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Euler–Poincaré Reduction on Principal Bundles

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Abstract. Let $\pi: P \to M$ be an arbitrary principal G-bundle. We give a full proof of the Euler–Poincaré reduction for a G-invariant Lagrangian $L: J^1P \to \mathbb{R}$ as well as the study of the second variation formula, the conservations laws, and study some of their properties.

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1. Introduction

In mechanics, the paradigm of Lagrangian reduction is the Euler–Poincaré reduction. This process can be summarized as follows. We begin with a Lie group G and a Lagrangian $L: TG \to \mathbb{R}$ invariant under the natural action of G on its tangent bundle. This naturally induces a function $l: TG/G \simeq \mathfrak{g} \to \mathbb{R}$ called the *reduced Lagrangian*. Moreover, the Euler–Lagrange equations for L for curves on G are equivalent to a new kind of equations for l for the reduced curves in the Lie algebra $\mathfrak{g}$. These equations are known as the *Euler–Poincaré* equations; see, for example, [1] and [10] for an exposition of this topic.

The first attempt to extend these ideas to field theories has been done in [3]. In this case, the analogue of the Euler–Poincaré reduction deals with a Lagrangian defined on the first jet bundle of a principal bundle $L: J^1P \to \mathbb{R}$, invariant under the natural action induced by the structure group G on J^1P. The reduced variational problem now takes place on $C := (J^1P)/G$, the bundle of connections, and has a nice geometrical interpretation in terms of connections. In [3], the authors studied reduction and reconstruction only for matrix groups as well as some examples in local coordinates. In this Letter we complete [3] with a proof for general Lie groups and we add several facts that are important for a future general theory of Lagrangian covariant reduction. Firstly, we show that the reduced problem is a variational problem with constrains in the algebra of admissible variations, which turns out to be the gauge algebra. Secondly, we prove that the Euler–Poincaré equations are nothing but the Noether conservation law in terms of the reduced space C. We also prove Noether's theorem for reduced symmetries. Finally, we write the second variation formula for the reduced problem and we study the example of harmonic mappings in detail.

Although the covariant Euler–Poincaré equations represent an unavoidable first step to a complete theory of Lagrangian reduction on fiber bundles, we would nevertheless like to emphasize its role in the framework of the variational calculus with constrains. The admissible variations of the reduced problem as well as the compatibility condition given in Theorem 2 are typical restrictions in the theory of constrains. The deduction of the Euler–Poincaré equations for the reduced Lagrangian l from the free variational principle for L can be used backwards as sometimes the free variational problem is easier to handle with than the reduced problem.

In this Letter the summation convention of repeated indices will be used. The space of smooth sections of a bundle $E \to M$ will be denoted by $C^\infty(E)$.

2. Preliminaries

Let $\pi: P \to M$ denote an arbitrary right principal bundle with structure group G. The group G acts on the tangent bundle TP by lifts, that is, $X \cdot g := T_e R_g X$, for $X \in TP$, where R_g denotes the right action of the element $g \in G$ on P. The quotient space $(TP)/G$ is a smooth vector bundle over M. The bundle of connections $p: C = C(P) \to M$ is by definition the sub-bundle of $\mathrm{Hom}(TM, (TP)/G)$ defined by those elements $\Gamma_x: T_xM \to ((TP)/G)_x$ that satisfy $\pi_* \circ \Gamma_x = \mathrm{Id}_{T_xM}$ (see, for example, [4, 5]). As is well known, C is an affine bundle modeled over the vector bundle $T^*M \otimes \mathrm{ad}P \to M$, where $\mathrm{ad}P := (P \times \mathfrak{g})/G$ is the adjoint bundle, that is, the associated bundle to $P \to M$ with respect to the adjoint representation of G on $\mathfrak{g}$; the right G-action on $P \times \mathfrak{g}$ is given by $g \cdot (z, \xi) = (z \cdot g, \mathrm{Ad}_{g^{-1}}\xi)$ for $z \in P$ and $\xi \in \mathfrak{g}$. Each global section $\sigma_\Gamma: M \to C$ corresponds bijectively to a principal connection Γ on the bundle P.

Let $J^1P \to P$ be the bundle of jets of local sections of π. The group G acts naturally on J^1P by $(j^1_x s) \cdot g = j^1_x(R_g \circ s)$, for any $j^1_x s \in J^1P$ and $g \in G$. It is well known that the quotient manifold $(J^1P)/G$ can be identified with the bundle C of connections of P. The projection $q: J^1P \to C$ is a principal G-fiber bundle isomorphic to the bundle p^*P (cf. [5]).

An automorphism of the bundle P is a diffeomorphism $\Phi: P \to P$ equivariant with respect to the right action of G; *i.e.*, such that $\Phi(u \cdot g) = \Phi(u) \cdot g$, for all $u \in P$ and $g \in G$. Automorphisms are necessarily fiber preserving maps, that is, every automorphism Φ naturally induces a diffeomorphism $\varphi: M \to M$ satisfying $\pi \circ \Phi = \varphi \circ \pi$. Denote by $\mathrm{Aut}P$ the infinite-dimensional Lie group of all automorphisms of the principal bundle P and define the *gauge group* $\mathrm{Gau}P$ to be the Lie subgroup consisting of those automorphisms that cover the identity on M.

The Lie algebra aut$P$ of the group Aut$P$ is the algebra of G-invariant vector fields on P, that is, vector fields $X \in \mathfrak{X}(P)$ such that $(R_g)_* X = X$, for all $g \in G$. Such G-invariant vector fields are also called *infinitesimal automorphisms*. The subalgebra gau$P$ of vertical G-invariant vector fields is the Lie algebra of the gauge group Gau$P$. It is clear that aut$P$ can be seen as the space of sections of the bundle $(TP)/G$ and gau$P$ as the space of sections of the bundle $(VP)/G$, where $V := \ker T\pi$ is the vertical bundle of P. Since $(VP)/G$ is bundle isomorphic to ad$P$, one can identify gau$P \cong C^\infty(\mathrm{ad}P)$.

