K)}$ on $(M_0)_{(K)}$ may still be defined. Just as in Definition 9, it is defined by the bracket $\{\cdot,\cdot\}_{(K)}$ and the vector space of derivations $\hat{\Delta}_{(K)}$. As in Definition 10, it defines a Hamiltonian formalism on $(M_0)_{(K)}$. However, since $(M_0)_{(K)}$ may not be a smooth manifold, the corresponding Hamiltonian dynamics may not necessarily be expressible as a set of differential and algebraic equations. ♦

Later on, in Section 7, we will see that the assumptions in Theorem 1 are typically satisfied for the class of constrained mechanical systems defined in Example 1.

Under the assumptions in Theorem 1 it follows that the sets $P^{-1}(0) \cap M_{(K)}$ and $(M_0)_{(K)}$ are smooth manifolds. Now, assume that the conditions in Propositions 2 and 5 are satisfied. Then we can first restrict the Dirac structure D to a Dirac structure on $P^{-1}(0) \cap M_{(K)}$ and afterwards project to a Dirac structure on $(M_0)_{(K)}$. Following the same line of proof as in Section 4, under the heading "Regular reduction", one can show that the result of this *regular* reduction process is exactly the Dirac structure $D_{(K)}$ as defined in (6.11). For clarity, we will state the result in a theorem.

First we introduce the following notation. The manifold $N_{(K)} = P^{-1}(0) \cap M_{(K)}$ will play the role of N in Proposition 2. The restricted Dirac structure will be denoted by $D_{N_{(K)}}$. Its characteristic distribution is denoted by $\Delta_{N_{(K)}}$. Furthermore, denote by $E_{N_{(K)}}$ the vector subbundle of $TN_{(K)} \oplus T^*N_{(K)}$, whose local sections are defined by

$$\Gamma_{loc}(E_{N_{(K)}}) = \{(X,\alpha) \in \mathfrak{X}_{loc}(N_{(K)}) \oplus \Omega^1_{loc}(N_{(K)}) \mid \alpha = \pi^*_{(K)}\hat{\alpha} \text{ for some } \hat{\alpha} \in \Omega^1_{loc}((M_0)_{(K)})\}. \quad (6.12)$$

The vector bundle $E_{N_{(K)}}$ will play the role of E in Proposition 5. Now we can state the theorem.

Theorem 2. *Suppose that the conditions in Theorem 1 are satisfied. Assume that $D(x) \cap (T_xN_{(K)} \times T^*_xM)$, $x \in N_{(K)}$, has constant dimension on $N_{(K)}$. Then the Dirac structure D can be restricted to a Dirac structure $D_{N_{(K)}}$ on $N_{(K)}$ by using Proposition 2. Assume next that the following two conditions hold:*

1. *$V|_{N_{(K)}} + \Delta_{N_{(K)}}$ is a smooth vector subbundle[8] of $TN_{(K)}$, and*
2. *$D_{N_{(K)}} \cap E_{N_{(K)}}$ is a smooth vector subbundle of $TN_{(K)} \oplus T^*N_{(K)}$.*

[8]Notice that the distribution $V|_{N_{(K)}}$ is constant dimensional on $N_{(K)}$. Its dimension is equal to the codimension of K in G.

Then the Dirac structure $D_{N_{(K)}}$ projects to a Dirac structure on $(M_0)_{(K)}$ by using Proposition 5. The resulting Dirac structure is precisely $D_{(K)}$ given in (6.11).

Corollary 1. *Assume that all the conditions in Theorems 1 and 2 are satisfied. Then the implicit Hamiltonian system (M, D, H) reduces by means of regular reduction to an implicit Hamiltonian system on each of the pieces $(M_0)_{(K)}$, yielding precisely the system $\big((M_0)_{(K)}, D_{(K)}, H_{(K)}\big)$.*

Remark 16. The constant dimensionality conditions in Theorem 2 are technical conditions, needed for the two-step regular reduction scheme to work. Basically, Theorem 2 and Corollary 1 state that if these technical conditions are satisfied, then the result by regular reduction to the pieces equals the result of Theorem 1. Even in the case of regular reduction of an ordinary Poisson structure, the first of these three technical conditions will be needed in order to be able to describe the restriction to a Dirac structure on the level set N. (In this case, the other two conditions will be satisfied automatically.) The last two conditions specifically arise from including constraints (i.e. algebraic equations) in the formalism. For a detailed discussion on these technical conditions, we refer to [6], Chapter 5.1, or [7], Section 7. ♦

The restriction of the singular reduced system to the pieces. Finally, in the last part of this section we reconsider the singular reduced implicit Hamiltonian system (M_0, D_0, H_0), as defined in Sections 4 and 5. Assume that the conditions in Theorem 1 are satisfied. Then we prove that the implicit Hamiltonian systems $((M_0)_{(K)}, D_{(K)}, H_{(K)})$ are precisely obtained by restricting the singular reduced implicit Hamiltonian system (M_0, D_0, H_0) to each of the pieces $(M_0)_{(K)}$.

First we show that the inclusion map $\iota^0_{(K)} : (M_0)_{(K)} \to M_0$ in Figure 1 is a Poisson map. Consider arbitrary $f_0, h_0 \in C^\infty(M_0)$, together with their restrictions $\bar{f}_0, \bar{h}_0 \in W^\infty((M_0)_{(K)})$, cf. Remark 13. Let $f, h \in C^\infty(M)^G$ be such that $f_0 \circ \pi = f|_{P^{-1}(0)}$ and $h_0 \circ \pi = h|_{P^{-1}(0)}$. Notice that by (6.6) we have that $\bar{f}_0 \circ \pi_{(K)} = f|_{P^{-1}(0)\cap M_{(K)}}$ and $\bar{h}_0 \circ \pi_{(K)} = h|_{P^{-1}(0)\cap M_{(K)}}$. By definition of the brackets (see (4.4) and (6.7)) and by the commutativity of the diagram in Figure 1 we have the following:

$$\{\bar{f}_0, \bar{h}_0\}_{(K)} \circ \pi_{(K)} = \{f, h\} \circ \iota_{(K)} = \{f, h\} \circ \iota \circ \tilde{\iota}_{(K)} = \{f_0, h_0\}_0 \circ \pi \circ \tilde{\iota}_{(K)} = \{f_0, h_0\}_0 \circ \iota^0_{(K)} \circ \pi_{(K)} \quad (6.13)$$

Since $\pi_{(K)}$ is surjective it follows that

$$\{\bar{f}_0, \bar{h}_0\}_{(K)} = \{f_0, h_0\}_0|_{(M_0)_{(K)}}. \quad (6.14)$$

Hence the inclusion map $\iota^0_{(K)}$ is a Poisson map and it follows that the generalized Poisson bracket $\{\cdot,\cdot\}_{(K)}$ is precisely the restriction of the generalized Poisson bracket $\{\cdot,\cdot\}_0$ to $(M_0)_{(K)}$.

