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On a maximal torus in the volume-preserving diffeomorphism group of the finite cylinder

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Abstract: Let $\mathcal{T}$ be the collection of all H^s ($s > 2$) diffeomorphisms η_ϕ of the cylindrical surface $M := S^1 \times [p, q]$, where η_ϕ rotates each 'horizontal' circle $S^1 \times z$ rigidly by an angle $\phi = \phi(z)$ which is a real valued H^s function. Let M be given the flat metric g induced from the Euclidean metric of $\mathbb{R}^3$. We shall prove in this paper that (1) Topologically, $\mathcal{T}$ is a real, infinite-dimensional, smooth, path-connected and closed submanifold of $\mathrm{Diff}_{\mathrm{vol}}$ relative to the H^s topology. (2) Algebraically, $\mathcal{T}$ is a maximal Abelian subgroup of $\mathrm{Diff}_{\mathrm{vol}}$, it is equal to its centralizer in Diff, and its Weyl group in $\mathrm{Diff}_{\mathrm{vol}}$ is $\mathbb{Z}_2$. (3) Geometrically, with respect to the g-induced right-invariant L^2 metric $\langle \cdot , \cdot \rangle$, $\mathcal{T}$ is a totally geodesic and flat Riemannian submanifold of $\mathrm{Diff}_{\mathrm{vol}}$; we also identify its normal bundle.

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1. Background on Hodge decompositions

Let (M, g) be a smooth m-dimensional compact oriented Riemannian manifold with smooth boundary ∂M. Denote the volume element on M by dV (locally, $(dV)_x = \sqrt{\det(g_{ij}(x))}\, d^m x$), that on ∂M by dS. There exists a smooth boundaryless m-dimensional compact oriented Riemannian manifold $(N, \tilde{g})$ and an isometric embedding $i : (M, g) \to (N, \tilde{g})$. This $(N, \tilde{g})$ is known as a smooth Riemannian double of (M, g). To simplify the notation, we shall regard M as a subset of N, think of $\tilde{g}$ as an extension of the metric from M to N, in which case it can be denoted by g as well. We list some examples of Riemannian doubles: if M is a flat 2-disc, then N is a smooth sphere-like surface with one of its polar caps flattened; if M is a cylindrical surface of finite height in Euclidean 3-space, then N can be chosen as a smooth torus-like surface with a 'straightened' portion.

Let $\mathcal{X}$ denote the vector space of all Sobolev H^s ($s > \frac{1}{2}m + 1$) vector fields on M, not necessarily tangent to ∂M at boundary points. On $\mathcal{X}$, one has the usual positive-definite g-induced

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L_2 inner product

$$\langle X, Y\rangle := \int_M g(X, Y)\, dV. \tag{1.1}$$

There are two important subspaces of $\mathfrak{X}$: $\mathfrak{X}_{\parallel}^{\text{div}}$ (divergence-free H^s vector fields on M which are tangent to ∂M at boundary points) and Grad (vector fields on M which are the gradients of globally defined H^{s+1} functions). Hodge theory gives the following L_2 orthogonal direct sum decomposition:

$$\mathfrak{X} = \mathfrak{X}_{\parallel}^{\text{div}} \oplus \text{Grad}, \quad \text{with } H^s \text{ closed summands.} \tag{1.2}$$

Computationally, this means that given any H^s vector field X on M, we have

$$X = X_{\parallel}^{\text{div}} + \text{grad } p_x =: \bar{P}X + \bar{G}X \tag{1.3}$$

where $X_{\parallel}^{\text{div}} \in \mathfrak{X}_{\parallel}^{\text{div}}$ and the potential p_x is a solution of the Neumann problem

$$\nabla^2 p_x = \text{div}\, X \text{ on } M, \quad \frac{\partial p_x}{\partial n} = g(X, n_{\text{out}}) \text{ at } \partial M. \tag{1.4}$$

Here, n_{out} is the unit outward normal and $\nabla^2 := \nabla^i \nabla_i$ is the Laplacian defined by the metric g.

There is some geometry behind (1.2) that we would like to point out. Let Emb denote the smooth Hilbert manifold [6] of all H^s embeddings of M into N. At each typical point η of Emb, the tangent space T_η Emb consists of objects of the form $X \circ \eta$, where X is an H^s vector field on N defined at the image points of the map η. In particular, at each diffeomorphism η of M, we have $T_\eta \text{Emb} = \mathfrak{X} \circ \eta$. Let Diff_{vol} denote the group of H^s volume-preserving diffeomorphisms of M; it is a submanifold of Emb. At each $\eta \in \text{Diff}_{\text{vol}}$, we have $T_\eta \text{Diff}_{\text{vol}} = \mathfrak{X}_{\parallel}^{\text{div}} \circ \eta$.

Certain vector fields play a significant role in the ensuing discussion. For Diff_{vol}, these are the right-invariant vector fields X^R, whose value at the identity is $X \in \mathfrak{X}_{\parallel}^{\text{div}}$, and the value at any η is $X \circ \eta$. For Emb, these are the rigid (as right-invariance does not make sense here) vector fields X^R such that $X^R(\eta) = X \circ \eta$, where X is independent of η and globally defined on N.

We define the metric

$$\langle X \circ \eta, Y \circ \eta\rangle := \int_{\eta(M)} [g(X, Y)]_{\eta(x)}\, (dV)_x \tag{1.5}$$

on Emb. A change-of-variables argument shows that when η is a volume-preserving diffeomorphism, we have $\langle X\circ\eta, Y\circ\eta\rangle = \langle X, Y\rangle$ for any two vector fields (not necessarily divergence-free) $X, Y \in \mathfrak{X}$. Two consequences of this fact are immediate. First, the metric defined in (1.5) is right-invariant when restricted to the submanifold Diff_{vol}. Secondly, (1.2) induces a $\langle\cdot, \cdot\rangle$-orthogonal direct sum decomposition of T_η Emb at each $\eta \in \text{Diff}_{\text{vol}}$:

$$T_\eta \text{Emb} = T_\eta \text{Diff}_{\text{vol}} \oplus (T_\eta \text{Diff}_{\text{vol}})^\perp = (\mathfrak{X}_{\parallel}^{\text{div}} \circ \eta) \oplus (\text{Grad} \circ \eta); \tag{1.6}$$

equivalently,

$$\begin{aligned} X \circ \eta = (X \circ \eta)_{\parallel} + (X \circ \eta)^\perp &:= (X_{\parallel}^{\text{div}}) \circ \eta + (\text{grad } p_x) \circ \eta \\ &=: (\bar{P}X) \circ \eta + (\bar{G}X) \circ \eta. \end{aligned} \tag{1.7}$$

The operator $\bar{P}_\eta$ which $\langle \cdot, \cdot \rangle$- orthogonally projects T_η Emb onto T_η $\mathrm{Diff}_{\mathrm{vol}}$ is thus defined as

$$\bar{P}_\eta(X \circ \eta) = (\bar{P}X) \circ \eta. \tag{1.8}$$

On Emb, there exists a unique torsion-free connection $\bar{\nabla}$ which is compatible with the metric (1.5). Likewise, restricting this metric to $\mathrm{Diff}_{\mathrm{vol}}$ induces the corresponding Levi-Civita connection $\bar{\nabla}^v$. As is standard in Riemannian Geometry, this induced connection can be obtained by applying the projection operator $\bar{P}$ to $\bar{\nabla}$. It has been shown (see Ebin–Marsden [2]) that $\bar{\nabla}$, when acting on rigid vector fields, has the following functorial property

$$\big(\bar{\nabla}_{X^R} Y^R\big)(\eta) = (\nabla_X Y) \circ \eta\,; \tag{1.9a}$$

that is,

$$\bar{\nabla}_{X^R} Y^R = \big(\nabla_X Y\big)^R. \tag{1.9b}$$

Here, ∇ is the Levi-Civita connection on N which corresponds to the extended metric g. In this paper, we shall not need a formula for the action of $\bar{\nabla}$ on arbitrary vector fields, though it can be deduced using the method of connectors (see Ebin–Marsden [2]; their treatment of the boundaryless case extends to manifolds with boundary, as claimed, and the resulting affine connection is both torsion-free and $\langle \cdot, \cdot \rangle$-compatible).