Given a local trivialization $U \times G \subset P$ and an element $B \in \mathfrak{g}$, we define the vector field $\tilde{B}$ as the infinitesimal generator of the flow $((x, g), t) \mapsto (x, \exp(tB)g)$. If $\{B_1, \ldots, B_m\}$, $m = \dim G$, is a basis of the Lie algebra $\mathfrak{g}$ of G, the vector fields $\{\tilde{B}_1, \ldots, \tilde{B}_m\}$ form a local basis of gau$P$ as a $C^\infty(U)$-module. If in addition, U is the coordinate domain of $(x^1, \ldots, x^n)$, $n = \dim M$, the local expression of an infinitesimal automorphism $X \in$aut$P$ is given by $X = f^j \partial/\partial x^j + g^\alpha \tilde{B}_\alpha$ where $f^j, g^\alpha \in C^\infty(U)$. Moreover, the chart (x^j) induces a coordinate system (x^j, A^α_j), $1 \leqslant j \leqslant n$, $1 \leqslant \alpha \leqslant m$, on $p^{-1}(U) \subset C$ by the condition

$$\Gamma_x\big(\partial/\partial x^j\big) = \partial/\partial x^j + A^\alpha_j(\Gamma_x)\tilde{B}_\alpha, \quad \text{for all} \quad \Gamma_x\colon T_xM \to ((TP)/G)_x,$$

where the right-hand side of the previous formula is understood as an element of $((TP)/G)_x$ since it is a G-invariant vector field along $\pi^{-1}(x)$, $x \in U$.

Given an automorphism $\Phi \in \mathrm{Aut}P$, its 1-jet lift $\Phi^{(1)}\colon J^1P \to J^1P$ is defined by $\Phi^{(1)}(j^1_x s) = j^1_{\varphi(x)}(\Phi \circ s \circ \varphi^{-1})$. The automorphism Φ also defines a unique diffeomorphism $\Phi_C\colon C \to C$ such that for every section $\sigma_\Gamma\colon M \to C$ of the bundle of connections, we have $\Phi_C \circ \sigma_\Gamma = \sigma_{\Phi(\Gamma)} \circ \varphi$, where Γ is the connection on P associated to σ_Γ and $\Phi(\Gamma)$ is the image of Γ by Φ (cf. [9, p. 79]). The map $\Phi \mapsto \Phi_C$ (resp. $\Phi \mapsto \Phi^{(1)}$) from Aut$P$ to Diff C (resp. Diff J^1P) is a homomorphism of groups which induces a Lie algebra homomorphism aut$P \ni X \mapsto X_C \in \mathfrak{X}(C)$ (resp. aut$P \ni X \mapsto X^{(1)} \in \mathfrak{X}(J^1P)$).

Finally, we review some basic facts concerning the calculus of variations on fiber bundles. A *Lagrangian density* is a fiber bundle morphism $\mathcal{L}\colon J^1P \to \bigwedge^n T^*M$ (the line bundle of n forms on M). We assume that M is oriented by a volume form υ, which shall be fixed throughout the paper. Then we can write $\mathcal{L} = L\upsilon$, where $L\colon J^1P \to \mathbb{R}$ is a function called the *Lagrangian*. Let $U \subset M$ be an open set with $\overline{U}$ compact. For a (local) section $s\colon \overline{U} \subset M \to P$ of π the *action* is defined by $\mathcal{S}(s) := \int_U L(j^1s)\upsilon$.

The section s is said to be *critical* if for every vertical variation s_ε, $\varepsilon \in \mathbb{R}$, with $s_0 = s$ and $s|_{\partial U} = s_0|_{\partial U}$, we have $\delta\mathcal{S} = \mathrm{d}/\mathrm{d}\varepsilon\big|_{\varepsilon=0}\mathcal{S}(s_\varepsilon) = 0$. Vertical means that the vector field $\mathrm{d}/\mathrm{d}\varepsilon|_{\varepsilon=0}s_\varepsilon$ is π-vertical. Nonvertical variations can be also considered. As is well known, s is critical if and only if it satisfies the Euler–Lagrange equations $\mathcal{EL}(L)(s) = 0$ (cf. [6]).

It is a basic fact that $\mathrm{d}/\mathrm{d}\varepsilon|_{\varepsilon=0}\mathcal{S}(s_\varepsilon)$ does not depend on the variation s_ε itself but only on the vector field $\delta s = \mathrm{d}/\mathrm{d}\varepsilon|_{\varepsilon=0}s_\varepsilon$ along s defined by it. For that reason, it is not necessary to check that s is critical with respect to all variations s_ε. It is enough to check $\mathrm{d}/\mathrm{d}\varepsilon|_{\varepsilon=0}\mathcal{S}(s_\varepsilon) = 0$ for a suitable subclass of variations such that every vertical vector field along s can be obtained as the derivative of a variation belonging to this class.

3. Euler–Poincaré Reduction

Assume that the Lagrangian $L: J^1P \to \mathbb{R}$ is G-invariant, that is, $L(j^1_x s \cdot g) = L(j^1_x s)$, for all $g \in G$. Let $l: (J^1P)/G = C \to \mathbb{R}$ be the function defined by L on the quotient. For a (local) section $s: U \subset M \to P$, define the section $\sigma: U \to C$ of the bundle of connections by $\sigma(x) = q(j^1_x s)$, $x \in U$, where q is the projection of the principal bundle $J^1P \to (J^1P)/G = C$.

The symmetries given by the group G can be used to reduce the Euler–Lagrange equations of L. These reduced equations are called the Euler–Poincaré equations of the reduced Lagrangian l on C. Before giving the precise statement of this result we introduce some necessary notation

We fix a connection $\mathcal{H}$ on the bundle P. The section $\sigma_{\mathcal{H}}$ of $C \to M$ associated to $\mathcal{H}$ allows us to identify C with the vector bundle it is modeled over, that is, we obtain a diffeomorphism $\Upsilon^{\mathcal{H}}: C \to T^*M \otimes \mathrm{ad}P$, $\Upsilon^{\mathcal{H}}(\Gamma_x) = \Gamma_x - \sigma_{\mathcal{H}}(x)$, $\Gamma_x \in C_x$. Given another section $\sigma : M \to C$, we define $\sigma^{\mathcal{H}} := \Upsilon^{\mathcal{H}} \circ \sigma: M \to T^*M \otimes \mathrm{ad}P$.

