ELSEVIER

Journal of Geometry and Physics 32 (1999) 160–188

JOURNAL OF GEOMETRY AND PHYSICS

Non-linear stability of singular relative periodic orbits in Hamiltonian systems with symmetry

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Received 1 October 1998; received in revised form 29 March 1999

Abstract

We generalize the sufficient condition for the stability of relative periodic orbits in symmetric Hamiltonian systems presented in [J.-P. Ortega, T.S. Ratiu, J. Geom. Phys. 32 (1999) 131–159] to the case in which these orbits have non-trivial symmetry. We also describe a block diagonalization, similar in philosophy to the one presented in [J.-P. Ortega, T.S. Ratiu, Nonlinearity 12 (1999) 693–720] for relative equilibria, that facilitates the use of this result in particular examples and shows the relation between the stability of the relative periodic orbit and the orbital stability of the associated singular reduced periodic orbit. © 1999 Elsevier Science B.V. All rights reserved.

Subj. Class.: Dynamical systems

1991 MSC: 34D35; 58F05

Keywords: Non-linear stability; Periodic orbits in Hamiltonian systems; Stability

1. Introduction

Let $(M, \omega, G, \mathbf{J} : M \rightarrow \mathfrak{g}^*, h : M \rightarrow \mathbb{R})$ be a Hamiltonian dynamical system with symmetry. We assume that the Lie group G with Lie algebra $\mathfrak{g}$ acts properly on the smooth symplectic manifold (M, ω) and that the G-action admits an equivariant momentum map $\mathbf{J} : M \rightarrow \mathfrak{g}^*$; $\mathfrak{g}^*$ denotes the dual space of $\mathfrak{g}$. Recall that a point $m \in M$ is called a *relative*

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PII: S0393-0440(99)00024-8

periodic point (RPP) of the Hamiltonian system if there is a $\tau > 0$ and an element $g \in G$ such that

$$F_{t+\tau}(m) = g \cdot F_t(m) \quad \text{for any } t \in \mathbb{R},$$

where F_t is the flow of the Hamiltonian vector field X_h. The set

$$\gamma(m) := \{F_t(m) \mid t > 0\}$$

is called a *relative periodic orbit* (RPO) through m. The constant $\tau > 0$ is its *relative period* and the group element $g \in G$ is its *phase shift*.

As we already saw in [24], in a Hamiltonian system like the one we are dealing with, the existence of a symmetry gives rise to drift phenomena, making non-trivial the choice of a definition of stability, since the obvious option, orbital stability, becomes too restrictive. The most natural thing to do is to imitate the notion of stability relative to a subgroup introduced by Patrick [27] for relative equilibria.

Definition 1.1. If G' is a Lie subgroup of G, the RPP m is *G'-stable*, or *stable modulo G'*, if for any G'-invariant open neighborhood V of the set $G' \cdot \{F_t(m)\}_{t>0}$, there is an open neighborhood $U \subseteq V$ of m such that $F_t(U) \subset V$, for any $t > 0$.

In [24], we described a sufficient condition to study the G_μ-stability of RPPs in some particular examples (the *symmetric energy-integrals method*). However, the hypothesis of this theorem required the regularity of μ and the freeness of the G-action. We will generalize this result to the case in which the RPP m has non-trivial symmetry. The main complications arising in this situation come from the impossibility to use symplectic reduction [19], and even though singular reduction tools [6,22,29] are available, many simplifying features of the regular case, as for instance the equivalence between the G_μ-stability of the RPO and the orbital stability of the reduced periodic orbit, are destroyed.

Since singular reduction, slices, invariant Poincaré sections, etc., will be key ingredients in our proofs we will dedicate two sections to briefly review the concepts and results that will be used later on. In Section 4, we present the statement and the proof of the symmetric energy-integrals method in the singular case and, as a corollary, an improvement of the energy–momentum and the energy-Casimir methods. In Section 5 we construct a block diagonalization of the stability form based on the isotypic decomposition of a linear representation of a compact Lie group that generalizes to the singular case the link between the G_μ-stability of the RPO and the orbital stability of the singular reduced periodic orbit. It should be emphasized that this block diagonalization is purely kinematical and similar in philosophy to the one presented in [23] for relative equilibria, which should not be confused with the dynamical block diagonalization of Lewis and coworkers [16,28].

2. Singular reduction and relative periodic orbits

Singular reduction is a topic that has been developed for the last 15 years. The first studies on the structure of the symplectic reduced spaces [19] in the singular case are in the works of Arms et al. [4], Otto [25], and Arms et al. [2]. The idea of using normal forms to describe these spaces as *stratified spaces* was first introduced by Sjamaar and Lerman [29] in the compact case, and by Bates and Lerman [6] in the case of proper actions. These and other reduction schemes are compared in [3]. The proofs of the results cited below can be found in these references or in [22], where the point of view of point reduction, mostly used in our discussion, is explained in detail.

Let $(M, \omega, G, \mathbf{J} : M \to \mathfrak{g}^*, h : M \to \mathbb{R})$ be a Hamiltonian dynamical system whose symmetry is given by the Lie group G acting properly on M. The Hamiltonian $h \in C^\infty(M)$ is G-invariant and $\mathbf{J}$ is assumed to be equivariant. Under these conditions, we say that M is a *Hamiltonian G-space*. If $m \in M$ is such that $\mathbf{J}(m) = \mu$ is a regular value of $\mathbf{J}$ whose coadjoint isotropy subgroup G_μ acts freely and properly on the manifold $\mathbf{J}^{-1}(\mu)$, it is well known [19] that the space $M_\mu := \mathbf{J}^{-1}(\mu)/G_\mu$ is a symplectic manifold and that the dynamics induced by h reduces naturally to Hamiltonian dynamics on $\mathbf{J}^{-1}(\mu)/G_\mu$. Below we present the manifolds that generalize M_μ when the given point m has non-trivial symmetry, that is, $H := G_m \neq \{e\}$. Recall that the properness of the action implies that H is compact. The notation that we will use is standard in the theory of group actions.

Proposition 2.1. *Let H and K be closed subgroups of G such that $H \subset K \subset G$. The sets*

$$\begin{aligned} M_{(H)} &= \{z \in M \mid G_z \text{ is conjugate to } H\}, \\ M_{(H)}^K &= \{z \in M \mid G_z \text{ is conjugate to } H \text{ in } K\}, \\ M^H &= \{z \in M \mid H \subseteq G_z\}, \\ M_H &= \{z \in M \mid H = G_z\} = M^H \cap M_{(H)} \end{aligned}$$

are submanifolds of M. M_H is an open submanifold of M^H. $M_{(H)}$ is called the (H)-orbit type manifold. If M is symplectic, M_H and M^H are symplectic submanifolds of M. Also, for any $m \in M_H$, if $\Phi : G \times M \to M$ denotes the group action, the tangent space to M_H is given by

$$T_m M_H = \{v_m \in T_m M \mid T_m \Phi_h \cdot v_m = v_m, \forall h \in H\} = T_m M^H.$$

Proof. See for example [6,7,13,26]. □

If V is a representation space of H then the H-fixed point space V^H is a vector subspace of V. If, in addition, V is symplectic and H acts canonically, then V^H is a symplectic subspace of V (see [13]). Thus, the last claim of Proposition 2.1 can be written as $T_m M_H = T_m M^H = (T_m M)^H$, the last action being the linearized action on the tangent bundle.

Also, if H and K are closed subgroups of G such that $H \subset K \subset G$, we will denote by

$$N(H) = \{n \in G \mid nHn^{-1} = H\},$$
$$N_K(H) = \{n \in K \mid nHn^{-1} = H\} = N(H) \cap K,$$

the normalizers of H in G and K, respectively.

We now introduce the singular reduced spaces.

Theorem 2.2. *Let (M, ω) be a Hamiltonian G-space with G acting properly on M. Let $\mathbf{J} : M \to \mathfrak{g}^*$ be the corresponding equivariant momentum map. For $m \in M$ let $\mu := \mathbf{J}(m) \in \mathfrak{g}^*$ and $H := G_m$ be the isotropy subgroup of m which, by the equivariance of $\mathbf{J}$, is a subgroup of G_μ, the coadjoint isotropy subgroup of G at $\mu \in \mathfrak{g}^*$. Then:*

(i) *The set $\mathbf{J}^{-1}(\mu) \cap M_H$ is a submanifold of M_H, and hence of M. Analogously, $\mathbf{J}^{-1}(\mu) \cap M_{(H)}^{G_\mu}$ is a submanifold of $M_{(H)}^{G_\mu}$, and therefore of M.*

(ii) *The set $M_\mu^{(H)} := (\mathbf{J}^{-1}(\mu) \cap M_{(H)}^{G_\mu})/G_\mu$ has a unique quotient differentiable structure such that the canonical projection*

$$\pi_\mu^{(H)} : \mathbf{J}^{-1}(\mu) \cap M_{(H)}^{G_\mu} \longrightarrow M_\mu^{(H)}$$

is a surjective submersion. Endowed with this differentiable structure, $M_\mu^{(H)}$ is diffeomorphic to $(\mathbf{J}^{-1}(\mu) \cap M_H)/(N_{G_\mu}(H)/H)$.

(iii) *There is a unique symplectic structure $\omega_\mu^{(H)}$ on $M_\mu^{(H)}$ characterized by*

$$i_\mu^{(H)*}\omega = \pi_\mu^{(H)*}\omega_\mu^{(H)},$$

where $i_\mu^{(H)} : \mathbf{J}^{-1}(\mu) \cap M_{(H)}^{G_\mu} \hookrightarrow M$ is the natural inclusion.

Proof. See [29,22]. □

The spaces introduced in Theorem 2.2 are suitable to reduce the dynamics induced by G-invariant Hamiltonians. For the proof see [22].

Theorem 2.3. *Let (M, ω) be a Hamiltonian G-space with G acting properly on M and admitting an equivariant momentum map $\mathbf{J} : M \to \mathfrak{g}^*$. Let $h : M \to \mathbb{R}$ be a G-invariant Hamiltonian, that is, $h \circ \Phi_g = h$ for any $g \in G$. Then, using the notation of Theorem 2.2:*

(i) *The flow F_t of X_h leaves the connected components of $\mathbf{J}^{-1}(\mu) \cap M_{(H)}^{G_\mu}$ invariant and commutes with the G_μ-action, so it induces a flow F_t^μ on $M_\mu^{(H)}$ that is characterized by*

$$\pi_\mu^{(H)} \circ F_t = F_t^\mu \circ \pi_\mu^{(H)}.$$

(ii) *The flow F_t^μ is Hamiltonian on $M_\mu^{(H)}$, with Hamiltonian function $h_\mu^{(H)} : M_\mu^{(H)} \to \mathbb{R}$ defined by*

$$h_\mu^{(H)} \circ \pi_\mu^{(H)} = h \circ i_\mu^{(H)}.$$

The vector fields X_h *and* $X_{h_\mu^{(H)}}$ *are* $\pi_\mu^{(H)}$*-related. We will call* $h_\mu^{(H)}$ *the reduced Hamiltonian.*

(iii) *Let* $k : M \to \mathbb{R}$ *be another* G*-invariant function. Then* $\{h, k\}$ *is also* G*-invariant and*

$$\{h, k\}_\mu^{(H)} = \{h_\mu^{(H)}, k_\mu^{(H)}\}_{M_\mu^{(H)}},$$

where $\{\,,\}_{M_\mu^{(H)}}$ *denotes the Poisson bracket induced by the symplectic structure in* $M_\mu^{(H)}$.