Second, recall the construction of $\hat{\Delta}$ in Section 4 and the construction of $\hat{\Delta}_{(K)}$ given in this section. The vector space of derivations $\hat{\Delta}$ on $C^\infty(M_0)$ restricts to a vector space, say $\hat{\Delta}|_{(M_0)_{(K)}}$, of derivations on $W^\infty\big((M_0)_{(K)}\big)$. Indeed, a derivation $\hat{X}$ in $\hat{\Delta}$ is the projection of a projectable vector field $X \in \Gamma_{loc}(\Delta)$. As explained before, this vector field projects to a vector field $\hat{X}'$ on $(M_0)_{(K)}$, which induces a derivation $\hat{X}'$ on $W^\infty\big((M_0)_{(K)}\big)$. Commutativity of the diagram in Figure 1, together with Remark 13, ensures that $\hat{X}'$ is the restriction of $\hat{X}$ to $W^\infty((M_0)_{(K)})$. By construction it is immediately clear that $\hat{\Delta}|_{(M_0)_{(K)}}$ and $\hat{\Delta}_{(K)}$ are equal. Indeed, both are derived from the same basis $X_1, \ldots, X_m \in \Gamma_{loc}(\Delta)$ of projectable local sections spanning Δ.

We conclude that the Dirac structure $D_{(K)}$ equals precisely the restriction of the singular reduced Dirac structure D_0 to the piece $(M_0)_{(K)}$.

Next, we show that the solutions $\gamma(t)$ of the singular reduced implicit Hamiltonian system (M_0, D_0, H_0) leave the pieces $(M_0)_{(K)}$ invariant and restrict to solutions of the implicit Hamiltonian systems $\left((M_0)_{(K)}, D_{(K)}, H_{(K)}\right)$. Consider a solution $\gamma(t)$ of (M_0, D_0, H_0), as stipulated in Definition 10. Recall the construction of $\hat{\Delta}$ in Section 4 and let $\hat{X}_1, \ldots, \hat{X}_m$ denote a basis of $\hat{\Delta}$. Then, everywhere locally along the integral curve $\gamma(t)$, there exist locally defined functions $c_0^1, \ldots, c_0^m \in C^\infty(M_0)$ such that

$$\hat{X}(\gamma(t)) - \{\cdot, H_0\}_0(\gamma(t)) = c_0^1(\gamma(t))\hat{X}_1(\gamma(t)) + \ldots + c_0^m(\gamma(t))\hat{X}_m(\gamma(t)) \in \hat{\Delta}(\gamma(t)). \tag{6.15}$$

Consider the derivation $\hat{Y}$ on $C^\infty(M_0)$ defined by

$$\hat{Y} = \{\cdot, H_0\}_0 + c_0^1\hat{X}_1 + \ldots + c_0^m\hat{X}_m. \tag{6.16}$$

This is the projection of a local vector field $Y = \{\cdot, H\} + c^1X_1 + \ldots + c^mX_m$ on M. Here $c^j \in C^\infty(M)^G$ are such that $c_0^j \circ \pi = c^j|_{P^{-1}(0)}$ and $X_1, \ldots, X_m \in \Gamma_{loc}(\Delta)$ form a basis of projectable local sections spanning Δ. Since the flow of this vector field commutes with the G-action, it preserves the submanifold $P^{-1}(0) \cap M_{(K)}$, cf. (6.9). It follows that the flow corresponding to the integral curve $\gamma(t)$ preserves the pieces $(M_0)_{(K)}$ and therefore $\gamma(t)$ restricts to a smooth curve $\bar{\gamma}(t)$ on $(M_0)_{(K)}$. The vector field Y restricts to a vector field $\hat{Y}'$ on $(M_0)_{(K)}$. By construction, it follows that

$$\hat{Y}' = \{\cdot, H_{(K)}\}_{(K)} + \bar{c}_0^1\hat{X}_1' + \ldots + \bar{c}_0^m\hat{X}_m'. \tag{6.17}$$

Here $\bar{c}_0^j = c_0^j|_{(M_0)_{(K)}}$, and $\hat{X}_j' \in \Gamma_{loc}\left(\hat{\Delta}_{(K)}\right)$ is the restriction of $\hat{X}_j$ to $(M_0)_{(K)}$, for $j = 1, \ldots, m$. The curve $\bar{\gamma}(t)$ is an integral curve of $\hat{Y}'$. To see this, notice that

$$\frac{d}{dt}\bar{f}_0(\bar{\gamma}(t)) = \frac{d}{dt}f_0(\gamma(t)) = \hat{X}[f_0](\gamma(t)) = \hat{Y}[f_0](\gamma(t)) = \hat{Y}'[\bar{f}_0](\bar{\gamma}(t)), \quad \forall \bar{f}_0 \in W^\infty\left((M_0)_{(K)}\right), \tag{6.18}$$

where we used (6.15). Since $W^\infty\left((M_0)_{(K)}\right)$ is dense in $C^\infty\left((M_0)_{(K)}\right)$, the result follows. Furthermore, it follows from (6.17) that everywhere locally along the integral curve $\gamma(t)$,

$$\hat{Y}'(\bar{\gamma}(t)) - J_{(K)}(\bar{\gamma}(t))dH_{(K)}(\bar{\gamma}(t)) = \hat{Y}'(\bar{\gamma}(t)) - \{\cdot, H_{(K)}\}_{(K)}(\bar{\gamma}(t)) \in \Gamma_{loc}\left(\hat{\Delta}_{(K)}\right)(\bar{\gamma}(t)). \tag{6.19}$$

Also, from (5.3) it follows that

$$\hat{Z}'[H_{(K)}](\bar{\gamma}(t)) = 0, \quad \forall \hat{Z}' \in \Gamma_{loc}\left(\hat{\Delta}_{(K)}\right). \tag{6.20}$$

This means that $\bar{\gamma}(t)$ is a solution of the implicit Hamiltonian system $\left((M_0)_{(K)}, D_{(K)}, H_{(K)}\right)$. Concluding, we have proved the following result.

Proposition 8. *Assume that the conditions in Theorem 1 are satisfied. Then the implicit Hamiltonian system $\left((M_0)_{(K)}, D_{(K)}, H_{(K)}\right)$ is exactly the restriction of the singular reduced implicit Hamiltonian system (M_0, D_0, H_0) to the piece $(M_0)_{(K)}$. A solution $\gamma(t)$ of (M_0, D_0, H_0), with $\gamma(0) \in (M_0)_{(K)}$, preserves the piece $(M_0)_{(K)}$ and restricts to a solution $\bar{\gamma}(t)$ of $((M_0)_{(K)}, D_{(K)}, H_{(K)})$.*

Finally, we remark that since the pieces $(M_0)_{(K)}$ are smooth manifolds, each implicit Hamiltonian system $\left((M_0)_{(K)}, D_{(K)}, H_{(K)}\right)$ can be written as a set of differential and algebraic equations (DAE). The singular reduced implicit Hamiltonian system (M_0, D_0, H_0) can thus be written as a collection of DAEs, one (set of differential and algebraic equations) on each piece.