For every $\sigma \in \Gamma(C)$, we define $\delta l/\delta\sigma: T^*M \otimes \mathrm{ad}P \to \mathbb{R}$ as the vertical derivative of l along σ, i.e., $(\delta l/\delta\sigma)(Y) := \mathrm{d}/\mathrm{d}\varepsilon|_{\varepsilon=0} l(\sigma(x) + \varepsilon Y)$, for $Y \in T^*_x M \otimes (\mathrm{ad}P)_x$. Consequently, $\delta l/\delta\sigma$ can be seen as a section of the dual bundle $TM \otimes (\mathrm{ad}P)^*$, that is, a vector field with values in the dual bundle $(\mathrm{ad}P)^*$.

Since $(\mathrm{ad}P)^*$ is an associated bundle to P with respect to the coadjoint action of G on $\mathfrak{g}^*$, the connection $\mathcal{H}$ induces a covariant differential $\nabla^{\mathcal{H}}: C^\infty(\bigwedge^r T^*M \otimes (\mathrm{ad}P)^*) \to C^\infty(\bigwedge^{r+1} T^*M \otimes (adP)^*)$, for all r. For every $\mathcal{X} \in C^\infty(TM \otimes (\mathrm{ad}P)^*)$, there exist a unique section $\mathrm{div}^{\mathcal{H}}\mathcal{X}$ of the bundle $(\mathrm{ad}P)^*$ such that

$$\pounds^{\mathcal{H}}_{\mathcal{X}}\upsilon = \upsilon \otimes \mathrm{div}^{\mathcal{H}}\mathcal{X}, \tag{1}$$

where the covariant Lie derivative is defined by $\pounds^{\mathcal{H}}_{\mathcal{X}}\upsilon = \nabla^{\mathcal{H}} i_{\mathcal{X}}(\upsilon) + i_{\mathcal{X}}\nabla^{\mathcal{H}}\upsilon = \nabla^{\mathcal{H}} i_{\mathcal{X}}(\upsilon)$, and υ is the fixed volume form on M. This operator verifies the following property (*cf.* [3]): $\mathrm{div}(\langle\mathcal{X}, \eta\rangle) = \langle\mathrm{div}^{\mathcal{H}}\mathcal{X}, \eta\rangle + \langle\mathcal{X}, \nabla^{\mathcal{H}}\eta\rangle$, for any $\eta \in C^\infty(\mathrm{ad}P)$; $\langle\,,\rangle$ is the natural pairing between dual spaces and div stands for the usual divergence operator on M.

Given $\sigma^{\mathcal{H}} \in C^\infty(T^*M \otimes \mathrm{ad}P)$, define the operator $\mathrm{ad}^*_{\sigma^{\mathcal{H}}}: TM \otimes (\mathrm{ad}P)^* \to (\mathrm{ad}P)^*$ as the pairing of T^*M and TM and the natural coadjoint operator in the $(\mathrm{ad}P)^*$ part.

THEOREM 1 (Reduction). *Let $\pi: P \to M$ be a principal G-fiber bundle over a manifold M with a volume form υ and let $L: J^1P \to \mathbb{R}$ be a G-invariant Lagrangian. Let $l: C \to \mathbb{R}$ be the mapping defined by L on the quotient. For an open set $U \subset M$ with $\bar{U}$ compact and a section $s: \bar{U} \to P$ of π, define $\sigma: U \to C$ by*

$\sigma(x) = q(j_x^1 s)$, where $q\colon J^1P \to C = (J^1P)/G$ is the canonical projection. Then, for every principal connection $\mathcal{H}$ on P, the following are equivalent:

(1) *the variational principle $\delta \int_U L(j_x^1 s)v = 0$ holds, for vertical variations δs along s with compact support,*

(2) *the local section $s\colon U \to P$ satisfies the Euler–Lagrange equations for L,*

(3) *the variational principle $\delta \int_U l(\sigma(x))v = 0$ holds, using variations of the form*

$$\delta\sigma = \nabla^{\mathcal{H}}\eta - [\sigma^{\mathcal{H}}, \eta], \tag{2}$$

where $\eta\colon U \to \mathrm{ad}P$ is any section of the adjoint bundle with compact support,

(4) *the Euler–Poincaré equations hold:*

$$\mathcal{EP}(l)(\sigma) := \mathrm{div}^{\mathcal{H}} \frac{\delta l}{\delta\sigma} + \mathrm{ad}^*_{\sigma^{\mathcal{H}}} \frac{\delta l}{\delta\sigma} = 0. \tag{3}$$

Proof. The equivalence (1) $\Leftrightarrow$ (2) is a standard result of the calculus of variations. For (1) $\Leftrightarrow$ (3), it is enough to prove that for a given infinitesimal vertical variation $\delta s = \mathrm{d}/\mathrm{d}\varepsilon|_{\varepsilon=0} s_\varepsilon$, the expression of the reduced variation $\delta\sigma = \mathrm{d}/\mathrm{d}\varepsilon|_{\varepsilon=0} q(j^1 s_\varepsilon)$ is given by (2). Given δs along s, let $X \in \mathrm{gau}P\big|_U$ be the unique gauge vector field such that $\delta s = X|_{s(M)}$. By means of the identification $\mathrm{gau}P \simeq C^\infty(\mathrm{ad}P)$, X determines a section η of the adjoint bundle. To prove that, $\delta\sigma = \nabla^{\mathcal{H}}\eta - [\sigma^{\mathcal{H}}, \eta]$, the problem being local, we work in a local trivialization $V \times G$ of the principal G-bundle P where $V \subset U$ is open in M. In this trivialization one can write $s(x) = (x, g(x))$ for a certain smooth mapping $g\colon V \to G$. This local trivialization induces an identification of the restriction $C|_V$ of the affine bundle C with the trivial vector bundle $T^*V \otimes \mathfrak{g}$. The section $\sigma = q(j^1 s)$ has therefore the expression $\sigma(x) = T_{g(x)}R_{g(x)^{-1}}(T_x g)$.

