Nonlinearity **16** (2003) 1119–1149 PII: S0951-7715(03)38972-8

The second grade fluid and averaged Euler equations with Navier-slip boundary conditions

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Received 28 June 2002, in final form 13 March 2003
Published 15 April 2003
Online at stacks.iop.org/Non/16/1119

Recommended by P Deift

Abstract
We study the equations governing the motion of second grade fluids in a bounded domain of $\mathbb{R}^d$, $d = 2, 3$, with Navier-slip boundary conditions with and without viscosity (averaged Euler equations). We show global existence and uniqueness of H^3 solutions in dimension two. In dimension three, we obtain local existence of H^3 solutions for arbitrary initial data and global existence for small initial data and positive viscosity. We close by finding Liapunov stability conditions for stationary solutions for the averaged Euler equations similar to the Rayleigh–Arnold stability result for the classical Euler equations.

Mathematics Subject Classification: 76A05, 76D03, 76B03, 76E30, 35Q35

Dedicated To Doina Cioranescu on the occasion of her 60th birthday

Introduction

The second grade fluid equations have been considered in the literature, for instance by Dunn and Rajagopal [16] (see also Joseph [29] for a discussion of such fluids). These fluids are a particular class of non-Newtonian fluids of special differential type. Rivlin and Ericksen [35] developed in 1955 the theory of isotropic materials for which at time t the stress tensor T depends on the velocity, velocity gradient, and their higher time derivatives (evaluated also at time t)

$$T = -pI + F(A_1, A_2, \ldots, A_k),$$

where the scalar function p is the pressure, F is an isotropic function, and A_i are the Rivlin–Ericksen tensors defined by

$$A := A_1 = L + L^{\mathrm{T}}, \qquad L = \nabla u,$$

$$A_j := \dot{A}_{j-1} + L^{\mathrm{T}} A_{j-1} + A_{j-1} L, \qquad j = 2, \ldots, k,$$

where the dot denotes the material derivative. If the function F is a polynomial of degree k, the fluid is said to be of grade k. The constitutive law of a fluid of second grade is

$$T = -pI + \nu A_1 + \alpha_1 A_2 + \alpha_2 A_1^2,$$

where α_1 and α_2 are normal stress moduli.

Dunn and Fosdick [15] (see also Fosdick and Rajagopal [20]) considered the thermodynamics and stability (from the point of view of physics) of these fluids and showed that ν, α_1, and α_2 have to verify

$$\nu \geqslant 0 \qquad \text{and} \qquad \alpha_1 + \alpha_2 = 0$$

as a consequence of the Clausius–Duhem inequality. Moreover, the minimality of the Helmholtz free energy at the equilibrium forces the additional condition $\alpha := \alpha_1 > 0$. If $\alpha = 0$ then one recovers the constitutive law for the Navier–Stokes equations.

With these restrictions, the equation of motion of a second grade fluid becomes:

$$\partial_t v - \nu \Delta u + u \cdot \nabla v + \sum_j v_j \nabla u_j = -\nabla p + F, \qquad \operatorname{div} u = 0, \tag{1}$$

where $v := u - \alpha \Delta u$. These equations come with several boundary conditions:

- Dirichlet

$$u = 0 \qquad \text{on } \partial U, \tag{2}$$

- Navier-slip

$$u \cdot n = 0, \quad (D(u) \cdot n)^{\tan} = 0 \qquad \text{on } \partial U, \tag{3}$$

  where n is the outward unit normal vector field to the boundary ∂U, the symmetric matrix $D(u)$ defined by $D(u) := (\nabla u + (\nabla u)^{\mathrm{T}})/2$ is the deformation tensor, and the upper index tan denotes the tangential component of a vector based at ∂U relative to the orthogonal decomposition given by the tangent and normal spaces to the boundary at that point;
- mixed, that is, on different connected components one has Dirichlet and Navier-slip boundary conditions.

Equations (1) with Dirichlet boundary conditions (2) have received a lot of attention. Cioranescu and Ouazar [10] introduced a special Galerkin basis in order to obtain global existence and uniqueness of H^3 solutions in two dimensions and local existence and uniqueness in three dimensions for simply connected domains. These authors proved the existence of solutions by Galerkin's method in the basis of the eigenfunctions of the operator curl curl $(u - \alpha \Delta u)$. This choice of basis is optimal because it allows one to prove the existence of solutions with minimal restrictions on the data and the domain. The global existence for small initial data in three dimensions is shown in Galdi and Sequeira [21] and Cioranescu and Girault [9]. The study of these equations with Navier-slip boundary conditions is the subject of this paper. We note that these special boundary conditions are very different from the Dirichlet boundary conditions; they change dramatically the properties of the equations. For example, it is well-known that for boundaryless oriented smooth Riemannian manifolds, the solutions of the Navier–Stokes equations converge as the viscosity tends to zero to the solutions of the Euler equations (see Ebin and Marsden [17]). As opposed to this, it is also well known that this result is false for Dirichlet boundary conditions in all dimensions due to the fact that the zero limit viscosity produces boundary layers. Nevertheless, for Navier-slip boundary conditions on two-dimensional bounded simply connected domain the above vanishing zero viscosity result holds (see Clopeau *et al* [11]). Also, the Navier–Stokes equations become controllable with Navier-slip boundary conditions (see Coron [12]).

Recently, equations (1) were discovered in a very different and unexpected manner. Camassa and Holm [7] propose the following new model for shallow water:

$$u_t + 2ku_x - u_{xxt} + 3uu_x = 2u_x u_{xx} + u_x u_{xxx}.$$

Camassa and Holm [7] and Kouranbaeva [31] show that these equations for $k = 0$ are the spatial representation of the geodesic spray on the diffeomorphism group relative to the right invariant H^1 metric. Misiolek [34] proves the same result for $k \neq 0$ on the Bott–Virasoro group. Motivated by these results, Holm *et al* [25, 26] introduce the H^1 geodesic equations on the group of volume preserving diffeomorphisms of an oriented compact Riemannian manifold relative to the weak right invariant H^1_α Riemannian metric whose value at the identity is given by

$$\|u\|_{H^1_\alpha} = \left(\|u\|^2_{L^2} + \alpha\|\nabla u\|^2_{L^2}\right)^{1/2}.$$

The spatial representation of the resulting geodesic spray is called the α-Euler or averaged Euler equation and it coincides with the second grade fluids equation for vanishing viscosity. For the geometric interpretation, the well-posedness result for Dirichlet boundary conditions of these equations see Marsden *et al* [32] and Shkoller [39], and for the limit of zero viscosity see [32] and Busuioc [5]. The study of these equations on closed compact manifolds can be found in Shkoller [38]. For the infinite time of existence result in two dimensions see Busuioc [5] (for the case of $\mathbb{R}^2$) and Shkoller [39] (for the case of a bounded domain).

As for the above shallow water equations, we mention the following results. Shkoller [38] has shown global well-posedness in $H^s(\mathbb{T})$ if $s > 3/2$. On the other hand, Himonas and Misiolek [22] have shown that these equations are not locally well-posed for initial data in $H^s(\mathbb{T})$ if $s < 3/2$. These results suggest that $s = 3/2$ is the critical index for well-posedness. However, the existence of H^1 global weak solutions has been established by Xin and Zhang [44]. In a recent paper, Johnson [28] discusses various models for water waves, including the Camassa–Holm model.

Let us note that another viscous variant of the α-Euler equation, called the α-Navier–Stokes equation, was introduced by Chen *et al* [8]. This equation is similar to equation (1) where the term $-\nu\Delta u$ has been replaced by $-\nu\Delta v$. Global well-posedness of solutions for this equation in three dimensions was proved by Foias *et al* [19] for the periodic case and by Busuioc [6] for the case of a bounded domain. A slightly different equation is considered by Marsden and Shkoller [33] who prove global well-posedness of solutions in a three-dimensional bounded domain. This result is further completed by Coutand *et al* [13].

An interesting feature of these equations is that both the second grade fluid equation and the α-Navier–Stokes equation are equivalent to the Navier–Stokes equation when $\alpha = 0$. The problem of convergence of solutions to those of the Navier–Stokes equation as $\alpha \to 0$ is treated by Foias *et al* [19], Marsden and Shkoller [33], and Coutand *et al* [13] for the α-Navier–Stokes equation and by Iftimie [23, 24] for the second grade fluid equation.

Another method to obtain the particular form of the second grade fluid with zero viscosity is by averaging the Euler equations. We briefly recall some of the most important results for the classical Euler equations.

- The global time existence of the solutions to the two-dimensional Euler equations has been proved by Wolibner [43] and Kato [30] for classical solutions and by Yudovich [45] for generalized solutions (e.g. that include vortex patches).
- Yudovich [45] also proved uniqueness of flows with vorticity ω in L^∞. This result has been extended for unbounded vorticities all of whose L^p norms have a controlled growth rate (not very much stronger than linear) in Yudovich [46] (see also Vishik [42]). These are the strongest known uniqueness results for the two-dimensional Euler equations and they are all based on the energy method. No other method is known to prove uniqueness.

- Arnold [3, 4] gave Liapunov stability estimates in the L^2 norm on the vorticity or the H^1 norm on the velocity for stationary solutions for the two-dimensional Euler equations (see also Holm *et al* [27] for further applications of this method).

The point raised by the previous results for the two-dimensional incompressible Euler equations is that, whereas long time existence and Liapunov stability estimates are formulated in the H^1 norm on the velocity, uniqueness is unknown in this space.

A result contained in the theorems of sections 3, 4, and 5 is that for the inviscid second grade fluid equations (i.e. the averaged Euler equations) with Navier boundary conditions, all these results hold in the *same* space.

We will prove the following results.

Theorem 1. *Consider equation (1) in a bounded domain $U \subset \mathbb{R}^d$, $d = 2, 3$, with Navier-slip boundary conditions (3). Let $u_0 \in H^3(U)$ be a divergence free initial condition verifying the Navier-slip boundary conditions and let $F \in L^1(0, \infty; H^1)$ denote the external forcing. Then there exists a unique H^3 solution $u \in L^\infty(0, T^*; H^3)$ where the time of existence T^* is estimated as follows:*

(a) if $d = 2$ then $T^* = +\infty$,
(b) if $d = 3$ and $\|u_0\|_{H^3(U)} + \|F\|_{L^1(0,\infty;H^1)} \leqslant D_1\nu$ then $T^* = +\infty$,
(c) if $d = 3$ then

$$T^* \geqslant \frac{D_2}{\|u_0\|_{H^3(U)} + \|F\|_{L^1(0,\infty;H^1)}},$$

for some constants $D_1 > 0$ and $D_2 > 0$.

Theorem 2. *Let q_e be the potential vorticity of a stationary solution of the two-dimensional averaged Euler equations in a simply connected domain U and denote by ψ_e the corresponding stream function given as the unique solution of the boundary value problem*

$$q = -(1 - \alpha\Delta)\Delta\psi \qquad \text{in } U$$

with boundary conditions

$$\psi|_{\partial U} = 0 \qquad \text{and} \qquad 4n_1n_2\partial_1\partial_2\psi + \left(\partial_2^2\psi - \partial_1^2\psi\right)\left(n_2^2 - n_1^2\right)|_{\partial U} = 0,$$

where (n_1, n_2) is the outward unit normal. The velocity field u is given in terms of the stream function by $u = (\partial_2\psi, -\partial_1\psi)$. Assume that $\nabla\psi_\mathrm{e}/\nabla q_\mathrm{e} = f(q_\mathrm{e})$, where $f : \mathbb{R} \to \mathbb{R}$ is a smooth function.

- *If there are constants $a, A > 0$ such that*

  $$-A \leqslant \frac{\nabla\psi_\mathrm{e}}{\nabla q_\mathrm{e}} \leqslant -a.$$

  then the stationary solution is stable in the H^3-norm on perturbations of the velocity field.
- *If there are constants $a, A > 0$ such that*

  $$a \leqslant \frac{\nabla\psi_\mathrm{e}}{\nabla q_\mathrm{e}} \leqslant A$$

  and $a > k_{\min}^{-2}$, where $k_{\min}^2$ is the minimal eigenvalue of $-(1 - \alpha\Delta)\Delta$ on the space of solutions of the above boundary value problem with $q \in L^2$, then the stationary solution is Liapunov stable in the L^2-norm on perturbations of the potential vorticity or in the H^3 norm on the perturbations of the velocity.

- *The eigenfunctions ψ_λ of the operator $-(1-\alpha\Delta)\Delta$ with the boundary conditions given above are stationary solutions. The ground states ψ_{λ_0} are non-isolated global constrained minima of $\int_U q^2\,\mathrm{d}x\,\mathrm{d}y$ on the level sets of the kinetic energy of the fluid; they are Liapunov stable as a family relative to the seminorm on perturbations of the potential vorticity q which is the L^2-norm on the orthogonal complement of the first eigenfunction.*

Before giving the full proofs of these theorems, we want to point out some key difficulties. In the proof of well-posedness we shall adapt the strategy used in [9, 10]. There are several complications. First, one needs a different proof of the regularity result:

$$\operatorname{curl}(u-\alpha\triangle u)\in L^2 \implies u\in H^3$$

for the Navier-slip boundary conditions because in [9, 10] the authors use special construction that is necessary for simply connected domains. Here, we do not use this hypothesis on the domain. Second, the general elliptic results of Agmon *et al* [1] do not seem to be directly applicable, because the operator $\operatorname{curl}(1-\alpha\triangle)$ is of order three. Moreover, there are problems related to the compatibility of the boundary conditions and the uniqueness of the solutions. We shall use the fact that this operator is the composition of two elliptic operators, curl and $1-\alpha\triangle$ on divergence free vector fields. This implies the verification of certain boundary conditions for each of these operators. There are two possibilities: either $u-\alpha\triangle u$ verifies the boundary conditions for curl (in our case this is the tangency boundary condition), or $\operatorname{curl} u$ verifies the boundary conditions for $1-\alpha\triangle$ (in our case, zero on the boundary). It will be shown that at least one of these conditions (depending on the dimension) is 'almost' verified in the case of the Navier-slip boundary conditions:

- in dimension two (but not three) we have that '$\operatorname{curl} u = 0$' on the boundary;
- in dimension three we have that '$n\cdot\triangle u = 0$' on the boundary.