Next we turn to $\mathrm{Diff}_{\mathrm{vol}}$. Here, we have the Levi-Civita connection $\bar{\nabla}^v$ and the second fundamental form $\bar{S}$. Since $\mathrm{Diff}_{\mathrm{vol}}$ is a Lie-group-like space, it is known (see Marsden et al. [7]) that the values of $\bar{\nabla}^v$ on right-invariant vector fields determine its values on arbitrary vector fields. As for the value of $(\bar{\nabla}^v_{X^R} Y^R)(\eta)$, $X, Y \in \mathfrak{X}^{\mathrm{div}}_{\parallel}$, $\eta \in \mathrm{Diff}_{\mathrm{vol}}$, we proceed as follows: Extend X, Y, which are defined on M, to all of N; the resulting vector fields, denoted U, V, are typically no longer divergence-free away from M. Form the rigid vector fields U^R, V^R on Emb and compute $\bar{\nabla}_{U^R} V^R$; in view of (1.9), the result is $(\nabla_U V)^R$. Since η is a diffeomorphism of M, we have $(\nabla_U V) \circ \eta = (\nabla_X Y) \circ \eta$, hence $(\bar{\nabla}_{U^R} V^R)(\eta) = (\nabla_X Y) \circ \eta$. Applying the projection operator $\bar{P}_\eta$ (see (1.8)) gives

$$\big(\bar{\nabla}^v_{X^R} Y^R\big)(\eta) = \big(\bar{P}\nabla_X Y\big) \circ \eta \tag{1.10a}$$

at any $\eta \in \mathrm{Diff}_{\mathrm{vol}}$; equivalently,

$$\bar{\nabla}^v_{X^R} Y^R = \big(\bar{P}\nabla_X Y\big)^R. \tag{1.10b}$$

This exhibits the expected right-invariance of $\bar{\nabla}^v$; but it also shows that $\bar{\nabla}^v$ is no longer functorial when acting on right-invariant vector fields. By definition, the value of the second fundamental form on two arbitrary tangent vectors $X \circ \eta$, $Y \circ \eta$ is $[(\bar{\nabla}_{U^R} V^R)(\eta)]^\perp$. In view of the above discussion and (1.7), we have

$$\bar{S}(X \circ \eta, Y \circ \eta) = \big[\big(\nabla_X Y\big) \circ \eta\big]^\perp =: \big(\bar{G}\nabla_X Y\big) \circ \eta, \tag{1.11}$$

where the operator $\bar{G}$ was defined in (1.2)–(1.4).

As a digression, we shall discuss the relationship between the curvature tensors $\bar{R}^v$ and $\bar{R}$ of $\mathrm{Diff}_{\mathrm{vol}}$ and Emb, respectively. Take arbitrary tangent vectors $X \circ \eta$, $Y \circ \eta$, $Z \circ \eta \in T_\eta\, \mathrm{Diff}_{\mathrm{vol}}$, where $X, Y, Z \in \mathfrak{X}_{\|}^{\mathrm{div}}$. Our convention of the curvature is that

$$R^v(X \circ \eta, Y \circ \eta)Z \circ \eta = \left[\bar{\nabla}^v_{X^R}\bar{\nabla}^v_{Y^R}Z^R - \bar{\nabla}^v_{Y^R}\bar{\nabla}^v_{X^R}Z^R - \bar{\nabla}^v_{(\bar{\mathcal{L}}^v_{X^R}Y^R)}Z^R\right](\eta).$$

Now, Lie differentiation $\bar{\mathcal{L}}^v$ on $\mathrm{Diff}_{\mathrm{vol}}$ is functorial on right-invariant vector fields, that is: $\bar{\mathcal{L}}^v_{X^R}Y^R = (\mathcal{L}_X Y)^R$. Property (1.10) then implies that

$$R^v(X \circ \eta, Y \circ \eta)Z \circ \eta = P\big[\nabla_X(P\nabla_Y Z) - \nabla_Y(P\nabla_X Z) - \nabla_{(\mathcal{L}_X Y)}Z\big] \circ \eta. \tag{1.12}$$

To compute $R(X\circ\eta, Y\circ\eta)Z\circ\eta$, we first find rigid vector fields on Emb which extend the tangent vectors $X\circ\eta$, $Y\circ\eta$, and $Z\circ\eta$. This is accomplished by extending X, Y, Z from M to all of N, getting U, V, W (these are typically no longer divergence-free away from M). Since the image of (the diffeomorphism) η is M, we have $U^R(\eta) = U \circ \eta = X \circ \eta$, etc. Then $R(X \circ \eta, Y \circ \eta)Z \circ \eta$, being tensorial, is equal to $[\bar{\nabla}_{U^R}\bar{\nabla}_{V^R}W^R - \bar{\nabla}_{V^R}\bar{\nabla}_{U^R}W^R - \bar{\nabla}_{(\bar{\mathcal{L}}_{U^R}V^R)}W^R](\eta)$. It can be shown from first principles that Lie differentiation $\mathcal{L}$ on Emb is functorial on rigid vector fields: $\mathcal{L}_{U^R}V^R = (\mathcal{L}_U V)^R$. This fact, together with repeated use of (1.9), gives $R(X \circ \eta, Y \circ \eta)Z \circ \eta = [\nabla_U\nabla_V W - \nabla_V\nabla_U W - \nabla_{(\mathcal{L}_U V)}W] \circ \eta$. Since the image of η is M, the above can be rewritten as

$$\begin{aligned} R(X \circ \eta, Y \circ \eta)Z \circ \eta &= \big[\nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{(\mathcal{L}_X Y)}Z\big] \circ \eta \\ &= [R(X, Y)Z] \circ \eta. \end{aligned} \tag{1.13}$$

Comparing (1.12) and (1.13), and using (1.7), gives the Gauss curvature equation:

$$R^v(X \circ \eta, Y \circ \eta)Z \circ \eta = P\big[R(X, Y)Z - \nabla_X(G\nabla_Y Z) + \nabla_Y(G\nabla_X Z)\big] \circ \eta. \tag{1.14}$$

Next, we shall take a more refined look at (1.2).

Let d be the exterior differential on M and let $\delta := (-1)^{m(k+1)+1} * d *$ denote the co-differential (on k-forms) with respect to the metric g. Here, $*$ is the Hodge dual associated with g. Our conventions are as follows: vector indices are up; co-vector indices are down; tensor indices are raised (through the operation $\sharp$) and lowered (through $\flat$) with the metric $g = g_{ij}\, dx^i \otimes dx^j$. Thus, for example,

$$\operatorname{div} X = -\,\delta\, X^\flat.$$

If we apply $\flat$ to the decomposition (1.2), we will see that the Grad piece corresponds to those 1-forms which are globally d-exact and not subjected to any boundary condition. The $\mathfrak{X}_{\|}^{\mathrm{div}}$ piece can be split further into two parts, corresponding to 1-forms which are globally δ-exact, and harmonic 1-forms. There are boundary conditions which must be imposed on these two $\langle\cdot\,,\cdot\rangle$-orthogonal summands of $\mathfrak{X}_{\|}^{\mathrm{div}}$. This decomposition of $\mathfrak{X}_{\|}^{\mathrm{div}}$, together with (1.2), constitutes one of several Hodge decompositions for manifolds with boundary. In particular, we would like to mention that the harmonic piece in Morrey [8] has no boundary conditions at all, while that in Gilkey [4] has fewer boundary conditions than we do.

We give the details of this splitting of $\mathfrak{X}_{\|}^{\mathrm{div}}$, for surfaces with boundary. On 2-manifolds, divergence-free vector fields can be locally represented by stream functions as follows. The condition $\operatorname{div} X = 0$ says that $*d*X^\flat = 0$, hence $*X^\flat$ is a closed 1-form. Unless $H^1(M, \mathbb{R}) = 0$, $*X^\flat$ is in general only locally exact, say, $*X^\flat = df$ for some 'stream function' f defined on a coordinate chart. Since $** = (-1)^{k(m-k)}$ when acting on k-forms of a Riemannian m-manifold, the above gives

$$X^\flat = - * df, \quad \text{or} \quad X_i = -f_{,l}\, g^{lk}\, \epsilon_{ki}; \tag{1.15}$$

equivalently,

$$X = -(*df)^\sharp, \quad \text{or} \quad X^i = -f_{,l}\, \epsilon^{li}. \tag{1.16}$$

Here,

$$\epsilon_{ki} \ := \ {}_{[ki]}\sqrt{g}, \qquad \epsilon^{li} \ := \ \frac{[li]}{\sqrt{g}} \tag{1.17}$$

is the Levi-Civita tensor (*not* the Levi-Civita tensor density), $[ki]$ denotes the totally anti-symmetric symbol with $[12] = 1$, and $\sqrt{g}$ means $\sqrt{\det(g_{ij})}$.