For the sake of simplicity, in our next result we will require the normalizer $N(H)$ of H in G to be compact. With this assumption, we can give a very useful characterization of the singular reduced spaces that will be used frequently later on. The following construction is based on the fact that the Lie group $L := N(H)/H$, whose Lie algebra we denote by $\mathfrak{l}$, acts freely and properly on M_H and that $\mathbf{J}(M_H) \subset (\mathfrak{g}^*)^H$, where we are consistent with the notation introduced in Proposition 2.1, that is, $(\mathfrak{g}^*)^H$ denotes the H-fixed vectors in $\mathfrak{g}^*$ under the coadjoint action. By compactness of $N(H)$, we can find $\mathrm{Ad}_{N(H)}$-invariant inner products on $\mathfrak{g}$ and on $\mathfrak{g}^*$ relative to which we have the orthogonal decompositions

$$\mathrm{Lie}(N(H)) = \mathfrak{h} \oplus \mathfrak{p} \quad \text{and} \quad \mathfrak{g}^* = \mathfrak{h}^* \oplus \mathfrak{r}^* \tag{2.1}$$

for some subspaces $\mathfrak{p} \subset \mathrm{Lie}(N(H)) \subset \mathfrak{g}$ and $\mathfrak{r}^* \subset \mathfrak{g}^*$. If $\lambda \in \mathfrak{l}^*$, let $\bar{\lambda} \in \mathfrak{p}$ be such that $\lambda = T_e\pi(\bar{\lambda})$, where $\pi : N(H) \to N(H)/H$ is the canonical projection onto the quotient. Then the linear map $\lambda \in \mathfrak{l} \mapsto \bar{\lambda} \in \mathfrak{p}$ is well defined, L-equivariant, has range equal to $[(\mathfrak{h}^\circ)^H]^*$ (the vector subspace of H-fixed vectors in the annihilator $\mathfrak{h}^\circ$ of $\mathfrak{h}$ in $\mathfrak{g}^*$), and is injective, so it defines an L-equivariant isomorphism $\Lambda : \mathfrak{l} \to [(\mathfrak{h}^\circ)^H]^*$ whose dual map

$$\Lambda^* : (\mathfrak{h}^\circ)^H \longrightarrow \mathfrak{l}^*$$

is hence also an L-equivariant isomorphism. Let

$$\rho : (\mathfrak{g}^*)^H \longrightarrow (\mathfrak{h}^\circ)^H$$

be the natural L-equivariant projection associated to the orthogonal decomposition (2.1). The L-action on M_H is canonical and has an associated equivariant momentum map $\mathbf{K}_L$ given by the expression

$$\mathbf{K}_L(z) = (\Lambda^* \circ \rho)(\mathbf{J}\,|_{M_H}(z)), \quad z \in M_H. \tag{2.2}$$

If $\mathbf{J}(m) = \mu$, we will write $\mu = \mu_{\mathfrak{h}^*} + \mu_{\mathfrak{r}^*}$ for the decomposition of μ according to the splitting (2.1) and will define $\lambda_0 := \Lambda^*(\mu_{\mathfrak{r}^*}) = \mathbf{K}_L(m)$.

Theorem 2.4. *If* $N(H)$ *is compact, the reduced symplectic space* $(M_\mu^{(H)}, \omega_\mu^{(H)})$ *is naturally symplectomorphic to the usual symplectic reduced space* $(\mathbf{K}_L^{-1}(\lambda_0)/L_{\lambda_0}, \omega_{\lambda_0})$*, defined by the* L*-action on* M_H.

Proof. See [22]. We will give here only the key ideas of the proof which will be used later on. One begins by showing that $\mathbf{J}|_{M_H} : M_H \to \mathfrak{g}^*$ is a subimmersion (a constant rank map). Therefore, $(\mathbf{J}|_{M_H})^{-1}(\mu) = \mathbf{J}^{-1}(\mu) \cap M_H$ is a smooth submanifold of the symplectic manifold M_H. Next, one proves that $\lambda_0 \in \mathfrak{l}^*$ is a regular value for the L-momentum map $\mathbf{K}_L : M_H \to \mathfrak{l}^*$. Then one shows that $\mathbf{K}^{-1}{}_L(\lambda_0) = \mathbf{J}^{-1}(\mu) \cap M_H$ and that $L_{\lambda_0} = N_{G_\mu}(H)/H$ to conclude that

$$\mathbf{K}_L^{-1}(\lambda_0)/L_{\lambda_0} = (\mathbf{J}^{-1}(\mu) \cap M_H)/(N_{G_\mu}(H)/H).$$

The space on the right-hand side of this expression, as we pointed out before, is diffeomorphic to $M_\mu^{(H)}$. □

Since $\ker T_m(\mathbf{J}|_{M_H}) = \ker T_m\mathbf{J} \cap T_m M_H$ and $T_m(\mathbf{J}^{-1}(\mu) \cap M_H) = T_m(\mathbf{K}_L^{-1}(\lambda_0)) = \ker T_m\mathbf{K}_L$, we conclude from the proof above that

$$\ker T_m\mathbf{J} \cap T_m M_H = T_m(\mathbf{J}^{-1}(\mu) \cap M_H) = T_m(\mathbf{K}_L^{-1}(\lambda_0)) = \ker T_m\mathbf{K}_L. \tag{2.3}$$

Remark 2.5. *If the condition on the compactness of $N(H)$ is dropped, there are still global models for the singular reduced spaces of the kind introduced in Theorem* 2.4*; however, the result is more complicated since the momentum map in* (2.2) *is not equivariant and the reduction has to be carried out by correcting the coadjoint action with the cocycle given by the non-equivariance of $\mathbf{K}_L$.*

Using these results on singular reduction we now see how the term relative periodic orbit is justified, that is, the projection of an RPO over the (singular) reduced space gives us a periodic orbit. We make these remarks more precise in the following theorem.

Theorem 2.6. *Let $m \in M$ such that $\mathbf{J}(m) = \mu$, and $G_m =: H$. In the conditions of Theorem* 2.2*, the following statements are equivalent:*

(i) *the point m is a RPP;*

(ii) *there is a constant $\tau > 0$ and $g \in N_{G_\mu}(H)$ such that*

$$F_{t+\tau}(m) = g \cdot F_t(m) \quad \text{for any} \quad t \in \mathbb{R},$$

where F_t is the flow of X_h;

(iii) *there is a constant $\tau > 0$ and a unique element $l \in N_{G_\mu}(H)/H$ such that*

$$F_{t+\tau}(m) = l \cdot F_t(m) \quad \text{for any} \quad t \in \mathbb{R},$$

where F_t is the flow of X_h;

(iv) *the point $[m]_\mu^{(H)} := \pi_\mu^{(H)}(m)$ is a periodic point of $(M_\mu^{(H)}, \omega_\mu^{(H)}, h_\mu^{(H)})$.*

Proof. *(i) ⇒ (ii).* If m is a RPP, there is a $\tau > 0$ such that $F_\tau(m) = g \cdot m$. Applying $\mathbf{J}$ to both sides of this equality and recalling Noether's theorem and the equivariance of $\mathbf{J}$, one obtains that $\mu = g \cdot \mu$, that is, $g \in G_\mu$. Also, the G-invariance of the Hamiltonian h implies the G-equivariance of the flow F_t for any t and hence $G_m = G_{F_t(m)}$; in particular, for $t = \tau$ one has

$$H := G_m = G_{F_\tau(m)} = G_{g\cdot m} = gG_m g^{-1} = gHg^{-1},$$

which implies that $g \in N(H)$, and hence $g \in N_{G_\mu}(H)$.

(ii) $\Leftrightarrow$ *(iii)*. Take $l = gH$. The uniqueness of l is a consequence of the freeness of the action of $N(H)/H$, and hence of $N_{G_\mu}(H)/H$ on M_H.

(ii) $\Rightarrow$ *(iv)*. If, with the notation of Theorem 2.2, we apply $\pi_\mu^{(H)}$ on both sides of the equality $F_{t+\tau}(m) = g \cdot F_t(m)$ and recall that $g \in G_\mu$, we obtain that

$$\pi_\mu^{(H)}(F_{t+\tau}(m)) = \pi_\mu^{(H)}(g \cdot F_t(m)) = \pi_\mu^{(H)}(F_t(m)),$$

or equivalently

$$F_{t+\tau}^\mu([m]_\mu^{(H)}) = F_t^\mu([m]_\mu^{(H)}),$$

where F_t^μ is the flow of the Hamiltonian vector field on $M_\mu^{(H)}$ defined by the reduced Hamiltonian function $h_\mu^{(H)}$. This shows that $[m]_\mu^{(H)}$ is a periodic point of $(M_\mu^{(H)}, \omega_\mu^{(H)}, h_\mu^{(H)})$.

(iv) $\Rightarrow$ *(i)*. By hypothesis, there is a $\tau > 0$ such that $F_{t+\tau}^\mu([m]_\mu^{(H)}) = F_t^\mu([m]_\mu^{(H)})$ for any t. Thus

$$\pi_\mu^{(H)}(F_{t+\tau}(m)) = \pi_\mu^{(H)}(F_t(m)) \quad \text{for any} \quad t.$$

In particular, for $t = 0$, $(\pi_\mu^{(H)} \circ F_\tau)(m) = \pi_\mu^{(H)}(m)$, hence there exists an element $g \in G_\mu$ such that $F_\tau(m) = g \cdot m$. Thus, if t is arbitrary,

$$F_{t+\tau}(m) = (F_t \circ F_\tau)(m) = F_t(g \cdot m) = g \cdot F_t(m),$$

as required. □

3. Slices, tubes, and Poincaré maps

In this section we introduce the local tools that we will utilize in the handling of RPOs and G-spaces in general. We begin with some standard definitions and results in Lie theory, whose proofs can be found for example in [7,26].

Definition 3.1. Let M be a manifold and G a Lie group acting properly on M. Let $m \in M$ and denote $G_m := H$. A *tube* about the orbit $G \cdot m$ is a G-invariant diffeomorphism

$$\varphi : G \times_H A \longrightarrow U,$$

with U a G-invariant neighborhood of $G \cdot m$ and A some manifold in which H acts. The twist-product $G \times_H A$ is defined as the orbit space corresponding to the free and proper action of H on $G \times A$ (Theorem 3.4 guarantees the compactness of H) by $h \cdot (g, a) = (gh, h^{-1} \cdot a)$. The manifold $G \times_H A$ is naturally a G-space relative to the left action $g' \cdot [g, a] = [g'g, a]$.

Definition 3.2. Let $m \in M$ be as in Definition 3.1. Let S be a submanifold of M such that $m \in S$ and $H \cdot S = S$. We say that S is a *slice* at m if the map

$$\begin{aligned} G \times_H S &\longrightarrow U, \\ [g, s] &\longmapsto g \cdot s \end{aligned}$$

is a tube about $G \cdot m$ for some G-invariant open neighborhood of $G \cdot m$.

Theorem 3.3. *Let $m \in M$ be as in Definition 3.1 and S be a submanifold of M such that $m \in S$. Then the following statements are equivalent:*

(i) *There is a tube $\varphi : G \times_H A \longrightarrow U$ about $G \cdot m$ such that $\varphi[e, A] = S$.*

(ii) *S is a slice at m.*

(iii) *$G \cdot S$ is an open neighborhood of $G \cdot m$ and there is an equivariant retraction*

$$r : G \cdot S \longrightarrow G \cdot m$$

such that $r^{-1}(m) = S$.

Theorem 3.4 (Slice theorem). *In the conditions of Definition 3.1, there is a slice for the G-action at m. The isotropy group $G_m =: H$ is compact and M/G is Hausdorff.*

Remark 3.5. *One way to construct the slice (see [7,26]) consists roughly (we omit here some technical details) of taking a H-invariant Riemannian metric g on M (always available by the compactness of H) and letting $A = T_m(G \cdot m)^{\perp}$, where $\perp$ denotes orthogonality with respect to g. The G-equivariant map*

$$\begin{aligned} \varphi : G \times_H A &\longrightarrow U, \\ [h, a] &\longmapsto h \cdot \exp_g a \end{aligned}$$

is a tube around the orbit $G \cdot m$.

The Slice theorem, just quoted, and the Tube lemma in point set topology ([21], Lemma 5.8, p. 169) imply the following corollary that we state for future reference.

Corollary 3.6. *Let G be a compact Lie group that acts on the manifold M and let $m \in M$. Any open neighborhood V of the orbit $G \cdot m$ contains a G-invariant open neighborhood of $G \cdot m$.*

We now introduce some local tools for the study of RPOs. Proofs for these results and additional information can be found in [[9,10]].

Definition 3.7. Let $X \in \mathfrak{X}(M)$ be a G-equivariant vector field on the G-manifold M. A G-invariant local transversal section of X at $m \in M$ is a G-invariant submanifold S of codimension 1 with $m \in S$ such that for all $z \in S$, $X(z)$ is not contained in $T_z S$.

If $m \in M$ is a RPP with relative period $\tau > 0$, phase shift $g \in G$, and S is a G-invariant local transversal section at m, then a G-equivariant Poincaré map of the RPP m is a mapping $\Theta : W_0 \to W_1$ satisfying:

(RPM1) $W_0, W_1 \subset S$ are open G-invariant neighborhoods of m in S and Θ is a G-equivariant diffeomorphism;

(RPM2) there is a continuous G-invariant function, called the *period function*, such that for all $z \in W_0$, $(z, \tau - \delta(z)) \in \mathcal{D}_X$, and $\Theta(z) = F(z, \tau - \delta(z))$. The open set $\mathcal{D}_X \subset \mathcal{M} \times \mathbb{R}$ is the domain of the flow $F : \mathcal{D}_X \subset \mathcal{M} \times \mathbb{R} \to \mathcal{M}$ of X;

(RPM3) if $t \in (0, \tau - \delta(z))$, then $F(z, t) \notin W_0$.

Theorem 3.8 (Existence and uniqueness of G-equivariant Poincaré maps). *Let m, M, and X be as in Definition* 3.7.