As a last remark, we note that these conditions do not hold for Dirichlet boundary conditions, hence the strategy employed in our proof is specific to the Navier-slip boundary conditions.

1. Notation and preliminary results

We denote by ∇u the matrix $\nabla u = (\partial_j u_i)_{i,j}$ and by $\operatorname{curl} u$ the scalar

$$\operatorname{curl} u := \partial_1 u_2 - \partial_2 u_1$$

if $d = 2$ or the vector field

$$\operatorname{curl} u := \begin{pmatrix}\partial_2 u_3 - \partial_3 u_2\\ \partial_3 u_1 - \partial_1 u_3\\ \partial_1 u_2 - \partial_2 u_1\end{pmatrix}$$

if $d = 3$. The symmetric matrix $D(u)$ is defined by

$$D(u) := \tfrac{1}{2}(\nabla u + (\nabla u)^{\mathrm{T}}).$$

The exterior product is given by

$$a\times b := \begin{pmatrix}a_2b_3 - a_3b_2\\ a_3b_1 - a_1b_3\\ a_1b_2 - a_2b_1\end{pmatrix}. \tag{4}$$

Throughout the paper, C will denote a generic constant whose value will not be specified; it can change from one inequality to another. The notation $K, K_0, K_1, \ldots$ is reserved for constants whose values do not change from one equation to the other.

We now state for the sake of completeness several classical results that will be used in the proofs below.

Lemma 1 (Korn). *There exists a constant K such that, for every vector field $u \in H^1(U)$, the following inequality holds:*

$$\|u\|_{L^2(U)} + \|D(u)\|_{L^2(U)} \leqslant \|u\|_{H^1(U)} \leqslant K(\|u\|_{L^2(U)} + \|D(u)\|_{L^2(U)}).$$

Lemma 2 (Gronwall). *Let $f, g : [0, T) \to \mathbb{R}$ satisfy the following conditions: f is differentiable, g is continuous, $f, g \geqslant 0$. Let a an arbitrary constant such that $f' + af \leqslant g$. Then*

$$f(t) \leqslant f(0)\mathrm{e}^{-at} + \int_0^t g(\tau)\mathrm{e}^{-a(t-\tau)}\,\mathrm{d}\tau.$$

Lemma 3. *Let f be a H^2 divergence free vector field verifying the Navier-slip boundary conditions (3) on the boundary of U and g an H^1 vector field tangent to the boundary of U. Then*

$$\int_U \Delta f \cdot g = -2\int_U D(f)D(g).$$

Proof. Recall that $D(f) = \frac{1}{2}(\nabla f + (\nabla f)^{\mathrm{T}})$. By definition

$$\begin{aligned}
\int_U \Delta f \cdot g &= \int_U \sum_{i,j} \partial_i^2 f_j\, g_j \\
&= \int_U \sum_{i,j} \partial_i(\partial_i f_j + \partial_j f_i) g_j - \int_U \sum_{i,j} \partial_i \partial_j f_i\, g_j \\
&= 2\int_U \sum_{i,j} \partial_i D(f)_{i,j} g_j - \int_U \sum_{i,j} \partial_i \partial_j f_i g_j \\
&= -2\int_U D(f) \cdot \nabla g + 2\int_{\partial U} \sum_{i,j} n_i D(f)_{i,j} g_j - \int_U \sum_j \partial_j(\operatorname{div} f)\, g_j \\
&= -2\int_U D(f)D(g) + 2\int_{\partial U} (D(f)n) \cdot g - \int_U g \cdot \nabla(\operatorname{div} f),
\end{aligned}$$

where we have used Stokes formula and the fact that $D(f)$ is a symmetric matrix. The hypotheses on f and g now imply the desired result. □

We now prove some 'boundary identities' implied by the Navier-slip boundary conditions. We will assume that the exterior normal n, defined *a priori* only on the boundary of U, is smoothly extended inside U by a vector field again denoted by n.

We start with the two-dimensional case. It is easy to check that if the boundary is a straight line and the Navier-slip conditions are verified, then $\operatorname{curl} u$ vanishes on the boundary. If the boundary is not a straight line, we will prove that the trace of $\operatorname{curl} u$ on the boundary can be expressed in terms of u only. More precisely, the following holds.

Proposition 1. *Let u be an H^2 vector field verifying the Navier-slip boundary conditions (3) in a bounded domain $U \subset \mathbb{R}^2$. The following identity holds:*

$$\operatorname{curl} u = 2u \cdot \frac{\partial n}{\partial \tau} \qquad \text{on } \partial U,$$

where $\partial/\partial\tau$ denotes the tangential derivative given by $\partial/\partial\tau = n_1\partial_2 - n_2\partial_1$.

Proof. Recall the Navier-slip conditions

$$u \cdot n = 0, \quad (D(u) \cdot n)^{\tan} = 0 \qquad \text{on } \partial U.$$

The second condition may be written in the form:

$$(D(u) \cdot n) \cdot n^{\perp} = 0, \tag{5}$$

where $n^{\perp} = (-n_2, n_1)$. Since

$$2D(u) = \begin{pmatrix} 2\partial_1 u_1 & \partial_1 u_2 + \partial_2 u_1 \\ \partial_1 u_1 + \partial_2 u_2 & 2\partial_2 u_2 \end{pmatrix},$$

we deduce from (5) that

$$2n_1 n_2(\partial_2 u_2 - \partial_1 u_1) + \left(n_1^2 - n_2^2\right)(\partial_1 u_2 + \partial_2 u_1) = 0 \qquad \text{on } \partial U. \tag{6}$$

We now use the first condition $u_1 n_1 + u_2 n_2 = 0$ and apply the tangential derivative $\partial/\partial\tau$ to infer that

$$\begin{aligned} 0 &= (n_1\partial_2 - n_2\partial_1)(u_1 n_1 + u_2 n_2) \\ &= (-n_2\,\partial_1 u_1 + n_1\,\partial_2 u_1)n_1 + (-n_2\,\partial_1 u_2 + n_1\,\partial_2 u_2)n_2 + u \cdot \frac{\partial n}{\partial\tau}, \end{aligned}$$

or, equivalently,

$$n_1 n_2(\partial_2 u_2 - \partial_1 u_1) + n_1^2\,\partial_2 u_1 - n_2^2\,\partial_1 u_2 + u \cdot \frac{\partial n}{\partial\tau} = 0. \tag{7}$$

Subtracting twice (7) from (6) yields

$$\left(n_1^2 + n_2^2\right)(\partial_1 u_2 - \partial_2 u_1) = 2u \cdot \frac{\partial n}{\partial\tau},$$

which is the desired conclusion since $n_1^2 + n_2^2 = 1$ and $\operatorname{curl} u = \partial_1 u_2 - \partial_2 u_1$. □

We now study the three-dimensional case. As in the two-dimensional case, we note that if the boundary is planar, then the vector field v is tangent to the boundary. If the boundary is not planar, we prove that the trace of $v \cdot n$ on the boundary can be expressed in terms of u and ∇u only.

Proposition 2. *Let u be an H^3 divergence free vector field verifying the Navier-slip boundary conditions in a bounded domain $U \subset \mathbb{R}^3$ and let $v = u - \alpha\Delta u$. The following identity holds:*

$$v \cdot n = G(u, \nabla u, n) \qquad \text{on } \partial U, \tag{8}$$

where $G(u, \nabla u, n)$ is a linear combination of u and ∇u with coefficients depending only on n and its derivatives.

Proof. We write the Navier-slip boundary conditions under the form

$$u \cdot n = 0, \quad D(u) \cdot n = \lambda\, n \qquad \text{on } \partial U,$$

where λ is a scalar function defined on ∂U. We now express $D(u)$, the deformation tensor, in the form

$$D(u) = (\nabla u)^{\mathrm{T}} + \tfrac{1}{2}\Omega(u), \tag{9}$$

where

$$\Omega(u) = (\partial_j u_i - \partial_i u_j)_{i,j} = \begin{pmatrix} 0 & \partial_2 u_1 - \partial_1 u_2 & \partial_3 u_1 - \partial_1 u_3 \\ \partial_1 u_2 - \partial_2 u_1 & 0 & \partial_3 u_2 - \partial_2 u_3 \\ \partial_1 u_3 - \partial_3 u_1 & \partial_2 u_3 - \partial_3 u_2 & 0 \end{pmatrix}.$$

Since

$$\omega = \operatorname{curl} u = \nabla \times u = \begin{pmatrix} \partial_2 u_3 - \partial_3 u_2 \\ \partial_3 u_1 - \partial_1 u_3 \\ \partial_1 u_2 - \partial_2 u_1 \end{pmatrix} = \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix},$$

it follows that

$$\Omega(u) = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}. \tag{10}$$

The second part of the Navier-slip conditions along with (9) implies that

$$(\nabla u)^{\mathrm{T}} n = \lambda n - \tfrac{1}{2}\Omega(u)\, n.$$

An easy computation using the form (10) of $\Omega(u)$ shows that

$$\Omega(u)\, n = \omega \times n, \tag{11}$$

so

$$(\nabla u)^{\mathrm{T}} n = \lambda n - \tfrac{1}{2}\omega \times n. \tag{12}$$

We now use that $u \cdot n$ and its tangential derivatives vanish on the boundary.

$$0 = (n \times \nabla)(u \cdot n) = \sum_i (n \times \nabla)(u_i\, n_i) = \sum_i u_i\, (n \times \nabla)\, n_i + \sum_i n_i\, (n \times \nabla)\, u_i.$$

The first component of $\sum_i n_i\, (n \times \nabla)\, u_i$ is

$$\sum_i (n_2\, \partial_3 u_i\, n_i - n_3\, \partial_2 u_i\, n_i) = n_2((\nabla u)^{\mathrm{T}} n)_3 - n_3((\nabla u)^{\mathrm{T}} n)_2 = [n \times ((\nabla u)^{\mathrm{T}} n)]_1.$$

Similar computations for the other components show that

$$\sum_i n_i\, (n \times \nabla)\, u_i = n \times ((\nabla u)^{\mathrm{T}} n).$$

In view of (12) we infer that

$$-\sum_i u_i\, (n \times \nabla) n_i = \lambda n \times n - \frac{1}{2} n \times (\omega \times n) = \frac{1}{2}(\omega \times n) \times n. \tag{13}$$

But it is a simple computation to verify that

$$(\omega \times n) \times n = -\omega + n\, (\omega \cdot n).$$

According to (13) this yields that

$$\omega - n\, (\omega \cdot n) = -(\omega \times n) \times n = 2 \sum_i u_i\, (n \times \nabla)\, n_i.$$

We now take the exterior product with n:

$$n \times \omega = 2n \times \sum_i u_i\, (n \times \nabla)\, n_i. \tag{14}$$

Next, we have that ω is divergence free and this implies after some computations that $\Delta u = -\operatorname{curl} \omega$. We infer that

$$v \cdot n = u \cdot n - \alpha\, n \cdot \Delta u = \alpha\, n \cdot \operatorname{curl} \omega. \tag{15}$$

From the well-known identity $(a \times b) \times (a \times c) = a\, (a \cdot (b \times c))$ we deduce that

$$(n \times \nabla) \times (n \times \omega) = n\, (n \cdot (\nabla \times \omega)) + H(\omega, n)$$

where $H(\omega, n)$ is linear in ω (does not involve any derivative of ω). Since the operator $n \times \nabla$ contains only tangential derivatives, we deduce from (14) that

$$\begin{aligned}(n \times \nabla) \times \left(2n \times \sum_i u_i (n \times \nabla) n_i\right) &= (n \times \nabla) \times (n \times \omega)\\ &= n(n \cdot (\nabla \times \omega)) + H(\omega, n).\end{aligned}$$

Taking the scalar product with n and using that $n \cdot n = 1$ we now get that

$$n \cdot \operatorname{curl} \omega = n \cdot \left((n \times \nabla) \times \left(2n \times \sum_i u_i (n \times \nabla) n_i\right)\right) - n \cdot H(\omega, n).$$

The conclusion now follows from relation (15). □

Remark 1. It is easy to see from the two previous propositions that in both dimensions, the following relation holds:

$$(\Omega n)_i = \sum_j u_j f_{ij}(n), \tag{16}$$

where Ω denotes the matrix $\Omega = \Omega(u) = (\partial_j u_i - \partial_i u_j)_{i,j}$ and $f_{ij}(n)$ some polynomial functions on n and its derivatives. This follows trivially in dimension two from proposition 1 while in dimension three is a direct consequence of relations (11) and (14).

