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CONTROLLABILITY OF POISSON SYSTEMS*

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Abstract. Sufficient conditions for the controllability of affine nonlinear control systems on Poisson manifolds are given. The important special case when the Poisson manifold is the reduced space of a symplectic manifold by a free Lie group action is studied. The controllability of the reduced system is linked to that of the given affine nonlinear system. Several examples illustrating the theory are also presented.

Key words. controllability, symplectic manifold, Poisson manifold, reduction, weak positive Poisson stability

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1. Introduction.

The phase space of classical conservative mechanical systems is usually described by a Poisson manifold $(P, \{\cdot,\cdot\})$. The dynamics on P, subject to external forces, can often be written in the form of an affine nonlinear control system as

$$\dot{x} = X(x) + \sum_{i=1}^{m} Y_i(x) u_i \,,$$

where the drift vector field X is a complete vector field tangent to the symplectic leaves of P and also preserves the symplectic volume on each one of them, $Y_1, \ldots, Y_m \in \mathfrak{X}(P)$ are smooth complete vector fields on P, the control $u := (u_1, \ldots, u_m) : (0, \infty) \longrightarrow B \subset \mathbb{R}^m$ is a measurable function, and B is a bounded subset of $\mathbb{R}^m$.

Deciding the controllability of nonlinear control systems is usually a difficult problem that has generated a large body of literature. As opposed to linear control systems, the Lie algebra rank condition is not sufficient for proving controllability of a nonlinear control system. Nevertheless, there is a link between nonlinear controllability and linear controllability given by the following well-known result: If the linearization of a nonlinear system at an equilibrium is controllable, then the nonlinear system is locally controllable. For nonlinear systems without drift, various characterizations of controllability based on Chow's theorem were obtained. These were generalized to nonlinear systems with drift in terms of the Lie algebra generated by the control vector fields. Significant results were obtained by Hermann [14], Haynes and Hermes [13], Brockett [8], Lobry [28], Sussmann and Jurdjevic [45], Krener [23], and others. Sufficient conditions for controllability of nonlinear systems satisfying the Lie algebra rank condition were obtained by Lobry [29] in the case of Poisson stable systems. This work was generalized by Jurdjevic and Quinn [18] and Bonnard [6] to the case when

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only the drift vector field is required to be Poisson stable. This method was applied by Crouch [11] to the study of spacecraft attitude control problems. In order to analyze the controllability of spacecraft systems, which include attitude-orbit coupling terms and are controlled only by attitude controllers using either reaction wheels or gas jets, Lian, Wang, and Fu [26] replaced the condition on the drift vector field to be Poisson stable with the less stringent condition of weak positive Poisson stability. More precisely, they showed that if an affine nonlinear control system verifies the Lie algebra rank condition and the drift vector field is weakly positively Poisson stable, then the system is controllable.

The problem of controllability for nonlinear systems that are invariant under the action of a Lie group was studied by San Martin and Crouch [42], Jurdjevic and Kupka [17] and, in a more general setting of fiber bundles, by Nijmeijer and van der Schaft [38] (see also Grizzle and Marcus [12] and Sánchez de Alvarez [43]). Other results concerning different aspects of the relation between the given and the reduced control system can be found in Jalnapurkar and Marsden [15], [16] and Bloch, Leonard, and Marsden [5].

The aim of this paper is to give sufficient conditions for the controllability of an affine control system on a Poisson manifold. For the case when the manifold is the cotangent bundle of a Lie group, this problem was studied by Manikonda and Krishnaprasad [30]; it was this paper that has inspired the present generalization. The strategy of the proof of the main results is to give topological conditions that guarantee that the drift vector field is weakly positively Poisson stable (WPPS). In order to do this, we will use the Poincaré recursion theorem for the dynamics of the drift vector field restricted to each symplectic leaf. We will prove that if one can find a continuous function $f : P \to \mathbb{R}$ that is constant on the flow of X and is such that either f restricted to each symplectic leaf is a proper function or f is a proper function from P to $\mathbb{R}$ and all symplectic leaves are closed and embedded submanifolds of P, then X is WPPS. There is a relatively subtle technical point in the proof of this theorem: The topology of a symplectic leaf is stronger than the relative topology induced by the ambient space P; that is, every open set in the induced topology on the leaf is also open in the immersed topology of the leaf, but there exist open sets in the immersed topology of the leaf that are not open in the induced topology. This immediately implies that there are subsets in the leaf which are compact in the induced topology but are not compact in the immersed topology on the leaf.

As an important case of this first result, we will study the situation when the Poisson manifold is the reduced space of a symplectic manifold by a free proper Lie group action which also admits a momentum map. We will show that if the momentum map is proper and the reduced affine nonlinear system verifies the Lie algebra rank condition, then it is controllable. Similarly, if the momentum map is not proper but the Lie group is compact and there is a proper map $f : M/G \to \mathbb{R}$ which is constant along the trajectories of the reduced drift vector field and in addition the reduced affine nonlinear system verifies the Lie algebra rank condition, then the system is controllable. We will also give the relation between the controllability of the reduced and initial affine nonlinear systems.

The paper ends with some examples of underactuated affine nonlinear control systems; for this, some useful technical lemmas implying the properness of functions are also given.

After this paper was submitted the authors were made aware of the work of Manikonda and Krishnaprasad [31] (a preliminary version of these results can be

found in Krishnaprasad and Manikonda [24]) with which the present work has a certain amount of overlap. We shall mention in the text explicitly where this is the case and compare their results to ours.

2. Controllability and Poisson stability. In this section we shall present a controllability result for affine nonlinear control systems on a general Poisson manifold. We begin by reviewing the classical definitions and results that will be used later on by adopting the terminology in the standard textbook of Nijmeijer and van der Schaft [39].

Let M be a smooth n-dimensional connected manifold and

$$\dot{x} = X(x) + \sum_{i=1}^{m} Y_i(x) u_i \tag{2.1}$$

an affine nonlinear control system on M, where $X, Y_1, \ldots, Y_m \in \mathfrak{X}(M)$ are smooth complete vector fields on M, the control $u := (u_1, \ldots, u_m) : (0, \infty) \longrightarrow B \subset \mathbb{R}^m$ is a measurable function, and B is a bounded subset of $\mathbb{R}^m$. We will denote by $\mathcal{L}$ the Lie subalgebra of $\mathfrak{X}(M)$ generated by the vector fields $X, Y_1, \ldots, Y_m$.

Definition 2.1. *The system* (2.1) *satisfies the* Lie algebra rank condition (LARC) *if* $\operatorname{span} \mathcal{L}(x) = T_x M$ *for every* $x \in M$, *where* $\mathcal{L}(x) := \{Z(x) \mid Z \in \mathcal{L}\}$.

Definition 2.2. *The system* (2.1) *is* controllable *if for any two points* $x_I, x_F \in M$ *there is a control* u *which takes the system from point* $x = x_I$ *at time* $t = t_I \in \mathbb{R}$ *to the point* $x = x_F$ *at time* $t = t_F \in \mathbb{R}$, *that is, if for a certain choice of the function* u *there is an integral curve* $x(t)$ *of* (2.1) *that begins at* x_I *and ends at* x_F *in finite time.*

It is well known that for a nonlinear control system without drift (i.e., $X = 0$), the LARC implies controllability. This is Chow's theorem [9]. For the general case $X \neq 0$, the situation is more complicated and, in general, the LARC is not sufficient to guarantee controllability. A lot of work was done in this direction and we will review below only the results relevant for our purposes.

In what follows we shall need a condition, called weak positive Poisson stability, on the drift vector field X. In order to understand how this concept appeared in the literature we shall quickly relate it to standard notions in the theory of dynamical systems. The next three definitions were introduced originally in Nijmeijer and van der Schaft [39], Lobry [28], [29], and Lian, Wang, and Fu [26]. Let $X \in \mathfrak{X}(M)$ be a smooth complete vector field on M and $\{\Phi_t\}_{t \in \mathbb{R}}$ its flow.