We can identify $\mathrm{ad}P|_V$ with $V \times \mathfrak{g}$ by means of the identification $\{(x, g), B\}_G \mapsto (x, Ad_g B)$, where $\{p, B\}_G$ represents the class of (p, B) in $\mathrm{ad}P = (P \times \mathfrak{g})/G$. Then $\eta \in C^\infty(adP)$ can be written as $\eta(x) = (x, \xi(x))$ for certain mapping $\xi\colon V \to \mathfrak{g}$ and the infinitesimal variation δs is thus obtained as the derivative of the variation $s_\varepsilon(x) = (x, \exp(\varepsilon\xi(x))g(x))$. Indeed for $\varepsilon = 0$ we have $s_0 = s$ and the derivative of s_ε with respect to ε is the gauge vector field X along the section s which coincides, by construction, to δs. Therefore

$$\begin{aligned}
\delta\sigma(x) &= \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\bigg|_{\varepsilon=0} \sigma_\varepsilon(x) = \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\bigg|_{\varepsilon=0} T_{\exp(\varepsilon\xi(x))g(x)} R_{g(x)^{-1}\exp(-\varepsilon\xi(x))} T_x(\exp(\varepsilon\xi)g) \\
&= \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\bigg|_{\varepsilon=0} T_{\exp(\varepsilon\xi(x))g(x)} R_{g(x)^{-1}\exp(-\varepsilon\xi(x))} (T_{\exp(\varepsilon\xi(x))} R_{g(x)} T_x(\exp(\varepsilon\xi)) + \\
&\quad + T_{g(x)} L_{\exp(\varepsilon\xi(x))} T_x g) \\
&= \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\bigg|_{\varepsilon=0} T_{\exp(\varepsilon\xi(x))} R_{\exp(-\varepsilon\xi(x))} T_x(\exp(\varepsilon\xi)) + \\
&\quad + \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\bigg|_{\varepsilon=0} \mathrm{Ad}_{\exp(\varepsilon xi(x))} T_{g(x)} R_{g(x)^{-1}} T_x g.
\end{aligned}$$

Using the formula for the exponential map at $\varepsilon\xi(x)$ (see, for example [13]), it follows that the first summand equals $T_x\xi$.

The second summand equals $[\xi(x), \sigma(x)]$. Thus we get

$$\begin{aligned}\delta\sigma(x) &= T_x\xi - [\sigma(x), \xi(x)] = T_x\xi - [\omega^{\mathcal{H}} \circ T_x s, \xi(x)] - [\sigma(x) - \omega^{\mathcal{H}} \circ T_x s, \xi(x)] \\ &= \nabla^{\mathcal{H}}\xi(x) - [\sigma^{\mathcal{H}}(x), \xi(x)],\end{aligned}$$

where $\omega^{\mathcal{H}}$ is the connection one-form of $\mathcal{H}$. This is equivalent to $\delta\sigma = \nabla^{\mathcal{H}}\eta - [\sigma^{\mathcal{H}}, \eta]$.

For the equivalence (3) $\Leftrightarrow$ (4), we have

$$\begin{aligned}\delta\int_M l(\sigma(x))\upsilon &= \int_M \left\langle \frac{\delta l}{\delta\sigma}, \delta\sigma \right\rangle \upsilon = \int_M \left\langle \frac{\delta l}{\delta\sigma}, \nabla^{\mathcal{H}}\eta - [\sigma^{\mathcal{H}}, \eta] \right\rangle \upsilon \\ &= \int_M \left(\operatorname{div}\left\langle \frac{\delta l}{\delta\sigma}, \eta \right\rangle - \left\langle \operatorname{div}^{\mathcal{H}} \frac{\delta l}{\delta\sigma}, \eta \right\rangle - \left\langle \operatorname{ad}^*_{\sigma^{\mathcal{H}}} \frac{\delta l}{\delta\sigma}, \eta \right\rangle \right) \upsilon \\ &= -\int_M \left\langle \operatorname{div}^{\mathcal{H}} \frac{\delta l}{\delta\sigma} + \operatorname{ad}^*_{\sigma^{\mathcal{H}}} \frac{\delta l}{\delta\sigma}, \eta \right\rangle \upsilon,\end{aligned}$$

where we used of the properties of the divergence operator $\operatorname{div}^{\mathcal{H}}$, and, for the last step, we have $\int_M \operatorname{div}\langle \delta l/\delta\sigma, \eta\rangle\upsilon = 0$ by the Stokes Theorem. Hence, η being arbitrary, $\delta\int_M l(\sigma(x))\upsilon = 0$ if and only if (3) holds. □

THEOREM 2 (Reconstruction). *Let $\pi: P \to M$ be a principal G-fiber bundle over a simply connected manifold M with a volume form υ and let $L: J^1P \to \mathbb{R}$ be a G-invariant Lagrangian. A solution σ of the Euler–Poincaré equations (3) comes from a solution s of the original variational problem defined by L if and only if σ is a flat connection with trivial holonomy. If σ satisfies this compatibility condition, we can reconstruct a family of sections s^g, $g \in G$, as the integral leaves of σ.*

As the holonomy of any flat connection is locally trivial, we always have the local equivalence

$$\mathcal{EL}(L)(s) = 0 \Leftrightarrow \begin{cases} \mathcal{EP}(l)(\sigma) = 0, \\ \operatorname{Curv}(\sigma) = 0. \end{cases}$$

Proof. If σ is the composition of j^1s with the projection $q: J^1P \to C$, for a certain section s of π, then σ is a flat connection. This follows from the geometrical meaning of q. Indeed, given $j^1_x s$, the element $q(j^1_x s) \in C$ is the G-distribution $\mathcal{H} \subset TP$ whose fiber $\mathcal{H}_p$ at $p \in P$ is the vector space complementary to the vertical space $V_pP \subset T_pP$, for $p \in \pi^{-1}(x) \subset P$, such that $\mathcal{H}_{s(x)} = T_x s(T_x M)$. Then it is clear that σ is flat and its integral leaves are the sections $R_g \circ s$, $g \in G$. For the converse, every flat connection with trivial holonomy defines a foliation of P such that every leaf is a section s of π. It is obvious that $\sigma = q(j^1 s)$ and from Theorem 1, s is a solution of $\mathcal{EL}(L)(s) = 0$ if and only if $\mathcal{EP}(l)(\sigma) = 0$. □

PROPOSITION 3. *Let $s\colon U \to P$ be a local section of π and let δs an arbitrary variation of s (not necessarily vertical). Let $X \in \mathrm{aut}P|_U$ be the unique G-invariant vector field on $\pi^{-1}(U)$ such that $X|_{s(U)} = \delta s$. Then we have $\delta\sigma = (X_C)|_{\sigma(U)}$.*