We now give a simple identity that will be used to pass from boundary integrals to integrals on U. It is a direct consequence of the Stokes formula that, for any H^1 function g,

$$\int_{\partial U} g = \sum_i \int_{\partial U} n_i (n_i g) = \sum_i \int_U \partial_i (n_i g) = \sum_i \int_U n_i \partial_i g + \int_U g \operatorname{div} n. \tag{17}$$

Of special interest in the proof is the following modified Stokes problem

$$\begin{aligned}&u - \alpha \Delta u = f + \nabla p, \qquad \operatorname{div} u = 0,\\ &u \text{ verifies the Navier-slip boundary conditions on } \partial U.\end{aligned} \tag{18}$$

Concerning the existence and the regularity of the solution (u, p) to this problem we refer to [36] for the following result.

Theorem 3. *Suppose that $f \in H^m(U)^3$, $m \in \mathbb{N}$. There exists a unique (up to a constant for p) solution $(u, p) \in (H^{m+2}(U)^3, H^{m+1}(U))$ and there is a constant $K(m)$ such that*

$$\|u\|_{H^{m+2}} + \|p\|_{H^{m+1}} \leqslant K(m) \|f\|_{H^m}.$$

Ščadilov and Solonnikov [36] actually do not consider exactly problem (18) but the Stokes problem

$$\begin{aligned}&-\alpha \Delta u = f + \nabla p, \qquad \operatorname{div} u = 0,\\ &u \text{ verifies the Navier-slip boundary conditions on } \partial U.\end{aligned} \tag{19}$$

whose variational formulation is

$$2\alpha \langle D(u), D(v)\rangle_{L^2} = \langle f, v\rangle_{L^2} \qquad \forall v$$

on the space of divergence free H^1 vector fields tangent to the boundary endowed with the scalar product $\langle D(u), D(v)\rangle_{L^2}$. In fact, it may happen that a non-zero vector field h exists in this space and such that its deformation tensor $D(h)$ vanishes. As a consequence, the result of [36] has to assume a compatibility condition on f ($\langle f, h\rangle_{L^2} = 0$) and the estimate on the solution u actually holds in an equivalence class which takes into account that h is a non-zero solution of the homogeneous problem. In our case these problems do not appear as the variational formulation is

$$\langle u, v\rangle_{L^2} + 2\alpha\langle D(u), D(v)\rangle_{L^2} = \langle f, v\rangle_{L^2} \qquad \forall v \tag{20}$$

and there is no non-zero vector field h such that $\|h\|_{L^2}^2 + 2\alpha\|D(h)\|_{L^2}^2 = 0$. The existence and the uniqueness of an H^1 solution u of (20) is immediate from the Lax–Milgram lemma and the regularity stated in theorem 3 follows exactly as in [36]. We also note that theorem 3 follows directly from the result of [36], too. Indeed, for instance in the case $m = 0$ if h is a solution of the homogeneous version of (19), then if we take as test function $v = h$ in (20) it follows that $\langle f - u, h = 0\rangle_{L^2}$. Thus, a solution of (18) is also a solution of (19) with f replaced by $f - u$ and $f - u$ verifies the compatibility condition. Since we know that $f - u \in L^2(U)$ it follows from the result of [36] that $u \in H^2(U)$. We finally note that even though the result of [36] is stated only for the case $m = 0$ it is remarked at the end of that paper that the general case (and $W^{m,p}$ estimates as well) follows from the general theory of elliptic systems as in [41].

As an immediate consequence of theorem 3 we obtain the following proposition.

Proposition 3. *Let U be a bounded domain and $m \geqslant 2$ be a positive integer. Then there is a constant $K > 0$ such that*

$$\|u\|_{H^m(U)} \leqslant K\|u - \alpha\Delta u\|_{H^{m-2}(U)}$$

for all divergence free H^m vector fields u verifying the Navier-slip boundary conditions (3).

Proof. Simply apply theorem 3 with $p = 0$ and $f = u - \alpha\Delta u$. □

We now recall some regularity results that will be used in connection with the Navier-slip boundary conditions. The first result is very classical and concerns the regularity of the Laplacian with Dirichlet boundary conditions.

Proposition 4. *Let U be a bounded domain, $m \geqslant 1$ and g be a function such that $g - \alpha\Delta g \in H^{m-2}(U)$ and $g|_{\partial U} \in H^{m-1/2}(\partial U)$. Then $g \in H^m(U)$ and there is a constant $K > 0$ such that*

$$\|g\|_{H^m(U)} \leqslant K\|g - \alpha\Delta g\|_{H^{m-2}(U)} + K\|g\|_{H^{m-1/2}(\partial U)}.$$

The next result is proved in Foias and Temam [18].

Proposition 5. *Let U be a bounded domain and m be a positive integer. If u is a L^2 vector field such that* $\operatorname{curl} u \in H^{m-1}(U)$, $\operatorname{div} u \in H^{m-1}(U)$ *and $u \cdot n|_{\partial U} \in H^{m-1/2}(\partial U)$ then $u \in H^m(U)$ and there is a constant $K > 0$ such that*

$$\|u\|_{H^m(U)} \leqslant K\|u\|_{L^2(U)} + K\|\operatorname{curl} u\|_{H^{m-1}(U)} + K\|\operatorname{div} u\|_{H^{m-1}(U)} + K\|u \cdot n\|_{H^{m-1/2}(\partial U)}$$

for all such vector fields u.

We now recall that for a function $f \in H^m(U)$, $m \geqslant 1$, the trace $f|_{\partial U}$ belongs to $H^{m-1/2}(\partial U)$ and that there is a constant $K > 0$ such that

$$\|f\|_{H^{m-1/2}(\partial U)} \leqslant K\|f\|_{H^m(U)}. \tag{21}$$

We shall use later the following well-known interpolation inequality: there is a constant $K > 0$ such that, for every function $f \in H^3(U)$, one has that

$$\|f\|_{H^2(U)}^2 \leqslant K\|f\|_{H^3(U)}\|f\|_{H^1(U)}. \tag{22}$$

We close this section with a regularity result that will be essential in later proofs.

Proposition 6. *There is a constant $K > 0$ such that for every divergence free vector field $u \in H^3(U)$ which verifies the Navier-slip boundary conditions (3) on ∂U the following inequality holds:*

$$\|u\|_{H^3(U)} \leqslant K\|\operatorname{curl}(u - \alpha\Delta u)\|_{L^2(U)} + K\|u\|_{H^1(U)}. \tag{23}$$

Proof. Let $v = u - \alpha\Delta u$. We start with the two-dimensional case. According to proposition 4 we have that

$$\|\operatorname{curl} u\|_{H^2(U)} \leqslant C\|\operatorname{curl} v\|_{L^2(U)} + C\|\operatorname{curl} u\|_{H^{3/2}(\partial U)}.$$

By proposition 1 and relation (21) we get

$$\|\operatorname{curl} u\|_{H^2(U)} \leqslant C\|\operatorname{curl} v\|_{L^2(U)} + C\left\|u\cdot\frac{\partial n}{\partial\tau}\right\|_{H^{3/2}(\partial U)} \leqslant C\|\operatorname{curl} v\|_{L^2(U)} + C\left\|u\cdot\frac{\partial n}{\partial\tau}\right\|_{H^2(U)}.$$

Since $H^2(U)$ is an algebra we further deduce that

$$\begin{aligned}\|\operatorname{curl} u\|_{H^2(U)} &\leqslant C\|\operatorname{curl} v\|_{L^2(U)} + C\|u\|_{H^2(U)}\left\|\frac{\partial n}{\partial\tau}\right\|_{H^2(U)}\\ &\leqslant C\|\operatorname{curl} v\|_{L^2(U)} + C\|u\|_{H^2(U)}.\end{aligned} \tag{24}$$

We now use proposition 5 with $m = 3$ to obtain

$$\|u\|_{H^3(U)} \leqslant C\|u\|_{L^2(U)} + C\|\operatorname{curl} u\|_{H^2(U)} + C\|\operatorname{div} u\|_{H^2(U)} + C\|u\cdot n\|_{H^{5/2}(\partial U)}.$$

Since u is divergence free and tangent to the boundary, we infer from (24) that

$$\|u\|_{H^3(U)} \leqslant C\|\operatorname{curl} v\|_{L^2(U)} + C\|u\|_{H^2(U)}. \tag{25}$$

From the interpolation inequality (22) we get that

$$C\|u\|_{H^2(U)} \leqslant C(\|u\|_{H^3(U)}\|u\|_{H^1(U)})^{1/2} \leqslant \tfrac{1}{2}\|u\|_{H^3(U)} + C\|u\|_{H^1(U)}.$$

The two previous relations now imply that

$$\|u\|_{H^3(U)} \leqslant C\|\operatorname{curl} v\|_{L^2(U)} + C\|u\|_{H^1(U)}$$

which completes the proof in the two-dimensional case.

We will now prove that a similar relation holds in the case of the dimension three. We apply proposition 5 with $m = 1$ to obtain

$$\|v\|_{H^1(U)} \leqslant C\|v\|_{L^2(U)} + C\|\operatorname{curl} v\|_{L^2(U)} + C\|\operatorname{div} v\|_{L^2(U)} + C\|v\cdot n\|_{H^{1/2}(\partial U)}. \tag{26}$$

Since v is divergence free, we deduce from proposition 2, (26) and (21) that

$$\|v\|_{H^1(U)} \leqslant C\|v\|_{L^2(U)} + C\|\operatorname{curl} v\|_{L^2(U)} + C\|G(u, \nabla u, n)\|_{H^1(U)}.$$

We know that G contains only u and its first derivatives and hence we can conclude that

$$\|v\|_{H^1(U)} \leqslant C\|\operatorname{curl} v\|_{L^2(U)} + C\|u\|_{H^2(U)}.$$

Next, proposition 3 implies that

$$\|u\|_{H^3(U)} \leqslant C\|v\|_{H^1(U)} \leqslant C\|\operatorname{curl} v\|_{L^2(U)} + C\|u\|_{H^2(U)}.$$

This relation is similar to relation (25). Therefore, relation (23) follows as in the case $d = 2$. □

2. *A priori* estimates

The aim of this section is to prove some H^3 *a priori* estimates. Estimates of the same nature appear already in [9, 10] but for Dirichlet boundary conditions and for $\nu > 0$. One cannot extend directly these estimates to the Navier-slip boundary conditions, especially if one wants to include the case $\nu = 0$.

As a first step, we deduce H^1 estimates. These estimates are well-known in the case of Dirichlet boundary conditions but the extension to the case of Navier-slip boundary conditions must be carefully checked.

2.1. *A priori* H^1 *estimates*

Multiplying by u the equation (1) and integrating over U we obtain

$$\int_U \partial_t v \cdot u - \int_U \nu\Delta u \cdot u + \int_U u \cdot \nabla v \cdot u + \int_U \sum_j v_j \nabla u_j \cdot u = -\int_U \nabla p \cdot u + \int_U F \cdot u. \quad (27)$$

To estimate the first term we use lemma 3:

$$\begin{aligned}\int_U \partial_t v \cdot u &= \int_U \partial_t u \cdot u - \alpha \int_U \partial_t(\Delta u)\cdot u \\ &= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_U |u|^2 + 2\alpha\int_U D(\partial_t u)\cdot D(u) \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\int_U |u|^2 + \alpha\int_U |D(u)|^2\right).\end{aligned} \quad (28)$$

Note that the quantity in parenthesis is equivalent to the H^1 norm (see lemma 1).

The second term of (27) is immediately estimated by lemma 3:

$$-\nu\int_U \Delta u \cdot u = 2\nu\int_U |D(u)|^2. \quad (29)$$

The two nonlinear terms of (27) are seen to vanish after some integration by parts using that $\operatorname{div} u = 0$:

$$\begin{aligned}\int_U u\cdot\nabla v\cdot u + \int_U \sum_j v_j \nabla u_j \cdot u &= \int_U \sum_{i,j}(u_i u_j \partial_i v_j + u_i \partial_i u_j v_j) \\ &= \int_U \sum_{i,j} \partial_i(u_i u_j v_j) \\ &= \int_{\partial U} (u\cdot n)\,(u\cdot v) \\ &= 0.\end{aligned} \quad (30)$$

Since u is divergence free and tangent to the boundary, the pressure term of (27) is also vanishing:

$$-\int_U \nabla p\cdot u = \int_U p\cdot \operatorname{div} u - \int_{\partial U} p\, n\cdot u = 0. \quad (31)$$

As for the forcing term, we simply estimate

$$\int_U F\cdot u \leqslant \|F\|_{L^2}\|u\|_{L^2} \leqslant \|F\|_{L^2}\left(\|u\|_{L^2}^2 + 2\alpha\|D(u)\|_{L^2}^2\right)^{1/2}. \quad (32)$$

We now obtain from relations (28)–(32) and (27) that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\|u\|_{L^2}^2+\alpha\|D(u)\|_{L^2}^2\right)+2\nu\|D(u)\|_{L^2}^2\leqslant\|F\|_{L^2}\big(\|u\|_{L^2}^2+2\alpha\|D(u)\|_{L^2}^2\big)^{1/2}.$$

As $\nu\geqslant 0$, we deduce from the previous relation after simplification that

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(\|u\|_{L^2}^2+2\alpha\|D(u)\|_{L^2}^2\big)^{1/2}\leqslant\|F\|_{L^2}.$$

Upon integration

$$\big(\|u(t)\|_{L^2}^2+2\alpha\|D(u(t))\|_{L^2}^2\big)^{1/2}\leqslant(\|u_0\|_{L^2}^2+2\alpha\|D(u_0)\|_{L^2}^2\big)^{1/2}+\int_0^t\|F\|_{L^2}.$$

