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and T. Ratiu

Reduction, symmetry, and phases in mechanics

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Contents

Abstract

Various holonomy phenomena are shown to be instances of the reconstruction procedure for mechanical systems with symmetry. We systematically exploit this point of view for fixed systems (for example with controls on the internal, or reduced, variables) and for slowly moving systems in an adiabatic context. For the latter, we obtain the phases as the holonomy for a connection which synthesizes the Cartan connection for moving mechanical systems with the Hannay-Berry connection for integrable systems. This synthesis allows one to treat in a natural way examples like the ball in the slowly rotating hoop and also non-integrable mechanical systems.

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Introduction

This paper is concerned with the interpretation of the Hannay-Berry phase for classical mechanical systems as the holonomy of a connection on a bundle associated with the given problem. The techniques apply to the quantum case in the spirit of Aharonov and Anandan [1987], Anandan [1988] and Simon [1983] using the well known fact that quantum mechanics can be regarded as an instance of classical mechanics (see for instance Abraham and Marsden [1978]). In carrying this out there are a number of interesting new issues beyond that found in Hannay [1985] and Berry [1984], [1985] that arise. Already this is evident for the example of the ball in the hoop discussed in Berry [1985]; some remarks on this example are discussed in §1 below. For slowly varying integrable systems and for some aspects of the nonintegrable case, progress was made already by Golin, Knauf, and Marmi [1989] and Montgomery [1988]. The situation for the integrable case has been generalized to the context of families of Lagrangian manifolds by Weinstein [1989a,b]. However, these do not satisfactorily cover even the ball in the hoop example. For this and other examples, there is need for a development of the formulation, and it is the purpose of this paper to give one, following the line of investigation initiated by these papers. One of the crucial new ingredients in the present paper is the introduction of a connection that is associated to the movement of a classical system that we term the ***Cartan connection***. It is related to the theory of classical spacetimes that was developed by Cartan [1923] (see for example, Marsden and Hughes [1983] for an account). Another ingredient is the systematic use of symmetry and reduction, which are the key concepts needed to generalize to the nonintegrable case. In fact it is through the reconstruction process that the holonomy enters.

The paper begins in **§1** with some simple examples. The purpose is to give an idea of the Cartan connection. The first example is the ball in the hoop. The second example is the problem of two coupled rigid bodies to illustrate some of the ideas involved in reconstruction (here there are no slowly varying parameters, but there is still holonomy). We also give the Aharonov-Anandan formula for quantum mechanics (given in detail in **§4**) and a resume of slowly varying integrable systems from Golin, Knauf, and Marmi [1989] and Montgomery [1988]. Finally, we give the example of reconstructing the motion of a freely spinning rigid body.

§§**2** and **3** deal with the general theory of reconstruction. Given a phase space P and a symmetry group G, we show how to reconstruct the dynamics on P from dynamics on the reduced spaces. If $\mathbf{J} : G \to \mathfrak{g}^*$ is an equivariant momentum map for the G-action, the reduced space is $P_\mu = \mathbf{J}^{-1}(\mu)/G_\mu$, where G_μ is the coadjoint isotropy at μ. This reconstruction is done using a choice of connection on the principal G_μ-bundle $\mathbf{J}^{-1}(\mu) \to P_\mu$ (assuming the action is free). In case P is a cotangent bundle, there is a family of natural choices of connections depending on a choice of metric on the configuration space and on a choice of transverse cross section to the G_μ -orbit. We shall later refer to this one as the ***mechanical connection***. Another one is built out of the canonical one-form and applies when G_μ is abelian. The mechanical connection connection was defined by Guichardet [1984] and is closely related to connections defined by Smale [1970] and Kummer [1981]. For the case of cotangent bundles of semisimple Lie groups, the first includes the second as a special case.

We treat both the Lagrangian and Hamiltonian cases, since in the former the procedure is considerably more concrete because the Euler-Lagrange equations are of second order. In these sections there are no slowly varying parameters, but there are connections and holonomy. The connections are combined with the Hannay-Berry construction in §**14**.

§**4** gives,for the convenience of the reader, background material on Ehresmann connections, curvature and holonomy that is needed for the paper. It is illustrated with the Aharonov-Anandan formula and other examples of mechanical systems—such as the top in a gravitational field and coupled planar rigid bodies in §**5**. In §§**6**, **7** and **8** we present the basic defining properties and the existence and uniqueness of the Hannay-Berry connection. A crucial aspect of the construction is to take a given connection and average it relative to the action of a group G. This action also defines a parametrized momentum map **I**, which plays an important role. We generalize the theory developed in Montgomery [1988], and Golin, Knauf, and Marmi [1989] for trivial bundles with symplectic fibers and the standard connection to nontrivial fiber bundles whose fibers are Poisson manifolds and with a connection compatible with this structure. It is important to allow a nontrivial connection at this stage, even if the bundle is trivial, in order to deal with moving systems, like the ball in the hoop. For moving systems, the nontrivial connection used is the *Cartan connection* described in §**11**. §**9** gives another way to look at the Hannay-Berry connection by utilizing the momentum map for the group G. §**10** studies the important case of slowly moving integrable systems. This is the case that motivated the development in Montgomery [1988], Golin, Knauf, and Marmi [1989], and Weinstein [1989a,b]. We generalize this to our context.

§**12** presents a general construction for inducing connections on a tower of two bundles: $E \to F \to M$ with a given a connection on $E \to M$ and a family of fiberwise connections on $E \to F$. This is applied in §**13** with $E = \mathbf{I}^{-1}(\mu)$, $F = \mathbf{I}^{-1}(\mu)/G_\mu$, and M the parameter space. This is a parametrized version of the bundle of reduced spaces. This construction allows us to glue together the Hannay-Berry connection and the connection on the bundle $\mathbf{J}^{-1}(\mu) \to P_\mu$ used in §§**2** and **3** to obtain a connection on $\mathbf{I}^{-1}(\mu) \to \mathbf{I}^{-1}(\mu)/G_\mu$. The holonomy of this synthesized connection gives the desired phase changes in many of the equations.

In this paper, there are three lines of investigation one can focus on if desired. We regard §4 on Ehresmann connections as necessary background for all three. The three lines are:

1 Reconstruction ideas: §**1C, D, F**,§**2, 3, 5, 13**

2 Adiabatic phases and moving systems: §**1A, B, E, 6, 7, 8, 9, 10, 11, 12**

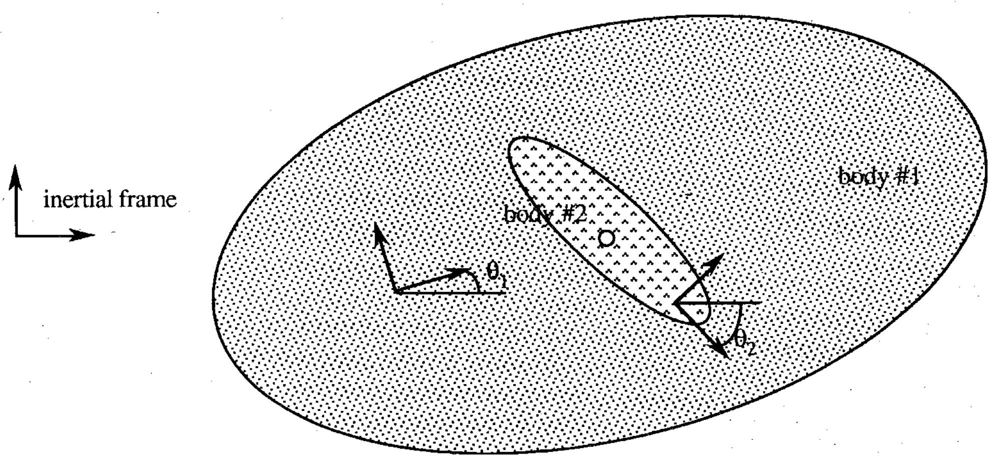
3 Synthesis and future directions: §**13, 14.**

§1 *Some Examples*

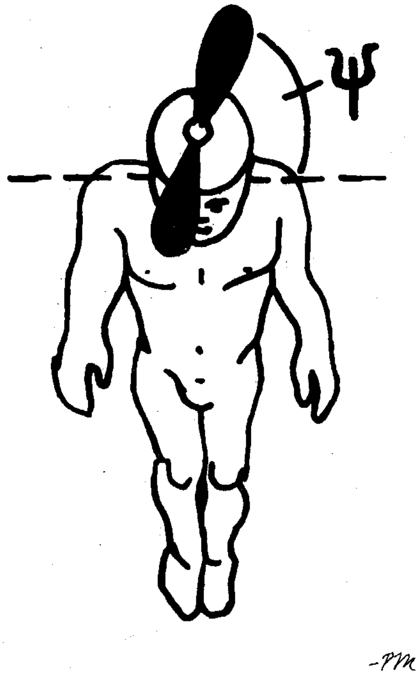
In this section we present some elementary examples exhibiting the general geometric features that will be discussed in the body of the paper. They focus on the ideas of reconstruction of dynamics and the "phases" obtained when reconstruction is performed on a closed loop. In this case, we shall distinguish between a geometric and a dynamic phase. Such phenomena naturally occur in Hamiltonian systems depending on a parameter, for example, in moving systems or in integrable systems depending on a "slow" parameter. The reader will find additional examples in **§5.** In particular, the rotating top in a gravitational field (the heavy top) and the dynamics of a system of planar coupled rigid bodies are treated there. The formula for the phase of the system of coupled planar rigid bodies is first computed by hand in **§5E**, so this can be read as part of the present section if desired.

Before beginning any serious examples, we will give an elementary example—Elroy's beanie—which still illustrates many of the interesting features of more complicated examples. In general, the theory and examples in this work can be divided into two types—those involving adiabatic phenomena and those that are "pure mechanical" or "pure reconstruction". Our first main example on moving systems in **§1A** is of the adiabatic type, while Elroy's beanie is purely mechanical.

Example—Elroy's Beanie Consider two planar rigid bodies joined by a pin joint *at their center of masses*. Let I_1 and I_2 be their moments of inertia, and θ_1, and θ_2 be the angle they make with a fixed inertial direction, as in the figure.



Elroy's Beanie



Elroy and his beanie

Conservation of angular momentum states that $I_1\dot{\theta}_1 + I_2\dot{\theta}_2 = \mu$ = constant in time, where the overdot means time derivative. The ***shape space*** of a system is the space whose points give the shape of the system. In this case, shape space is the circle S^1 parametrized by the ***hinge angle*** $\psi = \theta_2 - \theta_1$. We parametrize the configuration space of the system not by θ_1 and θ_2 but by $\theta = \theta_1$ and ψ. Conservation of angular momentum reads

$$I_1\dot{\theta} + I_2(\dot{\theta} + \dot{\psi}) = \mu\,; \quad \text{that is,} \quad d\theta + \frac{I_2}{I_1 + I_2}\,d\psi = \frac{\mu}{I_1 + I_2}\,dt\,. \tag{1}$$

The left hand side of (1) is the *mechanical connection* discussed in detail in §**2.4.** Suppose that the beanie (body #2) goes through one full revolution so that ψ increases from 0 to 2π. Suppose, moreover, that the total angular momentum is zero: $\mu = 0$. From (1) we see that the entire configuration undergoes a net rotation of

$$\Delta\theta = -\frac{I_2}{I_1 + I_2}\int_0^{2\pi} d\psi = -\left(\frac{I_2}{I_1 + I_2}\right)2\pi . \qquad (2)$$

This is the amount by which Elroy rotates, each time his beanie goes around once.

Notice that the result (2) is independent of the detailed dynamics and only depends on the fact that angular momentum is conserved and the beanie goes around once. In particular, we get the same answer even if there is a "hinge potential" hindering the motion or if there is a control present in the joint. Also note that if Elroy wants to rotate by $-2\pi k \frac{I_2}{I_1 + I_2}$ radians, where k is an integer, all he needs to do is spin his beanie around k times, then reach up and stop it. By conservation of angular momentum, he will stay in that orientation after stopping the beanie.

Here is a geometric interpretation of this calculation. The connection we used is $A_{mech} = d\theta + \frac{I_2}{I_1 + I_2}\, d\psi$. This is a flat connection for the trivial principal S^1-bundle $\pi : S^1 \times S^1 \to S^1$ given by $\pi(\theta, \psi) = \psi$. Formula (2) is the holonomy of this connection, when we traverse the base circle, $0 \le \psi \le 2\pi$. (We note that this is the same connection that appears in the Aharonov-Bohm effect.)

§1A *Moving systems*

Begin with a reference configuration Q and a Riemannian manifold $\mathcal{S}$. Let M be a space of embeddings of Q into $\mathcal{S}$ and let m_t be a curve in M. If a particle in Q is following a curve $q(t)$, and if we imagine the configuration space Q moving by the motion m_t, then the path of the particle in $\mathcal{S}$ is given by $m_t(q(t))$. Thus, its velocity in $\mathcal{S}$ is given by the time derivative:

$$T_{q(t)}m_t \cdot \dot{q}(t) + \mathcal{Z}_t(m_t(q(t))) \qquad (1)$$

where $\mathcal{Z}_t$, defined by $\mathcal{Z}_t(m_t(q)) = \frac{d}{dt} m_t(q)$, is the time dependent vector field (on $\mathcal{S}$ with domain $m_t(Q)$) generated by the motion m_t and $T_{q(t)}m_t \cdot w$ is the derivative (tangent) of the map m_t at the point $q(t)$ in the direction w. To simplify the notation, we write

$$\mathit{m}_t = T_{q(t)}m_t \qquad \text{and} \qquad \mathit{q}(t) = m_t(q(t)).$$

Consider a Lagrangian on TQ of the form kinetic minus potential energy. Using (1), we thus choose

$$L_{m_t}(q, v) = \frac{1}{2}\, \| \mathit{m}_t \cdot v + \mathcal{Z}_t(\mathit{q}(t)) \|^2 - V(q) - U(\mathit{q}(t)) \qquad (2)$$

where V is a given potential on Q and U is a given potential on $\mathcal{S}$.

Put on Q the (possibly time dependent) metric induced by the mapping m_t. In other words, we choose the metric on Q that makes m_t into an isometry for each t. In many examples of mechanical systems, such as the ball in the hoop given below, m_t is already a restriction of an isometry to a submanifold of $\mathcal{S}$, so the metric on Q in this case is in fact time independent. Now we take the Legendre transform of (2), to get a Hamiltonian system on T^*Q. Recall (see, for example, Abraham and Marsden [1978] or Arnold [1978]), that the Legendre transformation is given by $p = \frac{\partial L}{\partial v}$. Taking the derivative of (2) with respect to v in the direction of w gives:

$$p\cdot w = \langle m_t\cdot v + Z_t(q(t)), m_t\cdot w\rangle_{q(t)} = \langle m_t\cdot v + Z_t(q(t))^T, m_t\cdot w\rangle_{q(t)} \tag{3a}$$

where $p\cdot w$ means the natural pairing between the covector $p \in T^*_{q(t)}Q$ and the vector $w \in T_{q(t)}Q$, $\langle\ ,\ \rangle_{q(t)}$ denotes the metric inner product on the space $\mathcal{S}$ at the point $q(t)$ and ^{T} denotes the tangential projection to the space $m_t(Q)$ at the point $q(t)$. Recalling that the metric on Q, denoted $\langle\ ,\ \rangle_{q(t)}$ is obtained by declaring m_t to be an isometry, (3a) gives

$$p\cdot w = \langle v + m_t^{-1}Z_t(q(t))^T, w\rangle_{q(t)} \quad \textit{i.e.,} \quad p = (v + m_t^{-1}Z_t(q(t))^T)^\flat \tag{3b}$$

where $^\flat$ denotes the index lowering operation at q(t) using the metric on Q. The (in general time dependent) Hamiltonian is given by the prescription $H = p\cdot v - L$, which in this case becomes

$$\begin{aligned} H_{m_t}(q, p) &= \tfrac{1}{2}\|p\|^2 - \mathcal{P}(Z_t) - \tfrac{1}{2}\|Z_t^\perp\|^2 + V(q) + U(q(t)) \\ &= H_0(q, p) - \mathcal{P}(Z_t) - \tfrac{1}{2}\|Z_t^\perp\|^2 + U(q(t)), \end{aligned} \tag{4}$$

where $H_0(q, p) = \frac{1}{2}\|p\|^2 + V(q)$, the time dependent vector field $Z_t \in \mathfrak{X}(Q)$ is defined by $Z_t(q) = m_t^{-1}[Z_t(m_t(q))]^T$, the momentum function $\mathcal{P}(Y)$ is defined by $\mathcal{P}(Y)(q, p) = p\cdot Y(q)$ for $Y \in \mathfrak{X}(Q)$, and where $Z_t^\perp$ denotes the orthogonal projection of Z_t to $m_t(Q)$. Even though the Lagrangian and Hamiltonian are time dependent, we recall that the Euler-Lagrange equations for L_{m_t} are equivalent to Hamilton's equations for H_{m_t}. These give the correct equations of motion for this moving system. (An interesting example of this is fluid flow on the rotating earth, where it is important to consider the fluid with the motion of the earth superposed, rather than the motion relative to an observer.)

Let G be a Lie group that acts on Q. (For the ball in the hoop, this will be the dynamics of H_0 itself). We assume for the general theory that H_0 is G-invariant. Assuming the "averaging principle" (*cf.* Arnold [1978], for example) we replace H_{m_t} by its G-average,

$$\langle H_{m_t}\rangle(q, p) = \tfrac{1}{2}\|p\|^2 - \langle\mathcal{P}(Z_t)\rangle - \tfrac{1}{2}\langle\|Z_t^\perp\|^2\rangle + V(q) + \langle U(q(t))\rangle. \tag{5}$$

where $\langle\cdot\rangle$ denotes the G-average. This principle is hard to justify in general and is probably only justified for torus actions for integrable systems. We will only use it in this case in examples, so we proceed with the form (5). Furthermore, we shall discard the term $\frac{1}{2}\langle \| Z_t^{\perp} \|^2 \rangle$; we assume it is small compared to the rest of the terms. Thus, define

$$\mathcal{H}(q, p, t) = \frac{1}{2}\|p\|^2 - \langle \mathcal{P}(Z_t)\rangle + V(q) + \langle U(q(t))\rangle = H_0(q, p) - \langle \mathcal{P}(Z_t)\rangle + \langle U(q(t))\rangle. \qquad (6)$$

The dynamics of $\mathcal{H}$ on the extended space $T^*Q \times M$ is given by the vector field

$$(X_{\mathcal{H}}, Z_t) = \left(X_{H_0} - X_{\langle \mathcal{P}(Z_t)\rangle} + X_{\langle U \circ m_t \rangle}, Z_t\right). \qquad (7)$$

The vector field

$$\mathrm{hor}(Z_t) = \left(-X_{\langle \mathcal{P}(Z_t)\rangle}, Z_t\right) \qquad (8)$$

has a natural interpretation as the horizontal lift of Z_t relative to a connection, which we shall call the ***Hannay-Berry connection induced by the Cartan connection***; see **§11** and **§12**, especially Theorem **11.3**. The holonomy of this connection is interpreted as the Hannay-Berry phase of a slowly moving constrained system. Let us give a few more details for the case of the ball in the rotating hoop.

§1B *The ball in the rotating hoop*

In the following example, we follow some ideas of J. Anandan.

Consider Figure **1B-1** which shows a hoop (not necessarily circular) on which a bead slides without friction. As the bead is sliding, the hoop is slowly rotated in its plane through an angle $\theta(t)$ and angular velocity $\boldsymbol{\omega}(t) = \dot{\theta}(t)\,\mathbf{k}$. Let s denote the arc length along the hoop, measured from a reference point on the hoop and let $\mathbf{q}(s)$ be the vector from the origin to the corresponding point on the hoop; thus the shape of the hoop is determined by this function $\mathbf{q}(s)$. The unit tangent vector is $\mathbf{q}'(s)$ and the position of the reference point $\mathbf{q}(s(t))$ relative to an inertial frame in space is $R_{\theta(t)}\mathbf{q}(s(t))$, where R_θ is the rotation in the plane of the hoop through an angle θ.

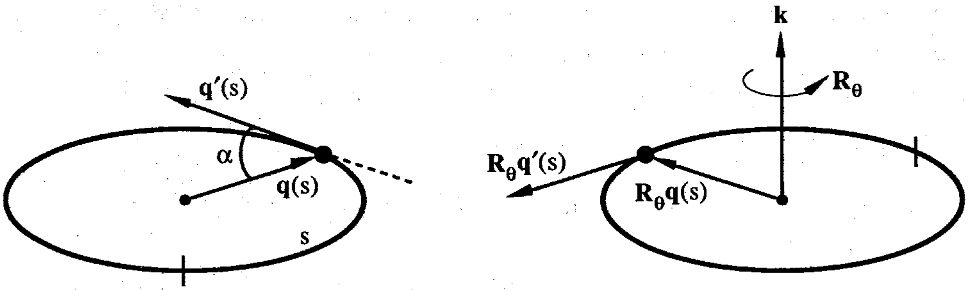


Figure 1B-1

The configuration space is diffeomorphic to the circle $Q = S^1$ with length L the length of the hoop. The Lagrangian $L(s, \dot{s}, t)$ is simply the kinetic energy of the particle; *i.e.*, since

$$\frac{d}{dt} R_{\theta(t)} \mathbf{q}(s(t)) = R_{\theta(t)} \mathbf{q}'(s(t))\, \dot{s}(t) + R_{\theta(t)}[\omega(t) \times \mathbf{q}(s(t))]\,,$$

we set

$$L(s, \dot{s}, t) = \frac{1}{2} m \,\|\, \mathbf{q}'(s)\, \dot{s} + \omega \times \mathbf{q}(s) \,\|^2. \tag{1}$$

The Euler-Lagrange equations

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{s}} = \frac{\partial L}{\partial s}$$

become

$$\frac{d}{dt} m[\dot{s} + \mathbf{q}' \cdot (\omega \times \mathbf{q})] = m[\dot{s}\mathbf{q}'' \cdot (\omega \times \mathbf{q}) + \dot{s}\mathbf{q}' \cdot (\omega \times \mathbf{q}') + (\omega \times \mathbf{q}) \cdot (\omega \times \mathbf{q}')]$$

since $\|\mathbf{q}'\|^2 = 1$. Therefore

$$\ddot{s} + \mathbf{q}'' \cdot (\omega \times \mathbf{q})\dot{s} + \mathbf{q}' \cdot (\dot{\omega} \times \mathbf{q}) = \dot{s}\mathbf{q}'' \cdot (\omega \times \mathbf{q}) + (\omega \times \mathbf{q}) \cdot (\omega \times \mathbf{q}')$$

i.e.,

$$\ddot{s} - (\omega \times \mathbf{q}) \cdot (\omega \times \mathbf{q}') + \mathbf{q}' \cdot (\dot{\omega} \times \mathbf{q}) = \mathbf{0}. \tag{2}$$

The second and third terms in (2) are the centrifugal and Euler forces respectively. We rewrite (2) as

$$\ddot{s} = \omega^2\, \mathbf{q} \cdot \mathbf{q}' - \dot{\omega}\, q \sin \alpha \tag{3}$$

where α is as in Figure 1-1 and $q = \|\mathbf{q}\|$. From (3), Taylor's formula with remainder gives

$$s(t) = s_0 + \dot{s}_0 t + \int_0^t (t - t')\left\{\omega(t')^2\, \mathbf{q} \cdot \mathbf{q}'(s(t')) - \dot{\omega}(t') q(s(t')) \sin \alpha(s(t'))\right\} dt'. \tag{4}$$

Now ω and $\dot{\omega}$ are assumed small with respect to the particle's velocity, so by the averaging theorem (see, e.g. Hale [1969]), the s-dependent quantities in (4) can be replaced by their averages around the hoop:

$$s(T) \approx s_0 + \dot{s}_0 T + \int_0^T (T - t')\left\{\omega(t')^2 \frac{1}{L}\int_0^L \mathbf{q}\cdot\mathbf{q}'\, ds - \dot{\omega}(t')\frac{1}{L}\int_0^L q(s)\sin\alpha\, ds\right\} dt' . \qquad (5)$$

Aside The essence of the averaging can be seen as follows. Suppose $g(t)$ is a rapidly varying function and $f(t)$ is slowly varying on an interval $[a,b]$. Over one period of g, say $[\alpha, \beta]$, we have

$$\int_\alpha^\beta f(t)g(t)dt \approx \int_\alpha^\beta f(t)\bar{g}\, dt \qquad (6)$$

where $\bar{g} = \frac{1}{\beta-\alpha}\int_\alpha^\beta g(t)dt$ is the average of g. The error in (6) is

$$\int_\alpha^\beta f(t)(g(t) - \bar{g})dt$$

which is less than $(\beta - \alpha) \times$ (variation of f) $\times$ constant $\leq$ constant $|f'|(\beta-\alpha)^2$. If this is added up over $[a, b]$ one still gets something small as the period of $g \to 0$. ♦

The first integral in (5) over s vanishes and the second is $2A$ where A is the area enclosed by the hoop. Now integrate by parts:

$$\int_0^T (T - t')\,\dot{\omega}(t')\, dt' = -T\omega(0) + \int_0^T \omega(t')dt' = -T\omega(0) + 2\pi, \qquad (7)$$

assuming the hoop makes one complete revolution in time T. Substituting (7) in (5) gives

$$s(T) = s_0 + \dot{s}_0 T + \frac{2A}{L}\omega_0 T - \frac{4\pi A}{L}. \qquad (8)$$

The initial velocity of the ball *relative to the hoop* is $\dot{s}_0$, while that *relative to the inertial frame* is (see (1)),

$$v_0 = \mathbf{q}'(0)\cdot[\mathbf{q}'(0)\,\dot{s}_0 + \omega_0 \times \mathbf{q}(0)] = \dot{s}_0 + \omega_0\, q(s_0)\sin\alpha(s_0). \qquad (9)$$

Now average (8) and (9) over the *initial conditions* to get

$$\boxed{\langle s(T) - s_0 - v_0 T\rangle = -\frac{4\pi A}{L}} \qquad (10)$$

which means that *on average*, the shift in position is by $\frac{4\pi A}{L}$ between the rotated and nonrotated hoop. Note that if $\omega_0 = 0$ (the situation assumed by Berry [1985]) then averaging over initial conditions is not necessary. This process of averaging over the initial conditions that we naturally encounter in this example is related to the recent work of Golin and Marmi [1989] on experimental procedures to measure the phase shift.

This extra length $\frac{4\pi A}{L}$ is sometimes called the ***Hannay-Berry phase***. Expressed in angular measure, it is $\frac{8\pi^2 A}{L^2}$. In §**11B** we show, using the Cartan connection, how to realize this answer as the holonomy of the associated Hannay-Berry connection.

§1C *Coupled planar pendula*

We return now to an example similar to Elroy's beanie, with which we began. Consider two coupled pendula in the plane moving under the influence of a potential depending on the hinge angle between them. Let r_1, r_2 be the distances from the joint to their centers of mass and let θ_1 and θ_2 be the angles formed by the straight lines through the joint and their centers of mass relative to an inertial coordinate system fixed in space, as in Figure **1C-1**. The Lagrangian of this system is

$$L = \frac{1}{2} m_1 r_1^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 r_2^2 \dot{\theta}_2^2 - V(\theta_1 - \theta_2)$$

and is therefore of the form kinetic minus potential energy, where the kinetic energy is given by the metric on $\mathbb{R}^2$

$$ds^2 = m_1 r_1^2 \, d\theta_1^2 + m_2 r_2^2 \, d\theta_2^2 \quad .$$

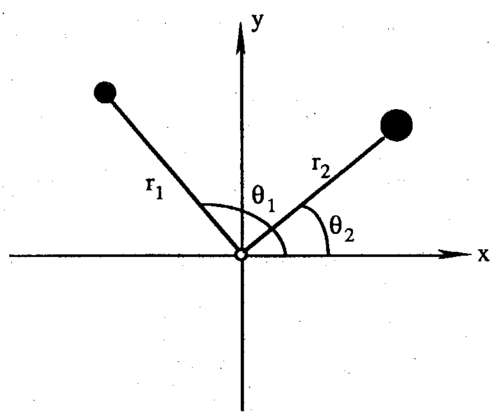


Figure 1C-1

To illustrate the ideas, we look at the special case $m_i = r_i = 1$ so that

$$L = \frac{1}{2}(\dot\theta_1^2 + \dot\theta_2^2) - V(\theta_1 - \theta_2)$$

and

$$ds^2 = d\theta_1^2 + d\theta_2^2 = \frac{1}{2}d(\theta_1 - \theta_2)^2 + \frac{1}{2}d(\theta_1 + \theta_2)^2 \quad .$$

The group S^1 acts on configuration space $\mathbb{T}^2 = \{(\theta_1, \theta_2)\}$ by $\theta \cdot (\theta_1, \theta_2) = (\theta + \theta_1, \theta + \theta_2)$ so L is invariant under this action. Letting $\varphi = (\theta_1 + \theta_2)/\sqrt{2}$ and $\psi = (\theta_1 - \theta_2)/\sqrt{2}$, we see that $\theta \cdot (\varphi, \psi) = (\varphi + \theta, \psi)$ and hence that the induced momentum map for the lifted action $J : T^*\mathbb{T}^2 \to \mathbb{R}$ is given by $J(\varphi, \psi, p_\varphi, p_\psi) = p_\varphi$. Therefore the reduced space $J^{-1}(\mu)/S^1$ is diffeomorphic to $T^*S^1 = \{(\psi, p_\psi)\}$ with the canonical symplectic structure. The Hamiltonian on $T^*\mathbb{T}^2$ is

$$H(\varphi, \psi, p_\varphi, p_\psi) = \frac{1}{2}(p_\varphi^2 + p_\psi^2) + V(\sqrt{2}\,\psi)$$

and the reduced Hamiltonian is

$$H_\mu(\psi, p_\psi) = \frac{1}{2}p_\psi^2 + V(\sqrt{2}\,\psi) \quad .$$

The equations of motion for H are

$$\dot\varphi = p_\varphi, \quad \dot p_\varphi = 0 \tag{1}$$

$$\dot\psi = p_\psi, \quad \dot p_\psi = -\sqrt{2}\,V'(\sqrt{2}\,\psi)\,. \tag{2}$$

Equations (2) are Hamilton's equations for H_μ on the reduced space.

Assume that we have solved (2) with initial conditions (ψ_0, p_{ψ_0}) and are given the initial conditions $(\varphi_0, \psi_0, p_{\varphi_0} = \mu, p_{\psi_0})$ of a solution for the system (1), (2). To find the solution for $(\varphi(t), \psi(t), p_\varphi(t), p_\psi(t))$ of (1), (2) we proceed in two steps:

Step 1 *Consider the curve* $d(t) = (\varphi_0, \psi(t), \mu, p_\psi(t))$.

Step 2 *Solve the equation* $\theta'(t) = \mu$, $\theta(0) = 0$ *yielding* $\theta(t) = \mu t$.

Then the solution to (1), (2) *is given by* $c(t) = \theta \cdot d(t) = (\varphi_0 + \mu t, \psi(t), \mu, p_\psi(t))$.

This method is quite general and applies to all Hamiltonian systems. We will discuss it in §**2** and §**3**. To get a feeling of what is happening, we make some remarks. The principal S^1-bundle $\mathbf{J}^{-1}(\mu) = \{(\varphi, \psi, \mu, p_\psi)\} \to T^*S^1$, $(\varphi, \psi, \mu, p_\psi) \mapsto (\psi, p_\psi)$ has a connection whose horizontal space at any point is generated by the vector fields $\left\{\frac{\partial}{\partial\psi}, \frac{\partial}{\partial p_\psi}\right\}$. Then the curve $d(t)$ in **Step 1** is simply the horizontal lift of the integral curve of the reduced system $(\psi(t), p_\psi(t))$

through $(\varphi_0, \psi_0, \mu, p_{\psi_0})$. Note that this connection on $J^{-1}(\mu) \to T^*S^1$ is the pull-back of the connection on $\mathbb{T}^2 \to S^1$, $(\varphi, \psi) \mapsto \psi$, whose horizontal space at any point is generated by $\frac{\partial}{\partial\psi}$, by the map which is the restriction of the cotangent bundle projection $T^*\mathbb{T}^2 \to \mathbb{T}^2$ to $J^{-1}(\mu)$. Relative to this connection and identifying $T^*\mathbb{T}^2$ with $T\mathbb{T}^2$ using the kinetic energy metric $ds^2 = d\varphi^2 + d\psi^2$, μ is the generator of the vertical part of this curve; note $\left\| \frac{\partial}{\partial\varphi} \right\|^2 = \left\| \frac{\partial}{\partial\psi} \right\|^2 = 1$. Thus the differential equation in **Step 2** is on the group S^1 and has right hand side given by the generator of the vertical part of the horizontally lifted curve in **Step 1**. Roughly, this describes the method of ***reconstruction of dynamics***. We shall explain this in §**2** and address the specific case of Lagrangian systems in §3, circumventing the use of the connection in **Step 1**.

§1D *Coupled bodies, linkages and optimal control*

The above example can be generalized to the case of coupled rigid bodies. Already the case of a single rigid body in space is an interesting example that will be discussed in §**1G** below. For several coupled rigid bodies, the dynamics is quite complex. For instance for bodies in the plane, the dynamics is known to be chaotic, despite the presence of stable relative equilibria. See Oh, Sreenath, Krishnaprasad, and Marsden [1989]. Berry phase phenomena for this type of example are quite interesting and are related to some of the work of Wilczek and Shapere on locomotion in micro-organisms. (See, for example, Shapere and Wilczek [1987]). In this problem, control of the system's *internal variables* can lead to phase changes in the *external variables*. These choices of variables are related to the variables in the reduced and the unreduced phase spaces, as we shall see. In this setting one can formulate interesting questions of optimal control such as "when a cat falls and turns itself over in mid flight (all the time with zero angular momentum!) does it do so with optimal efficiency in terms of say energy expended?"

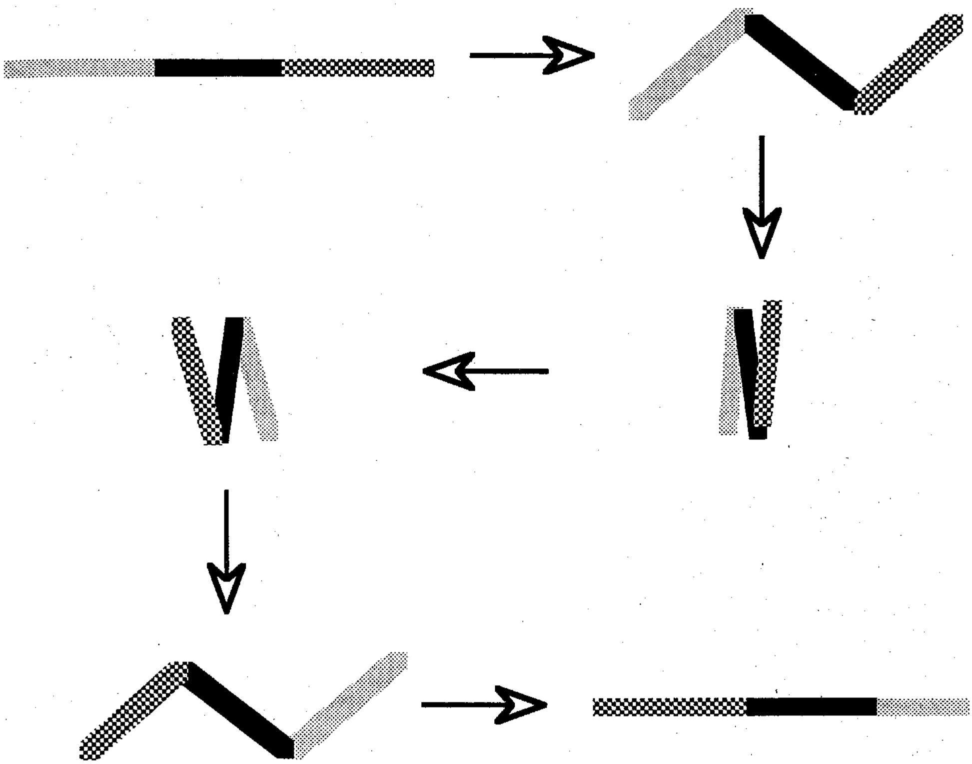


Figure 1D-1

There are interesting answers to these questions that are related to the dynamics of Yang-Mills particles moving in the associated gauge field of the problem. See Montgomery [1989] and references therein. This bundle approach to mechanics will be a theme developed in this work as well.

We give two simple examples of how this works. Additional details will be given for this type of example in §5. First, consider three coupled bars (or coupled planar rigid bodies) linked together with pivot (or pin) joints, so the bars are free to rotate relative to each other. Assume the bars are moving freely on the plane with no external forces and that the angular momentum is zero. However, assume that the joint angles can be controlled with, say, motors in the joints. Figure **1D-1** shows how the joints can be manipulated, each one going through an angle of 2π and yet the overall assemblage rotates through and angle π. A formula for the reconstruction phase applicable to examples of this type is given in Krishnaprasad [1989].