7 Constrained mechanical systems

The assumptions in Theorem 1 are typically satisfied for the class of constrained mechanical systems described in Example 1. Usually, the symmetries of these systems are lifted from symmetries on the configuration space. Consider the constraint distribution $\Lambda = \ker A^T(q)$ on Q, where $A^T(q)$ is defined as in (2.21). Let G be a Lie group acting properly on the configuration space Q, with action denoted by $\phi_Q : G \times Q \to Q$. Suppose that the action leaves Λ invariant, that is, $(\phi_Q(g))^* \Lambda \subset \Lambda$ for all $g \in G$. (Equivalently, the action leaves the codistribution on Q defined by the one-forms $\alpha_1, \ldots, \alpha_k$ invariant.) Let $\mathfrak{g}$ be the Lie algebra of G. Then the infinitesimal generators of the action satisfy the condition $L_{\xi_Q}\Gamma_{loc}(\Lambda) \subset \Gamma_{loc}(\Lambda)$, $\forall \xi \in \mathfrak{g}$.

The action ϕ_Q of G on Q lifts to an action of G on $M = T^*Q$, denoted by $\phi : G \times M \to M$. It is defined by $\phi_g : M \to M, \quad \phi_g = T^*(\phi_Q(g^{-1}))$, for every $g \in G$; see e.g. [1]. Let (q,p) denote local coordinates for M. Then a straightforward calculation shows that the local expression of the infinitesimal generators ξ_M corresponding to the action ϕ is

$$\xi_M(q,p) = \begin{pmatrix} \xi_Q(q) \\ -\frac{\partial \xi_Q}{\partial q}(q)^T p \end{pmatrix}, \quad \xi \in \mathfrak{g}. \tag{7.1}$$

Let $\omega = dq \wedge dp$ denote the canonical symplectic form on M. It is well know that the lifted action ϕ leaves the symplectic form invariant; see e.g. [1], Corollary 4.2.11. Furthermore, it can easily be checked that $\phi_g^* \Delta \subset \Delta$, where Δ is given by (2.23). Hence G is a symmetry Lie group of the type (4.3). The infinitesimal generators satisfy the condition $L_{\xi_M}\Gamma_{loc}(\Delta) \subset \Gamma_{loc}(\Delta)$, $\forall \xi \in \mathfrak{g}$. Hence $\mathfrak{g}$ defines a symmetry Lie algebra of the type (4.2).

Furthermore, a standard result shows that the action ϕ has an Ad^*-equivariant momentum map $P : M \to \mathfrak{g}^*$; again see [1], Corollary 4.2.11. It is defined by $\langle P(\alpha_q), \xi \rangle = \alpha_q(\xi_Q(q))$ for $\alpha_q \in T_q^*Q$ and $\xi \in \mathfrak{g}$, or, in local coordinates, $P(q,p)(\xi) = p^T \xi_Q(q)$. Hence P satisfies the first condition in (6.1). Finally, we assume that the symmetry group acts "horizontally", that is, its infinitesimal generators satisfy the kinematic constraints: $\xi_Q \in \Lambda$, $\forall \xi \in \mathfrak{g}$. Then, as follows from (2.23), also the second condition in (6.1) is satisfied.

In the case of *regular* reduction the following additional assumption is made: the action of G on Q is free and proper. By definition, freeness means that for every $q \in Q$ the map $g \mapsto \phi_Q(g,q)$, $g \in G$, is one-to-one. (Notice that this implies that the action of G on Q has no fixed points.) In particular, $\xi_Q(q) \neq 0$, for all $q \in Q$ and $\xi \in \mathfrak{g} \setminus \{0\}$. Then it follows immediately from (2.23) and (7.1) that the last condition in Theorem 1 holds, namely $\Delta \cap V = 0$. Furthermore, in this case there is only one piece, corresponding to the identity element e in G. Indeed, we have that $M_{(e)} = G \cdot M_e = G \cdot M = M$. Hence, the one and only piece is given by $(M_0)_{(e)} = (P^{-1}(0) \cap M_{(e)})/G = P^{-1}(0)/G = M_0$, which is precisely the whole reduced space (being a smooth manifold). So Theorems 1 and 2 collapse to the usual regular reduction theorem. Furthermore, since we do reduction at $\mu = 0$ it is well known that the reduced space M_0 is symplectomorphic to the cotangent bundle $T^*(M/G)$, see e.g. [1, 17, 21, 31]. Regular reduction of constrained mechanical systems has been explicitly carried out by various authors. See e.g. [5], [6] (Chapter 5.3), [9, 35, 37].

However, in the case of *singular* reduction we do *not* assume that the G action on Q is free. In particular, at certain points $q \in Q$ the stabilizer group G_q is nonzero.[9] In that case we need an

[9]Indeed, recall that this is exactly the idea behind the orbit type decomposition given in Section 6.

extra condition to make sure that $\Delta \cap V = 0$. This condition is the following: For every point $(q,p) \in T^*Q$ we have that

$$\operatorname{Im}\left(-\frac{\partial \xi_Q}{\partial q}(q)^T p\right) \cap \operatorname{Im} A(q) = 0, \quad \forall \xi \in \mathfrak{g}_q. \tag{7.2}$$

Here $\mathfrak{g}_q$ denotes the Lie algebra of the stabilizer group G_q. If this condition holds then we can conclude that $\Delta \cap V = 0$. In that case all the assumptions in Theorem 1 are satisfied, and hence we can reduce the constrained mechanical system to a dynamical system on each of the pieces.

7.1 The spherical pendulum

As a basic example of singular reduction we will treat the spherical pendulum in this section. The technique of singular reduction applied to the spherical pendulum was first presented in [3]. See also [11]. In these texts the system was embedded as a constrained system in $T\mathbb{R}^3$. The manifold $Q = \mathbb{R}^3$ represents the configuration space of the pendulum, and $q \in \mathbb{R}^3$ describes the position of the pendulum's mass in Cartesian coordinates. The pendulum is assumed to have length and mass equal to 1. The tangent bundle $T\mathbb{R}^3$ is a symplectic manifold by pulling back the standard symplectic form on the cotangent bundle by the Euclidean metric. The constrained phase space was defined by $TS^2 = \{(q,v) \in T\mathbb{R}^3 \mid \|q\| = 1 \text{ and } q^T v = 0\}$. This is a cosymplectic submanifold of $T\mathbb{R}^3$. Hence the symplectic form on $T\mathbb{R}^3$ restricts to a symplectic form on TS^2. Using modified Dirac brackets the dynamics of the spherical pendulum was calculated as an *unconstrained* (i.e., explicit) Hamiltonian system on TS^2. Next, this unconstrained Hamiltonian system was reduced at the singular value zero of the momentum map, leading to an unconstrained reduced Hamiltonian system on the singular reduced space $(TS^2)_0$.