By Korn's lemma 1, the norm $f\mapsto(\|f\|_{L^2}^2/2+\alpha\|D(f)\|_{L^2}^2)^{1/2}$ is equivalent to the H^1 norm. We conclude that

$$\|u(t)\|_{H^1}\leqslant C\|u_0\|_{H^1}+C\int_0^t\|F\|_{L^2}. \tag{33}$$

2.2. *L^2 estimates of* curl v

We now go to the estimate of the H^3 norm of u. In view of the above H^1 estimates and of proposition 6, it sufficient to estimate the L^2 norm of curl v. We will use the following notation:

$$\operatorname{curl} u=\omega_u,\qquad \operatorname{curl} v=\omega_v.$$

Applying the curl to (1), it is easy to see that ω_v verifies the following relation:

$$\partial_t\omega_v-\nu\Delta\omega_u+u\cdot\nabla\omega_v-(d-2)\omega_v\cdot\nabla u=\operatorname{curl} F\qquad(d=2,3).$$

We will now estimate the L^2 norm of curl v. Multiplying the above relation by ω_v yields the following relation:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_U|\omega_v|^2\underbrace{-\nu\int_U\Delta\omega_u\cdot\omega_v}_{I_1}+\underbrace{\int_U u\cdot\nabla\omega_v\cdot\omega_v}_{I_2}-(d-2)\underbrace{\int_U\omega_v\cdot\nabla u\cdot\omega_v}_{I_3}$$
$$=\underbrace{\int_U\operatorname{curl} F\cdot\omega_v}_{I_4}. \tag{34}$$

We will now estimate each of these terms. To estimate I_1 we first note that $\alpha\Delta\omega_u=\omega_u-\omega_v$ so that

$$\begin{aligned}I_1=&-\frac{\nu}{\alpha}\int_U(\omega_u-\omega_v)\cdot\omega_v=\frac{\nu}{2\alpha}\int_U\big(|\omega_v|^2+|\omega_v-\omega_u|^2-|\omega_u|^2\big)\\&\geqslant\frac{\nu}{2\alpha}\int_U\big(|\omega_v|^2-|\omega_u|^2\big)\geqslant\frac{\nu}{2\alpha}\|\omega_v\|_{L^2}^2-\frac{\nu}{2\alpha}\|u\|_{H^1}^2.\end{aligned} \tag{35}$$

Since u is divergence free and tangent to the boundary, the term I_2 vanishes:

$$I_2=\int_U u\cdot\nabla\omega_v\cdot\omega_v=\frac{1}{2}\int_U u\cdot\nabla\big(|\omega_v|^2\big)=-\frac{1}{2}\int_U\operatorname{div} u\,|\omega_v|^2+\frac{1}{2}\int_{\partial U}u\cdot n\,|\omega_v|^2=0. \tag{36}$$

The next term is trivially estimated by the Sobolev embedding $H^2\hookrightarrow L^\infty$

$$|I_3|\leqslant\|\omega_v\|_{L^2}^2\|\nabla u\|_{L^\infty}\leqslant K_1\|\omega_v\|_{L^2}^2\|u\|_{H^3}, \tag{37}$$

where K_1 is the constant from the above mentioned Sobolev embedding.

The last term is bounded by

$$|I_4| \leqslant \|\text{curl } F\|_{L^2} \|\omega_v\|_{L^2}. \tag{38}$$

Relations (35)–(38) and (34) yield

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\omega_v\|_{L^2}^2 + \frac{\nu}{\alpha}\|\omega_v\|_{L^2}^2 \leqslant \frac{\nu}{\alpha}\|u\|_{H^1}^2 + 2K_1(d-2)\|\omega_v\|_{L^2}^2\|u\|_{H^3} + 2\|\text{curl } F\|_{L^2}\|\omega_v\|_{L^2}. \tag{39}$$

3. Proof of the existence theorem

This section presents the proof of theorem 1.

3.1. Global estimates, $d = 2, 3$

We will now deduce some global *a priori* estimates. We assume that

$$(d-2)\|u\|_{H^3} \leqslant \frac{\nu}{4K_1\alpha}. \tag{40}$$

Under this hypothesis, we deduce from (39) that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\omega_v\|_{L^2}^2 + \frac{\nu}{2\alpha}\|\omega_v\|_{L^2}^2 \leqslant \frac{\nu}{\alpha}\|u\|_{H^1}^2 + 2\|\text{curl } F\|_{L^2}\|\omega_v\|_{L^2}. \tag{41}$$

If $\nu = 0$ it suffices to simplify $\|\omega_v\|_{L^2}$ and to integrate the above relation to obtain

$$\|\omega_v(t)\|_{L^2} \leqslant \|\omega_v(0)\|_{L^2} + \int_0^t \|\text{curl } F\|_{L^2}. \tag{42}$$

If $\nu > 0$ we apply Gronwall's lemma in (41) to infer that

$$\begin{aligned}
\|\omega_v(t)\|_{L^2}^2 \leqslant{}& \|\omega_v(0)\|_{L^2}^2 \mathrm{e}^{-(\nu/2\alpha)t} + \frac{\nu}{\alpha}\int_0^t \|u(\tau)\|_{H^1}^2 \mathrm{e}^{-(\nu/2\alpha)(t-\tau)}\mathrm{d}\tau \\
&+ 2\int_0^t \|\text{curl } F(\tau)\|_{L^2}\|\omega_v(\tau)\|_{L^2}\, \mathrm{e}^{-(\nu/2\alpha)(t-\tau)}\mathrm{d}\tau \\
\leqslant{}& \|\omega_v(0)\|_{L^2}^2 + \frac{\nu}{\alpha}\sup_{[0,t]}\|u\|_{H^1}^2 \int_0^t \mathrm{e}^{-(\nu/2\alpha)\tau}\mathrm{d}\tau + 2\sup_{[0,t]}\|\omega_v\|_{L^2}\int_0^t \|\text{curl } F\|_{L^2} \\
\leqslant{}& \|\omega_v(0)\|_{L^2}^2 + 2\sup_{[0,t]}\|u\|_{H^1}^2 + 2\sup_{[0,t]}\|\omega_v\|_{L^2}\int_0^t \|\text{curl } F\|_{L^2} \\
\leqslant{}& \|\omega_v(0)\|_{L^2}^2 + C\|u_0\|_{H^1}^2 + C\Big(\int_0^t \|F\|_{L^2}\Big)^2 + 2\sup_{[0,t]}\|\omega_v\|_{L^2}\int_0^t \|\text{curl } F\|_{L^2},
\end{aligned}$$

where we have applied the H^1 *a priori* estimates (33). We now take the supremum on $[0, T]$ to obtain

$$\begin{aligned}
\sup_{[0,T]}\|\omega_v\|_{L^2}^2 \leqslant{}& C\|u_0\|_{H^3}^2 + C\Big(\int_0^{\mathrm{T}} \|F\|_{L^2}\Big)^2 + 2\sup_{[0,T]}\|\omega_v\|_{L^2}\int_0^{\mathrm{T}} \|\text{curl } F\|_{L^2} \\
\leqslant{}& C\|u_0\|_{H^3}^2 + C\Big(\int_0^{\mathrm{T}} \|F\|_{L^2}\Big)^2 + \frac{1}{2}\sup_{[0,T]}\|\omega_v\|_{L^2}^2 + 2\Big(\int_0^{\mathrm{T}} \|\text{curl } F\|_{L^2}\Big)^2.
\end{aligned}$$

We deduce the following *a priori* estimates for $\|\omega_v\|_{L^2}$

$$\sup_{[0,T]}\|\omega_v\|_{L^2} \leqslant C\|u_0\|_{H^3} + C\int_0^{\mathrm{T}} \|F\|_{H^1} \tag{43}$$

and we remark from (42) that the same relation holds also if $\nu = 0$.

Combining this with the H^1 estimate (33) we now deduce some H^3 estimates for u. Recall that (43) is valid only if the inequality

$$(d-2)\|u\|_{H^3} \leqslant \frac{\nu}{4K_1\alpha} \tag{44}$$

is verified. We infer from (25), (43), and (33) that

$$\sup_{[0,t]} \|u\|_{H^3} \leqslant C \sup_{[0,t]} \|u\|_{H^1} + C \sup_{[0,t]} \|\mathrm{curl}\, v\|_{L^2} \leqslant K_2\|u_0\|_{H^3} + K_2 \int_0^t \|F\|_{H^1} \tag{45}$$

for some constant $K_2 > 1$.

If $d = 2$ inequality (44) is always verified. Therefore (45) implies

$$\|u\|_{L^\infty(0,\infty;H^3)} \leqslant K_2\|u_0\|_{H^3} + K_2\|F\|_{L^1(0,\infty;H^1)}, \tag{46}$$

which proves the first part of theorem 1.

If $d = 3$ and $\nu > 0$, to obtain global estimates we must first ensure that (44) holds at time $t = 0$, that is

$$\|u_0\|_{H^3} < \frac{\nu}{4K_1\alpha}.$$

We must also ensure that relation (45) implies the strict inequality of (44), that is

$$K_2\|u_0\|_{H^3} + K_2 \int_0^\infty \|F\|_{H^1} < \frac{\nu}{4K_1\alpha}. \tag{47}$$

If these two relations are verified, it is easy to see that (44) and hence (46) globally holds. Indeed, arguing by contradiction, suppose that this not true and let T be the first time for which the equality in (44) holds:

$$\|u(T)\|_{H^3} = \frac{\nu}{4K_1\alpha}.$$

Then (44) holds on $[0, T]$ and so does (45). From (45) and (47) we deduce that $\|u(T)\|_{H^3} < \nu/4K_1\alpha$, which is a contradiction. We conclude that if

$$\|u_0\|_{H^3} + \int_0^\infty \|F\|_{H^1} \leqslant \frac{\nu}{8K_1K_2\alpha}$$

then (46) holds also in the three-dimensional case and thus the second part of theorem 1 is proved.

3.2. *Local estimates, $d = 3$*

We now prove some local in time estimates for $\|u\|_{H^3}$ in the three-dimensional case.

We first consider the case $\nu > 0$ and apply Gronwall's lemma 2 in (39) to deduce:

$$\begin{aligned}
\|\omega_v(t)\|_{L^2}^2 \leqslant{}& \|\omega_v(0)\|_{L^2}^2 \mathrm{e}^{-(\nu/2\alpha)t} + \frac{\nu}{\alpha}\int_0^t \|u(\tau)\|_{H^1}^2 \mathrm{e}^{-(\nu/2\alpha)(t-\tau)}\mathrm{d}\tau \\
&+ 2K_1 \int_0^t \|\omega_v(\tau)\|_{L^2}^2 \|u(\tau)\|_{H^3} \mathrm{e}^{-(\nu/2\alpha)(t-\tau)}\mathrm{d}\tau \\
&+ 2\int_0^t \|\mathrm{curl}\, F(\tau)\|_{L^2} \|\omega_v(\tau)\|_{L^2} \mathrm{e}^{-(\nu/2\alpha)(t-\tau)}\mathrm{d}\tau \\
\leqslant{}& C\|u_0\|_{H^3}^2 + 2\sup_{[0,t]} \|u\|_{H^1}^2 + Ct\sup_{[0,t]} \|u\|_{H^3}^3 + 2\sup_{[0,t]} \|\omega_v\|_{L^2} \int_0^t \|\mathrm{curl}\, F\|_{L^2} \\
\leqslant{}& C\|u_0\|_{H^3}^2 + C\Big(\int_0^t \|F\|_{L^2}\Big)^2 + Ct\sup_{[0,t]} \|u\|_{H^3}^3 + C\sup_{[0,t]} \|u\|_{H^3} \int_0^t \|\mathrm{curl}\, F\|_{L^2};
\end{aligned}$$

in the last inequality (33) was used. A similar inequality holds if $\nu = 0$. This follows directly by integrating (39) or simply by remarking that in the above computations the constants are independent of ν so we can let $\nu \to 0$.

According to (33) and proposition 6 we further infer that

$$\|u(t)\|_{H^3}^2 \leqslant K_3\|u_0\|_{H^3}^2 + K_3\Big(\int_0^t \|F\|_{L^2}\Big)^2 + K_3 t \sup_{[0,t]} \|u\|_{H^3}^3 + K_3 \sup_{[0,t]} \|u\|_{H^3} \int_0^t \|\operatorname{curl} F\|_{L^2} \tag{48}$$

for some constant $K_3 > 1$.

Set now

$$M = \max\left(2\sqrt{K_3}\|u_0\|_{H^3}\,,\ 2\sqrt{K_3}\int_0^\infty \|F\|_{L^2}\,,\ 4K_3\int_0^\infty \|\operatorname{curl} F\|_{L^2}\right) \tag{49}$$

and define

$$T_1 = \sup\{t \mid \|u(t)\|_{H^3} \leqslant M\}.$$

We now prove that $T_1 > 1/4K_3M$. If $T_1 = +\infty$ there is nothing to prove. If $T_1 < +\infty$ we deduce by continuity and by the definition of T_1 that

$$\|u(T_1)\|_{H^3} = M \qquad \text{and} \qquad \|u(t)\|_{H^3} \leqslant M \qquad \forall t \in [0, T_1]. \tag{50}$$

For the rest of this section we will assume that $t \in [0, T_1]$. We apply (48) with $t = T_1$ and deduce from (50) that

$$M^2 = \|u(T_1)\|_{H^3}^2 \leqslant K_3\|u_0\|_{H^3}^2 + K_3\left(\int_0^\infty \|F\|_{L^2}\right)^2 + K_3M\int_0^\infty \|\operatorname{curl} F\|_{L^2} + K_3T_1M^3.$$

The definition (49) of M implies that each of the first three terms on the right-hand side of the above relation is bounded by $M^2/4$. Thus

$$M^2 = \|u(T_1)\|_{H^3}^2 \leqslant \tfrac{3}{4}M^2 + K_3T_1M^3,$$

that is,

$$T_1 > \frac{1}{4K_3M}.$$

We conclude that the following local *a priori* estimates hold

$$\|u\|_{L^\infty(0,1/4K_3M;H^3)} \leqslant M,$$

where M is defined in (49). This proves the third part of theorem 1.