Proof. We prove that if $X \in \mathrm{aut}P$, then the vector field $X^{(1)}$ is an infinitesimal automorphism of the principal fiber bundle $q\colon J^1P \to C$ which projects onto the vector field $X_C \in \mathfrak{X}(C)$. This is equivalent to prove that, for any $\Phi \in \mathrm{Aut}P$, $\Phi^{(1)}$ is an automorphism of $q\colon J^1P \to C$ projecting onto the diffeomorphism Φ_C. The G-equivariance of $\Phi^{(1)}$ with respect to the action of G on J^1P is easily obtained from the definition of $\Phi^{(1)}$. As for the projection of $\Phi^{(1)}$, given an element $j^1_x s \in J^1P$, we can define the horizontal subspace $H_{s(x)} := T_x s(T_x M) \subset T_{s(x)}P$ complementary to the vertical subspace $V_{s(x)}P$. We have that $\Phi^{(1)}(j^1_x s) = j^1_{\varphi(x)}(\Phi \circ s \circ \varphi^{-1})$ defines the horizontal subspace $T_{\varphi(x)}(\Phi \circ s \circ \varphi^{-1})(T_{\varphi(x)}M) = T_{s(x)}\Phi(H_{s(x)})$. If we denote by $\Gamma_x \in C$ the projection of $j^1_x s$ by q, then it is clear that $\Phi^{(1)}(j^1_x s)$ defines the element $\Phi_C(\Gamma_x)$, that is, $\Phi_C \circ q = q \circ \Phi^{(1)}$, thus finishing the proof. □

Remark 4. In Theorem 1, point 3, a particular expression of the reduced variation $\delta\sigma$ is shown. In a coordinate system (x^j, A^α_j) on C as defined in Section 2, taking $\mathcal{H}$ to be the trivial connection with $A^\alpha_j(\mathcal{H}) = 0$, for all α, j, we can thus deduce the local expression of the vector field X_C. If $X = g^\alpha \tilde{B}_\alpha$, we have $X_C = (\partial g^\alpha/\partial x^j + c^\alpha_{\beta\gamma} g^\beta A^\gamma_j)\partial/\partial A^\alpha_j$, where $c^\alpha_{\beta\gamma}$ are the structure constants of the Lie algebra $\mathfrak{g}$. This is the classical expression of X_C given in the literature (cf. [5, 7]).

By Theorem 1 and Proposition 3 we can characterize the reduced problem in a geometric way. Namely, the reduced problem is a zero order variational problem on the space of connections defined by the Lagrangian $l\colon C \to \mathbb{R}$ with constraints on the space of possible variations. The allowed infinitesimal variations along a section σ of $C \to M$ are the gauge vector fields X_C, $X \in \mathrm{gau}P$, on C.

The set of permitted variations form a Lie algebra with respect to the Lie bracket. We now show that this Lie algebra can be enlarged up to $\mathrm{aut}P$, but always dealing with sections σ satisfying the local compatibility condition $\mathrm{Curv}(\sigma) = 0$ stated in Theorem 2. More precisely:

PROPOSITION 5. *Let $U \subset M$ be an open set with $\bar{U}$ compact, let $s\colon \bar{U} \to P$ be a (local) section of π, and let $\sigma\colon U \to C$, $\sigma(x) = q(j^1_x s)$, be the induced section on the bundle of connections (that is, σ is a connection with* $\mathrm{Curv}(\sigma) = 0$ *by Theorem 2). The following are equivalent:*

(1) *s is a critical section of the variational problem defined by L*
(2) *the variational principle $\delta \int_U l(\sigma(x))v = 0$ holds, using variations of the form $\delta\sigma = (X_C)|_{\sigma(U)}$, $X \in \mathrm{aut}P$.*

Proof. It is a well-known fact of the calculus of variations that a critical section s for vertical variations is also critical for arbitrary variations and vice-versa. For

(1) $\Rightarrow$ (2), given an arbitrary variation δs along s, let X be the vector field in aut$P$ such that $X|_{s(U)} = \delta s$. From Proposition 3, $\delta\sigma = (X_C)|_{\sigma(U)}$, thus obtaining (2). For the converse, if σ is critical for $X \in \mathrm{aut}P$, then X is necessarily critical for $X \in \mathrm{gau}P$. Thus, σ satisfies the condition of Theorem 1, point 3, and hence s must be also critical. □

4. Conservation Laws

4.1. THE EULER–POINCARÉ EQUATION AS A CONSERVATION LAW

A vector field $X \in \mathfrak{X}(P)$ is said to be an infinitesimal symmetry of a Lagrangian density $\mathcal{L} = L\upsilon$ if $\pounds_{X^{(1)}}(L\upsilon) = 0$, or equivalently, if $\pounds_{X^{(1)}}\Theta_{\mathcal{L}} = 0$, where $\pounds$ stands for the Lie derivative and $\Theta_{\mathcal{L}}$ is Poincaré–Cartan form defined by $\mathcal{L}$ (see, for example, [6]). Along a critical section s, Noether's Theorem yields a conservation law $\mathrm{d}((j^1s)^* i_{X^{(1)}}\Theta_{\mathcal{L}}) = 0$ on M, for every such infinitesimal symmetry $X^{(1)}$.

We now consider a G-invariant Lagrangian. Hence, the infinitesimal generator $B^* \in \mathfrak{X}(P)$ of the action defined by any $B \in \mathfrak{g}$ is a infinitesimal symmetry of L. We can thus define a $\mathfrak{g}^*$-valued $(n-1)$-form $\mathbf{J}$ on J^1P by $\mathbf{J}(B) = i_{(B^*)^{(1)}}\Theta_{\mathcal{L}}$, $B \in \mathfrak{g}$ which is called the *Noether current*. It satisfies the following condition: for every critical section s we have $\mathrm{d}((j^1s)^*\mathbf{J}) = 0$ on M.

PROPOSITION 6. *The* $\mathfrak{g}$-*valued* $(n-1)$-*form* $\mathbf{J}$ *is a tensorial form of the coadjoint type on* J^1P (see [9, II.5]). *Therefore, it defines a* $(n-1)$-*form* J *on* C *taking values in the coadjoint bundle* $p^*(\mathrm{ad}P)^*$.

Using the isomorphism between TM *and* $\bigwedge^{n-1} T^*M$ *given by the volume form* υ, J *can be interpreted as a section* $\bar{J}$ *of the bundle* $p^*(TM \otimes (\mathrm{ad}P)^*)$. *Then* $\bar{J}$ *coincides with the vertical derivative* $\delta l := \mathrm{d}^v l$ *of the reduced Lagrangian* $l\colon C \to \mathbb{R}$.