To illustrate the theory developed in this paper, we will take a different point of view and treat both the unreduced and the reduced system as constrained, i.e. *implicit*, Hamiltonian systems. We describe the spherical pendulum as a constrained Hamiltonian system on the configuration space $Q = \mathbb{R}^3$. The kinematic constraint is holonomic and is given by

$$\alpha(q)\dot{q} = 0, \quad \text{with } \alpha(q) = q_1 dq_1 + q_2 dq_2 + q_3 dq_3 \in T_q^* Q, \tag{7.3}$$

which integrates to

$$q_1^2 + q_2^2 + q_3^2 = 1 \tag{7.4}$$

(the length of the pendulum). The implicit Hamiltonian system will be reduced at the singular value zero of the momentum map. The result is an *implicit* Hamiltonian system on the singular reduced space. The constraint is integrable (i.e. holonomic) and will be shown to give rise to a *Casimir* function of the singular reduced system. Our presentation here is based in part on the calculations and analysis performed in the above mentioned references. However, the implicit point of view, as well as the corresponding calculations regarding the projection of the holonomic constraint to the singular reduced space, are new. Furthermore, we will decompose the singular reduced space into pieces and calculate the dynamics on each of these pieces. We will show that these dynamics generate the usual equations of motion for a planar pendulum. Surprisingly, the reduction of the dynamics to the pieces and their correspondence to the equations of motion for a planar pendulum does not seem to have been reported in the literature before, at least to our knowledge.

Consider the Lie group S^1 acting on Q by rotations about the vertical q_3-axis:

$$(q_1, q_2, q_3) \overset{\theta \in S^1}{\longmapsto} (q_1 \cos\theta - q_2 \sin\theta, q_1 \sin\theta + q_2 \cos\theta, q_3). \tag{7.5}$$

Since S^1 is compact, its action is automatically proper. The infinitesimal generator is given by

$$\xi_Q(q) = -q_2 \frac{\partial}{\partial q_1} + q_1 \frac{\partial}{\partial q_2} \ . \tag{7.6}$$

Denote the holonomic constraint (7.4) by the zero level set of the function $C(q)$, where $C(q) := q_1^2 + q_2^2 + q_3^2 - 1$. Since $L_{\xi_Q} C = i_{\xi_Q} dC = 0$, the constraint is invariant under the S^1 action. Moreover the action is horizontal. The cotangent bundle $M := T^*Q \simeq \mathbb{R}^6$, with fiber coordinates $(p_1, p_2, p_3) \in T_q^*Q$, is equipped with the canonical symplectic form $\omega = dq_1 \wedge dp_1 + dq_2 \wedge dp_2 + dq_3 \wedge dp_3$. The action on Q lifts to an action on the cotangent bundle given by

$$\begin{aligned}(q_1, q_2, q_3, p_1, p_2, p_3) \overset{\theta \in S^1}{\longmapsto} & (q_1 \cos\theta - q_2 \sin\theta, q_1 \sin\theta + q_2 \cos\theta, q_3, \\ & p_1 \cos\theta - p_2 \sin\theta, p_1 \sin\theta + p_2 \cos\theta, p_3).\end{aligned} \tag{7.7}$$

The infinitesimal generator is given by

$$\xi_M(q, p) = -q_2 \frac{\partial}{\partial q_1} + q_1 \frac{\partial}{\partial q_2} - p_2 \frac{\partial}{\partial p_1} + p_1 \frac{\partial}{\partial p_2} \ . \tag{7.8}$$

The corresponding momentum map is defined by $P(q, p) = q_1 p_2 - q_2 p_1$, representing the angular momentum of the pendulum about the q_3-axis. The Hamiltonian of the system is defined by the total energy of the pendulum,

$$H(q, p) = \frac{1}{2}(p_1^2 + p_2^2 + p_3^2) + q_3. \tag{7.9}$$

Clearly, the Hamiltonian is invariant under the S^1 action (7.7). The vector field X spanning Δ can be calculated as $X = -\omega^\sharp(\pi_Q^* \alpha)$, where α is given by (7.3). This yields

$$\Delta = \text{span} \ \left\{ X := q_1 \frac{\partial}{\partial p_1} + q_2 \frac{\partial}{\partial p_2} + q_3 \frac{\partial}{\partial p_3} \right\}. \tag{7.10}$$

From (7.6) it follows that the stabilizer group of the action on Q is nonzero if and only if $q_1 = q_2 = 0$. In other words, every point on the q_3-axis is a fixed point of the rotation. At these points we have that $\xi_M = -p_2 \frac{\partial}{\partial p_1} + p_1 \frac{\partial}{\partial p_2}$ and $\Delta = \text{span} \ \{q_3 \frac{\partial}{\partial p_3}\}$. Hence, condition (7.2) is satisfied and we can conclude that $\Delta \cap V = 0$. Thus all conditions in Theorem 1 are satisfied and we can perform the reduction process.

Reduction to the singular reduced space. We do reduction at $\mu = 0$, which is clearly a singular value of P. To see this, notice that dP vanishes at the points where $q_1 = q_2 = p_1 = p_2 = 0$, which lie in $P^{-1}(0)$. In order to describe the singular reduced space $M_0 = P^{-1}(0)/S^1$, we need to find the algebra of S^1-invariant polynomials on $\mathbb{R}^6$. According to [3, 11, 43] this algebra is generated by

$$\begin{aligned} &\sigma_1 = q_3, \qquad &&\sigma_3 = p_1^2 + p_2^2 + p_3^2, \qquad &&\sigma_5 = q_1^2 + q_2^2, \\ &\sigma_2 = p_3, \qquad &&\sigma_4 = q_1 p_1 + q_2 p_2, \qquad &&\sigma_6 = q_1 p_2 - q_2 p_1. \end{aligned}$$

Notice that σ_6 equals the momentum map P. These polynomials satisfy the (in-)equalities

$$\sigma_4^2 + \sigma_6^2 = \sigma_5(\sigma_3 - \sigma_2^2), \qquad \sigma_3 \geq 0, \qquad \sigma_5 \geq 0. \tag{7.11}$$

The *Hilbert map* for the S^1 action is defined by

$$\sigma : \mathbb{R}^6 \to \mathbb{R}^6, \quad (q,p) \mapsto (\sigma_1(q,p), \ldots, \sigma_6(q,p)). \tag{7.12}$$

According to [3, 11] the singular reduced space M_0 can be identified with $\sigma(P^{-1}(0))$. Hence we conclude that M_0 is given by the semialgebraic variety

$$M_0 = \{(\sigma_1, \ldots, \sigma_5) \in \mathbb{R}^5 \mid \sigma_4^2 = \sigma_5(\sigma_3 - \sigma_2^2),\ \sigma_3 \geq 0,\ \sigma_5 \geq 0\}. \tag{7.13}$$

The singular reduced Poisson bracket $\{\cdot,\cdot\}_0$ on M_0 can easily be calculated as follows. First, calculate all the pairwise brackets $\{\sigma_i, \sigma_j\}$ on $\mathbb{R}^6$ (where, $\{\cdot,\cdot\}$ denotes the standard Poisson bracket defined by ω). This yields the following table, see [3, 11]:

$\{\sigma_i, \sigma_j\}$	σ_1	σ_2	σ_3	σ_4	σ_5	σ_6
σ_1	0	1	$2\sigma_2$	0	0	0
σ_2	-1	0	0	0	0	0
σ_3	$-2\sigma_2$	0	0	$-2(\sigma_3 - \sigma_2^2)$	$-4\sigma_4$	0
σ_4	0	0	$2(\sigma_3 - \sigma_2^2)$	0	$-2\sigma_5$	0
σ_5	0	0	$4\sigma_4$	$2\sigma_5$	0	0
σ_6	0	0	0	0	0	0