3.3. Passing to the limit

We now show how to use the previous estimates to get the existence of the solution. We will use the Galerkin method with a special basis, following the lines of the proof of [9, 10]. Set

$$X_3 = \{u \in H^3(U)^d;\ \operatorname{div} u = 0, u \text{ verifies the Navier-slip boundary conditions on } \partial U\},$$
$$X_1 = \text{closure of } X_3 \text{ for the } H^1 \text{ norm}$$

endowed with the following scalar products

$$\langle u, \phi\rangle_{X_1} = \langle u, \phi\rangle_{L^2} + 2\alpha\langle D(u), D(\phi)\rangle_{L^2}$$
$$\langle u, \phi\rangle_{X_3} = \langle \operatorname{curl}(u - \alpha\Delta u), \operatorname{curl}(\phi - \alpha\Delta\phi)\rangle_{L^2} + \langle u, \phi\rangle_{X_1}.$$

The Korn lemma 1 insures that the norm of X_1 is equivalent to the standard H^1 norm while proposition 6 implies that the norm of X_3 is equivalent to the H^3 norm. Since the embedding

$X_3 \hookrightarrow X_1$ is compact, from classical results it follows that there exists a set of eigenfunctions $\{e_i\}_{i\geqslant 1}$ of the problem

$$e_i \in X_3 \quad \text{and} \quad \langle u, e_i\rangle_{X_3} = \lambda_i \langle u, e_i\rangle_{X_1} \qquad \forall u \in X_3, \tag{51}$$

where $\lambda_i > 0$ and $\{e_i\}$ is an orthonormal basis of X_1. We immediately infer from the definition of the X_3 scalar product that

$$\langle \operatorname{curl}(u - \alpha\Delta u), \operatorname{curl}(e_i - \alpha\Delta e_i)\rangle_{L^2} = \mu_i \langle u, e_i\rangle_{X_1} \qquad \forall u \in X_3, \tag{52}$$

where $\mu_i = \lambda_i - 1 \geqslant 0$.

Before going any further, we need to know that the eigenfunctions e_i are slightly more regular than H^3. Actually, it is clear that problems of type (51) are elliptic and C^∞ regularity for the solutions e_i follows. On the other hand, we only need H^4 regularity and the argument for H^4 is not complicated. Therefore, we prefer to include it here in order to keep the paper as self-contained as possible.

We proceed as in [9, 10]. Let $f \in H^1(U)^3$. We solve the modified Stokes problem (18) to find a solution $u \in X_3$, $p \in H^1(U)$. Since $\operatorname{curl}\nabla p = 0$ we first obtain that $\operatorname{curl}(u - \alpha\Delta u) = \operatorname{curl} f$. Next, we integrate by parts and get

$$\langle u, e_i\rangle_{X_1} = \langle e_i, u - \alpha\Delta u\rangle_{L^2} = \langle e_i, f + \nabla p\rangle_{L^2} = \langle e_i, f\rangle_{L^2} + \langle e_i, \nabla p\rangle_{L^2}.$$

Since e_i is divergence free and tangent to the boundary we have that $\langle e_i, \nabla p\rangle_{L^2} = 0$ so $\langle u, e_i\rangle_{X_1} = \langle e_i, f\rangle_{L^2}$. It now follows from (52) that

$$\langle \operatorname{curl}(e_i - \alpha\Delta e_i), \operatorname{curl} f\rangle_{L^2} = \mu_i \langle e_i, f\rangle_{L^2} \qquad \forall f \in H^1(U)^d. \tag{53}$$

We now choose $f \in H_0^1(U)^d$ and integrate by parts in the first term to obtain

$$-\langle \Delta(e_i - \alpha\Delta e_i), f\rangle_{H^{-1},H_0^1} = \mu_i \langle e_i, f\rangle_{L^2} \qquad \forall f \in H_0^1(U)^d,$$

that is,

$$\Delta(e_i - \alpha\Delta e_i) = -\mu_i e_i. \tag{54}$$

Let us first consider the two-dimensional case. We go back to (53) and integrate by parts to obtain

$$\langle \operatorname{curl}(e_i - \alpha\Delta e_i), n \times f\rangle_{H^{-1/2}(\partial U),H^{(1/2)}(\partial U)} = 0$$

and, since $f|_{\partial U}$ can be chosen arbitrary in $H^{1/2}(\partial U)$, we infer that

$$\operatorname{curl}(e_i - \alpha\Delta e_i) = 0 \qquad \text{on } \partial U \quad (\text{in the sense of } H^{-1/2}(\partial U)). \tag{55}$$

Applying the curl operator to (54) we also get

$$\Delta\operatorname{curl}(e_i - \alpha\Delta e_i) = -\mu_i \operatorname{curl} e_i \in H^2(U). \tag{56}$$

Relations (55), (56) and the regularity results for the Dirichlet problem for the Laplacian imply that $\operatorname{curl}(e_i - \alpha\Delta e_i) \in H^4(U) \subset H^1(U)$. Applying again the same regularity result and proposition 1 we first get $\operatorname{curl} e_i|_{\partial U} \in H^{5/2}(\partial U)$ and then $\operatorname{curl} e_i \in H^3(U)$. From proposition 5 we finally deduce that $e_i \in H^4(U)$.

We now turn to the three-dimensional case. The boundary terms appearing after integrating by parts in (53) are in this case $\langle n \times \operatorname{curl}(e_i - \alpha\Delta e_i), f\rangle_{H^{-1/2}(\partial U),H^{1/2}(\partial U)}$ so that

$$n \times \operatorname{curl}(e_i - \alpha\Delta e_i) = 0 \qquad \text{on } \partial U \quad (\text{in the sense of } H^{-1/2}(\partial U)). \tag{57}$$

The vector field $\operatorname{curl}(e_i - \alpha\Delta e_i)$ verifies the boundary conditions (57), is divergence free, and its curl belongs to $H^1(U)$:

$$\operatorname{curl}\operatorname{curl}(e_i - \alpha\Delta e_i) = -\Delta(e_i - \alpha\Delta e_i) = \mu_i e_i \in H^1(U).$$

We can therefore deduce from the regularity results of [2] that $\operatorname{curl}(e_i - \alpha\Delta e_i) \in H^1(U)$. Next, proposition 2 implies that $n \cdot (e_i - \alpha\Delta e_i)|_{\partial U} \in H^{3/2}(\partial U)$. Therefore, from proposition 5 it follows that $e_i - \alpha\Delta e_i \in H^2(U)$ and once again proposition 3 implies $e_i \in H^4(U)$ in the three-dimensional case as well.

We now continue with the Galerkin method. Let

$$u^\ell(t,x) = \sum_{i=1}^{\ell} \alpha_i(t) e_i(x) \qquad \text{and} \qquad v^\ell(t,x) = u^\ell - \alpha\Delta u^\ell$$

be the solutions of the following system of ODEs for $\alpha_1, \ldots, \alpha_n$:

$$\langle \partial_t v^\ell, e_i \rangle_{L^2} + \left\langle -\nu\Delta u^\ell + u^\ell \cdot \nabla v^\ell + \sum_j v_j^\ell \nabla u_j^\ell - F, e_i \right\rangle_{L^2} = 0 \qquad \forall i \in \{1, \ldots, \ell\}. \tag{58}$$

Multiplying this equation by α_i and summing on i gives

$$\langle \partial_t v^\ell, u^\ell \rangle_{L^2} + \left\langle -\nu\Delta u^\ell + u^\ell \cdot \nabla v^\ell + \sum_j v_j^\ell \nabla u_j^\ell - F, u^\ell \right\rangle_{L^2} = 0,$$

which is equivalent to the H^1 *a priori* estimates (27). We next show that the L^2 estimates for $\operatorname{curl} v^\ell$ follow by multiplying (58) by $\alpha_i\mu_i$ and summing on i. Indeed, we have from relation (52) that

$$\begin{aligned} \sum_i \alpha_i\mu_i \langle \partial_t v^\ell, e_i \rangle_{L^2} &= \sum_i \alpha_i\mu_i \left\langle \partial_t u^\ell, e_i \right\rangle_{X_1} \\ &= \sum_i \alpha_i \left\langle \partial_t \operatorname{curl}(u^\ell - \Delta u^\ell), \operatorname{curl}(e_i - \alpha\Delta e_i) \right\rangle_{L^2} \\ &= \left\langle \partial_t \operatorname{curl}(u^\ell - \Delta u^\ell), \operatorname{curl}(u^\ell - \alpha\Delta u^\ell) \right\rangle_{L^2}. \end{aligned} \tag{59}$$

On the other hand, if we set

$$g^\ell = -\nu\Delta u^\ell + u^\ell \cdot \nabla v^\ell + \sum_j v_j^\ell \nabla u_j^\ell - F$$

then $g^\ell \in H^1(U)$ since $e_i \in H^4(U)$ and by proposition 3 we can construct $G^\ell \in X_3$ the solution to the modified Stokes problem

$$G^\ell - \alpha\Delta G^\ell = g^\ell + \nabla p^\ell.$$

Since $\langle \nabla p^\ell, e_i \rangle_{L^2} = 0$ we infer from relation (52) that

$$\begin{aligned} \mu_i \langle g^\ell, e_i \rangle_{L^2} &= \mu_i \left\langle G^\ell - \alpha\Delta G^\ell, e_i \right\rangle_{L^2} \\ &= \mu_i \left\langle G^\ell, e_i \right\rangle_{X_1} \\ &= \left\langle \operatorname{curl}(G^\ell - \alpha\Delta G^\ell), \operatorname{curl}(e_i - \alpha\Delta e_i) \right\rangle_{L^2} \\ &= \left\langle \operatorname{curl} g^\ell, \operatorname{curl}(e_i - \alpha\Delta e_i) \right\rangle_{L^2}. \end{aligned}$$

Consequently,

$$\sum_i \alpha_i\mu_i \langle g^\ell, e_i \rangle_{L^2} = \langle \operatorname{curl} g^\ell, \operatorname{curl}(u^\ell - \alpha\Delta u^\ell) \rangle_{L^2}$$

and we can deduce from (59) that

$$\langle \partial_t \operatorname{curl} v^\ell, \operatorname{curl} v^\ell \rangle_{L^2} + \langle \operatorname{curl} g^\ell, \operatorname{curl} v^\ell \rangle_{L^2} = 0,$$

which corresponds to relation (34).

The global and local estimates of the previous sections therefore hold for u^ℓ and the announced existence result now follows in a standard manner (see [9]).

4. Proof of the uniqueness theorem

This section presents the proof of the uniqueness part of theorem 1.

4.1. The uniqueness argument.

Let u and $\tilde{u}$ be two solutions with the same initial data and belonging to $L^\infty(0, T; H^3(U))$. Set $w = \tilde{u} - u$. Subtracting the equations for u and $\tilde{u}$ yields:

$$\partial_t(w - \alpha\Delta w) - \nu\Delta w + \tilde{u}\cdot\nabla\tilde{v} - u\cdot\nabla v + \sum_j \tilde{v}_j\nabla\tilde{u}_j - v_j\nabla u_j = \nabla(\tilde{p} - p). \tag{60}$$

We now multiply by w and integrate in space and time. According to the calculations of the H^1 *a priori* estimates and taking into account that $w(0) = 0$, we obtain that

$$\begin{aligned}\frac{1}{2}\|w(t)\|_{L^2}^2 + \alpha\|D(w(t))\|_{L^2}^2 + 2\nu\int_0^t \|D(w)\|_{L^2}^2 = -\int_0^t \Bigg(&\underbrace{\int_U w\cdot\nabla v\cdot w}_{I_1} \\ &+ \underbrace{\int_U u\cdot\nabla(\tilde{v} - v)\cdot w}_{I_2} + \int_U w\cdot\nabla(\tilde{v} - v)\cdot w + \underbrace{\sum_j\int_U v_j\nabla w_j\cdot w}_{I_3} \\ &+ \underbrace{\int_U(\tilde{v}_j - v_j)\nabla u_j\cdot w}_{I_4} + \int_U(\tilde{v}_j - v_j)\nabla w_j\cdot w\Bigg).\end{aligned} \tag{61}$$

The same computations as for the H^1 *a priori* estimates show that:

$$\int_U (w\cdot\nabla(\tilde{v} - v)\cdot w + (\tilde{v}_j - v_j)\nabla w_j\cdot w) = 0.$$

We shall prove in the appendix the following assertion.

Assertion 1. Each of the integrals I_1, I_2, I_3 and I_4 on the right-hand side of (61) can be written as the sum of terms of one of the following two forms:

form 1 $\quad \displaystyle\int_U \mathcal{D}(w)\mathcal{D}(w)\mathcal{D}(u)g(n)$

or

form 2 $\quad \displaystyle\int_U w_i\mathcal{D}(w)\mathcal{D}^2(u)g(n),$

where $g(n)$ denotes some polynomial in n and its derivatives. For a vector field f the notation $\mathcal{D}^m(f)$ denotes a derivative of order $\leqslant m$ of some component of f.