(i) *There exists a G-invariant local transversal section S and a G-equivariant Poincaré map $\Theta : W_0 \to W_1$ for $m \in M$.*

(ii) *If $\Theta : W_0 \to W_1$ is a G-equivariant Poincaré map for m in the G-invariant local transversal section S and similarly $\Theta' : W_0' \to W_1'$ for $m' := F_{t_0}(m)$ in S', then Θ and Θ' are locally G-equivariantly conjugate, that is, there are G–invariant open neighborhoods W_2 of $m \in S$, W_2' of $m' \in S'$, and a G-equivariant diffeomorphism $\mathcal{H} : (W_2 \to W_2')$ such that $W_2 \subset W_0 \cap W_1$, $W_2' \subset W_0' \cap W_1'$, and the diagram*

$$\begin{array}{ccc} \Theta^{-1}(W_2) \cap W_2 & \xrightarrow{\Theta} & W_2 \cap \Theta(W_2) \\ \mathcal{H} \downarrow & & \downarrow \mathcal{H} \\ W_2' & \xrightarrow{\Theta'} & S' \end{array}$$

commutes.

For future reference we quote here a lemma due to Patrick (see [27] for a proof).

Lemma 3.9. *Let A and B be bilinear forms on a finite dimensional vector space. Suppose that A is positive semidefinite and that B is positive definite on* $\ker A$. *Then there exists $r > 0$ such that $A + \epsilon B$ is positive definite for all $\epsilon \in (0, r)$.*

4. The symmetric energy-integrals method

We shall work generally on a Poisson manifold, that is, a manifold M whose space of functions $C^\infty(\mathcal{M})$ admits a bracket $\{\cdot\,,\cdot\}$ relative to which it is a Lie algebra and the Leibniz identity holds in each argument. The Hamiltonian vector field X_h given by a function $h \in C^\infty(\mathcal{M})$ is defined (as a derivation) by the relation $X_h = \{\cdot\,, h\}$. The elements of the center of the Lie algebra $(C^\infty(\mathcal{M}), \{\cdot\,,\cdot\})$ are called *Casimir functions*. The triplet $(M, \{\cdot\,,\cdot\}, h)$ is called a *Poisson system*. Any Poisson manifold is partitioned into symplectic leaves, which are connected immersed symplectic manifolds of M by the inclusion map, whose own Poisson bracket coincides with the given one on M. The tangent space at m to a leaf consists of all vectors that are equal to the value of some Hamiltonian vector field at m. The symplectic leaves are invariant under the flow of any Hamiltonian vector field.

We can state now the main result on the stability of RPPs.

Theorem 4.1 (The symmetric energy-integrals method). *Let $(M, \{\cdot,\cdot\}, G, \mathbf{J} : M \to \mathfrak{g}^*, h : M \to \mathbb{R})$ be a Hamiltonian system with a symmetry given by the Lie group G acting properly on M. Assume that the Hamiltonian $h \in C^\infty(M)$ is G-invariant and that $\mathbf{J}$ is equivariant. Let $m \in M$ be a RPP such that $\mathbf{J}(m) = \mu \in \mathfrak{g}^*$ and the coadjoint isotropy subgroup G_μ is compact. Then, if there is a set of G_μ-invariant conserved quantities $C_1, \ldots, C_n \in C^\infty(M)$, for which*

$$\mathbf{d}(C_1 + \cdots + C_n)(m) = 0,$$

and

$$\mathbf{d}^2(C_1 + \cdots + C_n)(m)|_{W\times W}$$

is definite for some (and hence for any) W such that

$$\begin{aligned}&\ker \mathbf{d}C_1(m) \cap \cdots \cap \ker \mathbf{d}C_n(m) \cap \ker T_m\mathbf{J}\\ &\quad = W \oplus (\operatorname{span}\{X_h(m)\} + T_m(G_\mu \cdot m)),\end{aligned} \tag{4.1}$$

then m is a G_μ-stable RPP. If $\dim W = 0$, then m is always a G_μ-stable RPP. In what follows, the matrix $\mathbf{d}^2(C_1 + \cdots + C_n)(m)|_{W\times W}$ will be referred to as the stability form.

Proof. We first prove the case $W \neq \{0\}$ and we begin by showing that the result does not depend on the choices of m in the RPO and W. Indeed, if $\mathbf{d}(C_1 + \cdots + C_n)(m) = 0$ and F_t is the flow of the Hamiltonian vector field X_h, then for any $t > 0$ and any $v, w \in T_mM$ we have

$$\begin{aligned}&\mathbf{d}(C_1 + \cdots + C_n)(F_t(m))(T_mF_t(v), T_mF_t(w))\\ &\quad = F_t^*(\mathbf{d}(C_1 + \cdots + C_n)(m))(v, w)\\ &\quad = \mathbf{d}(F_t^*(C_1 + \cdots + C_n))(m)(v, w)\\ &\quad = \mathbf{d}(C_1 + \cdots + C_n)(m)(v, w),\end{aligned}$$

since $F_t^* \circ \mathbf{d} = \mathbf{d} \circ F_t^*$ and $C_1, C_2, \ldots, C_n$ are invariant under F_t. If W is a complement to $(\operatorname{span}\{X_h(m)\} + T_m(G_\mu \cdot m))$ in $\ker \mathbf{d}C_1(m) \cap \cdots \cap \ker \mathbf{d}C_n(m) \cap \ker T_m\mathbf{J}$, then for any $t > 0$, $T_mF_t(W)$ is a complement to $(\operatorname{span}\{X_h(F_t(m))\} + T_m(G_\mu \cdot F_t(m)))$ in $\ker \mathbf{d}C_1(F_t(m)) \cap \cdots \cap \ker \mathbf{d}C_n(F_t(m)) \cap \ker T_{F_t(m)}\mathbf{J}$. Moreover, $\mathbf{d}^2(C_1 + \cdots + C_n)(m)|_{W\times W}$ is definite iff $\mathbf{d}^2(C_1 + \cdots + C_n)(F_t(m))|_{T_mF_t\cdot W \times T_mF_t\cdot W}$ is definite, since the conservation of $C_1, \ldots, C_n$, implies that for any $v, w \in T_mM$:

$$\begin{aligned}&\mathbf{d}^2(C_1 + \cdots + C_n)(F_t(m))(T_mF_t(v), T_mF_t(w))\\ &\quad = \mathbf{d}^2(F_t^*C_1 + \cdots + F_t^*C_n)(m)(v, w)\\ &\quad = \mathbf{d}^2(C_1 + \cdots + C_n)(m)(v, w).\end{aligned}$$

The statement of the theorem does therefore not depend on the choice of the point m in the RPO.

The choice of W is also irrelevant since $\mathbf{d}^2(C_1 + \cdots + C_n)(m)(v, w) = 0$ whenever $v \in \text{span}\{X_h(m)\} + T_m(G_\mu \cdot m)$. Indeed if we take without loss of generality $v = X_h(m) + \xi_M(m) = X_h(m) + X_{\mathbf{J}^\xi}(m)$ with $\xi \in \mathfrak{g}_\mu$, then

$$\begin{aligned}\mathbf{d}^2(C_1 + \cdots + C_n)(m)(v, w) &= w[(X_h + X_{\mathbf{J}^\xi})[C_1 + \cdots + C_n]] \\ &= w[\{C_1, h\} + \cdots + \{C_n, h\} + \{C_1, \mathbf{J}^\xi\} + \cdots + \{C_n, \mathbf{J}^\xi\}] = 0,\end{aligned}$$

since the functions C_i, for $i \in \{1, \ldots, n\}$, are G_μ-invariant conserved quantities for the evolution induced by h and therefore $\{C_i, h\} = 0$, and for any $z \in M$, $\{C_i, \mathbf{J}^\xi\}(z) = \mathbf{d}C_i(z) \cdot \xi_M(z) = 0$ by G_μ-invariance.

We now construct a G_μ-invariant local transversal section for X_h at m with the help of the Slice theorem. Since G_μ is closed in G and G acts properly on M, so does G_μ. Therefore, there is a G_μ-invariant neighborhood of $G_\mu \cdot m$ that can be represented as a G_μ-space by

$$Y_\mu = G_\mu \times_H B,$$

with B a H-vector space constructed as in Remark 3.5, and where the point m is represented by $[e, 0]$. For times t small enough, the flow F_t is represented in these coordinates by $F_t[e, 0] = [g(t), b(t)]$, where $g(t) \in G_\mu$, and $b(t) \in B^H$. Indeed, by the G-equivariance of F_t, $H = G_{[e,0]} = G_{[g(t),b(t)]}$, hence $[g(t), b(t)] \in (Y_\mu)_H = N_{G_\mu}(H) \times_H B^H$, which implies that $g'(0) \in \text{Lie}(N_{G_\mu}(H))$ and $b := b'(0) \in B^H$. Notice that since m is not a relative equilibrium, $b \neq 0$ necessarily. The subspace span$\{b\}$ of $B^H \subset B$ is H-invariant in B. The compactness of H and a standard result in representation theory of compact Lie groups (see for example Proposition 2.1 in [11]) guarantees the existence of a H-invariant subspace $B_I \subset B$ such that

$$B = \text{span}\{b\} \oplus B_I.$$

The set $G_\mu \times_H B_I$ is a submanifold of Y_μ, and by construction, there is a G_μ-invariant neighborhood T of $m \equiv [e, 0]$ in $G_\mu \times_H B_I$ such that for any $z \in T$, $X_h(z) \notin T_zT$, that is, T is a G_μ-invariant local transversal section to X_h at m. We now define S as the submanifold of T given by

$$S := (H \times_H B_I) \cap T,$$

and prove the following lemma.

Lemma 4.2. *With the notation previously introduced, the submanifold S satisfies*

$$T_mM = T_mS \oplus T_m(G_\mu \cdot m) \oplus \text{span}\{X_h(m)\}.$$

Proof. Note first that the sum $T_m(G_\mu \cdot m) \oplus \text{span}\{X_h(m)\}$ is indeed direct since there is no $\xi \in \mathfrak{g}_\mu$ for which $X_h(m) = \xi_M(m)$, for this equality is equivalent to m being a relative equilibrium (see [1]) which we assume is not the case; m is assumed to be a genuine RPP.

Second, we show that the sum $T_mS + (T_m(G_\mu \cdot m) \oplus \text{span}\{X_h(m)\})$ is also direct by showing that $(T_m(G_\mu \cdot m) \oplus \text{span}\{X_h(m)\}) \cap T_mS = \{0\}$. Indeed, since m is identified with

$[e, 0]$ we conclude that the $\mathfrak{h}$-action on $\mathfrak{g}_\mu \times B$ (which is the linearization of the H-action at $(e, 0)$) is just the translation by $\mathfrak{h} \times \{0\}$ in $\mathfrak{g}_\mu \times B$. Thus, we can identify $T_m M \cong \mathfrak{g}_\mu/\mathfrak{h} \times B$, $T_m S \cong \{0\} \times B_I$, and $T_m(G_\mu \cdot m) \cong (\mathfrak{g}_\mu/\mathfrak{h}) \times \{0\}$. Finally, $X_h(m)$ is represented by a pair of the form $(\xi + \eta, b)$, for some $\eta \in \mathfrak{h}$, $\xi \in \mathrm{Lie}(N_{G_\mu}(H)) \subset \mathfrak{g}_\mu$, and $b \notin B_I$. Thus, an arbitrary vector $w \in T_m(G_\mu \cdot m) \oplus \mathrm{span}\{X_h(m)\}$ is represented by $(\rho + \eta, b)$, for $\eta \in \mathfrak{h}$, $\rho \in \mathfrak{g}_\mu$, and $b \in B$. However, if this vector is also in $T_m S$, then this representative must lie in $\{0\} \times B_I$, that is, $\rho = -\eta \in \mathfrak{h}$ and $b \in B_I \cap \mathrm{span}\{b\} = \{0\}$. Therefore, this representative must be an element of $\mathfrak{h} \times \{0\}$ which in the quotient is the zero vector; we showed that $w = 0$.

Third, $\dim(T_m S) = \dim B - 1 = \dim M - \dim G_\mu + \dim H - 1$, $\dim(T_m(G_\mu \cdot m)) = \dim G_\mu - \dim H$, and $\dim(\mathrm{span}\{X_h(m)\}) = 1$, hence

$$\dim(T_m S \oplus T_m(G_\mu \cdot m) \oplus \mathrm{span}\{X_h(m)\}) = \dim M = \dim(T_m M),$$

which concludes the proof. ∇

We now define

$$Z := T_m S \cap \ker T_m \mathbf{J} \cap \ker \mathbf{d}C_1(m) \cap \cdots \cap \ker \mathbf{d}C_n(m).$$

The reduction lemma ($T_m(G_\mu \cdot m) = \ker T_m\mathbf{J} \cap T_m(G \cdot m)$; see [1]), Noether's theorem, and the choice of the functions $C_1, \ldots, C_n$ imply that

$$T_m(G_\mu \cdot m) + \mathrm{span}\{X_h(m)\} \subset \ker \mathbf{d}C_1(m) \cap \cdots \cap \ker \mathbf{d}C_n(m) \cap \ker T_m\mathbf{J}.$$

This inclusion and Lemma 4.2 allow us to write

$$\begin{aligned}
&\ker \mathbf{d}C_1(m) \cap \cdots \cap \ker \mathbf{d}C_n(m) \cap \ker T_m\mathbf{J} \\
&\quad = T_m M \cap \ker \mathbf{d}C_1(m) \cap \cdots \cap \ker \mathbf{d}C_n(m) \cap \ker T_m\mathbf{J} \\
&\quad = (T_m S \cap \ker T_m\mathbf{J} \cap \ker \mathbf{d}C_1(m) \cap \cdots \cap \ker \mathbf{d}C_n(m)) \\
&\qquad \oplus (T_m(G_\mu \cdot m) + \mathrm{span}\{X_h(m)\}) \\
&\quad = Z \oplus (T_m(G_\mu \cdot m) + \mathrm{span}\{X_h(m)\}),
\end{aligned}$$

hence Z is one of the spaces that satisfy the defining conditions of W in the statement of the theorem.