There is another perspective. Note that the Levi-Civita tensor is the area form on the surface, and hence gives rise to a symplectic structure $\omega := \frac{1}{2}\omega_{ij}\, dx^i \wedge dx^j$, where

$$\omega_{ij} \ := \ \epsilon_{ij} \ = \ {}_{[ij]}\sqrt{g}. \tag{1.18}$$

The surface also has a natural almost complex structure

$$J \ := \ \sharp \circ * \circ \flat, \tag{1.19}$$

in terms of which (1.16) reads $X = -J \operatorname{grad} f$. These structures are compatible with the Riemannian metric, namely,

$$g(\cdot\,,\cdot) \ = \ \omega(\cdot\,, J\cdot). \tag{1.20}$$

It is straightforward to check that, with respect to ω, the quantity $-(*df)^\sharp$ is simply the Hamiltonian vector field of f; to reflect this fact, we shall thereafter denote it by X_f. Thus,

$$X_f \ = \ -J \operatorname{grad} f. \tag{1.21}$$

In $\mathfrak{X}_\parallel^{\text{div}}$, there is a subspace $\mathcal{H}_\parallel$, consisting of all globally Hamiltonian vector fields which are tangent to ∂M at boundary points. Any generic element of $\mathcal{H}_\parallel$ is of the form X_h, where h is a H^{s+1} function defined globally on M and is constant on each connected component $(\partial M)_i$, $i = 1, 2, \ldots, p$, of ∂M. The reason for this constancy is as follows: since both n_{out} (the unit outward pointing normal at ∂M, with M regarded as a submanifold of its Riemannian double N) and grad h are orthogonal to X_h (resp., by hypothesis and (1.16)), and since we are in two dimensions, they must be collinear. Thus the directional derivative of h along ∂M must be zero, and hence h is constant on each $(\partial M)_i$. This subspace $\mathcal{H}_\parallel$ is the analogue of globally δ-exact 1-forms in Hodge theory; but we re-iterate here that ours obey boundary conditions.

Let $\mathcal{Y}_\parallel$ denote the $\langle\cdot\,,\cdot\rangle$-orthogonal complement of $\mathcal{H}_\parallel$ in $\mathfrak{X}_\parallel^{\text{div}}$. Properties that characterize a generic element $Y \in \mathcal{Y}_\parallel$ can be deduced as follows. By (1.20), $g(X_h, Y) = \omega(X_h, JY) = (dh)(JY) = \operatorname{div}(hJY) - h \operatorname{div}(JY)$. So

$$\begin{aligned}\langle X_h, Y\rangle &= \int_M g(X_h, Y)\, dV \\ &= -\int_M h \operatorname{div}(JY)\ dV + \sum_{i=1}^{p} C_i \int_{(\partial M)_i} g(JY, n_{\text{out}})\, dS,\end{aligned}$$

where C_i is the constant value of h on the connected submanifold $(\partial M)_i$. The right hand side must vanish, in particular, for all H^{s+1} 'bump' functions h which are supported in the interior of M; thus $\operatorname{div}(JY) = 0$ on M. Now that the term $-\int_M h \operatorname{div}(JY)\, dV$ has dropped out, that linear combination of boundary integrals must vanish for arbitrary choices of the constants C_i; hence each boundary integral must be zero. Thus $Y \in \mathcal{Y}_\|$ if and only if

$$\operatorname{div} Y = 0 \text{ on } M, \tag{1.22a}$$

$$\operatorname{div}(JY) = 0 \text{ on } M, \tag{1.22b}$$

$$g(Y, n_{\text{out}}) = 0 \text{ at } \partial M, \tag{1.22c}$$

$$\int_{(\partial M)_i} g(JY, n_{\text{out}})\, dS = 0, \quad i = 1, 2, \ldots, p. \tag{1.22d}$$

In the language of differential forms, the above can be re-stated as

$$\delta Y^\flat = 0 \text{ on } M, \tag{1.23a}$$

$$dY^\flat = 0 \text{ on } M, \tag{1.23b}$$

$$Y^\flat(n_{\text{out}}) = 0 \text{ at } \partial M, \tag{1.23c}$$

$$\int_{(\partial M)_i} (*Y^\flat)(n_{\text{out}})\, dS = 0, \quad i = 1, 2, \ldots, p. \tag{1.23d}$$

From Gilkey ([4, pp. 243–246]), one learns that the $Y^\flat$'s satisfying (1.23a)–(1.23c) are in fact solutions of an elliptic problem, namely, the zero-modes of the Laplacian $d\delta + \delta d$ on 1-forms with suitable boundary conditions. Consequently, our subspace $\mathcal{Y}_\|$, whose elements having to satisfy additional constraints (1.23d) corresponding to the boundary components, is finite dimensional; it is the analogue of harmonic 1-forms in Hodge theory.

For the remainder of this section, we shall verify that the two $\langle\cdot\,,\cdot\rangle$-orthogonal summands $\mathcal{H}_\|$ and $\mathcal{Y}_\|$ indeed do exhaust $\mathfrak{X}_\|^{\text{div}}$; that is,

$$\mathfrak{X}_\|^{\text{div}} = \mathcal{H}_\| \oplus \mathcal{Y}_\|. \tag{1.24}$$

Take any $X \in \mathfrak{X}_\|^{\text{div}}$; we would like to demonstrate how it can be decomposed as $X = X_h + Y$, where $X_h \in \mathcal{H}_\|$ and $Y \in \mathcal{Y}_\|$. Observe that applying J and then div to this equation yields $\operatorname{div}(JX) = \nabla^2 h + \operatorname{div}(JY)$, because $JX_h = \operatorname{grad} h$. This motivates us to solve for h through the Poisson equation

$$\nabla^2 h = \operatorname{div}(JX) \tag{1.25}$$

with Dirichlet boundary conditions

$$h|_{(\partial M)_i} \equiv \text{constant } C_i, \quad i = 1, 2, \ldots, p, \tag{1.26}$$

and then define Y as

$$Y := X - X_h. \tag{1.27}$$

It is easy to see that three of the defining properties of $\mathcal{Y}_\|$, namely (1.22a)–(1.22c), follow respectively from (1.27), (1.25), and (1.26). It turns out that in order for Y to satisfy (1.22d) as well, one must choose the constants C_i judiciously.

To this end, let $G(x, y)$ denote the Dirichlet Green's function with singularity at x, that is,

$$\nabla_y^2 G(x, y) = \delta^x(y) \text{ on } M, \tag{1.28}$$

and

$$G(x, y) = 0 \quad \text{if } x \text{ or } y \in \partial M. \tag{1.29}$$

The subscript on ∇_y^2 indicates differentiation with respect to y. Similarly, in what follows, let $\partial G(x, y)/\partial n_y$ denote the normal derivative of $G(x, y)$ with respect to the y variable. The solution of (1.25), (1.26) is then

$$h(x) = F(x) + \sum_{j=1}^{p} C_j \, K_j(x), \tag{1.30}$$

where, for the purpose of facilitating certain calculations below, some familiar quantities have been abbreviated as $F(x)$ and $K_j(x)$:

$$F(x) := \int_M G(x, y) \, [\mathrm{div}(JX)]_y \, (dV)_y, \tag{1.31}$$

$$K_j(x) := \int_{(\partial M)_j} \frac{\partial G(x, y)}{\partial n_y} \, (dS)_y. \tag{1.32}$$

Here, we recognize that K_j is harmonic and has zero boundary data except on $(\partial M)_j$, where it has constant value 1. Also, F has zero boundary data and its Laplacian equals that of h.

Substituting (1.27) into (1.22d) gives, for each i, the statement

$$\int_{(\partial M)_i} \frac{\partial h}{\partial n} \, dS \;=\; \int_{(\partial M)_i} g(JX, n_{\mathrm{out}}) \, dS \tag{1.33}$$

which, through (1.30), becomes

$$\sum_{j=1}^{p} C_j \int_{(\partial M)_i} \frac{\partial K_j}{\partial n} \, dS \;=\; \int_{(\partial M)_i} \left[g(JX, n_{\mathrm{out}}) - \frac{\partial F}{\partial n} \right] dS. \tag{1.34}$$

Let us first consider the case $p = 1$; that is, ∂M is connected. Then (1.34) reads

$$C \int_{\partial M} \frac{\partial K}{\partial n} \, dS \;=\; \int_{\partial M} \left[g(JX, n_{\mathrm{out}}) - \frac{\partial F}{\partial n} \right] dS.$$

The left hand side is zero because K, being a harmonic function on M with constant boundary value 1, is constant on M. The right hand side also vanishes; in fact, $\int_{\partial M} [g(JX, n_{\mathrm{out}}) - \partial F/\partial n] \, dS = \int_M [\mathrm{div}(JX) - \nabla^2 F] \, dV$, and, (1.30) together with (1.25) imply that $\nabla^2 F = \mathrm{div}(JX)$. Thus C, the constant boundary value of our function h, is arbitrary. This simply reflects the fact that adding a global constant to h would not change the Hamiltonian vector field X_h. It also says that $Y := X - X_h$ automatically satisfies (1.22d). Hence the decomposition (1.24) is valid for the case $p = 1$.