Let us suppose for the moment that this is proved. By the Hölder inequality and the Sobolev embedding $H^2 \hookrightarrow L^\infty$ which hold in both two- and three-dimensional cases, we deduce that a term of *form 1* can be bounded as follows:

$$\begin{aligned}\left|\int_U \mathcal{D}(w)\mathcal{D}(w)\mathcal{D}(u)g(n)\right| &\leqslant C\|\mathcal{D}(w)\|_{L^2}\|\mathcal{D}(w)\|_{L^2}\|\mathcal{D}(u)\|_{L^\infty} \\ &\leqslant C\|w\|_{H^1}^2\|\mathcal{D}(u)\|_{H^2} \\ &\leqslant C\|w\|_{H^1}^2\|u\|_{H^3}.\end{aligned}$$

To estimate a term of the type in *form 2*, we will use the Sobolev embedding

$$H^1(U) \hookrightarrow L^p(U) \qquad \text{for all } p \in [2, 6]$$

again true in both the two- and three-dimensional cases. We obtain:

$$\begin{aligned} \left| \int_U w_i \mathcal{D}(w) \mathcal{D}^2(u) g(n) \right| &\leqslant C \|w_i\|_{L^6} \|\mathcal{D}(w)\|_{L^2} \|\mathcal{D}^2(u)\|_{L^3} \\ &\leqslant C \|w_i\|_{H^1} \|w\|_{H^1} \|\mathcal{D}^2(u)\|_{H^1} \\ &\leqslant C \|w\|_{H^1}^2 \|u\|_{H^3}. \end{aligned}$$

Inserting these bounds in (61), neglecting the viscosity term, and using the Korn lemma 1 yields

$$\|w(t)\|_{H^1}^2 \leqslant C \int_0^t \|w\|_{H^1}^2 \|u\|_{H^3}.$$

Gronwall's lemma now implies that $w = 0$ and the uniqueness result follows.

5. Liapunov stability in the two-dimensional inviscid case

In this section we shall determine Liapunov stability conditions for stationary solutions of the two-dimensional averaged Euler equations

$$\dot{q} = \{\psi, q\} := (\partial_1 \psi)(\partial_2 q) - (\partial_2 \psi)(\partial_1 q), \tag{62}$$

in a domain $U \subset \mathbb{R}^2$, where ψ is the stream function for the velocity field u, that is, $u = (\partial_2 \psi, -\partial_1 \psi)$ and $q := \omega_v = -(1 - \alpha\Delta)\Delta\psi$ is the potential vorticity. As usual, $\omega_u = -\Delta\psi$ denotes the vorticity of the velocity field u. Equation (62) is directly obtained from (1) by setting $\nu = 0$, working in two dimensions, and replacing u and v by their expressions given above in terms of the stream function and the potential vorticity.

The method of proof for the stability estimates is identical to that in Arnold [3, 4]. To simplify the exposition we shall assume that the domain U is simply connected. The general case can be dealt with as in Holm *et al* [27] by keeping track of the circulations on the various components of the boundary.

Before starting work on the stability we shall make several preliminary remarks that will be used in the following.

5.1. An elliptic boundary value problem

We are now concerned with the existence of functions ψ_μ, solutions of the following eigenvalue problem:

$$-\Delta(\psi_\mu - \alpha\Delta\psi_\mu) = \mu\psi_\mu. \tag{63}$$

The boundary conditions are the following:

$$\psi_\mu = 4n_1 n_2 \partial_1 \partial_2 \psi_\mu + \left(\partial_2^2 \psi_\mu - \partial_1^2 \psi_\mu\right)\left(n_2^2 - n_1^2\right) = 0 \qquad \text{on } \partial U. \tag{64}$$

We will show the following proposition.

Proposition 7. *There exists a increasing sequence of eigenvalues* $0 < k_0^2 \leqslant k_1^2 \leqslant k_2^2 \leqslant \cdots$ *of problem (63) and (64). Corresponding to each eigenvalue* k_i^2 *there is an eigenfunction* ϕ_i *such that the set* $\{\phi_0, \phi_1, \ldots\}$ *form an orthogonal basis of* L^2*. Moreover, the first eigenvalue* k_0^2 *is simple.*

Proof. We could deduce this proposition directly from the special basis used in the Galerkin method, section 3.3. However, we prefer to prove this proposition directly as we will use later some of the properties that follow. We first prove that, for given $f \in L^2(U)$, the equation

$$\begin{aligned} &-\Delta(\psi - \alpha\Delta\psi) = f, \qquad \psi \in H^4(U), \\ &\psi = 4n_1n_2\partial_1\partial_2\psi + \left(\partial_2^2\psi - \partial_1^2\psi\right)\left(n_2^2 - n_1^2\right) = 0 \qquad \text{on } \partial U \end{aligned} \tag{65}$$

is equivalent to the variational problem:

$$\begin{aligned} &\psi \in H^2(U) \cap H_0^1(U) \\ &a(\psi, \varphi) = \langle f, \varphi\rangle, \qquad \forall \varphi \in H^2(U) \cap H_0^1(U), \end{aligned} \tag{66}$$

where

$$a(\psi, \varphi) = \langle u_\psi, u_\varphi\rangle_{L^2} + 2\alpha\langle D(u_\psi), D(u_\varphi)\rangle_{L^2}$$

and

$$u_\psi = (\partial_2\psi, -\partial_1\psi), \qquad u_\varphi = (\partial_2\varphi, -\partial_1\varphi).$$

We will also define $\omega_\psi = -\Delta\psi = \operatorname{curl} u_\psi$. We first prove that for smooth functions φ and ψ (no boundary conditions assumed yet) the following identity holds true:

$$a(\psi, \varphi) = \langle -\Delta(\psi - \alpha\Delta\psi), \varphi\rangle_{L^2} + \int_{\partial U} \frac{\partial\psi}{\partial n}\varphi - \alpha\int_{\partial U} \Delta\frac{\partial\psi}{\partial n}\,\varphi + 2\alpha\int_{\partial U} (D(u_\psi)\cdot n)\cdot u_\varphi. \tag{67}$$

One has that

$$\langle -\Delta(\psi - \alpha\Delta\psi, \varphi\rangle_{L^2} = -\int_U \Delta\psi\,\varphi + \alpha\int_U \Delta\Delta\psi\,\varphi.$$

An immediate integration by parts implies that

$$-\int_U \Delta\psi\,\varphi = \int_U \nabla\psi\cdot\nabla\varphi - \int_{\partial U}\frac{\partial\psi}{\partial n}\,\varphi = \int_U u_\psi\cdot u_\varphi - \int_{\partial U}\frac{\partial\psi}{\partial n}\,\varphi.$$

Next,

$$\int_U \Delta\Delta\psi\,\varphi = -\int_U \nabla\,\Delta\psi\cdot\nabla\varphi + \int_{\partial U}\Delta\,\frac{\partial\psi}{\partial n}\,\varphi = -\int_U \Delta u_\psi\cdot u_\varphi + \int_{\partial U}\Delta\,\frac{\partial\psi}{\partial n}\,\varphi.$$

We have proved in the course of the proof of lemma 3 an identity that implies here that

$$-\int_U \Delta u_\psi\cdot u_\varphi = 2\int_U D(u_\psi)\cdot D(u_\varphi) - 2\int_{\partial U}(D(u_\psi)\cdot n)\cdot u_\varphi,$$

where we have used that $\operatorname{div} u_\varphi = 0$. We infer that

$$\begin{aligned} \langle -\Delta(\psi - \alpha\Delta\psi), \varphi\rangle_{L^2} &= \langle u_\psi, u_\varphi\rangle_{L^2} + 2\alpha\langle D(u_\psi), D(u_\varphi)\rangle_{L^2} \\ &\quad - \int_{\partial U}\frac{\partial\psi}{\partial n}\,\varphi + \alpha\int_{\partial U}\Delta\frac{\partial\psi}{\partial n}\,\varphi - 2\alpha\int_{\partial U}(D(u_\psi)\cdot n)\cdot u_\varphi, \end{aligned}$$

which is the desired relation.

We can now proceed with the proof of the equivalence between (65) and (66). Let ψ be a solution of (65). Then it is a easy computation to show that u_ψ verifies the Navier-slip boundary conditions. For $\varphi \in H^2(U) \cap H_0^1(U)$ we have that u_φ is tangent to the boundary, so we deduce from (67) that

$$a(\psi, \varphi) = \langle -\Delta(\psi - \alpha\Delta\psi, \varphi\rangle = \langle f, \varphi\rangle,$$

so relation (66) follows. Let now ψ be a solution of (66) and consider first an arbitrary element φ of $H_0^2(U)$. Then we obtain from (67) that

$$\langle f, \varphi\rangle = a(\psi, \varphi) = \langle -\Delta(\psi - \alpha\Delta\psi), \varphi\rangle_{H^{-2}(U), H_0^2(U)} \qquad \forall\varphi \in H_0^2(U). \tag{68}$$

Note that although (67) is proved for smooth functions ψ, φ the relation above is shown in a standard manner by passing to the limit on a sequence of smooth approximations of ψ and φ. It follows from (68) that

$$-\Delta(\psi - \alpha\Delta\psi) = f \qquad \text{in } U. \tag{69}$$

Next, we note that for $\varphi \in H^2(U) \cap H_0^1(U)$ we have that $u_\varphi \in H^1(U)^2$, $\operatorname{div} u_\varphi = 0$ and u_φ is tangent to the boundary. And vice versa, for a given H^1 divergence free vector field u tangent to the boundary, since U is simply connected we can construct $\tilde{\varphi} \in H^2(U)$ such that $u = (\partial_2\tilde{\varphi}, -\partial_1\tilde{\varphi})$. As $u \cdot n = 0$ we obtain that $\partial\tilde{\varphi}/\partial\tau = 0$ on ∂U, i.e. $\tilde{\varphi} = K = \text{const}$ on ∂U. Then $\varphi = \tilde{\varphi} - K$ belongs to $H^2 \cap H_0^1$ and $u = u_\varphi$. Therefore, there is an isomorphism from $H^2(U) \cap H_0^1(U)$ to the space of H^1 divergence free vector fields tangent to the boundary given by $\varphi \leftrightarrow u_\varphi$.

We conclude that the variational formulation (66) is equivalent to the following variational formulation: for all H^1 vector fields v tangent to the boundary:

$$\langle u_\psi, v\rangle_{L^2} + 2\alpha\langle D(u_\psi), D(v)\rangle_{L^2} = \langle f, \varphi\rangle_{L^2} = \langle F, v\rangle_{L^2},$$

where we used that U is simply connected to construct $F \in H^1(U)^2$ and $\varphi \in H^2(U) \cap H_0^1(U)$ such that $f = \operatorname{curl} F$ and $v = u_\varphi$.

This formulation is similar to relation (20) so by theorem 3 we deduce that $u_\psi \in H^3(U)$ which implies that $\psi \in H^4(U)$. It remains to prove the boundary conditions in (65). It follows from (66), (67) and (69) that

$$\int_{\partial U} (Du_\psi \cdot n) \cdot u_\varphi = 0$$

for all H^1 divergence free vector fields v tangent to the boundary. We readily deduce that

$$(D(u_\psi) \cdot n)\, n^\perp = 0,$$

which implies after a simple calculation the boundary conditions in (65).

It is a very simple matter to solve (66). The bilinear form $a(\varphi, \varphi)$ is coercive on $H^2(U) \cap H_0^1(U)$. Indeed from lemma 1 we obtain that $a(\varphi, \varphi) \sim \|u_\varphi\|_{H^1(U)} \sim \|\nabla\varphi\|_{H^1(U)}$ and $\|\varphi\|_{L^2(U)}$ is also controlled by $a(\varphi, \varphi)$ since we have the Poincaré inequality $\|\varphi\|_{L^2(U)} \leqslant C\|\nabla\varphi\|_{L^2(U)} \leqslant Ca(\varphi, \varphi)$. We conclude that $a(\varphi, \varphi)$ is equivalent to the H^2 norm of φ and this completes the proof of the well-posedness of problem (66).