Let now f_1 and f_2 be the functions defined by

$$\begin{aligned}
f_1 &= (C_1 - C_1(m)) + \cdots + (C_n - C_n(m)), \\
f_2 &= (C_1 - C_1(m))^2 + \cdots + (C_n - C_n(m))^2 + \|\mathbf{J} - \mu\|^2,
\end{aligned}$$

where in f_2, the norm is associated to some $\mathrm{Ad}^*_{G_\mu}$-invariant inner product in $\mathfrak{g}^*$ (always available by the compactness of G_μ); this makes f_2 G_μ-invariant. Since f_1 and $C_1 + \cdots + C_n$ differ by a constant, the hypothesis of the theorem implies that $\mathbf{d}f_1(m) = \mathbf{d}f_2(m) = 0$, and that the form $\mathbf{d}^2 f_1|_{Z \times Z}$ is definite. Taking into account that

$$\mathbf{d}^2 f_1(m)|_{Z \times Z} = (\mathbf{d}^2 f_1(m)|_{T_m S \times T_m S})|_{Z \times Z} = \mathbf{d}^2(f_1|_S)(m)|_{Z \times Z},$$

the hypothesis of the theorem implies that $\mathbf{d}^2(f_1|_S)(m)|_{Z \times Z}$ is definite.

We now prove that Z is the kernel of $\mathbf{d}^2(f_2|_S)(m)$. It is easy to see that if $v_1, v_2 \in T_m S$ then

$$\begin{aligned}&\mathbf{d}^2(f_2|_S)(m)(v_1, v_2)\\&\quad = 2[(\mathbf{d}C_1(m)\cdot v_2)(\mathbf{d}C_1(m)\cdot v_1) + \cdots + (\mathbf{d}C_n(m)\cdot v_2)(\mathbf{d}C_n(m)\cdot v_1)\\&\qquad + \|T_m\mathbf{J}\cdot v_1\|\|T_m\mathbf{J}\cdot v_2\|].\end{aligned}$$

Then, $v_1 \in \ker \mathbf{d}^2(f_2|_S)(m)$ iff for any $v_2 \in T_m S$, we have that

$$\begin{aligned}&(\mathbf{d}C_1(m)\cdot v_2)(\mathbf{d}C_1(m)\cdot v_1) + \cdots + (\mathbf{d}C_n(m)\cdot v_2)(\mathbf{d}C_n(m)\cdot v_1)\\&\quad + \|T_m\mathbf{J}\cdot v_1\|\|T_m\mathbf{J}\cdot v_2\| = 0.\end{aligned}$$

In particular, for $v_1 = v_2$, this identity implies that $\mathbf{d}C_1(m)\cdot v_1 = \cdots = \mathbf{d}C_n(m)\cdot v_1 = \|T_m\mathbf{J}\cdot v_1\| = 0$ and hence $v_1 \in \ker T_m\mathbf{J} \cap \ker \mathbf{d}C_1(m) \cap \cdots \cap \ker \mathbf{d}C_n(m) \cap T_m S = Z$. Conversely, if $v_1 \in Z = T_m S \cap \ker \mathbf{d}C_1(m) \cap \cdots \cap \ker \mathbf{d}C_n(m) \cap \ker T_m\mathbf{J}$ the above relation is satisfied trivially for all $v_1 \in T_m S$. Therefore,

$$Z = \ker \mathbf{d}^2(f_2|_S)(m).$$

Using these remarks, Lemma 3.9 guarantees the existence of some $a > 0$ for which the function f defined by

$$f := af_1 + f_2 \tag{4.2}$$

is such that $\mathbf{d}^2(f|_S)(m)$ is positive definite. Note that f is a G_μ-invariant integral of the motion such that $f(m) = 0$. Shrinking T if necessary, the Morse lemma allows us to choose S such that $f \geq 0$ on S. We now prove the following lemma.

Lemma 4.3. *The submanifold S is a slice at $m \in T \subset M$ for the G_μ-action on T.*

Proof. By Theorem 3.3 (iii), it is enough to prove that $G_\mu \cdot S$ is an open neighborhood of $G_\mu \cdot m$ in T and that there is a G_μ-equivariant retraction $r : G_\mu \cdot S \to G_\mu \cdot m$ such that $r^{-1}(m) = S$. Without loss of generality we may take $S = H \times_H B_l$, and $T = G_\mu \times_H B_l$. Clearly $G_\mu \cdot S = T$, which is trivially open in T, and the equivariant retraction that we need is

$$\begin{aligned} r : G_\mu \cdot S = T &\longrightarrow G_\mu \cdot m,\\ [g, b] &\longmapsto [g, 0] \equiv g \cdot m.\end{aligned}$$

The map r is clearly well-defined, it is G_μ-equivariant, and $r^{-1}(m) = \{[h, b] \in T \mid h \in H, b \in B_l\} = S$. ∇

Notice that Theorem 3.3 (i) guarantees that the G_μ-invariant local transversal section T is locally diffeomorphic to $G_\mu \times_H S \equiv G_\mu \cdot S$.

We now recall that, by Theorem 2.6 (ii), the RPP m has a phase shift in G_μ. The dynamic orbit through m can therefore be considered as a RPO associated to the G_μ symmetry of the system, and taking T as a G_μ-invariant local transversal section, Theorem 3.8 guarantees the existence of a G_μ-equivariant Poincaré section Θ, with G_μ-invariant open sets W_0, $W_1 \subset T$, and G_μ-invariant period function $\tau : W_0 \to \mathbb{R}$.

With all these tools we will prove the G_μ-stability of m. Let V be an arbitrary G_μ-invariant open neighborhood of $G_\mu \cdot \{F_t(m)\}_{t>0}$. The positive definiteness of $\mathbf{d}^2(f|_S)(m)$ and the Morse lemma guarantee the existence of certain $\epsilon > 0$ such that

$$f^{-1}[0, \epsilon) \cap S \subset V \cap W_0 \cap W_1, \tag{4.3}$$

with $f^{-1}[0, \epsilon) \cap S$ an open subset of S. Notice that the G_μ-invariance of f implies that $f^{-1}[0, \epsilon)$ is G_μ-invariant, in particular H-invariant. This allows us to define the open submanifold A of T as

$$A := G_\mu \times_H (f^{-1}[0, \epsilon) \cap S) \equiv G_\mu \cdot (f^{-1}[0, \epsilon) \cap S).$$

The G_μ-invariance of V, W_0 and W_1 and (4.3) guarantee that

$$A = G_\mu(f^{-1}[0, \epsilon) \cap S) \subset G_\mu \cdot (V \cap W_0 \cap W_1) = V \cap W_0 \cap W_1. \tag{4.4}$$

We now show that if $\pi : M \to M/G_\mu$ is the continuous canonical projection of M onto the Hausdorff quotient topological space, then the closed orbit γ in M/G_μ corresponding to the RPP m, that is $\gamma = \pi(\{F_t(m)\}_{t\geq 0})$, is orbitally stable.

The G_μ-invariance of V, W_0, W_1, T, and A allows us to define $\widehat{V} = V/G_\mu = \pi(V)$, and analogously, $\widehat{W_0}$, $\widehat{W_1}$, $\widehat{T}$, and $\widehat{A}$. Also, let $\widehat{\Theta}$, $\widehat{\delta}$ and F_t^μ be the continuous maps uniquely determined by the equalities: $\widehat{\Theta} \circ \pi = \pi \circ \Theta$, $\widehat{\delta} \circ \pi = \delta$, and $F_t^\mu \circ \pi = \pi \circ F_t$, with F_t the G-equivariant Hamiltonian flow of X_h. Note that by construction $\widehat{A}$, $\widehat{W_0}$, and $\widehat{W_1}$ are included in $\widehat{T}$ and that for any $[z] = \pi(z) \in \widehat{W_0}$, $\widehat{\Theta}([z]) = F^\mu([z], \tau - \widehat{\delta}([z])) \in \widehat{W_1}$.

We now see how the G_μ-invariance of f guarantees that if $[z] \in \widehat{A}$, then $\widehat{\Theta}([z]) \in \widehat{A}$. Indeed, if $z = l \cdot a$ with $l \in G_\mu$ and $a \in f^{-1}[0, \epsilon) \cap S \subset W_0 \cap W_1 \cap S$, then $\Theta(z) = F_{\tau-\delta(z)}(z) = l \cdot F_{\tau-\delta(l\cdot a)}(a)$. By the G_μ-invariance of δ, $F_{\tau-\delta(l\cdot a)}(a) = F_{\tau-\delta(a)}(a)$, and as $a \in W_0$, $F_{\tau-\delta(a)}(a) \in W_1 \subset T$ necessarily. Since by Lemma 4.3, $T \equiv G_\mu \times_H S$, there are elements $n \in G_\mu$, and $b \in S$ such that $F_{\tau-\delta(a)}(a) = n \cdot b$. Now, since f is a G_μ-invariant conserved quantity

$$f(\Theta(z)) = f(F_{\tau-\delta(z)}(z)) = f(z) = f(a);$$

but, at the same time

$$f(\Theta(z)) = f(F_{\tau-\delta(z)}(z)) = f(ln \cdot b) = f(b) = f(a),$$

which guarantees that $b \in f^{-1}[0, \epsilon) \cap S \subset W_0 \cap W_1 \cap S$ and hence $\widehat{\Theta}([z]) = [b] \in \widehat{A}$.

Note that expression (4.4) implies that

$$\widehat{A} \subset \widehat{W_0} \cap \widehat{W_1} \cap \widehat{V},$$

since

$$\widehat{A} = \pi(A) \subset \pi(W_0 \cap W_1 \cap V) \subset \pi(W_0) \cap \pi(W_1) \cap \pi(V) = \widehat{W_0} \cap \widehat{W_1} \cap \widehat{V}.$$

Also, the G_μ-invariance of V (resp. A) guarantees that $\widehat{V}$ (resp. $\widehat{A}$) is open in the quotient topology of M/G_μ: $\widehat{V}$ is open iff $\pi^{-1}(\widehat{V})$ is open in M. We show that $\pi^{-1}(\widehat{V}) = V$. Indeed, it is always true that $V \subset \pi^{-1}(\pi(V)) = \pi^{-1}(\widehat{V})$. Conversely, if $z \in \pi^{-1}(\widehat{V})$, $\pi(z) = \pi(v)$ for some $v \in V$, hence $z = l \cdot v$ for some $l \in G_\mu$. The G_μ-invariance of V guarantees that $z \in V$, and therefore $V = \pi^{-1}(\widehat{V})$.

Notice that since $\gamma = F^\mu([0, \tau], [m])$ and F^μ is continuous, γ is a compact subset of M/G_μ. This fact, together with the openness of $\widehat{V}$, implies that the number $D_{\widehat{V}}$ defined by

$$D_{\widehat{V}} = \inf\{d([x], \gamma) | [x] \in \mathrm{Cl}\widehat{V} \setminus \widehat{V}\}$$

is never zero, where d is the distance function associated to a metric on M/G_μ that induces its quotient topology and $\mathrm{Cl}\widehat{V}$ is the point set topological closure of the set $\widehat{V}$. This metric always exists and can be constructed as follows: take a G_μ-invariant Riemannian metric on M (always available by the compactness of G_μ). This makes M into a metric space whose metric topology coincides with the topology of M (see [8, Proposition 10.6.2]). With this metric, G_μ acts by isometries, which implies that the distance function drops to M/G_μ, endowing it with a metric structure.