Definition 2.3. *A point* $x \in M$ *is called* positively Poisson stable *for* $X \in \mathfrak{X}(M)$ *if for any* $T > 0$ *and any neighborhood* V_x *of* x *there exists a time* $t > T$ *such that* $\Phi_t(x) \in V_x$. *The vector field* $X \in \mathfrak{X}(M)$ *is called* positively Poisson stable *if the set of positively Poisson stable points of* X *is dense in* M.

Definition 2.4. *A point* $x \in M$ *is called a* nonwandering point *of* $X \in \mathfrak{X}(M)$ *if for any* $T > 0$ *and for any neighborhood* V_x *of* x *there exists a time* $t > T$ *such that* $\Phi_t(V_x) \cap V_x \neq \emptyset$.

Let Γ_X be the set of all nonwandering points of X, usually called the *nonwandering set* of X. The following result and its proof can be found in Lian, Wang, and Fu [26].

Theorem 2.5. *The nonwandering set of a positively Poisson stable vector field* X *is the entire manifold* M, *that is,* $\Gamma_X = M$.

Proof. For a given $x \in M$ one needs to prove that for any neighborhood V_x of x and for any $T > 0$ there exists a time $t > T$ such that $\Phi_t(V_x) \cap V_x \neq \emptyset$. Let

$\underline{S_X}$ denote the set of positively Poisson stable points of $X \in \mathfrak{X}(M)$. By definition, $\overline{S_X} = M$. Thus there is a positively Poisson stable point y in V_x. This implies that for all $T > 0$ there is a time $t > T$ such that $\Phi_t(y) \in V_x$. Hence $\Phi_t(V_x) \cap V_x \neq \emptyset$. So x is nonwandering for X. Since x was arbitrary, we get $\Gamma_X = M$. $\square$

Positive Poisson stability of a vector field is hence a sufficient condition for the nonwandering set to be the entire manifold. Since the converse is not true, one introduces a weaker definition.

DEFINITION 2.6. *A vector field is called WPPS if its nonwandering set equals M (i.e., $\Gamma_X = M$).*

A natural question that arises now is the following: When is a vector field X on a manifold WPPS? In order to answer this question, we will recall the Poincaré recursion theorem (for a proof see, e.g., Abraham and Marsden [1], Abraham, Marsden, and Ratiu [2]).

Let (M, Ω) be a manifold with a volume form Ω. Let $\mathcal{B}$ denote the collection of Borel sets on M, that is, the σ-algebra generated by the open (or closed, or compact) subsets of M. Then there exists a unique Borel measure m_Ω on $\mathcal{B}$ such that for every continuous function f with compact support

$$\int_M f dm_\Omega = \int_M f\Omega. \tag{2.2}$$

For K a compact and U an open subset of M, we have $m_\Omega(K) < \infty$ and $m_\Omega(U) > 0$. If we consider on M a vector field whose flow preserves the volume form (i.e., $\Phi_t^*\Omega = \Omega$), then $m_\Omega(\Phi_t(A)) = m_\Omega(A)$ for any measurable subset A of M.

THEOREM 2.7 (Poincaré recursion theorem). *Let (M, Ω) be a manifold with a volume form Ω and m_Ω the associated Borel measure. Let X be a time-independent, complete vector field such that its flow $\{\Phi_t\}_{t\in\mathbb{R}}$ preserves the volume. Suppose A is a measurable subset of M with $0 < m_\Omega(A) < \infty$ which is also invariant under the flow of X. Then for each measurable subset B of A with $m_\Omega(B) > 0$ and for any $T > 0$, there exists $t > T$ such that $\Phi_t(B) \cap B \neq \emptyset$.*

An immediate consequence is the following proposition.

PROPOSITION 2.8. *Let (M, Ω) be a compact manifold with a volume form Ω and X a time-independent vector field such that its flow preserves the volume form. Then X is a WPPS vector field.*

The link between the WPPS condition and controllability is given by the following theorem which is due to Lian, Wang, and Fu [26]. Earlier versions of this theorem, where the hypothesis required X to be Poisson stable, are due to Lobry [29], Bonnard [6], and Crouch [11].

THEOREM 2.9. *Suppose that X is a WPPS vector field. Then the system* (2.1) *is controllable if and only if the LARC holds.*

We now state our first result on controllability of an affine nonlinear control system on a Poisson manifold. Recall that a finite dimensional Poisson manifold is a smooth manifold P whose ring of smooth real-valued functions $C^\infty(P)$ is endowed with a Lie algebra structure $\{\cdot,\cdot\}$ satisfying the Leibniz identity in every factor. Thus, if $h \in C^\infty(P)$, the derivation $\{\cdot, h\}$ defines a vector field X_h on P, called the *Hamiltonian vector field* induced by the *Hamiltonian function* h, that is, $\langle df, X_h\rangle = \{f, h\}$ for any $f \in C^\infty(P)$. The vector fields $\{X_h \mid h \in C^\infty(P)\}$ define a singular integrable distribution whose integral manifolds are symplectic immersed submanifolds whose Poisson bracket coincides with the given one on P; these integral manifolds are called the *symplectic leaves* of P. For further information on Poisson manifolds see, for

example, Libermann and Marle [27], Marsden [34], Marsden and Ratiu [35], and Puta [41].

Note that the topology of a symplectic leaf is stronger than the topology induced by the ambient manifold P, that is, every open set in the induced topology on the leaf is also open in the immersed topology but there exist open sets in the immersed topology of the leaf that are not open in the induced topology. Therefore, there are compact subsets in the induced topology of the leaf that are not compact in the immersed topology of the leaf. In the next proof one needs to come to grips with this problem.

THEOREM 2.10. *Let $(P, \{\cdot,\cdot\})$ be a connected Poisson manifold and*

$$\dot{x} = X(x) + \sum_{i=1}^{m} Y_i(x)u_i$$

an affine nonlinear control system such that the drift vector field X is tangent to the symplectic leaves of P and also preserves the symplectic volume on each one of them. Let $f : P \to \mathbb{R}$ be a continuous function that is constant on the flow of X. Assume that one of the following hypotheses holds:

(i) *f restricted to each symplectic leaf is a proper function.*

(ii) *f is a proper function and all symplectic leaves are closed and embedded submanifolds of P.*

Then X is WPPS. If the system also verifies the LARC, then it is controllable.

Proof. Let x_0 be an arbitrary point of P and U_{x_0} an arbitrary open neighborhood of x_0 in P. Denote by L_{x_0} the symplectic leaf containing x_0 and let $c_0 = f(x_0)$. There are two possibilities: $f(L_{x_0}) = c_0$, or $f(L_{x_0}) = I$, where I is a nondegenerate connected interval in $\mathbb{R}$.

Assume first that $f(L_{x_0}) = c_0$. Under hypothesis (i), $f_{|L_{x_0}} : L_{x_0} \to \mathbb{R}$ is a proper function, so $L_{x_0} = f^{-1}_{|L_{x_0}}(c_0)$ is compact. By Proposition 2.8 it follows that X restricted to the leaf L_{x_0} is a WPPS vector field, which implies that x_0 is a nonwandering point for the flow ϕ_t of X on L_{x_0}. Thus for any $T > 0$ there exists $t > T$ such that $(L_{x_0} \cap U_{x_0}) \cap \phi_t(L_{x_0} \cap U_{x_0}) \neq \emptyset$ which, in particular, implies that $U_{x_0} \cap \phi_t(U_{x_0}) \neq \emptyset$. Since x_0 and U_{x_0} were arbitrary, it follows that X is WPPS on P. Under hypothesis (ii), $f : P \to \mathbb{R}$ is a proper function, so $f^{-1}(c_0)$ is compact in P. Since $L_{x_0} \subset f^{-1}(c_0)$ and L_{x_0} is closed and embedded in P by hypothesis, it follows that L_{x_0} is compact in P. As before, applying Proposition 2.8, we obtain that X is WPPS.