A second example is the dynamics of linkages; see §5E for more details. This type of example is consideres in Krishnaprasad [1990], Yang and Krishnaprasad [1989], and Krishnaprasad and Yang [1990], including comments on the relation with the three manifold theory of Thurston. Here one considers a linkage of rods, say four rods linked by pivot joints as in Figure **1D-2.** The system is free to rotate without external forces or torques, but there are assumed to be torques at the joints. When one turns the small "crank" the whole assemblage turns even though the angular momentum, as in the previous example, stays zero.

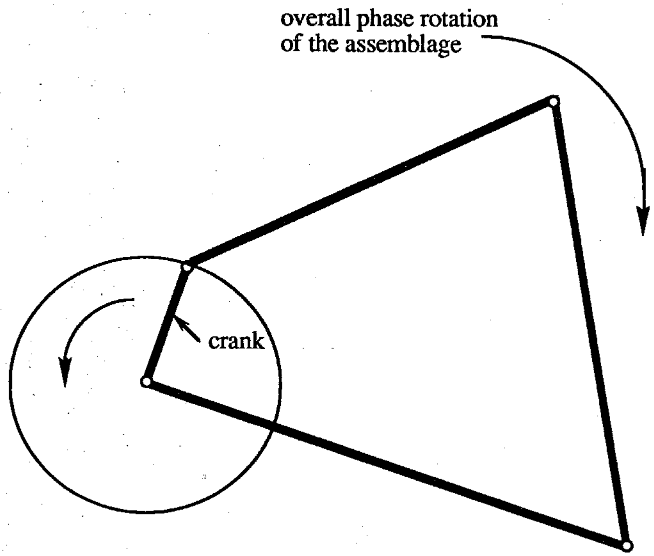


Figure 1D-2

§1E *Quantum mechanics*

The original motivation for geometic phases came from quantum mechanics. Here the important contributions historically were by Kato in 1950 (for the quantum adiabatic theorem), Longuet-Higgins in 1958 for anomalous spectra in rotating molecules, Berry [1984] who first saw the geometry of the phenomena for a variety of systems, and Simon [1983] who explicitly realized the phases as the holonomy of the Chern-Bott connection. For more information on quantum mechanical phases, and for the references quoted, see the collection of papers in Shapere and Wilczek [1988].

For the purposes of the present work, the paper of Aharonov and Anandan [1987] plays an important role. They got rid of the adiabaticity and showed that the phase for a closed loop in projectivized complex Hilbert space is the exponential of the symplectic area of a two-dimensional manifold whose boundary is the given loop. The symplectic form in question is naturally induced on the projective space from the canonical symplectic form of complex Hilbert space (minus the imaginary part of the inner product) via reduction, as in Abraham and Marsden [1978]. We shall show in §4 that this formula is the holonomy of the closed loop relative to a principal S^1-connection on complex Hilbert space and is a particular case of the holonomy formula in principal bundles with abelian structure group.

Littlejohn [1988] has shown that the Bohr-Sommerfeld and Maslov phases of semi-classical mechanics can be viewed as incarnations of Berry's phase. To do this he notes that Gaussian wave-packets define an embedding of classical phase space into Hilbert space, then uses the Aharonov-Anandan point of view on phases, together with the variational formulation of quantum mechanics. The quantum-classical relation between the phases is also considered in Hannay [1985], Anandan [1988], and Weinstein [1989a,b].

§1F *Integrable systems*

Consider an integrable system with action-angle variables $(I_1, I_2, ..., I_n, \theta_1, \theta_2, ..., \theta_n)$ and with a Hamiltonian $H(I_1, I_2, \ldots I_n, \theta_1, \theta_2, \ldots \theta_n; m)$ that depends on a parameter $m \in M$. Let c be a loop based at a point m_0 in M. We want to compare the angular variables in the torus over m_0, once the system is slowly changed as the parameters undergo the circuit c. Since the dynamics in the fiber varies as we move along c, even if the actions vary by a negligible amount, there will be a shift in the angle variables due to the frequencies $\omega^i = \partial H/\partial I^i$ of the integrable system; correspondingly, one defines

$$\textbf{\textit{dynamic phase}} = \int_0^1 \omega^i(I, c(t))\, dt \ .$$

Here we assume that the loop is contained in a neighborhood whose standard action coordinates are defined. In completing the circuit c, we return to the same torus, so a comparison between the angles makes sense. The actual shift in the angular variables during the circuit is the dynamic phase plus a correction term called the ***geometric phase***. One of the key results is that this geometric phase is the holonomy of an appropriately constructed connection called the ***Hannay-Berry connection*** on the torus bundle over M which is constructed from the action-angle variables. The corresponding angular shift, computed by Hannay [1985], is called ***Hannay's angles***, so the actual phase shift is given by

$$\boxed{\Delta\theta = \text{dynamic phases} + \text{Hannay's angles}\ .}$$

The geometric construction of the Hannay-Berry connection for classical systems is given in terms of momentum maps and averaging in Golin, Knauf, and Marmi [1989] and Montgomery [1988].

In this paper we will put this geometry into a more general context and will synthesise it with our work on connections associated with moving systems.

§1G *The Rigid Body*

The motion of a rigid body is a geodesic with respect to a left-invariant Riemannian metric (the inertia tensor) on $\mathbf{SO}(3)$. The corresponding phase space is $P = T^*\mathbf{SO}(3)$ and the momentum map $\mathbf{J} : P \to \mathbb{R}^3$ for the *left* $\mathbf{SO}(3)$ action is *right* translation to the identity. We identify $\mathbf{so}(3)^*$ with $\mathbf{so}(3)$ via the Killing form and identify $\mathbb{R}^3$ with $\mathbf{so}(3)$ via the map $v \mapsto \hat{v}$ where $\hat{v}(w) = v \times w$, $\times$ being the standard cross product. Points in $\mathbf{so}(3)^*$ are regarded as the left reduction of $T^*\mathbf{SO}(3)$ by $\mathbf{SO}(3)$ and are the angular momenta as seen from a *body-fixed* frame. The reduced spaces $\mathbf{J}^{-1}(\mu)/G_\mu$ are identified with spheres in $\mathbb{R}^3$ of Euclidean radius $\|\mu\|$, with their symplectic form $\omega_\mu = -dS/\|\mu\|$ where dS is the standard area form on a sphere of radius $\|\mu\|$ and where G_μ consists of rotations about the μ-axis. The trajectories of the reduced dynamics are obtained by intersecting a family of homothetic ellipsoids (the energy ellipsoids) with the angular momentum spheres. In particular, all but at most four of the reduced trajectories are periodic. These four exceptional trajectories are the well known homoclinic trajectories.

Suppose a reduced trajectory $\Pi(t)$ is given on P_μ, with period T. *After time* T, *by how much has the rigid body rotated in space*? The spatial angular momentum is $\pi = \mu = g\Pi$, which is the conserved value of $\mathbf{J}$. Here $g \in \mathbf{SO}(3)$ is the attitude of the rigid body and Π is the body angular momentum. If $\Pi(0) = \Pi(T)$ then $\mu = g(0)\Pi(0) = g(T)\Pi(T)$ and so $g(T)^{-1}\mu = g(0)^{-1}\mu$ *i.e.*, $g(T)g(0)^{-1}$ is a rotation about the axis μ. We want to compute the angle of this rotation.

To answer this question, let $c(t)$ be the corresponding trajectory in $\mathbf{J}^{-1}(\mu) \subset P$. Identify $T^*\mathbf{SO}(3)$ with $\mathbf{SO}(3) \times \mathbb{R}^3$ by left trivialization, so $c(t)$ gets identified with $(g(t), \Pi(t))$. Since the reduced trajectory $\Pi(t)$ closes after time T, we recover the fact that $c(T) = g\,c(0)$ for some $g \in G_\mu$. Here, $g = g(T)g(0)^{-1}$ in the preceding notation. Thus, we can write

$$g = \exp[(\Delta\theta)\zeta] \tag{1}$$

where $\zeta = \mu/\|\mu\|$ identifies $\mathfrak{g}_\mu$ with $\mathbb{R}$ by $a\zeta \mapsto a$, for $a \in \mathbb{R}$. Let D be one of the two spherical caps on S^2 enclosed by the reduced trajectory, Λ be the corresponding oriented solid angle, *i.e.*, $|\Lambda| = (\text{area } D)/\|\mu\|^2$, and let H_μ be the energy of the reduced trajectory. All norms are taken relative to the Euclidean metric of $\mathbb{R}^3$. We shall prove below that modulo 2π, we have

$$\boxed{\Delta\theta = \frac{1}{\|\mu\|}\left\{\int_D \omega_\mu + 2H_\mu T\right\} = -\Lambda + \frac{2H_\mu T}{\|\mu\|}\,.} \tag{2}$$

(The special case of this formula for a symmetric free rigid body was given by Hannay [1985] and Anandan [1988], formula (20)).

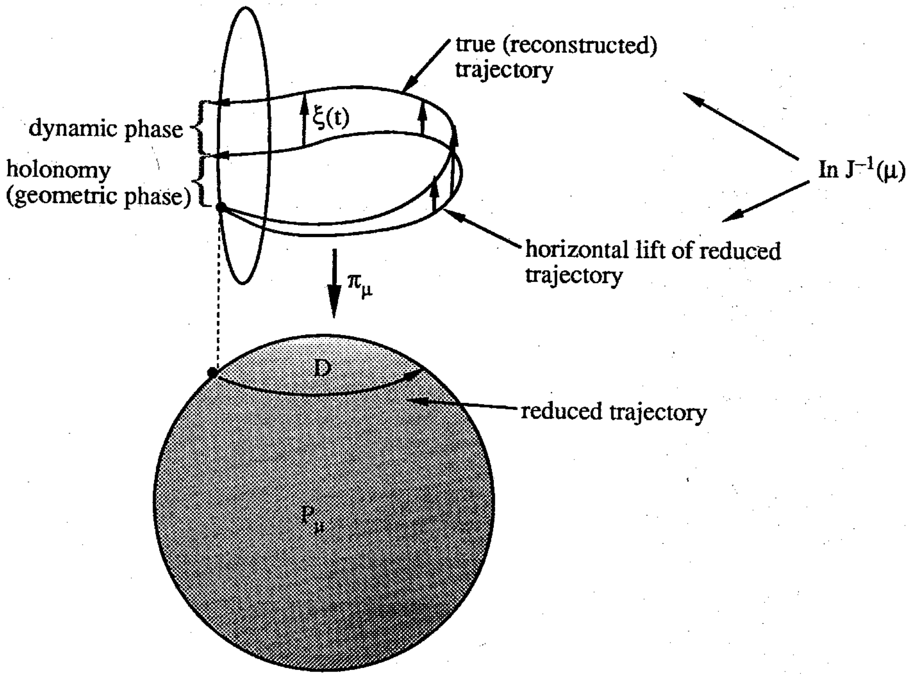


Figure 1G-1 For $G_\mu = S^1$, (log holonomy) $= \frac{1}{\|\mu\|}\int_D \omega_\mu$, (log dynamic phase) $= \frac{1}{\|\mu\|}\int_0^T \xi(t)\,dt$, where, T = period of reduced trajectory and ω_μ = reduced symplectic form.

To prove (2), we choose the connection one-form on $\mathbf{J}^{-1}(\mu)$ to be (see Proposition **2.2**)

$$A = \frac{1}{\|\mu\|}\theta_\mu \tag{3}$$

where θ_μ is the pull back to $\mathbf{J}^{-1}(\mu)$ of the canonical one-form θ on $T^*\mathbf{SO}(3)$. The curvature of A as a two-form on the base P_μ, the sphere of radius $\|\mu\|$ in $\mathbb{R}^3$, is given by

$$-\frac{1}{\|\mu\|}\omega_\mu = \frac{1}{\|\mu\|^2}\, dS\ . \tag{4}$$

The first terms in (2) represent the *geometric phase, i.e.,* the holonomy of the reduced trojectory with respect to this connection. By Corollary **4.2**, the logarithm of the holonomy (modulo 2π) is given as minus the integral over D of the curvature, *i.e.,* it equals

$$\frac{1}{\|\mu\|}\int_D \omega_\mu = -\frac{1}{\|\mu\|^2}\ (\text{area D}) = -\Lambda\ (\text{mod } 2\pi) \tag{5}$$

The second terms in (2) represent the dynamic phase. By the algorithm of Proposition **2.1** it is calculated in the following way. First one horizontally lifts the reduced closed trojectory $\Pi(t)$ to $\mathbf{J}^{-1}(\mu)$ relative to the connection (3). This horizontal lift is easily seen to be (identity, $\Pi(t)$) in the left trivalization of $T^*\mathbf{SO}(3)$ as $\mathbf{SO}(3) \times \mathbb{R}^3$. Second, one computes

$$\xi(t) = (A \cdot X_H)(\Pi(t))\ . \tag{6}$$

Since in coordinates

$$\theta_\mu = \sum_i p_i\, dq^i \text{ and } X_H = \sum_i p^i \frac{\partial}{\partial q^i} + \frac{\partial}{\partial p} \text{ terms}$$

for $p^i = \sum_j g^{ij} p_j$, g^{ij} being the inverse of the Riemannian metric g_{ij} on $\mathbf{SO}(3)$, we get

$$(\theta_\mu \cdot X_H)(\Pi(t)) = \sum_i p_i p^i = 2H(\text{identity, } \Pi(t)) = 2H_\mu, \tag{7}$$

where H_μ is the value of the energy on S^2 along the integral curve $\Pi(t)$. Consequently,

$$\xi(t) = \frac{2H_\mu}{\|\mu\|}\,\zeta \tag{8}$$

Third, since $\xi(t)$ is independent of t, the solution of the equation

$$\dot g = g\xi = \frac{2H_\mu}{\|\mu\|}\, g\zeta \quad \text{is} \quad g(t) = \exp\left(\frac{2H_\mu t}{\|\mu\|}\,\zeta\right)$$

so that the dynamic phase equals

$$\Delta\theta_d = \frac{2H_\mu}{\|\mu\|}\, T \ \ (\text{mod } 2\pi) \tag{9}$$

Formulas (5) and (9) prove (2). Note that (2) is independent of which spherical cap one chooses amongst the two bounded by $\Pi(t)$. Indeed, the solid angles on the unit sphere defined by the two caps add to 4π, which does not change formula (2).

§2 *Reconstruction of Dynamics for Hamiltonian Systems*

This section presents a reconstruction method for the dynamics of a given Hamiltonian system from that of the reduced system. The method and formulas found here are applied in the next section to Lagrangian systems.

§2A *General Considerations*

We begin with the abstract reconstruction method. Let P be a Poisson manifold on which a Lie group acts in a Hamiltonian manner and has a momentum map $\mathbf{J} : P \to \mathfrak{g}^*$; here $\mathfrak{g}$ is the Lie algebra of G and $\mathfrak{g}^*$ is its dual. For a weakly regular value $\mu \in \mathfrak{g}^*$ of $\mathbf{J}$, assuming that the ***reduced space*** $P_\mu := \mathbf{J}^{-1}(\mu)/G_\mu$ is a smooth manifold with the canonical projection a surjective submersion, P_μ a Poisson manifold (see Marsden and Ratiu [1986] for the general theory of Poisson reduction). Given f, h : $P_\mu \to \mathbb{R}$, lift them to $\mathbf{J}^{-1}(\mu)$ by π_μ, then extend them to G-invariant functions on $\mathbf{J}^{-1}(\mathcal{O}_\mu)$, where $\mathcal{O}_\mu$ is the coadjoint orbit of μ in $\mathfrak{g}^*$, and then extend these functions arbitrarily to $\bar{f}, \bar{h} : P \to \mathbb{R}$. The Poisson bracket of f and h in the Poisson structure of P_μ is defined by $\{f, h\} \circ \pi_\mu = \{\bar{f} | \mathcal{O}_\mu, \bar{h} | \mathcal{O}_\mu\}$. If P is symplectic, then P_μ is also symplectic (see Marsden and Weinstein [1974] and Abraham and Marsden [1978], Chapter 4). If $H : P \to \mathbb{R}$ is a G-invariant Hamiltonian it induces a Hamiltonian $H_\mu : P_\mu \to \mathbb{R}$ and the flow of the Hamiltonian vector field X_{H_μ} on P_μ is the G_μ-quotient of the flow of X_H on $\mathbf{J}^{-1}(\mu)$.

Assume that an integral curve $c_\mu(t)$ of X_{H_μ} on P_μ is known. For $p_0 \in \mathbf{J}^{-1}(\mu)$, we search for the corresponding integral curve $c(t) = F_t(p_0)$ of X_H such that $\pi_\mu(c(t)) = c_\mu(t)$, where $\pi_\mu : \mathbf{J}^{-1}(\mu) \to P_\mu$ is the projection. Pick a smooth curve d(t) in $\mathbf{J}^{-1}(\mu)$ such that $d(0) = p_0$ and $\pi_\mu(d(t)) = c_\mu(t)$. Write $c(t) = \Phi_{g(t)}(d(t))$ for some curve g(t) in G_μ to be determined. We have

$$X_H(c(t)) = c'(t) = T_{d(t)}\Phi_{g(t)}(d'(t)) + T_{d(t)}\Phi_{g(t)} \cdot \left(T_{g(t)}L_{g(t)^{-1}}(g'(t))\right)_P(d(t)). \qquad (1)$$

Since $\Phi_g^* X_H = X_{\Phi_g^* H} = X_H$, (1) gives

$$d'(t) + \left(T_{g(t)}L_{g(t)^{-1}}(g')(t)\right)_P(d(t)) = T\Phi_{g(t)^{-1}} X_H(\Phi_{g(t)}(d(t))) = (\Phi_{g(t)}^* X_H)(d(t)) = X_H(d(t)). \quad (2)$$

This is an equation for g(t) written in terms of d(t) only. We solve it in two steps:

Step 1 *Find* $\xi(t) \in \mathfrak{g}_\mu$ *such that*

$$\xi(t)_P(d(t)) = X_H(d(t)) - d'(t); \tag{3}$$

Step 2 *With* $\xi(t)$ *determined, solve the non-autonomous ordinary differential equation on* G_μ:

$$g'(t) = T_eL_{g(t)}(\xi(t)), \quad \text{with } g(0) = e. \tag{4}$$

Step **1** is typically of an algebraic nature; in coordinates, for matrix Lie groups, (3) is just a matrix equation. We show later how $\xi(t)$ can be explicitly computed if a connection is given on $\mathbf{J}^{-1}(\mu) \to P_\mu$. Step **2** gives an answer "in quadratures" if G is abelian as we shall see in formula (5) below. In general (4) cannot be solved explicitly and represents the main technical difficulty in the reconstruction method. With $g(t)$ determined, the desired integral curve $c(t)$ is given by $c(t) = \Phi_{g(t)}(d(t))$. The same construction works on P/G, even if the G-action does not admit a momentum map.

Step **2** can be carried out *explicitly* when G is abelian. Here the connected component of the identity of G is a cylinder $\mathbb{R}^p \times \mathbb{T}^{k-p}$ and the exponential map $\exp(\xi_1, ..., \xi_k) = (\xi_1, ..., \xi_p, \xi_{p+1}(\text{mod } 2\pi), ..., \xi_k(\text{mod } 2\pi))$ is onto, so we can write $g(t) = \exp \eta(t)$, $\eta(0) = 0$. Therefore $\xi(t) = T_{g(t)}L_{g(t)^{-1}}(g'(t)) = \eta'(t)$ since η' and η commute, *i.e.*, $\eta(t) = \int_0^t \xi(s)ds$. Thus the solution of (4) in Step **2** when G is abelian is

$$g(t) = \exp\left(\int_0^t \xi(s)ds\right) \tag{5}$$

This reconstruction method depends on the choice of $d(t)$. With additional structure, $d(t)$ can be chosen in a *natural geometric* way. What is needed is a way of lifting curves on the base of a principal bundle to curves in the total space. We do this using connections. One can object at this point that at the moment, reconstruction involves integrating one ordinary differential equation, whereas introducing a connection will involve integration of *two* ordinary differential equations, one for the horizontal lift and one for constructing the solution of (4) from it. However, for the determination of phases, there are some situations in which the phase can be computed without actually solving either equation, so one actually solves *no* differential equations; a specific case is the rigid body, discussed in §**1G**. In other circumstances, one can compute the horizontal lift explicitly (see Marsden, Ratiu, and Raugel [1990]). However, in general, without such added information, it is true that the number of equations in principle is two rather than one.

Suppose that $\pi_\mu : \mathbf{J}^{-1}(\mu) \to P_\mu$ is a principal G_μ-bundle with a ***connection*** A. This means that A is a $\mathfrak{g}_\mu$-valued one-form on $\mathbf{J}^{-1}(\mu) \subset P$ satisfying i $A_p \cdot \xi_P(p) = \xi$ for $\xi \in \mathfrak{g}_\mu$

ii $L_g^* A = \mathrm{Ad}_g \circ A$ for $g \in G_\mu$.

Let $d(t)$ be the ***horizontal lift*** of c_μ through p_0; *i.e.*, $A \cdot d'(t) = 0$, $\pi_\mu \circ d = c_\mu$, and $d(0) = p_0$.

2.1 Theorem *Suppose* $\pi_\mu : \mathbf{J}^{-1}(\mu) \to P_\mu$ *is a principal* G_μ*-bundle with connection* A. *Let* c_μ *be an integral curve of the reduced dynamical system on* P_μ. *Then the corresponding curve* c *(through* $p_0 \in \pi_\mu^{-1}(c_\mu(0))$*) of the system on* P *is determined as follows:*

i *Horizontally lift* c_μ *to form the curve* d *in* $\mathbf{J}^{-1}(\mu)$ *through* p_0.

ii *Set* $\xi(t) = A \cdot X_H(d(t))$, *so that* $\xi(t)$ *is a curve in* $\mathfrak{g}_\mu$.

iii *Solve the equation* $\dot{g}(t) = g(t) \cdot \xi(t)$.

Then $c(t) = g(t) \cdot d(t)$ *is the integral curve of the system on* P *with initial condition* p_0.

Suppose c_μ is a closed curve; thus, both c and d reintersect the same fiber. Write

$$d(1) = g \cdot d(0) \quad \text{and} \quad c(1) = h \cdot c(0)$$

for $g, h \in G_\mu$. Note that

$$h = g(1)\, g . \tag{6}$$

The Lie group element g (or the Lie algebra element $\log g$) is called the ***geometric phase***. It is the holonomy of the path c_μ with respect to the connection A and has the important property of being parametrization independent. The Lie group element $g(1)$ (or $\log g(1)$) is called the ***dynamic phase***.

For compact or semi-simple G, G_μ is generically abelian. The computation of $g(1)$ and g are then relatively easy, as was indicated above.

§2B *Cotangent Bundle with G_μ one Dimensional*

We now discuss the case in which $P = T^*Q$, and G acts on Q and therefore on P by cotangent lift. In this case the momentum map is given by the formula

$$J(\alpha_q) \cdot \xi = \alpha_q \cdot \xi_Q(q) = \xi_{T^*Q}(\alpha_q) \lrcorner\, \theta(\alpha_q),$$

where $\xi \in \mathfrak{g}_\mu$, $\alpha_q \in T_q^*Q$, $\theta = \sum_i p_i\, dq^i$ is the canonical one-form and $\lrcorner$ is the interior product.

Assume G_μ is the circle group or the line. Pick a generator $\zeta \in \mathfrak{g}_\mu$, $\zeta \neq 0$. For instance, one can choose the shortest ζ such that $\exp(2\pi\zeta) = 1$. Identify $\mathfrak{g}_\mu$ with the real line via $\omega \mapsto \omega\zeta$. Then a connection one-form is a standard one-form on $\mathbf{J}^{-1}(\mu)$.

2.2 Proposition *Suppose* $G_\mu \cong S^1$ or $\mathbb{R}$. *Identify* $\mathfrak{g}_\mu$ *with* $\mathbb{R}$ *via a choice of generator* ζ. *Let* θ_μ *denote the pull-back of the canonical one-form to* $\mathbf{J}^{-1}(\mu)$. *Then*

$$\boxed{A = \frac{1}{\langle\mu, \zeta\rangle}\theta_\mu \otimes \zeta}$$

is a connection one-form on $\mathbf{J}^{-1}(\mu) \to P_\mu$. *Its curvature as a two-form on the base* P_μ *is*

$$\boxed{\Omega = -\frac{1}{\langle\mu, \zeta\rangle}\,\omega_\mu ,}$$

where ω_μ *is the reduced symplectic form on* P_μ.

Proof Since G acts by cotangent lift, it preserves θ, and so θ_μ is preserved by G_μ and therefore A is G_μ-invariant. Also, $A\cdot\zeta_P = [\zeta_P \lrcorner\, \theta / \langle\mu, \zeta\rangle]\zeta = [\mathbf{J}^\zeta/\langle\mu, \zeta\rangle]\zeta = \zeta$. This verifies that A is a connection. The calculation of its curvature is straightforward. (See §**4** and note that $\omega = -\,\mathbf{d}\theta$ in our conventions.) ■

Remarks 1 The result of Proposition **2.2** holds for any *exact* symplectic manifold. We shall use this in §5A.

2 In the next section we shall show how to construct a connection on $\mathbf{J}^{-1}(\mu) \to P_\mu$ in general. For the case $Q = G$ it includes the connection in **2.2**. For $Q = SO(3)$ this recovers the connection for the rigid body. We will return to this point shortly.

§2C *Cotangent Bundles - General Case*

If G_μ is not abelian, the formula for A given above does not satisfy the second axiom of a connection. However, if the bundle $Q \to Q/G_\mu$ has a connection, we will show below how this induces a connection on $\mathbf{J}^{-1}(\mu) \to (T^*Q)_\mu$. To do this, we recall the cotangent bundle reduction theorem of Satzer, Marsden and Kummer (see Abraham and Marsden [1978], §**4.3** and Kummer [1981]).

Assume the Lie group G acts freely on the left on Q; lift this to a symplectic action on T^*Q. The momentum map of this lift is $\mathbf{J}(\alpha_q)\cdot\xi = \alpha_q\cdot\xi_Q(q)$, where $\alpha_q \in T_q^*Q$, $\xi \in \mathfrak{g}$, and

$\xi_Q(q) = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} (\exp \varepsilon\xi \cdot q)$ is the infinitesimal generator of the G-action on Q defined by ξ. Assume that μ is a weakly regular value and that $J^{-1}(\mu)/G_\mu = (T^*Q)_\mu$ is a smooth manifold and that the canonical projection $\pi_\mu : J^{-1}(\mu) \to (T^*Q)_\mu$ is a surjective submersion. Let $\mathfrak{g}_\mu = \{\eta \in \mathfrak{g} \mid (\mathrm{ad}\,\eta)^*\mu = 0\}$ be the isotropy Lie algebra at μ; *i.e.*, the Lie algebra of G_μ, and denote by $\mu' = (\mu \mid \mathfrak{g}_\mu) \in \mathfrak{g}_\mu^*$, the restriction of μ to $\mathfrak{g}_\mu$. Assume that $\rho_\mu : Q \to Q/G_\mu$ is a principal bundle and $\gamma \in \Omega^1(Q; \mathfrak{g}_\mu)$ is a left connection one-form on Q, *i.e.*, $\gamma(\eta_Q) = \eta$ for all $\eta \in \mathfrak{g}_\mu$ and γ is G_μ-equivariant: $\gamma_{g\cdot q}(g \cdot v) = \mathrm{Ad}_g(\gamma(v))$ for all $g \in G_\mu$, $q \in Q$, $v \in T_qQ$.

2.3 Theorem *Let* curv(γ) *be the curvature of* γ *and let* B *be the pull-back by the cotangent bundle projection* $T^*(Q/G_\mu) \to Q/G_\mu$ *of the two form on* Q/G_μ *induced by the* μ'*-component* $\mu'\cdot\mathrm{curv}(\gamma) \in \Omega^2(Q)$ *of* curv(γ); *thus* $\mu'\cdot\mathrm{curv}(\gamma)$ *is a closed real-valued two-form on* Q *and* B *is a closed two-form on* $T^*(Q/G_\mu)$. *Endow* $T^*(Q/G_\mu)$ *with the symplectic form* $\omega - B$, *where* ω *is the canonical two-form of the cotangent bundle. Then* $(T^*Q)_\mu$ *is symplectically embedded in* $(T^*(Q/G_\mu), \omega - B)$ *and its image is a vector subbundle with base* Q/G_μ. *This embedding is onto if and only if* $\mathfrak{g} = \mathfrak{g}_\mu$.

Denote by $J_\mu : T^*Q \to \mathfrak{g}_\mu^*$ the induced momentum map, *i.e.*, $J_\mu(\alpha_q) = J(\alpha_q) \mid \mathfrak{g}_\mu$. From the proof of the theorem, which is ultimately based on a momentum shift in the fiber, it follows that this diagram commutes:

$$
\begin{array}{ccccccccc}
J^{-1}(\mu) & \longrightarrow & J_\mu^{-1}(\mu') & \xrightarrow{t_\mu} & J^{-1}(0) & \longrightarrow & T^*Q & \xrightarrow{\pi} & Q \\
\downarrow \pi_\mu & & & & \downarrow & & & & \downarrow \rho_\mu \\
(T^*Q)_\mu & & \longrightarrow & & T^*(Q/G_\mu) & & \longrightarrow & & Q/G_\mu \\
[\alpha_q] & & \longmapsto & & [\alpha_q - \mu'\cdot\gamma_q(\cdot)] & & \longmapsto & & [q]
\end{array}
$$

where $t_\mu(\alpha_q) = \alpha_q - \mu'\cdot\gamma_q(\cdot)$ is fiber translation by the μ'-component of the connection form and where $[\alpha_q - \mu'\cdot\gamma_q(\cdot)]$ means the element of $T^*(Q/G_\mu)$ determined by $\alpha_q - \mu'\cdot\gamma_q(\cdot)$. Call the composition of the two maps on the bottom of this diagram $\sigma : [\alpha_q] \in (T^*Q)_\mu \mapsto [q] \in Q/G_\mu$. A natural way of inducing a connection on $\mathbf{J}^{-1}(\mu) \to (T^*Q)_\mu$ is to be consistent with this theorem, *i.e.*, by pull-back in the diagram above.

2.4 Corollary *The connection one-form* $\gamma \in \Omega^1(Q; \mathfrak{g}_\mu)$ *induces a connection one-form* $\tilde{\gamma}^\mu \in \Omega^1(\mathbf{J}^{-1}(\mu); \mathfrak{g}_\mu)$ *by pull-back:* $\tilde{\gamma}^\mu = (\pi \circ t_\mu)^* \gamma$, *i.e.*,

$$\tilde{\gamma}^\mu(\alpha_q) \cdot U_{\alpha_q} = \gamma(q) \cdot T_{\alpha_q}\pi(U_{\alpha_q}), \quad \textit{for } \alpha_q \in T_q^*Q, \; U_{\alpha_q} \in T_{\alpha_q}(T^*Q).$$

Similarly $\mathrm{curv}(\tilde{\gamma}^\mu) = (\pi \circ t_\mu)^* \, \mathrm{curv}(\gamma)$ *and in particular the* μ'*-component of the curvature of this connection equals* B, *the pull-back of* $\mu' \cdot \mathrm{curv}(\gamma)$.

The proof is a direct verification.

2.5 Corollary *Assume that* $\rho_\mu : Q \to Q/G_\mu$ *is a principal* G_μ*-bundle with a connection* $\gamma \in \Omega^1(Q; \mathfrak{g}_\mu)$. *If* H *is a* G*-invariant Hamiltonian on* T^*Q *inducing the Hamiltonian* H_μ *on* $(T^*Q)_\mu$ *and* $c_G(t)$ *is an integral curve of* X_{H_μ}, *denote by* d(t) *a horizontal lift of* $c_G(t)$ *in* $\mathbf{J}^{-1}(\mu)$ *relative to the natural connection of Corollary* **2.4** *and let* $q(t) = \pi(d(t))$ *be the base integral curve of* c(t). *Then* $\xi(t)$ *of step* **ii** *in Theorem* **2.1** *is given by*

$$\xi(t) \;=\; \gamma(q(t)) \cdot \mathbb{F}H(d(t)),$$

where $\mathbb{F}H : T^*Q \to TQ$ *is the fiber derivative of* H, *i.e.*, $\mathbb{F}H(\alpha_q) \cdot \beta_q \;=\; \left.\frac{d}{dt}\right|_{t=0} H(\alpha_q + t\beta_q)$.

Proof $\tilde{\gamma}^\mu \cdot X_H \;=\; \gamma \cdot T\pi \cdot X_H \;=\; \gamma \cdot \mathbb{F}H$. ■

If Q carries a G-invariant Riemannian metric $\langle\!\langle\, , \rangle\!\rangle$ define an associated connection $\gamma_{\mathrm{mech}} \in \Omega^1(Q; \mathfrak{g}_\mu)$ by declaring the horizontal space at any $q \in Q$ to be the orthogonal complement of the vertical space. Assume also that the Hamiltonian H is of the form kinetic energy with respect to the metric $\langle\!\langle\, , \rangle\!\rangle$ plus a G-invariant potential energy.

2.6 Corollary *Under these hypotheses, step* **ii***in Proposition* **2.1** *is equivalent to*
ii′ $\xi(t) \in \mathfrak{g}_\mu$ *is given by*

$$\xi(t) = \gamma_{\mathrm{mech}}(q(t)) \cdot d(t)^\#, \quad \textit{where } d(t) \in T^*_{q(t)}Q.$$

Proof Apply Corollary **2.4** and use the fact that $\mathbb{F}H(\alpha_q) = \alpha_q^\#$. (In coordinates, $\frac{\partial H}{\partial p_\mu} = g^{\mu\nu} p_\nu$).

■

In this case, γ_{mech} is closely related to the ***mechanical connection*** that appears in the work of Smale [1970], Guichardet [1984], Wilczek and Shapere [1988], Montgomery [1988,90a,b], Lewis, Marsden, Simo and Posbergh [1989] and Simo, Lewis and Marsden [1989]. It is given explicitly as follows: let $\mathbb{I}_\mu(q): \mathfrak{g}_\mu \to \mathfrak{g}_\mu^*$ be defined by $\mathbb{I}_\mu(\zeta)(\eta) = \langle \zeta_Q(q), \eta_Q(q)\rangle$ be the μ-***locked inertia tensor.*** This name is used because for coupled rigid (or rigid-flexible) structures, $\mathbb{I}_\mu(q)$ is the inertia tensor of the system with the joints locked in the configuration q, thereby forming a rigid body. Then $\gamma_{\mathrm{mech}}: TQ \to \mathfrak{g}_\mu$ is

$$\boxed{\gamma_{\mathrm{mech}}(v_q) = \mathbb{I}_\mu(q)^{-1}\mathbf{J}(v_q{}^\flat)}$$

where $v_q{}^\flat$ is the one-form corresponding to the vector v_q via the metric. Shapere and Wilzcek call this the "master formula".

Besides choosing the connection γ_{mech} to be defined by the *metric* orthogonal to the G_μ-orbits, one can use other complements. Here is one such:

2.7 Corollary *If* $Q = G$, *dim* $G_\mu = 1$, *and* ζ *is a generator of* $\mathfrak{g}_\mu$, *the one-form* $\gamma_R \in \Omega^1(G)$ *given by*

$$\gamma_R(g) = \frac{1}{\langle \mu, \zeta \rangle} T_g^* R_{g^{-1}}(\mu)$$

induces via the procedure in Corollery **2.4** *the connection* $A \in \Omega^1(\mathbf{J}_L^{-1}(\mu))$ *given in Proposition* **2.2**, *where* $\mathbf{J}_L(\alpha_g) = T_e^* R_g(\alpha_g)$. *Here,* $\langle \mu, \zeta \rangle$ *is the natural pairing between* μ *and* ζ.

Proof The axioms of a left principal G_μ-connection, namely $\gamma_R(\zeta_G) = \zeta$ and $\gamma_R(hg)(T_g L_h(v_g)) = \gamma_R(g)(v_g)$ for all $h \in G_g$ and $v_g \in T_g G$ are straightforward-verifications; recall that $\zeta_G(g) = T_e R_g(\zeta)$. Moreover, for $\alpha_g \in \mathbf{J}_L^{-1}(\mu)$ and $U_{\alpha_g} \in T_{\alpha_g}(\mathbf{J}_L^{-1}(\mu))$, we have

$$\begin{aligned}\tilde{\gamma}_R^\mu(\alpha_g)\cdot U_{\alpha_g} &= \gamma_R(g)\cdot T_{\alpha_g}\pi(U_{\alpha_g}) \\ &= \frac{1}{\langle \mu, \zeta \rangle} T_g^* R_{g^{-1}}(\mu)\cdot T_{\alpha_g}\pi(U_{\alpha_g}) = A(\alpha_g)\cdot U_{\alpha_g}\end{aligned}$$

since $\alpha_g \in \mathbf{J}_L^{-1}(\mu)$ is necessarily equal to $T_g^* R_{g^{-1}}(\mu)$. ■

§3 *Reconstruction of Dynamics for Lagrangian Systems*

In this section we reconstruct the dynamics of a given Lagrangian system with symmetry from the reduced dynamics. We begin by recalling the basic facts about Lagrangian systems.