Proof. First we remark that, given $B \in \mathfrak{g}$, the infinitesimal generator $B^\bullet$ of the action on the principal fiber bundle $q\colon J^1P \to C$ coincides with the lift $(B^*)^{(1)}$. On the other hand, we have

$$\pounds_{(A^*)^{(1)}}\mathbf{J}(B) = \pounds_{(A^*)^{(1)}} i_{(B^*)^{(1)}}\Theta_{\mathcal{L}} = i_{([A,B]^*)^{(1)}}\Theta_{\mathcal{L}} + i_{(B^*)^{(1)}}\pounds_{(A^*)^{(1)}}\Theta_{\mathcal{L}} = \mathbf{J}([A, B]),$$

or shortly, $\pounds_{A^\bullet}\mathbf{J} = \mathrm{ad}^*_A \circ \mathbf{J}$, that is, $\mathbf{J}$ is of the coadjoint type.

We work now in a trivialization of P given by $U \times V$, U open in M, V open in G, whose coordinates shall be denoted by (x^i, y^α). Taking the local expression of the Poincaré–Cartan form (see, for example, [6])

$$\Theta_{\mathcal{L}} = \frac{\partial L}{\partial y^\alpha_i}(\mathrm{d}y^\alpha - y^\alpha_j\, \mathrm{d}x^j) \wedge \upsilon_i + L\upsilon,$$

where $\upsilon_i = i_{\partial/\partial x^i}\upsilon$, we see that $\mathbf{J}$ is horizontal with respect to the projection $p_1\colon J^1P \to M$ and, in particular, with respect to $q\colon J^1P \to C$. Then $\mathbf{J}$ is a tensorial form of the coadjoint type and, consequently, it defines a $(n-1)$-form J on C taking values in the coadjoint bundle associated to the principal bundle $q\colon J^1P \to C$ [9, II.5].

This coadjoint bundle is isomorphic to $p^*(\mathrm{ad}P)^*$. Let now V be a neighborhood of the identity e in G and denote by (y^α), $1 \leqslant \alpha \leqslant m$, the normal coordinate system induced by the exponential on G and the given basis $\{B_1, \dots, B_m\}$ of $\mathfrak{g}$. Then, along the section $j^1 s_0$, with $s_0(x) = (x, e)$, it is easy to check that $\mathbf{J}$, seen as a $\mathfrak{g}^*$-valued $(n-1)$-form, has the expression $\mathbf{J} = (\partial L/\partial y_i^\alpha) v_i \otimes B^\alpha$. The identification between TM and $\bigwedge^{n-1} T^*M$ given by the volume form υ gives the expression of the associated $\mathfrak{g}^*$-valued vector field on J^1P

$$\bar{\mathbf{J}} = \frac{\partial L}{\partial y_i^\alpha} \frac{\partial}{\partial x^i} \otimes B^\alpha.$$

which naturally induces the section $\bar{J}$ of $p^*(TM \otimes (\mathrm{ad}P^*)$. A standard computation shows that this section has the expression $(\partial l/\partial A_i^\alpha)(\partial/\partial x^i) \otimes \tilde{B}^\alpha$ which coincides with the vertical derivative of l. (In the previous expression $\tilde{B}^\alpha$ is the dual basis of sections of $(\mathrm{ad}P)^*$ induced by the basis of sections $\tilde{B}_\alpha$ of $(\mathrm{ad}P)$.) □

Once we have projected the Noether current to C and identified it with δl, we are going to study the conservation law defined by it.

PROPOSITION 7. *The Noether conservation law for the infinitesimal symmetries B^*, $B \in \mathfrak{g}$, along a critical section s is equivalent to the equation* $\mathrm{div}^\sigma \delta(l/\delta\sigma) = 0$, *where* $\sigma = q(j^1 s)$, *and* div^σ *is the covariant divergence operator defined by the connection σ (see formula (1)). That is, we have the Euler–Poincaré equation of l for $\mathcal{H} = \sigma$.*

Proof. For any critical section s, we have $\mathrm{d}((j^1 s)^* \mathbf{J}) = 0$ which is easily seen to be equivalent to

$$\mathrm{d}^\sigma(\sigma^*(J)) = 0, \tag{4}$$

where $\mathrm{d}^\sigma = \nabla^\sigma$ denotes the covariant differential defined by the connection σ. With the identification between TM and $\bigwedge^{n-1} T^*M$ given by the volume form υ and the definition of div^σ, (see formula (1)), the expression (4) is equivalent to $\mathrm{div}^\sigma(\sigma^*(\bar{J})) = 0$. The proof is completed by recalling that $\bar{J} = \delta l$ and that $\sigma^*(\delta l) = \delta l/\delta\sigma$. □

COROLLARY 8. *The Euler–Lagrange equations of a G-invariant Lagrangian are equivalent to the Noether conservation law of the G-symmetry plus the compatibility condition* $\mathrm{Curv}\sigma = 0$.

4.2. THE SYMMETRIES OF THE REDUCED LAGRANGIAN l

We say that a vector field on C is an *infinitesimal symmetry of the reduced Lagrangian l* if

(1) it is of the form X_C with $X \in \mathrm{aut}P$, and
(2) $\pounds_{X_C}(l\upsilon) = 0$.

Condition (1) says that the infinitesimal symmetry must belong to the algebra of admissible variations of the variational problem, which is, by virtue of Proposition 5, the algebra of all vector fields of the form X_C, with $X \in \mathrm{aut}P$. Condition 2) can be written as $X_C[l] + l\mathrm{div}(X') = 0$, where X' is the projection of X onto M and $X_C[l] = \pounds_{X_C} l = \langle \mathrm{d}l, X_C \rangle$ is the derivative of l in the direction X_C.

PROPOSITION 9. *If σ is a solution of the Euler–Poincaré equations satisfying the compatibility condition* $\mathrm{Curv}(\sigma) = 0$ *and X_C, $X \in \mathrm{aut}P$, is an infinitesimal symmetry of l, the following conservation holds*

$$\mathrm{div}\left\langle \frac{\delta l}{\delta\sigma}, Y \right\rangle \upsilon + \pounds_{X'}((l \circ \sigma)\upsilon) = 0,$$

where X' is the projection of X onto M and $Y \in \Gamma(\mathrm{ad}P) = \mathrm{gau}P$ is the vertical part of X with respect to the connection $\mathcal{H}^\sigma$ defined by σ.