Now assume that $f(L_{x_0}) = I$, where I is a nondegenerate connected interval. Then, without loss of generality (replacing x_0 with another point in the leaf, if necessary), we can assume that c_0 lies in the interior of I and hence there is an $\varepsilon > 0$ such that $[-\varepsilon + c_0, c_0 + \varepsilon] \subset I$. The set $K := L_{x_0} \cap f^{-1}([-\varepsilon + c_0, c_0 + \varepsilon])$ is compact in L_{x_0} in hypothesis (i) because $f_{|L_{x_0}}$ is proper and in hypothesis (ii) because L_{x_0} is closed and embedded in P and $f^{-1}([-\varepsilon + c_0, c_0 + \varepsilon])$ is compact in P. This implies that $m_{L_{x_0}}(K) < \infty$, where $m_{L_{x_0}}$ is the Borel measure associated to the symplectic volume form on L_{x_0}. Also, K contains an open set of L_{x_0}, for example, $f^{-1}_{|L_{x_0}}((-\varepsilon + c_0, c_0 + \varepsilon))$, and thus $m_{L_{x_0}}(K) > 0$. By the Poincaré recursion theorem (see Theorem 2.7), for any $T > 0$ there exists a $t > T$ such that $(K \cap U_{x_0}) \cap \phi_t(K \cap U_{x_0}) \neq \emptyset$ which, in particular, implies that $U_{x_0} \cap \phi_t(U_{x_0}) \neq \emptyset$. Consequently, x_0 is a nonwandering point of the flow of X. Since x_0 was arbitrary, it follows that X is WPPS on P.

If the control system also satisfies the LARC, Theorem 2.9 implies that it is controllable. $\square$

It should be noted that in hypothesis (ii) there are two hypotheses on the symplectic leaves of P: They need to be embedded and closed. It can happen, even in the Lie–Poisson case, that the leaves are embedded but not closed. For example, the Poisson manifold $\mathbb{R}^2$ with the bracket given by $\{f,g\}(x,y) = y(f_x g_y - f_y g_x)$ has the upper and the lower half plane as open two-dimensional symplectic leaves and the points on the x-axis as the zero-dimensional leaves.

An important case in which the drift vector field X satisfies the hypotheses of the theorem is when $X = X_h$ for some Hamiltonian function $h \in C^\infty(P)$. Indeed, X_h is always tangential to the leaves and it preserves the symplectic volume on each leaf by the Liouville theorem.

Note that if P is a Poisson manifold, in order for the above affine nonlinear control system to verify the LARC it is necessary that at least one of the vector fields $Y_1, \ldots, Y_m \in \mathfrak{X}(P)$ be non-Hamiltonian.

Theorem 2.10 immediately implies both Theorems 4.3 and 4.11 in Manikonda and Krishnaprasad [31]. Finally, it should be noted that this theorem applies in the particular, but important, case of Lie–Poisson systems. In addition, Theorem 2.10(i) can handle Poisson manifolds with nonembedded and nonclosed symplectic leaves, such as the Kirillov example of the dual of a five-dimensional semidirect product Lie group with coadjoint orbits that accumulate on themselves (see Kirillov [22] or Marsden and Ratiu [35] for a discussion of this Lie group and its coadjoint orbits).

3. Controllability of reduced systems. In this section we shall study the important case of the previous theorem when the Poisson manifold is the reduction of a symplectic manifold by a compact Lie group action. In this particular case we can give sufficient topological conditions that imply the hypotheses of Theorem 2.10. To do this, we begin with a quick review of some standard results on symplectic reduction necessary in the subsequent proofs; detailed expositions of this subject can be found in standard textbooks such as Abraham and Marsden [1], Libermann and Marle [27], Marsden [34], Marsden and Ratiu [35], Ortega and Ratiu [40], and Puta [41].

Consider a $2n$-dimensional connected symplectic manifold (M,ω) on which there is a free proper symplectic action of a Lie group G. Denote by $\{\cdot,\cdot\}_\omega$ the Poisson bracket on M defined by the symplectic form ω. Then the orbit space M/G is a smooth Poisson manifold and the projection

$$\pi : (M, \{\cdot,\cdot\}_\omega) \longrightarrow \left(M/G, \{\cdot,\cdot\}_{M/G}\right)$$

is a Poisson surjective submersion. If, in addition, the Lie group G is compact, then π is a closed proper map. (Proofs of these statements can be found in, e.g., Abraham and Marsden [1], Abraham, Marsden, and Ratiu [2], Bredon [7], Kawakubo [19], Libermann and Marle [27], and Ortega and Ratiu [40].)

Suppose that the free and proper G-action on M admits an associated momentum map $J : M \longrightarrow \mathfrak{g}^*$. If the momentum map is not equivariant with respect to the coadjoint action of G on $\mathfrak{g}^*$, then there is a $\mathfrak{g}^*$-valued group one-cocycle σ on G such that $\sigma(g) = J(g \cdot m) - \mathrm{Ad}^*_{g^{-1}} J(m)$ for every $m \in M$ and $g \in G$, where Ad^* denotes the coadjoint representation of G on $\mathfrak{g}^*$. (The connectedness of M is needed to show that the right-hand side is independent of m.) Defining the affine action of G on $\mathfrak{g}^*$ by $g \cdot \mu := \mathrm{Ad}^*_{g^{-1}} \mu + \sigma(g)$, the momentum map $J : M \to \mathfrak{g}^*$ becomes now equivariant relative to the given action on M and the just defined affine action on $\mathfrak{g}^*$. The *Marsden–Weinstein reduction theorem* states that if $\mu \in \mathfrak{g}^*$ is a value of J, then the smooth quotient manifold $M_\mu := J^{-1}(\mu)/G_\mu$ is symplectic with symplectic form

ω_μ characterized by

$$\pi_\mu^* \omega_\mu = i_\mu^* \omega,$$

where G_μ denotes the isotropy subgroup of μ under the affine action, $i_\mu : J^{-1}(\mu) \longrightarrow M$ is the inclusion, and $\pi_\mu : J^{-1}(\mu) \longrightarrow M_\mu$ is the projection. (For a proof, see the original paper Marsden and Weinstein [36], or Abraham and Marsden [1], Libermann and Marle [27], Marsden [34], and Puta [41].) The symplectic manifolds (M_μ, ω_μ) will be called *point reduced spaces*.

These point reduced spaces M_μ can be understood in a natural way as symplectic leaves of the Poisson manifold $(M/G, \{\cdot,\cdot\}_{M/G})$. Indeed, the smooth map $j_\mu : M_\mu \longrightarrow M/G$ naturally defined by the commutative diagram

$$\begin{array}{ccc} J^{-1}(\mu) & \xrightarrow{\quad i_\mu \quad} & M \\ \Big\downarrow \pi_\mu & & \Big\downarrow \pi \\ M_\mu & \xrightarrow{\quad j_\mu \quad} & M/G \end{array}$$

is a Poisson injective immersion. Moreover, the j_μ-images in M/G of the connected components of the symplectic manifolds (M_μ, ω_μ) are its symplectic leaves (see Manikonda and Krishnaprasad [30] or Ortega and Ratiu [40]).

Observe that, in general, j_μ is only an injective immersion. So the topology of the image of j_μ, homeomorphic to the topology of M_μ, is stronger than the subspace topology induced by the ambient space M/G. This image topology on $j_\mu(M_\mu)$ is called the *immersed topology.* As in the previous section, we draw attention to the fact that we can have a subset of $j_\mu(M_\mu)$ which is compact in the induced topology from M/G and not compact in the immersed topology. A key point in the proof of Theorem 3.4 on controllability stated in what follows is to give sufficient and easily verifiable conditions under which these two topologies coincide.