§3A *Lagrangian Systems*

If Q is a manifold, $L : TQ \to \mathbb{R}$ is a smooth function, and $\tau : TQ \to Q$ the projection, let $\mathbb{F}L : TQ \to T^*Q$ be the ***fiber derivative*** of L given by

$$\mathbb{F}L(v) \cdot w = \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} L(v + \varepsilon w),$$

for $v, w \in T_qQ$. If Ω denotes the canonical symplectic structure on T^*Q, let $\Omega_L = (\mathbb{F}L)^*\Omega$, a closed two form on TQ. If Ω_L is a (weak) symplectic form, we call L ***regular***. In local charts this is equivalent to the second derivative in the fiber variable to be a (weakly) non-degenerate bilinear form. If $\mathbb{F}L$ is a diffeomorphism, L is called ***hyperregular*** and $\mathbb{F}L$ the ***Legendre transformation.*** Returning to a general Lagrangian L, let $A(v) = \mathbb{F}L(v) \cdot v$ be the *action* of L and $E = A - L$ the ***energy*** of L. A vector field $Z \in \mathfrak{X}(TQ)$ is called a ***Lagrangian system***, if $\mathbf{i}_Z\Omega_L = dE$, where $\mathbf{i}_Z$ denotes the interior product (or contraction) with Z. If L is regular, then Z is a second order equation, *i.e.*, $T\tau \circ Z =$ identity on TQ, where $\tau : TQ \to Q$ is the tangent bundle projection. In general, if we assume that Z is second order, then locally Lagrange's equations hold:

$$\frac{d}{dt}(\mathbf{D}_2L(q(t), \dot{q}(t))) = \mathbf{D}_1L(q(t), \dot{q}(t))$$

where $\dot{q}(t) = dq(t)/dt$ and $\mathbf{D}_1$ and $\mathbf{D}_2$ denote the partial Fréchet derivatives. Moreover, E is conserved by the flow of Z. *In this section we will assume that Z is a second order equation.*

If L is hyperregular, then $Z = X_E$ is the Hamiltonian vector field relative to Ω_L defined by E. Then $H = E \circ (\mathbb{F}L)^{-1} : T^*Q \to \mathbb{R}$ defines a Hamiltonian system X_H on T^*Q whose flow is conjugate by $\mathbb{F}L$ to that of X_E on TQ. The fiber derivative $\mathbb{F}H : T^*Q \to TQ$ of H (defined in Corollary **2.5**) is the inverse of $\mathbb{F}L$. Conversely, if $H : T^*Q \to \mathbb{R}$ is hyperregular, *i.e.*, $\mathbb{F}H : T^*Q \to TQ$ is diffeomorphism, let $\Theta(X_H)$ be the ***action*** of H, where Θ is the canonical one-form on T^*Q. Our convention is $\Omega = -\mathbf{d}\Theta$. Let $A = \Theta(X_H) \circ (\mathbb{F}H)^{-1}$, $E = H \circ (\mathbb{F}H)^{-1}$, and $L =$

A – E. Then L is the Lagrangian system inducing H and the above prescription defines a bijective correspondence between hyperregular Lagrangians on TQ and hyperregular Hamiltonians on T^*Q. Moreover, the base integral curves (*i.e.*, the projections of the integrals curves on TQ and T^*Q onto Q) of X_E and X_H coincide. See, *e.g.*, Abraham and Marsden [1978] for the proofs.

Let a Lie group G act on Q and that L is invariant under the lifted G-action to TQ. The Legendre transformation $\mathbb{F}L$ is equivariant relative to this action and the cotangent lifted action on T^*Q, and so A, E, Ω_L, and Z are G-invariant. Assuming that (TQ)/G is a smooth manifold with the projection $\rho : TQ \to (TQ)/G$ a surjective submersion, E induces a smooth function E_G and Z a smooth vector field Z_G on (TQ)/G; the flow of Z_G conserves E_G. The question we shall address in this section is the following: given an integral curve $c_G(t)$ of Z_G, $c_G(0) = [v_q]$ and $v_q \in T_qQ$, construct the integral curve c(t) of Z satisfying $c(0) = v_q$.

§3B *Reconstruction for* Q = G

We begin with the simplest case: Q = G with a left-invariant Lagrangian $L : TG \to \mathbb{R}$, $L(T_gL_h(v_g)) = L(v_g)$, where $h, g \in G$, $v_g \in T_gG$, $L_h : G \to G$ denotes left-translation by h, $L_h(k) = hk$, and $T_gL_h : T_gG \to T_{hg}G$ is its tangent map.

3.1 Proposition *Let* $L : TG \to \mathbb{R}$ *be a left-invariant Lagrangian such that its Lagrangian vector field* $Z \in \mathfrak{X}(TG)$ *is a second order equation. Let* $Z_G \in \mathfrak{X}(\mathfrak{g})$ *be the induced vector field on* $(TG)/G \approx \mathfrak{g}$ *and let* $\xi(t)$ *be an integral curve of* Z_G. *If* $g(t) \in G$ *is the solution of the non-autonomous ordinary differential equation* $\dot{g}(t) = T_eL_{g(t)}\xi(t)$, $g(0) = e$, *and* $g \in G$ *then* $v(t) = T_eL_{gg(t)}\xi(t)$ *is the integral curve of* Z *satisfying* $v(0) = T_eL_g\,\xi(0)$ *and* v(t) *projects to* $\xi(t)$, *i.e.*, $TL_{\tau(v(t))^{-1}}\, v(t) = \xi(t)$.

Proof Let v(t) be the integral curve of Z satisfying $v(0) = T_eL_g\xi(0)$ for a given element $\xi(0) \in \mathfrak{g}$. Since $\xi(t)$ is the integral curve of Z_G whose flow is conjugated to the flow of Z by left-translation, we have $TL_{\tau(v(t))^{-1}}\, v(t) = \xi(t)$. If $h(t) = \tau(v(t))$, since Z is a second order equation, we have

$$v(t) = \dot{h}(t) = T_eL_{h(t)}\,\xi(t) \quad \text{and} \quad h(0) = \tau(v(0)) = g$$

so that letting $g(t) = g^{-1}h(t)$ we get $g(0) = e$ and

$$\dot{g}(t) = TL_{g^{-1}}\,\dot{h}(t) = TL_{g^{-1}}TL_{h(t)}\,\xi(t) = TL_{g(t)}\,\xi(t).$$

This determines g(t) uniquely from $\xi(t)$ and so

$$v(t) = T_eL_{h(t)}\,\xi(t) = T_eL_{gg(t)}\,\xi(t). \quad \blacksquare$$

In general, Z_G is not Hamiltonian. However, if L is hyperregular, then the Legendre transformation $\mathbb{F}L$ will induce a Poisson structure on $\mathfrak{g}$. In fact, in this case, we can reconstruct the dynamics of the induced Hamiltonian system.

3.2 Corollary *Let* $L : TG \to \mathbb{R}$ *be a left-invariant hyperregular Lagrangian,* $H : T^*G \to \mathbb{R}$ *the induced Hamiltonian, and* $h : \mathfrak{g}^*_- \to \mathbb{R}$ *the induced Hamiltonian on the dual* $\mathfrak{g}^*$ *of* $\mathfrak{g}$ *endowed with the Lie-Poisson structure;* $h = H \mid \mathfrak{g}^*$. *Let* $\mu(t)$ *be an integral curve of* $X_h(\mu) = \mathrm{ad}\left(\frac{\delta h}{\delta \mu}\right)^* \mu$, $\mu(0) = T_e^*L_g(\alpha_g)$, $\alpha_g \in G$ *fixed. If* $\xi(t) = (\mathbb{F}L)^{-1}\mu(t)$ *and* $g(t)$ *is the solution of* $\dot{g}(t) = T_eL_{g(t)}\,\xi(t)$, $g(0) = e$, *then* $\alpha(t) = T^*L_{(gg(t))^{-1}}\,\mu(t)$ *is the integral curve of* X_H *on* T^*G, $\alpha(0) = \alpha_g$ *and* $\alpha(t)$ *projects to* $\mu(t)$, *i.e.,* $T_e^*L_{\pi(\alpha(t))} = \mu(t)$; $\pi : T^*G \to G$ *denotes the cotangent bundle projection.*

The proof is a consequence of the previous result and the fact that $\mathbb{F}L$ commutes with left translation: $\mathbb{F}L \circ TL_g = T^*L_{g^{-1}} \circ \mathbb{F}L$.

§3C *General* Q

In §3B we studied reconstruction for the case of Lie-Poisson reduction for a given invariant Lagrangian. In the general case, instead we take a symplectic reduction viewpoint. Given is $L : TQ \to \mathbb{R}$, G-invariant, $Z \in \mathfrak{X}((TQ)$ its Lagrangian vector field which we assume to be a second order equation. The lift of the G-action to TQ induces an equivariant momentum map $\mathbf{J} : TQ \to \mathfrak{g}^*$ given by

$$\mathbf{J}(v_q)\cdot\xi = \mathbb{F}L(v_q)\cdot\xi_Q(q) \tag{1}$$

for $v_q \in T_qQ$, $\xi \in \mathfrak{g}$. Let $\mu \in \mathfrak{g}^*$ be a weakly regular value of $\mathbf{J}$ and assume that $\rho_\mu : Q \to Q/G_\mu$, $TQ \to (TQ)/G_\mu$ are principal G_μ-bundles, where G_μ is the coadjoint isotropy group at μ. By conservation of $\mathbf{J}$, G_μ acts on $\mathbf{J}^{-1}(\mu)$ so we can form the reduced space $(TQ)_\mu = \mathbf{J}^{-1}(\mu)/G_\mu$. Given is $v_q \in \mathbf{J}^{-1}(\mu)$ and an integral curve $c_\mu(t)$ of the induced vector field $Z_\mu \in \mathfrak{X}((TQ)_\mu)$, $c_\mu(0) = [v_q]$; we want to explicitly find the integral curve $c(t)$ satisfying $c(0) = v_q$. For this purpose we formulate the tangent bundle version of Theorem **2.3** of the previous section.

3.3 Proposition *Let* $\mu' = \mu | \mathfrak{g}_\mu$ *and* $\mathbf{J}^\mu : TQ \to \mathfrak{g}_\mu^*$ *be given by* $\mathbf{J}^\mu(v_q)\cdot\eta = \mathbb{F}L(v)\cdot\eta_Q(q)$ *for all* $\eta \in \mathfrak{g}_\mu$. *Assume that there is a vector field* $Y_\mu \in \mathfrak{X}(Q)$ *such that* $Y_\mu(Q) \subset (\mathbf{J}^\mu)^{-1}(\mu')$ *and* Y_μ *is G-equivariant:* $Y_\mu(g\cdot q) = g\cdot Y_\mu(q)$. *Then there is an embedding* $\varphi_\mu : (TQ)_\mu = \mathbf{J}^{-1}(\mu)/G_\mu \to T(Q/G_\mu)$ *whose range is a vector subbundle with base* Q/G_μ. *This embedding is onto if and only if* $\mathfrak{g} = \mathfrak{g}_\mu$. *If* $L : TQ \to \mathbb{R}$ *is a G-invariant Lagrangian whose Lagrangian vector field* $Z \in \mathfrak{X}(TQ)$ *is a second order equation, let* $c(t)$ *denote an integral curve of* Z *and let* $q(t) = \tau(c(t))$ *be the corresponding base integral curve, where* $\tau : TQ \to Q$ *is the tangent bundle projection. Let* $q_\mu(t) = \rho_\mu(q(t))$ *be the corresponding curve in* Q/G_μ *and* $\tilde{Y}_\mu \in \mathfrak{X}((Q/G_\mu)$ *the vector field induced by* Y_μ. *Then the integral curve* $c_\mu(t)$ *of* Z_μ *covered by* $c(t)$ *is* $\varphi_\mu^{-1}(q_\mu'(t) - \tilde{Y}_\mu(q_\mu(t)))$. *The curve* $q_\mu(t)$ *is the image of* $c_\mu(t)$ *by* Y_μ *followed by the projection* $T(Q/G_\mu) \to Q/G_\mu$.

Proof The first part of the proof is a restatement of the standard proof of Theorem **2.3**. Indeed, Y_μ induces, by equivariance, the vector field $\tilde{Y}_\mu$ on Q/G_μ: $\tilde{Y}_\mu \circ \rho_\mu = T\rho_\mu \circ Y_\mu$. Define the projection $\tau_\mu : (TQ)_\mu \to Q/G_\mu$ by $\tau_\mu([v_q]) = [q]$, so that $\rho_\mu \circ \tau = \tau_\mu \circ \pi_\mu$, where, as usual, $\pi_\mu : \mathbf{J}^{-1}(\mu) \to (TQ)_\mu$ is the canonical projection. Let $t_\mu : (\mathbf{J}^\mu)^{-1} \to (\mathbf{J}^\mu)^{-1}(0)$ be given by $t_\mu(v_q) = v_q - Y_\mu(q)$ and let $\varphi_\mu : (TQ)_\mu \to T(Q/G_\mu)$ be the induced map, $\varphi_\mu \circ \pi_\mu = T\rho_\mu \circ t_\mu$, defined on the set $\mathbf{J}^{-1}(\mu)$. Then φ_μ is an embedding and it is easy to see that it is onto iff $\mathfrak{g} = \mathfrak{g}_\mu$ by comparing $(\mathbf{J}^\mu)^{-1}(0)$ with $\mathbf{J}^{-1}(0)$.

For the second part, let $c_\mu(t) = \pi_\mu(c(t))$ be the integral curve of Z_μ covered by $c(t)$. Then $q_\mu(t) = \rho_\mu(q(t)) = (\rho_\mu \circ \tau)(c(t)) = (\tau_\mu \circ \pi_\mu)(c(t)) = \tau_\mu(c_\mu(t))$. Let us show that $q_\mu'(t) - \tilde{Y}_\mu(q_\mu(t))$ is in the range of φ_μ. We have

$$
\begin{aligned}
q_\mu'(t) - \tilde{Y}_\mu(q_\mu(t)) &= T\rho_\mu(q'(t)) - q_\mu'(t) - \tilde{Y}_\mu(\rho_\mu(q(t)) \\
&= T\rho_\mu(q'(t)) - T\rho_\mu(Y_\mu(q(t)) = T\rho_\mu(q'(t) - Y_\mu(q(t))).
\end{aligned}
$$

Since Z is a second order equation, $q'(t) = c(t)$. By conservation of $\mathbf{J}$, $c(t) \in \mathbf{J}^{-1}(\mu)$ for all t, so $q'(t) - Y_\mu(q(t)) = t_\mu(q'(t))$ and thus

$$
q_\mu'(t) - \tilde{Y}_\mu(q_\mu(t)) = (T\rho_\mu \circ t_\mu)(q'(t)) = (\varphi_\mu \circ \pi_\mu)(q'(t)) = (\varphi_\mu \circ \pi_\mu)(c(t)) = \varphi_\mu(c_\mu(t)).
$$

Therefore, the integral curve $c_\mu(t)$ covered by $c(t)$ equals $c_\mu(t) = \varphi_\mu^{-1}(q_\mu'(t) - \tilde{Y}_\mu(q_\mu(t)))$ and the proposition is proved. ■

In general, Z_μ is *not* a second order equation. However, its integral curves are still uniquely determined by the corresponding base integral curves *i.e.*, by their projections on Q/G_μ. We next turn to the question of reconstruction of dynamics. We are given a base integral curve $q_\mu(t)$ of Z_μ determined by the initial conditions $q_\mu(0)$ and $q_\mu'(0)$ and an equivariant vector field Y_μ with values in $(J^\mu)^{-1}(\mu')$, implementing the embedding of $(TQ)_\mu$ into $T(Q/G_\mu)$. We want to reconstruct the integral curve $c(t) = q'(t)$ of Z covering $c_\mu(t) = \varphi_\mu^{-1}(q_\mu(t) - \tilde{Y}_\mu(q_\mu(t)))$. For this purpose, let $\gamma \in \Omega^1(Q; \mathfrak{g}_\mu)$ be a connection for the principal G_μ-bundle $\rho_\mu : Q \to Q/G_\mu$. Horizontally lift $q_\mu(t)$ to a curve $q_h(t)$ in Q with $q_h(0) = q(0)$:

$$\rho_\mu(q_h(t)) = q_\mu(t) \quad \text{and} \quad \gamma(q_h(t))\cdot q_h'(t) = 0. \tag{2}$$

Define the induced connection $\gamma^\mu \in \Omega^1(J^{-1}(\mu); \mathfrak{g}_\mu)$ by

$$\boxed{\gamma^\mu(V_{v_q}) = \gamma(q)\cdot T\tau(V_{v_q}) \tag{3}}$$

for $v_q \in T_qQ$ and $V_{v_q} \in T_{v_q}(TQ)$. (Note that if L is hyperregular, *i.e.*, $\mathbb{F}L : TQ \to T^*Q$ is a diffeomorphism, then $(\mathbb{F}L)^* \tilde{\gamma}^\mu = \gamma^\mu$, for $\tilde{\gamma}^\mu$ the connection given in Corollary **2.4**). In Step **i** of the reconstruction procedure we horizontally lift $c_\mu(t)$ to a curve $d(t)$ in $J^{-1}(\mu)$ with $d(0) = c(0)$: $\pi_\mu(d(t)) = c_\mu(t)$ and $\gamma^\mu(d(t)) \cdot d'(t) = 0$. To determine $d(t)$, we begin by showing that $\tau(d(t)) = q_h(t)$. Indeed, $\tau(d(t))$ covers $q_\mu(t)$: $\rho_\mu(\tau(d(t))) = (\tau_\mu \circ \pi_\mu)(d(t)) = \pi_\mu(c_\mu(t)) = q_\mu(t)$. Moreover, since $(\tau \circ d)'(t) = T\tau(q'(t))$, the horizontality condition on $d(t)$ becomes $\gamma(\tau(d(t)))\cdot(\tau \circ d)'(t) = 0$, *i.e*, $\tau(d(t))$ is horizontal. Finally, since $\tau(d(0)) = \tau(c(0)) = q(0) = q_h(0)$, the equality $\tau(d(t)) = q_h(t)$ follows by uniqueness of horizontal lift.

Next, we show that $q_h'(t)$ is the γ-horizontal part of $d(t)$. Indeed, since $\pi_\mu(d(t)) = c_\mu(t) = \varphi_\mu^{-1}(q_\mu'(t) - \tilde{Y}_\mu(q_\mu(t)))$ by Proposition **3.3**, we have

$$\begin{aligned} q_\mu'(t) - \tilde{Y}_\mu(q_\mu(t)) &= (\varphi_\mu \circ \pi_\mu)(c(t)) = (\varphi_\mu \circ \pi_\mu)(d(t)) = (T\rho_\mu \circ t_\mu)(d(t)) \\ &= T\rho_\mu(d(t) - Y_\mu(q_h(t))) = T\rho_\mu(d(t)) - \tilde{Y}_\mu(\rho_\mu(q_h(t))) \\ &= T\rho_\mu(d(t)) - \tilde{Y}_\mu(q_\mu(t)), \end{aligned}$$

whence

$$q_\mu'(t) = T\rho_\mu(d(t)).$$

Since $T\rho_\mu(q_h'(t)) = q_\mu'(t)$, it follows that $d(t) - q_h'(t)$ is vertical, so $q_h'(t)$ is the horizontal part of $d(t)$ and thus there is a unique $\xi(t) \in \mathfrak{g}_\mu$ such that

$$d(t) = q_h'(t) + \xi(t)_Q(q_h(t)). \tag{4}$$

Note that $\xi(0)_Q(q)$ is the vertical part of the initial condition v_q. The only remaining condition on $d(t)$ is that it lie in $\mathbf{J}^{-1}(\mu)$, *i.e.*, that

$$\mathbb{F}L(q_h'(t) + \xi(t)_Q(q_h(t)))\cdot\eta_Q(q_h(t)) = \mu\cdot\eta \tag{5}$$

for all $\eta \in \mathfrak{g}$. This condition uniquely determines $\xi(t)$ by the formula $\xi(t) = \gamma(q_h(t))\cdot d(t)$. We have proved the following:

3.4 Corollary *Under the hypotheses and notations of Proposition* **3.3**, *let* $q_\mu(t)$ *be the base integral curve of* $Z_\mu \in \mathfrak{X}((TQ)_\mu)$ *with initial conditions* $q_\mu(0)$, $q_\mu'(0)$ *and* $c_\mu(t) = \varphi_\mu^{-1}(q_\mu'(t) - \tilde{Y}_\mu(q_\mu(t)))$ *the corresponding integral curve of* Z_μ. *Let* $\gamma \in \Omega^1(Q; \mathfrak{g}_\mu)$ *be a connection on the principal* G_μ*-bundle* $\rho_\mu : Q \to Q/G_\mu$. *Then the integral curve* $c(t) = q'(t)$ *of* $Z \in \mathfrak{X}(TQ)$ *covering* $c_\mu(t)$ *with initial condition* $v_q \in \pi_\mu^{-1}(c_\mu(0))$ *is found in the following way:*

i *horizontally lift* $q_\mu(t)$ *to a curve* $q_h(t)$ *in* Q, $q_h(0) = q$;

ii *determine* $\xi(t) \in \mathfrak{g}_\mu$ *from the system*

$$\mathbb{F}L(q_h'(t) + \xi(t)_Q(q_h(t)))\cdot\eta_Q(q_h(t)) = \mu\cdot\eta$$

for all $\eta \in \mathfrak{g}$; *this implies that the horizontal part of the initial condition* v_q *is* $q_h'(0)$ *and the vertical part is* $\xi(0)_Q(q)$;

iii *solve the non-autonomous ordinary differential equation* $g'(t) = T_eL_{g(t)}\xi(t)$ *with initial condition* $g(0) = e$ *on the Lie subgroup* G_μ.

Then the base integral curve $q(t)$ *of* Z *with initial conditions* $q(0) = q$, $q'(0) = v_q$ *is given by* $q(t) = g(t)\cdot q_h(t)$ *and the integral curve of* Z *with initial condition* v_q *is* $q'(t) = g(t)\cdot(q_h'(t) + \xi(t)_Q(q_h(t)))$.

The vector field $Y_\mu \in \mathfrak{X}(Q)$ postulated in the hypotheses of Proposition **3.3** can be chosen consistent with the connection γ^μ given by (3). Namely, define

$$Y_\mu(g) = \mathbb{F}L(\mu'\cdot\gamma(g)), \tag{6}$$

i.e., $Y_\mu(g)$ is the Legendre transform of the μ'-component of the connection $\gamma \in \Omega^1(Q; \mathfrak{g}_\mu)$. This choice for hyperregular Lagrangians (*i.e.*, if $\mathbb{F}L$ is a vector bundle isomorphism between TQ and T^*Q) allows one to pass freely between the Lagrangian and Hamiltonian point of views both at the unreduced and reduced levels. We shall use this remark in §**4D**.

§3D *Simple Mechanical Systems*

To specialize this corollary to the case of Lagrangians of kinetic minus potential type, let $(Q, \langle\,,\rangle)$ be a Riemannian manifold with positive definite, G-invariant metric $\langle\,,\rangle$ and let $V : Q \to \mathbb{R}$ be a G-invariant potential. Then the classical Lagrangian $L(v_q) = \frac{1}{2}\|v_q\|^2 - V(q)$ is G-invariant. The metric induces a principal connection on $\rho_\mu : Q \to Q/G_\mu$ by declaring the horizontal subbundle to equal the orthogonal complement of the vertical subbundle. The condition determining $\xi(t) \in \mathfrak{g}_\mu$ becomes

$$\langle q_h'(t) + \xi(t)_Q(q_h(t)),\ \eta_Q(q_h(t))\rangle = \mu\cdot\eta \tag{1}$$

for all $\eta \in \mathfrak{g}$. Condition (1) implies that $q_h'(0)$ is the horizontal part and $\xi(0)_Q(q)$ the vertical part of the initial condition $v_q \in T_qQ$. Split $\mathfrak{g} = \mathfrak{g}_\mu \oplus E$ for some complement E of $\mathfrak{g}_\mu$. If in (1) η is taken to lie in $\mathfrak{g}_\mu$, then by the definition of the connection,

$$\langle \xi(t)_Q(q_h(t)),\ \eta_Q(q_h(t))\rangle = \mu\cdot\eta \tag{2}$$

for all $\eta \in \mathfrak{g}_\mu$, an equality uniquely determining $\xi(t)_Q(q_h(t))$ for every t, and hence by freeness of the G_μ-action, uniquely determining $\xi(t) \in \mathfrak{g}_\mu$. There are still $\dim G - \dim G_\mu$ equations to be satisfied; these hold automatically by the previous corollary. We have thus proved the following:

3.5 Corollary *In the hypotheses of Proposition* **3.3**, *let* $L(v_q) = \frac{1}{2}\|v_q\|^2 - V(q)$, *where* $\langle\,,\rangle$ *is a* G*-invariant positive definite Riemannian metric on* Q *and* $V : Q \to \mathbb{R}$ *is* G*-invariant. Let* $Z \in \mathfrak{X}(TQ)$ *be the Lagrangian vector field of* L *and let* $Z_\mu \in \mathfrak{X}((TQ)_\mu)$ *be the induced vector field on the reduced space. Let* $v_q \in J^{-1}(\mu)$ *and let* $q_\mu(t)$ *be the base integral curve of* Z_μ *with initial conditions* $\pi_\mu(v_q)$. *Then the integral curve* $c(t)$ *of* Z *with initial condition* $c(0) = v_q$ *is found in the following way:*

- **i** *endow the principal* G_μ*-bundle* $\rho_\mu : Q \to Q/G_\mu$ *with the connection* γ_{mech} *whose horizontal subbundle is the orthogonal complement relative to* $\langle\,,\rangle$ *of the vertical subbundle;*
- **ii** *horizontally lift* $q_\mu(t)$ *to a curve* $q_h(t)$ *in* Q *satisfying* $q_h(0) = q$;
- **iii** *determine* $\xi(t) \in \mathfrak{g}_\mu$ *from the algebraic system* $\langle \xi(t)_Q(q_h(t)),\ \eta_Q(q_h(t))\rangle = \mu\cdot\eta$ *for all* $\eta \in \mathfrak{g}_\mu$; *this implies that* $q_h'(0)$ *and* $\xi(0)_Q(q)$ *are the horizontal and vertical parts of the initial condition* v_q;
- **iv** *solve the non-autonomous equation* $g'(t) = T_eL_{g(t)}\,\xi(t),\ g(0) = e$ *in* G_μ.

Then $q(t) = g(t)\cdot q_h(t)$ *is the base integral curve of* Z *with initial conditions* $q(0) = q$, $q'(0) = v_q$ *and* $c(t) = q'(t) = g(t)\cdot(q_h'(t) + \xi(t)_Q(q_h(t)))$ *is the integral curve of* Z *with initial conditon* $c(0) = v_q$.

It is easy to check that the Legendre transform of $c(t)$ gives the integral curve of the associated Hamiltonian system as obtained in **2.1** relative to the connection in **2.6.**

There are several important situations when step **iv** can be carried out explicitly.

a If G_μ is abelian, the equation $g'(t) = T_e L_{g(t)}\, \xi(t)$, $g(0) = e$, has solution given by $g(t) = \exp \int_0^t \xi(s)ds$. If $G_\mu = S^1$, $\xi(s)$ can be explicitly determined from condition **iii** and hence the reconstruction method of Corollary **3.5** has an explicit solution. Indeed, if we denote by $\zeta \in \mathfrak{g}_\mu$ a generator of $\mathfrak{g}_\mu$, then $(a(t)\zeta)_Q = a(t)\zeta_Q$, so that by **ii** we get $\mu\cdot\zeta = \langle a(t)\zeta_Q(q_h(t)), \zeta_Q(q_h(t))\rangle = a(t)\, \| \zeta_Q(q_h(t)) \|^2$ and hence

$$a(t) = \frac{\mu \cdot \zeta}{\| \zeta_Q(q_h(t)) \|^2} .$$

Thus writing $g(t) = \exp(\theta(t)\zeta)$, the solution of the equation in **iv** is given by

$$\theta(t) = (\mu\cdot\zeta)\int_0^t \frac{ds}{\| \zeta_Q(q_h(s)) \|^2} .$$

3.6 Corollary *Let* $(Q, \langle\, ,\rangle)$ *be a Riemannian manifold with a* G*-invariant metric and* $V : Q \to \mathbb{R}$ *be a* G*-invariant function. Assume* G_μ *equals* S^1 *or* $\mathbb{R}$ *and let* ζ *be a generator of* $\mathfrak{g}_\mu$. *If* $v_q \in \mathbf{J}^{-1}(\mu) = \{u \in TQ \mid \langle u, \zeta_Q(q)\rangle = \mu\cdot\zeta\}$ *and* q_μ *is the base integral curve of the induced vector field* $Z_\mu \in \mathfrak{X}((TQ)_\mu)$ *with initial conditions* $\pi_\mu(v_q)$, *the base integral curve and solution of the Lagrangian vector field* Z *given by* $L(v_q) = \frac{1}{2}\|v_q\|^2 - V(q)$ *with initial condition* v_q *is given by*

$$q(t) = \exp(\theta(t)\zeta)\cdot q_h(t) \quad \textit{and} \quad c(t) = q'(t) = \exp(\theta(t)\zeta)\cdot\left(q_h'(t) + \frac{\mu\cdot\zeta}{\|\zeta_Q(q_h(t))\|^2}\,\zeta_Q(q_h(t))\right), \tag{4}$$

where

$$\theta(t) = (\mu\cdot\zeta)\int_0^t \frac{ds}{\| \zeta_Q(q_h(s)) \|^2} \tag{5}$$

and $q_h(t)$ *is the horizontal lift of* $q_\mu(t)$ *satisfying* $q_h(0) = q$.

b A second case in which the equation $g'(t) = T_eL_{g(t)}\,\xi(t)$ can be solved, even if G_μ is non-abelian when $\xi(t) = \xi(0) = \xi$ is constant. Then $g(t) = \exp(t\xi)$ is the solution. This holds if $\mathfrak{g}$ admits a non-degenerate metric $(\,,\,)$ satisfying

$$(\xi, \eta) = \langle \xi_Q(q), \eta_Q(q)\rangle \tag{6}$$

for any $q \in Q$. This condition is prohibitively strong. For example, it implies that $(\,,\,)$ defines a positive definite metric on $\mathfrak{g}$. Denoting by $\Phi_g : Q \to Q$ the action of G on Q, this relation implies by G-invariance of the metric $\langle\,,\,\rangle$ on Q that:

$$\begin{aligned}(\mathrm{Ad}_g\,\xi, \mathrm{Ad}_g\,\eta) &= \langle(\mathrm{Ad}_g\,\xi)_Q(q), (\mathrm{Ad}_g\eta)_Q(q)\rangle = \langle\Phi^*_{g^{-1}}\,\xi_Q(q), \Phi^*_{g^{-1}}\eta_Q(q)\rangle \\ &= \langle T\Phi_g(\xi_Q(g^{-1}\cdot q)),\ T\Phi_g(\eta_Q(g^{-1}\cdot q))\rangle = \langle\xi_Q(g^{-1}\cdot q), \eta_Q(g^{-1}\cdot q)\rangle = (\xi, \eta),\end{aligned}$$

i.e., $(\,,\,)$ must in addition be bi-invariant. This excludes semisimple Lie algebras of non-compact type. What is even worse, if G is compact and we are interested in classical Lagrangian systems on TG defined by left-invariant metrics on G, this condition forces the metric $\langle\,,\,\rangle$ to be bi-invariant: $(\xi, \eta) = \langle\xi_G(e), \eta_G(e)\rangle = \langle\xi, \eta\rangle$. This hypothesis thus fails for the free rigid body (unless it is spherically symmetric). A class of compact Lie group actions where such a condition holds occurs in Kaluza-Klein theories; see Montgomery [1989]. Using condition **iii** in Corollary **3.5**, condition (6) implies that $\xi(t)$ is constant: $(\xi(t), \eta) = \langle\xi(t)_Q\,(q_h(t)), \eta_Q(q_h(t))\rangle = \mu\cdot\eta$ and therefore $\xi(t) = \xi$, where $(\xi, \cdot\,) = \mu\,|\,\mathfrak{g}_\mu$.

3.7 Corollary *Let* $(Q, \langle\,,\,\rangle)$ *be a Riemannian manifold with a* G*-invariant metric and a* G*-invariant Lagrangian.* $L = \frac{1}{2}\|v_q\|^2 - V(q)$. *Assume* $\mathfrak{g}$ *carries a positive definite bi-invariant metric* $(\,,\,)$ *satisfying* $(\xi, \eta) = \langle\xi_Q(q), \eta_Q(q)\rangle$ *for all* $q \in Q$. *Then the solution of the Lagrangian vector field on* TQ *with initial condition* $v_q \in \mathbf{J}^{-1}(\mu)$ *is determined from the solution of the reduced system by following steps* **i - iv** *in Corollary* **3.5** *with* $g(t) = \exp(t\xi)$ *for* $\xi \in \mathfrak{g}_\mu$ *determined by* $(\xi, \cdot\,) = \mu\,|\,\mathfrak{g}_\mu$.

c Another class of problems where $g'(t) = T_eL_{g(t)}\xi(t)$ is explicitly solvable is as follows. We begin by trying to find a real valued function $f(t)$ such that $g(t) = \exp(f(t)\xi(t))$ solves this equation. Since $g(0) = e$, we require $f(0) = 0$. We have

$$g'(t) = (T_{f(t)\xi(t)}\exp)(f'(t)\,\xi(t) + f(t)\,\xi'(t)).$$

From the formula $(T_\eta\exp)\cdot\eta = T_eL_{\exp\eta}\cdot\eta$, we get

$$T_eL_{\exp f(t)\xi(t)}\,\xi(t) = \frac{1}{f(t)}\,T_eL_{\exp f(t)\xi(t)}(f(t)\,\xi(t))$$

$$= \frac{1}{f(t)}\,(T_{f(t)\xi(t)}\exp)(f(t)\,\xi(t)) \;=\; (T_{f(t)\xi(t)}\exp)(\xi(t))$$

so that $g'(t) = T_eL_{g(t)}\,\xi(t)$ for $g(t) = \exp f(t)\,\xi(t)$ is satisfied if and only if

$$f'(t)\,\xi(t) + f(t)\,\xi'(t) = \xi(t).$$

This requires $\xi'(t) = \alpha(t)\,\xi(t)$ in which case $f(t)$ is given by

$$f(t) = \int_0^t \exp\left[\int_t^s \alpha(r)dr\right] ds\,.$$

This procedure can be generalized as follows. Consider $f(t) = \sum_{i=1}^{N} f_i(t)\xi^{(i)}(t)$ and assume

i that it commutes with $\xi(t)$ for all t, and

ii $\xi^{(N+1)}(t) = \sum_{i=1}^{N} \alpha_i(t)\,\xi^{(i)}(t)$, for some functions $\alpha_0(t), \ldots, \alpha_N(t)$.

Then proceeding as before and using the formula $(T_\eta\exp)(\xi) = T_eL_{\exp\eta}(\xi)$ for $[\xi, \eta] = 0$ one is led to the system of ordinary differential equations with variable coefficients $\mathbf{f}' = \mathbf{A}\mathbf{f} + \mathbf{v}$, where

$$\mathbf{f} = \begin{bmatrix} f_0 \\ \cdot \\ \cdot \\ \cdot \\ f_N \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ \cdot \\ \cdot \\ 0 \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} 0 & 0 & \cdots & 0 & -\alpha_0(t) \\ -1 & 0 & \cdots & 0 & -\alpha_1(t) \\ 0 & -1 & \cdots & 0 & -\alpha_2(t) \\ 0 & 0 & \cdots & -1 & -\alpha_N(t) \end{bmatrix}.$$

Here $g = \exp(f_1\xi^1 + \ldots + f_N\xi^N)$ is sometimes called the ***Magnus expansion.***

d Another method by which one can treat $g' = T_eL_g\xi$ was pointed out by P S. Krishnaprasad. If G_μ is solvable, this method leads to explicit solutions. Write

$$g \;=\; \exp(f_1x_1)\exp(f_2x_2)\ldots\exp(f_nx_n)$$

for a suitable basis $\{x_i\}$ of $\mathfrak{g}_\mu$. For solvable G_μ, Wei and Norman [1984] showed that the f_i's can be obtained by quadrature. For $\mathbf{SO}(3)$ it leads to some nonlinear equations for f_1, f_2, f_3 which do not seem to be integrable by quadratures.

§4 *Ehresmann Connections and Holonomy*

In this section, we review some of the relevant facts from the theory of connections that will be needed. The exposition is by no means complete and is provided for the convenience of the reader. For applications to phases, the context of principal G-bundles is not adequate for all the examples, so we work in the larger framework of Ehresmann connections. These connections were introduced in Ehresman [1950]. Since our needs are modest, we do not attempt to survey the vast modern literature on connections, but we do make some historical remarks below.