When $p \geqslant 2$, it is useful to view (1.34) in the matrix form

$$AC \;=\; B, \tag{1.35}$$

where A is a $p \times p$ matrix with

$$A_{ij} := \int_{(\partial M)_i} \frac{\partial K_j}{\partial n}\, dS, \quad 1 \leqslant i, j \leqslant p. \tag{1.36}$$

C is a column vector with entries $C_1, \dots, C_p$, and B is a column vector with entries

$$B_i := \int_{(\partial M)_i} \left[g(JX, n_{\text{out}}) - \frac{\partial F}{\partial n} \right] dS, \quad 1 \leqslant i \leqslant p. \tag{1.37}$$

The square matrix A has some interesting properties which we now deduce. First, recall that for each fixed j, the function K_j defined in (1.32) is a harmonic function on M with constant value 1 on $(\partial M)_j$ and 0 on the other boundary components $(\partial M)_i$, $i \neq j$. By the strong maximum principle, the values of K_j are strictly between 0 and 1 in the interior of M. It is then clear that $\partial K_j/\partial n(x) \geqslant 0$ if $x \in (\partial M)_j$, and is $\leqslant 0$ if $x \in (\partial M)_i$, $i \neq j$. However, since ∂M is smooth, these inequalities are actually strict (see for example, lemma 3.4 of [3]). Consequently, the matrix A has strictly positive diagonal entries, and strictly negative off-diagonal entries. Furthermore, summing along the jth column gives $\int_{\partial M}(\partial K_j/\partial n)\, dS$, which vanishes by the divergence theorem and the fact that K_j is harmonic. Likewise, summing along the ith row gives $\int_{(\partial M)_i}[\partial(K_1 + \cdots + K_p)/\partial n]\, dS$, which vanishes because $K_1 + \cdots + K_p$ is a harmonic function on M with constant boundary value 1, and hence is identically 1 everywhere. We summarize:

$$A_{ii} > 0, \tag{1.38}$$

$$A_{ij} < 0, \quad i \neq j, \tag{1.39}$$

$$\text{all column sums of } A \text{ are } 0, \tag{1.40}$$

$$\text{all row sums of } A \text{ are } 0. \tag{1.41}$$

For later use, we also record the fact that the column sum of B is zero:

$$B_1 + \cdots + B_p = 0. \tag{1.42}$$

Indeed, $B_1 + \cdots + B_p = \int_{\partial M}[g(JX, n_{\text{out}}) - \partial F/\partial n]\, dS = \int_M[\text{div}(JX) - \nabla^2 F]\, dV$; but from (1.30) and (1.25), one sees that $\nabla^2 F = \text{div}(JX)$.

We shall solve the system (1.35) by row-reducing the augmented matrix $(A|B)$ to $(R|\tilde{B})$, where R is the row-reduced echelon form of A. In view of (1.40) and (1.42), the last row of $(R|\tilde{B})$ must consist entirely of zeroes. We shall soon show that the remaining rows of R constitute a $(p-1) \times p$ matrix which has 1's on the main diagonal, -1's on the last column, and 0's elsewhere. Consequently the system (1.35) is consistent and the solution set is given by

$$C_p = C, \quad C_i = \tilde{B}_i + C, \quad i = 1, \dots, p-1, \quad C \text{ arbitrary}. \tag{1.43}$$

The structure exhibited by (1.43) shows that the function h can only be determined up to a global constant. This is to be expected since the decomposition $X = X_h + Y$ concerns the Hamiltonian vector field X_h, and not the function h per se.

The above solution determines the constant boundary values of h in such a way that $Y := X - X_h$ is in $\mathcal{Y}_\parallel$. Thus the decomposition (1.24) is valid for smooth compact 2-dimensional Riemannian manifolds whose boundary is smooth and consists of a finite number of connected components. Together with (1.2), we obtain a Hodge decomposition for this class of surfaces:

$$\mathcal{X} = \text{Grad} \oplus \mathcal{H}_{\|} \oplus \mathcal{Y}_{\|}. \tag{1.44}$$

The summands on the right hand side are respectively the exact, co-exact, and harmonic pieces that one expects from any Hodge decomposition. We hasten to point out, however, that our so-called harmonic piece $\mathcal{Y}_{\|}$ obeys four conditions, namely (1.22a)–(1.22d) [equivalently, (1.23a)–(1.23d)]. If it were to obey only the first three conditions [that is, (1.22a)–(1.22c)], then (see Gilkey [4]) it would be isomorphic to the first simplicial cohomology of M.

The remainder of this section will be concerned with a proof (due to Auchmuty [1]) of the aforementioned structure of R, namely, the row-reduced echelon form of the coefficient matrix A in (1.35).

We have noted that since the column sums of A are zero, the last row of the $p \times p$ matrix R must consist entirely of zeroes. By the rank-nullity theorem, this implies that the null space of A is at least 1-dimensional. As we shall see, the proof reduces to showing that the nullity of A is exactly 1. Recall our assertion above that the remaining rows of R constitute a $(p-1) \times p$ matrix Q which has 1's on the main diagonal, -1's on the last column, and 0's elsewhere. Such a structure is automatic if A has rank equal to $p-1$, for then R will have exactly $p-1$ nonzero rows and, by the criteria for row reduced echelon forms, the first $p-1$ columns of Q would have to form the $(p-1) \times (p-1)$ identity matrix. This, together with the observation that row reduction preserves the zero row sums of A, would in turn explain why the last column of Q consists entirely of -1's.

Therefore, it suffices to check that A has rank $p-1$ or, equivalently, that its nullity is 1. To this end, we first write A as

$$A = D - E. \tag{1.45}$$

Here, D is a diagonal matrix consisting of the diagonal entries of A; by (1.38), these are all positive numbers. The matrix E has 0's on the diagonal, and its off-diagonal entries are the absolute values of those of A; in view of (1.39), the off-diagonal entries of E are all positive also. It is easy to see that the nullity of A is 1 if and only if $\lambda = 1$ is a geometrically simple eigenvalue of $D^{-1}E$. Since the null space of A is non-trivial, we know that 1 is an eigenvalue of $D^{-1}E$. The question is whether it is geometrically simple.

The matrix $D^{-1}E$ has 0's on its diagonal and all positive entries elsewhere, so the above question can be studied using one version of Perron's theorem, namely, that for irreducible matrices with all non-negative entries. However, in order to avoid the concept of irreducibility, we choose to work with $I + D^{-1}E$ instead, which has 2 as an eigenvalue and is amenable to a treatment by Perron's theorem for matrices with all positive entries. Note that 1 is a geometrically simple eigenvalue of $D^{-1}E$ if and only if 2 is a geometrically simple eigenvalue of $I + D^{-1}E$.

Perron's theorem (see, for example, [5]) says that the spectral radius ρ of $I + D^{-1}E$ is one of its algebraically (hence geometrically) simple eigenvalue(s). Since the row sums of A are zero [see (1.41)], D and E must have the same row sums, from which one finds that the row sums of $I + D^{-1}E$ are all equal to 2. Now, ρ is majorized by any matrix norm of $I + D^{-1}E$, in particular by its maximum row sum (of absolute values) matrix norm, hence $\rho \leqslant 2$. But 2 is in the spectrum of $I + D^{-1}E$, so $\rho = 2$ and we are done.

Let us summarize our efforts in this section with the following:

Theorem 1. *Let (M, g) be a smooth compact 2-dimensional Riemannian manifold whose boundary is smooth and consists of a finite number of connected components. Let $\mathcal{X}$ denote the vector space of all Sobolev H^s $(s > \frac{1}{2}m + 1 = 2)$ vector fields on M, not necessarily tangent to ∂M at boundary points. Then the L^2 orthogonal Hodge decomposition*

$$\mathcal{X} = \mathrm{Grad} \oplus \mathcal{H}_{\|} \oplus \mathcal{Y}_{\|}$$

holds, with $\mathcal{Y}_{\|}$ given by (1.22) *or* (1.23).

2. A refined Hodge decomposition for finite cylindrical surfaces

For the rest of this paper, let M be a cylindrical surface of finite height in Euclidean 3-space. In other words, $M = S^1 \times [p, q]$ and a convenient parametrization is $(\theta, z) \mapsto (\cos\theta, \sin\theta, z)$, with $0 \leqslant \theta \leqslant 2\pi$, $p \leqslant z \leqslant q$, and $p < q$. In this section, we show that for such M, the $\mathcal{Y}_{\|}$ piece in the Hodge decomposition (1.44) is zero. Then we demonstrate that the globally Hamiltonian piece $\mathcal{H}_{\|}$ can be split further into two $\langle\cdot\,,\cdot\rangle$-orthogonal summands $\mathfrak{t}$ and $\mathfrak{r}$, both closed in the H^s topology.