The proof of the next proposition requires compactness of G.

PROPOSITION 3.1. *Suppose that the free symplectic compact G-action on (M, ω) admits a momentum map $J : M \longrightarrow \mathfrak{g}^*$. Then the symplectic leaves of $(M/G, \{\cdot,\cdot\}_{M/G})$ are closed sets.*

Proof. Since $J^{-1}(\mu)$ is closed in M and $\pi : M \longrightarrow M/G$ is a closed map (because G is compact), the set $j_\mu(M_\mu) = \pi(J^{-1}(\mu))$ is closed in the topology of M/G. Therefore, the connected components of $j_\mu(M_\mu)$, which are the symplectic leaves of M/G, are also closed in the topology of M/G. $\square$

We return now to the general case with G noncompact. Up to now we have regarded the symplectic leaves of $(M/G, \{\cdot,\cdot\}_{M/G})$ as the j_μ-images of the connected components of M_μ. However, as sets,

$$j_\mu(M_\mu) = J^{-1}(\mathcal{O}_\mu)/G,$$

where $\mathcal{O}_\mu \subset \mathfrak{g}^*$ is the orbit through μ relative to the affine action of G on $\mathfrak{g}^*$. The set $M_{\mathcal{O}_\mu} := J^{-1}(\mathcal{O}_\mu)/G$ is called *the orbit reduced space* associated to the orbit $\mathcal{O}_\mu$. The smooth manifold structure (and hence the topology) on $M_{\mathcal{O}_\mu}$ is the one that makes $j_\mu : M_\mu \longrightarrow M_{\mathcal{O}_\mu}$ into a diffeomorphism.

The group one-cocycle σ induces by derivation a real-valued Lie algebra two-cocycle $\Sigma : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ which can be shown to equal $\Sigma(\xi, \eta) = J^{[\xi,\eta]}(m) - \{J^\xi, J^\eta\}(m)$

for every $m \in M$ and $\xi, \eta \in \mathfrak{g}$; $J^\xi : M \to \mathbb{R}$ denotes the ξ-component of J, that is, $J^\xi(m) := \langle J(m), \xi \rangle$. Denote by $\xi_{\mathfrak{g}^*}(\nu) := -\mathrm{ad}^*_\xi \nu + \Sigma(\xi, \cdot)$ the infinitesimal generator of the affine action of G on $\mathfrak{g}^*$, for $\nu \in \mathfrak{g}^*$, where ad^* denotes the dual of the adjoint representation ad of $\mathfrak{g}$ on $\mathfrak{g}$ defined by $\mathrm{ad}_\xi \eta := [\xi, \eta]$, for $\xi, \eta \in \mathfrak{g}$. The affine action orbit $\mathcal{O}_\mu$ carries two symplectic forms given by

$$\omega^{\pm}_{\mathcal{O}_\mu}(\nu)(\xi_{\mathfrak{g}^*}(\nu), \eta_{\mathfrak{g}^*}(\nu)) = \pm\langle \nu, [\xi, \eta]\rangle \mp \Sigma(\xi, \eta), \tag{3.1}$$

for any $\xi, \eta \in \mathfrak{g}$. They are the natural modifications of the usual Kirillov–Kostant–Souriau symplectic forms on coadjoint orbits. For the proofs of the statements above see Abraham and Marsden [1], Libermann and Marle [27], Ortega and Ratiu [40], and Puta [41]; the formulation used above is that of Ortega and Ratiu [40].

The next theorem characterizes the symplectic form and the Hamiltonian dynamics on $M_{\mathcal{O}_\mu}$.

THEOREM 3.2 (symplectic orbit reduction). *Assume that the free proper symplectic action of the Lie group G on the symplectic manifold (M, ω) admits an associated momentum map $J : M \longrightarrow \mathfrak{g}^*$.*

(i) *On $J^{-1}(\mathcal{O}_\mu)$ there is a unique immersed smooth manifold structure such that $\pi_{\mathcal{O}_\mu} : J^{-1}(\mathcal{O}_\mu) \longrightarrow M_{\mathcal{O}_\mu}$ is a surjective submersion, where $M_{\mathcal{O}_\mu}$ is endowed with the manifold structure making j_μ into a diffeomorphism. This smooth manifold structure does not depend on the choice of μ in the orbit $\mathcal{O}_\mu$. If $J^{-1}(\mathcal{O}_\mu)$ is a submanifold of M in its own right, then the immersed topology and the induced topology on $M_{\mathcal{O}_\mu}$ coincide.*

(ii) *$M_{\mathcal{O}_\mu}$ is a symplectic manifold with the symplectic form $\omega^{\maltese}_{\mathcal{O}_\mu}$ uniquely characterized by the relation*

$$i^*_{\mathcal{O}_\mu}\omega = \pi^*_{\mathcal{O}_\mu}\omega^{\maltese}_{\mathcal{O}_\mu} + J^*_{\mathcal{O}_\mu}\omega^{+}_{\mathcal{O}_\mu},$$

where $J_{\mathcal{O}_\mu}$ is the restriction of J to $J^{-1}(\mathcal{O}_\mu)$, $i_{\mathcal{O}_\mu} : J^{-1}(\mathcal{O}_\mu) \hookrightarrow M$ is the inclusion, and $\omega^{+}_{\mathcal{O}_\mu}$ is the $+$orbit symplectic form on $\mathcal{O}_\mu$ given by (3.1).

(iii) *Let H be a G-invariant function on M, and define $\widetilde{H} : M/G \longrightarrow \mathbb{R}$ by $H = \widetilde{H} \circ \pi$. Then the Hamiltonian vector field X_H is also G-invariant and hence induces a vector field on M/G which coincides with the Hamiltonian vector field $X_{\widetilde{H}}$. Moreover, the flow of $X_{\widetilde{H}}$ leaves the symplectic leaves $M_{\mathcal{O}_\mu}$ of M/G invariant. This flow restricted to the symplectic leaves is again Hamiltonian relative to the symplectic form $\omega^{\maltese}_{\mathcal{O}_\mu}$ and the Hamiltonian function $\widetilde{H}_{\mathcal{O}_\mu}$ given by*

$$\widetilde{H}_{\mathcal{O}_\mu} \circ \pi_{\mathcal{O}_\mu} = H \circ i_{\mathcal{O}_\mu}.$$

The proof of this theorem in the regular case and when $\mathcal{O}_\mu$ is an embedded submanifold of $\mathfrak{g}^*$ can be found in Marle [32], Kazhdan, Kostant, and Sternberg [20], and Marsden [33]. For the general case, when $\mathcal{O}_\mu$ is not a submanifold of $\mathfrak{g}^*$, see Ortega and Ratiu [40]. Here is the main idea of the proof. Consider for each value $\mu \in \mathfrak{g}^*$ of J the G-equivariant bijection

$$s : G \times_{G_\mu} J^{-1}(\mu) \to J^{-1}(\mathcal{O}_\mu),$$

$$[g, m] \mapsto g \cdot m,$$

where $G \times_{G_\mu} J^{-1}(\mu) := (G \times J^{-1}(\mu))/G_\mu$, the G_μ-action being the diagonal action. Endow $J^{-1}(\mathcal{O}_\mu)$ with the smooth manifold structure that makes the bijection s into a diffeomorphism. Then $J^{-1}(\mathcal{O}_\mu)$ with this smooth structure is an immersed submanifold of M. This is the manifold structure on $J^{-1}(\mathcal{O}_\mu)$ used in the statement of Theorem 3.2.