§4A *Ehresmann Connections*

Let $\pi : E \to M$ be a surjective submersion, $V = \ker T\pi$ the ***vertical subbundle*** of TE and $\mathfrak{X}_{vert}(E; M)$ its space of sections, elements of which are called ***vertical vector fields.*** An ***Ehresmann connection*** on $\pi : E \to M$ is a smooth subbundle H of TE called the ***horizontal subbundle*** such that $H \oplus V = TE$. The space of sections of H, denoted by $\mathfrak{X}_{hor}(E; M)$ is the space of ***horizontal vector fields.*** Since $T\pi \,|\, H : H \to TM$ is an isomorphism on every fiber it has a fiberwise inverse called the ***horizontal lift operator*** $\mathrm{hor}_p : T_{\pi(p)}M \to T_pE$ for all $p \in E$, *i.e.*, $\mathrm{hor}_p = (T_p\pi \,|\, H_p)^{-1}$. Since H is a smooth subbundle of TE, the horizontal lift defines a linear map $\mathrm{hor} : \mathfrak{X}(M) \to \mathfrak{X}_{hor}(E; M)$ by $(\mathrm{hor}\, X)(p) = \mathrm{hor}_p(X(p))$.

In general, a ***lift*** of a vector $v \in T_mM$ is a vector field lift v along (and in general not tangent to) $\pi^{-1}(m)$ such that $T_m\pi(\mathrm{lift}\, v) = v$ and $v \mapsto \mathrm{lift}\, v$ is linear. Given $X \in \mathfrak{X}(M)$, define $\mathrm{lift}\, X : E \to TE$ by $(\mathrm{lift}\, X)(p) = (\mathrm{lift}\, X)(\pi(p))(p)$ and note that $(\mathrm{lift}\, X)(p) \in T_pE$. We say that the lift is ***smooth*** if $\mathrm{lift}\, X \in \mathfrak{X}(E)$ for all $X \in \mathfrak{X}(M)$. With these definitions, we see that an Ehresmann connection is alternatively defined by a smooth lift $\mathrm{hor} : \mathfrak{X}(M) \to \mathfrak{X}(E)$ such that if $H_p = \mathrm{hor}_p(T_{\pi(p)}M)$, then $H = \bigcup_{p \in E} H_p$ is a smooth subbundle of TE complementary to V.

For any $u \in T_pE$, let $u = u^{hor} + u^{vert}$ be its decomposition into its horizontal and vertical parts. The vertical projection $\gamma(p) : T_pE \to V_p$ given by $\gamma(p)(u) = u^{vert}$ defines a smooth V-valued one-form $\gamma \in \Omega^1(E; V)$, called the ***connection one form***. This form satisfies γ_p = identity on V_p. Conversely, given γ with γ_p = identity on V_p, it uniquely determines a smooth horizontal lift by

$$\mathrm{hor}_p\, v = \tilde{v} - \gamma(p)(\tilde{v}) \qquad (1)$$

where $v \in T_{\pi(p)}M$ and $\tilde{v} \in T_pE$ is an arbitrary vector satisfying $T_e\pi(\tilde{v}) = v$. Thus *an Ehresmann connection is equivalently described by a horizontal subbundle, a horizontal lift, or a vector bundle valued connection one-form which is the identity on the vertical subbundle.*

If $\alpha \in \Omega^k(E)$, its ***covariant exterior derivative*** $\mathbf{D}\alpha \in \Omega^{k+1}(E)$ is defined by

$$\mathbf{D}\alpha(X_0, \ldots, X_k) = \mathbf{d}\alpha(X_0^{hor}, \ldots, X_k^{hor}), \tag{2}$$

where $X_0, \ldots, X_k \in \mathfrak{X}(E)$ and X_i^{hor} is the horizontal part of X, taken pointwise. In terms of pointwise operations, we can write, with a minor abuse of notation,

$$(\mathbf{D}\alpha(m)(v_0, \ldots, v_k))(p) = \mathbf{D}\alpha(p)(\mathrm{hor}_p v_0, \ldots, \mathrm{hor}_p v_k), \quad \text{for } m = \pi(p). \tag{3}$$

If $\lambda \in \Omega^k(E; V)$ is a V-valued k-form, where V is the vertical subbundle of TE, the ***covariant derivative*** $\mathbf{D}\lambda \in \Omega^{k+1}(E; V)$ is defined using ***Cartan's formula***:

$$\begin{aligned}\mathbf{D}\lambda(X_0, \ldots, X_k) = &\sum_{i=0}^{k} (-1)^i [X_i^{hor}, \lambda(X_0^{hor}, \ldots, \check{X}_i, \ldots, X_k^{hor})]^{vert} \\ &+ \sum_{0 \le i < j \le k} (-1)^{i+j} \lambda([X_i^{hor}, X_j^{hor}], X_0^{hor}, \ldots, \check{X}_i, \ldots, \check{X}_j, \ldots, X_k^{hor})\end{aligned} \tag{4}$$

where $X_0, \ldots, X_k \in \mathfrak{X}(E)$ and $\check{}$ above a vector field means that it is deleted. As above, the covariant derivative of λ can be thought of as a (k + 1)-form on M with values in $\mathfrak{X}_{vert}(E; M)$, or equivalently, as a (k + 1)-linear skew symmetric map $\tilde{\mathbf{D}}\lambda : \mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M) \to \mathfrak{X}_{vert}(E; M)$, namely

$$\tilde{\mathbf{D}}\lambda(Y_0, \ldots, Y_k) = \mathbf{D}\lambda(\mathrm{hor}\, Y_0, \ldots, \mathrm{hor}\, Y_k). \tag{5}$$

If $\gamma \in \Omega^1(E; V)$ is a connection one-form, then $\mathbf{D}\gamma = \Omega \in \Omega^2(E; V)$ is called the ***curvature*** of γ. Sometimes we shall write $\mathrm{curv}(\gamma)$ for the curvature of γ. Since γ annihilates horizonal vectors, (4) gives

$$\Omega(X, Y)(p) = -\gamma(p)([X^{hor}, Y^{hor}](p)) \tag{6}$$

and its induced map on the base $\bar{\Omega} : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}_{vert}(E; M)$ is given by $\bar{\Omega}(U, W) = \Omega(\mathrm{hor}\, U, \mathrm{hor}\, W) = \mathrm{hor}[U, W] - [\mathrm{hor}\, U, \mathrm{hor}\, W]$; this follows from the identity $[\mathrm{hor}\, U, \mathrm{hor}\, W]^{hor} = \mathrm{hor}[U, W]$ which we now prove. By definition of the horizontal lift, $[\mathrm{hor}\, U, \mathrm{hor}\, W]^{hor}$ is horizontal and $T\pi([\mathrm{hor}\, U, \mathrm{hor}\, W]^{hor}) = T\pi([\mathrm{hor}\, U, \mathrm{hor}\, W]) = [U, W]$ because $\mathrm{hor}\, U$ and U

are π-related and π-relatedness is bracket preserving. Note that Ω and $\bar{\Omega}$ determine each other uniquely. We just saw how Ω induces $\bar{\Omega}$. Conversely, let $\bar{\Omega}$ be given by (6). This determines Ω uniquely on horizontal vectors and so declaring $\mathbf{i}_Z\Omega = 0$ for all $Z \in \mathfrak{X}_{vert}(E; M)$ determines Ω on $\mathfrak{X}(E) \times \mathfrak{X}(E)$. The formula for Ω follows from the definition of $\bar{\Omega}$ and the fact that $\gamma(p)$ is the vertical projection $T_pE \to V_p$.

Bianchi's identity states that $\mathbf{D}\Omega = 0$. This is proved by using the definition of the covariant derivative and the remark that *the Lie bracket of any vertical with any projectable vector field is again vertical*; a vector field on E is ***projectable*** if it is π-related to some vector field on M. For example, the bracket of any horizontally lifted vector field with a vertical vector field is again vertical.

Let $m(t)$, $t \in [0, 1]$ be a smooth path in M. A ***horizontal lift*** of $m(t)$ is a smooth path $p(t)$ in E such that $\pi(p(t)) = m(t)$ and the tangent vector $\dot{p}(t)$ to $p(t)$ is horizontal for every $t \in [0, 1]$. *If* $\pi : E \to M$ *is a locally trivial fiber bundle, then given a smooth path* $m(t)$, *in* M *with* $m_0 = m(0), 0 \le t \le 1$ *and* $p_0 \in \pi^{-1}(m(0))$, *there is a unique locally defined horizontal lift* $p(t)$ *of* $m(t)$ *satisfying* $p(0) = p_0$. If $p(t)$ can be extended for all $t \in [0, 1]$, we call the connection ***complete***. We will generally assume this is the case; if E is a locally trivial finite dimensional fiber bundle with structure group acting transitively on the fibers and if the Ehresman connection is invariant under the action, then the connection is complete using the following local existence argument, compactness of [0, 1] and the fact that the time of existence is independent of the position on a given fiber. In particular, completeness holds for principal connections. To prove local existence and uniqueness of horizontal lifts, it suffices to do so for the case of a trivial bundle $E = M \times F$ with $\pi : M \times F \to M$ the projection. The equality

$$\gamma(m, f)(u, v) = v + \lambda(m, f)(u) \tag{7}$$

for $m \in M$, $f \in F$, $u \in T_mM$, and $v \in T_fF$ defines a bijective correspondence between connection one-forms $\gamma \in \Omega^1(M \times F; V)$ and smooth sections of the bundle $L(TM, TF) \to M \times F$ of vector bundle maps from TM to TF with base $M \times F$. Thus (u, v) is horizontal iff $v = -\lambda(m, f)(u)$, so that if $m(t)$ is a path in M, define $p(t) = (m(t), f(t))$ with $f(t)$ the solution of the time dependent differential equation

$$\frac{df(t)}{dt} = -\lambda(m(t), f(t))\dot{m}(t) \tag{8}$$

with initial condition $f(0) = f_0$, where $p_0 = (m_0, f_0)$. By local existence and uniqueness for differential equations this defines $f(t)$ (and hence $p(t)$) for small t. If the path $m(t)$ is constant and equal to m_0, then $p(t)$ is also constant and equal to p_0. Let $m(t)$ be a general path, $m(0) =$

m_0 and $m(1) = m_1$. If $e_p(t)$ is the horizontal lift of $m(t)$ satisfying $e_p(0) = p \in \pi^{-1}(m_0)$, then the map sending p to $e_p(s)$ defines, by uniqueness of horizontal lift, a bijection from $\pi^{-1}(m_0)$ to $\pi^{-1}(m_1)$. By smooth dependence of solutions of ordinary differential equations on initial conditions, it follows that this map is a diffeomorphism; it is called the ***parallel transport operator.***

Historical Remarks

We thank Arthur Fischer for the following remarks. An Ehresmann connection γ on a fiber bundle $\pi : E \to M$ induces an Ehresmann connection $T\gamma$ on the tangent bundle $T\pi : TE \to TM$ in a functorial way. (This is due to Kobayashi [1957] for the principal bundle case and to Vilms [1967] for the vector bundle case). More interestingly, the connection γ also induces a connection γ^B on the vertical bundle $v_E : VE \to E$, called the ***Berwald connection*** (see Yano and Okubo [1961], Vilms [1968], and Tong Van Duc [1975].) The original work of Berwald [1926, 1933, 1939] was for the case of $E = TM$. In particular, formula (4) above may be viewed as the covariant exterior derivative with respect to the Berwald connection γ^B induced by the given Ehresmann connection. ♦

§4B *Holonomy*

Now let $m(t)$, $t \in [0, 1]$ be a closed path in N, $m(0) = m_0$. The diffeomorphism of $\pi^{-1}(m_0)$ onto itself given by parallel transport along $m(t)$ is called the ***holonomy of the path*** $m(t)$. It is easy to see that parallel transport sends juxtaposition of loops based at m_0 into the composition of diffeomorphisms of $\pi^{-1}(m_0)$. Thus the holonomy operation is a group homomorphism of $\mathcal{L}(m_0)$, the loop group at m_0, to the diffeomorphism group of $\pi^{-1}(m_0)$; its image $\mathcal{H}(m_0)$ is called the ***holonomy group at*** m_0. It is straightforward to see that if M is connected all holonomy groups are conjugate: $\mathcal{H}(m_0)$ and $\mathcal{H}(m_1)$ are conjugate by the parallel transport along any path connecting m_0 to m_1. Thus if M is connected, we speak of $\mathcal{H}$, the ***holonomy group of the connection.***

If $\Phi : E \times G \to E$ is a (by tradition, right) Lie group action of G on E and the connection γ is G-invariant, *i.e.*, if $H_{p \cdot g} = T_p \Phi_g(H_p)$ for all $p \in E$, $g \in G$, then parallel transport is equivariant. This happens if $\pi : E \to M$ is a principal G-bundle and $\gamma \in \Omega^1(E; \mathfrak{g})$ is a principal connection: horizontal lift commutes with the G-action. Here we have identified the vertical bundle V with $M \times \mathfrak{g}$, which is done by using the group action. This map is G-equivariant when we take the action on $E \times \mathfrak{g}$ to be the diagonal one:

$$(e, \xi) \cdot g = (e \cdot g, \mathrm{Ad}_{g^{-1}} \xi).$$

Consequently, the G-equivariance of λ as a $\mathfrak{g}$-valued one-form means that

$$g^*\gamma = \mathrm{Ad}_g \circ \gamma .$$

The holonomy can be realized more explicitly in the principal bundle case by fixing a point $p \in \pi^{-1}(m_0)$. Associate to every holonomy operator $\varphi \in \mathcal{H}(m_0)$ the group element g of G by $\varphi(p) = p \cdot g$. We will call this the holonomy *measured from* p. If we measured the holonomy from $p' = p \cdot g_1$ instead, we would find that g gets replaced by $g_1^{-1} g g_1$. This is because φ commutes with right multiplication.

In what follows, we consider the case of principal bundles for which the computation of the holonomy of a path is theoretically very simple.

4.1 Proposition *Let* $\pi : E \to M$ *be a principal* G*-bundle and* $\gamma \in \Omega^1(E; \mathfrak{g})$ *a principal connection. Let* c *be a closed path in* M *which is contained in the open set* U. *Suppose* $s : U \subset M \to E$ *is a local section and set* $a = s^*\gamma$ *so that* a *is a* $\mathfrak{g}$*-valued one-form on* U. *Let* g(t) *be the solution to*

$$\frac{dg(t)}{dt} = -a\left(\frac{dc}{dt}\right) \cdot g(t)$$

with $g(0) = 1$. *Then the holonomy of* c, *measured from* $s(c(0))$ *is* $g(1)$.

Remark The solution g to this equation is written $\mathcal{P}\exp\left(-\int_c a\right)$ in the physics literature, where $\mathcal{P}$ denotes "path ordering".

Proof The section s induces a local trivialization of E in the usual way: write $e = s(x) \cdot g$. Let $(c(t), g(t))$ be the "coordinates" of the horizontal lift $\bar{c}$ of c in this trivialization. By the transformation law for connections,

$$\gamma(x, g) = g^{-1}(a + \mathbf{d})g = g^{-1}ag + g^{-1}\mathbf{d}g$$

in this trivialization. Here we have used matrix notation for simplicity: that is, we write $g^{-1}ag = \mathrm{Ad}_{g^{-1}} \circ a$. The condition that a curve $\bar{c}$ be horizontal is that $\bar{c}^*\gamma = 0$ *i.e.,* $\gamma\left(\frac{d\bar{c}}{dt}\right) = 0$. In our coordinates this reads

$$g^{-1}\left(a\left(\frac{dc}{dt}\right) \cdot g + \frac{dg}{dt}\right) = 0 \quad \textit{i.e.,} \quad \frac{dg}{dt} = -a\left(\frac{dc}{dt}\right) \cdot g(t) . \blacksquare$$

Now assume that G is abelian; then it is a cylinder $\mathbb{R}^k \times \mathbb{T}^{s-k}$ and in particular each g is of the form $\exp \eta$. Therefore $TL_{g^{-1}} \dot{g} = \dot{\eta}$ so, by the proposition, the holonomy element is

$$g(1) = \exp \eta(1) = \exp\left(-\int_0^1 (s^*\gamma)(m(\lambda)) \cdot m'(\lambda) d\lambda\right) = \exp\left(-\iint s^* \, \mathbf{d}\gamma\right) \qquad (10)$$

where the last equality is obtained by Stokes' theorem and $\iint$ denotes the integral over the two dimensional submanifold of M whose boundary is the loop $m(t), t \in [0, 1]$. Such a surface may not always exist in which case the last equality in (10) is dropped in the expression of the holonomy. For example, a circle which is one of the generators of a torus does not bound any surface on the torus. This case actually occurs in the dynamics of three coupled rigid bodies (see §5). Returinng to the case when m(t) does bound a surface, the structure equations imply that $\mathbf{d}\gamma = \mathbf{D}\gamma = \Omega$, the curvature of γ. Thus, thinking of the curvature as a two-form on the base with values in the adjoint bundle, which for abelian groups is trivial, we obtain the well-known result:

4.2 Corollary *If* $\pi : E \to M$ *is a principal* G-*bundle with* G *abelian and* $\gamma \in \Omega^1(E; \mathfrak{g})$ *is a principal connection, then the holonomy of the closed path* m(t) *in* M *is given by the group element*

$$\text{holonomy} = \exp\left(-\iint \mathbf{d}\gamma\right) = \exp\left(-\iint \Omega\right), \qquad (11)$$

where the integral is taken over any two-dimensional submanifold in M *whose boundary is* m(t).

For the S^1-connection on $\mathbf{J}^{-1}(\mu) \to T^*Q_\mu$ given in Proposition **2.2** the holonomy of a closed loop is given by

$$\exp\left(\frac{1}{\langle \mu, \xi \rangle} \iint \omega_\mu\right);$$

the integral being taken over the two-manifold in M whose boundary is the given loop. Returning to the reconstruction formula for the free rigid body given in Example **1F**, note that the first term in (**1F**.2) is the holonomy of the closed integral curve on S^2.

§5 *Reconstruction Phases*

In this section we give a number of examples of how to compute phases that arise from pure reconstruction (*i.e.*, without adiabaticity) using the theory of the preceding section. The free rigid body was already indicated in the introduction. Here we consider a number of other mechanical systems. In addition, we compute the rigid body phase using another choice of connection and compute the phase for the heavy top. In particular, we see that the nice choice of connection for the free rigid body is not so convenient for the heavy top, primarily because it does not drop down to configuration space. We also compute phases for some other examples, including coupled planar rigid bodies.

§5A *Quantum Mechanics*

Classical holonomy can be used for the computation of the geometric Berry phase in quantum mechanics, using the well known fact that quantum mechanics is a special case of classical mechanics, and the Schrödinger equation is a special case of Hamilton's equations. These points are discussed in Abraham and Marsden [1978] and references therein. In this spirit, we have:

5.1 Proposition (Aharonov and Anandan [1987]) *The holonomy of a loop in projective complex Hilbert space is the exponential of twice the symplectic area of any two-dimensional submanifold whose boundary is the given loop.*

Proof Let $\mathcal{H}$ be a complex Hilbert space and $M = \mathbb{P}\mathcal{H}$, the space of complex lines in $\mathcal{H}$. In Corollary **4.2**, let $E = \mathcal{H}$, $G = S^1$ and use the connection A defined by

$$A(\psi)\cdot\varphi = \mathrm{Re}\,\langle -i\psi, \varphi\rangle = -\mathrm{Im}\,\langle \psi, \varphi\rangle$$

where $\varphi, \psi \in \mathcal{H}$ and $\langle\ ,\ \rangle$ is the Hermitian inner product. The curvature is the differential of A and equals

$$\Omega(\varphi, \psi) = 2\mathrm{Im}\,\langle \varphi, \psi\rangle = -2\omega(\varphi, \psi)$$

where $\omega \in \Omega^2(\mathcal{H})$ is the usual symplectic form on complex Hilbert space. Therefore $-\iint \Omega = 2\iint \overline{\omega}$ = twice the symplectic area of the 2-manifold whose boundary is the given loop, where $\overline{\omega}$ is the reduced symplectic form on projective Hilbert space $\mathbb{P}\mathcal{H}$. ■

Remarks

a The result of Proposition **5.1** can be derived from Proposition **2.2** (generalized to exact symplectic manifolds). Take $P = \mathcal{H}$ with the symplectic form $\omega(\varphi, \psi) = -\operatorname{Im}\langle\varphi, \psi\rangle$, and the S^1-action the multiplication of a vector by a scalar of the form $e^{i\theta}$. Then $G = G_\mu = S^1$ for any $\mu \in \mathfrak{g}^* = \mathbb{R}$. The generator of $\mathfrak{g}$ is taken to be 1. If $\xi \in \mathbb{R}$, then $\xi_{\mathcal{H}}(\varphi) = i\xi\varphi$ for any $\varphi \in \mathcal{H}$ so that the momentum map $\mathbf{J} : \mathcal{H} \to \mathbb{R}$ is $\mathbf{J}(\varphi) = -\|\varphi\|^2/2$. The symplectic form ω is exact and $\omega = -\mathbf{d}\theta$, where $\theta(\varphi)\cdot\psi = \frac{1}{2}\operatorname{Im}\langle\varphi, \psi\rangle$. The level set $\mathbf{J}^{-1}\left(-\frac{1}{2}\right)$ is the sphere in $\mathcal{H}$ of radius 1 and $P_\mu = \mathbb{P}\mathcal{H}$. Thus the connection A is precisely the one used in the proof of Proposition **5.1**.

b The connection A used in the proof of **5.1** is the connection used in Simon [1985]. ♦

§5B *Phases for simple mechanical systems with* S^1*-symmetry*

Let $(Q, \langle\cdot,\cdot\rangle)$ be a Riemannian manifold, G a Lie group acting by isometries on Q and assume $H(\alpha_q) = \frac{1}{2}\|\alpha_q\|^2 + V(q)$ is a classical Hamiltonian, where $V : Q \to \mathbb{R}$ is a G-invariant potential energy and $\|\alpha_q\|$ denotes the norm of the bundle metric induced by $\langle\cdot,\cdot\rangle$ on T^*Q, *i.e.*, $\|\alpha_q\| = \|\alpha_q{}^\#\|$, where $^\# : T^*Q \to TQ$ is the index raising operation induced by the metric. The momentum map $\mathbf{J} : T^*Q \to \mathfrak{g}^*$ is $\mathbf{J}(\alpha_q)\cdot\xi = \alpha_q\cdot\xi_Q(q)$ for any $\xi \in \mathfrak{g}$. Let $\mu \in \mathfrak{g}^*$ and assume throughout this section that $G_\mu = S^1$ and that $\rho_\mu : Q \to Q/G_\mu$ and $\pi_\mu : \mathbf{J}^{-1}(\mu) \to (T^*Q)_\mu$ are principal S^1-bundles. Let ζ be a generator of $\mathfrak{g}_\mu$ implementing the isomorphism of $\mathfrak{g}_\mu$ with $\mathbb{R}$. Let $\bar{c}_\mu(t)$ be an integral curve of the reduced system with Hamiltonian H_μ on $(T^*Q)_\mu$ and assume it is periodic with period T. If $\bar{c}(t)$ denotes the reconstructed integral curve lying in the level set $\mathbf{J}^{-1}(\mu)$, then

$$\bar{c}(t) = \exp(\varphi\zeta)\cdot\bar{c}(0) \tag{1}$$

for some angle φ, called the *total phase* of the integral curve $\bar{c}(t)$. In this section we shall compute φ in two ways, by choosing the connection A of Proposition **2.2** and the connection γ^μ_{mech} of Corollary **2.4** where $\gamma = \gamma_{\text{mech}}$ is the mechanical connection associated to the given metric. See the discussion surrounding Corollary **2.6**.

We begin with the connection $A \in \Omega^1(\mathbf{J}^{-1}(\mu))$ given by

$$A = \frac{1}{\mu\cdot\zeta}\theta_\mu \tag{2}$$

where θ_μ is the pull-back of the canonical one-form θ of T^*Q to $J^{-1}(\mu)$. Let $d_A(t)$ denote the A-horizonal lift of $\bar{c}_\mu(t)$ and $q_A(t) = \pi(d_A(t))$ its base curve, where $\pi : T^*Q \to Q$ is the canonical projection. By Theorem **2.1**, $c(t) = \exp(\psi(t)\zeta)\cdot d_A(t)$, where $\dot\psi(t) = (A\cdot X_H)(d_A(t)) = \theta(d_A(t))\cdot X_H(d_A(t))/(\mu\cdot\zeta)$. Thus the base integral curve $g(t)$ of $c(t)$ equals $g(t) := \pi(c(t)) = \exp(\psi(t)\zeta)\cdot d_A(t)$, whence by G-invariance of V, we get $V(q_A(t)) = V(q(t)) = \bar{V}_\mu(q_\mu(t))$, where $\bar{V}_\mu : Q/G_\mu \to \mathbb{R}$ is induced by V via $V = \bar{V}_\mu \circ \rho_\mu$ and $q_\mu(t)$ is the projection of $\bar{c}_\mu(t) \in (T^*Q)_\mu$ to Q/G_μ (see Proposition **2.2**). Thus,

$$\theta(d_A(t))\cdot X_H(d_A(t)) = \|d_A(t)\|^2 = 2H(d_A(t)) - 2V(q_A(t)) = 2H_\mu - 2\bar{V}_\mu(q_\mu(t))$$

where H_μ is the constant value of the reduced Hamiltonian on the integral curve $\bar{c}_\mu(t)$. Therefore

$$\psi(t) = \frac{1}{\mu\cdot\zeta}\left(2H_\mu t - 2\int_0^t \bar{V}_\mu(q(s))ds\right) . \tag{3}$$

Since the curvature of the connection (2) is $-\omega_\mu/(\mu\cdot\zeta)$, where ω_μ is the reduced symplectic form on $(T^*Q)_\mu$, Corollary **4.2** shows that the holonomy is the exponential of

$$\left(\frac{1}{\mu\cdot\zeta}\iint_D \omega_\mu\right)\zeta \tag{4}$$

where D is a two-dimensional surface in $(T^*Q)_\mu$ whose boundary is $\bar{c}_\mu(t)$. As we discussed in §**4B** such a surface might not always exist, but we shall assume it does, for convenience, in the sequel. Consequently, by the reconstruction method of Theorem **2.1**, the total phase of the integral curve $\bar{c}(t)$ in $J^{-1}(\mu)$ equals

$$\varphi = \frac{1}{\mu\cdot\zeta}\iint_D \omega_\mu + \frac{2(H_\mu - \langle\bar{V}_\mu\rangle)T}{\mu\cdot\zeta} \tag{5a}$$

where

$$\langle\bar{V}_\mu\rangle = \frac{1}{T}\int_0^t \bar{V}_\mu(q_\mu(t))\,dt \tag{5b}$$

is the average of the potential V on the base integral curve $q_\mu(t)$. Note that if $Q = \mathbf{SO}(3)$, $V = 0$, and H is the kinetic energy of a free rigid body, (5) reduces to formula (2) in §**1G** by choosing $\zeta = \mu/\|\mu\|$ and where we have identified $\mathbf{so}(3)$ and its dual with $\mathbb{R}^3$ in the usual manner (using the Killing form).

Next we turn to the total phase computation using the connection γ^μ_{mech} induced by the mechanical connection γ_{mech}. We begin by translating the results of Corollary **3.6** to the cotangent bundle setting. The notation will be that of §3C and §3D, namely, L denotes a classical Lagrangian with energy $E(v_q) = \frac{1}{2}\|v_q\|^2 + V(q)$, $\mathbb{F}L = \flat : TQ \to T^*Q$ is the corresponding Legendre transform which is a vector bundle isomorphism, $c_\mu(t)$ is an integral curve of the reduced system on $(TQ)_\mu$, and $c(t)$ is a reconstructed integral curve on the μ-level set of the momentum map. We shall fix the mechanical connection $\gamma_{mech} \in \Omega^1(Q)$ inducing both the connections $\bar{\gamma}^\mu_{mech}$ (in Corollary **2.4**) and γ^μ_{mech} (see (3) in §3C). In addition, we shall use in the cotangent bundle reduction theorem both in cotangent (Proposition **2.3**) and tangent (Proposition **3.3**) formulations, the embeddings into $T^*(Q/G_\mu)$ and $T(Q/G_\mu)$ respectively as being given by momentum or velocity shifts induced by γ (see closing comments in §3C). We let $H = E \circ (\mathbb{F}L)^{-1}$ be the corresponding Hamiltonian system on T^*Q. Since $(\mathbb{F}L)^*\bar{\gamma}^\mu_{mech} = \gamma^\mu_{mech}$ as remarked in §3C, $\mathbb{F}L$ will transport all information from the Lagrangian side to the Hamiltonian side. For example, if $(\mathbb{F}L)_\mu : (TQ)_\mu \to (T^*Q)_\mu$ is the induced diffeomorphism, we conclude that $\bar{c}_\mu(t) = \mathbb{F}L(c_\mu(t))$ is an integral curve of the reduced Hamiltonian system on $(T^*Q)_\mu$, that $\bar{c}(t) = \mathbb{F}L(c(t))$ is the reconstructed integral curve on $J^{-1}(\mu)$, and that $\bar{d}(t) = \mathbb{F}L(d(t))$ is the $\bar{\gamma}^\mu_{mech}$-horizontal lift of $\bar{c}_\mu(t)$ iff $d(t)$ is the γ^μ_{mech}-horizontal lift of $\bar{c}_\mu(t)$. Thus by Corollary **3.6** we have

$$\bar{c}(t) = \exp(\theta(t)\zeta)\cdot\left(q_h'(t) + \frac{\mu\cdot\zeta}{\|\zeta_Q(q_h(t))\|^2}\,\zeta_Q(q_h(t))\right)^\flat \tag{6}$$

with

$$\theta(t) = (\mu\cdot\zeta)\int_0^t \frac{ds}{\|\zeta_Q(q_h(s))\|^2}\,, \tag{7}$$

where $q_h(t) \in Q$ is the γ_{mech}-horizonal lift of the projection $q_\mu(t) \in Q/G_\mu$ of the reduced integral curve $\bar{c}_\mu(t) \in (T^*Q)_\mu \subset T^*(Q/G_\mu)$.

Let us determine the $\bar{\gamma}^\mu_{mech}$-holonomy of the periodic curve $\bar{c}_\mu(t)$ with period T. In the proof of Corollary **3.4** we showed that the γ^μ_{mech}-horizontal lift of $c_\mu(t)$ is the curve $d(t) = q_h'(t) + (\mu\cdot\zeta)\zeta_Q(q_h(t))/\|\zeta_Q(q_h(t))\|^2$ (see formula (4) in §3C). Thus since $\bar{d}(t) = d(t)^\flat$ and the group action is by isometries, the holonomy of $\bar{c}_\mu(t)$ measured from $\bar{c}(0)$ equals the holonomy of $c_\mu(t)$ measured from $c(0)$. Let $k \in G_\mu = S^1$ be the holonomy of $q_\mu(t)$ measured from $q(0)$, *i.e.*,

$$q_h(T) = k\cdot q_h(0) = k\cdot q(0)\,. \tag{8}$$

Since G_μ is abelian and $\zeta \in \mathfrak{g}_\mu$, the vector field ζ_Q is equivariant and we get

$$\begin{aligned} d(T) &= q_h'(T) + \frac{\mu\cdot\zeta}{\|\zeta_Q(q_h(t))\|^2}\zeta_Q(q_h(T)) \\ &= k\cdot q_h'(0) + \frac{\mu\cdot\zeta}{\|k\cdot\zeta_Q(q_h(0))\|^2}k\cdot\zeta_Q(q_h(0)) \\ &= k\cdot d(0) \end{aligned} \tag{9}$$

i.e., the holonomy of $\bar{c}_\mu(t)$ measured from $c(0)$ equals the holonomy of $q_\mu(t)$ measured from $q(0) = \tau(c(0)) = \pi(\bar{c}(0))$. Letting $k = \exp(\chi\zeta)$, we conclude from Theorem **2.1** and (7) that the total phase equals

$$\varphi = \chi + (\mu\cdot\zeta)\int_0^T \frac{ds}{\|\zeta_Q(q_h(s))\|^2}\,. \tag{10}$$

We collect formulas (5) and (10) in the following :

5.2 Proposition *Given is a simple mechanical system on* T^*Q *with Hamiltonian* H. *Assume that for* $\mu \in \mathfrak{g}^*$, $G_\mu = S^1$, *that* $J^{-1}(\mu) \to (T^*Q)_\mu$ *and* $Q \to Q/G_\mu$ *are principal* S^1-*bundles and that* $\gamma_{mech} \in \Omega^1(Q)$ *is the connection on the second bundle whose horizontal subbundle is the orthogonal complement of the vertical bundle. Let* $\bar{c}_\mu(t) \in J^{-1}(\mu)$ *be a periodic integral curve with period* T *of the reduced system with energy* H_μ *and let* $q_\mu(t) \in Q/G_\mu$ *be its base integral curve. Then the total phase of the reconstructed integral curve on the level set* $J^{-1}(\mu)$ *measured from* $\bar{c}(0)$ *equals*

$$\boxed{\varphi = \frac{1}{\mu\cdot\zeta}\iint_D \omega_\mu + \frac{2(H_\mu - \langle\bar{V}_\mu\rangle)T}{\mu\cdot\zeta} = \chi + (\mu\cdot\zeta)\int_0^T \frac{ds}{\|\zeta_Q(q_h(s))\|^2}}$$

where ζ *is a generator of* $\mathfrak{g}_\mu$, ω_μ *is the reduced symplectic form on* $(T^*Q)_\mu$, V *is the potential energy of* H, $\langle\bar{V}_\mu\rangle$ *is the average over* $q_\mu(t)$ *of the induced function* $\bar{V}_\mu : Q/G_\mu \to \mathbb{R}$, $q_h(t)$ *is the* γ_{mech}-*horizontal lift to* Q *of* $q_\mu(t)$, *and* χ *is the* γ_{mech}-*holonomy of* $q_\mu(t)$ *measured from* $q(0) := \pi(\bar{c}(0))$. *Here,* D *is a two-dimensional surface in* $(T^*Q)_\mu$ *whose boundary is* $\bar{c}_\mu(t)$ *and whose existence is assumed. The first terms in the two formulas are the geometric and the second the dynamic phases.*

§5C *Phases for the Free Rigid Body*

As we already remarked, if $Q = \mathbf{SO}(3)$ is endowed with the left invariant Riemannian metric whose value at the identity is $\langle x, y\rangle = x\cdot\mathbb{I}y$, $\mathbb{I} = \mathrm{diag}(I_1, I_2, I_3)$, $I_i > 0$, $i = 1, 2, 3$, $V = 0$, and $\zeta = \mu/\|\mu\|$, the total phase formula given by the first equality in Proposition **5.2** is equal to formula (2) in §**1.F**, namely

$$\boxed{\varphi = -\Lambda + \frac{2H_\mu T}{\|\mu\|}, \qquad (1)}$$

where H_μ is the energy of the solution curve $\Pi(t)$ on the sphere, T is its period, and Λ is the oriented solid angle it bounds. We shall derive below a similar formula but using the connection γ^μ_{mech} which is given by the second equality in Proposition **5.2**.