Let us first record some preliminaries. On M, we have the local coordinates θ and z. The two connected components of the boundary are then described by $z = p$ [for $(\partial M)_1$] and $z = q$ [for $(\partial M)_2$]. At $(\partial M)_1$ and $(\partial M)_2$, we have n_{out} respectively equal to $-\partial_z$ and $+\partial_z$. The Euclidean metric of $\mathbb{R}^3$ induces a flat metric g on M. In fact, all Christoffel symbols vanish. The area form on M is $d\theta \wedge dz$. Let $Y = Y^\theta\,\partial_\theta + Y^z\,\partial_z$ be any vector field on M, then

$$Y^\flat = Y^\theta\, d\theta + Y^z\, dz, \tag{2.1}$$

$$*Y^\flat = -Y^z\, d\theta + Y^\theta\, dz, \tag{2.2}$$

$$dY^\flat = (-\partial_z Y^\theta + \partial_\theta Y^z)\, d\theta \wedge dz, \tag{2.3}$$

$$\delta Y^\flat := - * d * Y^\flat = -(\partial_\theta Y^\theta + \partial_z Y^z), \tag{2.4}$$

and

$$\triangle Y^\flat := (d\delta + \delta d) Y^\flat = -(\nabla^2 Y^\theta)\, d\theta - (\nabla^2 Y^z)\, dz, \tag{2.5a}$$

with

$$\nabla^2 := \partial_\theta^2 + \partial_z^2. \tag{2.5b}$$

Now consider any $Y \in \mathcal{Y}_{\|}$. From $\delta Y^\flat = 0$ and $dY^\flat = 0$ [respectively (1.23a) and (1.23b)], we see that $\triangle Y^\flat = 0$ which, in view of (2.5), implies that the components Y^θ and Y^z of Y are both harmonic functions on the cylinder M. Since Y is tangent to ∂M [see (1.23c)], we have $Y^z = 0$ at all boundary points; being a harmonic function, Y^z must then be identically zero on M. Next, $dY^\flat = 0$ [(1.23b)] and (2.3) give $-\partial_z Y^\theta + \partial_\theta Y^z = 0$ which, upon the use of $Y^z = 0$, reduces to $\partial_z Y^\theta = 0$; in particular, the harmonic function Y^θ has normal derivative equal to zero at all points of ∂M, and hence is a constant, say C, on M. Thus any Y satisfying the conditions (1.23a)–(1.23c) must be of the form $Y = C\,\partial_\theta$, or equivalently $Y^\flat = C\, d\theta$, where C is an arbitrary constant. Conversely, we see from (2.1), (2.3), and (2.4) that any such Y will satisfy those conditions. At the same time, let us recall that since the circle is a deformation retract of our M, the cohomology group $H^1(M, \mathbb{R})$ is generated by one element, say, $d\theta$. These observations verify, explicitly for cylindrical surfaces of finite height, the claim (proven in [4]) in Section 1 that the vector space of all Y's satisfying the first three criteria of $\mathcal{Y}_\parallel$ is isomorphic to the first simplicial cohomology group of M. To complete the proof that

$$\mathcal{Y}_\parallel = 0, \tag{2.6}$$

we substitute $Y = C\,\partial_\theta$, equivalently $*Y^\flat = C\,dz$ [see (2.2)], into the remaining condition (1.23d), and find that $C = 0$.

In view of (1.24), what we have just shown implies indirectly that all divergence-free vector fields on M which are tangent to the boundary must be globally Hamiltonian. That is,

$$\mathcal{X}_\parallel^{\mathrm{div}} = \mathcal{H}_\parallel. \tag{2.7}$$

This can also be checked directly. In fact, given any $X \in \mathcal{X}_\parallel^{\mathrm{div}}$, the (unique up to a constant) globally Hamiltonian function h which generates it satisfies $\operatorname{grad} h = JX$, and is therefore given by a line integral of JX; here, J is the almost complex structure defined in (1.19). The well-definedness of h, as well as its constancy on the two boundary components, follow from the properties of X, the topology of M, and suitable applications of Stokes' theorem.

Next, we decompose $\mathcal{H}_\parallel$ into two $\langle\cdot\,,\cdot\rangle$-orthogonal summands.

In $\mathcal{H}_\parallel$, there is a subspace $\mathfrak{t}$ consisting of elements of the form X_f, where $f = f(z)$ (no dependence on θ; that is, axially symmetric about the z-axis). More explicitly,

$$\mathfrak{t} := \left\{ X \in \mathcal{H}_\parallel : X = X_f = \frac{df}{dz}\,\partial_\theta \text{ for some } f = f(z) \right\}. \tag{2.8}$$

This $\mathfrak{t}$ is a maximal Abelian subalgebra of $\mathcal{H}_\parallel$. We shall prove more, namely that the centralizer of $\mathfrak{t}$ in the Lie algebra $\mathcal{X}$ of all vector fields is $\mathfrak{t}$ itself. Indeed, the Lie bracket between an arbitrary $Y := Y^\theta\,\partial_\theta + Y^z\,\partial_z \in \mathcal{X}$ and X_f is

$$\left(\ddot{f}\,Y^z - \dot{f}\,\partial_\theta Y^\theta\right)\partial_\theta - \left(\dot{f}\,\partial_\theta Y^z\right)\partial_z,$$

where the dots denote differentiation with respect to z. This vanishes for all f if and only if $\partial_\theta Y^z = 0$ and $\ddot{f}\,Y^z - \dot{f}\,\partial_\theta Y^\theta = 0$. The first relation says that Y^z is a function of z only, hence the same is true of $\partial_\theta Y^\theta$. But this is incompatible with the periodicity of Y^θ unless $\partial_\theta Y^\theta$ actually vanishes. Therefore Y^θ depends only on z and, in view of the second relation above, Y^z must in fact be zero. Lastly, since $s > \frac{1}{2}m + 1$, $Y^\theta(z)$ is a C^1 function and can be rewritten as the z derivative of a function on $[p, q]$; in other words, $Y \in \mathfrak{t}$.

Secondly, we show that $\mathfrak{t}$ is its own normalizer in $\mathcal{H}_\parallel$. Let $X_h = (\partial_z h)\partial_\theta - (\partial_\theta h)\partial_z \in \mathcal{H}_\parallel$ be arbitrary. Its Lie bracket with any $X_f \in \mathfrak{t}$ is again an element of $\mathfrak{t}$ if and only if $\partial_\theta^2 h = 0$. That is, h has the form $a(z) + b(z)\theta$; but its periodicity forces $b(z)$ to vanish. So $X_h \in \mathfrak{t}$.

In order to find the conditions which characterize $\mathfrak{r}$, the subspace in $\mathcal{H}_\parallel$ which is orthogonal to $\mathfrak{t}$, let us first observe that for any two elements $X_h, X_{\tilde{h}}$ in $\mathcal{H}_\parallel$, we have $g(X_h, X_{\tilde{h}}) = \omega(X_h, JX_{\tilde{h}}) = (dh)(\operatorname{grad}\tilde{h}) = \operatorname{div}[h(\operatorname{grad}\tilde{h})] - h\nabla^2\tilde{h}$. Thus

$$\langle X_h, X_{\tilde{h}}\rangle = \int_{\partial M} h\,\frac{\partial\tilde{h}}{\partial n}\,dS - \int_M h\,\nabla^2\tilde{h}\,dV. \tag{2.9}$$

It follows that $\mathfrak{r}$ must consist of elements X_k such that $\int_{\partial M} f \cdot (\partial k/\partial n)\, dS - \int_M f \nabla^2 k\, dV$ vanishes for all $f = f(z)$. In coordinates, this reads

$$\begin{aligned} C_1 \int_0^{2\pi} (-\partial_z k)|_{z=p}\, d\theta + C_2 \int_0^{2\pi} (\partial_z k)|_{z=q}\, d\theta \\ - \int_p^q f(z) \Big[\int_0^{2\pi} (\nabla^2 k)(\theta, z)\, d\theta \Big] dz \\ = 0 \end{aligned} \tag{2.10}$$

for all $f = f(z)$ and constants $C_1 = f(p)$, $C_2 = f(q)$. Using arguments similar to those which led to (1.22a)–(1.22d), we see from (2.10) that X_k is an element of $\mathfrak{r}$ if and only if

$$\int_0^{2\pi} (\nabla^2 k)(\theta, z)\, d\theta = 0 \quad \text{for all } z \in [p, q], \tag{2.11a}$$

and

$$\int_0^{2\pi} (\partial_z k)(\theta, z)\, d\theta = 0 \quad \text{for } z = p, q. \tag{2.11b}$$