In the particular case in which $J^{-1}(\mathcal{O}_\mu)$ is a smooth submanifold of M in its own right, this manifold structure coincides with the one induced by the mapping s described previously since in this situation the bijection s becomes a diffeomorphism relative to the a priori given smooth manifold structure on $J^{-1}(\mathcal{O}_\mu)$.

If μ is a regular value of J and $\mathcal{O}_\mu$ is an embedded submanifold of $\mathfrak{g}^*$, then J is transverse to $\mathcal{O}_\mu$ and hence $J^{-1}(\mathcal{O}_\mu)$ is automatically an embedded submanifold of M.

The following result is important for our work through its consequences.

PROPOSITION 3.3 (bifurcation lemma). *Let (M, ω) be a symplectic manifold and G a Lie group acting symplectically on M (not necessarily freely). Suppose also that the action has an associated momentum map $J : M \longrightarrow \mathfrak{g}^*$. For any $m \in M$,*

$$(\mathfrak{g}_m)^\circ = \mathrm{range}(T_m J),$$

where $\mathfrak{g}_m = \{\xi \in \mathfrak{g} \mid \xi_M(m) = 0\}$ is the Lie algebra of the isotropy subgroup $G_m = \{g \in G \mid g \cdot m = m\}$ and $(\mathfrak{g}_m)^\circ = \{\mu \in \mathfrak{g}^* \mid \mu_{|\mathfrak{g}_m} = 0\}$ denotes the annihilator of $\mathfrak{g}_m$ in $\mathfrak{g}^*$.

An immediate consequence of this is the fact that when the action of G is free, then every value $\mu \in \mathfrak{g}^*$ of the momentum map J is a regular value of J.

Now we give the setting for the controllability result on M/G. Let G be a Lie group acting freely properly and symplectically on a $2n$-dimensional connected symplectic manifold (M, ω). Suppose that the action admits an associated momentum map $J : M \longrightarrow \mathfrak{g}^*$. Consider on M the affine nonlinear control system

$$\dot{x} = X_H(x) + \sum_{i=1}^{m} Y_i(x)\, u_i\,, \tag{3.2}$$

where X_H is a complete Hamiltonian vector field with G-invariant Hamiltonian H, the smooth vector fields $Y_1, \ldots, Y_m \in \mathfrak{X}(M)$ are assumed to be G-invariant and complete, and the control $u := (u_1, \ldots, u_m) : (0, \infty) \longrightarrow B \subset \mathbb{R}^m$ is a measurable function with values in a bounded subset B of $\mathbb{R}^m$. Then the system (3.2) will naturally induce the affine nonlinear control system on $(M/G, \{\cdot,\cdot\}_{M/G})$,

$$\dot{\widetilde{x}} = X_{\widetilde{H}}(\widetilde{x}) + \sum_{i=1}^{m} \widetilde{Y}_i(\widetilde{x})\, u_i\,, \tag{3.3}$$

where $X_{\widetilde{H}}$ is the Hamiltonian vector field with respect to the Poisson bracket $\{\cdot,\cdot\}_{M/G}$ and Hamiltonian function $\widetilde{H}$ given by $H = \widetilde{H} \circ \pi$, for $\pi : M \longrightarrow M/G$ the canonical projection. The following theorem generalizes Theorem 4.11 in Manikonda and Krishnaprasad [31] in the sense that it can deal with noncompact Lie group actions and nonequivariant momentum maps.

THEOREM 3.4. *Suppose that the system* (3.3) *verifies the LARC.*

(i) *If the momentum map $J : M \longrightarrow \mathfrak{g}^*$ is proper, then the system* (3.3) *is controllable.*

(ii) *If the Lie group G is compact and if there exists a continuous proper map $f : M/G \longrightarrow \mathbb{R}$ which is constant along the trajectories of $X_{\widetilde{H}}$, then the system* (3.3) *is controllable.*

Proof. The strategy to prove the controllability of (3.3) is to show that $X_{\widetilde{H}}$ is WPPS and then the conclusion follows from Theorem 2.9.

(i) As subsets of M/G, the symplectic leaves are $M_{\mathcal{O}_\mu}$ or, equivalently, $j_\mu(M_\mu)$ and the symplectic form is given by $\omega^{\maltese}_{\mathcal{O}_\mu}$. Because J is a proper map, the set $J^{-1}(\mu)$ is a compact submanifold of M. Thus M_μ is a compact manifold, which implies that the injective immersion j_μ is in fact an embedding. So the immersed topology and the induced topology on $M_{\mathcal{O}_\mu}$ coincide and, therefore, the symplectic leaves are compact embedded submanifolds of M/G.

The vector field $X_{\widetilde{H}}$ is tangent to the leaves $M_{\mathcal{O}_\mu}$ of M/G and is Hamiltonian on each of them relative to the symplectic form $\omega^{\maltese}_{\mathcal{O}_\mu}$. In particular, its flow preserves the Liouville volume on each leaf. Since the leaves are compact, the restriction of the vector field $X_{\widetilde{H}}$ to every leaf is WPPS by Proposition 2.8. Thus each point of every leaf is a nonwandering point of the flow of $X_{\widetilde{H}}$; that is, the nonwandering set of $X_{\widetilde{H}}$ equals P. Thus $X_{\widetilde{H}}$ is WPPS.

(ii) For compact G, the coadjoint orbits are submanifolds of $\mathfrak{g}^*$ and J is transverse to the coadjoint orbits that lie in its image (since by hypothesis, the action is free). So $J^{-1}(\mathcal{O}_\mu)$ is a submanifold of M in its own right and by Theorem 3.2(i) the immersed topology and the induced topology on the symplectic leaves $M_{\mathcal{O}_\mu}$ of M/G coincide. By Proposition 3.1, these leaves are also closed. So we are in the hypotheses of Theorem 2.10(ii) and the result follows. ∎

The relationship between the controllability of the reduced system (3.3) and the initial system (3.2) is given by the following corollary, also contained in Theorem 4.11 of Manikonda and Krishnaprasad [31].

COROLLARY 3.5. *Suppose that the initial system* (3.2) *verifies the LARC and the hypotheses in Theorem* 3.4(ii). *Then the system* (3.2) *is also controllable.*

Proof. Since the vector fields X_H and $X_{\widetilde{H}}$ are π-related, the function $f \circ \pi$ is a constant of the motion for X_H. This function is proper as a composition of two proper maps; π is proper because G is compact. We are in the hypotheses of Theorem 2.10(ii) since M is a symplectic manifold and hence its symplectic leaves, when thinking of M as a Poisson manifold, are its connected components. ∎

Remark 3.6. Note that for the controllability of (3.3) it is not necessary for the vector fields $\widetilde{Y}_i \in \mathfrak{X}(M/G)$ to be induced by some G-invariant vector fields on M.

4. Examples. We will illustrate the theory with several examples. In all of them we will use the following well-known lemmas to prove the properness of the integrals of motion.

LEMMA 4.1. *Let $f : \mathbb{R}^n \to \mathbb{R}^k$ be a continuous function. Then f is proper if and only if*

$$\lim_{\|x\|\to\infty} \|f(x)\| = +\infty.$$

Proof. Suppose that f is proper. If $\lim_{\|x\|\to\infty} \|f(x)\| \neq +\infty$, there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ and a constant $M > 0$ such that $\|x_n\| \to \infty$ and $\|f(x_n)\| \leq M$. Thus $\{x_n\}_{n\in\mathbb{N}}$ lies in the inverse image by f of the closed ball of radius M, which is a compact set in $\mathbb{R}^n$ because f is assumed to be proper. Hence $\{x_n\}_{n\in\mathbb{N}}$ contains a convergent subsequence. However, $\|x_n\| \to \infty$, which is a contradiction.