Notice that the bracket of σ_6 with any other invariant polynomial vanishes. Hence, the bracket $\{\cdot,\cdot\}_0$ on $M_0 = \sigma(P^{-1}(0)) = \sigma(\sigma_6^{-1}(0))$ is simply given by deleting the last row and column in the table above. Secondly, we calculate the vector space $\hat{\Delta}$ of derivations on $C^\infty(M_0)$. Recall from Section 4 that we need to show that the spanning vector field X in (7.10) projects to a derivation $\hat{X}$ on $C^\infty(M_0)$. By Definition 8, a function $f_0(\sigma_1, \ldots, \sigma_5)$ on M_0 is smooth if and only if there exists a S^1-invariant function $f(\sigma_1, \ldots, \sigma_6)$ such that $f(\sigma_1, \ldots, \sigma_5, 0) = f_0(\sigma_1, \ldots, \sigma_5)$. Now,

$$\begin{aligned}
X[f]\Big|_{\sigma_6=0} &= \sum_{i=1}^{3} \frac{\partial f}{\partial p_i} q_i \Big|_{\sigma_6=0} \\
&= \sum_{i=1}^{3}\sum_{j=1}^{6} \frac{\partial f}{\partial \sigma_j}\frac{\partial \sigma_j}{\partial p_i} q_i \Big|_{\sigma_6=0} \\
&= \left(\frac{\partial f}{\partial \sigma_2} q_3 + \frac{\partial f}{\partial \sigma_3} 2(q_1p_1 + q_2p_2 + q_3p_3) + \frac{\partial f}{\partial \sigma_4}(q_1^2 + q_2^2)\right)\Big|_{\sigma_6=0} \\
&= \left(\frac{\partial f}{\partial \sigma_2}\sigma_1 + \frac{\partial f}{\partial \sigma_3} 2(\sigma_4 + \sigma_1\sigma_2) + \frac{\partial f}{\partial \sigma_4}\sigma_5\right)\Big|_{\sigma_6=0} \\
&= \frac{\partial f_0}{\partial \sigma_2}\sigma_1 + \frac{\partial f_0}{\partial \sigma_3} 2(\sigma_4 + \sigma_1\sigma_2) + \frac{\partial f_0}{\partial \sigma_4}\sigma_5 \\
&=: \hat{X}[f_0].
\end{aligned} \tag{7.14}$$

Hence, the vector field X projects to a derivation $\hat{X}$ on $C^\infty(M_0)$, defined by the last two lines of (7.14). This derivation spans a one-dimensional vector space of derivations on $C^\infty(M_0)$ denoted by $\hat{\Delta}$. As in Definition 9, the singular reduced Dirac structure is defined as the pair $(\{\cdot,\cdot\}_0, \hat{\Delta})$. The dynamics of the implicit Hamiltonian system (M_0, D_0, H_0), with $H_0 = \frac{1}{2}\sigma_3 + \sigma_1$, is described in Definition 10.

Some remarks are in order. First of all, consider the smooth function $\mathfrak{C} = \sigma_4^2 - \sigma_5(\sigma_3 - \sigma_2^2)$ on $\mathbb{R}^5$. It can easily be seen that $\mathfrak{C}$ is a Casimir function of the bracket $\{\cdot,\cdot\}_0$. Indeed, a simple calculation shows that $\{\sigma_i, \mathfrak{C}\}_0 = 0$ for all $i = 1, \ldots, 5$. Furthermore, another calculation shows that we also have that $\hat{X}[\mathfrak{C}] = 0$. This implies that $\mathfrak{C}$ is a Casimir function of the singular reduced system. As a consequence we can conclude that any integral curve of (M_0, D_0, H_0) leaves the level set $\mathfrak{C}^{-1}(0)$ invariant, and hence defines an integral curve on M_0. That is, we have directly checked that the singular dynamics on M_0 is indeed well defined.

Secondly, recall that the kinematic constraint defined by (7.3) is holonomic, with integral (7.4). Next, we show that the constraint on the singular reduced space, defined by the derivation $\hat{\Delta}$, is also holonomic. That is, it can be "integrated" to a Casimir function. Indeed, the function $C(q) = q_1^2 + q_2^2 + q_3^2 - 1$ defined earlier induces on M_0 the function $C_0(\sigma_1, \ldots, \sigma_5) := \sigma_5 + \sigma_1^2 - 1$. As can easily be checked, $\hat{X}[C_0] = 0$. Moreover, we also have

$$\{\cdot, C_0\}_0 = -2\begin{pmatrix}0 & \sigma_1 & 2(\sigma_4 + \sigma_1\sigma_2) & \sigma_5 & 0\end{pmatrix}^T = -2\hat{X} \in \hat{\Delta}. \tag{7.15}$$

This implies that C_0 is a *Casimir function of the singular reduced dynamics* since it satisfies the condition in Definition 7. (Notice, however, that we are considering a topological Dirac structure in this case!) In particular, the level set $C_0^{-1}(0)$ is invariant under the integral curves of (M_0, D_0, H_0). If we would restrict the implicit dynamical system to the constraint space $M_0^c = C_0^{-1}(0)$, then we would obtain the unconstrained singular reduced Hamiltonian system studied in [3, 11].

Reduction to the pieces. Next, we calculate the pieces. Since S^1 is one-dimensional, its only connected subgroups are e (the identity element, $\theta = 0$) and S^1 itself. Furthermore, S^1 is Abelian. The manifold of *orbit type* S^1 is exactly given by the fixed points of the action on $\mathbb{R}^6$. That is,

$$M_{(S^1)} = \{(q,p) \in \mathbb{R}^6 \mid q_1 = q_2 = 0 \text{ and } p_1 = p_2 = 0\}. \tag{7.16}$$

The manifold of orbit type e is just the complement, i.e., $M_{(e)} = \mathbb{R}^6 \setminus M_{(S^1)}$. The action on $M_{(S^1)}$ is trivial, whereas the action on $M_{(e)}$ is free and proper (and hence, regular). Factoring out the group action yields the following.

$$(M_0)_{(S^1)} = \sigma\big(P^{-1}(0) \cap M_{(S^1)}\big) = \{(\sigma_1, \ldots, \sigma_5) \in \mathbb{R}^5 \mid \sigma_3 - \sigma_2^2 = 0,\ \sigma_4 = 0,\ \sigma_5 = 0\}. \tag{7.17}$$

This is a smooth 2-dimensional manifold, being the graph of the function $(\sigma_1, \sigma_2) \mapsto (\sigma_1, \sigma_2, \sigma_2^2, 0, 0)$.