We begin by explicitly computing the connection γ_{mech} on the principal S^1-bundle $\rho_\mu : \mathbf{SO}(3) \to \mathbf{SO}(3)/S^1 \approx S^2$, $\rho_\mu(g) = g^{-1}\mu$. If $u_g \in T_g\mathbf{SO}(3)$, since ζ is the generator of $\mathfrak{g}_\mu$, $\zeta = \mu/\|\mu\|$ we can write the horizontal plus vertical decomposition of u_g as

$$u_g = v_g + \lambda\hat{\zeta}_{\mathbf{SO}(3)}(g)) = v_g + \lambda\hat{\zeta}g\,.$$

Since v_g is horizontal, *i.e.*, orthogonal in the Riemannian metric of $\mathbf{SO}(3)$ to $\hat{\zeta}g$, we conclude that $\langle u_g, \hat{\zeta}g\rangle = \lambda\langle\hat{\zeta}g, \hat{\zeta}g\rangle$, or using left-invariance, $\langle g^{-1}\cdot u_g, g^{-1}\hat{\zeta}g\rangle = \lambda\langle g^{-1}\hat{\zeta}g, g^{-1}\hat{\zeta}g\rangle$, whence $\lambda = (g^{-1}\cdot u_g)^{\check{}}\cdot\mathbb{I}g^{-1}\zeta/g^{-1}\zeta\cdot\mathbb{I}g^{-1}\zeta$, where $\check{}: \mathbf{so}(3) \to \mathbb{R}^3$ is the inverse of the Lie algebra homomorphism

$$x \in \mathbb{R}^3 \mapsto \hat{x} = \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix} \in \mathbf{so}(3)\,.$$

Therefore, the connection form $\gamma_{\text{mech}} \in \Omega^1(\mathbf{SO}(3))$ is given by

$$\gamma_{\text{mech}}(g)(u_g) = \frac{(g^{-1}\cdot u_g)^{\check{}}\cdot\mathbb{I}g^{-1}\zeta}{g^{-1}\zeta\cdot\mathbb{I}g^{-1}\zeta}\,. \qquad (2)$$

To compute the dynamic phase, we need to determine the horizontal lift $g_h(t) \in \mathbf{SO}(3)$ of the closed path $\Pi(t)$ on S^2, the sphere of radius $\|\mu\|$. By (1), $g_h(t)$ is the horizontal lift of $\Pi(t)$ iff

$$g_h(t)^{-1}\mu = \Pi(t) \text{ and } (g_h(t)^{-1}\dot{g}_h(t))^{\check{}}\cdot\mathbb{I}g_h(t)^{-1}\zeta = 0\,. \qquad (3)$$

However, the square of the norm of $\hat{\zeta}_{SO(3)}(g_h(t)) = \hat{\zeta}g_h(t)$ in the $SO(3)$ metric is by left-invariance and (3), $g_h(t)^{-1}\zeta\cdot\mathbb{I}g_h(t)^{-1}\zeta = \frac{1}{\|\mu\|^2}\Pi(t)\cdot\mathbb{I}\Pi(t)$, so that the dynamic phase in the second formula of Proposition **5.2** equals

$$\|\mu\|^3\int_0^T \frac{dt}{\Pi(t)\cdot\mathbb{I}\Pi(t)} \,. \tag{4}$$

To determine the geometric phase, we need to compute the curvature of the connection (2). On $SO(3)$ this equals $\mathbf{d}\gamma_{mech}$. By Cartan's formula, $\mathbf{d}\gamma_{mech}(X, Y) = X[\gamma_{mech}\cdot Y] - Y[\gamma_{mech}\cdot X] - \gamma_{mech}\cdot[X, Y]$ for any two vector fields $X, Y \in \mathfrak{X}(SO(3))$. If $X(g) = T_eL_g\hat{\mathbf{x}}$, $\mathbf{x} \in \mathbb{R}^3$ we have

$$(\gamma_{mech}\cdot X)(g) = \frac{(g^{-1}\cdot X(g))^{\vee}\cdot\mathbb{I}g^{-1}\zeta}{g^{-1}\zeta\cdot\mathbb{I}g^{-1}\zeta} = \frac{\mathbf{x}\cdot\mathbb{I}g^{-1}\zeta}{g^{-1}\zeta\cdot\mathbb{I}g^{-1}\zeta}$$

so that if $Y(g) = T_eL_g\hat{\mathbf{y}}$, we get

$$\begin{aligned} Y[\gamma_{mech}\cdot X](g) &= \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0}(\gamma_{mech}X)(g\exp\varepsilon\hat{\mathbf{y}}) \\ &= -\frac{1}{(g^{-1}\zeta\cdot\mathbb{I}g^{-1}\zeta)^2}\Big[(\mathbf{x}\cdot\mathbb{I}\hat{\mathbf{y}}g^{-1}\zeta)(g^{-1}\zeta\cdot\mathbb{I}g^{-1}\zeta) - 2(\mathbf{x}\cdot\mathbb{I}g^{-1}\zeta)(g^{-1}\zeta\cdot\mathbb{I}\hat{\mathbf{y}}g^{-1}\zeta)\Big]. \end{aligned}$$

Therefore

$$\begin{aligned} &\mathbf{d}\gamma_{mech}(g)(T_eL_g\hat{\mathbf{x}}, T_eL_g\hat{\mathbf{y}}) \\ &= -\frac{1}{(g^{-1}\zeta\cdot\mathbb{I}g^{-1}\zeta)^2}\Big\{(g^{-1}\zeta\cdot\mathbb{I}g^{-1}\zeta)\big[\mathbb{I}\mathbf{y}\cdot(\mathbf{x}\times g^{-1}\zeta) - \mathbb{I}\mathbf{x}\cdot(\mathbf{y}\times g^{-1}\zeta) + (\mathbf{x}\times\mathbf{y})\cdot\mathbb{I}g^{-1}\zeta\big] \\ &\qquad - 2(\mathbf{y}\cdot\mathbb{I}g^{-1}\zeta)((\mathbf{x}\times g^{-1}\zeta)\cdot\mathbb{I}g^{-1}\zeta) + 2(\mathbf{x}\cdot\mathbb{I}g^{-1}\zeta)((\mathbf{y}\times g^{-1}\zeta)\cdot\mathbb{I}g^{-1}\zeta)\Big\}. \end{aligned} \tag{5}$$

The curvature Ω as a two-form on the base S^2, the sphere of radius $\|\mu\|$, is given by the condition $\rho_\mu^*\Omega = \mathbf{d}\gamma_{mech}$. Since

$$T_g\rho_\mu(T_eL_g\hat{\mathbf{x}}) = g^{-1}\mu\times\mathbf{x} = \|\mu\|\,g^{-1}\zeta\times\mathbf{x}, \tag{6}$$

we get $\Omega(g^{-1}\mu)(g^{-1}\mu\times\mathbf{x}, g^{-1}\mu\times\mathbf{y}) = \mathbf{d}\gamma_{mech}(g)(T_eL_g\hat{\mathbf{x}}, T_eL_g\hat{\mathbf{y}})$ so that letting $\Pi = g^{-1}\mu$, one has $\|\Pi\| = \|\mu\|$ and (5) yields

$$\begin{aligned} \Omega(\Pi)(\Pi\times\mathbf{x}, \Pi\times\mathbf{y}) &= -\frac{\|\mu\|}{(\Pi\cdot\mathbb{I}\Pi)^2}\Big\{(\Pi\cdot\mathbb{I}\Pi)[\mathbb{I}\mathbf{y}\cdot(\mathbf{x}\times\Pi) - \mathbb{I}\mathbf{x}\cdot(\mathbf{y}\times\Pi) + (\mathbf{x}\times\mathbf{y})\cdot\mathbb{I}\Pi] \\ &\qquad - 2(\mathbf{y}\cdot\mathbb{I}\Pi)((\mathbf{x}\times\Pi)\cdot\mathbb{I}\Pi) + 2(\mathbf{x}\cdot\mathbb{I}\Pi)((\mathbf{y}\times\Pi)\cdot\mathbb{I}\Pi)\Big\}. \end{aligned}$$

Up to a factor of 2, the last two terms equal

$$((\Pi \times \mathbb{I}\Pi)\cdot\mathbf{y})(\mathbb{I}\Pi\cdot\mathbf{x}) - ((\Pi \times \mathbb{I}\Pi)\cdot\mathbf{x})(\mathbb{I}\Pi\cdot\mathbf{y})$$

$$= ((\Pi \times \mathbb{I}\Pi) \times \mathbb{I}\Pi)\cdot(\mathbf{y} \times \mathbf{x})$$

$$= (\|\mathbb{I}\Pi\|^2\Pi - (\Pi\cdot\mathbb{I}\Pi)\mathbb{I}\Pi)\cdot(\mathbf{x} \times \mathbf{y})$$

so that Ω becomes

$$\Omega(\Pi)(\Pi \times \mathbf{x}, \Pi \times \mathbf{y}) = -\frac{\|\mu\|}{(\Pi\cdot\mathbb{I}\Pi)^2}\Big\{(\Pi\cdot\mathbb{I}\Pi)\Pi\cdot(\mathbb{I}\mathbf{y} \times \mathbf{x} - \mathbb{I}\mathbf{x} \times \mathbf{y})$$

$$- (\Pi\cdot\mathbb{I}\Pi)((\mathbf{x} \times \mathbf{y})\cdot\mathbb{I}\Pi) + 2\,\|\mathbb{I}\Pi\|^2\Pi\cdot(\mathbf{x} \times \mathbf{y})\Big\}.$$

However,

$$\Pi\cdot(\mathbb{I}\mathbf{y} \times \mathbf{x} - \mathbb{I}\mathbf{x} \times \mathbf{y}) - \mathbb{I}\Pi\cdot(\mathbf{x} \times \mathbf{y}) = -(\mathrm{tr}\mathbb{I})\Pi\cdot(\mathbf{x} \times \mathbf{y})$$

so that

$$\Omega(\Pi)(\Pi \times \mathbf{x}, \Pi \times \mathbf{y}) = -\frac{\|\mu\|}{(\Pi\cdot\mathbb{I}\Pi)^2}\Big[2\|\mathbb{I}\Pi\|^2 - (\Pi\cdot\mathbb{I}\Pi)(\mathrm{tr}\mathbb{I})\Big]\Pi\cdot(\mathbf{x} \times \mathbf{y})$$

$$= -\frac{2\|\mathbb{I}\Pi\|^2 - (\Pi\cdot\mathbb{I}\Pi)(\mathrm{tr}\mathbb{I})}{(\Pi\cdot\mathbb{I}\Pi)^2}\, dS(\Pi)(\Pi \times \mathbf{x}, \Pi \times \mathbf{y}) \tag{7}$$

where $dS(\Pi)(\Pi \times \mathbf{x}, \Pi \times \mathbf{y}) = \|\mu\|\Pi\cdot(\mathbf{x} \times \mathbf{y})$ is the area element on the sphere of radius $\|\mu\|$. Thus the total phase of the integral curve starting at $c(0) \in \mathbf{J}^{-1}(\mu)$, reconstructed from the periodic orbit $\Pi(t)$ of period T on the sphere of radius $\|\mu\|$ is (mod 2π) equal to

$$\varphi = \iint_D \frac{2\|\mathbb{I}\Pi\|^2 - (\Pi\cdot\mathbb{I}\Pi)(\mathrm{tr}\mathbb{I})}{(\Pi\cdot\mathbb{I}\Pi)^2}\, ds + \|\mu\|^3 \int_0^T \frac{dt}{\Pi(t)\cdot\mathbb{I}\Pi(t)} \tag{8}$$

where D is the spherical cap bounded by $\Pi(t)$. The first term is the geometric and the second the dynamic phase. The right hand sides of (1) and (8) are equal since they both represent the total phase of the same integral curve. We see here how the split of the total phase into geometric and dynamic phases is entirely dependent on the chosen connection. The same phenomenon will be discussed for the ball in the hoop example (see **§1B**) in **§12B**.

§5D *Phases for the heavy top*

The notations and conventions are as in the previous example, except that the Hamiltonian equals

$$H(\alpha_h) = \frac{1}{2}\|\alpha_h\|^2 + Mg\ell\mathbf{k}\cdot h\chi\,, \quad \alpha_h \in T_h^*\mathbf{SO}(3)\,, \tag{1}$$

where $\mathbf{k}$ is the unit vector of the spatial Oz-axis, g is the gravitational acceleration pointing in the negative direction of Oz, M is the total mass of the body, the fixed point about which the body is moving is the origin, and χ is the unit vector of the straight line segment of length ℓ connecting the origin to the center of mass of the body. The Hamiltonian is left-invariant under the rotations about the spatial Oz-axis and the corresponding momentum map is $\mathbf{J} : T^*\mathbf{SO}(3) \to \mathbb{R}$, $\mathbf{J}(h, \Pi) = h\Pi\cdot\mathbf{k}$, where $T^*\mathbf{SO}(3)$ is identified with $\mathbf{SO}(3) \times \mathbb{R}^3$ via left translations. The reduced spaces $\mathbf{J}^{-1}(\mu)/S^1$ are generically cotangent bundles of spheres with the symplectic form equal to the canonical form plus a magnetic term, or, equivalently coadjoint orbits in $\mathbf{so}(3)^* = \mathbb{R}^3 \times \mathbb{R}^3$, $\mathcal{O} = \{(\Pi, \Gamma) \mid \Pi\cdot\Gamma = \mu, \|\Gamma\|^2 = 1\}$; the map $\mathbf{J}^{-1}(\mu) \to \mathcal{O}$ is given by $\alpha_h \mapsto (T_e^*L_h(\alpha_h) = \Pi$, where $h^{-1}\mathbf{k} = \Gamma$, as in Marsden, Ratiu, and Weinstein [1984]. After an appropriate momentum shift (see Proposition **2.3**), the cotangent bundle projection $\Pi : \mathcal{O} \to S^2$ of $\mathcal{O}$ onto the sphere of radius 1, is given by $\pi(\Pi, \Gamma) = \Gamma$.

Let $(\Pi(t), \Gamma(t))$ be a periodic orbit of period T of the heavy top equations. Then considering the connection $\gamma \in \Omega^1(\mathbf{SO}(3))$ whose horizontal bundle is the orthogonal (relative to the rigid body metric) of the vertical bundle, all considerations of the previous example apply and we get the total phase formula

$$\boxed{\varphi = \iint_D \frac{2\|\mathbb{I}\Gamma\|^2 - (\Gamma\cdot\mathbb{I}\Gamma)(\mathrm{tr}\mathbb{I})}{(\Gamma\cdot\mathbb{I}\Gamma)^2}\,ds + \int_0^T \frac{dt}{\Gamma\cdot\mathbb{I}\Gamma}} \tag{2}$$

where D is the spherical cap on the unit sphere inclosed by the closed curve $\Gamma(t)$, $t \in [0,T]$.

Next, let us recompute the total phase using the connection A of Proposition **2.2**. Since in the left trivialization, $\mathbf{J}^{-1}(\mu) = \{(h, \Pi) \mid h\Pi\cdot\mathbf{k} = \Pi\cdot\Gamma = \mu\}$ and $\theta(h, \Pi)(h\hat{\mathbf{x}}, \lambda) = \Pi\cdot\mathbf{x}$, the connection one-form (2) in this case becomes

$$A(h,\Pi)(h\hat{\mathbf{x}}, \lambda) = \frac{1}{\mu}\,\Pi\cdot\mathbf{x}\,. \tag{3}$$

The A-horizontal lift of the closed curve $(\Pi(t), \Gamma(t))$ is $(h(t), \Pi(t)) \in \mathbf{J}^{-1}(\mu)$, where $h(t)$ is uniquely determined by the conditions

$$h(t)^{-1}\mathbf{k} = \Gamma(t) \text{ and } (h(t)^{-1}\dot{h}(t))\check{}\cdot\Pi(t) = 0\,. \tag{4}$$

Therefore, the cotangent bundle projection of this horizontal lift is h(t) and hence

$$V(h(t)) = Mg\ell \mathbf{k}\cdot h\chi = Mg\ell\Gamma(t)\cdot\chi . \tag{5}$$

Thus, by Proposition **5.2**, the dynamic phase is

$$\frac{1}{\mu}\left(2H_\mu T - 2Mg\ell\int_0^t \Gamma(t)\cdot\chi dt\right) \tag{6}$$

The geometric phase is obtained in the following way. Let $\mathcal{D}$ be a two-dimensional surface in $T^*\mathbf{SO}(3)_\mu = \mathcal{O} \cong TS^2$ bounded by the integral curve $(\Pi(t), \Gamma(t))$ and let

$$\begin{aligned}\omega_\mu(\Pi, \Gamma)((\Pi\times\mathbf{x}+\Gamma\times\mathbf{y}, \Gamma\times\mathbf{x}), (\Pi\times\mathbf{x}'+\Gamma\times\mathbf{y}', \Gamma\times\mathbf{x}'))\\ = -\Pi\cdot(\mathbf{x}\times\mathbf{x}') - \Gamma\cdot(\mathbf{x}\times\mathbf{y}' - \mathbf{x}'\times\mathbf{y})\end{aligned} \tag{7}$$

be the orbit symplectic form on $\mathcal{O}$. Then the geometric phase equals

$$\frac{1}{\mu}\iint_{\mathcal{D}}\omega_\mu . \tag{8}$$

By (6) and (8), the total phase equals

$$\varphi = \frac{1}{\mu}\iint_{\mathcal{D}}\omega_\mu + \frac{1}{\mu}\left(2H_\mu T - 2Mg\ell\int_0^T \Gamma(t)\cdot\chi dt\right). \tag{9}$$

Again, as in §**5B**, two distinct formulas, (2) and (9), are obtained for the total phase.

§5E *Phases for Coupled Planar Rigid Bodies*

In this section we compute the phase for another mechanical system with S^1 symmetry, namely a system of coupled rigid bodies in the plane. We thank P.S. Krishnaprasad for informing us of this example; see Krishnaprasad [1990] and Krishnaprasad and Yang [1990] for more information. Following Krishnaprasad's lecture at the Geometric Phases workshop at MSI-Cornell University (October 10-13, 1989), we first calculate the phase "by hand" without using any of the general theory, and then we shall show that the formula so obtained is a special case of formula (10) in §**5B** (see Proposition **5.2**).

We consider n bodies forming a chain, as in Figure **5E-1**. The center of mass motion has been assumed to have been eliminated, so the configuration space $Q = \mathbb{T}^n = S^1 \times \ldots \times S^1$ is the product of n copies of the circle. The i^{th} circle describes the orientation of body i with respect to an inertial frame. We let $q^i \in S^1$ denote this angle for the i^{th} body and note that from these angles, the configuration of the bodies in the plane is known.

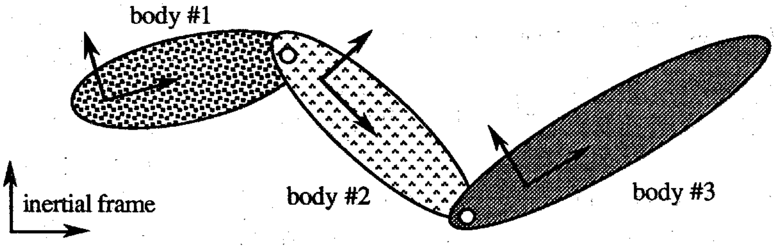


Figure 5E-1

As shown in Sreenath *et. al.*[1988], the equations for the dynamics of coupled planar rigid bodies are the Euler-Lagrange equations for a Lagrangian of the following form:

$$L = \frac{1}{2} \sum_{i,j=1}^{n} \omega^i \mathbb{J}_{ij} \omega^j \tag{1}$$

where $\omega^i = \dot{q}^i$. The matrix $\mathbb{J}_{ij}$ is a positive definite inertia-type matrix that depends in a nontrivial way on the angles q^i. The Hamiltonian $H : T^*Q \to \mathbb{R}$ corresponding to (1) is obtained by the usual Legendre transform, $p_i = \mathbb{J}_{ij}\omega^j$, giving

$$H = \frac{1}{2} \sum_{i,j=1}^{n} p_i \mathbb{J}^{ij} p_j \tag{2}$$

where $\mathbb{J}^{ij}$ is the inverse matrix of $\mathbb{J}_{ij}$.

Imagine two types of motion. First, we consider free motion according to the Lagrangian or Hamiltonian system just described and second, the motion where the joints are controlled with a torque T_i exerted on body i by body $i+1$ by, say, a motor at the joint between the bodies. Here, $i = 1, \ldots n-1$ and the torques are *internal*, so in either case, the angular momentum of the overall system is conserved. Corresponding to overall rotations of the system, the group S^1 acts on Q by the diagonal action $\varphi \cdot (q^1, q^2, \ldots, q^n) = (q^1 + \varphi, q^2 + \varphi, \ldots, q^n + \varphi)$ and hence on the tangent and the cotangent spaces by the tangent and the cotangent lift. The Lagrangian and the Hamiltonian are both invariant under this action, as is reflected by the invariance of the matrix $\mathbb{J}$. In fact, $\mathbb{J}$ depends only on the phase differences, $\theta^i = q^{i+1} - q^i$. Notice that the phase differences parametrize the *shape space* $S = Q/S^1 = \mathbb{T}^{n-1}$. The reduced space is therefore T^*S. The unreduced motion occurs on the space of q^i's and the reduced motion on the space of θ^i's. Note that for $n = 3$, the shape space is the two torus, so possible difficulties with a loop on S not being the boundary of a surface in S can certainly occur. Identify the Lie algebra of S^1 with $\mathbb{R}$; if $\xi \in \mathbb{R}$, then the infinitesimal generator is given by $\xi_Q(q) = \xi \sum_{i=1}^{n} \frac{\partial}{\partial q^i}$. The associated momentum map is $\mathbf{J}(q, p)\xi = \xi \sum_{i=1}^{n} p_i = \xi e \cdot \mathbb{J}\omega$, where e is the column vector consisting of n 1's. Write

$$\omega = \begin{bmatrix} \omega^1 \\ \omega^2 \\ \vdots \\ \omega^n \end{bmatrix} = \omega^1 \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ \dot\theta_1 \\ \dot\theta_1 + \dot\theta_2 \\ \vdots \\ \dot\theta_1 + \ldots + \dot\theta_{n-1} \end{bmatrix} := \omega^1 e + M\dot\theta, \tag{2}$$

which defines the matrix M. Suppose that the unreduced motion takes place with angular momentum $\mu = \mathbf{J}(q, p)\xi = \xi e \cdot \mathbb{J}\omega$. Substitution from (2) gives

$$\mu = \xi e \cdot (\mathbb{J}\omega_1 e) + \xi e \cdot \mathbb{J}M\dot\theta$$

so that

$$\dot q^1 = \omega^1 = \frac{\mu}{\xi e \cdot (\mathbb{J}e)} - \frac{\mathbb{J}M\dot\theta}{\xi e \cdot (\mathbb{J}e)} \tag{3}$$

Integration of (3) along a given curve c(t), $0 \le t \le T$ in shape space gives

$$\Delta q^1 = \mu \int_0^T \frac{dt}{\xi e \cdot (\mathbb{J}e)} - \int_0^T \frac{\mathbb{J}M\dot\theta}{\xi e \cdot (\mathbb{J}e)} \tag{4}$$

(A choice of ξ corresponds to a choice of the unit of time.) Note that the second term of (4) can be written as an integral over the shape space curve

$$\int_c \frac{\mathbb{J}Md\theta}{\xi e \cdot (\mathbb{J}e)} \tag{5}$$

where we regard $\mathbb{J}Md\theta$ as a one form on S. In particular, (5) is parametrization independent, which is a hallmark of a *geometric phase*. The first term of (4) corresponds to the *dynamic phase* and is parametrization dependent. Both integrals are regarded as integrals over a curve in shape space S. From (4) we can construct the change in q^1, and from this and the supposed known changes in the θ^i, get the phase changes in the remaining variables, thus *reconstructing the dynamics explicitly in terms of quadratures*, with or without internal torques. We are, of course, most interested in these formulas when the curve in shape space is closed, but at this point it can be a general curve. To be even more explicit, for the case of two bodies, and using the formula for $\mathbb{J}$ from Sreenath *et. al.*[1988], one finds that (5) equals

$$\int \frac{I_2 + \varepsilon d_1 d_2 \sin\theta}{I_1 + I_2 + 2\varepsilon d_1 d_2 \sin\theta}\, d\theta\,, \tag{6}$$

where θ is the single joint angle, I_1 and I_2 are *augmented* moments of inertia of the two bodies (see Sreenath, *et. al.* [1988] for the definition), $\varepsilon = \frac{m_1 m_2}{m_1 + m_2}$ is the reduced mass and d_i are the distances from the center of mass to the hinge point for body i. We remark that (6) can be integrated explicitly (see for instance Frohlich [1979]), although the corresponding formula for n bodies is rather complicated.

Formula (4) can be derived from our Proposition **5.2**. Indeed, we assert that *the formula* (10) *in Section* **5B** *gives formula* (4) *above as a special case*. To see this, first note that the connection γ_{mech} is given from the remarks following Corollary **3.6** by

$$\gamma_{mech} = \frac{e \cdot \mathbb{J} dq}{e \cdot \mathbb{J} e} \tag{7}$$

Indeed, it is easily verified that (7) defines a connection one form and it is clear from its expression that its horizontal subspace (*i.e.,* the kernel of (7)) is the (metric) $\mathbb{J}$-orthogonal complement to the S^1- orbits, that is to the space spanned by the vector e. The general formula from Proposition **5.2** is

$$\varphi = \chi + (\mu \cdot \zeta) \int_0^T \frac{ds}{\|\zeta_Q(q_h(s))\|^2}\,. \tag{8}$$

In this formula, the phase φ is the phase in the S^1 fiber for the reconstructed motion. In our case, the projection $Q \to S$ has a fiber which we identify with the phase of the first body, namely with Δq^1. By definition, the term χ is the holonomy for the connection γ_{mech}. This is most easily computed using the formula (10) of §**4.B**. Indeed, here one interprets the matrix M as the derivative of a global section s for the bundle $Q \to S$ and so the pull-back of the form (7) is just

$$s^* \gamma_{mech} = \frac{\mathbb{J} M d\theta}{\xi e \cdot (\mathbb{J} e)}$$

whose integral is the second term of (4). Thus, the second term of (4) is exactly the geometric phase, that is, the first term of (8). To deal with the second term of (8), take $\zeta = \xi$ and use the fact that $\xi_Q(q) = \xi \sum_{i=1}^{n} \frac{\partial}{\partial q^i} = \xi e$, as we have seen. Thus,

$$\|\zeta_Q(q_h(s))\|^2 = \|\xi_Q(q_h(s))\|^2 = \xi^2\| e \|^2 = \xi^2 e\cdot \mathbb{J}e,$$

recalling that $\mathbb{J}$ depends on the joint angles only, so that this expression does not require the computation of the horizontal lift q_h , but depends only on the reduced variables. Thus, it is clear that the second term of (8) gives the first term of (4), so our assertion is proved.

When the reduced curve bounds a surface in S (which need not happen if the curve has non trivial homology class on the torus—that is, if it winds around the torus nontrivially) then the holonomy can be converted to a surface integral of the curvature of the mechanical connection.

Formula (5) of section **5B** provides an alternative formula for the phase *in the case there are no torques*. (Torques are not allowed in that formula since it involves the energy of the reduced curve and with internal or external torques, this need not be conserved.) To include the possibility of torques, one replaces the term $2H_\mu T$ with the integral of $2H_\mu$ along the reduced curve. That is, we have

$$\Delta q^1 = \frac{1}{\mu\xi}\left(\iint_D \omega_\mu + 2\langle H_\mu\rangle T\right), \tag{9}$$

where

$$\langle H_\mu\rangle = \frac{1}{T}\int_0^t H_\mu(\theta(t))\, dt\,.$$

In this formula, note that the reduced symplectic form is on $T^*S = T^*\mathbb{T}^{n-1}$ and the surface D is chosen to span the curve (θ, p_θ), where p_θ is the conjugate momentum to the joint velocities $\dot{\theta}$. Note that the computation of the double integral in (9) involves the magnetic terms of the reduction, which involves the computation of the curvature of the mechanical connection. Also, the reduced energy H_μ is rather complicated. It is computed in Sreenath *et.al.* [1988]. One substitutes (3) in (1). This expression involves the amended potential for the reduction.

It seems, therefore, that for examples like coupled rigid bodies, the approach using the mechanical connection rather than the canonical one form connection A gives the most tractable results.

§6 *Averaging Connections*

The purpose of this section is to define the Hannay-Berry connection by the averaging process. The main properties of such connections are given here and will be shown to characterize the connections in the next section.

§6A *Families of Actions*

Let $\pi : E \to M$ be a Poisson fiber bundle, *i.e.*, π is a surjective submersion, all fibers are Poisson manifolds, and the transition functions are Poisson maps. Let G be a Lie group. A ***family of Hamiltonian G-actions*** on E is a smooth (left) G-action on E such that each fiber $\pi^{-1}(m)$ is invariant under the action and the action restricted to each fiber is Hamiltonian, *i.e.*, it is Poisson and it admits a fiberwise momentum map $\mathbf{I} : E \to \mathfrak{g}^*$. This means that for each $\xi \in \mathfrak{g}$, we have

$$\xi_E(p) = X_{I^\xi}(p) \tag{1}$$

where $\xi_E(p) = \frac{d}{d\varepsilon}\Big|_{t=0} (\exp \varepsilon\xi \cdot p)$ is the infinitesimal generator of the action defined by ξ and $X_{I^\xi}(p)$ is the Hamiltonian vector field on the fiber through p defined by the function $I^\xi : E \to \mathbb{R}$ restricted to this fiber. Here, I^ξ denotes the real valued function defined by $I^\xi(p) = \mathbf{I}(p)\cdot\xi$, for $p \in E$ and $\xi \in \mathfrak{g}$. Since the action on each fiber is Hamiltonian, the symplectic leaves of the fiber are G-invariant. Also, note that the Casimirs are G-invariant.

An Ehresmann connection on $\pi : E \to M$ is called ***Poisson***, if its horizontal lift is a Poisson bracket derivation, *i.e.*,

$$(\mathrm{hor}\, Z)[\{f, h\}] = \{(\mathrm{hor}\, Z)[f], h\} + \{f, (\mathrm{hor}\, Z)[h]\} \tag{2}$$

for all $f, h : E \to \mathbb{R}$ and $Z \in \mathfrak{X}(M)$. Here, the bracket means that the functions are restricted to the fibers and the bracket is computed fiberwise. Equivalently, (2) says that

$$\mathbf{D}\{f, h\}\cdot X = \{\mathbf{D}f\cdot X, h\} + \{f, \mathbf{D}h\cdot X\} \tag{2'}$$

for all $X \in \mathfrak{X}(E)$, where $\mathbf{D}$ is the covariant differentiation defined by hor. As we shall see later, our two main examples, the Cartan connection and the Hannay-Berry connection built from it (or from another suitable connection) do satisfy the condition (2).

If the fibers are symplectic, this notion is the same as that of a symplectic connection. Symplectic connections have been studied by Lichnerowicz [1978], Hess [1981], Marsden, Ratiu and Raugel [1990], and Guillemin and Lerman [1989].

At present, these two structures -- the connection and the G-action -- have no relationship to each other. The goal of this section is to show how a given connection can be altered so that

$$\mathbf{DI} = 0 \tag{3}$$

holds. (In doing so, we may also have to alter $\mathbf{I}$ by adding to it fiberwise Casimirs.)

6.1 Theorem: Averaging of Connections *Let* $\pi : E \to M$ *be a fiber bundle and* $\gamma \in \Omega^1(E, V)$ *an Ehresmann connection. Suppose the compact Lie group* G *acts on* E *by bundle transformations, not necessarily covering the identity. Then the average* $\langle\gamma\rangle$ *of* γ *is also an Ehresmann connection. Moreover the G-action commutes with the action of parallel translation with respect to* $\langle\gamma\rangle$.

Proof Since γ is vector-bundle valued we have to be careful in defining its average. If g is a bundle automorphism we define the ***pull-back*** $g^*\gamma$ of γ by

$$(g^*\gamma)(e)\cdot v = Tg^{-1}\cdot\gamma(g\cdot e)\cdot Tg\cdot v\ .$$

This works because Tg maps V to V. Define the ***average*** of γ by

$$\langle\gamma\rangle = \frac{1}{|G|}\int_G (g^*\gamma)\, dg \tag{4}$$

where dg is a Haar measure and $|G|$ is the total volume of G.

To check that the V-valued one form $\langle\gamma\rangle$ is a connection one-form, we need only check that it is the identity on the vertical bundle V. Indeed, if $v \in V$ then $g^*\gamma\cdot v = v$, so that

$$\langle\gamma\rangle\cdot v = \frac{1}{|G|}\int_G (g^*\gamma)\cdot v\, dg = \frac{1}{|G|}\int_G v\, dg = v.$$

One also checks that

$$g^*\langle\gamma\rangle = \langle\gamma\rangle \quad \text{for } g \in G.$$

It follows that G takes $\langle\gamma\rangle$-horizontal spaces to $\langle\gamma\rangle$-horizontal subspaces. Now let $c(t)$ be a curve in M and $\bar{c}$ a horizontal lift of c. It follows that $g\cdot\bar{c}$ is a horizontal lift of $g\cdot c$. (Since G acts by bundle automorphisms it induces an action of G on M.) Thus the G-action commutes with parallel translation. (Note that this Proposition still holds if $\pi : E \to M$ is just a submersion.) ■

§6B *The Hannay-Berry Connection*

We now return to the setting where G defines a family of Hamiltonian G-actions. Fix a Poisson Ehresmann connection with horizontal lift hor_0 and corresponding covariant differentiation operator $\mathbf{D}_0$.

6.2 Definition *The* ***Hannay-Berry*** (HB) ***connection*** *induced by the Poisson-Ehresmann connection* hor_0 *is the Ehresmann connection on* $\pi : E \to M$ *obtained by averaging* hor_0 *according to equation* (4). *We will let* $\mathbf{D}$ *denote its covariant derivative,* hor *its horizontal lift, and* $\gamma \in \Omega^1(E, \ker T\pi)$ *its connection one-form.*

If λ is any tensor field defined along (as opposed to on) a G-invariant submanifold of E, its ***average*** $\langle\lambda\rangle$ is the smooth G-invariant tensor field of the same type defined along the same submanifold by

$$\langle\lambda\rangle = \frac{1}{|G|}\int_G (\Phi_g^* \lambda)\, dg ,$$

where $\Phi : G \times E \to E$ is the G-action on E. Note that $\langle\lambda\rangle$ is a G-invariant tensor field.

6.3 Proposition *Suppose* G *is compact and connected. Then the* HB *connection satisfies the following properties:*

i *It is Poisson.*

ii *If* $v \in T_m M$ *then its* HB *horizontal lift is given by* $\mathrm{hor}\, v = \langle \mathrm{hor}_0\, v\rangle$.

iii $\mathrm{hor}\, Z = \mathrm{hor}_0 Z + X_{K\cdot Z}$, *for a smooth function* $K\cdot Z : E \to \mathbb{R}$ *and* $Z \in \mathfrak{X}(M)$.

iv *The connection one-form of the* HB *connection is given by*

$$\gamma(v) = \gamma_0(v) - X_{K\cdot T\pi(v)}(p), \quad \text{for } v \in T_p E . \tag{7}$$

v $\langle K\cdot Z\rangle$ *is a fiberwise Casimir function.*

vi $\mathbf{D}\langle\lambda\rangle = \langle \mathbf{D}\lambda\rangle = \langle d\lambda\rangle \circ \mathbf{P}_{\mathrm{hor}}$, *for any* $\lambda \in \Omega^k(E)$, $k = 0, 1, \ldots,$ *where* $\mathbf{P}_{\mathrm{hor}}$ *is the horizontal projection relative to* hor.

vii $\mathbf{D}I^{\xi}\cdot \mathrm{hor}\, Z = \langle \mathbf{D}_0 I^{\xi}\cdot \mathrm{hor}\, Z\rangle$ *is a Casimir for each* $\xi \in \mathfrak{g}$ and $Z \in \mathfrak{X}(M)$. *Moreover,* $\mathbf{D}I = \langle \mathbf{D}_0 I\rangle$ *and hence* $\mathbf{D}I = 0$ *iff* $\langle \mathbf{D}_0 I\rangle = 0$.

Remarks

1 The function K in property **iii** can, in many examples, be constructed using symmetry properties. See §**8**.

2 Property **vi** holds for any averaged connection. The rest of the properties are consequences of the following general principle:

> If E has structure group $\mathcal{G}$, and both γ_0 and G preserve this structure, then $\langle\gamma_0\rangle$ also preserves this structure. In the HB case, $\mathcal{G}$ is the group of Hamiltonian automorphisms of the fiber.

Instead of trying to prove this general principle, we will prove proposition **6.3** "by hand." The proof of part **iv** of proposition **6.3** will follow from a lemma which we prove first. ♦

6.4 Lemma *Let* $\pi : E \to M$ *be a Poisson fiber bundle and* hor_0 *be the horizontal lift of a Poisson-Ehresmann connection. Then*

$$[\mathrm{hor}_0 Z, X_f] = X_{df \cdot \mathrm{hor}_0 Z}$$

In particular, if E *is also endowed with a family of Hamiltonian* G*-actions with parametrized momentum map* $\mathbf{I} : E \to \mathfrak{g}^*$, *then*

$$[\mathrm{hor}_0 Z, \xi_E] = X_{d\mathbf{I}^\xi \cdot \mathrm{hor}_0 Z}$$

for all $Z \in \mathfrak{X}(M)$ and $\xi \in \mathfrak{g}$.

Proof For any $g : E \to \mathbb{R}$ we have

$$\begin{aligned} [\mathrm{hor}_0 Z, X_f][g] &= (\mathrm{hor}_0 Z)[X_f[g]] - X_f[(\mathrm{hor}_0 Z)[g]] \\ &= (\mathrm{hor}_0 Z)[\{g, f\}] - \{(\mathrm{hor}_0 Z)[g], f\} \\ &= \{g, (\mathrm{hor}_0 Z)[f]\} = X_{(\mathrm{hor}_0 Z)[f]}[g] \end{aligned}$$

and the first formula is proved. The second follows from the first and the equality $\xi_E = X_{\mathbf{I}^\xi}$. ■

Proof of 6.3

ii Let $v \in T_m M$. hor v is defined by the two conditions: $T\pi \cdot \mathrm{hor}\, v = v$, $\langle\gamma\rangle \cdot \mathrm{hor}\, v = 0$. Since $T\pi \cdot \mathrm{hor}\, v = v$ and G is fiber-preserving, $\langle \mathrm{hor}_0 v\rangle$ automatically satisfies the first condition. Therefore

$$\langle \mathrm{hor}_0 v\rangle = \mathrm{hor}_0\, v + Y \tag{5}$$

where Y is some vertical vector field on E_m. Since $\langle\langle \mathrm{hor}_0 v\rangle\rangle = \langle \mathrm{hor}_0 v\rangle$, we have $\langle Y\rangle = 0$.