Conversely, assume that $\lim_{\|x\|\to\infty} \|f(x)\| = +\infty$, and let $K \subset \mathbb{R}^k$ be a compact subset. The set $f^{-1}(K)$ is closed since f is continuous. To conclude that $f^{-1}(K)$ is compact we shall show that it is also bounded. If not, there would exist a sequence $\{x_n\}_{n\in\mathbb{N}} \subset f^{-1}(K)$ such that $\|x_n\| \to \infty$. By hypothesis, $\|f(x_n)\| \to \infty$, which contradicts the fact that $f(x_n) \in K$, which is bounded. □

LEMMA 4.2. *Let M, N, and P be Hausdorff topological spaces. Let $f : M \to N$ and $g : N \to P$ be two continuous functions. If $g \circ f : M \to P$ is proper, then f is also proper.*

Proof. Let $K \subset M$ be a compact subset. Then $g(K)$ is compact in P and hence $(g\circ f)^{-1}(g(K))$ is compact in M. Since $f^{-1}(K) \subset (g\circ f)^{-1}(g(K))$ is closed, it follows that it is also compact. □

Example 1. We will study the controllability of the Hamiltonian system describing the motion of a hollow rigid body and a particle oscillating in it; the body moves about a fixed point which is also the equilibrium position of the particle that oscillates along one of the principal axes of inertia. The description of this system and the proof of its nonintegrability by the method of Ziglin can be found in Christov [10].

The equations of motion are

$$\begin{aligned}
\dot{x}_1 &= \frac{x_2x_3}{C} - \frac{x_2x_3}{B+mx_4^2},\\
\dot{x}_2 &= \frac{x_1x_3}{A+mx_4^2} - \frac{x_1x_3}{C},\\
\dot{x}_3 &= \frac{x_1x_2}{B+mx_4^2} - \frac{x_1x_2}{A+mx_4^2},\\
\dot{x}_4 &= x_5,\\
\dot{x}_5 &= \frac{x_1^2x_4}{(A+mx_4^2)^2} + \frac{x_2^2x_4}{(B+mx_4^2)^2} - \frac{\sigma x_4}{m},
\end{aligned}$$

where $A > B > C$ are the principal moments of inertia, σ is the stiffness of the spring, and m is the mass of the particle.

This is a Hamiltonian system with phase space $so(3)^* \times \mathbb{R}^2$. The Poisson bracket is the product of the Lie–Poisson bracket on $so(3)^*$ with the Poisson bracket induced by the symplectic form $m dx_4 \wedge dx_5$ on $\mathbb{R}^2$. The Hamiltonian is given by

$$H = \frac{1}{2}\left(\frac{x_1^2}{A+mx_4^2} + \frac{x_2^2}{B+mx_4^2} + \frac{x_3^2}{C} + \sigma x_4^2 + mx_5^2\right).$$

It is easy to see that the symplectic leaves are embedded closed manifolds: Every four-dimensional leaf is the product of a sphere with $\mathbb{R}^2$, and the two-dimensional leaf is $\mathbb{R}^2$. One can easily check that the Hamiltonian H is a proper function.

Consider the underactuated control system with torques

$$\begin{aligned}
\dot{x}_1 &= \frac{x_2x_3}{C} - \frac{x_2x_3}{B+mx_4^2} + u_1,\\
\dot{x}_2 &= \frac{x_1x_3}{A+mx_4^2} - \frac{x_1x_3}{C} + u_2,\\
\dot{x}_3 &= \frac{x_1x_2}{B+mx_4^2} - \frac{x_1x_2}{A+mx_4^2} + u_3,\\
\dot{x}_4 &= x_5,\\
\dot{x}_5 &= \frac{x_1^2x_4}{(A+mx_4^2)^2} + \frac{x_2^2x_4}{(B+mx_4^2)^2} - \frac{\sigma x_4}{m} + u_4,
\end{aligned}$$

where the control $u := (u_1, u_2, u_3, u_4) : (0, \infty) \to B \subset \mathbb{R}^4$ is a measurable function with values in a bounded subset $B \subset \mathbb{R}^4$. The vector fields $\frac{\partial}{\partial x_1}$, $\frac{\partial}{\partial x_2}$, $\frac{\partial}{\partial x_3}$, $\frac{\partial}{\partial x_5}$, $[X_H, \frac{\partial}{\partial x_5}]$ verify the LARC and, as a result of Theorem 2.10(ii), we obtain that the above system is controllable.

Example 2. In this example we follow the presentation of the geometric structure in Adams and Ratiu [3]. The motion of three point vortices for an ideal inviscid incompressible fluid in the plane is given by the equations

$$\begin{cases} \dot{x}_j = -\frac{1}{2\pi} \sum\limits_{\substack{i=1 \\ i \neq j}}^{3} \Gamma_i (y_j - y_i)/r_{ij}^2, \\ \dot{y}_j = -\frac{1}{2\pi} \sum\limits_{\substack{i=1 \\ i \neq j}}^{3} \Gamma_i (x_j - x_i)/r_{ij}^2, \end{cases} \tag{4.1}$$

$j = 1, 2, 3$, where $r_{ij}^2 = (x_i - x_j)^2 + (y_i - y_j)^2$ and Γ_1, Γ_2, and Γ_3 are nonzero constants, the circulations given by the corresponding point vortices. These equations are defined on $\mathbb{R}^6$ after eliminating all the diagonals $\{(x_i, y_i) = (x_j, y_j)\}$ for $i \neq j$. Kirchhoff [21] noted that (4.1) can be written in the form

$$\Gamma_j \frac{dx_j}{dt} = \frac{\partial H}{\partial y_j},$$
$$\Gamma_j \frac{dy_j}{dt} = -\frac{\partial H}{\partial x_j},$$

where

$$H(x_1, x_2, x_3, y_1, y_2, y_3) = -\frac{1}{4\pi} \sum_{\substack{i=1 \\ i \neq j}}^{3} \Gamma_i \Gamma_j \log r_{ij}$$

is the Hamiltonian and the symplectic form is given by

$$\Omega = \sum_{i=1}^{3} \Gamma_i dx_i \wedge dy_i. \tag{4.2}$$

In what follows it is convenient to identify $\mathbb{R}^2$ with $\mathbb{C}$ by the map $(x, y) \mapsto x + \sqrt{-1}y$. The special Euclidean group $SE(2) := \{(e^{\sqrt{-1}\theta}, w) \mid \theta \in \mathbb{R}, w \in \mathbb{C}\}$ acts on $\mathbb{C}$ by $(e^{\sqrt{-1}\theta}, w) \cdot z := e^{\sqrt{-1}\theta} z + w$. This action is not free. The diagonal action of $SE(2)$ on $\mathbb{C}^3$ is free on the open invariant subset $\mathbb{C}^3 \setminus \{(z, z, z) \mid z \in \mathbb{C}\}$ that contains the open invariant subset $S := \mathbb{C}^3 \setminus \{(z_1, z_2, z_3) \mid z_i \neq z_j \text{ for } i \neq j\}$ on which the three point vortex problem is defined. It can be easily verified that this action is proper on S. This action has an associated nonequivariant momentum map $J : \mathbb{R}^6 \equiv \mathbb{C}^3 \to \mathbb{R}^3$ relative to the symplectic form (4.2) given by

$$J(\mathbf{x}, \ \mathbf{y}) = \left(-\frac{1}{2} \sum_{i=1}^{3} \Gamma_i (x_i^2 + y_i^2), \ \sum_{i=1}^{3} \Gamma_i y_i, \ -\sum_{i=1}^{3} \Gamma_i x_i \right).$$

If the vortex strengths Γ_1, Γ_2, and Γ_3 have the same signs, then, by applying Lemmas 4.2 and 4.1, it follows that J is a proper map and we are in the hypotheses of Theorem 3.4(i).