To calculate the second piece, notice that

$$P^{-1}(0) \cap M_{(e)} = \{(q,p) \in \mathbb{R}^6 \mid q_1p_2 - q_2p_1 = 0 \text{ and } q_1^2 + q_2^2 + p_1^2 + p_2^2 > 0\} \tag{7.18}$$

which implies

$$\begin{aligned}(M_0)_{(e)} &= \sigma\big(P^{-1}(0) \cap M_{(e)}\big)\\ &= \{(\sigma_1, \ldots, \sigma_5) \in \mathbb{R}^5 \mid \sigma_4^2 = \sigma_5(\sigma_3 - \sigma_2^2),\ \sigma_3 \geq 0,\ \sigma_5 \geq 0,\ \sigma_5 + \sigma_3 - \sigma_2^2 > 0\}.\end{aligned} \tag{7.19}$$

This can equivalently be written as (notice the inequalities)

$$(M_0)_{(e)} = \{(\sigma_1, \ldots, \sigma_5) \in \mathbb{R}^5 \mid \sigma_4^2 = \sigma_5(\sigma_3 - \sigma_2^2),\ \sigma_3 - \sigma_2^2 \geq 0,\ \sigma_5 \geq 0,\ \sigma_5 + \sigma_3 - \sigma_2^2 > 0\}. \quad (7.20)$$

The piece $(M_0)_{(e)}$ is a connected smooth 4-dimensional manifold. To see this, consider two charts. The first is defined by the subset

$$U_1 = \{(\sigma_1, \ldots, \sigma_5) \in \mathbb{R}^5 \mid \sigma_4^2 = \sigma_5(\sigma_3 - \sigma_2^2),\ \sigma_3 - \sigma_2^2 > 0,\ \sigma_5 \geq 0\}. \quad (7.21)$$

On U_1, define the bijection $(\sigma_1, \sigma_2, \sigma_3, \sigma_4) \in \mathbb{R} \times \mathbb{R} \times (0, \infty) \times \mathbb{R} \overset{u_1}{\longmapsto} \left(\sigma_1, \sigma_2, \sigma_3, \sigma_4, \frac{\sigma_4^2}{\sigma_3 - \sigma_2^2}\right) \in U_1$.

Secondly, define the subset

$$U_2 = \{(\sigma_1, \ldots, \sigma_5) \in \mathbb{R}^5 \mid \sigma_4^2 = \sigma_5(\sigma_3 - \sigma_2^2),\ \sigma_3 - \sigma_2^2 \geq 0,\ \sigma_5 > 0\}. \quad (7.22)$$

On U_2, define the bijection $(\sigma_1, \sigma_2, \sigma_4, \sigma_5) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times (0, \infty) \overset{u_2}{\longmapsto} (\sigma_1, \sigma_2, \frac{\sigma_4^2}{\sigma_5} + \sigma_2^2, \sigma_4, \sigma_5) \in U_2$.

Clearly, U_1 and U_2 overlap and the overlap map $u_2^{-1} \circ u_1$ is smooth. Hence, (U_1, u_1), (U_2, u_2) form an atlas of $(M_0)_{(e)}$, and it follows that $(M_0)_{(e)}$ is a smooth 4-dimensional manifold. Its connectedness is a direct verification. Finally, notice that $(M_0)_{(e)}$ is open and dense in M_0, as expected from the general theory (see [12], Theorem 2.7, and [34], Theorem 5.9).

Dynamics on the pieces. Next, we calculate the dynamics on the pieces. We start with the two-dimensional piece $(M_0)_{(S^1)}$. In the original coordinates this piece is defined by the equations $q_1 = q_2 = 0$ and $p_1 = p_2 = 0$. It is clear that the implicit dynamical system on this piece, together with the holonomic constraint (7.4), restricts to the points $(0, 0, \pm 1, 0, 0, 0)$. These are the equilibria, where $(0, 0, -1, 0, 0, 0)$ represents the stable downward and $(0, 0, 1, 0, 0, 0)$ the unstable upright equilibrium position of the pendulum.

To calculate the dynamics on the second piece $(M_0)_{(e)}$ we first restrict our attention to the chart (U_1, u_1). We need to calculate the Poisson bracket and the characteristic distribution on U_1. Recall that the bracket $\{\cdot, \cdot\}_0$ on M_0 is given by

$\{\sigma_i, \sigma_j\}_0$	σ_1	σ_2	σ_3	σ_4	σ_5
σ_1	0	1	$2\sigma_2$	0	0
σ_2	-1	0	0	0	0
σ_3	$-2\sigma_2$	0	0	$-2(\sigma_3 - \sigma_2^2)$	$-4\sigma_4$
σ_4	0	0	$2(\sigma_3 - \sigma_2^2)$	0	$-2\sigma_5$
σ_5	0	0	$4\sigma_4$	$2\sigma_5$	0

Now, the Poisson bracket $\{\cdot, \cdot\}_{U_1}$ on U_1 is simply given by deleting the last row and column from the table above. Next, consider the vector space $\hat{\Delta}$ of derivations on $C^\infty(M_0)$, spanned by the derivation $\hat{X}$ defined in (7.14). It induces the vector field

$$\hat{X}'_{U_1} = \sigma_1 \frac{\partial}{\partial \sigma_2} + 2(\sigma_4 + \sigma_1 \sigma_2) \frac{\partial}{\partial \sigma_3} + \frac{\sigma_4^2}{\sigma_3 - \sigma_2^2} \frac{\partial}{\partial \sigma_4} \quad (7.23)$$

on U_1 which spans the one-dimensional characteristic distribution $\hat{\Delta}_{U_1}$ on U_1. Finally, the Hamiltonian H_0 restricts to the function $H_{U_1} = \frac{1}{2}\sigma_3 + \sigma_1$ on U_1. The dynamics defined in Definition 10, restricted to the chart (U_1, u_1), then yields the following set of differential and algebraic equations:

$$\dot{\sigma}_1 = \sigma_2 \tag{7.24a}$$
$$\dot{\sigma}_2 = -1 + \sigma_1\lambda \tag{7.24b}$$
$$\dot{\sigma}_3 = -2\sigma_2 + 2(\sigma_4 + \sigma_1\sigma_2)\lambda \tag{7.24c}$$
$$\dot{\sigma}_4 = \sigma_3 - \sigma_2^2 + \frac{\sigma_4^2}{\sigma_3 - \sigma_2^2}\lambda \tag{7.24d}$$
$$0 = \sigma_4 + \sigma_1\sigma_2. \tag{7.24e}$$

Here, $\lambda \in \mathbb{R}$ denotes the Lagrange multiplier, determined by the condition that the algebraic constraint (7.24e) be satisfied at all times. Notice that (7.24e) is defined by the condition $\hat{X}'_{U_1}[H_{U_1}] = 0$ and corresponds in original coordinates to the holonomic constraint (7.3).