In order to solve the eigenvalue problem (63), (64), we denote by T the operator

$$T : L^2(U) \longrightarrow H^2(U) \cap H_0^1(U) \qquad f \longmapsto T(f) = \psi,$$

where ψ is the solution of (66). Then if S denotes the embedding operator $S : H^2(U) \cap H_0^1(U) \longrightarrow L^2(U)$, the operator S is compact, so $S \circ T$ is compact and self adjoint:

$$\langle S \circ T\, f, \tilde{f}\rangle_{L^2} = \langle T\, f, \tilde{f}\rangle_{L^2} = \langle \tilde{f}, T\, f\rangle_{L^2} = a(T\,\tilde{f}, T\, f)$$

and similarly

$$\langle S \circ T\,\tilde{f}, f\rangle_{L^2} = a(T\tilde{f}, T\, f) = \langle \tilde{f}, S \circ Tf\rangle_{L^2}.$$

By general functional theory (see for instance [37]) we deduce that $S \circ T$ possesses a decreasing sequence of eigenvalues $\rho_0 \geqslant \rho_1 \geqslant \cdots$ and a corresponding sequence of eigenfunctions $f_0, f_1, \ldots$ which form an orthogonal basis of $L^2(U)$ and $H^2(U) \cap H_0^1(U)$

$$T\, f_i = \rho_i\, f_i.$$

We also know that the first eigenvalue is simple. We now set

$$\phi_i = T\, f_i \qquad \text{and} \qquad k_i^2 = \frac{1}{\rho_i}$$

and deduce that

$$a(\phi_i, \varphi) = \langle f_i, \varphi\rangle_{L^2} = k_i^2\langle T f_i, \varphi\rangle_{L^2} = k_i^2\langle \phi_i, \varphi\rangle_{L^2},$$

which shows that k_i^2, respectively ϕ_i, are eigenvalues, respectively eigenfunctions, for the problem (63), (64). It is also clear that for each eigenvalue, respectively eigenfunction, of problem (63), (64) we can construct a corresponding eigenvalue, respectively eigenvector, of the operator $S \circ T$ and proposition 7 follows. □

5.2. The energy-Casimir set-up

Equations (62) are Hamiltonian relative to the Lie–Poisson bracket on q as explained in [25,26] relative to the Hamiltonian given by the energy

$$H(q) = \frac{1}{2}\int_U q\psi \,\mathrm{d}x\,\mathrm{d}y. \tag{70}$$

Unlike the case for the Euler equations, the scalar vorticity ω_u is not advected by the flow, but the potential vorticity $q = \omega_v$ is. This immediately implies that

$$C(q) = \int_U \Phi(q)\,\mathrm{d}x\,\mathrm{d}y \tag{71}$$

is a Casimir function, so, in particular, is constant on the flow of (62). This integral plays the same role for the averaged Euler equations as the entrophy played in the inviscid two-dimensional Euler equations.

Stationary solutions q_e of (62) are characterized by the condition $\nabla q_e || \nabla\psi_e$. In particular, all eigenfunctions of the operator $-(1-\alpha\Delta)\Delta$ are stationary solutions of (62).

We proceed now with the energy-Casimir method to determine classes of Liapunov stable stationary solutions q_e. We search among the stationary solutions for those which are critical points of the functional $H + C$, that is,

$$0 = D(H+C)(q_e)\cdot\delta q = \int_U (\psi_e + \Phi'(q_e))\delta q\,\mathrm{d}x\,\mathrm{d}y,$$

which is equivalent to

$$\psi_e = -\Phi'(q_e), \tag{72}$$

a condition consistent with $\nabla q_e||\nabla\psi_e$. For such solutions we shall search for Liapunov stability conditions. To do this, remark that along any solution $q(t)$ of (62), the function $H(q(t)) + C(q(t))$ is constant. Therefore, if $\delta(q(t)$ denotes a perturbation of the stationary solution q_e, the function $(H+C)(q_e + \delta q(t))$ is constant in time. We can thus conclude that the function

$$\begin{aligned}(H+C)&(q_e+\delta q(t)) - (H+C)(q_e)\\ &= (H+C)(q_e+\delta q(t)) - (H+C)(q_e) - D(H+C)(q_e)\cdot\delta q(t)\end{aligned}$$

is also constant in time. A simple computation shows that this equals

$$\frac{1}{2}\int_U \delta q(t)\delta\psi(t) + \int_U (\Phi(q_e+\delta q(t) - \Phi(q_e) - \Phi'(q_e)\delta q(t))\,\mathrm{d}x\,\mathrm{d}y. \tag{73}$$

5.3. Proof of the first part of theorem 2

Assume now that there are constants $a, A > 0$ such that $a < \Phi''(s) < A$ for all s. Such real valued functions of a real variable can easily be found and this condition is independent of (72). By Taylor's theorem we have then

$$\tfrac{1}{2}a(\delta q(t))^2 \leqslant \Phi(q_e+\delta q(t)) - \Phi(q_e) - \Phi'(q_e)\delta q(t) \leqslant \tfrac{1}{2}A(\delta q(t))^2,$$

so that denoting

$$\|q\|^2 := \frac{1}{2}\left(\int_U q\psi \,\mathrm{d}x\,\mathrm{d}y + \|q\|^2_{L^2}\right), \tag{74}$$

we conclude from (73)

$$\begin{aligned}
\min(1,a)\|\delta q(t)\|^2 &\leqslant \frac{1}{2}\int_U \delta q(t)\delta\psi(t)\,\mathrm{d}x\,\mathrm{d}y + \frac{a}{2}\int_U (\delta q(t))^2\mathrm{d}x\,\mathrm{d}y \\
&\leqslant \frac{1}{2}\int_U \delta q(t)\delta\psi(t)\,\mathrm{d}x\,\mathrm{d}y + \int_U (\Phi(q_\mathrm{e}+\delta q(t)) - \Phi(q_\mathrm{e}) - \Phi'(q_\mathrm{e})\delta q(t))\,\mathrm{d}x\,\mathrm{d}y \\
&= (H+C)(q_\mathrm{e}+\delta q(t)) - (H+C)(q_\mathrm{e}) - D(H+C)(q_\mathrm{e})\cdot\delta q(t) \\
&= \frac{1}{2}\int_U \delta q(0)\delta\psi(0)\,\mathrm{d}x\,\mathrm{d}y + \int_U (\Phi(q_\mathrm{e}+\delta q(0) - \Phi(q_\mathrm{e}) - \Phi'(q_\mathrm{e})\delta q(0))\,\mathrm{d}x\,\mathrm{d}y \\
&\leqslant \frac{1}{2}\int_U \delta q(0)\delta\psi(0)\,\mathrm{d}x\,\mathrm{d}y + \frac{A}{2}\int_U (\delta q(0))^2\mathrm{d}x\,\mathrm{d}y \\
&\leqslant \max(1,A)\|\delta q(0)\|^2,
\end{aligned}$$

which proves the Liapunov stability of such stationary solutions in the norm (74). This norm is, however, equivalent to the H^3 norm of the velocity field u.

To see what these conditions mean in terms of the stationary solution itself, we take the gradient of (72) to get $\nabla\psi_\mathrm{e} = -\Phi''(q_\mathrm{e})\nabla q_\mathrm{e}$, that is, we require boundedness of the proportionality factor between the gradients of ψ_e and q_e:

$$-A \leqslant \frac{\nabla\psi_\mathrm{e}}{\nabla q_\mathrm{e}} \leqslant -a < 0. \tag{75}$$

This proves the first part of theorem 2.

5.4. Proof of the second part of theorem 2

Next we analyse the case $\Phi''(s) < 0$. Recall that the stream function ψ is the unique solution of the elliptic problem on the simply connected domain U with smooth boundary ∂U

$$-(1-\alpha\Delta)\Delta\psi = q \qquad \text{on } U \tag{76}$$

with the boundary conditions

$$\psi = 4n_1n_2\partial_1\partial_2\psi + \left(n_2^2 - n_1^2\right)\left(\partial_2^2\psi - \partial_1^2\psi\right) = 0 \qquad \text{on } \partial U. \tag{77}$$

As was shown in proposition 7, this is a well-posed elliptic problem with unique solution $\psi \in H^4(U)$ for $q \in L^2(U)$. Define the space of functions

$$\mathcal{F} := \{\psi \in H^4(U) \mid \psi \text{ satisfies (76) and (77) with } q \in L^2(U)\}.$$

The following Poincaré type inequality will be useful later on.

Lemma 4. *Let $k^2_{\min}$ be the minimal strictly positive eigenvalue of $-(1-\alpha\Delta)\Delta$ on $\mathcal{F}$. Then*

$$\int_U q[(1-\alpha\Delta)\Delta]^{-1}q \geqslant -k_{\min}^{-2}\|q\|^2_{L^2}. \tag{78}$$

Proof. Let k_i^2 be the eigenvalues of $-(1-\alpha\Delta)\Delta$, $i = 0, 1, \ldots,$ with $k_0^2 = k^2_{\min}$ and let $\{\phi_i\}$ be an L^2 orthonormal basis of eigenfunctions, that is,

$$-(1-\alpha\Delta)\Delta\phi_i = k_i^2\phi_i, \qquad \text{with } \int_U \phi_i\phi_j\,\mathrm{d}x\,\mathrm{d}y = \delta_{ij}.$$

Thus $-k_i^{-2}$ are the eigenvalues of $[(1-\alpha\Delta)\Delta]^{-1}$, that is,

$$[(1-\alpha\Delta)\Delta]^{-1}\phi_i = -k_i^{-2}\phi_i, \qquad i = 0, 1, \ldots.$$

Setting $q = \sum_{i=0}^{\infty} c_i\phi_i$ we conclude

$$\begin{aligned}\int_U q[(1-\alpha\Delta)\Delta]^{-1}q\,\mathrm{d}x\,\mathrm{d}y &= \sum_{i,j=0}^{\infty} c_i c_j \int_U \phi_i[(1-\alpha\Delta)\Delta]^{-1}\phi_j\,\mathrm{d}x\,\mathrm{d}y \\ &= -\sum_{i,j=0}^{\infty} c_i c_j k_j^{-2}\int_U \phi_i\phi_j\,\mathrm{d}x\,\mathrm{d}y = -\sum_{j=0}^{\infty} k_j^{-2}c_j^2 \\ &\geqslant -k_{\min}^{-2}\sum_{j=0}^{\infty} c_j^2 = -k_{\min}^{-2}\|q\|_{L^2}^2\end{aligned}$$

since $k_j^{-2} \leqslant k_{\min}^{-2}$ for all $j = 0, 1, \ldots$. □

Assume that $0 < a \leqslant -\Phi''(s) \leqslant A < \infty$. Applying again Taylor's theorem, we get

$$\frac{a}{2}(\delta q(t))^2 \leqslant -\Phi(q_e + \delta q(t)) + \Phi(q_e) + \Phi'(q_e)\delta q(t) \leqslant \frac{1}{2}A(\delta q(t))^2,$$

so that, using (73)

$$\begin{aligned}&\frac{a}{2}\int_U (\delta q(t))^2\,\mathrm{d}x\,\mathrm{d}y - \frac{1}{2}\int_U \delta q(t)\delta\psi(t)\,\mathrm{d}x\,\mathrm{d}y \\ &\quad\leqslant \int_U (-\Phi(q_e+\delta q(t)) + \Phi(q_e) + \Phi'(q_e)\delta q(t))\,\mathrm{d}x\,\mathrm{d}y - \frac{1}{2}\int_U \delta q(t)\delta\psi(t)\,\mathrm{d}x\,\mathrm{d}y \\ &\quad= -(H+C)(q_e+\delta q(t)) + (H+C)(q_e) + D(H+C)(q_e)\cdot\delta q(t) \\ &\quad= \int_U (-\Phi(q_e+\delta q(0) + \Phi(q_e) + \Phi'(q_e)\delta q(0))\,\mathrm{d}x\,\mathrm{d}y - \frac{1}{2}\int_U \delta q(0)\delta\psi(0)\,\mathrm{d}x\,\mathrm{d}y \\ &\quad\leqslant \frac{A}{2}\int_U (\delta q(0))^2\,\mathrm{d}x\,\mathrm{d}y - \frac{1}{2}\int_U \delta q(0)\delta\psi(0)\,\mathrm{d}x\,\mathrm{d}y \\ &\quad\leqslant \frac{A}{2}\|\delta q(0)\|_{L^2}^2.\end{aligned}$$

Combining this with lemma 4 we get

$$\begin{aligned}A\|\delta q(0)\|_{L^2}^2 &\geqslant a\|\delta q(t)\|_{L^2}^2 - \int_U \delta q(t)\delta\psi(t)\,\mathrm{d}x\,\mathrm{d}y \\ &= a\|\delta q(t)\|_{L^2}^2 + \int_U \delta q(t)[(1-\alpha\Delta)\Delta]^{-1}\delta q(t)\,\mathrm{d}x\,\mathrm{d}y \\ &\geqslant a\|\delta q(t)\|_{L^2}^2 - k_{\min}^{-2}\|\delta q(t)\|_{L^2}^2 \\ &= \left(a - k_{\min}^{-2}\right)\|\delta q(t)\|_{L^2}^2.\end{aligned}$$

This inequality shows that if $a - k_{\min}^{-2} > 0$ we get Liapunov stability in the L^2 norm on the potential vorticity q.

Since the flow rate is also preserved, that is, $\delta\psi|_{\partial U} = 0$ and $\delta\psi$ also satisfies the second boundary condition in (77), the elliptic problem (76), (77) determines the H^4 norm of $\delta\psi$ in terms of the L^2-norm on δq and hence the H^3-norm of the perturbation of the velocity field is determined by the L^2 norm of δq. Thus, the Liapunov stability in this case is again relative to H^3 perturbations of the velocity field.

This proves the second part of theorem 2.

5.5. Plane parallel shear flow

As an example, let us take plane parallel shear flow. The domain U is an infinite strip bounded by the Ox axis and the line $y = Y$. The velocity field is given by $u(x, y) = (u_1(y), 0)$. Then the vorticity is the scalar $-u_1'(y)$, the potential vorticity has the expression $q_e(y) = -(1-\alpha\partial_y^2)u_1'(y)$, and the stability condition (75) becomes

$$-A \leqslant \frac{u_1(y)}{-(1-\alpha\partial_y^2)u_1''(y)} \leqslant -a < 0 \qquad \text{for all } y \in [0, Y].$$

Assuming $u_1(y)$ to be of class C^4, this condition always holds as long as the denominator $-(1-\alpha\partial_y^2)u_1''(y)$ does not vanish, that is, Liapunov stability holds if $u_1(y)$ is a C^4 function satisfying $\partial^2 u_1(y) - \alpha\partial_y^4 u_1(y) \neq 0$ for all $y \in [0, Y]$. This is the analogue of the Rayleigh–Arnold no inflection point stability criterion for the two-dimensional Euler flow in a strip.