We now check that $\langle \mathrm{hor}_0 v\rangle$ satisfies the second condition. It is automatically G-invariant: $T\Phi_g \cdot \langle \mathrm{hor}_0 v\rangle_e = \langle \mathrm{hor}_0 v\rangle_{g \cdot e}$. Thus

$$\langle\gamma_0\rangle_e\cdot\langle\mathrm{hor}_0 v\rangle = \frac{1}{|G|}\int_G T\Phi_g^{-1}\cdot\gamma(g\cdot e)\cdot T\,\Phi_g\langle\mathrm{hor}_0 v\rangle_e\, dg$$

$$= \frac{1}{|G|}\int_G T\Phi_g^{-1}\cdot\gamma_0(g\cdot e)\cdot\langle\mathrm{hor}_0 v\rangle_{g\cdot e}\, dg$$

$$= \frac{1}{|G|}\int_G T\Phi_g^{-1}\cdot\gamma_0(g\cdot e)\cdot((\mathrm{hor}_0 v)_{g\cdot e} + Y_{g\cdot e})dg$$

$$= \frac{1}{|G|}\int_G T\Phi_g^{-1}\cdot Y_{g\cdot e}\, dg = \langle Y\rangle = 0,$$

and the second condition is satisfied.

iii Since G is compact and connected, the exponential map is onto. We can write $g = \exp\xi$ for some $\xi \in \mathfrak{g}$. By Lemma **6.4**,

$$\Phi_g^*(\mathrm{hor}_0 Z) - \mathrm{hor}_0 Z = \int_0^1 \frac{d}{dt}\Phi^*_{\exp t\xi}(\mathrm{hor}_0 Z)dt = \int_0^1 \Phi^*_{\exp t\xi}[\xi_E, \mathrm{hor}_0 Z]dt$$
$$= -\int_0^1 \Phi^*_{\exp t\xi} X_{\mathbf{d}I^\xi\cdot\mathrm{hor}_0 Z}\, dt = -\int_0^1 X_{\Phi^*_{\exp t\xi}((\mathbf{d}I^\xi\cdot\mathrm{hor}_0 Z)}dt = X_{f_Z(g)}$$

where

$$f_Z(g) = -\int_0^1 \Phi^*_{\exp t\xi}(\mathbf{d}I^\xi\cdot\mathrm{hor}_0 Z)dt : E \to \mathbb{R},$$

a smooth function on E. The function f_Z depends on $\xi \in \mathfrak{g}$. Since there can be more than one way to write $g = \exp\xi$, let $g = \exp\eta$ for some other $\eta \in \mathfrak{g}$ and denote by $h_Z(g)$ the corresponding smooth function on E. Therefore

$$\Phi_g^*(\mathrm{hor}\, Z_0) - \mathrm{hor}\, Z_0 = X_{f_Z(g)} = X_{h_Z(g)}$$

i.e., $f_Z(g) - h_Z(g)$ is fiberwise a Casimir and in particular is G-invariant. Therefore

$$Y = \mathrm{hor}\, Z - \mathrm{hor}_0 Z = \langle\mathrm{hor}_0 Z\rangle - \mathrm{hor}_0 Z = \frac{1}{|G|}\int_G \Phi_g^*(\mathrm{hor}\, Z_0)dg - \mathrm{hor}_0 Z$$
$$= \frac{1}{|G|}\int_G \left(\Phi_g^*(\mathrm{hor}_0 Z) - \mathrm{hor}_0 Z\right)dg = \frac{1}{|G|}\int_G X_{f_Z(g)}dg = X_{K\cdot Z}$$

where $K \cdot Z = \frac{1}{|G|} \int_G f_Z(g)\, dg$. Since any two possible $f_Z(g)$'s differ by a Casimir, the difference of their averages is again a Casimir, so any two possible $K \cdot Z$'s differ by a Casimir, which makes $X_{K \cdot Z}$ well defined.

i For each $Z \in \mathfrak{X}(M)$, $\mathrm{hor}_0 Z$ and $X_{K \cdot Z}$ are both Poisson derivations. Thus, hor Z, which is their sum by **iii**, is a Poisson derivation.

iv This is a restatement of **iii**.

v Since G acts canonically on the fiber and using the definition of the average,

$$0 = \langle\langle \mathrm{hor}_0 Z\rangle - \mathrm{hor}_0 Z\rangle = \langle \mathrm{hor}\, Z - \mathrm{hor}_0 Z\rangle = \langle X_{K \cdot Z}\rangle = X_{\langle K \cdot Z\rangle}.$$

Therefore $\langle K \cdot Z\rangle$ is fiberwise a Casimir.

vi We prove the result for $\lambda = f$, a function on E, the proof for general k-forms being similar. Letting $Z \in \mathfrak{X}(M)$ and using that hor Z is G-invariant, we get

$$\mathbf{D}\langle f\rangle \cdot \mathrm{hor}\, Z = \mathbf{d}\left(\frac{1}{|G|} \int_G \Phi_g^* f\, dg\right) \cdot \mathrm{hor}\, Z = \frac{1}{|G|} \int_G \Phi_g^*(\mathbf{d}f \cdot \mathrm{hor}\, Z) dg = \langle \mathbf{d}f \cdot \mathrm{hor}\, Z\rangle$$

$$= \frac{1}{|G|} \int_G \Phi_g^*(\mathbf{D}f \cdot \mathrm{hor}\, Z) dg = \left(\frac{1}{|G|} \int_G \Phi_g^*(\mathbf{D}f) dg\right) \mathrm{hor}\, Z = \langle \mathbf{D}f\rangle \cdot \mathrm{hor}\, Z\ .$$

We have proved that $\mathbf{D}\langle f\rangle \cdot \mathrm{hor}\, Z = \langle \mathbf{D}f\rangle \cdot \mathrm{hor}\, Z = \langle\, \mathbf{d}f \cdot \mathrm{hor}\, Z\rangle$. Since $\Phi_g^* \xi_E = \xi_E$ for any $\xi \in \mathfrak{g}$ and $g \in G$, we have

$$\langle \mathbf{D}f\rangle \cdot \xi_E = \frac{1}{|G|} \int_G \Phi_g^*(\mathbf{D}f) \cdot \xi_E\, dg = \frac{1}{|G|} \int_G \Phi_g^*(\mathbf{D}f \cdot \xi_E) dg = 0$$

since $\mathbf{D}f \cdot \xi_E = 0$. Therefore both $\mathbf{D}\langle f\rangle$ and $\langle \mathbf{D}f\rangle$ vanish for vertical vectors, proving the equality $\mathbf{D}\langle f\rangle = \langle \mathbf{D}f\rangle$. Finally, since $\langle \mathbf{D}_0 f\rangle \circ \mathbf{P}_{\mathrm{hor}}$ vanishes on vertical vector fields, the string of equalities in **vi** is proved, if we show that

$$\langle \mathbf{d}f\rangle \cdot \mathrm{hor}\, Z = \langle \mathbf{D}f \cdot \mathrm{hor}\, Z\rangle = \langle \mathbf{d}f \cdot \mathrm{hor}\, Z\rangle$$

for all $Z \in \mathfrak{X}(M)$. But this has been shown above.

vii Since hor Z is G-invariant, $[\mathrm{hor}\, Z, \xi_E] = 0$ for all $\xi \in \mathfrak{g}$. But $\xi_E = X_{I^\xi}$, so by lemma **6.4** we have $X_{\mathbf{D}I^\xi \cdot \mathrm{hor}\, Z} = 0$. Thus $\mathbf{D}I^\xi \cdot \mathrm{hor}\, Z$ is a fiberwise Casimir. Since G acts in a Hamiltonian fashion, it leaves fiberwise Casimirs invariant. Consequently by **iii**,

$$\begin{aligned} \mathbf{D}I^{\xi}\cdot\text{hor } Z &= \langle \mathbf{D}I^{\xi}\cdot\text{hor } Z\rangle = \langle \mathbf{d}I^{\xi}\cdot(\text{hor}_0 Z + X_{K\cdot Z})\rangle \\ &= \langle \mathbf{D}_0 I^{\xi}\cdot\text{hor}_0 Z\rangle + \langle \mathbf{d}I^{\xi}\cdot X_{K\cdot Z}\rangle = \langle \mathbf{D}_0 I\,\rangle\cdot\text{hor } Z - \langle \xi_E[K\cdot Z]\rangle \\ &= \langle \mathbf{D}_0 I^{\xi}\rangle\cdot\text{hor } Z - \xi_E\,[\langle K\cdot Z\rangle] = \langle \mathbf{D}_0 I^{\xi}\rangle\cdot\text{hor } Z. \end{aligned}$$

Thus $\mathbf{D}I^{\xi}$ and $\langle \mathbf{D}_0 I^{\xi}\rangle$ coincide on horizontal vectors relative to the HB connection. Since both vanish on vertical vectors, we get $\mathbf{D}I^{\xi} = \langle \mathbf{D}_0 I^{\xi}\rangle$. ■

§7 *Existence, Uniqueness, and Curvature of the Hannay-Berry Connection*

In this section we show that the HB connection is the unique Ehresmann connection on $\pi : E \to M$ satisfying three conditions. We also calculate its curvature. As in §5, we assume E has a family of Hamiltonian G-actions with a parametrized momentum map $\mathbf{I} : E \to \mathfrak{g}^*$ and let hor_0 denote the horizontal lift of a Poisson Ehresmann connection. The proofs in this section are almost verbatim from Montgomery [1988].

§7A *Characterization of the HB Connection*

7.1 Theorem *Let* hor *be the horizontal lift of an Ehresmann connection on* $\pi : E \to M$ *satisfying*

a $\mathbf{DI} = 0$, *where* $\mathbf{D}$ *is the covariant differentiation given by* hor ; *this says that parallel translation relative to* hor *preserves the level sets of* $\mathbf{I}$;

b $\mathrm{hor}\, Z = \mathrm{hor}_0 Z + X_{K\cdot Z}$ *for a smooth function* $K\cdot Z : E \to \mathbb{R}$, *where* $Z \mapsto K\cdot Z$ *is linear;*

c $\langle K\cdot v\rangle$ *is a Casimir function on* $\pi^{-1}(m)$ *where* $v \in T_m M$; *replacing* $K\cdot v$ *by* $K\cdot v - \langle K\cdot v\rangle$, *we can assume that this Casimir is zero.*

Then

i *such a connection is unique;*

ii *such a connection exists if and only if the 'adiabatic condition'*

$$\langle \mathbf{D}_0 \mathbf{I}\rangle = 0 \tag{A}$$

holds, in which case it is the Hannay-Berry connection, and its covariant differentiation on functions is given by

$$\mathbf{D}f\cdot u = \mathbf{D}_0 f\cdot u + \{f,\ K\cdot T\pi(u)\} \tag{1}$$

for any $u \in TE$.

Proof **i** Suppose there are two horizontal lifts hor_1, hor_2 satisfying **a**, **b** and **c**. By **b**, $\mathrm{hor}_1 v = \mathrm{hor}_0 v + X_{K_1\cdot v}$, $\mathrm{hor}_2 v = \mathrm{hor}_0 v + X_{K_2\cdot v}$. Thus $\mathrm{hor}_1 v - \mathrm{hor}_2 v = X_{K_1\cdot v - K_2\cdot v}$ so that by **a**, $\mathbf{dI}\cdot X_{K_1\cdot v - K_2\cdot v} = 0$. Therefore, for each $\xi \in \mathfrak{g}$,

$$0 = \mathbf{d}I^{\xi}\cdot X_{K_1\cdot v - K_2\cdot v} = -\xi_E[K_1\cdot v - K_2\cdot v]$$

and thus $K_1\cdot v - K_2\cdot v$ is G-invariant by connectedness of G. In particular $K_1\cdot v - K_2\cdot v = \langle K_1\cdot v - K_2\cdot v\rangle = \langle K_1\cdot v\rangle - \langle K_2\cdot v\rangle = 0$ by **c**. Therefore $K_1\cdot v = K_2\cdot v$ for any $v \in TM$ and the two connections are equal.

ii For any $\xi \in \mathfrak{g}$ and $Z \in \mathfrak{X}(M)$, **b** gives

$$0 = \mathbf{D}I^{\xi}\cdot \text{hor } Z = \mathbf{d}I^{\xi}\cdot \text{hor}_0 Z + \mathbf{d}I^{\xi}\cdot X_{K\cdot Z} = \mathbf{D}_0 I^{\xi}\cdot \text{hor } Z - \xi_E[K\cdot Z]$$

so that using **c** and proceeding as in **6.3vi**, we conclude

$$0 = \langle \mathbf{D}_0 I^{\xi}\rangle\cdot \text{hor } Z - \xi_E[\langle K\cdot Z\rangle] = \langle \mathbf{D}_0 I^{\xi}\rangle\cdot \text{hor } Z \;.$$

This shows that (A) is a necessary condition for existence.

Now assume, conversely, that $\langle \mathbf{D}_0 \mathbf{I}\rangle = 0$. The HB connection satisfies **a**, **b**, and **c** by proposition **6.3**.

Formula (1) is an immediate consequence of **6.3iii**.

Remark We call condition (A) the 'adiabatic condition' because in the context of a family of completely integrable systems this equality is the content of the classical adiabatic theorem. See also §**9**.

7.2 Corollary *If* G *is semisimple and* **I** *is equivariant the adiabatic condition* (A) *holds. Consequently, the* HB *connection satisfies property* **a**, **b**, *and* **c** *of Theorem* **7.1**.

Proof By properties **vi** and **vii** of Proposition **6.3**, $\mathbf{DI} = \langle \mathbf{DI}\rangle = \mathbf{D}\langle \mathbf{I}\rangle$. By equivariance $\langle \mathbf{I}\rangle^{\xi} = I^{\langle\xi\rangle}$ for every $\xi \in \mathfrak{g}$. Now $\langle\xi\rangle$ is an Ad-invariant vector. Since G is semi-simple this means $\langle\xi\rangle = 0$, consequently $\langle \mathbf{I}\rangle = 0$, and thus $\mathbf{DI} = 0$. (Recall that for semisimple Lie groups the adjoint representation is irreducible.) ■

7.3 Corollary *Suppose, for each* $v \in T_m M$ *there is a function* $\tilde{K}\cdot v : \pi^{-1}(m) \to \mathbb{R}$ *satisfying*

$$\mathbf{d}I^{\xi}\cdot \text{hor}_0 v + \{I^{\xi}, \tilde{K}\cdot v\} = 0 \tag{2}$$

for all $\xi \in \mathfrak{g}$. *Set* $K\cdot v = \tilde{K}\cdot v - \langle \tilde{K}\cdot v\rangle$ *and* $\text{hor } v = \text{hor}_0 v + X_{K\cdot v}$. *Assume* $\langle \mathbf{D}_0 \mathbf{I}\rangle = 0$. *Then* hor *defines the HB connection.*

Proof hor so defined clearly determines an Ehresmann connection satisfying properties **b** and **c** of **7.1**. To prove property **a**, fix $\xi \in \mathfrak{g}$ and compute for $Z \in \mathfrak{X}(M)$

$$\begin{aligned} \mathbf{D}I^{\xi}\cdot\mathrm{hor}\, Z &= \mathbf{d}I^{\xi}\cdot(\mathrm{hor}_0 Z + X_{K\cdot Z}) = \{I^{\xi}, K\cdot Z\} + \mathbf{d}I^{\xi}\cdot\mathrm{hor}_0 Z \\ &= \{I^{\xi}, \tilde{K}\cdot Z\} + \mathbf{d}I^{\xi}\cdot\mathrm{hor}_0 Z - \{I^{\xi}, \langle\tilde{K}\cdot Z\rangle\} = \xi_E[\langle\tilde{K}\cdot Z\rangle] = 0 \end{aligned}$$

by G-invariance of $\langle\tilde{K}\cdot Z\rangle$. By uniqueness, hor defines the HB connection. ■

§7B *Curvature of the Hannay-Berry Connection*

7.4 Corollary *In addition to the hypothesis of* **7.1**, *assume that the curvature* Ω_0 *of* hor_0 *is Hamiltonian i.e.,* $\bar{\Omega}_0(Z_1, Z_2) = X_{K\cdot(Z_1, Z_2)}$ *for some* $K\cdot(Z_1, Z_2) : E \to \mathbb{R}$. *Then the curvature* $\bar{\Omega} : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}_{\mathrm{vert}}(E; M)$ *of the* HB *connection is also Hamiltonian, and given by*

$$\bar{\Omega}(Z_1, Z_2) = X_{-\langle\{K\cdot Z_1,\, K\cdot Z_2\}\rangle + \langle K\cdot(Z_1, Z_2)\rangle} . \tag{3}$$

Proof By definition of curvature,

$$\bar{\Omega}(Z_1, Z_2) = \mathrm{hor}[Z_1, Z_2] - [\mathrm{hor}\, Z_1, \mathrm{hor}\, Z_2] .$$

Write $\mathrm{hor}\, Z_i = \mathrm{hor}_0 Z_i + X_{K\cdot Z_i}$, $i = 1, 2$, so that by Lemma **6.4**,

$$\begin{aligned} [\mathrm{hor}\, Z_1, \mathrm{hor}\, Z_2] &= [\mathrm{hor}_0 Z_1 + X_{K\cdot Z_1}, \mathrm{hor}_0 Z_2 + X_{K\cdot Z_2}] \\ &= [\mathrm{hor}_0 Z_1, \mathrm{hor}_0 Z_2] - X_{\mathbf{d}(K\cdot Z_1)\cdot \mathrm{hor}_0 Z_2} + X_{\mathbf{d}(K\cdot Z_2)\cdot \mathrm{hor}_0 Z_1} - X_{\{K\cdot Z_1,\, K\cdot Z_2\}} \\ &= [\mathrm{hor}_0 Z_1, \mathrm{hor}_0 Z_2] + X_{F_{12}} \end{aligned} \tag{4}$$

where

$$F_{12} = -\{K\cdot Z_1, K\cdot Z_2\} + \mathbf{d}(K\cdot Z_2)\cdot\mathrm{hor}_0 Z_1 - \mathbf{d}(K\cdot Z_1)\cdot\mathrm{hor}_0 Z_2 . \tag{5}$$

Let us show that $F_{12} - K\cdot(Z_1, Z_2)$ satisfies

$$\mathbf{d}I^{\xi}\cdot\mathrm{hor}_0[Z_1, Z_2] + \{I^{\xi}, F_{12} - K\cdot(Z_1, Z_2)\} = 0 \tag{6}$$

for all $\xi \in \mathfrak{g}$. Indeed

$$\mathbf{d}I^{\xi}\cdot\mathrm{hor}_0[Z_1, Z_2] + \{I^{\xi}, F_{12} - K\cdot(Z_1, Z_2)\} = \mathbf{d}I^{\xi}\cdot(\mathrm{hor}_0[Z_1, Z_2] - [\mathrm{hor}_0 Z_1, \mathrm{hor}_0 Z_2])$$

$$
\begin{aligned}
&+ \mathbf{d}I^{\xi}\cdot[\mathrm{hor}_0 Z_1, \mathrm{hor}_0 Z_2] + \{I^{\xi}, F_{12}\} - \{I^{\xi}, K\cdot(Z_1, Z_2)\} \\
&= (\mathrm{hor}_0 Z_1)[(\mathrm{hor}\, Z_2 - X_{K\cdot Z_2})[I^{\xi}]] - (\mathrm{hor}_0 Z_2)[(\mathrm{hor}\, Z_1 - X_{K\cdot Z_1})[I^{\xi}]] + \{I^{\xi}, F_{12}\} \\
&= -(\mathrm{hor}_0 Z_1)[\{I^{\xi}, K\cdot Z_2\}] + (\mathrm{hor}_0 Z_2)[\{I^{\xi}, K\cdot Z_1\}] + \{I^{\xi}, F_{12}\} \\
&= -\{(\mathrm{hor}\, Z_1)[I^{\xi}] - X_{K\cdot Z_1}[I^{\xi}], K\cdot Z_2\} - \{I^{\xi}, (\mathrm{hor}_0 Z_1)[K\cdot Z_2]\} + \\
&\qquad \{(\mathrm{hor}\, Z_2)[I^{\xi}] - X_{K\cdot Z_2}[I^{\xi}], K\cdot Z_1\} + \{I^{\xi}, (\mathrm{hor}_0 Z_2)[K\cdot Z_1]\} + \{I^{\xi}, F_{12}\} \\
&= \{\{I^{\xi}, K\cdot Z_1\}, K\cdot Z_2\} + \{\{K\cdot Z_2, I^{\xi}\}, K\cdot Z_1\} - \{I^{\xi}, (\mathrm{hor}_0 Z_1)[K\cdot Z_2] \\
&\qquad - (\mathrm{hor}_0 Z_2)[K\cdot Z_1] - F_{12}\} \\
&= -\{\{K\cdot Z_1, K\cdot Z_2\}, I^{\xi}\} - \{I^{\xi}, \{K\cdot Z_1, K\cdot Z_2\}\} = 0
\end{aligned}
$$

so (6) holds. Therefore, by **7.3**, $F_{12} - K\cdot(Z_1, Z_2) - \langle F_{12} - K\cdot(Z_1, Z_2)\rangle$ is the Hamiltonian function generating the HB connection for $[Z_1, Z_2]$, *i.e.*,

$$\mathrm{hor}[Z_1, Z_2] = X_{F_{12} - \langle F_{12}\rangle} - X_{K(Z_1, Z_2) - \langle K\cdot(Z_1, Z_2)\rangle} + \mathrm{hor}_0[Z_1, Z_2] \ .$$

By (4),

$$
\begin{aligned}
\bar{\Omega}(Z_1, Z_2) &= \mathrm{hor}[Z_1, Z_2] - [\mathrm{hor}\, Z_1, \mathrm{hor}\, Z_2] \\
&= X_{F_{12} - \langle F_{12}\rangle} - X_{K\cdot(Z_1, Z_2) - \langle K\cdot(Z_1, Z_2)\rangle} + \mathrm{hor}_0[Z_1, Z_2] - [\mathrm{hor}_0 Z_1, \mathrm{hor}_0 Z_2] - X_{F_{12}} \\
&= X_{-\langle F_{12}\rangle} - X_{K\cdot(Z_1, Z_2)} + X_{\langle K\cdot(Z_1, Z_2)\rangle} + \bar{\Omega}_0(Z_1, Z_2) \\
&= X_{-\langle F_{12}\rangle + \langle K\cdot(Z_1, Z_2)\rangle} \ .
\end{aligned}
$$

By **6.3vi** and G-invariance of $\mathrm{hor}\, Z_1$, the condition $\langle K\cdot Z_2\rangle = 0$ implies

$$
\begin{aligned}
0 &= \mathbf{D}\langle K\cdot Z_2\rangle\cdot \mathrm{hor}\, Z_1 = \langle \mathbf{D}(K\cdot Z_2)\rangle\cdot \mathrm{hor}\, Z_1 = \mathbf{d}\langle (K\cdot Z_2)\cdot \mathrm{hor}\, Z_1\rangle \\
&= \langle \mathbf{d}(K\cdot Z_2)\cdot \mathrm{hor}_0 Z_1 + \mathbf{d}(K\cdot Z_2)\cdot X_{K\cdot Z_1}\rangle \ ;
\end{aligned}
$$

i.e.,

$$\langle \mathbf{d}(K\cdot Z_2)\cdot \mathrm{hor}_0 Z_1\rangle = \langle \{K\cdot Z_1, K\cdot Z_2\}\rangle$$

and similarly

$$-\langle \mathbf{d}(K\cdot Z_1)\cdot \mathrm{hor}_0 Z_2\rangle = \langle \{K\cdot Z_1, K\cdot Z_2\}\rangle \ .$$

Therefore

$$
\begin{aligned}
\langle F_{12}\rangle &= -\langle \{K\cdot Z_1, K\cdot Z_2\}\rangle + \langle \mathbf{d}(K\cdot Z_2)\cdot \mathrm{hor}_0 Z_1\rangle - \langle \mathbf{d}(K\cdot Z_1)\cdot \mathrm{hor}_0 Z_2\rangle \\
&= -\langle \{K\cdot Z_1, K\cdot Z_2\}\rangle + \langle \{K\cdot Z_1, K\cdot Z_2\}\rangle + \langle \{K\cdot Z_1, K\cdot Z_2\}\rangle \\
&= \langle \{K\cdot Z_1, K\cdot Z_2\}\rangle \ . \qquad \blacksquare
\end{aligned}
$$

§7C *Hamiltonian One-forms*

From Theorem **7.1**, it follows that the HB connection is uniquely determined by the one-form on M given by $Z \in \mathfrak{X}(M) \mapsto K\cdot Z \in$ functions on E modulo fiber-Casimirs. This function is in general not G-invariant since hor_0 isn't. We shall call this one-form the ***Hamiltonian one-form of the HB connection.***

7.5 Theorem *Let* $\pi : E \to M$ *be a Poisson fiber bundle having a family of Hamiltonian* G-*actions with parametrized momentum map* $\mathbf{I} : E \to \mathfrak{g}^*$ *and Poisson-Ehresmann connection* hor_0. *Assume that the curvature* Ω_0 of hor_0 *is Hamiltonian, i.e.,* $\bar{\Omega}(Z_1, Z_2) = X_{K\cdot(Z_1,Z_2)}$ *for all* Z_1, $Z_2 \in \mathfrak{X}(M)$. *Let* $L(\mathfrak{g}, \mathcal{C}(E))$ *be the vector space of linear maps of the Lie algebra* $\mathfrak{g}$ *of* G *to the space* $\mathcal{C}(E)$ *of fiberwise Casimir functions. Then* $\langle \mathbf{D}_0\mathbf{I}\rangle$ *is the pull-back of a closed* $L(\mathfrak{g}, \mathcal{C}(E))$-*valued one-form on the base* M.

Proof We show that $\langle \mathbf{D}_0 I^\xi\rangle$ is a closed form for any $\xi \in \mathfrak{g}$. First, we prove that $\langle \mathbf{D}_0 I^\xi\rangle$ vanishes on vertical vectors $u \in \ker T_p\pi$. Indeed,

$$\langle \mathbf{D}_0 I^\xi\rangle\,(p)\cdot u \;=\; \frac{1}{|G|}\int_G \Phi_g^*\langle \mathbf{D}_0 I^\xi\rangle(p)\cdot u\, dg \;=\; \frac{1}{|G|}\int_G \mathbf{d}I^\xi(\Phi_g(p))\cdot \mathbf{P}_{\mathrm{hor}_0}(T_p\Phi_g(u))dg\,, \qquad (7)$$

where $\mathbf{P}_{\mathrm{hor}_0}$ is the horizontal projection defined by hor_0. Since u is vertical, so is $T_p\Phi_g(u)$ because $\pi\circ\Phi_g = \pi$. Therefore $\mathbf{P}_{\mathrm{hor}_0}(T_p\Phi_g(u)) = 0$. Thus the integrand in (7) is zero and hence $\langle \mathbf{D}_0 I^\xi\rangle(p)\cdot u = 0$.

Second, we show that $\mathbf{d}\langle \mathbf{D}_0\mathbf{I}\rangle$ vanishes on horizontal lifts. By **6.3vii**, for $Z_1, Z_2 \in \mathfrak{X}(M)$ we have

$$\begin{aligned}
\mathbf{d}\langle \mathbf{D}_0 I^\xi\rangle(\mathrm{hor}\, Z_1, \mathrm{hor}\, Z_2) &= (\mathrm{hor}\, Z_1)[\langle \mathbf{D}_0 I^\xi\rangle\cdot \mathrm{hor}\, Z_2] - (\mathrm{hor}\, Z_2)[\langle \mathbf{D}_0 I^\xi\rangle\cdot \mathrm{hor}\, Z_1] \\
&\qquad - \langle \mathbf{D}_0 I^\xi\rangle[\mathrm{hor}\, Z_1, \mathrm{hor}\, Z_2] \\
&= (\mathrm{hor}\, Z_1)[\mathbf{D}I^\xi\cdot \mathrm{hor}\, Z_1] - (\mathrm{hor}\, Z_2)[\mathbf{D}I^\xi\cdot \mathrm{hor}\, Z_1] \\
&= (\mathrm{hor}\, Z_1)[\mathbf{d}I^\xi\cdot \mathrm{hor}\, Z_2] - (\mathrm{hor}\, Z_2)[\mathbf{d}I^\xi\cdot \mathrm{hor}\, Z_1] \\
&= \mathbf{d}(\mathbf{d}I^\xi)(\mathrm{hor}\, Z_1, \mathrm{hor}\, Z_2) - \mathbf{d}I^\xi\cdot\Omega(\mathrm{hor}\, Z_1, \mathrm{hor}\, Z_2) \\
&= \{I^\xi, \langle\{K\cdot Z_1, K\cdot Z_2\}\rangle - \langle K\cdot Z_1, Z_2\rangle\rangle\} = 0
\end{aligned}$$

by G-invariance of the $\langle\{K\cdot Z_1, K\cdot Z_2\}\rangle - \langle K\cdot(Z_1, Z_2)\rangle$ and conservation of momentum maps.

By step one, $\mathbf{d}\langle \mathbf{D}_0\, I^\xi\rangle$ vanishes on any pair of vertical vector fields. Indeed, if V_1, V_2 are vertical vector fields, then so is $[V_1, V_2]$ and thus

$$\mathbf{d}\langle \mathbf{D}_0\, I^\xi\rangle(V_1, V_2) = V_1[\langle \mathbf{D}_0 I^\xi\rangle \cdot V_2] - V_2[\langle \mathbf{D}_0 I^\xi\rangle \cdot V_1] - \langle \mathbf{D}_0 I^\xi\rangle \cdot [V_1, V_2] = 0 .$$

Finally, let us show that $\mathbf{d}\langle \mathbf{D}_0 I^\xi\rangle$ vanishes on a pair formed by a horizontal and a vertical vector field. Indeed, for $Z \in \mathfrak{X}(M)$ and $V \in \mathfrak{X}_{\mathrm{vert}}(E; M)$,

$$\begin{aligned}\mathbf{d}\langle \mathbf{D}_0 I^\xi\rangle \cdot (\mathrm{hor}\, Z, V) &= (\mathrm{hor}\, Z)[\langle \mathbf{D}_0 I^\xi\rangle \cdot V] - V[\langle \mathbf{D}_0 I^\xi\rangle \cdot \mathrm{hor}\, Z] - \langle \mathbf{D}_0 I^\xi\rangle \cdot [\mathrm{hor}\, Z, V] \\ &= -V[\mathbf{D} I^\xi \cdot \mathrm{hor}\, Z] = -V[\mathbf{d} I^\xi \cdot \mathrm{hor}\, Z] .\end{aligned}$$

By **6.4**, $X_{\mathbf{d}I^\xi \cdot \mathrm{hor}\, Z} = [\mathrm{hor}\, Z, \xi_E] = 0$ since hor Z is G-invariant. Therefore $\mathbf{d}I^\xi \cdot \mathrm{hor}\, Z$ is a Casimir function on every fiber, i.e., as a function of E with values in $\mathcal{C}(E)$, it is constant. Consequently, as a $\mathcal{C}(E)$-valued one-form, $\mathbf{d}\langle \mathbf{D}_0 I^\xi\rangle \cdot (\mathrm{hor}\, Z, V) = 0$.

These four steps prove that:

i $\mathbf{d}\langle \mathbf{D}_0 I^\xi\rangle = 0$

ii $\langle \mathbf{D}_0 I^\xi\rangle$ is a one-form on E vanishing on vectors that are tangent to the fibers and $\langle \mathbf{D}_0 I^\xi\rangle \cdot \mathrm{hor}\, Z = \mathbf{d}I^\xi \cdot \mathrm{hor}\, Z$ is constant as a $\mathcal{C}(E)$-valued function on E , *i.e.*, $\langle \mathbf{D}_0 I^\xi\rangle$ is the pull-back of a one-form α_ξ on M with values in $\mathcal{C}(E)$.

Since π is a surjective submersion, **ii** implies that $\mathbf{d}\alpha_\xi = 0$. ■

7.6 Corollary *In a simply connected neighborhood* U *of any point* $m_0 \in M$ *there is a* $L(\mathfrak{g}, \mathcal{C}(E))$*-valued function* f *on* U *such that* $\mathbf{I}' = (\mathbf{I} + f \circ \pi) \mid \pi^{-1}(U)$ *satisfies* $\langle \mathbf{D}_0 \mathbf{I}'\rangle = 0$.

Proof Let α denote the $L(\mathfrak{g}, \mathcal{C}(E))$-valued one-form on M whose pull-back is $\langle \mathbf{D}_0 \mathbf{I}\rangle$ and define

$$f(m) = -\int_{m_0}^{m} \alpha$$

where the integral is taken over any path connecting m_0 to an arbitrary point $m \in U$. By simple connectivity of U , f is well defined, and so $\mathbf{d}f = -\alpha$ on U . Then $\mathbf{D}_0(f \circ \pi) = \mathbf{d}\pi^* f = \pi^* \mathbf{d}f = -\alpha = -\langle \mathbf{D}_0 I^\xi\rangle$. ■

In particular, if M is simply connected, the parametrized momentum map **I** can always be chosen such that $\langle \mathbf{D}_0\, \mathbf{I}\rangle = 0$. Thus, for Poisson fiber bundles with simply connected base, the HB connection preserves the level sets of a carefully chosen parametrized momentum map of the Hamiltonian G-action.

§8 *The Hannay-Berry Connection in the Presence of Additional Symmetry*

In this section we use symmetry to simplify the computation of the HB connection. Letting $\pi : E \to M$, $\mathbf{I} : E \to \mathfrak{g}^*$, and hor_0 be as in the previous sections, assume there is another Lie group H acting on the *left* on E leaving $\mathbf{I}$ invariant, *i.e.*, $\mathbf{I}(h \cdot p) = \mathbf{I}(p)$ for all $h \in H$ and $p \in E$. If $\eta \in \mathfrak{h}$, the Lie algebra of H, the vertical projection $\eta_E - P_{hor_0}\eta_E$ is a vector field tangent to the fibers of E; P_{hor_0} denotes, as usual, the horizontal projection relative to hor_0. We say that the H-action is ***Hamiltonian*** if there is a function $\mathbf{J} : E \to \mathfrak{h}^*$, called the ***momentum mapping*** such that

$$\eta_E - \mathbf{P}_{hor_0}\eta_E = X_{J^\eta} \tag{1}$$

where $J^\eta(p) = \mathbf{J}(\eta)(p)$. In particular, the G-action is Hamiltonian.

In the examples, H acts by bundle transformations on E. This means that the action covers an action on M. In the examples, the group G is isomorphic to the isotropy subgroup of a point m of M *i.e.*, $G = H_m$.

In $I^\xi(h \cdot p) = I^\xi(p)$, let $\xi \in \mathfrak{g}$, $h = \exp t\eta$, and take the time derivative at $t = 0$ to get

$$\begin{aligned} 0 &= \mathbf{d}I^\xi(p) \cdot \eta_E(p) = \mathbf{d}I^\xi(p) \cdot X_{J^\eta}(p) + \mathbf{d}I^\xi(p) \cdot \mathbf{P}_{hor_0}\eta_E(p) \\ &= \mathbf{d}I^\xi(p) \cdot \mathbf{P}_{hor_0}\eta_E(p) + \{I^\xi, J^\eta\}(p) \ . \end{aligned}$$

By **7.3**,

$$\boxed{K \cdot v_\eta = J^\eta - \langle J^\eta \rangle \ ,} \tag{2}$$

where $v_\eta = T_p\pi(\eta_E(p)) \in T_{\pi(p)}M$ has horizontal lift relative to hor_0 equal to $P_{hor_0}\eta_E(p)$. This will determine the Hannay-Berry connection if $T_{\pi(p)}M = T_p\pi(\{\eta_E(p) \mid \eta \in \mathfrak{g}\})$. We have proved:

8.1 Proposition *Let* $\pi : E \to M$ *be a Poisson fiber bundle with a family of Hamiltonian* G-*actions with parametrized momentum map* $\mathbf{I} : E \to \mathfrak{g}^*$ *and with a Poisson-Ehresmann connection* hor_0. *Assume* $\langle \mathbf{D}_0\mathbf{I} \rangle = 0$. *Let another Lie group* H *act on* E *in a Hamiltonian fashion with momentum map* $\mathbf{J} : E \to \mathfrak{h}^*$. *Assume that* H *leaves* $\mathbf{I}$ *invariant and that the tangent spaces to the* H-*orbits on* E *project by* $T\pi$ *onto the corresponding tangent space of* M. *Then the* HB *connection satisfies* (2).