The function $C_0|_{U_1} = \frac{\sigma_4^2}{\sigma_3-\sigma_2^2} + \sigma_1^2 - 1$ satisfies the condition $\hat{X}'_{U_1}[C_0|_{U_1}] = 0$. Moreover, $\{\cdot, C_0|_{U_1}\}_{U_1} = -2\hat{X}'_{U_1} \in \Gamma_{loc}(\hat{\Delta}_{U_1})$. Therefore it is a Casimir (in the sense of Definition 7) and hence a first integral of the system (7.24). The existence of this Casimir can be interpreted as the constraint (7.24e) being holonomic, or integrable. We can eliminate the Lagrange multiplier λ by differentiating (7.24e) once with respect to time. If we restrict attention to the zero level set of the Casimir $C_0|_{U_1}$ we obtain after a straightforward calculation:

$$0 = \dot{\sigma}_4 + \dot{\sigma}_1\sigma_2 + \sigma_1\dot{\sigma}_2 = \sigma_3 - \sigma_1 + \lambda. \tag{7.25}$$

This implies that $\lambda = \sigma_1 - \sigma_3$. Substituting this into (7.24) we obtain the *unconstrained* dynamics on $(C_0|_{U_1})^{-1}(0)$:

$$\dot{\sigma}_1 = \sigma_2, \quad \dot{\sigma}_2 = -1 + \sigma_1(\sigma_1 - \sigma_3), \quad \dot{\sigma}_3 = -2\sigma_2, \quad \dot{\sigma}_4 = \sigma_1 - \sigma_2^2 - \sigma_1^2(\sigma_1 - \sigma_3). \tag{7.26}$$

This is a set of nonlinear ordinary differential equations which can be integrated to obtain the trajectories of the pendulum restricted to the chart (U_1, u_1). Notice that the equations imply that

$$\dot{\sigma}_5 = \frac{d}{dt}(1 - \sigma_1^2) = -2\sigma_1\sigma_2. \tag{7.27}$$

Next, we calculate the dynamics on the second chart (U_2, u_2) of $(M_0)_{(e)}$. The Poisson bracket $\{\cdot,\cdot\}_{U_2}$ is simply given by deleting the third row and column in the table for $\{\cdot,\cdot\}_0$. In order to calculate $\hat{X}'_{U_2}$ it suffices to notice that because of the constraints (7.3) we have that $\sigma_4 + \sigma_1\sigma_2 = 0$. Hence the derivation $\hat{X}$ restricted to the smooth functions on the level set $(C_0|_{U_2})^{-1}(0)$ yields the vector field $\hat{X}'_{U_2} = \sigma_1\frac{\partial}{\partial\sigma_2} + \sigma_5\frac{\partial}{\partial\sigma_4}$. The Hamiltonian becomes $H_{U_2} = \frac{1}{2}\left(\frac{\sigma_4^2}{\sigma_5} + \sigma_2^2\right) + \sigma_1$. The dynamics on U_2, restricted to the level set $(C_0|_{U_2})^{-1}(0)$, can now be calculated to have the form

$$\dot{\sigma}_1 = \sigma_2, \qquad \dot{\sigma}_2 = -1 + \sigma_1\lambda, \qquad \dot{\sigma}_4 = \frac{\sigma_4^2}{\sigma_5} + \sigma_5\lambda, \qquad \dot{\sigma}_5 = 2\sigma_4, \tag{7.28}$$

together with the constraint $\sigma_4 + \sigma_1\sigma_2 = 0$. The Lagrange multiplier can be solved as $\lambda = \sigma_1 - \left(\frac{\sigma_4^2}{\sigma_5} + \sigma_2^2\right)$. Using the equality $\sigma_5 + \sigma_1^2 - 1 = 0$ it can easily be shown that on the level set $(C_0|_{(M_0)_{(e)}})^{-1}(0)$ the equations (7.28) equal the equations (7.26,7.27).

We conclude that the differential equations (7.26,7.27) describe the reduced dynamics of the spherical pendulum everywhere on the piece $(M_0)_{(e)}$, restricted to the level set $(C_0|_{(M_0)_{(e)}})^{-1}(0)$.

Remark 17. Notice that, by virtue of the constraint (7.24e), the first three equations in (7.26) decouple from the last one. It is instructive to compare the result with the Hamiltonian derivation on the singular reduced space $(TS^2)_0$ obtained in [11], Equation (27), p. 156. This derivation represents the Hamiltonian dynamics of the *unconstrained* reduced Hamiltonian system on the singular reduced space $(TS^2)_0$. (It was described as a singular reduced *implicit*, i.e. constrained, Hamiltonian system on $M_0 = (T^*\mathbb{R}^3)_0$ in the present paper.) After restriction of the Hamiltonian derivation in [11] to the open and dense piece in $(TS^2)_0$ it becomes a smooth vector field. Its integral curves are exactly described by the first three equations of (7.26). ♦

The planar pendulum. It is well known that a spherical pendulum, having angular momentum zero about the vertical axis, moves in a vertical plane and hence behaves as a planar pendulum. In this paragraph we show that the ordinary differential equations (7.26,7.27) indeed generate the equations of motion for a planar pendulum. Since the motion of the spherical pendulum is rotationally invariant we can restrict our attention to any arbitrary vertical plane through the origin. We shall choose $q_1 = p_1 = 0$. Parameterize q_2 and q_3 in terms of the angle ψ on a great circle through the north and south poles of the sphere $S^2 \subset Q$ as follows:

$$q_2(t) = \sin\psi(t), \quad q_3(t) = -\cos\psi(t). \tag{7.29}$$

(This defines the downward position of the pendulum at $\psi = 0$.) The corresponding S^1-invariant polynomials are given by

$$\begin{aligned} &\sigma_1(t) = -\cos\psi(t), \quad \sigma_2(t) = \dot\psi(t)\sin\psi(t), \quad \sigma_3(t) = \dot\psi^2(t), \\ &\sigma_4(t) = \dot\psi(t)\sin\psi(t)\cos\psi(t), \quad \sigma_5(t) = \sin^2\psi(t). \end{aligned} \tag{7.30}$$

After substitution into (7.26,7.27) the first and the last equation are trivial, while the other three yield

$$\ddot\psi\sin\psi = -\sin^2\psi, \quad \ddot\psi\dot\psi = -\dot\psi\sin\psi, \quad \ddot\psi\sin\psi\cos\psi = -\sin^2\psi\cos\psi. \tag{7.31}$$

Notice that the last equation in (7.31) can be obtained from the first by multiplication with $\cos\psi$. Hence we only need to consider the first two equations in (7.31).

Recall from (7.20) that on $(M_0)_{(e)}$ the condition $\sigma_5 + \sigma_3 - \sigma_2^2 > 0$ holds. For $\sigma_1, \ldots, \sigma_5$ as in (7.30) this yields

$$\sin^2\psi + \dot\psi^2(1 - \sin^2\psi) > 0. \tag{7.32}$$

Notice that the only points not satisfying (7.32) are those for which $\sin\psi = 0$ and $\dot\psi = 0$. But those are exactly the equilibria of the pendulum, which are described on the other piece $(M_0)_{(S^1)}$.

Now, multiply the first equation of (7.31) with $\sin\psi$ and the second equation with $\dot\psi(1 - \sin^2\psi)$. Adding the two results yields the following equation:

$$\ddot\psi\left(\sin^2\psi + \dot\psi^2(1 - \sin^2\psi)\right) = -\sin\psi\left(\sin^2\psi + \dot\psi^2(1 - \sin^2\psi)\right). \tag{7.33}$$

In view of (7.32), we can divide by the expression in the parenthesis to get the nonlinear differential equation

$$\ddot\psi = -\sin\psi, \tag{7.34}$$

which is recognized as the equation of motion for a planar pendulum under gravity moving in a vertical plane.