In Adams and Ratiu [3] it is shown that the quotient $S/SE(2)$ is diffeomorphic to $T := \mathbb{R}^3 \backslash (\{(0,0,c) \mid c \in \mathbb{R}\} \cup \{(a,0,0) \mid a \geq 0\})$, that the push forward of the quotient Poisson bracket on $S/SE(2)$ to T has the matrix

$$4 \begin{bmatrix} 0 & 2a_3 & -2a_2 \\ -2a_3 & 0 & 2a_1 - \|\mathbf{a}\| \\ 2a_2 & -2a_1 + \|\mathbf{a}\| & 0 \end{bmatrix},$$

and that the reduced Hamiltonian is

$$\begin{aligned} \widetilde{H}(a_1, a_2, a_3) = -\frac{1}{4\pi}(&\Gamma_1\Gamma_2 \log((a_3 + \|\mathbf{a}\|)/2) + \Gamma_1\Gamma_3 \log((-a_3 + \|\mathbf{a}\|)/2) \\ &+ \Gamma_2\Gamma_3 \log(-a_1 + \|\mathbf{a}\|)). \end{aligned}$$

Therefore, the reduced equations are

$$\begin{aligned} \dot{a}_1 &= \frac{2}{\pi}\left(\Gamma_1\Gamma_2 \frac{a_2}{(a_3 + \|\mathbf{a}\|)} - \Gamma_1\Gamma_3 \frac{a_2}{(-a_3 + \|\mathbf{a}\|)}\right), \\ \dot{a}_2 &= \frac{1}{\pi}\left(\Gamma_1\Gamma_2 \frac{(-2a_1 + a_3 + \|\mathbf{a}\|)}{(a_3 + \|\mathbf{a}\|)} + \Gamma_1\Gamma_3 \frac{(2a_1 + a_3 - \|\mathbf{a}\|)}{(-a_3 + \|\mathbf{a}\|)} + \Gamma_2\Gamma_3 \frac{a_3}{(-a_1 + \|\mathbf{a}\|)}\right), \\ \dot{a}_3 &= \frac{1}{\pi}\left(\Gamma_2\Gamma_3 \frac{a_2}{(-a_1 + \|\mathbf{a}\|)} - \Gamma_1\Gamma_2 \frac{a_2}{(a_3 + \|\mathbf{a}\|)} - \Gamma_1\Gamma_3 \frac{a_2}{(-a_3 + \|\mathbf{a}\|)}\right). \end{aligned}$$

Consider the reduced control system

$$\begin{aligned} \dot{a}_1 &= \frac{2}{\pi}\left(\Gamma_1\Gamma_2 \frac{a_2}{(a_3 + \|\mathbf{a}\|)} - \Gamma_1\Gamma_3 \frac{a_2}{(-a_3 + \|\mathbf{a}\|)}\right) + u_1, \\ \dot{a}_2 &= \frac{1}{\pi}\left(\Gamma_1\Gamma_2 \frac{(-2a_1 + a_3 + \|\mathbf{a}\|)}{(a_3 + \|\mathbf{a}\|)} + \Gamma_1\Gamma_3 \frac{(2a_1 + a_3 - \|\mathbf{a}\|)}{(-a_3 + \|\mathbf{a}\|)} + \Gamma_2\Gamma_3 \frac{a_3}{(-a_1 + \|\mathbf{a}\|)}\right) + u_2, \\ \dot{a}_3 &= \frac{1}{\pi}\left(\Gamma_2\Gamma_3 \frac{a_2}{(-a_1 + \|\mathbf{a}\|)} - \Gamma_1\Gamma_2 \frac{a_2}{(a_3 + \|\mathbf{a}\|)} - \Gamma_1\Gamma_3 \frac{a_2}{(-a_3 + \|\mathbf{a}\|)}\right) + u_3, \end{aligned}$$

where the control $u := (u_1, u_2, u_3) : (0, \infty) \to B \subset \mathbb{R}^3$ is a measurable function with values in a bounded subset $B \subset \mathbb{R}^3$. It is easy to check that the vector fields $X_{\widetilde{H}}, \frac{\partial}{\partial a_1}, \frac{\partial}{\partial a_2}, \frac{\partial}{\partial a_3}$ verify the LARC and, by Theorem 3.4(i), we conclude that this reduced system is controllable.

Example 3. The next example, whose geometric study can be found in Blaom [4], is the resonant three-wave interaction. This is a Hamiltonian system whose phase space is $\mathbb{R}^6 = \mathbb{C}^3$, equipped with the symplectic structure

$$\omega = \sum_{j=1}^{3} \frac{1}{s_j \gamma_j} dq_j \wedge dp_j,$$

where $s_1, s_2, s_3 \in \{-1, 1\}$ and $\gamma_1, \gamma_2, \gamma_3 \in \mathbb{R}$ are parameters subject to the constraint $\gamma_1 + \gamma_2 + \gamma_3 = 0$. We will restrict our attention to the particular case when $(s_1, s_2, s_3) = (1, 1, 1)$ and $(\gamma_1, \gamma_2, \gamma_3) = (1, 1, -2)$.

In standard coordinates on $\mathbb{C}^3$, $z_j := q_j + \sqrt{-1}p_j$, $j = 1, 2, 3$, the Hamiltonian is given by

$$H(z_1, z_2, z_3) = -\frac{1}{2}(\bar{z}_1 z_2 \bar{z}_3 + z_1 \bar{z}_2 z_3).$$

This Hamiltonian is invariant under the action of the compact Lie group $G \equiv S^1 \times S^1$ on P given by

$$(e^{i\theta_1}, e^{i\theta_2}) \cdot (z_1, z_2, z_3) = (e^{-i\theta_1} z_1, e^{-i(\theta_1+\theta_2)} z_2, e^{-i\theta_2} z_3), \qquad 0 \le \theta_j < 2\pi.$$

The momentum map for this action is $J : P \to \mathfrak{g}^* \cong \mathbb{R}^2$,

$$J(z_1, z_2, z_3) = \left(\frac{1}{2}\left(|z_1|^2 + |z_2|^2\right), \frac{1}{2}\left(|z_1|^2 - \frac{1}{2}|z_3|^2\right)\right).$$

This action is free on the open invariant subset $\left[(\mathbb{C}\backslash\{0\}) \times \mathbb{C} \times (\mathbb{C}\backslash\{0\})\right] \cup \left[\{0\} \times (\mathbb{C}\backslash\{0\}) \times (\mathbb{C}\backslash\{0\})\right] \cup \left[(\mathbb{C}\backslash\{0\}) \times (\mathbb{C}\backslash\{0\}) \times \{0\}\right]$. As in Blaom [4], we shall restrict the study of the resonant three-wave interaction to $S := (\mathbb{C}\backslash\{0\}) \times \mathbb{C} \times (\mathbb{C}\backslash\{0\})$, where the action is free. The smooth map $(z_1, z_2, z_3) \in S \mapsto (z_2\bar{z}_1\bar{z}_3/|z_1 z_3|, |z_1|, |z_3|) \in \mathbb{R}^2 \times (0, \infty)^2$ induces a diffeomorphism $S/G \approx \mathbb{R}^2 \times (0, \infty)^2$. The push forward by this diffeomorphism of the quotient Poisson bracket on S/G to $\{(q, p, a, b) \in \mathbb{R}^4 \mid q, p \in \mathbb{R}, a > 0, b > 0\} = \mathbb{R}^2 \times (0, \infty)^2$ has the expression

$$\begin{bmatrix} 0 & 1 & -\frac{p}{a} & 2\frac{p}{b} \\ -1 & 0 & \frac{q}{a} & -2\frac{q}{b} \\ \frac{p}{a} & -\frac{q}{a} & 0 & 0 \\ -2\frac{p}{b} & 2\frac{q}{b} & 0 & 0 \end{bmatrix}$$

and the reduced Hamiltonian is

$$\widetilde{H}(q, p, a, b) = -abq.$$

The reduced equations of motion are

$$\begin{cases} \dot{q} = \dfrac{qpb}{a} - 2\dfrac{qpa}{b}, \\ \dot{p} = ab - \dfrac{q^2 b}{a} + 2\dfrac{q^2 a}{b}, \\ \dot{a} = -pb, \\ \dot{b} = 2pa. \end{cases} \tag{4.3}$$

A constant of motion for the system (4.3) is given by the function $f : Q \to \mathbb{R}$, $f(q, p, a, b) = q^2 + p^2 + a^2 + b^2$, which is proper by Lemma 4.1.