We define the map:

$$\begin{aligned} D : \widehat{A} &\longrightarrow \mathbb{R}, \\ [z] &\longmapsto D([z]) := \max_{t \in [0, \tau - \widehat{\delta}([z])]} d(F_t^\mu([z]), \gamma). \end{aligned}$$

Note that $D([m]) = 0$. By the continuity of D, we can choose $\epsilon > 0$ (and therefore $\widehat{A}$) small enough so that $D([z]) < D_{\widehat{V}}/2$ for any $[z] \in \widehat{A}$. Define the open neighborhood $\widehat{U}$ of γ by

$$\widehat{U} := \{F_t^\mu([z']) \mid [z'] \in \widehat{A}, t \geq 0\}.$$

We shall prove below that $F_t^\mu(\widehat{U}) \subset \widehat{V}$ for all $t \geq 0$. In order to see this, note that, by construction, $\widehat{U}$ is invariant under the flow F_t^μ and hence the claim is proved if we show that $\widehat{U} \subset \widehat{V}$. Let us suppose the contrary, namely that there is an element $F_t^\mu([z']) \in \widehat{U}$, $[z'] \in \widehat{A}$ such that $F_t^\mu([z']) \notin \widehat{V}$. Without loss of generality we can assume that $t \in [0, \tau - \widehat{\delta}([z'])]$ which then implies that $d(F_t^\mu([z']), \gamma) \leq D([z']) < D_{\widehat{V}}/2$. However, since we assume that $F_t^\mu([z']) \notin \widehat{V}$, it follows that $d(F_t^\mu([z']), \gamma) \geq D_{\widehat{V}}$ by the definition of $D_{\widehat{V}}$. This contradiction guarantees that γ is orbitally stable. We now see how the orbital stability of γ implies the G_μ-stability of m, taking as the open set that we need $U = \pi^{-1}(\widehat{U})$, that is, $F_t(z) \in V$ for any positive time t and any $z \in U$: we know that $F_t^\mu([z]) \in \widehat{V} = \pi(V)$ for any positive t. Since F_t^μ is defined by the relation

$$\pi \circ F_t = F_t^\mu \circ \pi,$$

it follows that $\pi \circ F_t(z) = \pi(v)$ for some $v \in V$. Hence there exists some $h \in G_\mu$ such that $F_t(z) = h \cdot v$ but since V is G_μ-invariant, $h \cdot v \in V$, and therefore $F_t(U) \subset V$ as required. This proves the case $\dim W \neq 0$.

If $W = \{0\}$, the proof is completely analogous, but in this case one takes f_2 as f, given that $\{0\} = Z = \ker \mathbf{d}^2(f_2|_S)(m)$, and hence $\mathbf{d}^2(f_2|_S)(m)$ is positive definite. □

Example 4.4. The previous theorem applies to the collision solutions of the examples presented in [24] and that could not be treated there, given their singular nature:

(i) *The spherical pendulum.* If the angular momentum of the pendulum is equal to zero, the spherical pendulum becomes a planar pendulum. However, zero is not a regular value of the momentum map associated to the $SO(2)$ symmetry of the system, hence we need Theorem 4.1 to deal with the stability of these solutions that consist of an equilibrium, when the energy of the system equals $-mgl$, and a set of planar periodic orbits when the energy is bigger. These periodic orbits are not orbitally stable since a small perturbation on p_φ makes the system precess. However, as trivial RPOs (RPOs whose phase shift is the identity), they are S^1-stable, which follows from Theorem 4.1 by showing that in this case $W = \{0\}$.

(ii) *The Manev problem.* In this case the situation is analogous. When the angular momentum equals zero, we have a collision problem, that is, apart from the equilibrium solution, the particle describes an oscillatory motion on a straight line that goes through the origin. If we consider these motions as periodic orbits, they are not orbitally stable however, as trivial RPOs, Theorem 4.1 shows that they are $SO(3)$-stable ($W = \{0\}$).

The approach followed in the proof of Theorem 4.1 allows us to generalize the *energy–momentum method* (see [15,16,20,23,27,28]), that we briefly review. This method is designed to study the stability of *relative equilibria* in symmetric Hamiltonian systems. If $(M, \{\cdot,\cdot\}, h, G, \mathbf{J} : M \to \mathfrak{g}^*)$ is a Hamiltonian system with symmetry, in which G acts properly on M, recall that a relative equilibrium associated to the dynamics induced by the G-invariant Hamiltonian h is a point $m \in M$ such that the integral curve $m(t)$ of the Hamiltonian vector field X_h starting at m equals $\exp(t\xi) \cdot m$ for some $\xi \in \mathfrak{g}$, where $\exp : \mathfrak{g} \to G$ is the exponential map; any such ξ is called a *velocity* of the relative equilibrium. Note that if m has a non-trivial isotropy subgroup, ξ is not uniquely determined; this leads us to define the concept of *orthogonal velocity* of a relative equilibrium. If our relative equilibrium $m \in M$ is such that $H := G_m$ and $\mathbf{J}(m) = \mu \in \mathfrak{g}^*$, it can be proved (see [23]) that there is a unique $\lambda \in \mathrm{Lie}(N_{G_\mu}(H)/H) \subset \mathrm{Lie}(N(H)/H) := \mathrm{Lie}(L)$ such that

$$F_t(m) = \exp_L t\lambda \cdot m,$$

where F_t is the Hamiltonian flow of X_h. The properness of the G-action allows us to choose an Ad_H-invariant inner product in $\mathfrak{n}_\mu := \mathrm{Lie}(N_{G_\mu}(H))$ and hence we have an orthogonal direct sum decomposition

$$\mathfrak{n}_\mu = \mathfrak{h} \oplus \mathfrak{p}_\mu, \tag{4.5}$$

where $\mathfrak{p}_\mu$ is the orthocomplement of $\mathfrak{h}$ in $\mathfrak{n}_\mu$ relative to the inner product on $\mathfrak{n}_\mu$.

If we consider the quotient Lie group $N_{G_\mu}(H)/H$, the canonical projection

$$\pi : N_{G_\mu}(H) \to N_{G_\mu}(H)/H$$

is a surjective submersion and therefore $\ker T_e\pi = (T_e\pi)^{-1}([e]) = \mathfrak{h}$, which implies that

$$\mathrm{Lie}(N_{G_\mu}(H)/H) \simeq \mathfrak{n}_\mu/\mathfrak{h} \simeq \mathfrak{p}_\mu. \tag{4.6}$$

Let $\xi \in \mathfrak{p}_\mu \subset \mathfrak{n}_\mu$ be the unique image of $\lambda \in \mathrm{Lie}(N_{G_\mu}(H)/H)$ under the isomorphism in (4.6). Since π is a group homomorphism, we can write

$$F_t(m) = \exp_L t\lambda \cdot m = \exp t\xi \cdot m.$$

Definition 4.5. The unique element $\xi \in \mathfrak{p}_\mu$ just defined is called the *orthogonal velocity* of the relative equilibrium $m \in M$ relative to the splitting (4.5).

Due to the possibility of group drift, the definition of stability that one uses when dealing with relative equilibria is similar in spirit to the case of the RPOs.

Definition 4.6. Let $(M, \{\cdot,\cdot\}, h, G, \mathbf{J} : M \to \mathfrak{g}^*)$ be a Hamiltonian system with symmetry and let G' be a subgroup of G. A relative equilibrium $m \in M$ is called *G'-stable*, or *stable modulo G'*, if for any G'-invariant open neighborhood V of the orbit $G' \cdot m$, there is an open neighborhood $U \subseteq V$ of m such that if F_t is the flow of the Hamiltonian vector field X_h and $u \in U$, then $F_t(u) \in V$ for all $t \geq 0$.

The energy–momentum method, as stated in [23], is described in the following theorem.

Theorem 4.7. *Let $(M, \{\cdot,\cdot\}, h)$ be a Poisson system with a symmetry given by the Lie group G acting properly on M in a globally Hamiltonian fashion, with associated equivariant momentum map $\mathbf{J} : M \to \mathfrak{g}^*$. Assume that the Hamiltonian $h \in C^\infty(M)$ is G-invariant. Let $m \in M$ be a relative equilibrium such that $\mathbf{J}(m) = \mu \in \mathfrak{g}^*$, $\mathfrak{g}^*$ admits an $\mathrm{Ad}^*{}_{G_\mu}$-invariant inner product, $H := G_m$, and $\xi \in \mathrm{Lie}(N_{G_\mu}(H))$ is its orthogonal velocity relative to a given Ad_H-invariant splitting. If the quadratic form*

$$\mathbf{d}^2(h - \mathbf{J}^\xi)(m)|_{W \times W}$$

is definite for some (and hence for any) subspace W such that

$$\ker T_m\mathbf{J} = W \oplus T_m(G_\mu \cdot m),$$

then m is a G_μ-stable relative equilibrium. If $\dim W = 0$, then m is always a G_μ-stable relative equilibrium.

This can be generalized as follows.

Theorem 4.8 (Generalized energy-momentum-method). *Let $(M, \{\cdot,\cdot\}, G, \mathbf{J} : M \to \mathfrak{g}^*, h : M \to \mathbb{R})$ be a Poisson system with a symmetry given by the Lie group G acting properly on M. Assume that the Hamiltonian $h \in C^\infty(M)$ is G-invariant and that $\mathbf{J}$ is equivariant. Let $m \in M$ be a relative equilibrium such that $\mathbf{J}(m) = \mu \in \mathfrak{g}^*$, G_μ is compact, $H := G_m$, and $\xi \in \mathrm{Lie}(N_{G_\mu}(H))$ is its orthogonal velocity, relative to a given Ad_H-invariant splitting. If there is a set of G_μ-invariant conserved quantities $C_1, \ldots, C_n : M \to \mathbb{R}$ for which*

$$\mathbf{d}(h - \mathbf{J}^\xi + C_1 + \cdots + C_n)(m) = 0,$$

and

$$\mathbf{d}^2(h - \mathbf{J}^\xi + C_1 + \cdots + C_n)(m)|_{W \times W}$$

is definite for some (and hence for any) W such that

$$\ker \mathbf{d}C^1(m) \cap \cdots \cap \ker \mathbf{d}C^n(m) \cap \ker T_m\mathbf{J} = W \oplus T_m(G_\mu \cdot m),$$

then m is a G_μ-stable relative equilibrium. If $\dim W = 0$, *then m is always a G_μ-stable relative equilibrium.*

Proof. We first suppose that $W \neq \{0\}$. One shows as in Theorem 4.1 that the result does not depend on the choice of m in the relative equilibrium. The choice of W is also irrelevant since

$$\mathbf{d}^2(h - \mathbf{J}^\xi + C_1 + \cdots + C_n)(m)(v, w) = 0,$$

whenever $v \in T_m(G_\mu \cdot m)$, and $w \in \ker T_m\mathbf{J} \cap \ker \mathbf{d}C_1(m) \cap \cdots \cap \ker \mathbf{d}C_n(m)$. Indeed, if we take $v = \eta_M(m)$ with $\eta \in \mathfrak{g}_\mu$, then

$$\begin{aligned}\mathbf{d}^2(h - \mathbf{J}^\xi + C_1 + \cdots + C_n)(m)(v, w) &= w[X_{\mathbf{J}^\eta}[h - \mathbf{J}^\xi + C_1 + \cdots + C_n]] \\ &= w[\{h, \mathbf{J}^\eta\} - \mathbf{J}^{[\xi,\eta]} + \{C_1, \mathbf{J}^\eta\} + \cdots + \{C_n, \mathbf{J}^\eta\}] \\ &= w[\mathbf{J}^{[\xi,\eta]}] = 0,\end{aligned}$$

where we used the G_μ-invariance of $h, C_1, \ldots, C_n$, and that $w \in \ker T_m\mathbf{J}$.

Let now $T := G_\mu \times_H W$ be a tube around the orbit $G_\mu \cdot m$ associated to the Hamiltonian action of G_μ on M. We denote by S the submanifold of T given by

$$S := H \times_H W.$$

It can be easily shown (see Lemma 4.2) that

$$T_mM = T_mS \oplus T_m(G_\mu \cdot m). \tag{4.7}$$

Now let it be

$$Z := T_mS \cap \ker T_m\mathbf{J} \cap \ker \mathbf{d}C_1(m) \cap \cdots \cap \ker \mathbf{d}C_n(m). \tag{4.8}$$

Since $T_m(G_\mu \cdot m) \subset \ker T_m\mathbf{J} \cap \ker \mathbf{d}C_1(m) \cap \cdots \cap \ker \mathbf{d}C_n(m)$, by (4.7) and (4.8), we have that

$$\ker T_m\mathbf{J} \cap \ker \mathbf{d}C_1(m) \cap \cdots \cap \ker \mathbf{d}C_n(m) = Z \oplus T_m(G_\mu \cdot m);$$

hence Z satisfies the requirements of W in the statement of the theorem.

We now introduce *Patrick velocity map*. We start by recalling a lemma whose proof can be found in [23].