Note that the denominator is identically zero if $u_1(y) = \alpha C_1 e^{y/\sqrt{\alpha}} + \alpha C_2 e^{-y/\sqrt{\alpha}} + C_3 y + C_4$, for arbitrary constants $C_1, C_2, C_3, C_4 \in \mathbb{R}$; thus $u_1(y)$ is not allowed to belong to this family of functions.

5.6. Ground states

In this section, we follow the method in [14] to find a Liapunov stable *family* of stationary solutions for the averaged Euler equations in two dimensions, called ground states.

As discussed previously, a family of stationary solutions is given by eigenfunctions of the operator $-(1-\alpha\Delta)\Delta$, that is,

$$q_\lambda = -(1-\alpha\Delta)\Delta\psi_\lambda = \lambda\psi_\lambda$$

with the boundary conditions (77). It can be checked directly that these potential vorticities q_λ are critical points of $C_2 - \lambda H$, where $C_2(q)$ is the Casimir functional induced by the quadratic function, that is,

$$C_2(q) = \frac{1}{2}\int_U q^2 \, dx \, dy.$$

Let $0 < \lambda_0 < \lambda_1 \leqslant \cdots$ be the eigenvalues of $-(1-\alpha\Delta)\Delta$ with the boundary conditions (77). The first eigenvalue λ_0 is strictly positive and simple (see proposition 7). Denote by $\mathcal{E}_i$ the eigenspace corresponding to the eigenvalue λ_i, by $\Pi_i : L^2(U) \to \mathcal{E}_i$ the orthogonal projection onto the space $\mathcal{E}_i$, and by $\Pi_i^\perp : L^2(U) \to \mathcal{E}_i^\perp$ the orthogonal projection onto the orthogonal complement $\mathcal{E}_i^\perp$ of $\mathcal{E}_i$.

The stationary solutions q_{λ_0} are called *ground states*, because they are the global constrained minima of C_2 on the level sets of the Hamiltonian H. To prove this, fix one such $q_{\lambda_0} = -(1-\alpha\Delta)\Delta\psi_{\lambda_0} = \lambda_0\psi_{\lambda_0}$, denote $h_{\lambda_0} := H(q_{\lambda_0})$, and consider a perturbation $\delta\psi$ satisfying the boundary conditions (77) which implies that $\psi_{\lambda_0} + \delta\psi$ also satisfies the same boundary conditions. Let $\delta q = -(1-\alpha\Delta)\Delta\delta\psi$ and assume that $H(q_{\lambda_0} + \delta q) = h_{\lambda_0}$, that is,

$$\int_U \psi_{\lambda_0}\delta q \, dx \, dy + \frac{1}{2}\int_U \delta q \delta\psi \, dx \, dy = 0. \tag{79}$$

The multiplication of this equality by λ_0 and the definition of ψ_{λ_0} gives

$$\int_U q_{\lambda_0}\delta q \, dx \, dy = -\frac{\lambda_0}{2}\int_U \delta q \delta\psi \, dx \, dy.$$

which implies

$$C_2(q_{\lambda_0} + \delta q) - C_2(q_{\lambda_0}) = \frac{1}{2}\int_U \delta q(\delta q - \lambda_0\delta\psi) \, dx \, dy \geqslant \frac{\lambda_1 - \lambda_0}{2\lambda_1} \left\| \Pi_0^\perp \delta q \right\|_{L^2}^2. \tag{80}$$

The last inequality was obtained by an eigenfunction expansion (and the condition $H(q_{\lambda_0} + \delta q) = h_{\lambda_0}$ was not used in deriving it). In particular, $C_2(q_{\lambda_0} + \delta q) \geqslant C_2(q_{\lambda_0})$ for any

$\delta q = -(1-\alpha\Delta)\Delta\delta\psi$ such that $H(q_{\lambda_0}+\delta q) = h_{\lambda_0}$. Therefore, $C_2(q_{\lambda_0})$ is the minimal value of $C_2(q)$ subject to the condition $H(q) = h_{\lambda_0}$. In addition, (80) shows that the inequality $C_2(q_{\lambda_0}+\delta q) \geqslant C_2(q_{\lambda_0})$ becomes an equality if and only if $\delta q \in \mathcal{E}_0$. Since λ_0 is simple, there exists a constant $a \in \mathbb{R}$ such that $\delta q = a\psi_{\lambda_0}$. Equality (79) implies then

$$a\int_U \psi_{\lambda_0}^2 \,\mathrm{d}x\,\mathrm{d}y + \frac{1}{2}\int_U \delta q\delta\psi \,\mathrm{d}x\,\mathrm{d}y = 0.$$

However, both integrals are positive and the first one is strictly positive, whence $a = 0$, that is, $\delta q = 0$ thereby showing that the minimum is unique.

For an arbitrary perturbation δq, we have

$$\begin{aligned}\frac{1}{2}\left\|\Pi_0^{\perp}\delta q\right\|_{L^2}^2 &\geqslant \frac{1}{2}\int_U \delta q(\delta q - \lambda_0\delta\psi)\,\mathrm{d}x\,\mathrm{d}y\\ &= (C_2-\lambda_0 H)(q_{\lambda_0}+\delta q) - (C_2-\lambda_0 H)(q_{\lambda_0})\\ &= (C_2-\lambda_0 H)(q_{\lambda_0}+\delta q) - (C_2-\lambda_0 H)(q_{\lambda_0}) - D(C_2-\lambda_0 H)(q_{\lambda_0})\cdot\delta q\\ &\geqslant \frac{\lambda_1-\lambda_0}{2\lambda_1}\left\|\Pi_0^{\perp}\delta q\right\|_{L^2}^2. \end{aligned} \tag{81}$$

As in the previous sections, (81) and the constancy in time of $(C_2-\lambda_0 H)(q_{\lambda_0}+\delta q) - (C_2-\lambda_0 H)(q_{\lambda_0})$ show that the solutions q_{λ_0} are stable in the seminorm $\left\|\Pi_0^{\perp}\delta q\right\|$ on perturbations of the potential vorticity. The degeneracy direction of this seminorm is precisely $\mathcal{E}_0$, thus proving that the ground states q_{λ_0} are Liapunov stable as a family with a drift in the direction of the first eigenfunction possible. This proves the third part of theorem 2.

Acknowledgments

We would like to thank D Cioranescu, D Iftimie, J Marsden, and G Raugel for advice and comments. TSR was partially supported by the European Commission and the Swiss Federal Government through funding for the Research Training Network 'Mechanics and Symmetry in Europe' (MASIE) as well as the Swiss National Science Foundation.

Appendix. Some technical calculations

We give in this appendix the proof of assertion 1 (section 4.1).

The integral I_3 is clearly of *form 2*. We next integrate by parts in I_1 and use that div $w = 0$ to obtain

$$I_1 = \sum_{i,j}\int_U w_i\,\partial_i v_j\,w_j = -\sum_{i,j}\int_U w_i\,v_j\partial_i w_j + \sum_{i,j}\int_{\partial U}(n\cdot w)\,(v\cdot w).$$

Since w is tangent to the boundary, the boundary integral vanishes. The other integral is of *form 2* so the conclusion also follows for the term I_1.

For I_4 we write

$$I_4 = \sum_j\int_U w_j\nabla u_j\cdot w - \alpha\sum_j\int_U \Delta w_j\nabla u_j\cdot w.$$

The first term of the right-hand side is of *form 1* (and *form 2* as well). For the second term we write

$$\Delta w_j = \sum_k \partial_k^2 w_j = \sum_k \partial_k(\partial_k w_j - \partial_j w_k) = \sum_k \partial_k\Omega_{jk}(w),$$

and integrate by parts to obtain that:

$$\sum_j \int_U \Delta w_j \, \nabla u_j \cdot w = \sum_{i,j,k} \int_U \partial_k \Omega_{jk}(w) \, \partial_i u_j \, w_i$$
$$= -\sum_{i,j,k} \int_U \Omega_{jk}(w) \, \partial_i \partial_k u_j \, w_i - \sum_{i,j,k} \int_U \Omega_{jk}(w) \, \partial_i u_j \, \partial_k w_i$$
$$+ \sum_{ijk} \int_{\partial U} n_k \, \Omega_{jk}(w) \, \partial_i u_j \, w_i .$$

The first, respectively second, term of the right-hand side is of *form 2*, respectively *form 1*. To complete the estimate of I_4 it is sufficient to estimate the last boundary integral. To do that, we first use (16) to replace $\sum_k n_k \, \Omega_{jk}(w) = (\Omega(w) \cdot n)_j$ by $H_j(w, n)$, where we have denoted by $H_j(w, n)$ the right-hand side of (16) with u replaced by w and i replaced by j, that is,

$$H_j(w, n) = \sum_l w_l \, f_{jl}(n)$$

and we next use relation (17) to return to an integral on U:

$$\sum_{i,j} \int_{\partial U} H_j(w, n) \partial_i u_j \, w_i = \sum_{i,j} \int_U H_j(w, n) \, \partial_i u_j \, w_i \, \mathrm{div}\, n + \sum_{i,j,k} \int_U n_k \, \partial_k \, [H_j(w, n) \partial_i u_j \, w_i].$$

After expanding the last term we clearly obtain that the right-hand side is a sum of terms of *form 1* or *form 2*. This completes the estimate for I_4.

The last term to bound is I_2. Integrating by parts yields

$$I_2 = \int_U u \cdot \nabla(\tilde{v} - v) \cdot w = -\int_U u \cdot \nabla w \cdot (\tilde{v} - v)$$
$$= -\int_U u \cdot \nabla w \cdot w + \alpha \int_U u \cdot \nabla w \cdot \Delta w = \alpha \sum_{i,j} \int_U u_i \, \partial_i w_j \, \Delta w_j .$$

We write as above $\Delta w_j = \sum_k \partial_k \Omega_{jk}(w)$ and integrate by parts to obtain

$$I_2 = -\alpha \underbrace{\sum_{i,j,k} \int_U \partial_k u_i \, \partial_i w_j \, \Omega_{jk}(w)}_{J_1} - \alpha \underbrace{\sum_{i,j,k} \int_U u_i \, \partial_i \partial_k w_j \, \Omega_{jk}(w)}_{J_2}$$
$$+ \alpha \underbrace{\sum_{i,j} \int_{\partial U} u_i \, \partial_i w_j \, (\Omega(w) \, n)_j}_{J_3} .$$

Now, J_1 is of *form 1*. For J_2 we write

$$J_2 = \sum_{i,j,k} \int_U u_i \, \partial_i \partial_k w_j \, (\partial_k w_j - \partial_j w_k) = \underbrace{\sum_{i,j,k} \int_U u_i \, \partial_i \partial_k w_j \, \partial_k w_j}_{J_4} - \underbrace{\sum_{i,j,k} \int_U u_i \, \partial_i \partial_k w_j \, \partial_j w_k}_{J_5} .$$

The term J_4 vanishes. As for J_5 we integrate by parts to get

$$\begin{aligned} J_5 &= \sum_{i,j,k} \int_U u_i\, \partial_i \partial_k w_j\, \partial_j w_k \\ &= -\sum_{i,j,k} \int_U u_i\, \partial_k w_j\, \partial_i \partial_j w_k + \sum_{i,j,k} \int_{\partial U} u_i\, n_i\, \partial_k w_j\, \partial_j w_k \\ &= -\sum_{i,j,k} \int_U u_i\, \partial_j w_k\, \partial_i \partial_k w_j = -J_5, \end{aligned}$$

where we have used that $\sum_i u_i\, n_i = u \cdot n = 0$ on ∂U and interchanged the indices j and k in the summation. We infer that $J_5 = 0$ so $J_2 = 0$, too. The last term to estimate is the boundary integral J_3. As previously, we return to an integral on U by using relation (17) and after replacing $\big(\Omega(w)\, n\big)_j$ by $H_j(w, n)$ we get

$$\begin{aligned} J_3 &= \sum_{i,j} \int_U u_i\, \partial_i w_j\, H_j(w,n)\, \mathrm{div}\, n + \sum_{i,j,k} \int_U n_k\, \partial_k(u_i\, \partial_i w_j\, H_j(w,n)) \\ &= \sum_{i,j} \int_U u_i\, \partial_i w_j\, H_j(w,n)\, \mathrm{div}\, n + \sum_{i,j,k} \int_U n_k\, \partial_k u_i\, \partial_i w_j\, H_j(w,n) \\ &\quad + \sum_{i,j,k} \int_U n_k\, u_i\, \partial_i w_j\, \partial_k H_j(w,n) + \sum_{i,j,k} \int_U n_k\, u_i\, \partial_i \partial_k w_j\, H_j(w,n). \end{aligned}$$

Clearly the first three terms on the right-hand side are of *form 1* or *form 2*. As for the last term, another integration by parts implies that

$$\begin{aligned} &\sum_{i,j,k} \int_U n_k\, u_i\, \partial_i \partial_k w_j\, H_j(w,n) \\ &\qquad = -\sum_{i,j,k} \int_U \partial_i n_k\, u_i\, \partial_k w_j\, H_j(w,n) - \sum_{i,j,k} \int_U n_k\, u_i\, \partial_k w_j\, \partial_i H_j(w,n), \end{aligned}$$

where the boundary terms vanish because $u \cdot n$ on ∂U. Now, the right-hand side has the first term of *form 2* and the second term of *form 1*. This completes the proof for the integral I_2.

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