As in Montgomery [1988], formula (2) is useful for computing the HB connection for many interesting systems. We now set the stage to illustrate the usefulness of this idea; the results will be summarized in Proposition **8.2**. Assume that $E = P \times \mathcal{O}_\zeta$, where $\mathcal{O}_\zeta$ is an adjoint orbit of a compact Lie group G and ζ is a regular element, so the adjoint isotropy subgroup G_ζ is a maximal torus in G. Assume P is a Poisson manifold and endow E with the trivial connection whose horizontal subbundle is $0 \times \mathcal{O}_\zeta \subset T(P \times \mathcal{O}_\zeta)$. This connection is a Poisson-Ehresmann connection on $P \times \mathcal{O}_\zeta$. Let $\langle \, , \rangle$ be a bi-invariant metric on $\mathfrak{g}$ and use it to identify $\mathfrak{g}^*$ and $\mathfrak{g}$. Assume that the G-action has an *equivariant* momentum map $\mathbf{J} : P \to \mathfrak{g}^*$ and define $\mathbf{I} : P \times \mathcal{O}_\zeta \to \mathfrak{g}_\zeta$ by

$$\langle \mathbf{I}(p, \eta), \lambda \rangle = \langle \mathbf{J}(p), i_\eta(\lambda) \rangle, \tag{3}$$

where $\lambda \in \mathfrak{g}_\zeta$ and $i_\eta : \mathfrak{g}_\zeta \to \mathfrak{g}$ is given by $i_\eta(\lambda) = \mathrm{Ad}_h \lambda$, where $h \in G_\zeta$ is chosen to satisfy $\eta = \mathrm{Ad}_h \zeta$. Such an $h \in G_\zeta$ always exists since $\eta \in \mathcal{O}_\zeta$ and the definition of i_η is independent of h since G_ζ is abelian. Since G_ζ is a torus, the functions $p \mapsto \langle \mathbf{I}(p, \eta), \xi \rangle$, $\xi \in \mathfrak{g}_\zeta$ generate a torus action on P, depending smoothly on η. This action, together with the trivial action on $\mathcal{O}_\zeta$ defines an action of G_ζ on $P \times \mathcal{O}_\zeta$ with parametrized momentum map $\mathbf{I}$. Since $i_{g \cdot \eta}(\lambda) = g \cdot i_\eta(\lambda)$ for any $\lambda \in \mathfrak{g}_\zeta$ and $g \in G$, the bi-invariance of the metric $\langle \, , \rangle$ and the *equivariance* of $\mathbf{J}$ imply that $\mathbf{I}$ *is invariant under the* G-*action on* $P \times \mathcal{O}_\zeta$; the momentum map of this action is $(p, \eta) \mapsto \mathbf{J}(p) - \eta$.

We can apply Proposition **8.1** to this situation, with H there replaced by G, and G by G_ζ. We begin by verifying the adiabatic condition $\langle \mathbf{D}_0 \mathbf{I} \rangle = 0$. Since any vector tangent to $\mathcal{O}_\zeta$ at h is of the form $[\xi, \eta]$ for some $\xi \in \mathfrak{g}$, then for $\lambda \in \mathfrak{g}_\zeta$ and $v_p \in T_pP$, we get

$$\langle \mathbf{D}_0 \mathbf{I}(p, \eta) \cdot (v_p, [\xi, \eta]), \lambda \rangle = \langle \mathbf{dI}(p, \eta) \cdot (0, [\xi, \eta]), \lambda \rangle = \langle \mathbf{J}(p), [\xi, i_\eta(\lambda)] \rangle = \langle \mathrm{Ad}_{h^{-1}} [\mathbf{J}(p), \xi], \lambda \rangle,$$

whence

$$\mathbf{D}_0 \mathbf{I}(p, \eta) \cdot (v_p, [\xi, \eta]) = \mathrm{Ad}_{h^{-1}} \mathbb{P}_\zeta [\mathbf{J}(p), \xi] \tag{4}$$

where $\mathrm{Ad}_h \zeta = \eta$ and $\mathbb{P}_\zeta : \mathfrak{g} \to \mathfrak{g}_\zeta$ is the orthogonal projection relative to the metric $\langle , \rangle$. Taking the G_ζ-average of (4), we get

$$\langle \mathbf{D}_0 \mathbf{I} \rangle (p, \eta) \cdot (v_p, [\xi, \eta]) = \langle \mathbb{P}_\zeta [\mathbf{J}(p), \xi] \rangle . \tag{5}$$

Writing $\xi = \xi_1 + \xi_2$, where $\xi_1 \in \mathfrak{g}_\zeta$ and $\xi_2 \in \mathfrak{g}_\zeta^\perp$, we conclude that $[\mathbf{J}(p), \xi] = [\mathbf{J}(p), \xi_2]$, since $\mathfrak{g}_\zeta$ is abelian and both $\mathbf{J}(p)$ and ξ_1 belong to $\mathfrak{g}_\zeta$. Therefore, $\mathbb{P}_\zeta [\mathbf{J}(p), \xi_2] = 0$ since $[\mathfrak{g}_\zeta, \mathfrak{g}_\zeta^\perp] \subset \mathfrak{g}_\zeta^\perp$ and we conclude that $\langle \mathbf{D}_0 \mathbf{I} \rangle = 0$.

By (2), the Hamiltonian one-form of the HB connection for $p \in P$, $\eta \in \mathcal{O}_\zeta$ and $\lambda \in \mathfrak{g}$, is given by

$$(K \cdot \lambda_{\mathcal{O}_\zeta})(p, \eta) = J^\lambda(p, \eta) - \langle J^\lambda \rangle (p, \eta) . \tag{6}$$

However

$$\langle J^\lambda\rangle(p,\eta) = \frac{1}{|G_\eta|}\int_{G_\eta}(\Phi_g^* J^\lambda)(p,\eta)\,dg = \frac{1}{|G_\eta|}\int_{G_\eta} J^\lambda(g\cdot p, \mathrm{Ad}_g\eta)\,dg$$
$$= \frac{1}{|G_\eta|}\int_{G_\eta} J^\lambda(g\cdot p)\,dg - \eta = \left\langle J(p), \frac{1}{|G_\eta|}\int_{G_\eta}\mathrm{Ad}_{g^{-1}}\lambda\,dg\right\rangle - \eta\,.$$

If $\lambda = \lambda_1 + \lambda_2$ where $\lambda_1 \in \mathfrak{g}_\eta$ and $\lambda_2 \in \mathfrak{g}_\eta^\perp$, then $\mathrm{Ad}_{g^{-1}}\lambda_1 = \lambda_1$ and $\mathrm{Ad}_{g^{-1}}\lambda_2 \in \mathfrak{g}_\eta^\perp = \mathfrak{g}_\zeta^\perp$, so that

$$\langle J^\lambda\rangle(p,\eta) = \langle J(p), \lambda_1\rangle - \eta\ . \tag{7}$$

Therefore, by (6) and (7),

$$\left(K\cdot\lambda_{\mathcal{O}_\zeta}(\eta)\right)(p,\eta) = \langle J(p), \mathbb{P}_\eta^\perp\lambda\rangle \tag{8}$$

where $\mathbb{P}_\eta^\perp : \mathfrak{g} \to \mathfrak{g}_\eta^\perp$ is the orthogonal projection relative to the metric $\langle\,,\rangle$. Therefore by **6.3** the horizontal lift of the HB connection has the expression

$$(\mathrm{hor}\,\lambda_{\mathcal{O}_\zeta}(\eta))(p,\eta) = \left(X_{J\mathbb{P}_\eta^\perp(\lambda)}, \lambda_{\mathcal{O}_\zeta}(\eta)\right) \tag{9}$$

and the connection one-form is

$$\gamma(p,\eta)\cdot(v_p, \lambda_{\mathcal{O}_\zeta}(\eta)) = v_p - X_{J\mathbb{P}_\eta^\perp(\lambda)}(p)\ , \tag{10}$$

for all $v_p \in T_pP$ and $\lambda_{\mathcal{O}_\zeta}(\eta) \in T_\eta\mathcal{O}_\zeta$. The curvature is given by the Hamiltonian vector field whose Hamiltonian is the average of $\{K\cdot\lambda_{\mathcal{O}_\zeta}(\eta),\ K\cdot\lambda'_{\mathcal{O}_\zeta}(\eta)\} = \{J^{\mathbb{P}_\eta^\perp\lambda}, J^{\mathbb{P}_\eta^\perp\lambda'}\} = J^{[\mathbb{P}_\eta^\perp\lambda\ ,\ \mathbb{P}_\eta^\perp\lambda']}$ (see Corollary **7.4**), *i.e.*,

$$\bar{\Omega}(\lambda_{\mathcal{O}_\zeta}(\eta), \lambda'_{\mathcal{O}_\zeta}(\eta))(p,\eta) = -\left(X_{\langle J[\mathbb{P}_\eta^\perp\lambda, \mathbb{P}_\eta^\perp\lambda']\rangle}(p), 0\right). \tag{11}$$

Finally, let us compute the holonomy of a given closed path $\eta(t)$ in $\mathcal{O}_\zeta$, where $\eta(0) = \eta(1) = \eta_0$. It is a diffeomorphism of the fiber P in $\pi : P\times\mathcal{O}_\zeta \to \mathcal{O}_\zeta$ obtained by parallel transport along the path $\eta(t)$. To compute it, consider the principal G_ζ-bundle $\sigma : G \to \mathcal{O}_\zeta$, $\sigma(g) = \mathrm{Ad}_g\zeta$ and endow it with the canonical connection $\gamma_c \in \Omega^1(G; \mathfrak{g}_\zeta)$ given by

$$\gamma_c(g)\cdot v_g = \mathbb{P}_\zeta T_g L_{g^{-1}}(v_g)\ , \tag{12}$$

where $v_g \in T_gG$ and $\mathbb{P}_\zeta : \mathfrak{g} \to \mathfrak{g}_\zeta$ is the orthogonal projection relative to the bi-invariant metric $\langle\,,\rangle$. The horizontal lift of this connection is given by

$$(\mathrm{hor}_c\lambda_{O_\zeta}(\eta))(g) = T_eR_g(\mathbb{P}_\eta^\perp\lambda), \quad \text{for } \eta = \mathrm{Ad}_g\zeta . \tag{13}$$

The curvature of this connection, thought of as a one-form on the base O_ζ with values in $\mathfrak{g}_\zeta$ (the adjoint bundle is trivial since G_ζ is abelian), is therefore given by

$$\mathrm{curv}(\gamma_c)(\eta)(\lambda_{O_\zeta}(\eta), \lambda'_{O_\zeta}(\eta)) = \mathbb{P}_\zeta\mathrm{Ad}_{g^{-1}}[\mathbb{P}_\eta^\perp\lambda, \mathbb{P}_\eta^\perp\lambda'] = \mathrm{Ad}_{g^{-1}}(\mathbb{P}_\eta[\lambda, \lambda'] - [\mathbb{P}_\eta\lambda, \mathbb{P}_\eta\lambda']) \tag{14}$$

where $\eta = \mathrm{Ad}_g\zeta$. According to Corollary **4.2** the holonomy of the standard connection γ_c on $\sigma : G \to O_\zeta$ is given by $h_c = \exp\left(-\iint \mathrm{curv}(\gamma_c)\right) \in G_\zeta$, where the integral is taken over any two-mainfold in O_ζ whose boundary is given by the closed path $\eta(t)$. Let us rephrase this in terms of horizontal lifts of curves. By (11), if $g(t)$ is the horizontal lift of $\eta(t)$ in G, then it must satisfy

$$g'(t) = T_e R_{g(t)}\mathbb{P}_{\eta(t)}^\perp\lambda(t) \tag{15}$$

where $\eta(t)$ is the solution of $\dot\eta(t) = [\lambda(t), \eta(t)]$, and the holonomy $h_c \in G_\zeta$ is characterized by

$$g(1) = h_c g(0) . \tag{16}$$

We shall use this to explicitly compute the holonomy of the closed path $\eta(t)$ in $\pi : P \times O_\zeta \to O_\zeta$. By (9), if $(p(t), \eta(t))$, $p(0) = p_0$, $h(0) = \eta_0$, is the horizontal lift of the path $\eta(t)$ we must have

$$\dot p(t) = X_{J\mathbb{P}_{\eta(t)}^\perp\lambda(t)}(p(t)), \qquad p(0) = p_0 .$$

But by (15), $\mathbb{P}_{\eta(t)}^\perp\lambda(t) = T_{g(t)}R_{g(t)^{-1}}\dot g(t)$, where $g(t)$ is the horizontal lift of $\eta(t)$ in the canonical bundle $\sigma : G \to O_\zeta$. Therefore, by definition of the momentum map, $p(t)$ satisfies the differential equation

$$\dot p(t) = \left(T_{g(t)}R_{g(t)^{-1}}\dot g(t)\right)_P (p(t)), \quad p(0)) = p_0 . \tag{17}$$

Now consider the curve $g(t)g(0)^{-1} \cdot p_0$, where $g(t)$ is the solution of (15). Note that at $t = 0$ this curve passes through p_0. Using the general formula $T_g(\Phi_\cdot(p))(v_g) = (T_gR_{g^{-1}}(v_g))_P(g\cdot p)$, where $\Phi : G \times P \to P$ is the given G-action, we see that

$$\frac{d}{dt} g(t)g(0)^{-1}\cdot p_0 = (T_{g(t)}R_{g(t)^{-1}}g'(t))_P\, (g(t), g(0)^{-1}\cdot p_0) \tag{18}$$

i.e., by (17) and (18), $p(t)$ and $g(t)g(0)^{-1}\cdot p_0$ satisfy the same differential equation, so, by uniqueness, $p(t) = g(t)g(0)^{-1}\cdot p_0$ and hence by (16), $p(1) = g(1)g(0)^{-1}\cdot p_0 = h_c\cdot p_0$. We have proved the following:

8.2 Proposition *Let* P *be a Poisson manifold and* G *a compact Lie group acting on* P *with equivariant momentum map* $\mathbf{J} : P \to \mathfrak{g}^*$. *Let* $\zeta \in \mathfrak{g}$ *be a regular element and consider the torus action on* P *whose momentum map for each* $\eta \in \mathcal{O}_\zeta$ *is given by*

$$\langle \mathbf{I}(p, \eta), \lambda \rangle = \langle \mathbf{I}(p), i_\eta(\lambda) \rangle , \ \lambda \in \mathfrak{g}_\zeta$$

where $i_\eta : \mathfrak{g}_\zeta \to \mathfrak{g}$ *is given by* $i_\eta(\lambda) = \mathrm{Ad}_h(\lambda)$ *for* $h \in G_\zeta$ *satisfying* $\eta = \mathrm{Ad}_h \zeta$. *Let* G_ζ *act trivially on the adjoint orbit* $\mathcal{O}_\zeta$. *Then the trivial connection on* $\pi : P \times \mathcal{O}_\zeta \to \mathcal{O}_\zeta$ *induces the* HB *connection whose Hamiltonian one-form, horizontal lift and connection form are given respectively by*

$$(K\cdot[\lambda, \eta])(p, \eta) = \langle \mathbf{J}(p), \mathbb{P}_\eta^\perp \lambda \rangle$$

$$(\mathrm{hor}[\lambda, \eta])(p, \eta) = \left(X_{\mathbf{J}\, \mathbb{P}_\eta^\perp(\lambda)}(p), [\lambda, \eta] \right)$$

$$\gamma(p, \eta)\cdot(v_p, [\lambda, \eta]) = v_p - X_{\mathbf{J}\, \mathbb{P}_\eta^\perp(\lambda)}(p)$$

for $\lambda \in \mathfrak{g}$, $\eta \in \mathcal{O}_\zeta$, $p \in P$, $v_p \in T_pP$. *The curvature of the Hannay-Berry connection, as a two-form on the base with values in* $\mathfrak{g}_\zeta$ *is given by*

$$\bar{\Omega}([\lambda, \eta], [\lambda', \eta])(p, \eta) = \left(- X_{\langle \mathbf{J} [\mathbb{P}_\eta^\perp(\lambda), \mathbb{P}_\eta^\perp(\lambda')] \rangle}(p), 0 \right).$$

The Hannay-Berry connection preserves the level sets of $\mathbf{I}$. *The holonomy of the closed path* $\eta(t)$ *in* $\mathcal{O}_\zeta$ *is the diffeomorphism of* P *given by the action of the group element* h_c *representing the holonomy of* $\eta(t)$ *in the canonical bundle* $\sigma : G \to \mathcal{O}_\zeta$.

Remark When the G_ζ-action is free, there is an alternative description of the bundle and the HB connection on $\mathbf{I}^{-1}(\mu)$ within $P \times \mathcal{O}_\zeta$. Consider the two projections

$$\mathcal{O}_\zeta \xleftarrow{p_1} \mathbf{I}^{-1}(\mu) \xrightarrow{p_2} \mathbf{I}^{-1}(\mu)/G .$$

By equivariance, $p_1(p_2^{-1}(x)) = \mathcal{O}_\zeta$ for each $x \in \mathbf{I}^{-1}(\mu)/G$. By freeness, $p_2^{-1}(x) \cong G$. Thus, restricting p_1 to $p_2^{-1}(x)$, we have the homogeneous bundle $G \to \mathcal{O}_\zeta = G/G_\zeta$. Consequently,

$$\mathbf{I}^{-1}(\mu) = \bigcup_{x \in \mathbf{I}^{-1}(\mu)} p_2^{-1}(x)$$

is a bundle of homogeneous bundles. In the case of the Foucault pendulum, $\mathbf{I}^{-1}(\mu)/G$ is a point, so that $\mathbf{I}^{-1}(\mu) \cong G = \mathbf{SO}(3)$.

§9 *The Hannay-Berry Connection on Level Sets of the Momentum Map*

By property **a** of Theorem **7.1**, the HB connection induces a connection on any level set of I. Indeed, if $\mu \in \mathfrak{g}^*$ is a (weakly) regular value, then $T_p(\mathbf{I}^{-1}(\mu)) = \ker \mathbf{dI}(p)$, and property **a** states that $\mathbf{dI}(p)$ vanishes on horizontal vectors, i.e. the Hannary-Berry horizontal lift at a point of $\mathbf{I}^{-1}(\mu)$ is necessarily tangent to $\mathbf{I}^{-1}(\mu)$. The proof of Theorem **7.1 i** shows that this connection is unique with the properties **b** and **c**.

9.1 Proposition *The* HB *connection induces a unique connection on each submanifold* $\mathbf{I}^{-1}(\mu)$ *(for* μ *a weakly regular value) satisfying properties* **b** *and* **c** *of Theorem* **7.1**.

There is another way to describe this connection. Fix a weakly regular value $\mu \in \mathfrak{g}^*$ of $\mathbf{I}$ and let $\pi_\mu : \mathbf{I}^{-1}(\mu) \to M$ be the restriction of π to $\mathbf{I}^{-1}(\mu)$. Let us assume that π_μ is surjective. This is in general not the case. If $G = \mathbb{T}^n$, Golin et al. [1989] prove that π_μ is surjective; see also Proposition **10.6** where we sketch their proof. Let us assume throughout this section that π_μ is surjective. It is also a submersion : $T\pi$ restricted to any horizontal subspace is an isomorphism onto the corresponding tangent space to M and, as we just argued, $\ker \mathbf{dI}(p)$ contains the horizontal space at p. The group G does not act on $\mathbf{I}^{-1}(\mu)$, but the coadjoint isotropy subgroup G_μ does. For $Z \in \mathfrak{X}(M)$, define

$$\mathrm{hor}_\mu Z = \langle \mathrm{hor}_0 Z \rangle_\mu := \frac{1}{|G_\mu|} \int_{G_\mu} \Phi_g^*(\mathrm{hor}_0 Z) dg \qquad (1)$$

where $|G_\mu|$ is the volume of G_μ in the induced Haar measure dg on G_μ from the one given on G. As before, we assume that $\pi : E \to M$ is endowed with a Poisson-Ehresmann connection hor_0. *A priori*, it is not clear that $\mathrm{hor}_\mu Z$ is a vector field tangent to $\mathbf{I}^{-1}(\mu)$. To show this, let $p \in \mathbf{I}^{-1}(\mu)$, $m = \pi(p)$, $v \in T_m M$, and $\xi \in \mathfrak{g}_\mu$, the Lie algebra at G_μ. By equivariance of $\mathbf{I}$ (if it is not equivariant, work with the induced affine action of G on $\mathfrak{g}^*$ — see Abraham and Marsden [1978, Proposition 4.2.7] for this procedure), we have

$$\begin{aligned} \mathbf{I}^{g^{-1}\cdot\xi}(p) &= \mathbf{I}(p)\left(\mathrm{Ad}_{g^{-1}}\mathbf{I}(p)\right)\cdot\xi = \mathrm{Ad}^*_{g^{-1}}\mu\cdot\xi = \mu\cdot\xi \\ &= \mathbf{I}(p)(\xi) = \mathbf{I}^\xi(p) = \mathbf{I}^\xi(g\cdot p) = (\Phi_g^*\mathbf{I}^\xi)(p) , \end{aligned}$$

so that

$$\mathbf{dI}^\xi(p)\cdot(\mathrm{hor}_\mu v)(p) = \frac{1}{|G_\mu|}\int_{G_\mu} \mathbf{dI}^\xi\cdot\Phi_g^*(\mathrm{hor}_0 v))(p) dg$$

$$= \frac{1}{|G_\mu|}\int_{G_\mu} \mathbf{d}I^{g^{-1}\cdot\xi}(p)\cdot\Phi_g^*(\mathrm{hor}_0 v)(p)dg$$

$$= \frac{1}{|G_\mu|}\int_{G_\mu} \Phi_g^*(\mathbf{D}_0 I^\xi\cdot\mathrm{hor}_\mu v)(p)dg$$

$$= \frac{1}{|G_\mu|}\int_{G_\mu} (\Phi_g^*(\mathbf{D}_0 I^\xi)\cdot\mathrm{hor}_\mu v)(p)dg = \langle \mathbf{D}_0 I^\xi\rangle_\mu(p)\cdot(\mathrm{hor}_\mu v)(p) .$$

Therefore, denoting by $\mathbf{D}_\mu$ the covariant derivative induced by hor_μ, we get the analog of Proposition **6.3 vii**,

$$\mathbf{D}_\mu \mathbf{I} = \langle \mathbf{D}_0 \mathbf{I}\rangle_\mu . \tag{2}$$

Thus $\langle \mathbf{D}_0\mathbf{I}\rangle_\mu = 0$, iff $\mathbf{D}_\mu \mathbf{I} = 0$ and $\mathrm{hor}_\mu v$ is tangent to $\mathbf{I}^{-1}(\mu)$.

The verification of properties **b** and **c** is done as in Proposition **6.3** with G replaced by G_μ, E by $\mathbf{I}^{-1}(\mu)$, and G-averages by G_μ-averages. Therefore by Proposition **9.1**, (1) defines the unique connection on $\mathbf{I}^{-1}(\mu) \to M$ satisfying properties **b** and **c** of Theorem **7.1**, since if $\langle K \cdot Z\rangle_\mu = 0$, by Fubini's theorem on bundles, it follows that $\langle K\cdot Z\rangle = 0$. We have proved the following:

9.2 Theorem *In the hypotheses of Theorem* **7.1**, *assume in addition that* $\mathbf{I}^{-1}(\mu) \to M$ *is onto for all weakly regular values* μ *and* $\langle \mathbf{D}_0\mathbf{I}\rangle_\mu = 0$ *for all* $\mu \in \mathfrak{g}^*$. *Then* (1) *defines the induced* HB *connection on* $\mathbf{I}^{-1}(\mu) \to M$ *for all weakly regular values* μ *of* $\mathbf{I}$.

Remark If $\mathbf{I}^{-1}(\mu) \to M$ is a principal G_μ-bundle, then the HB connection is a principal connection, since the horizontal lift is manifestly G_μ-invariant. Such a situation occurs for slowly varying integrable systems discussed in the next section.

§10 *Case I: Bundles with the Canonical Connection; Integrable Systems and Hannay's Angle*

In this section we treat a large class of Poisson fiber bundles which come equipped with a Poisson-Ehresmann connection and study the induced HB connection.

A ***bundle of symplectic manifolds*** is a fiber bundle $\pi : E \to M$ all of whose fibers are symplectic and whose transition functions are symplectic. Gotay et al. [1980] gave conditions guaranteeing the existence of a presymplectic (*i.e.*, closed) form ω on E whose pull-back to each fiber is the given fiber symplectic form. We assume that such a *closed* two-form ω on E is given and call $\pi : (E, \omega) \to M$ a ***coherent bundle of symplectic manifolds.***

As remarked in Gotay et al. [1983], *any coherent bundle of symplectic manifolds comes equipped with an Ehresmann connection.* Indeed, if $V = \ker T\pi$ is the vertical bundle of TE, then each fiber V_p of V, where $p \in E$, is a symplectic vector space. Define the subbundle H to be the ω-orthogonal complement to V

$$H_p = V_p^\omega = \{u \in T_pE \mid \omega(p)(u, v) = 0 \text{ for all } v \in V\};$$

$H = \bigcup_{p \in E} H_p$ is a subbundle of TE since the rank of ω is constant on connected components of M. We claim that $TE = H \oplus V$. To prove this, note that if $u \in H \cap V$ then $\omega(u, v) = 0$ for all $v \in V$, so that $u = 0$ by non-degeneracy of ω on V and thus $H \cap V = \{0\}$. Now consider the map $T_pE \to T_p^*E \to V_p^*$, the first arrow being $u \mapsto \omega(u, \cdot)$ and the second, the restriction of a linear functional on T_pE on the subspace V_p. Since V_p is symplectic, this linear map $T_pE \to V_p^*$ is onto. The kernel of this map equals $V_p^\omega = H_p$ and thus $\dim T_pE - \dim H_p = \dim V_p^* = \dim V_p$. This proves the claim. Thus, H is the horizontal subbundle of an Ehresmann connection on E. Let hor_0 denote its horizontal lift: $(\mathrm{hor}_0 w)(p) = (T_p\pi)^{-1}w$ for $w \in T_{\pi(p)}M$. We denote by $\mathbf{D}_0$ the covariant differentiation defined by hor_0.

A function $f : E \to \mathbb{R}$, defines a ***vertical Hamiltonian vector field*** X_f by

$$j_m^* \, df = \mathbf{i}_{X_f} j_m^* \, \omega, \tag{1}$$

where $\pi(p) = m$ and j_m is the fiber inclusion $\pi^{-1}(m) \to E$. We note that this is not the usual Hamiltonian vector field corresponding to the presymplectic form ω. The ***vertical Poisson bracket*** is defined by $\{f, h\} = \omega(X_f, X_h)$. One easily checks that:

10.1 Proposition *A projectable vector field $Y \in \mathfrak{X}(E)$ is a Poisson bracket derivation if and only if the Lie derivative $\pounds_Y\omega$ vanishes on any pair of vertical vectors.*

Now let us assume that $Y = \mathrm{hor}_0 Z$ for some $Z \in \mathfrak{X}(M)$. By Proposition **10.1**, $\mathrm{hor}_0 Z$ is a Poisson bracket derivation if and only if $(\pounds_{\mathrm{hor}_0 Z}\omega)(X_f, X_h) = 0$ for any $f, h : E \to \mathbb{R}$. Since ω is closed we have

$$\begin{aligned}(\pounds_{\mathrm{hor}_0 Z}\,\omega)(X_f, X_h) &= (\mathbf{d}\mathbf{i}_{\mathrm{hor}_0 Z}\,\omega)(X_f, X_h) \\ &= X_f[\omega(\mathrm{hor}_0 Z, X_h)] - X_h[\omega(\mathrm{hor}_0 Z, X_f)] - \omega(\mathrm{hor}_0 Z, [X_f, X_h]) = 0 \qquad (2)\end{aligned}$$

since ω vanishes on every pair of horizontal and vertical vector fields; the last term is also zero since $[X_f, X_h] = -X_{\{f,h\}}$ is vertical. Thus we have proved the following:

10.2 Proposition *The canonical connection* hor_0 *of the coherent bundle of symplectic manifolds* $\pi : (E, \omega) \to M$ *is a Poisson-Ehresmann connection.*

10.3 Proposition *Let* Ω_0 *denote the curvature of the canonical connection* hor_0. *Then for any* $Z_1, Z_2 \in \mathfrak{X}(M)$ *we have*

$$\bar{\Omega}_0(Z_1, Z_2) = -\,\omega(\mathrm{hor}_0 Z_1, \mathrm{hor}_0 Z_2)\ . \qquad (3)$$

Proof For $Z_1, Z_2 \in \mathfrak{X}(M)$ and $V \in \mathfrak{X}_{\mathrm{vert}}(E; M)$,

$$\begin{aligned}(\mathbf{i}_{\bar{\Omega}_0(Z_1, Z_2)}\omega)(V) &= \omega(\bar{\Omega}_0(Z_1, Z_2), V) \\ &= \omega(\mathrm{hor}_0[Z_1, Z_2], V) - \omega([\mathrm{hor}_0 Z_1, \mathrm{hor}_0 Z_2], V) \\ &= -\,\omega([\mathrm{hor}_0 Z_1, \mathrm{hor}_0 Z_2], V)\ .\end{aligned}$$

Now we use the formula:

$$\begin{aligned}(\mathbf{d}\omega)(A, B, C) &= A[\omega(B, C)] - B[\omega(A, C)] + C[\omega(A, B)] \\ &\quad - \omega([A, B], C) + \omega([A, C], B) - \omega([B, C], A) \qquad (4)\end{aligned}$$

with $A = \mathrm{hor}_0 Z_1$, $B = \mathrm{hor}_0 Z_2$, $C = V$. In (4), the first and second summands vanish. Also, since the bracket of a vertical with a horizontally lifted vector field is again vertical, the fifth and sixth terms also vanish. Since $\mathbf{d}\omega = 0$ by hypothesis, we get

$$V[\omega(\mathrm{hor}_0 Z_1, \mathrm{hor}_0 Z_2)] = \omega([\mathrm{hor}_0 Z_1, \mathrm{hor}_0 Z_2], V)\ ,$$

i.e.,

$$(\mathbf{i}_{\bar{\Omega}_0(Z_1, Z_2)}\omega)(V) = \mathbf{d}(\omega(\mathrm{hor}_0 Z_1, \mathrm{hor}_0 Z_2)) \cdot V\ . \ \blacksquare$$

If G is a compact Lie group defining a family of Hamiltonian G-actions on $\pi : (E, \omega) \to M$, then the action Φ restricted to every fiber $\pi^{-1}(m)$ preserves ω pulled back to $\pi^{-1}(m)$. If we denote by $\Phi^m : G \times \pi^{-1}(m) \to \pi^{-1}(m)$ the restriction of the G-action Φ to $\pi^{-1}(m)$, then this says that $(\Phi_g^m)^*\omega = \omega$ on $\pi^{-1}(m)$. In general, $\Phi_g^*\omega \neq \omega$ on E. Let hor_0 denote the horizontal lift of the connection induced by ω. Let hor denote the horizontal lift of the induced Hannay-Berry connection.

10.4 Theorem *Assume that there is an equivariant momentum map* $\mathbf{I}$ *in the presymplectic sense for the averaged form* $\langle\omega\rangle$ *on* E, *i.e.,*

$$\mathbf{d}\mathbf{I}^\xi = \mathbf{i}_{\xi_E}\langle\omega\rangle \tag{5}$$

for all $\xi \in \mathfrak{g}$. *Then the following hold.*

i $\mathbf{I} : E \to \mathfrak{g}^*$ *is a parametrized momentum map for the family of Hamiltonian* G-*actions on* E.

ii *There is a unique decomposition* $\langle\omega\rangle = \omega + \mathbf{d}\sigma$, *where* σ *is a one-form on* E *annihilating all vertical vectors.*

iii *The Hamiltonian one-form* $Z \mapsto K\cdot Z$ *of the* HB *connection is* $\sigma \circ \mathrm{hor}_0$.

iv *The horizontal distribution of the* HB *connection is the* $\langle\omega\rangle$*-orthogonal complement of the vertical subbundle.*

v *Let* $\tilde{\Omega}(X, Y) = -\langle\omega\rangle(X^h, Y^h)$, *where* $X, Y \in \mathfrak{X}(E)$ *and* X^h, Y^h *denote their horizontal parts relative to* hor. *The horizontal distribution of the* HB *connection equals the characteristic subbundle of* $\langle\omega\rangle + \tilde{\Omega}$, *i.e. a vector* $v \in T_pE$ *is horizontal if and only if* $\langle\omega\rangle(v, u) + \tilde{\Omega}(v, u) = 0$ *for all* $u \in T_pE$.

vi *The* HB *connection preserves the level sets of* $\mathbf{I}$, *i.e.,* $\mathbf{DI} = 0$.

vii *The adiabatic condition* $\langle\mathbf{D}_0\mathbf{I}\rangle = 0$ *holds.*

viii *The curvature of the* HB *connection is a Hamiltonian vector field with Hamiltonian function equal to* $\tilde{\Omega}(\mathrm{hor}\, Z_1, \mathrm{hor}\, Z_2)$ *for* $Z_1, Z_2 \in \mathfrak{X}(M)$, *i.e.,*

$$\bar{\Omega}(Z_1, Z_2) = X_{-\langle\omega\rangle(\mathrm{hor}\, Z_1,\, \mathrm{hor}\, Z_2)}\,.$$

Remarks 1 Properties **ii**, **iv**, **v**, and **vi** were obtained for the case of a trivial bundle with fiber an exact symplectic manifold and hor_0 the trivial connection by Golin, Knauf, and Marmi [1989].

2 If the G-action preserves ω, then $\mathbf{D} = \mathbf{D}_0$ for then $\omega = \langle\omega\rangle$.

Proof **i** We begin by showing that ω and $\langle\omega\rangle$ coincide on vertical vectors. Indeed, if $u, v \in V_p = \ker T_p\pi$ are vertical vectors, then $T_p\Phi_g(u) = T_p\Phi_g^m(u)$ and similarly for v, so that

$$\langle\omega\rangle(p)(u, v) = \frac{1}{|G|}\int_G (\Phi_g^*\omega)(p)(u, v)\, dg = \frac{1}{|G|}\int_G \omega(\Phi_g^m(p))(T_p\Phi_g^m(u), T_p\Phi_g^m(v))\, dg$$

$$= \frac{1}{|G|}\int_G ((\Phi_g^m)^*\omega)(p)(u, v)\, dg \;=\; \omega(p)(u, v)\ .$$

Thus, if $\mathbf{I}$ is the momentum map defined by $\langle\omega\rangle$ on E (in the presymplectic sense), then for any $\xi \in \mathfrak{g}$, $v \in V_p$ we have

$$\mathbf{dI}^\xi(p)(v) \;=\; \langle\omega\rangle(p)(\xi_E(p), v) \;=\; \omega(p)(\xi_E(p), v)$$

since both $\xi_E(p)$ and v are vertical. But this says that on the fiber through p, I^ξ is a momentum map of the G-action, i.e. $\mathbf{I}$ is a parametrized momentum map of the action.

ii Since G is compact, the exponential map is onto. So if $g \in G$, write $g = \exp\xi$ and hence

$$\Phi_g^*\omega - \omega \;=\; \int_0^1 \frac{d}{dt}\Phi^*_{\exp t\xi}\omega\, dt \;=\; \int_0^1 \Phi^*_{\exp t\xi}\pounds_{\xi_E}\,\omega\, dt \;=\; \int_0^1 \Phi^*_{\exp t\xi}\mathbf{d}\mathbf{i}_{\xi_E}\,\omega\, dt$$

$$= \mathbf{d}\left[\int_0^1 \Phi^*_{\exp t\xi}\mathbf{i}_{\xi_E}\,(\omega - \langle\omega\rangle)\, dt + \int_0^1 \Phi^*_{\exp t\xi}\mathbf{i}_{\xi_E}\,\langle\omega\rangle\, dt\right] \;=\; \mathbf{d}\sigma_g\ ,$$

since

$$\int_0^1 \Phi^*_{\exp t\xi}\mathbf{i}_{\xi_E}\,\langle\omega\rangle\, dt \;=\; \int_0^1 \Phi^*_{\exp t\xi}\,\mathbf{d}I^\xi\, dt \;=\; \mathbf{d}I^\xi\ ,$$

where we have written

$$\sigma_g = \int_0^1 \Phi^*_{\exp t\xi}\mathbf{i}_{\xi_E}(\omega - \langle\omega\rangle)\, dt \in \Omega^1(E)\ . \tag{6}$$

Since ω and $\langle\omega\rangle$ coincide on vertical vector fields, σ_g annihilates all vertical vectors; hence $\sigma = \langle\sigma\rangle$ will also annihilate all vertical vectors. Averaging we obtain: $\langle\omega\rangle - \omega = \mathbf{d}\sigma$.

iii In the proof of Proposition **6.3**, we gave an explicit formula for $K\cdot Z$, namely

$$K\cdot Z \;=\; \langle f_Z\rangle, \tag{7}$$

where

$$f_Z(g) \;=\; -\int_0^1 \Phi^*_{\exp t\xi}(\mathbf{d}I^\xi\cdot \mathrm{hor}_0 Z)\, dt \tag{8}$$

for $g = \exp\xi$. However, since

$$\mathbf{i}_{\xi_E}(\omega - \langle\omega\rangle)\cdot \mathrm{hor}_0 Z \;=\; -\,\mathbf{i}_{\xi_E}\langle\omega\rangle(\mathrm{hor}_0 Z) \;=\; -\,\mathbf{d}I^\xi\cdot \mathrm{hor}_0 Z\ ,$$

we see that (6) and (8) imply that $K\cdot Z = \sigma(\mathrm{hor}_0 Z)$.

iv We have seen in **i** that the presymplectic form $\langle\omega\rangle$ when restricted to each fiber coincides with ω, and hence is symplectic in each fiber. Therefore, $(E, \langle\omega\rangle)$ is a coherent bundle of symplectic manifolds and hence defines a canonical connection whose horizontal subbundle is $V^{\langle\omega\rangle}$, the $\langle\omega\rangle$-orthogonal complement of V. On the other hand, if Y is a vertical vector field, by **iii**, the fact that $\sigma(W) = 0$ for any vertical vector field W, and $\text{hor } Z = \text{hor}_0 Z + X_{K\cdot Z}$, we get

$$
\begin{aligned}
(\mathbf{i}_{\text{hor } Z}\langle\omega\rangle)(Y) &= \langle\omega\rangle(\text{hor } Z, Y) = \omega(\text{hor}_0 Z + X_{\sigma(\text{hor}_0 Z)}, Y) + \mathbf{d}\sigma(\text{hor } Z, Y) \\
&= \omega(X_{\sigma(\text{hor}_0 Z)}, Y) + (\text{hor } Z)[\sigma(Y)] - Y[\sigma(\text{hor } Z)] - \sigma([\text{hor } Z, Y]) \\
&= \mathbf{d}(\sigma(\text{hor}_0 Z))\cdot Y - Y[\sigma(\text{hor}_0 Z + X_{K\cdot Z})] \\
&= Y[\sigma(\text{hor}_0 Z)] - Y[\sigma(\text{hor}_0 Z)] = 0,
\end{aligned}
$$

which says that the horizontal subbundle of the HB connection is included in $V^{\langle\omega\rangle}$. Since its complement is V, it equals $V^{\langle\omega\rangle}$.

v Let $u, v \in T_pE$ and write $u = u_h + u_v$, $v = v_h + v_v$. Using **iv**, write

$$\langle\omega\rangle(u, v) - \tilde{\Omega}(u, v) = \omega(u_v, v_v)$$

which shows that the left hand side vanishes for all $u \in T_pE$ if and only if $v_v = 0$, *i.e.,* v is horizontal.

vi The statement is equivalent to

$$\mathbf{d}I^{\xi}\cdot\text{hor } Z = 0$$

for all $\xi \in \mathfrak{g}$, $Z \in \mathfrak{X}(M)$. By **vi** and the definition of $\mathbf{I}$ we get

$$\mathbf{d}I^{\xi}\cdot\text{hor } Z = \langle\omega\rangle(\xi_E, \text{hor } Z) = 0 .$$

vii By Proposition **6.3 vii**, $\mathbf{DI} = 0$ is equivalent to $\langle\mathbf{D}_0\mathbf{I}\rangle = 0$. The result now follows from **vi**.

viii The curvature formula is a direct consequence of Theorem **7.1**, Corollary **7.4**, and Proposition **10.3** provided we show

$$\langle\omega\rangle(\text{hor } Z_1, \text{hor } Z_2) = \langle\{K\cdot Z_1, K\cdot Z_2\}\rangle + \langle\omega(\text{hor}_0 Z_1, \text{hor}_0 Z_2)\rangle .$$

By G-invariance of $\text{hor } Z_1$ and $\text{hor } Z_2$ we have

$$
\begin{aligned}
\langle\omega\rangle(\text{hor } Z_1, \text{hor } Z_2) &= \frac{1}{|G|}\int_G \Phi_g^*(\omega(\text{hor } Z_1, \text{hor } Z_2))dg \\
&= \frac{1}{|G|}\int_G \Phi_g^*\{K\cdot Z_1, K\cdot Z_2\}dg + \frac{1}{|G|}\int_G \Phi_g^*(\omega(\text{hor}_0 Z_1, \text{hor}_0 Z_2))dg
\end{aligned}
$$

$$= \langle\{K\cdot Z_1, K\cdot Z_2\}\rangle + \langle\omega(\mathrm{hor}_0 Z_1, \mathrm{hor}_0 Z_2)\rangle \ . \ \blacksquare$$

There are important cases in which the hypothesis of this theorem is always satisfied.