Lemma 4.9. *Fix a splitting* (4.5) *and let $\xi \in \mathfrak{p}_\mu$ be the corresponding orthogonal velocity of the relative equilibrium $m \in M$ whose symmetry group is $H := G_m$. Then* $\mathrm{Ad}_h\xi = \xi$ *for any $h \in H$.*

Following a strategy identical to Lemma 4.3, it is easy to see that S is a slice at m for the G_μ-action; in other words $G_\mu \times_H S \equiv G_\mu \cdot S = T$ is an open G_μ-invariant neighborhood of the orbit $G_\mu \cdot m$ and there is a G_μ-equivariant retraction

$$\begin{aligned} r : G_\mu \cdot S &\longrightarrow G_\mu \cdot m, \\ [g, w] &\longmapsto g \cdot m. \end{aligned}$$

We define

$$\begin{aligned} \tilde{\Psi} : G_\mu \cdot m &\longrightarrow G_\mu \cdot \xi, \\ g \cdot m &\longmapsto \mathrm{Ad}_g \xi \end{aligned}$$

with ξ the orthogonal velocity of the relative equilibrium. The previous lemma guarantees that $\tilde{\Psi}$ is well–defined: if $g \cdot m = g' \cdot m$ then $g^{-1} g' \in H$ and therefore $g^{-1} g' \cdot \xi = \xi$ and so $g' \cdot \xi = g \cdot \xi$. We define Patrick's velocity map as $\Psi := \tilde{\Psi} \circ r : g \cdot z \in G_\mu \cdot S \mapsto \mathrm{Ad}_g \xi \in G_\mu \cdot \xi$. Note that $\Psi(m) = \tilde{\Psi}(m) = \xi$ and that for any $g \in G_\mu$ and any $z = g' \cdot z' \in G_\mu \cdot S$,

$$\Psi(g \cdot z) = \Psi(gg' \cdot z') = \mathrm{Ad}_{gg'} \xi = \mathrm{Ad}_g(\mathrm{Ad}_{g'} \xi) = \mathrm{Ad}_g \Psi(g' \cdot z') = \mathrm{Ad}_g \Psi(z).$$

Also, $\mathrm{Im}\Psi = G_\mu \cdot \xi$ and $\langle \mu, \Psi(z) \rangle = \langle \mu, \xi \rangle$, for any $z \in G_\mu \cdot S$.

Now let f_1 and f_2 be the functions defined by

$$\begin{aligned} f_1 &= (h - h(m)) + (\langle \mathbf{J}, \Psi \rangle - \langle \mu, \xi \rangle) + (C_1 - C_1(m)) + \cdots + (C_n - C_n(m)) \\ f_2 &= (C_1 - C_1(m))^2 + \cdots + (C_n - C_n(m))^2 + \|\mathbf{J} - \mu\|^2, \end{aligned}$$

where in f_2, the modulus is taken using the norm associated to some $\mathrm{Ad}^*_{G_\mu}$-invariant inner product in $\mathfrak{g}^*$ (always available by the compactness of G_μ) that makes f_2 a G_μ-invariant conserved quantity. Remark that f_1 is G_μ-invariant, but, in general, it is not conserved. Notice also that on S, $h - \mathbf{J}^\xi + C_1 + \cdots + C_n$ and $f_1|_S$ differ by a constant, which implies that $\mathbf{d}(f_1|_S)(m) = 0$ and $\mathbf{d}^2(f_1|_S)(m)$ is well defined. Moreover,

$$\mathbf{d}^2(f_1|_S)(m)|_{Z \times Z} = \mathbf{d}^2(h - \mathbf{J}^\xi + C_1 + \cdots + C_n)(m)|_{Z \times Z}.$$

Since Z satisfies the requirements of W, the hypotheses of the theorem guarantees that $\mathbf{d}^2(f_1|_S)(m)|_{Z \times Z}$ is definite. As we did in the proof of Theorem 4.1, it can be checked that Z is the kernel of $\mathbf{d}^2(f_2|_S)(m)$, hence by Lemma 3.9, there exists a positive constant a for which

$$f := af_1 + f_2$$

is such that $\mathbf{d}^2(f|_S)(m)$ is positive definite. Note that f is G_μ-invariant but, in general, it is not a constant of the motion since $\langle \mathbf{J}, \Psi \rangle$ is not conserved. In fact, for any $z \in S$,

$$\begin{aligned} f(F_t(z)) - f(z) &= \langle \mathbf{J}(F_t(z)), \Psi(F_t(z)) \rangle - \langle \mathbf{J}(z), \Psi(z) \rangle = \langle \mathbf{J}(z), \Psi(F_t(z)) - \Psi(z) \rangle \\ &= \langle \mathbf{J}(z) - \mu + \mu, \Psi(F_t(z)) - \xi \rangle \\ &= \langle \mathbf{J}(z) - \mu, \Psi(F_t(z)) - \xi \rangle + \langle \mu, \Psi(F_t(z)) \rangle - \langle \mu, \xi \rangle \\ &= \langle \mathbf{J}(z) - \mu, \Psi(F_t(z)) - \xi \rangle, \end{aligned}$$

where we used Noether's theorem, $\Psi(z) = \xi$ because $z \in S$, and $\langle \mu, \Psi(z) \rangle = \langle \mu, \xi \rangle$, for any $z \in G_\mu(S)$. Hence, for any $z \in S$,

$$\begin{aligned} 0 &\leq f(F_t(z)) \leq f(z) + a|\langle \mathbf{J}(z) - \mu, \Psi(F_t(z)) - \xi \rangle| \\ &\leq f(z) + a\|\mathbf{J}(z) - \mu\|(\|\Psi(F_t(z))\| + \|\xi\|) \\ &= f(z) + 2a\|\xi\|\|\mathbf{J}(z) - \mu\|, \end{aligned} \tag{4.9}$$

where we used that $\mathrm{Im}\Psi = G_\mu \cdot \xi$, and the G_μ-invariance of the norm $\|\cdot\|$.

With these tools, we prove the G_μ-stability of m. Let V be a G_μ-invariant open neighborhood of $G_\mu \cdot m$. Since $f(m) = 0$, by the positive definiteness of $\mathbf{d}^2(f|_S)(m)$ and the Morse lemma, there is an $\epsilon > 0$ such that

$$f^{-1}[0, \epsilon) \cap S \subset V \quad \text{and} \quad \overline{f^{-1}[0, \epsilon) \cap S} \subset S, \tag{4.10}$$

where $f^{-1}[0, \epsilon)$ is an open neighborhood of m in S. The continuity of f and $\mathbf{J}$ and Corollary 3.6 imply the existence of an open H-invariant neighborhood S' of m on S such that $S' \subset f^{-1}[0, \epsilon) \cap S$, and that for any $z \in S'$, $f(z) < \epsilon/2$ and $\|\mathbf{J}(z) - \mu\| < \epsilon/4a\|\xi\|$. We shall prove that

$$F_t(S') \subset f^{-1}[0, \epsilon) \cap G_\mu \cdot S \quad \text{for all positive} t. \tag{4.11}$$

Given this inclusion $U := \cup_{t \geq 0} F_t(G_\mu \cdot S') \subset f^{-1}[0, \epsilon) \cap G_\mu \cdot S \subset V$ is the neighborhood that we need to conclude stability, in other words, $F_t(U) \subset V$ for all $t \geq 0$.

We will show the inclusion (4.11) by contradiction. Suppose that (4.11) is false for some positive t, which implies the existence of a $z_0 \in S'$ such that

$$t_0 := \sup\{t \geq 0 | F_s(z_0) \in f^{-1}(0, \epsilon) \cap G_\mu \cdot S, \forall s \in [0, t)\} < \infty.$$

The point $p_0 := F_{t_0}(z_0) \notin f^{-1}[0, \epsilon) \cap G_\mu \cdot S$ by the openness of $f^{-1}[0, \epsilon) \cap G_\mu \cdot S$ and the definition of t_0. However, by construction $p_0 \in \overline{f^{-1}[0, \epsilon) \cap G_\mu \cdot S}$. Thus, there are sequences $\{z_i\} \subset S$ and $\{g_i\} \subset G_\mu$ such that $g_i \cdot z_i \to p_0$. Since $\{g_i \cdot z_i\} \subset f^{-1}[0, \epsilon) \cap G_\mu \cdot S$ and f is G_μ-invariant, the sequence $\{z_i\}$ actually belongs to $f^{-1}[0, \epsilon) \cap S$. Since S is relatively compact, we may assume that $z_i \to z \in \overline{f^{-1}[0, \epsilon) \cap S} \subset S$ by (4.10). At the same time, the properness of the G_μ action gives a subsequence of $\{g_i\}$ converging to some $g \in G_\mu$. Therefore, $F_{t_0}(z_0) = p_0 = g \cdot z \in G_\mu \cdot S$ so that we can apply inequality (4.9) to get

$$f(p_0) = f(F_{t_0}(z_0)) \leq f(z_0) + 2a\|\xi\|\|\mathbf{J}(z_0) - \mu\| < \epsilon,$$

since $z_0 \in S'$. We conclude that $p_0 \in f^{-1}[0, \epsilon) \cap G_\mu \cdot S$, which is a contradiction.

If $W = \{0\}$, then $\mathbf{d}^2(f_1|_S)(m)$ is definite. The theorem follows taking f_1 as f in the previous proof. □

Remark 4.10. *This method presents the advantage, with respect to the classical energy-momentum method, that* dim W, *and therefore the dimensionality of the stability form, is* *generically reduced by one each time we find a G_μ-invariant conserved quantity. This is particularly convenient at the time of applications.*

Notice that if in Theorem 4.8, we consider the case $G = \{e\}$, the relative equilibrium becomes an equilibrium and one has the following corollary.

Corollary 4.11 (Generalized energy-Casimir method). *Let $(M, \{\cdot,\cdot\}, h)$ be a Poisson system, and $m \in M$ be an equilibrium of the Hamiltonian vector field X_h. If there is a set of conserved quantities $C_1, \ldots, C_n \in C^\infty(M)$ for which*

$$\mathbf{d}(h + C_1 + \cdots + C_n)(m) = 0,$$

and

$$\mathbf{d}^2(h + C_1 + \cdots + C_n)(m)|_{W\times W},$$

is definite for W defined by

$$W = \ker \mathbf{d}C_1(m) \cap \cdots \cap \ker \mathbf{d}C_n(m),$$

then m is stable. If $W = \{0\}$, m is always stable.

Notice that this statement of the energy-Casimir method presents an improvement with respect to the classical one [5,14], since here the definiteness of $\mathbf{d}^2(h + C_1 + \cdots + C_n)(m)$ is required only on $W \times W$, while in the original version, this condition needs to be satisfied on the whole $T_mM \times T_mM$. This difference becomes apparent in the following example where the stability of the sleeping Lagrange top is studied. The classical treatment of this solution based on the energy-Casimir method requires the study of a 6×6 matrix (see [18]), however, using Corollary 4.11, a 4×4 matrix will suffice to obtain the classical stability condition.

Example 4.12 (Stability of the sleeping Lagrange top as a Poisson equilibrium). The Lagrange top can be described as a Poisson system on $\mathbb{R}^3 \times \mathbb{R}^3 \simeq \mathfrak{se}(3)$, by taking the Poisson structure given by the (−) Lie-Poisson bracket in $\mathfrak{se}(3)^*$. If we denote by (Π, Γ) the elements in $\mathbb{R}^3 \times \mathbb{R}^3$ this bracket, also called *the heavy top bracket*, has the form

$$\{F, G\}(\Pi, \Gamma) = -\Pi \cdot (\nabla_\Pi F \times \nabla_\Pi G) - \Gamma \cdot (\nabla_\Pi F \times \nabla_\Gamma G - \nabla_\Pi G \times \nabla_\Gamma F).$$

The Lagrange top Hamiltonian in these variables takes the form

$$h(\Pi, \Gamma) = \frac{1}{2}\left(\frac{\Pi_1^2 + \Pi_2^2}{I_1} + \frac{\Pi_3^2}{I_3}\right) + Mgl\Gamma_3,$$

where M is the total mass of the body, l the distance from the fixed point of the top to its center of mass and the inertia tensor of the body is $\mathbb{I} = \mathrm{diag}[I_1, I_1, I_3]$.