Proof. We decompose the vector field $X \in \mathrm{aut}P$ as $X = Y + Z$, where Y is vertical and Z horizontal with respect to $\mathcal{H}^\sigma$. We note that $Y \in \mathrm{gau}P$ and $Z \in \mathrm{aut}P$. Along the section σ, we have

$$\begin{aligned} 0 = \sigma^* \pounds_{X_C}(l\upsilon)|_\sigma &= \sigma^* \pounds_{Y_C}(l\upsilon) + \sigma^* \pounds_{Z_C}(l\upsilon) \\ &= \left\langle \frac{\delta l}{\delta\sigma}, \nabla^{\mathcal{H}} Y - [\sigma^{\mathcal{H}}, Y] \right\rangle + \sigma^* \pounds_{Z_C}(l\upsilon) \\ &= \left(\mathrm{div}\left\langle \frac{\delta l}{\delta\sigma}, Y \right\rangle - \langle \mathcal{EP}(l)(\sigma), Y \rangle + \sigma \circ Z_C(l) + (l \circ \sigma)\mathrm{div}(X') \right)\upsilon, \end{aligned}$$

where $Y \in \mathrm{gau}P$ is seen as a section of $\mathrm{ad}P$. Since σ is a critical section, it follows that $\mathcal{EP}(l)(\sigma) = 0$. Because $\mathrm{Curv}(\sigma) = 0$, there locally exists a section s of the bundle $P \to M$ such that $\sigma = q \circ j^1 s$. It is clear $\sigma \circ Z_C(l) = j^1 s \circ Z^{(1)}(L)$. As Z is horizontal with respect to $\mathcal{H}^\sigma$ and s is an integral leaf, we have $Z \circ s = Ts \circ X'$ and then $Z^{(1)} \circ j^1 s = T(j^1 s) \circ X'$ thus implying that $j^1 s \circ (Z^{(1)}[L]) = X'[L \circ j^1 s] = X'[l \circ \sigma]$. Then $(\sigma \circ (Z_C[l]) + (l \circ \sigma)\mathrm{div}(X'))\upsilon = \pounds_{X'}((l \circ \sigma)\upsilon)$ thus finishing the proof. □

5. The Second Variation

We now discuss the second variation formula (the Hessian) along a critical section σ of the Euler–Poincaré equations. As the variational problem defined by l is not free, the expression of the Hessian does not take its usual form (see, for example, [8]). For a complete description of the Hessian, one has to deal only with admissible variations and take into account their special structure. More precisely, given two admissible vertical variations $\delta\sigma_1$ and $\delta\sigma_2$ along a solution σ of (3), we define

$$\mathrm{Hess}^l_\sigma(\delta\sigma_1, \delta\sigma_2) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} \int_U l(\sigma + t\delta\sigma_1 + s\delta\sigma_2)\upsilon.$$

By Proposition 3, we know that $\delta\sigma_1 = X_C|_{\sigma(U)}$ and $\delta\sigma_2 = Y_C|_{\sigma(U)}$ for certain

$X, Y \in \mathrm{gau}P$. Then the previous formula easily yields

$$\mathrm{Hess}^l_\sigma(X_C, Y_C) = \int_U Y_C(X_C(l))v = \int_U \left\langle \frac{\delta}{\delta\sigma} X_C[l], Y_C \right\rangle v, \tag{5}$$

where $\langle\,,\rangle$ stands for the natural pairing between $TM \otimes (\mathrm{ad}P)^*$ and its dual.

PROPOSITION 10. *If σ is a solution of the Euler–Poincaré equations, then*

$$\mathrm{Hess}^l_\sigma(X_C, Y_C) = \mathrm{Hess}^l_\sigma(Y_C, X_C),$$

that is, Hess^l_σ *is a symmetric bilinear operator defined on the space of admissible vector fields along σ.*

Proof. As σ is a critical section, we have that $\int_U \langle \delta l/\delta\sigma, Z_C\rangle v = 0$, where $Z = [X, Y]$. The proof is complete by taking into account that $[X_C, Y_C] = [X, Y]_C$. □

The expression given in formula (5) is not very convenient. For example, the Jacobi fields, that is, the variations $\delta\sigma$ belonging to the kernel (or null-space) of the bilinear form Hess^l_σ, are not characterized by the condition $(\delta/\delta\sigma)X_C[l] = 0$ as the variations $\delta\sigma_2 = Y_C$ are not arbitrary but gauge fields on C. The true characterization of the Jacobi fields is as follows.

PROPOSITION 11. *The Hessian has the following expression*

$$\mathrm{Hess}^l_\sigma(X_C, Y_C) = \int_U \langle \mathcal{EP}(X_C[l]), Y\rangle v,$$

where $Y \in \mathrm{gau}P$ is seen as a section of $\mathrm{ad}P$ *with compact support in U.*

Accordingly, the Jacobi operator $\mathcal{J}\colon C^\infty(\mathrm{ad}P) \to C^\infty((\mathrm{ad}P)^*)$ of the reduced Lagrangian l is defined by $\mathcal{J}(X_C) = \mathcal{EP}(X_C[l])$, $X \in \mathrm{gau}P$. Hence a gauge vector field X_C on C belongs to the kernel of Hess^l_σ *if and only if $\mathcal{J}(X_C) = 0$.*

Proof. We have

$$\begin{aligned}\mathrm{Hess}^l_\sigma(X_C, Y_C) &= \int_U \left\langle \frac{\delta}{\delta\sigma} X_C[l], \nabla Y - [\sigma^{\mathcal{H}}, Y] \right\rangle v \\ &= \int_U \left(\left\langle \mathrm{div}^{\mathcal{H}}\left(\frac{\delta}{\delta\sigma} X_C[l] \right) + \mathrm{ad}^*_{\sigma^{\mathcal{H}}}\left(\frac{\delta}{\delta\sigma} X_C[l] \right), Y \right\rangle + \right. \\ &\qquad \left. + \mathrm{div}\left\langle \frac{\delta}{\delta\sigma} X_C[l], Y \right\rangle \right) v \\ &= \int_U \langle \mathcal{EP}(X_C[l]), Y\rangle v,\end{aligned}$$

since Y has compact support. As the section $Y \in C^\infty(\mathrm{ad}P)$ is arbitrary, the fundamental lemma of the calculus of variations gives the condition for the Jacobi vector fields. □