Note that (2.11a), together with the divergence theorem, implies that $\int_0^{2\pi} (\partial_z k)(\theta, p)\, d\theta = \int_0^{2\pi} (\partial_z k)(\theta, q)\, d\theta$. So (2.11b) only represents one constraint, not two, on the function k. Also, we should include the condition that k is constant on the boundary components of ∂M. A priori, its constant value on $(\partial M)_1$ [denoted, say, by C_p] may be different from that on $(\partial M)_2$ [denoted by C_q]. It is thus somewhat surprising to find that, on account of (2.11a) and (2.11b), these two constants are actually equal:

$$k(\cdot, p) \equiv C \equiv k(\cdot, q) \quad \text{for some constant } C. \tag{2.11c}$$

We shall derive (2.11c) by doing a Fourier expansion of $k(\theta, z)$ with z-dependent coefficients. Since k is well-defined on our cylindrical surface M, it is 2π-periodic. Also, our choice of function space implies, through the Sobolev embedding lemma, that all our vector fields are at least C^1, hence k is at least C^2. Consequently, k is pointwise equal to its Fourier series, whose z-dependent coefficients are all at least twice differentiable.

Given these remarks, we write

$$k(\theta, z) = \tfrac{1}{2} e(z) + \sum_{n=1}^{\infty} [a_n(z) \cos(n\theta) + b_n(z) \sin(n\theta)] \tag{2.12}$$

where, as usual,

$$\begin{aligned} e(z) &:= \frac{1}{\pi} \int_0^{2\pi} k(\theta, z)\, d\theta, \\ a_n(z) &:= \frac{1}{\pi} \int_0^{2\pi} k(\theta, z) \cos(n\theta)\, d\theta, \\ b_n(z) &:= \frac{1}{\pi} \int_0^{2\pi} k(\theta, z) \sin(n\theta)\, d\theta. \end{aligned}$$

We then find that

$$\partial_\theta k = \sum_{n=1}^{\infty} n[\, b_n(z)\cos(n\theta) - a_n(z)\sin(n\theta)\,], \tag{2.13}$$

$$\partial_z k = \tfrac{1}{2}\dot{e}(z) + \sum_{n=1}^{\infty} [\, \dot{a}_n(z)\cos(n\theta) + \dot{b}_n(z)\sin(n\theta)\,], \tag{2.14}$$

$$\nabla^2 k = \tfrac{1}{2}\ddot{e}(z) + \sum_{n=1}^{\infty} [\, (\ddot{a}_n - n^2 a_n)(z)\cos(n\theta) + (\ddot{b}_n - n^2 b_n)(z)\sin(n\theta)\,]. \tag{2.15}$$

Here, the dots on e, a_n, and b_n denote differentiation with respect to z. Substituting (2.15) into (2.11a), we find that $\ddot{e}(z) = 0$ for all $z \in [p, q]$. Likewise, substituting (2.14) into (2.11b) gives $\dot{e}(p) = 0 = \dot{e}(q)$. Thus the function $e(z)$ must be constant. On the other hand, inputting $k(\theta, p) \equiv C_p$ and $k(\theta, q) \equiv C_q$ into (2.12) gives $e(p) = 2C_p$, $e(q) = 2C_q$, and $a_n(z) = b_n(z) = 0$ at $z = p, q$. Putting all this together, we obtain (2.11c), and a refined version of (2.12):

$$k(\theta, z) = C + \sum_{n=1}^{\infty} [\, a_n(z)\cos(n\theta) + b_n(z)\sin(n\theta)\,], \tag{2.16a}$$

where

$$a_n(p) = b_n(p) = 0, \quad a_n(q) = b_n(q) = 0 \quad \text{for all } n. \tag{2.16b}$$

We have just shown that (2.11) implies (2.16); one can easily check that the converse is also true. Therefore (2.11) and (2.16) are equivalent characterizations of functions k such that $X_k \in \mathfrak{r}$. Incidentally, (2.16a) shows that the integral in (2.11b) actually vanishes for all $z \in [p, q]$, not just at $z = p, q$.

Let us check that $\mathfrak{r}$ is indeed complementary to $\mathfrak{t}$, in $\mathcal{H}_{\|}$. That is,

$$\mathcal{H}_{\|} = \mathfrak{t} \oplus \mathfrak{r}. \tag{2.17}$$

Such is not automatic because the summands are infinite dimensional. Take any $X_h \in \mathcal{H}_{\|}$, we shall produce a function $k(\theta, z)$ satisfying (2.11a)–(2.11c), and an axially symmetric function $f(z)$, such that

$$X_h = X_f + X_k. \tag{2.18}$$

Observe that applying J, then div, and then $\int_0^{2\pi}$ to both sides of (2.18) gives $\int_0^{2\pi} \nabla^2 h \, d\theta = \int_0^{2\pi} \nabla^2 f \, d\theta + \int_0^{2\pi} \nabla^2 k \, d\theta$. Motivated by (2.11a), we shall solve the equation

$$\int_0^{2\pi} \nabla^2 f \, d\theta = \int_0^{2\pi} \nabla^2 h \, d\theta \tag{2.19}$$

for the function $f = f(z)$. Since $\nabla^2 f = \ddot{f}(z)$ [see (2.5b)], this equation reduces to

$$\ddot{f}(z) = \frac{1}{2\pi}\int_0^{2\pi} (\nabla^2 h)(\theta, z)\, d\theta$$

which, upon integrating twice, yields the following two parameter family of candidates for f:

$$f(z) = \alpha + \beta(z - p) + \frac{1}{2\pi}\int_p^z \left\{ \int_p^x \left[\int_0^{2\pi} (\nabla^2 h)(\theta, y)\, d\theta \right] dy \right\} dx, \tag{2.20}$$

where α and β are the two constants of integration. They are respectively equal to the 'initial' data $f(p)$ and $\dot{f}(p)$. Leaving these constants arbitrary for the moment, we define

$$k = h - f. \tag{2.21}$$

Due to (2.19), this k satisfies (2.11a).

The constant α in (2.20) is to remain arbitrary because the Hamiltonian function f can only be determined up to a global constant. As for β, we shall fix it by insisting that our newly defined k satisfies (2.11b). That is, we set $\int_0^{2\pi} [\partial_z(h - f)](\theta, z)\, d\theta$ equal to zero at $z = p$ (as remarked before, doing the same at $z = q$ produces no extra information). This gives

$$\beta = \frac{1}{2\pi} \int_0^{2\pi} (\partial_z h)(\theta, p)\, d\theta. \tag{2.22}$$

Lastly, let A_p and A_q denote the constant values of h on $(\partial M)_1$ and $(\partial M)_2$. A straightforward calculation involving (2.20) and (2.22) shows that $f(q) = f(p) + A_q - A_p$, thus $A_q - f(q) = A_p - f(p)$, which is the expected (2.11c). This completes the proof of the decomposition (2.17). Alternatively, one can perform the splitting indicated by (2.17) using Fourier series. Such an approach is more amenable to the computations in later sections.

Let us conclude this section by showing that $\mathfrak{t}$ and $\mathfrak{r}$ are closed subspaces of $\mathcal{H}_\parallel$ relative to the H^s topology. We begin with the closedness of $\mathfrak{t}$. So, suppose one has a sequence $\{X_{f_i}\}$ in $\mathfrak{t}$ which converges, in the H^s norm, to some $X_h \in \mathcal{H}_\parallel$ [here, both h and the f_i's are H^{s+1}, with $s > \frac{1}{2}m + 1 = 2$]. By the Sobolev embedding lemma, the above convergence also occurs under the C^1 norm; in particular, it occurs under the sup norm and hence pointwise on M. Thus $(df_i/dz)\partial_\theta$ converges pointwise to $(\partial_z h)\partial_\theta - (\partial_\theta h)\partial_z$. This implies that h must be independent of θ; that is, $X_h \in \mathfrak{t}$.

To show that $\mathfrak{r}$ is H^s closed in $\mathcal{H}_\parallel$, we need to check that if any sequence $\{X_{k_i}\}$ in $\mathfrak{r}$ converges, in the H^s norm, to some $X_k \in \mathcal{H}_\parallel$, then the function k actually satisfies (2.11). Since X_k is already an element of $\mathcal{H}_\parallel$, it suffices to establish (2.11a) and (2.11b), for then (2.11c) follows. As before, our hypothesis on s implies that the convergence here is in fact C^1 for the vector fields and C^2 for their stream functions. Hence $\nabla^2 k_i$ and $\partial_z k_i$ converge uniformly to $\nabla^2 k$ and $\partial_z k$, respectively. One can therefore pass the limit under the integral sign, proving that k satisfies (2.11a) and (2.11b).