Consider now the underactuated reduced control system

$$\begin{cases} \dot{q} = \dfrac{qpb}{a} - 2\dfrac{qpa}{b} + u_1, \\ \dot{p} = ab - \dfrac{q^2 b}{a} + 2\dfrac{q^2 a}{b} + u_2, \\ \dot{a} = -pb, \\ \dot{b} = 2pa + u_3, \end{cases} \tag{4.4}$$

where the control $u := (u_1, u_2, u_3) : (0, \infty) \to B \subset \mathbb{R}^3$ is a measurable function with values in a bounded subset B. A short computation shows that the vector

fields $\{\frac{\partial}{\partial q}, \frac{\partial}{\partial p}, \frac{\partial}{\partial b}, [\frac{\partial}{\partial b}, [\frac{\partial}{\partial p}, X_{\widetilde{H}}]]\}$ generate at every point $(q,p,a,b) \in \mathbb{R}^2 \times (0,\infty)^2$ the tangent space $T_{(q,p,a,b)}(\mathbb{R}^2 \times (0,\infty)^2)$, which proves that the system (4.4) verifies the LARC. By Theorem 3.4(ii), the system (4.4) is controllable.

Example 4. We will study the controllability of the reduced system of two coupled planar rigid bodies. We take the description of the system given in Sreenath, Oh, Krishnaprasad, and Marsden [44]. After the reduction to the center of mass frame we have the configuration space $S^1 \times S^1$ with the diagonal action of S^1. The phase space is $T^*(S^1 \times S^1)$ with the canonical symplectic form of a cotangent bundle. The momentum map for the lifted action of S^1 is given by

$$J((\theta_1, \mu_1), (\theta_2, \mu_2)) = \mu_1 + \mu_2.$$

Krishnaprasad and Marsden [25] have shown that the reduced Poisson space is

$$P := T^*(S^1 \times S^1)/S^1 \cong S^1 \times \mathbb{R}^2$$

and, if we chose coordinates (θ, μ_1, μ_2) on P, the matrix of the Poisson bracket is given by

$$\begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}.$$

The reduced Hamiltonian is given by the formula

$$H = \frac{1}{2\triangle}(\widetilde{I}_2\mu_1^2 - 2\varepsilon\lambda(\theta)\mu_1\mu_2 + \widetilde{I}_1\mu_2^2),$$

where $\triangle = \widetilde{I}_1\widetilde{I}_2 - \varepsilon^2(\lambda(\theta))^2 > 0$ and

d_i	is the distance from the hinge to the center of mass of body $i = 1, 2$,
θ	is the joint angle from body 1 to body 2,
$\lambda(\theta)$	equals $d_1 d_2 \cos\theta$,
m_i	is the mass of body $i = 1, 2$,
ε	equals $m_1 m_2/(m_1 + m_2)$ (the reduced mass),
I_i	is the moment of inertia of body i about its center of mass, and
$\widetilde{I}_i$	equals $I_i + \varepsilon d_i^2$, $i = 1, 2$ (the augmented moments of inertia).

To apply Theorem 3.4 we need to show that H is a proper function. To do this, we need the following lemma.

LEMMA 4.3. *Let $f : K \to \mathbb{R}$ and $g : \mathbb{R}^n \to \mathbb{R}$ be two continuous functions, where K is compact and g is a proper function. Then the function $h : K \times \mathbb{R}^n \to \mathbb{R}$ given by $h(x,y) := f(x)g(y)$ is a proper function.*

Proof. We shall prove that $h^{-1}([a,b])$ is compact in $K \times \mathbb{R}^n$. Let $z_n := (x_n, y_n)$ be an arbitrary sequence in $h^{-1}([a,b])$. Since K is compact, we can assume that $\{x_n\}_{n\in\mathbb{N}}$ is convergent. Because $\{f(x_n)g(y_n)\}_{n\in\mathbb{N}} \subset [a,b]$ and $\{f(x_n)\}_{n\in\mathbb{N}}$ is bounded, the sequence $\{g(y_n)\}_{n\in\mathbb{N}}$ is also bounded and hence there are $a', b' \in \mathbb{R}$ such that $\{g(y_n)\}_{n\in\mathbb{N}} \subset [a', b']$. Therefore, $\{y_n\}_{n\in\mathbb{N}} \subset g^{-1}([a', b'])$, which is a compact set in $\mathbb{R}^n$ because g is a proper function. Consequently, there is a convergent subsequence of $\{y_n\}_{n\in\mathbb{N}}$. The corresponding subsequence of $\{z_n\}_{n\in\mathbb{N}}$ is convergent, which proves that $h^{-1}([a,b])$ is compact. $\square$

To apply this lemma we write H in the form

$$H = \frac{1}{2\triangle}\left(\left(\sqrt{\widetilde{I}_2}\,\mu_1 - \frac{\varepsilon\lambda(\theta)}{\sqrt{\widetilde{I}_2}}\mu_2\right)^2 + \left(\widetilde{I}_1 - \frac{\varepsilon^2\lambda^2(\theta)}{\widetilde{I}_2}\right)\mu_2^2\right).$$

Since

$$\widetilde{I}_1 - \frac{\varepsilon^2\lambda^2(\theta)}{\widetilde{I}_2} > 0,$$

the smooth change of variables $(\theta, \mu_1, \mu_2) \mapsto (\theta, X, Y)$, where

$$X := \sqrt{\widetilde{I}_2}\,\mu_1 - \frac{\varepsilon\lambda(\theta)}{\sqrt{\widetilde{I}_2}}\mu_2,$$

$$Y := \left(\widetilde{I}_1 - \frac{\varepsilon^2\lambda^2(\theta)}{\widetilde{I}_2}\right)^{1/2}\mu_2$$

transforms H to the function $\frac{1}{2\triangle}\left(X^2 + Y^2\right)$ with $\frac{1}{2\triangle}$ defined on S^1. This function is proper by Lemmas 4.1 and 4.3. Thus H is a proper integral of motion for the reduced system.

Now we consider the following underactuated reduced control system with torques u_1, u_2

$$\dot{\theta} = -\frac{\partial H}{\partial \mu_1} + \frac{\partial H}{\partial \mu_2},$$

$$\dot{\mu}_1 = \frac{\partial H}{\partial \theta} + u_1,$$

$$\dot{\mu}_2 = -\frac{\partial H}{\partial \theta} + u_2,$$

where the control $u := (u_1, u_2) : (0, \infty) \to B \subset \mathbb{R}^2$ is a measurable function with values in a bounded subset B. It is easy to see that the vector fields $[X_H, \frac{\partial}{\partial \mu_1}]$, $[X_H, \frac{\partial}{\partial \mu_2}]$, $\frac{\partial}{\partial \mu_1}$, $\frac{\partial}{\partial \mu_2}$ verify the LARC and, as a consequence of Theorem 3.4(ii), we obtain that the reduced control system above is controllable.

Using Corollary 3.5 one can study the controllability of the unreduced system of the two coupled planar rigid bodies by considering any system of controls that satisfies the LARC and reduces to a system of controls that also satisfies the LARC, for example, the one above.

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