It can be easily verified that the quantities

$$C_1 = \varphi_1(\|\Gamma\|^2), \quad C_2 = \varphi_2(\Pi \cdot \Gamma), \quad \text{and} \quad C_3 = \varphi_3(\Pi_3),$$

with φ_1, φ_2, and φ_3 arbitrary real smooth functions, are constants of the motion, that we will use in Corollary 4.11 to study the stability of the sleeping top solution, that is, the equilibrium solution given by

$$m \equiv (0, 0, \Pi_3, 0, 0, 1),$$

with Π_3 arbitrary. Let $f = h + C_1 + C_2 + C_3$. It is easy to see that

$$\mathbf{d}f(m) = \left(0, 0, \frac{\Pi_3}{I_3} + \varphi_2'(\Pi_3) + \varphi_3'(\Pi_3), 0, 0, Mgl + 2\varphi_1'(1) + \Pi_3\varphi_2'(\Pi_3)\right).$$

Hence taking φ_1, φ_2, and φ_3 such that

$$\varphi_1'(1) = -\frac{1}{2}(Mgl + k\Pi_3), \quad \varphi_2'(\Pi_3) = k, \quad \text{and} \quad \varphi_3'(\Pi_3) = -\frac{kI_3 + \Pi_3}{I_3},$$

with $k \in \mathbb{R}$ arbitrary, we have that $\mathbf{d}f(m) = 0$. With the notation of Corollary 4.11, it may be computed that

$$\begin{aligned} W &= \ker \mathbf{d}h(m) \cap \ker \mathbf{d}C_1(m) \cap \ker \mathbf{d}C_2(m) \cap \ker \mathbf{d}C_3(m) \\ &= \operatorname{span}\{(1,0,0,0,0,0), (0,1,0,0,0,0), (0,0,0,1,0,0), (0,0,0,0,1,0)\}. \end{aligned}$$

Moreover,

$$\mathbf{d}^2 f(m)|_{W\times W} = \begin{pmatrix} 1/I_1 & 0 & k & 0 \\ 0 & 1/I_1 & 0 & k \\ k & 0 & -Mgl - k\Pi_3 & 0 \\ 0 & k & 0 & -Mgl - k\Pi_3 \end{pmatrix},$$

whose eigenvalues are

$$\begin{aligned} \lambda_\pm = \frac{1}{2I_1}(1 - MglI_1 - kI_1\Pi_3 \\ \pm\sqrt{4I_1(I_1k^2 + Mgl + k\Pi_3) + (1 - MglI_1 - kI_1\Pi_3)^2}). \end{aligned}$$

The eigenvalues λ_+ and λ_- have both the same sign provided that

$$\Pi_3^2 > \left(I_1 k + \frac{Mgl}{k}\right)^2.$$

Since we are free in the choice of k, the optimal stability condition will occur when $(I_1k + Mgl//k)^2$ has a minimum with respect to k, which happens when $k = \sqrt{Mgl/I_1}$. Hence an upright sleeping top is stable provided that

$$\Pi_3^2 > 4MglI_1,$$

which is the classical stability condition for a fast top.

5. Block diagonalization and reduced periodic orbits

As we saw in Theorem 2.6 (iv), the projection of a RPO onto the reduced space produces a periodic orbit. In [24] we saw how, in the regular case, the G_μ-stability of the RPO is equivalent to the orbital stability of the corresponding periodic orbit in the associated symplectic reduced space. In this section we will construct a block diagonalization of the stability form based on the isotypic decomposition of a linear representation of a compact Lie group. In virtue of the blocking obtained, it will be easy to see the generalization to the singular case of the link between the stability of the RPO and its associated (singular) reduced periodic orbit.

In all that follows, we use the notation introduced in the statement of Theorem 4.1. In order to use the results on singular reduction introduced in Section 2 in its simplest form, we will assume that $N(H)$, the normalizer of the isotropy of the RPO, is a compact subgroup of G.

Let g be a H-invariant metric on M, always available by the compactness of H, and define

$$A := T_m(G_\mu \cdot m) \oplus \operatorname{span}\{X_h(m)\}.$$

Obviously,

$$T_m M = (T_m(G_\mu \cdot m) \oplus \operatorname{span}\{X_h(m)\}) \oplus A^\perp,$$

where $A^\perp$ is the orthogonal complement to A relative to the inner product induced by g on $T_m M$. Since $A \subset \ker \mathbf{d}C_1(m) \cap \cdots \cap \ker \mathbf{d}C_n(m) \cap \ker T_m\mathbf{J}$, we have that

$$\begin{aligned} &\ker \mathbf{d}C_1(m) \cap \cdots \cap \ker \mathbf{d}C_n(m) \cap \ker T_m\mathbf{J} \\ &\quad = (T_m(G_\mu \cdot m) \oplus \operatorname{span}\{X_h(m)\}) \\ &\quad \oplus(\ker \mathbf{d}C_1(m) \cap \cdots \cap \ker \mathbf{d}C_n(m) \cap \ker T_m\mathbf{J} \cap A^\perp) := A \oplus W, \end{aligned} \tag{5.1}$$

where we use W for $\ker \mathbf{d}C_1(m) \cap \cdots \cap \ker \mathbf{d}C_n(m) \cap \ker T_m\mathbf{J} \cap A^\perp$, since it is obviously one of the spaces mentioned in the hypothesis of Theorem 4.1 needed to construct the stability form.

Proposition 5.1. *The subspace $W = \ker \mathbf{d}C_1(m) \cap \cdots \cap \ker \mathbf{d}C_n(m) \cap \ker T_m\mathbf{J} \cap A^\perp$, constructed above using the H-invariant metric g, has the following properties:*

(i) *W is H-invariant as a subspace of $T_m M$, where H acts on W via the natural lifted action.*

(ii) *The vector subspace W^H of H-fixed vectors is naturally isomorphic to $W_\mu^{(H)}$, where $W_\mu^{(H)}$ is such that*

$$\begin{aligned} &\ker \mathbf{d}(C_1)_\mu^{(H)}([m]_\mu^{(H)}) \cap \cdots \cap \ker \mathbf{d}(C_n)_\mu^{(H)}([m]_\mu^{(H)}) \\ &\quad = W_\mu^{(H)} \oplus \operatorname{span}\{X_{h_\mu^{(H)}}([m]_\mu^{(H)})\}. \end{aligned}$$

The functions $(C_i)_\mu^{(H)}$ with $i \in \{1, \ldots, n\}$ are uniquely defined by the identity

$$(C_i)_\mu^{(H)} \circ \pi_\mu^{(H)} = C_i \circ 1_\mu^{(H)}.$$

Proof. (i) The space $\ker \mathbf{d}C_1(m) \cap \cdots \cap \ker \mathbf{d}C_n(m) \cap \ker T_m\mathbf{J}$ is clearly H-invariant by the G_μ-invariance of the functions C_i and the G-equivariance of $\mathbf{J}$. We therefore just need to show that $A^\perp$ is H-invariant. Let $v \in A^\perp$; by definition, for any $\xi \in \mathfrak{g}_\mu$, $g(m)(v, \xi_M(m) + X_h(m)) = 0$. The vector $k \cdot v$ for $k \in H$ behaves similarly because, by the H-invariance of g,

$$\begin{aligned} g(m)(k \cdot v, \xi_M(m) + X_h(m)) &= g(m)(v, k^{-1} \cdot \xi_M(m) + k^{-1} \cdot X_h(m)) \\ &= g(m)(v, (\mathrm{Ad}_{k^{-1}}\xi)_M(m) + X_h(m)) = 0, \end{aligned}$$

since $\mathrm{Ad}_{k^{-1}}\xi \in \mathfrak{g}_\mu$.

(ii) By Theorem 2.4, $(M_\mu^{(H)}, \omega_\mu^{(H)})$ is naturally symplectomorphic to the symplectic reduced space $(\mathbf{K}_L^{-1}(\lambda_0)/L_{\lambda_0}, \omega_{\lambda_0})$. Abusing the notation, we will denote by $\pi_\mu^{(H)}$ the surjective submersion

$$\begin{aligned} \pi_\mu^{(H)} : \mathbf{K}_L^{-1}(\lambda_0) = \mathbf{J}^{-1}(\mu) \cap M_H &\longrightarrow \mathbf{K}_L^{-1}(\lambda_0)/L_{\lambda_0} \\ &= (\mathbf{J}^{-1}(\mu) \cap M_H)/(N_{G_\mu}(H)/H), \end{aligned}$$

and by $i_\mu^{(H)}$ the injection

$$i_\mu^{(H)} : \mathbf{K}_L^{-1}(\lambda_0) \hookrightarrow M.$$

Since $T_{[m]_\mu^{(H)}}\pi_\mu^{(H)}$ is surjective with kernel $T_m(L_{\lambda_0} \cdot m)$, we can identify (non-canonically because of the choice of m) $T_{[m]_\mu^{(H)}} M_\mu^{(H)}$ with $T_m\mathbf{K}_L^{-1}(\lambda_0)/T_m(L_{\lambda_0} \cdot m)$, and since

$$\begin{aligned} W^H = W \cap T_m M_H &= \ker \mathbf{d}C_1(m) \cap \cdots \cap \ker \mathbf{d}C_n(m) \cap \ker T_m\mathbf{J} \cap A^\perp \cap T_m M_H \\ &= \ker \mathbf{d}C_1(m) \cap \cdots \cap \ker \mathbf{d}C_n(m) \cap A^\perp \cap T_m\mathbf{K}_L^{-1}(\lambda_0) \subset T_m\mathbf{K}_L^{-1}(\lambda_0), \end{aligned}$$

we can define the linear map

$$\begin{aligned} \Delta : W^H &\longrightarrow T_{[m]_\mu^{(H)}} M_\mu^{(H)} \simeq T_m\mathbf{K}_L^{-1}(\lambda_0)/T_m(L_{\lambda_0} \cdot m), \\ w &\longmapsto w + T_m(L_{\lambda_0} \cdot m) = T_m\pi_\mu^{(H)} \cdot w. \end{aligned}$$

We first show that Δ is injective: if $w \in W^H$ is such that $w + T_m(L_{\lambda_0} \cdot m) = T_m(L_{\lambda_0} \cdot m)$, then $w \in T_m(L_{\lambda_0} \cdot m)$ necessarily and hence there is an element $\xi \in \mathrm{Lie}(N_{G_\mu}(H)) \subset \mathfrak{g}_\mu$ such that $w = \xi_M(m)$. Since $W \cap (T_m(G_\mu \cdot m) \oplus \mathrm{span}\{X_h(m)\}) = \{0\}$, it follows that $w = 0$ and therefore Δ is injective and an isomorphism onto its image, which we will prove is a vector space $W_\mu^{(H)} := \Delta(W^H) \subset T_{[m]_\mu^{(H)}} M_\mu^{(H)}$ such that

$$\begin{aligned} &\ker \mathbf{d}(C_1)_\mu^{(H)}([m]_\mu^{(H)}) \cap \cdots \cap \ker \mathbf{d}(C_n)_\mu^{(H)}([m]_\mu^{(H)}) \\ &\quad = W_\mu^{(H)} \oplus \mathrm{span}\{X_{h_\mu^{(H)}}([m]_\mu^{(H)})\}. \end{aligned} \tag{5.2}$$

We show first that the sum $W_\mu^{(H)} + \mathrm{span}\{X_{h_\mu^{(H)}}([m]_\mu^{(H)})\}$ is direct, that is,

$$W_\mu^{(H)} \cap \mathrm{span}\{X_{h_\mu^{(H)}}([m]_\mu^{(H)})\} = \{0\}.$$

Suppose, without loss of generality, that there is a $v \in W^H$ such that $\Delta(v) = v + T_m(L_{\lambda_0} \cdot m) = X_{h_\mu^{(H)}}([m]_\mu^{(H)})$. By Theorem 2.3, $X_{h_\mu^{(H)}}([m]_\mu^{(H)}) = T_m\pi_\mu^{(H)} \cdot X_h(m)$. If we write $\Delta(v)$ as $T_m\pi_\mu^{(H)} \cdot v$, then $T_m\pi_\mu^{(H)}(X_h(m) - v) = 0$, and therefore there exists an element $\xi \in \mathrm{Lie}(N_{G_\mu}(H))$ such that $X_h(m) - v = \xi_M(m)$, hence $v = X_h(m) + \xi_M(m)$ and therefore $v \in A \cap W = \{0\}$.