Let us summarize:

Theorem 2. *Let $M := S^1 \times [p, q]$ be a cylindrical surface of finite height in Euclidean 3-space, endowed with the induced flat metric. Let $s > \frac{1}{2}m + 1 = 2$. Then every divergence-free, tangent to the boundary, H^s vector field on M is globally Hamiltonian:*

$$\mathcal{X}_\parallel^{\mathrm{div}} = \mathcal{H}_\parallel.$$

We also have the L^2 orthogonal decomposition

$$\mathcal{H}_\parallel = \mathfrak{t} \oplus \mathfrak{r},$$

where the summands are respectively defined by (2.8) *and* (2.11); *both are closed subspaces with respect to the H^s topology. Furthermore, $\mathfrak{t}$ is a maximal Abelian Lie subalgebra of $\mathcal{H}_\parallel$; it is equal to its centralizer in $\mathcal{X}$, and is also equal to its normalizer in $\mathcal{H}_\parallel$.*

Intuitively, this theorem allows us to think of $\mathcal{H}_{\|}$ as a compact Lie algebra with $\mathfrak{t}$ a maximal toral subalgebra and $\mathfrak{r}$ the sum of all nonzero root spaces. Strictly speaking this is, of course, false. Nevertheless, as we will show in the next section, the above analogy can be extended all the way to the group level.

3. A maximal torus

By Theorem 2, we see that for our cylindrical surface $M := S^1 \times [p, q]$, we have a $\langle \cdot , \cdot \rangle$-orthogonal decomposition

$$\mathcal{X}_{\|}^{\mathrm{div}} = \mathfrak{t} \oplus \mathfrak{r} \tag{3.1a}$$

into closed subspaces in the H^s topology. Since $\langle X \circ \eta, Y \circ \eta \rangle = \langle X, Y \rangle$ whenever η is a volume-preserving diffeomorphism, (3.1a) induces a $\langle \cdot , \cdot \rangle$-orthogonal direct sum decomposition (into H^s closed subspaces) of $T_\eta \operatorname{Diff}_{\mathrm{vol}}$ at each $\eta \in \operatorname{Diff}_{\mathrm{vol}}$:

$$T_\eta \operatorname{Diff}_{\mathrm{vol}} = \mathcal{X}_{\|}^{\mathrm{div}} \circ \eta = (\mathfrak{t} \circ \eta) \oplus (\mathfrak{r} \circ \eta). \tag{3.1b}$$

The corresponding $\langle \cdot , \cdot \rangle$-orthogonal decomposition of the tangent bundle is

$$T \operatorname{Diff}_{\mathrm{vol}} = \bar{\mathfrak{t}} \oplus \bar{\mathfrak{r}}, \tag{3.1c}$$

where the fibres of $\bar{\mathfrak{t}}$ and $\bar{\mathfrak{r}}$ at η are respectively $\mathfrak{t} \circ \eta$ and $\mathfrak{r} \circ \eta$. In this section we shall give, in the spirit of §1, a geometrical interpretation of (3.1).

Let $\mathcal{T}$ be the collection of all H^s diffeomorphisms η_ϕ of M of the form

$$\eta_\phi(\theta, z) = (\theta + \phi(z), z); \tag{3.2}$$

that is, η_ϕ *rigidly* rotates each horizontal circle $S^1 \times z$ by an angle ϕ which varies with z. Since the Jacobian determinant of each such η is equal to one, $\mathcal{T}$ is an Abelian subgroup of $\operatorname{Diff}_{\mathrm{vol}}$. As a metric space, this subgroup is path-connected. Indeed, given $\phi_0(z)$ and $\phi_1(z)$, the obvious 1-parameter family that connects them, namely $\phi_0 + t(\phi_1 - \phi_0)$, $0 \leqslant t \leqslant 1$, is a continuous curve in the H^s function space. In order to ascertain that the corresponding family in $\mathcal{T}$ is a continuous curve, it is more than sufficient to prove that $\mathcal{T}$ is a closed submanifold of $\operatorname{Diff}_{\mathrm{vol}}$ modeled on $\mathfrak{t}$ (see (2.8)), which in turn is isomorphic as a Hilbert space to $H^s([p, q], \mathbb{R})$, the H^s real valued functions on $[p, q]$.

We accomplish this through a digression that will make use of the following fact: if $\mathcal{A} \subset \mathcal{B} \subset \mathcal{C}$ are all manifolds, and if $\mathcal{A}$, $\mathcal{B}$ are both (closed) submanifolds of $\mathcal{C}$, then $\mathcal{A}$ must be a (closed) submanifold of $\mathcal{B}$. As an example, let $\mathcal{A} := \operatorname{Diff}_{\mathrm{vol}}$, $\mathcal{B} := \operatorname{Diff}$ (the full H^s diffeomorphism group; it is generated by $\mathcal{X}_{\|}$, namely vector fields tangent to ∂M but which are not necessarily divergence-free), and $\mathcal{C} := \operatorname{Emb}$ (the Hilbert manifold of H^s embeddings of M into its double). Then $\operatorname{Diff}_{\mathrm{vol}}$ is a closed submanifold of Diff. For our task at hand, $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ are respectively $\mathcal{T}$, $\operatorname{Diff}_{\mathrm{vol}}$, and Diff. The desired conclusion would follow if we could demonstrate that $\mathcal{T}$ is a closed submanifold of Diff.

To this end, consider the map $\Psi : H^s([p, q], \mathbb{R}) \to \operatorname{Diff}$ defined by $\phi \mapsto \eta_\phi$, where η_ϕ is given by (3.2). This is an injective group homomorphism whose range is $\mathcal{T}$. Relative to the usual flat metric on the cylinder M, the geodesics are helices, horizontal circles and vertical lines.

Thus, in the exponential chart at the identity on Diff, Ψ has the expression $\phi \mapsto \phi\, \partial_\theta$, which is clearly smooth. By right translation (also a smooth map on Diff), this chart at the identity defines an atlas on Diff, which shows that Ψ is smooth in every chart of this atlas and hence is smooth on its domain. The derivative of Ψ at the 'origin' of $H^s([p, q], \mathbb{R})$ is found to be given by $f(z) \mapsto f(z)\partial_\theta$, which is injective. Its range $\mathfrak{t}$ is closed by Theorem 2, together with the fact that $\mathfrak{X}_{\|}^{\mathrm{div}}$ is closed in $\mathfrak{X}_{\|}$. So Ψ is an injective immersion. $\mathcal{T}$ is also closed in Diff since it is closed in the above exponential chart at the identity and hence (by right translation) in every chart of Diff. Therefore, Ψ is an embedding and the tangent space of $\mathcal{T}$ at the identity is $\mathfrak{t}$.

Next, let us prove that $\mathcal{T}$ is a maximal Abelian subgroup of $\mathrm{Diff}_{\mathrm{vol}}$. Motivated by Theorem 2, we will show that the centralizer of $\mathcal{T}$ in Diff, the group of all H^s diffeomorphisms of M, equals $\mathcal{T}$ itself; that is, any diffeomorphism η (not necessarily volume-preserving) which commutes with every element in $\mathcal{T}$ must actually be in $\mathcal{T}$.

We begin by checking that any such η must map each horizontal circle $S^1 \times z$ to itself. Suppose not, then η maps some point (θ_0, z_0) to (θ_1, z_1), with $z_1 \neq z_0$. A contradiction can be derived as follows. Consider the point (θ_1, z_0). Since $z_0 \neq z_1$, there are elements A, B in $\mathcal{T}$ such that $A(\theta_0, z_0) = B(\theta_0, z_0) = (\theta_1, z_0)$ but $A(\theta_1, z_1) \neq B(\theta_1, z_1)$. However, $A(\theta_1, z_1) = (A \circ \eta)(\theta_0, z_0) = (\eta \circ A)(\theta_0, z_0) = \eta(\theta_1, z_0) = (\eta \circ B)(\theta_0, z_0) = B(\theta_1, z_1)$, which is inconsistent with one defining feature of A and B.

It remains to show that the action of the above η on $S^1 \times z$ is a rigid rotation. To see this, let us write $\eta(0, z) = (\phi(z), z)$, where ϕ is a H^s function of z only. For each constant α, let A_α denote the element in $\mathcal{T}$ such that $A_\alpha(\theta, z) := (\theta + \alpha, z)$. Then $\eta(\theta_0, z) = (\eta \circ A_{\theta_0})(0, z) = (A_{\theta_0} \circ \eta)(0, z) = A_{\theta_0}(\phi(z), z) = (\phi(z) + \theta_0, z)$. This holds for all θ_0, that is, $\eta(\theta, z) = (\theta + \phi(z), z)$. So $\eta \in \mathcal{T}$.