7.2 Non-horizontal symmetries and nonholonomic constraints

At the beginning of this section we indicated in which way the assumptions of Theorem 1 are usually satisfied by constrained mechanical systems. Perhaps the most restrictive assumption is that the symmetry group has to act horizontally. In other words, the infinitesimal generators have to satisfy the kinematic constraints. Clearly, this is the case in the example of the spherical pendulum. The "horizontality" condition is certainly not met in all known examples. In such cases, the momentum map P is not a section of the annihilator of Δ (i.e. the second condition in (6.1) does not hold). As a consequence, the momentum map is not constant along solutions of (M, D, H). Therefore, it does not make sense to reduce the system to $P^{-1}(0)/G$, since the solutions do not leave the level set $P^{-1}(0)$ invariant. On the other hand, the symmetries can still be factored out and the system can be reduced to the quotient space M/G. (We refer to this as a one-step reduction, as opposed to a two-step reduction where one first restricts to the level set $P^{-1}(0)$ and afterwards factors out the G-action.) Even if the G-action is not regular, one can take the quotient to obtain the singular reduced topological space M/G. Again, by considering the compact stabilizer subgroups of G, the space M/G can be decomposed into orbit types. It is clear that the basic procedure developed in this paper can be applied, *mutatis mutandis*, to such a one-step reduction.

In this light it is interesting to recall a theorem of Śniatycki [35], see also [6, 7]. This theorem states that the set of horizontal symmetries forms a *normal* Lie subgroup K of G. The horizontal momentum map $P_{\mathfrak{k}}$ is defined as the restriction of P to $\mathfrak{k}$, the Lie algebra of K. It is constant along solutions of the system. Hence the system can be reduced to the singular reduced space $P_{\mathfrak{k}}^{-1}(0)/K$ following the procedure developed in this paper. Afterwards a further reduction can be done by factoring out the remaining symmetries, defined by the symmetry Lie group G/K, using a one-step reduction procedure as indicated above.

Finally, we want to recall that the kinematic constraint was holonomic in the spherical pendulum example. This fact was observed again at the end of the singular reduction procedure by the existence of a non-trivial Casimir function (i.e., C_0). It is clear, however, that the reduction procedure works equally well for mechanical systems with nonholonomic kinematic constraints. The only difference is that after the reduction process the kinematic constraint is still nonholonomic and cannot be "integrated" to a Casimir function.

8 Conclusions

In this paper we have studied the singular reduction of implicit Hamiltonian systems admitting a symmetry Lie group together with a corresponding equivariant momentum map. The results extend the singular reduction theory developed in [3, 4, 11, 12, 13, 18, 29, 30, 34] for symplectic or Poisson Hamiltonian systems. The main result is a topological description of the reduced implicit Hamiltonian system using the definition of a topological Dirac structure. In particular, the reduced space is not assumed to be a smooth manifold. The dynamics corresponding to this system are defined and it is shown that the projectable solutions of the original system project to solutions of the singular reduced system. If the symmetry Lie group acts regularly (e.g. freely and properly) and the value of the momentum map is regular, then the singular reduced implicit Hamiltonian system equals the regular reduced implicit Hamiltonian system as described in [6, 7]. Finally, under certain conditions (given in Theorems 1 and 2), the singular reduced space can be decomposed into a set of smooth manifolds called pieces. It is shown that the singular reduced implicit Hamiltonian system restricts to a regular reduced implicit Hamiltonian system on each of these pieces. In order to illustrate the theory, an example of a holonomically constrained mechanical system, namely the spherical pendulum with zero angular momentum about the vertical axis, is worked out in detail. In particular, the reduced dynamics on the pieces is calculated. It is shown to correspond to the usual equation of motion for a planar pendulum under gravity moving in a vertical plane.

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List of notations	
M	smooth n-dimensional manifold
$TM \oplus T^*M$	smooth vector bundle over M with fibers $T_xM \times T_x^*M,\ x \in M$
$D \subset TM \oplus T^*M$	Dirac structure on M
$\mathfrak{X}_{loc}(M),\ \mathfrak{X}(M)$	space of smooth local, resp. global, vector fields on M
$\Omega^k_{loc}(M),\ \Omega^k(M)$	space of smooth local, resp. global, k-forms on M
$\mathfrak{D}_{loc},\ \mathfrak{D}$	space of smooth local, resp. global, sections of D
Δ	characteristic distribution corresponding to D, see (2.4)
$\Gamma_{loc}(\Delta)$	space of smooth local sections of Δ
Γ	codistribution corresponding to D, see (2.5)
$C^\infty(M)$	set of smooth functions on M
$C^\infty(M)^G$	set of smooth G-invariant functions on M
H	Hamiltonian or energy function
$\{\cdot,\cdot\}$	generalized Poisson bracket on M
J	generalized Poisson structure corresponding to $\{\cdot,\cdot\}$
ω	symplectic form, or nondegenerate two-form, on M
G	symmetry Lie group
G_x	stabilizer group of $x \in M$
$\mathfrak{g}$	Lie algebra of G
$\mathfrak{g}_x$	Lie algebra of G_x
$\phi : G \times M \to M$	Lie group action on M
ξ_M	infinitesimal generator corresponding to $\xi \in \mathfrak{g}$, see (3.1)
π	projection map
V	vertical subbundle of TM, i.e. $V = \ker T\pi$
$\mathfrak{V}_{loc}$	space of smooth local sections of V
G-projectable solution	integral curve in M of a projectable vector field $X \sim_\pi \hat{X}$, see §3
P	momentum map
ι	inclusion map
$W^\infty(N)$	set of Whitney smooth functions on the subset $N \subset M$
M_0	singular reduced space, i.e. $M_0 = P^{-1}(0)/G$
$C^\infty(M_0)$	smooth functions on the singular reduced space M_0, see Definition 8
$\{\cdot,\cdot\}_0$	generalized Poisson bracket on $C^\infty(M_0)$, see (4.4)
$\hat{\Delta}$	vector space of derivations on $C^\infty(M_0)$ induced by Δ
D_0	singular reduced (topological) Dirac structure on M_0
$M_{(K)}$	manifold of orbit type (K), see (6.2)
$(M_0)_{(K)}$	leaf of the decomposition of M_0 by orbit type, called piece
$W^\infty\left((M_0)_{(K)}\right)$	set of Whitney smooth functions on $(M_0)_{(K)}$
$C^\infty\left((M_0)_{(K)}\right)$	set of smooth functions on $(M_0)_{(K)}$
$\{\cdot,\cdot\}_{(K)}$	generalized Poisson bracket on $(M_0)_{(K)}$, see (6.7)
$J_{(K)}$	generalized Poisson structure corresponding to $\{\cdot,\cdot\}_{(K)}$
$\hat{\Delta}_{(K)}$	distribution on $(M_0)_{(K)}$ induced by Δ
$D_{(K)}$	reduced Dirac structure on $(M_0)_{(K)}$
$H_{(K)}$	reduced Hamiltonian on $(M_0)_{(K)}$