We now prove the equality in (5.2). Since every $\Delta(v) \in W_\mu^{(H)}$ is such that $v \in \ker \mathbf{d}C_1(m) \cap \ldots \cap \ker \mathbf{d}C_n(m) \cap \ker T_m\mathbf{J}$, this implies that for any $i \in \{1, \cdots, n\}$,

$$\mathbf{d}(C_i)_\mu^{(H)}([m]_\mu^{(H)}) \cdot \Delta(v) = T_m((C_i)_\mu^{(H)} \circ \pi_\mu^{(H)})(m) \cdot v = \mathbf{d}(C_i \circ i_\mu^{(H)})(m) \cdot v = 0,$$

and therefore,

$$W_\mu^{(H)} \oplus \mathrm{span}\{X_{h_\mu^{(H)}}([m]_\mu^{(H)})\} \subset \ker \mathbf{d}(C_1)_\mu^{(H)}([m]_\mu^{(H)}) \cap \cdots \cap \ker \mathbf{d}(C_n)_\mu^{(H)}([m]_\mu^{(H)}).$$

Conversely, let $[v]_\mu^{(H)} = T_m\pi_\mu^{(H)} \cdot v \in \ker \mathbf{d}(C_1)_\mu^{(H)}([m]_\mu^{(H)}) \cap \cdots \cap \ker \mathbf{d}(C_n)_\mu^{(H)}([m]_\mu^{(H)})$ with $v \in T_m\mathbf{K}_L^{-1}(\lambda_0) = \ker T_m\mathbf{J} \cap T_mM_H$. Clearly, this implies that $v \in \ker T_m\mathbf{J} \cap T_mM_H \cap \ker \mathbf{d}C_1(m) \cap \cdots \cap \ker \mathbf{d}C_n(m)$, and hence it can be uniquely decomposed as $v = v_1 + v_2$, with $v_1 \in A$ and $v_2 \in W$. Without loss of generality we assume that $v_1 = \xi_M(m) + X_h(m)$, for some $\xi \in \mathfrak{g}_\mu$. Since $v \in T_mM_H$, $k \cdot v = v$, for any $k \in H$, and hence

$$k \cdot \xi_M(m) + k \cdot X_h(m) + k \cdot v_2 = v_1 + v_2,$$

or equivalently,

$$(\mathrm{Ad}_k\xi)_M(m) + X_h(m) + k \cdot v_2 = \xi_M(m) + X_h(m) + v_2.$$

Since $\mathrm{Ad}_k\xi \in \mathfrak{g}_\mu$, the directness of the splitting (5.1) implies that $k \cdot \xi_M(m) = \xi_M(m)$ and that $k \cdot v_2 = v_2$ for all $k \in H$, and therefore $[v]_\mu^{(H)} = X_{h_\mu^{(H)}}([m]_\mu^{(H)}) + T_m\pi_\mu^{(H)} \cdot v_2$, with $v_2 \in T_mM_H \cap W = W^H$ which guarantees that $[v]_\mu^{(H)} = X_{h_\mu^{(H)}}([m]_\mu^{(H)}) + \Delta(v_2) \in \mathrm{span}\{X_{h_\mu^{(H)}}([m]_\mu^{(H)})\} \oplus W_\mu^{(H)}$. □

Definition 5.2. We call the subspace $W = \ker \mathbf{d}C_1(m) \cap \cdots \cap \ker \mathbf{d}C_n(m) \cap \ker T_m\mathbf{J} \cap A^\perp$, the *stability subspace* through the RPP $m \in M$, associated to the constants of the motion $C_1, \ldots, C_n$, and the H-invariant metric g on M.

Another tool that will be of great importance is the *isotypic decomposition* of the stability subspace W.

Theorem 5.3. *Let W be a stability subspace as in Definition* 5.2. *Then:*

(i) *Up to H-isomorphisms, there are a finite number of distinct (that is, not H-isomorphic) H-irreducible subspaces of W. Call these $U_1, \ldots, U_t$.*

(ii) *Define W_k to be the sum of all H-irreducible subspaces V of W such that V is H-isomorphic to U_k. Then*

$$W = W_1 \oplus \cdots \oplus W_t.$$

We say that the above direct sum decomposition is the H-isotypic decomposition of W and that W_k is the isotypic component of W of type U_k. By construction, this decomposition is unique.

(iii) *Let $A : W \to W$ be a H-equivariant linear mapping. Then $A(W_k) \subset W_k$ for $k = 1, \ldots, t$.*

Proof. This is a restatement of the isotypic decomposition theorem for compact Lie group representations in [11] (Theorems 2.5 and 3.5 of Chapter 12). The hypotheses are verified since there is a well-defined H-representation on W by Proposition 5.1 (i) and $H = G_m$ is compact, because the G-action on M is proper. □

We now state the main result of this section.

Theorem 5.4. *Assume the hypotheses and notations of Theorem* 4.1 *and Proposition* 5.1, *and that, in addition the group $N(H)$ is compact. Let W be the stability subspace through $m \in M$ associated to a H-invariant metric on M. Then, the stability form of the RPP m can be written as*

$$\mathbf{d}^2(C_1 + \cdots + C_n)(m)|_{W \times W} = \begin{pmatrix} \mathbf{d}^2((C_1)_\mu^{(H)} + \cdots + (C_n)_\mu^{(H)})([m]_\mu^{(H)})|_{W_\mu^{(H)} \times W_\mu^{(H)}} & 0 \\ 0 & \begin{matrix} A_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & A_r \end{matrix} \end{pmatrix}, \tag{5.3}$$

where $\mathbf{d}^2((C_1)_\mu^{(H)} + \cdots + (C_n)_\mu^{(H)})([m]_\mu^{(H)})|_{W_\mu^{(H)} \times W_\mu^{(H)}}$ is the orbital stability form associated to the periodic point $[m]_\mu^{(H)} \in M_\mu^{(H)}$, and $A_1, \ldots, A_r$ are the restrictions of $\mathbf{d}^2(C_1 + \cdots + C_n)(m)$ to the non-trivial isotypic components $W_1, \ldots, W_r$ of the stability subspace W.

Proof. We see first how the block structure in (5.3) is determined by the isotypic decomposition of W. The subspace W^H is the trivial isotypic component of W. Therefore, by Theorem 5.3, there exist subspaces $W_1, \ldots, W_r$ such that

$$W = W^H \oplus W_1 \oplus \cdots \oplus W_r, \tag{5.4}$$

is the isotypic decomposition of W. The H-invariance of $C_1 + \cdots + C_n$ implies that $\mathbf{d}^2(C_1 + \cdots + C_n)(m)|_{W \times W}$, considered as an automorphism of W, is H-equivariant. Part (iii) of Theorem 5.3 and (5.4) imply the block diagonal form of (5.3). □

It remains to be shown that the (1, 1)-block of (5.3) equals $\mathbf{d}^2((C_1)_\mu^{(H)} + \cdots + (C_n)_\mu^{(H)})([m]_\mu^{(H)})|_{W_\mu^{(H)} \times W_\mu^{(H)}}$. We will use the following technical lemma that was proved in [22,23].

Lemma 5.5. *Any $v \in W^H$ can be expressed as $v = (\mathrm{d}/\mathrm{d}t)|_{t=0} F_t^v(m)$, with F_t^v the Hamiltonian flow of a $N(H)$-invariant function $g_v \in C^\infty(M)$.*

The entries of the (1, 1)-block that we want to compute have the expressions

$$\mathbf{d}^2(C_1 + \cdots + C_n)(m)(v, w) \quad \text{for arbitrary} \quad v, w \in W^H.$$

By Lemma 5.5, there is a $N(H)$-invariant function g_v on M, whose Hamiltonian flow F_t^v satisfies $v = (\mathrm{d}/\mathrm{d}t)|_{t=0} F_t^v(m)$. We extend w to a vector field $\mathcal{W}$ along $F_t^v(m)$ by setting

$$\mathcal{W}(F_t{}^v(m)) = T_m F_t{}^v \cdot w.$$

By the definition of the Hessian we get

$$\begin{aligned}
&\mathbf{d}^2(C_1 + \cdots + C_n)(m)(v, w) = v[\mathcal{W}[C_1 + \cdots + C_n]] \\
&= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \mathcal{W}[C_1 + \cdots + C_n](F_t{}^v(m)) \\
&= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (\mathbf{d}C_1(F_t{}^v(m)) \cdot T_m F_t{}^v \cdot w + \cdots + \mathbf{d}C_n(F_t{}^v(m)) \cdot T_m F_t{}^v \cdot w). \quad (5.5)
\end{aligned}$$

Recall that the $(C_i)_\mu^{(H)}$ are defined by the relations $(C_i)_\mu^{(H)} \circ \pi_\mu^{(H)} = C_i \circ i_\mu^{(H)}$ with $i \in \{1, \ldots, n\}$. We identify again $(M_\mu^{(H)}, \omega_\mu^{(H)})$ with $(\mathbf{K}_L^{-1}(\lambda_0)/L_{\lambda_0}, \omega_{\lambda_0})$, and we take for $\pi_\mu^{(H)}$ and $i_\mu^{(H)}$ (abusing the notation) the canonical projection $\pi_\mu^{(H)} : \mathbf{K}_L^{-1}(\lambda_0) \to \mathbf{K}_L^{-1}(lo)/L_{\lambda_0}$ and immersion $i_\mu^{(H)} : \mathbf{K}_L^{-1}(\lambda_0) \hookrightarrow M$. If $v, w \in W^H$, the vector $T_m \pi_\mu^{(H)} \cdot w \in W_\mu^{(H)}$ can be extended, using F_t^v, to a vector field $\mathcal{W}_\mu^{(H)}$ along $\pi_\mu^{(H)}(F_t{}^v(m))$ by

$$\mathcal{W}_\mu^{(H)}(\pi_\mu^{(H)}(F_t{}^v(m))) = T_m(\pi_\mu^{(H)} \circ F_t{}^v) \cdot w.$$

Then, since $\mathbf{d}((C_1)_\mu^{(H)} + \cdots + (C_n)_\mu^{(H)})([m]_\mu^{(H)}) = 0$, we get

$$\begin{aligned}
&\mathbf{d}^2((C_1)_\mu^{(H)} + \cdots + (C_n)_\mu^{(H)})([m]_\mu^{(H)})(T_m\pi_\mu^{(H)} \cdot v, T_m\pi_\mu^{(H)} \cdot w) \\
&= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (\mathbf{d}(C_1)_\mu^{(H)}((\pi_\mu^{(H)} \circ f_t^v(m))(T_m(\pi_\mu^{(H)} \circ f_t^v) \cdot w) + \cdots \\
&\quad + \mathbf{d}(C_n)_\mu^{(H)}((\pi_\mu^{(H)} \circ f_t^v(m))(T_m(\pi_\mu^{(H)} \circ f_t^v) \cdot w)) \\
&= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (\mathbf{d}((C_1)_\mu^{(H)} \circ \pi_\mu^{(H)})(f_t^v(m)) \cdot T_m f_t^v \cdot w + \cdots \\
&\quad + \mathbf{d}((C_n)_\mu^{(H)} \circ \pi_\mu^{(H)})(f_t^v(m)) \cdot T_m f_t^v \cdot w) \\
&= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (\mathbf{d}(C_1 \circ i_\mu^{(H)})(f_t^v(m)) \cdot T_m f_t^v \cdot w + \cdots \\
&\quad + \mathbf{d}(C_n \circ i_\mu^{(H)})(f_t^v(m)) \cdot T_m f_t^v \cdot w)
\end{aligned}$$

$$= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (\mathbf{d}C_1(f_t^v(m)) \cdot T_m f_t^v \cdot w + \cdots + \mathbf{d}C_n(f_t^v(m)) \cdot T_m f_t^v \cdot w),$$

which coincides with expression (5.5). Since, by definition, $W_\mu^{(H)} = T_m \pi_\mu^{(H)} \cdot W^H$, the fact that

$$\begin{aligned} &\mathbf{b}^2(C_1 + \cdots + C_n)(m)(v, w) \\ &\quad = \mathbf{d}^2((C_1)_\mu^{(H)} + \cdots + (C_n)_\mu^{(H)})([m]_\mu^{(H)})(T_m \pi_\mu^{(H)} \cdot v, T_m \pi_\mu^{(H)} \cdot w), \end{aligned}$$

for arbitrary $v, w \in W^H$, proves the equality in (5.3).

5.1. Summary of the method

We have shown that taking the stability subspace $W = \ker \mathbf{d}C_1(m) \cap \cdots \cap \ker \mathbf{d}C_n(m) \cap \ker T_m \mathbf{J} \cap A^\perp$ through the RPP $m \in M$ associated to the constants of the motion $C_1, \ldots, C_n$, and the H-invariant metric g, the relative periodic point m is G_μ-stable if the symmetric matrix

$$\begin{pmatrix} \mathbf{d}^2((C_1)_\mu^{(H)} + \cdots + (C_n)_\mu^{(H)})([m]_\mu^{(H)})|_{W_\mu^{(H)} \times W_\mu^{(H)}} & 0 \\ 0 & \begin{matrix} A_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & A_r \end{matrix} \end{pmatrix} \tag{5.6}$$

is definite, where $A_1, \ldots, A_r$ are the restrictions of $\mathbf{d}^2(C_1 + \cdots + C_n)(m)$ to the non-trivial isotypic components $W_1, \ldots, W_r$ of the H-space W.

Summarizing, given a RPP $m \in M$, Theorems 4.1 and 5.4 guarantee that m is stable modulo G_μ if the following three conditions are satisfied:

(i) The bilinear form $\mathbf{d}^2((C_1)_\mu^{(H)} + \cdots + (C_n)_\mu^{(H)})([m]_\mu^{(H)})|_{W_\mu^{(H)} \times W_\mu^{(H)}}$ is definite, and therefore, the associated singular reduced periodic orbit is orbitally stable,

(ii) The bilinear forms $\mathbf{d}^2(C_1 + \cdots + C_n)(m)|_{W_i \times W_i}$ are definite, for any $i \in \{1, \ldots, r\}$,

(iii) All the definite bilinear forms in (i) and (ii) have the same sign.

Acknowledgements

We thank F. Fassò, J. Marsden, A. Weinstein, and the referees for critically listening to us and giving us advice on various aspects of this paper.

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