As a final algebraic property of $\mathcal{T}$, we will prove that its normalizer in $\mathrm{Diff}_{\mathrm{vol}}$ is a $\mathbb{Z}_2$ cover of itself. If $\eta = (\eta_1, \eta_2) \in \mathrm{Diff}_{\mathrm{vol}}$ is an element of the normalizer, then given every $A \in \mathcal{T}$ there is another element $B \in \mathcal{T}$ such that $\eta \circ A = B \circ \eta$. If we write $A(\theta, z) = (\theta + \phi(z), z)$, $B(\theta, z) = (\theta + \psi(z), z)$, then the above relation is equivalent to

$$\eta_1(\theta + \phi(z), z) = \eta_1(\theta, z) + \psi(\eta_2(\theta, z)) + 2k\pi \tag{3.3a}$$

for some integer k, and

$$\eta_2(\theta + \phi(z), z) = \eta_2(\theta, z). \tag{3.3b}$$

Equation (3.3b) implies that $\eta_2(\theta, z) = \eta_2(z)$. Since $\eta \in \mathrm{Diff}_{\mathrm{vol}}$ its Jacobian determinant equals one and thus $(\partial_\theta \eta_1)(\partial_z \eta_2) = 1$. Therefore $\partial_\theta \eta_1$ is a function of z alone, say $a(z)$, and hence $\eta_1(\theta, z) = a(z)\theta + b(z)$ for some function $b(z)$. The obvious condition $\eta_1(2\pi, z) = \eta_1(0, z) + 2\ell\pi$ for some integer ℓ now implies that $a(z) = \ell$. Substituting this updated form of η_1 into the Jacobian criterion gives $\eta_2 = z/\ell + C$, where C is a constant. Using the fact that all diffeomorphisms in question preserve boundaries, together with the structure of η_2, one can solve for the constants ℓ and C. This narrows η down to two possibilities:

$$\eta_+(\theta, z) \;=\; (\theta + b(z),\; z), \tag{3.4a}$$

which maps each boundary circle onto itself, or

$$\eta_-(\theta, z) \;=\; (-\theta + b(z),\; -z + p + q), \tag{3.4b}$$

which interchanges the two boundary circles.

In the first case, $\eta_+ \in \mathcal{T}$, so of course it commutes with $\mathcal{T}$. Let us consider the second scenario. Substituting (3.4b) into (3.3a) gives $-\phi(z) = \psi(-z + p + q) + 2k\pi$, which upon relabeling is equivalent to

$$\psi(z) = -\phi(-z + p + q) - 2k\pi. \tag{3.5}$$

Given any ϕ which defines $A \in \mathcal{T}$, one can use (3.5) to produce (say, with $k = 0$) a ψ and hence a $B \in \mathcal{T}$ such that $\eta_- \circ A = B \circ \eta_-$. Thus η_- is in the normalizer as well. From these deliberations, we see that the Weyl group of $\mathcal{T}$ in $\mathrm{Diff}_{\mathrm{vol}}$ is $\mathbb{Z}_2$.

The properties of $\mathcal{T}$ proved so far are reminiscent of those for the classical maximal tori in compact Lie groups. To further support this analogy, we now turn to the study of the geometry of $\mathcal{T}$ and show that, as a Riemannian submanifold of $(\mathrm{Diff}_{\mathrm{vol}}, \langle\cdot,\cdot\rangle)$, it is both flat and totally geodesic. Since the curvature and the second fundamental form are both tensorial, it suffices to verify that their values on right-invariant vector fields are zero.

To begin, let $\bar{\nabla}^t$ denote the Levi–Civita connection on $\mathcal{T}$ corresponding to the induced metric $\langle\cdot,\cdot\rangle$, and let $\bar{\mathcal{S}}$ denote the second fundamental form of $\mathcal{T}$ as a Riemannian submanifold of $(\mathrm{Diff}_{\mathrm{vol}}, \langle\cdot,\cdot\rangle)$. Even though $\mathcal{T}$ is infinite dimensional, the existence of $\bar{\nabla}^t$ is ensured by the $\langle\cdot,\cdot\rangle$-orthogonal decomposition (3.1). In fact, let P, Q denote arbitrary vector fields on $\mathcal{T}$ and let X, Y denote (respectively) their extensions to $\mathrm{Diff}_{\mathrm{vol}}$, then at each $\eta \in \mathcal{T}$, (3.1b) induces a splitting of $(\bar{\nabla}^v_X Y)(\eta)$ into two $\langle\cdot,\cdot\rangle$-orthogonal components:

$$(\bar{\nabla}^v_X Y)(\eta) \ =: \ (\bar{\nabla}^t_P Q)(\eta) \ + \ (\mathcal{S}_{PQ})(\eta), \tag{3.6}$$

where $(\bar{\nabla}^t_P Q)(\eta) \in (\mathbf{t} \circ \eta)$ and $(\mathcal{S}_{PQ})(\eta) \in (\mathfrak{r} \circ \eta)$.

Take two arbitrary elements X_{f_1} and X_{f_2} from $\mathbf{t}$. They generate right-invariant vector fields $(X_{f_1})^r$, $(X_{f_2})^r$ on $\mathcal{T}$, and right-invariant vector fields $(X_{f_1})^R$, $(X_{f_2})^R$ on $\mathrm{Diff}_{\mathrm{vol}}$. As explained in Section 1 [see (1.10a)], at any volume-preserving diffeomorphism, in particular at any $\eta \in \mathcal{T}$, the value of the quantity $[\,\bar{\nabla}^v_{(X_{f_1})^R}(X_{f_2})^R\,](\eta)$ is the projection of $\nabla_{X_{f_1}} X_{f_2}$ onto $\mathcal{H}_{\|}$, where ∇ is the Levi-Civita connection on our cylindrical surface M. Since all Christoffel symbols on M are zero, we have

$$\nabla_{X_{f_1}} X_{f_2} = \left(\frac{df_1}{dz}\right) \frac{\partial}{\partial\theta} \left(\frac{df_2}{dz}\right) \partial_\theta,$$

which vanishes because f_2 has no z-dependence. Thus $[\,\bar{\nabla}^v_{(X_{f_1})^R}(X_{f_2})^R\,](\eta) = 0$ and, in view of (3.6), we have

$$[\,\bar{\nabla}^t_{(X_{f_1})^r}(X_{f_2})^r\,](\eta) = 0, \tag{3.7}$$

and

$$[\,\bar{\mathcal{S}}_{(X_{f_1})^r(X_{f_2})^r}\,](\eta) = 0. \tag{3.8}$$

From (3.8), we see immediately that $\mathcal{T}$ is a totally geodesic Riemannian submanifold of $(\mathrm{Diff}_{\mathrm{vol}}, \langle\cdot,\cdot\rangle)$. Next, (3.7) says that the connection $\bar{\nabla}^t$ vanishes on right-invariant vector fields, hence so does its curvature tensor; therefore $\mathcal{T}$ is flat.

We have proved the following theorem:

Theorem 3. *Let $\mathcal{T}$ be the collection of all H^s ($s > \frac{1}{2}m + 1 = 2$) diffeomorphisms η_ϕ of the cylindrical surface $M := S^1 \times [p, q]$, where η_ϕ rotates each 'horizontal' circle $S^1 \times z$ rigidly by an angle $\phi = \phi(z)$ which is a real valued H^s function. Here, M is given the flat metric g induced from the Euclidean metric of $\mathbb{R}^3$. Then*

(1) *Topologically, $\mathcal{T}$ is a real, infinite-dimensional, smooth, path-connected and closed submanifold of* $\mathrm{Diff}_{\mathrm{vol}}$ *relative to the H^s topology.*

(2) *Algebraically, $\mathcal{T}$ is a maximal Abelian subgroup of* $\mathrm{Diff}_{\mathrm{vol}}$, *and its (formal) Lie algebra is* $\mathbf{t}$; *furthermore, $\mathcal{T}$ is equal to its centralizer in* Diff, *and its Weyl group in* $\mathrm{Diff}_{\mathrm{vol}}$ *is* $\mathbb{Z}_2$.

(3) *Geometrically, with respect to the g-induced right-invariant L^2 metric $\langle \cdot, \cdot \rangle$, $\mathcal{T}$ is a totally geodesic and flat Riemannian submanifold of* $\mathrm{Diff}_{\mathrm{vol}}$, *and its normal bundle is* $\mathfrak{r}$.

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