6. Example: Harmonic Mappings

Let (M, g) be a compact oriented Riemannian manifold, and let (G, h) be a Lie group equipped with a right invariant Riemannian metric. We identify the mappings $\phi: M \to G$ with the global sections of the trivial principal bundle $P = M \times G$ endowed with the trivial connection. For each $\phi \in C^\infty(M, G)$, we define the energy E on $C^\infty(M, G)$ by $E(\phi) = \int_M L(j^1\phi)\upsilon$, where $L(j^1\phi) = \frac{1}{2}\langle T\phi, T\phi\rangle_{g,h}$, and $\langle\cdot,\cdot\rangle_{g,h}$ is the induced metric on $T^*M \otimes TG$ by g and h. The Euler–Lagrange equations for this Lagrangian are given by $\mathrm{Tr}\nabla\phi = 0$, where ∇ is the induced Riemannian covariant derivative on $C^\infty(T^*M \otimes TG)$ and Tr is the trace defined by g. The solutions of this equation are called *harmonic mappings*. The reduced Lagrangian $l: T^*M \otimes \mathfrak{g} \to \mathbb{R}$ is $l(\sigma) = \frac{1}{2}\langle\sigma, \sigma\rangle_{g,h}$ where h is thought of as a metric on $\mathfrak{g}$. The Euler–Poincaré equations are

$$\langle \mathrm{d}^*\sigma, \cdot\rangle_h + \mathrm{ad}^*_\sigma\langle\sigma, \cdot\rangle_h = 0,$$

where $\mathrm{d}^* = *\mathrm{d}*$ is the codifferential. For the Jacobi operator, first we compute $\delta X_C[l]/\delta\sigma$ with $X_C = \delta\sigma = \mathrm{d}\eta + [\eta, \sigma]$, and obtain

$$\delta/\delta\sigma X_C[l] = \langle\delta\sigma, \cdot\rangle_{g,h} + \mathrm{ad}^*_\eta\langle\sigma, \cdot\rangle_{g,h}.$$

Hence the Jacobi operator reads

$$\langle \mathrm{d}^*\delta\sigma, \cdot\rangle_h + \mathrm{ad}^*_\eta\langle \mathrm{d}^*\sigma, \cdot\rangle_h + \mathrm{ad}^*_\sigma\langle\delta\sigma, \cdot\rangle_h + \mathrm{ad}^*_\sigma \mathrm{ad}^*_\eta\langle\sigma, \cdot\rangle_h.$$

Taking into account that σ is a critical section we have

$$\begin{aligned}&\langle \mathrm{d}^*\delta\sigma, \cdot\rangle_h + \mathrm{ad}^*_\sigma\langle\delta\sigma, \cdot\rangle_h + \mathrm{ad}^*_\sigma\mathrm{ad}^*_\eta\langle\sigma, \cdot\rangle_h - \mathrm{ad}^*_\eta\mathrm{ad}^*_\sigma\langle\sigma, \cdot\rangle_h \\ &\quad = \langle \mathrm{d}^*\delta\sigma, \cdot\rangle_h + \mathrm{ad}^*_\sigma\langle[\eta, \sigma], \cdot\rangle_h + \mathrm{ad}^*_{[\eta,\sigma]}\langle\sigma, \cdot\rangle_h + \mathrm{ad}^*_\sigma\langle \mathrm{d}\eta, \cdot\rangle_h.\end{aligned}$$

As η has compact support, we finally obtain $\mathrm{Hess}^l_\sigma(X_C, \cdot) = \int_M \langle\mathcal{J}(X_C), \cdot\rangle_h \upsilon$ with

$$\langle\mathcal{J}(X_C), \cdot\rangle_h = \langle \mathrm{d}^*\delta\sigma, \cdot\rangle_h + \mathrm{ad}^*_\sigma\langle[\eta, \sigma], \cdot\rangle_h + \mathrm{ad}^*_{[\eta,\sigma]}\langle\sigma, \cdot\rangle_h - \mathrm{ad}^*_{\mathrm{d}^*\sigma}\langle\eta, \cdot\rangle_h.$$

If the metric h is also left invariant, we have $\mathrm{ad}^*_A\langle B, \cdot\rangle_h + \mathrm{ad}^*_B\langle A, \cdot\rangle_h = 0$, for all $A, B \in \mathfrak{g}$, and the Euler–Poincaré equation and the Jacobi operator can be simplified to $\mathrm{d}^*\sigma = 0$ and $\mathcal{J}(X_C) = \mathrm{d}^*\delta\sigma$ respectively. Both equations, for this particular case of harmonic mappings into a Lie group with a bi-invariant metric, were obtained for the first time in [11], not from reduction techniques but by direct computation.

Assume now that G is Abelian. In this case, any right invariant metric is also left invariant. The Euler–Poincaré equation and the compatibility condition read $\mathrm{d}^*\sigma = 0$ and $\mathrm{d}\sigma = 0$ respectively, that is, σ is a $\mathfrak{g}$-valued harmonic 1-form on M. The Jacobi operator is $\mathcal{J}(X_C) = \mathrm{d}^*\delta\sigma = \mathrm{d}^*\mathrm{d}\eta = \Delta\eta$, where Δ is the Laplace–Beltrami operator. Note that $\mathcal{J}$ does not depend on the section σ. Then, the Jacobi fields are the vector valued harmonic maps $\eta: M \to \mathfrak{g}$. Moreover, the quadratic form of the Hessian reads

$$\mathrm{Hess}^l_\sigma(X_C, X_C) = \int_M \langle \mathcal{J}(X_C), X\rangle \upsilon = \int_M \langle \Delta\eta, \eta\rangle_h \upsilon = \int_M \langle \mathrm{d}\eta, \mathrm{d}\eta\rangle_{g,h} \upsilon \geqslant 0,$$

that is, Hess^l_σ is semi-positive. Hence, every harmonic map is stable.

Similarly, for G arbitrary and h bi-invariant, the solution $\sigma = 0$, corresponding to a constant map, also yields

$$\mathrm{Hess}^l_\sigma(X_C, X_C) = \int_M \langle \mathcal{J}(X_C), X\rangle \upsilon = \int_M \langle \Delta\eta, \eta\rangle_h \upsilon = \int_M \langle \mathrm{d}\eta, \mathrm{d}\eta\rangle_{g,h} \upsilon \geqslant 0,$$

thus obtaining the stability of this solution. We refer the reader, for example, to [12, § 5] for a classical proof of these results.

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