10.5 Proposition *Assume that the coherent bundle of symplectic manifolds* $\pi : (E, \omega) \to M$ *is exact, i.e., there is a one-form* θ *on* E *such that* $\omega = -\mathbf{d}\theta$. *Then any family of symplectic G-actions is Hamiltonian and its parametrized momentum map is given by*

$$I(p)\cdot\xi = \langle\theta\rangle(\xi_E) \ .$$

Proof $\langle\theta\rangle$ is G-invariant and hence $\pounds_{\xi_E}\langle\theta\rangle = 0$. This says that

$$\mathbf{i}_{\xi_E}\langle\omega\rangle = -\mathbf{i}_{\xi_E}\mathbf{d}\langle\theta\rangle = \mathbf{d}\mathbf{i}_{\xi_E}\langle\theta\rangle \ ,$$

i.e., $\mathbf{i}_{\xi_E}\langle\theta\rangle = I^{\xi}$. $\blacksquare$

The situation described in this section for $E = P \times M$, where (P, ω) is a given sympletic manifold, $\pi : P \times M \to M$ the canonical projection, using the presymplectic form on E obtained by pulling back ω by the projection of E to P, and hor_0 the trivial connection, has been treated in detail by Montgomery [1988] and by Golin, Knauf, and Marmi [1989]. The following proposition appears in these papers.

10.6 Proposition *Suppose that the parameter dependent Hamiltonian* $H : P \times M \to \mathbb{R}$ *defines a completely integrable system for each value of the parameter* $m \in M$ *and that* $\mathbf{I} : P \times M \to \mathbb{R}^n$ *is a parametrized set of global action variables,* $\dim P = 2n$. *Assume that the adiabatic condition* $\langle\mathbf{d}_M \mathbf{I}\rangle = 0$ *holds. Then the* HB *connection on* $P \times M$ *preserves the level sets of* $\mathbf{I}$. *If* $\mu \in \mathbb{R}^n$ *is a regular value of* $\mathbf{I}(\cdot, m_0)$ *for a fixed* $m_0 \in M$, *if* $\mathbf{I}^{-1}(\mu)$, $\mathbf{I}^{-1}(\mu) \cap (P \times \{m\})$ and M *are connected, and if the sets* $\mathbf{I}^{-1}(\mu) \cap (P \times \{m\})$ *are compact for all* $m \in M$, *then* $\mathbf{I}^{-1}(\mu) \to M$ *is a principal torus bundle and the* HB *connection is a principal connection.*

For loops c *in* M *based at* m_0 *(small enough in general, or arbitrary, if we guarantee that* $\mathbf{I}^{-1}(\mu) \to M$ *is a principal torus bundle), the holonomy of the* HB *connection coincides with* ***Hannay's angle*** [1985], *namely*

$$\Delta\theta = \int_c \langle\mathbf{d}_M \theta\rangle ,$$

where $(\theta_1, \ldots, \theta_n)$ *is the parameter dependent set of angle variables defined by* $\mathbf{I}$.

Proof Assume for the moment (as in §9) that the restriction of π to $\mathbf{I}^{-1}(\mu)$ covers M. We first prove the theorem under this additional hypothesis. The first part is a direct consequence of the general theory if we prove that $\mathbf{I}^{-1}(\mu) \to M$ is a principal torus bundle. Since $\mathbf{I}^{-1}(\mu)$ is compact, parallel translation in $\mathbf{I}^{-1}(\mu)$ is complete. Parallel translation along a path connecting m_0 to an arbitrary $m \in M$ defines a diffeomorphism of the fiber $\mathbf{I}^{-1}(\mu) \cap (P \times \{m_0\})$ with the fiber $\mathbf{I}^{-1}(\mu) \cap (P \times \{m\})$. Both fibers are orbits of the torus action on $P \times M$ defined by the integrability assumption of H and this diffeomorphism intertwines the torus action. Thus $\mathbf{I}^{-1}(\mu) \to M$ is a principal torus bundle.

Next we compute the horizontal lift of the HB connection induced by the trivial connection $(\mathrm{hor}_0 Z)(p, m) = (0, Z(m))$. Since hor is the horizontal lift for the averaged connection,

$$(\mathrm{hor}\, Z)[f](p) = \frac{1}{|G|}\int_G (\mathrm{hor}_0 Z)[f \circ \Phi_{g^{-1}}](g\cdot p)\, dg \tag{9}$$

for $Z \in \mathfrak{X}(M)$, $f : E \to \mathbb{R}$, $p \in E$, and $g \in G$. Apply (9) to our case taking f to be each of θ^i, I^i, x^a and $Z = \partial/\partial x^a$, where I^i are the coordinate representations of the action $\mathbf{I} : P \times M \to \mathfrak{g}^*$ for each fixed value of the parameter, and θ^i are the corresponding canonically conjugate variables, *i.e.*, the angle variables defined by $\mathbf{I}$. We get

$$\left(\mathrm{hor}\frac{\partial}{\partial x^a}\right)[I^i] = \left\langle\frac{\partial I^i}{\partial x^a}\right\rangle, \quad \left(\mathrm{hor}\frac{\partial}{\partial x^a}\right)[\theta^i] = \left\langle\frac{\partial \theta^i}{\partial x^a}\right\rangle, \text{ and } \left(\mathrm{hor}\frac{\partial}{\partial x^a}\right)[x^b] = \delta^{ab},$$

i.e.,

$$\mathrm{hor}\frac{\partial}{\partial x^a} = \left\langle\frac{\partial I^i}{\partial x^a}\right\rangle\frac{\partial}{\partial I^i} + \left\langle\frac{\partial \theta^i}{\partial x^a}\right\rangle\frac{\partial \theta^i}{\partial x^a}\frac{\partial}{\partial \theta^i} + \frac{\partial}{\partial x^a}$$

By hypothesis $\langle \partial I^i/\partial x^a\rangle = 0$, so that

$$\mathrm{hor}\frac{\partial}{\partial x^a} = \left\langle\frac{\partial \theta^i}{\partial x^a}\right\rangle\frac{\partial \theta^i}{\partial x^a}\frac{\partial}{\partial \theta^i} + \frac{\partial}{\partial x^a} .$$

Therefore, if c is a curve in M and if $(\theta^i(t), I^i(t), x^a(t))$ is the horizontal lift of c given in coordinates by $(x^a(t))$, $t \in [0, 1]$, then the differential equations defining the horizontal lift are

$$\frac{d\theta^i}{dt} = \left\langle\frac{\partial \theta^i}{\partial x^a}\right\rangle\frac{dx^a}{dt}, \quad \frac{dI^i}{dt} = 0 .$$

Integrate the first equation over the path c with initial condition $\theta^i(0) = \theta^i_0$ and get

$$\theta(1) = \theta_0 + \int_c \langle \mathbf{d}_M \theta\rangle .$$

Thus, if c is a loop based at m_0, the holonomy of this path is $\theta(1) - \theta_0 = \int_c \langle \mathbf{d}_M \theta \rangle$, which is the formula for Hannay's angles.

To complete the proof of the theorem, we address the issue of why $\mathbf{I}^{-1}(\mu)/\mathbb{T}^n$ is diffeomorphic to the parameter space M. The projection $\pi : P \times M \to M$ induces a map $\mathbf{I}^{-1}(\mu)/\mathbb{T}^n \to M$ which is smooth and injective since $\mathbf{I}^{-1}(\mu) \cap (P \times \{m\})$ consists of exactly one torus, as we have seen at the beginning of the proof, using parallel translation. This argument also shows that it is a local diffeomorphism, so all that remains to be shown is that it is surjective. This is equivalent to proving that the set of values of $\mathbf{I}$ is parameter independent. We begin by remarking that by the generalization of Poincaré's last geometric theorem, the tori for nearby values of the parameters intersect (see Arnold [1978], Appendix 3). To prove the theorem, we need the ***global*** version of this result, for our case, *i.e.*, if as assumed, μ is a regular value of $\mathbf{I}(\cdot, m_0) =: \mathbf{I}_{m_0}$ then we need to show that for *all* $m \in M$ the tori $\mathbf{I}_{m_0}^{-1}(\mu)$ and $\mathbf{I}_m^{-1}(\mu)$ intersect. If this is shown, then clearly μ is in the range of $\mathbf{I}_m$ and thus the range of the parametrized momentum map is parameter independent.

An ingenious proof of this fact was given by Golin, Knauf, and Marmi [1989]. We reproduce here their proof for the sake of completeness. Let $c(t), t \in [0, 1]$ be a smooth path in M connecting m_0 to m. Then the Hamiltonian one-form of the HB connection given by $c'(t)$, namely $K \cdot c'(t) = \sigma(c'(t))$ defines a time dependent Hamiltonian on P. Its evolution operator (flow) gives hence a parameter dependent symplectic transformation $\psi_t : P \to P$. It is this transformation, which by horizontal lift, maps the tori of different values of the parameters to each other, as we saw at the beginning of the proof. So, what one has to exclude is the situation that an orbit starting at a point on the torus $\mathbf{I}_{m_0}^{-1}(\mu)$ is undefined for a time $t_0 < 1$.

Let us argue by contradiction and assume that such an orbit and t_0 do exist. Then there is a smaller $t_1 < t_0$ such that the torus $\mathbf{I}_{c(t)}^{-1}(\mu)$ does not intersect $\mathbf{I}_{m_0}^{-1}(\mu)$ for otherwise, by compactness there would be an accumulation point $p \in \mathbf{I}_{m_0}^{-1}(\mu)$ with $\mathbf{I}(p, c(t_1)) = \mu$. By openness of regular values, there is an open neighborhood U of μ and $t_2 \in]t_1, t_0[$ such that $\mathbf{I}_{c(t)}^{-1}(\nu) \neq \varnothing$ for all $\nu \in U$ and $t \in [0, t_2]$. By Jost's theorem, P fibers locally as a torus bundle, *i.e.*, locally P is of the form $B \times \mathbb{T}^n$, for B an open ball in $\mathbb{R}^n$. By the Lagrangian embedding theorem, there is an embedding λ of this piece of P into $T^*\mathbb{T}^n$, which sends the torus $\mathbf{I}_{m_0}^{-1}(\mu)$ to the zero section. Since the set $S = \{p \in P \mid \mathbf{I}_{c(t)}^{-1}(p) \in U \text{ for some } t \in [0, t_2]\}$ is compact with non-empty interior, define a time dependent Hamiltonian H_t on $T^*\mathbb{T}^n$ with t-independent compact support $\lambda(S)$ by $H_t(\lambda(p)) = \sigma(c'(t))(p)$ for all p in some open subset of S given by shrinking U to a smaller neighborhood of μ. Let φ_t be the time dependent flow in $T^*\mathbb{T}^n$ generated by H_t. So φ_t is a family of Hamiltonian isotopies. The other tori $\mathbf{I}_{c(t)}^{-1}(\mu)$ are Lagrangian submanifolds of $T^*\mathbb{T}^n$ via the embedding λ. So, we reduced the problem to the study of the intersection of the Lagrangian manifolds $\mathbb{T}^n$ and $\varphi_t(\mathbb{T}^n)$ in $T^*\mathbb{T}^n$. Hofer [1985] has shown that the number of intersections of $\varphi_1(N)$ with N in T^*N for N compact is at least $CL(N)+1$, where $CL(N)$ is the cup-length of the homology ring of N; for generalizations see Laudenbach and Sikorav [1985]. This result implies for our case that $I_{m_0}^{-1}(\mu) \cap I_{c(t_2)}^{-1}(\mu)$ contains at least $n+1$ points, a contradiction. ■

Remarks

1 For the non-integrable case, we suspect that the image of the momentum map can be parameter dependent.

2 Connectivity of the fibers $I^{-1}(\mu) \cap (P \times \{m\})$ is automatic if P is compact. This was proved by Atiyah [1982] and Guillemin and Sternberg [1984] in the course of the proof of the convexity theorem for images of momentum maps given by toral actions. Moreover, the image of $I(\cdot,m)$ up to translation, depends only on the cohomology class of P. (V. Guillemin, private conversation.) We can fix the image by insisting that its center of mass is the origin: $\int_P I(\cdot,m) = 0$. Since the image is independent of the parameter, $I^{-1}(\mu)$ automatically projects onto all of M.

3 The adiabatic condition $\langle \mathbf{d}_M I \rangle = 0$ is not always satisfied. For example, if I defines global action variables on P and $f : M \to \mathbb{R}^n$, then $I + f \circ \pi$ generates the same toral action as I, but $\langle \mathbf{d}_M(I + f \circ \pi)\rangle = \langle \mathbf{d}_M I\rangle + \mathbf{d}_M f$.

4 If we deal with slowly varying one degree of freedom systems, $\langle \mathbf{d}_M I\rangle = 0$ implies that I is an adiabatic invariant (Arnold [1978], [1983]). This means that for a time scale of order $1/\varepsilon$, the actions vary by $O(\varepsilon)$ as $\varepsilon \to 0$; ε measures how slowly the system varies.* The proof of adiabatic invariance depends on the averaging method, and fails for higher degree of freedom systems. Instead, an almost adiabatic invariance holds. This means that for all initial conditions *except* for a set of measure $O(\varepsilon)$, adiabatic invariance holds. See Golin and Marmi [1989] or Arnold [1978] and especially references to Neistadt therein. ♦

Let us address this last question in more detail. By the Liouville-Arnold and Jost theorems for completely integrable systems whose level sets for the action are compact, we have generically a locally trivial fibration of phase space as a torus bundle. Now consider a parameter dependent family of Hamiltonian systems with parametrized actions $I : P \times M \to \mathbb{R}^n$. Restrict I to the open subset of P where P is $B \times \mathbb{T}^n$, for B an open ball in $\mathbb{R}^n$. Consider a basis $(c_1, \ldots, c_n)$ of the first homology of $\mathbb{T}^n$ ($\mathbb{T}^n$ thought of as being the torus at the point (p_0, m_0)), and choose them to depend smoothly by on (p, m), in a neighborhood of (p_0, m_0). Let θ be a one form on the neighborhood of (p_0, m_0) such that $\omega = -\mathbf{d}\theta$, *i.e.*, in the identification of this neighborhood

---

* It is a classical result of Kruskal, Littlewood and others, that for one degree of freedom systems, the variation of the action is actually $O(\varepsilon^n)$ for any integer n. In fact, it is believed to be exponentially small. See §51 of Landau and Lifschitz [1976] and references in Holmes, Marsden, and Scheurle [1988].

with $B \times \mathbb{T}^n$, θ is the canonical one-form. Then $I^i = \int_{c_i} \theta$ define the ***standard action variables*** (Arnold and Avez [1968], Arnold [1978], Abraham and Marsden [1978], Chapter 5). They are smooth in (p, m). The angle variables are constructed by integrating the flows of the actions I^i. Note that the standard actions I^i equal $\mathbf{i}_{\partial/\partial\theta^i}\langle\theta\rangle$, where $\partial/\partial\theta^i$ is the infinitesimal generator of the i-th standard basis vector in $\mathbb{R}^n$, thought of as the Lie algebra of $\mathbb{T}^n$. Combining Theorem **10.4vi** and Proposition **10.5** gives a simple proof of the following result announced in Montgomery [1988] and proved by topological arguments in Weinstein [1989b].

10.7 Proposition *For standard actions* $\mathbf{I}$, *the adiabatic condition* $\langle d_M \mathbf{I}\rangle = 0$ *holds. Any two actions* $\mathbf{I}$ and $\mathbf{I}'$ *satisfying this adiabatic condition are related by* $\mathbf{I}' = A\mathbf{I} + v$, *where* $A \in \mathbf{SL}(n, \mathbb{Z})$ *and* $v \in \mathbb{R}^n$ *are constant.*

The second part of **10.7** is a consequence of the Arnold-Liouville theorem.

Many integrable systems are not integrable by virtue of a global torus action *i.e.*, they do not admit global action variables. For example, the spherical pendulum does not—see Duistermaat [1980]. We cannot apply **10.6** to compute, or even guarantee the existence, of the HB connection for families of such integrable systems.

This situation is salvageable, however, since every integrable system admits a local torus action, provided we delete separatrices. This means that the phase space, after deletion, is covered by an atlas of charts $\psi_\alpha : W_\alpha \to U_\alpha \times \mathbb{T}^n$, where U_α is an open subset of $\mathbb{R}^n$, $\mathbb{T}^n$ is the n-torus, and the overlap maps have the form $(\psi_\alpha \circ \psi_\beta^{-1})(x, \theta) = (f(x), \varphi_{\alpha\beta}(x) \cdot \theta)$, where $\varphi_{\alpha\beta}(x)$ is an affine automorphism of the torus. (An affine automorphism is one induced by an affine transformation of $\mathbb{R}^n$: $\varphi_{\alpha\beta}(x) \cdot \theta = \Gamma_{\alpha\beta}(x) \cdot \theta + \lambda$, where $\Gamma_{\alpha\beta}(x) \in \mathbf{SL}(n, \mathbb{Z})$ and $\theta, \lambda \in \mathbb{T}^n = \mathbb{R}^n / \mathbb{Z}^n$). Since affine automorphisms are measure preserving, averaging over the torus is well-defined.

Families of integrable systems also admit local torus actions provided we delete, in addition to the separatrices, certain "bad" parameter values. These are values at which there do not exist any integrals in involution which are continuous in the parameters. (The "bad" parameter values for a family of linear oscillators are the ones for which there is a 1:1 resonance.) The charts φ_α on the remaining open dense set are parameter dependent action-angle variables. "Averaging" is implemented by averaging over the angles in such a chart, and is globally well-defined. Since the HB connection is constructed purely through averaging, it also is well-defined. It satisfies all the properties of Proposition **10.6**, provided one makes obvious notational changes (*e.g.*, the actions $\mathbf{I}$ are now local). For details see Montgomery [1988].

Remarks

Perhaps the most interesting feature of these connections is their holonomy around the set Σ of 'bad' parameter values. The set Σ is in a fairly strict sense, a monopole source for the connection.

In the simplest case, alluded to above, the parameter space $\mathcal{S}$ is the space of real symmetric matrices. In this case, Σ is the set of matrices with double eigenvalues, which has codimension 2. Taking M to be a circle surrounding Σ, the holonomy associated with going once around M is -1. This can be viewed as either a classical or a quantum effect—the quantum effect is called the ***Jahn-Teller effect.*** The relevant bundle is the frame bundle associated to the canonical real line bundle over $M = \mathbb{RP}^1$. This frame bundle is the boundary of the Möbius strip.

We get interesting and important examples by replacing $\mathcal{S}$ with the set $\mathcal{S}_{\mathbb{C}}$ of Hermitian matrices, or with the set $\mathcal{S}_{\mathbb{H}}$ of quaternionic symmetric matrices. The case of $\mathcal{S}_{\mathbb{C}}$ is in Berry's original paper (Berry [1984]). Here Σ has codimension 3, and represents a Dirac monopole. The case $\mathcal{S}_{\mathbb{H}}$ was recently discussed by Avron, Sadun, Segert, and Simon [1989]. Here, Σ has codimension 5 and represents an instanton.

In the quantum case, Σ always represents a set of parameter values where eigenvalues collide. Kiritsis [1987] performed a general analysis of the bundles resulting from Σ 's of various dodimensions, d. His methods are those of Steenrod obstruction theory and cohomology. His base space M is always a small sphere S^{d-1} surrounding Σ.

§11 *Case II: Cartan Connections and Moving Systems*

The set-up of the previous section is not adequate to treat moving systems in the following sense. Consider, for instance the ball in the hoop. If one wants to apply the previous theory, one can do so using a limiting process, starting from the premise that there is a two dimensional confining potential field, then moving this potential field, computing a phase, and then taking the limit as the potential becomes an infinitely sharp, confining the particle to the one dimensional hoop. However, this method does not obviously give a technique that enables one to handle the one dimensional system *directly*. Indeed, if one attempts to do this directly as a one dimensional integrable system according to the integrable prescription, then the phase is zero. (See §12C). In this section we introduce another class of connections based on *Cartan's theory of classical spacetimes* that enables one to handle examples like the ball in the hoop in a direct manner.

§11A *Cartan Connections*

Let $(\mathcal{S}, g)$ be a Riemannian manifold and M a space of embeddings of a given manifold Q into $\mathcal{S}$. We think of Q as a given body and of $\mathcal{S}$ as space. Below we define a connection on the trivial bundle $\pi : Q \times M \to M$. The vertical subbundle of $T(Q \times M)$ is the pull-back of TQ over $Q \times M$, *i.e.*, the vector bundle $TQ \times M \to Q \times M : (v_q, m) \mapsto (q, m)$ for $q \in Q$, $m \in M$, and $v_q \in T_qQ$. A tangent vector to M at m is a vector field over m, *i.e.*, a map $u_m : Q \to T\mathcal{S}$ such that $u_m(q) = T_{m(q)}\mathcal{S}$. Relative to the metric g on $\mathcal{S}$, orthogonally project $u_m(q)$ to $T_{m(q)}m(Q) \in (T_qm)(T_qQ)$ and denote this vector $u_m^T(q)$. In this way, we have defined another element of T_mM. Pull back $u_m^T(q)$ by Tm^{-1} to T_qQ:

$$\mathcal{U}(q) = Tm^{-1}(u_m^T(q))$$

which defines a vector field $\mathcal{U} \in \mathfrak{X}(Q)$. For $Z \in \mathfrak{X}(M)$, define $\mathcal{Z}^m(q) = Tm^{-1}(\mathcal{Z}(m)^T(q))$. The association $m \mapsto Z(m)^T$ defines a vector field on M denoted by Z^T and so for each $m \in M$,

$$\mathcal{Z}^m = Tm^{-1} \circ Z^T(m) \in \mathfrak{X}(Q) \, . \tag{1}$$

For moving systems, we usually take the embeddings to be restrictions of isometries of $\mathcal{S}$ to $\mathcal{S}$. However, in the general theory, this need not be the case. For instance, consider the embeddings corresponding to blowing up a sphere by rescaling.

11.1 Definition *The Cartan connection on* $\pi : Q \times M \to M$ *is given by the one-form* $\gamma_c \in \Omega^1(Q \times M\,;\, \ker T\pi)$ *defined by*

$$\gamma_c(q, m)(v_q, u_m) = (v_q + (Tm^{-1} \circ u_m^T)(q), 0) \ , \tag{2}$$

where u_m^T *is the pointwise* g-*orthogonal projection of* $u_m \in T_mM$ *on* $T(m(Q))$.

The horizontal subspace at (q, m) of this connection is given by

$$H_{(q,m)} = \ker \gamma_c(q, m) = \{(-(Tm^{-1} \circ u_m^T)(q), u_m) \mid u_m \in T_mM\}$$

and so $T(Q \times M) = H \oplus V$. Thus the Cartan connection defines an Ehresmann connection on $\pi : Q \times M \to M$. By (1) and the expression of the horizontal subspace, we see that the horizontal lift of a vector field $Z \in \mathfrak{X}(M)$ is given by

$$(\mathrm{hor}_c Z)(q, m) = (-(Tm^{-1} \circ Z^T(m))(q), Z_m)) \,. \tag{3}$$

The Cartan connection induces a connection γ_0 on $\rho : T^*Q \times M \to M$ as follows: If $\rho : T^*Q \times M \to M$ is the trivial bundle, then the fiber of the vertical subbundle $\ker T\rho$ at (α_q, m) is $T_{\alpha_q}(T^*Q) \times 0_m$. Define the ***induced Cartan connection*** $\gamma_0 \in \Omega^1(T^*Q \times M \,;\, \ker T\rho)$ by

$$\boxed{\gamma_0(\alpha_q, m)(U_{\alpha_q}, u_m) = (U_{\alpha_q} + X_{\mathcal{P}\cdot u_m}(\alpha_q), 0_m)} \tag{4}$$

where $\mathcal{P}u_m$ is the momentum function of $Tm^{-1} \circ u_m^T \in \mathfrak{X}(Q)$, *i.e.*,

$$(\mathcal{P}\cdot u_m)(\alpha_q) = \alpha_q \cdot (Tm^{-1} \circ u_m^T)(q) \ . \tag{5}$$

The horizontal lift of $Z \in \mathfrak{X}(M)$ relative to the Cartan connection is thus

$$\boxed{(\mathrm{hor}_0 Z)(\alpha_q, m) \;=\; (-X_{\mathcal{P}\cdot Z(m)}(\alpha_q), Z(m)) \,.} \tag{6}$$

11.2 Proposition *The induced Cartan connection on* $\rho : T^*Q \times M \to M$ *is a Poisson-Ehresmann connection.*

Proof This is immediate, since the first component of $\mathrm{hor}_0 Z$ is given by a Hamiltonian vector field and the Poisson bracket on $T^*Q \times M$ is taken fiberwise. ■

§11B *The Cartan-Hannay-Berry Connection*

11.3 Theorem *Assume that a Lie group* G *acts on* T^*Q *on the left with equivariant momentum map* $\mathbf{I} : T^*Q \to \mathfrak{g}^*$. *Then* G *defines a family of Hamiltonian actions on* $T^*Q \times M$ *by letting* G *act trivially on* M. *Its parametrized momentum map is simply* $\mathbf{I}$ *thought of as a function of two variables, independent of* M.

i *The adiabatic condition* $\langle \mathbf{D}_0 \mathbf{I} \rangle = 0$ *holds.*

ii *The induced Cartan connection on* $T^*Q \times M$ *defines the Hannay-Berry connection given by the connection one-form*

$$\boxed{\gamma(\alpha_q, m) \cdot (U_{\alpha_q}, u_m) = (U_{\alpha_q} + X_{\langle \mathcal{P} \cdot u_m \rangle}(\alpha_q), 0).}$$

Its horizontal lift for $Z \in \mathfrak{X}(M)$ *has the expression*

$$\boxed{(\text{hor } Z)(\alpha_q, m) = (-X_{\langle \mathcal{P} \cdot Z(m) \rangle}, Z(m)),}$$

and the Hamiltonian one-form is given by

$$\boxed{(K \cdot Z)(\alpha_q, m) = \mathcal{P} \cdot Z(m)(\alpha_q) - \langle \mathcal{P} \cdot Z(m) \rangle (\alpha_q)\ .}$$

Parallel transport of the Hannay-Berry connection preserves the level sets of $\mathbf{I}$.

Proof i For $\xi \in \mathfrak{g}$, $\alpha_q \in T_q^*Q$, $m \in M$, $U_{\alpha_q} \in T_{\alpha_q}(T^*Q)$ and $u_m \in T_m M$, we have

$$\begin{aligned} (\mathbf{D}_0 I^\xi)(\alpha_q, m) \cdot (U_{\alpha_q}, u_m) &= \mathbf{d}I^\xi(\alpha_q, m) \cdot \mathbf{P}_{\text{hor}_0}(U_{\alpha_q}, u_m) \\ &= \mathbf{d}I^\xi(\alpha_q, m) \cdot (-X_{\mathcal{P} \cdot \bar{u}_m}(\alpha_q), u_m) \\ &= -\{I^\xi, (\mathcal{P} \cdot u_m\}(\alpha_q) \end{aligned}$$

so that, since the G-action is symplectic on T^*Q, we get

$$\begin{aligned} \Phi_g^*(\mathbf{D}_0 I^\xi)(\alpha_q, m) \cdot (U_{\alpha_q}, u_m) &= (\mathbf{D}_0 I^\xi)(g \cdot \alpha_q, m) \cdot (g \cdot U_{\alpha_q}, u_m) \\ &= -\{I^\xi, (\mathcal{P} \cdot u_m)\}(g \cdot \alpha_q) \\ &= -g^*\{I^\xi, \mathcal{P} \cdot u_m\}(\alpha_q)\ , \end{aligned}$$

where Φ_g denotes the G-action on $T^*Q \times M$ and $g \cdot \alpha_q$, $g \cdot U_{\alpha_q}$, the corresponding actions on T^*Q and $T(T^*Q)$. Thus,

$$
\begin{aligned}
\langle \mathbf{D}_0 I^\xi\rangle(\alpha_q, m)\cdot(U_{\alpha_q}, u_m) &= \frac{1}{|G|}\int_G \Phi_g^*(\mathbf{D}_0 I^\xi)(\alpha_q, m)\cdot(U_{\alpha_q}, u_m)\, dg \\
&= -\frac{1}{|G|}\int_G \Phi_g^*\{I^\xi, \mathcal{P}\cdot u_m\}(\alpha_q)\, dg \\
&= -\langle\{I^\xi, \mathcal{P}\cdot u_m\}\rangle(\alpha_q) \quad .
\end{aligned}
$$

Part **i** of the proposition now follows from the following.

11.4 Lemma *Let* $\pi : E \to M$ *be a Poisson fiber bundle endowed with a family of Hamiltonian* G*-actions with equivariant parametrized momentum map* $\mathbf{I} : E \to \mathfrak{g}^*$ *. Then for any* $f : E \to \mathbb{R}$ *we have* $\langle\{I^\xi, f\}\rangle = 0$ *for all* $\xi \in \mathfrak{g}$ *.*

Proof of lemma For $g \in G$, denote by $g \cdot \xi = \mathrm{Ad}_g \xi$ the adjoint action of G on $\mathfrak{g}$. We prove the lemma first if G is abelian. Then $g \cdot \xi = \xi$ for all $g \in G$, $\xi \in \mathfrak{g}$ so that

$$
\begin{aligned}
\langle\{I^\xi, f\}\rangle &= \frac{1}{|G|}\int_G \Phi_g^*\{I^\xi, f\}\, dg = \frac{1}{|G|}\int_G \{\Phi_g^* I^\xi, \Phi_g^* f\}\, dg \\
&= \frac{1}{|G|}\int_G \{\Phi^{g\cdot\xi}, \Phi_g^* f\}\, dg = \frac{1}{|G|}\int_G \{I^\xi, \Phi_g^* f\}\, dg = \{I^\xi, \langle f\rangle\} = 0
\end{aligned}
$$

by G-invariance of $\langle f\rangle$ and conservation of momentum maps.

Next we prove the general case. Fix $\xi \in \mathfrak{g}$ and let T denote the maximal torus containing $\exp \xi$; this is always possible since G is assumed to be compact. By Fubini's theorem we get for any $\varphi : E \to \mathbb{R}$

$$
\begin{aligned}
\langle\varphi\rangle(p) &= \frac{1}{|G|}\int_G \varphi(g\cdot p)\, dg = \frac{1}{|G|}\int_{Tg\in G/T}\left(\int_{t\in T}(h\cdot p)dh\right)d(Tg) \\
&= \frac{1}{|G|}\int_{Tg\in G/T}\left(\int_{t\in T}(f\cdot g\cdot p)dp\right)d(Tg)
\end{aligned}
$$

where we denote by $dt, dh, d(Tg)$ the induced Haar measures on T, Tg and G/T respectively. Now apply this formula to $\varphi = \{I^\xi, f\}$. Then

$$
\varphi(t\cdot g\cdot p) = (\Phi_g^*\Phi_t^*\{I^\xi, f\})(p)
$$

and hence

$$
\langle\{I^\xi, f\}\rangle(p) = \frac{1}{|G|}\int_G \Phi_g^*\{I^\xi, f\})(p)\, dg = \frac{1}{|G|}\int_{Tg\in G/T}\left(\int_{t\in T}\Phi_t^*\{I^\xi, f\}(p)dt\right)d(Tg)
$$

The interior fiber integral is zero by the first step so that the entire integral vanishes. ▼

ii For $Z \in \mathfrak{X}(M)$,

$$(\mathrm{hor}\, Z)(\alpha_q, m) = \langle(\mathrm{hor}_0 Z)(\alpha_q, m)\rangle = \frac{1}{|G|}\int_G \Phi_g^*\Big(-X_{\mathcal{P}\cdot Z(m)}(\alpha_q), Z(m)\Big)dg$$

$$= (-X_{\langle\mathcal{P}\cdot Z(m)\rangle}(\alpha_q), Z(m)))$$

and hence

$$(\mathrm{hor}\, Z - \mathrm{hor}_0 Z)(\alpha_q, m) = \big(X_{\mathcal{P}\cdot Z(m) - \langle\mathcal{P}\cdot Z(m)\rangle}, 0)\big) .$$

In view of **i**, the Hannay-Berry connection has parallel transport preserving the level sets of **I**. ■

§12 *The Cartan Angles; the Ball in the Hoop and the Foucault Pendulum*

In this section we compute the holonomy of the Hannay-Berry connection induced by a Cartan connection. We shall treat two examples in detail: the ball in the hoop and the Foucault pendulum. For the ball in the hoop example we will show that the Hannay angle formula $\int_c \langle \mathbf{d}_M \theta \rangle$ does *not* give the correct phase shift if we take the parameter space to be the space of frequencies.

§12A *Cartan Angles*

Recall that the holonomy of a closed loop relative to an Ehresmann connection is the diffeomorphism of the fiber given by parallel translation. In the case of the Hannay-Berry connection induced by a Cartan connection, the fiber is T^*Q. Thus if $c(t)$ is a closed loop of embeddings of Q in $\mathcal{S}$ the differential equations for the horizontal lift of $c(t)$ in T^*Q are Hamilton's equations for the Hamiltonian $\langle \mathcal{P}(Tc(t)^{-1} \circ c'(t)^T) \rangle$; see Theorem **11.3.**

Marsden [1981] proved that there is a neighborhood V of G_μ in $T^*G_\mu \times (T^*Q)_\mu \times \mathcal{O}_\mu$ and a neighborhood U of $G_\mu \cdot \alpha_q$ in T^*Q and a symplectic diffeomorphism $F : V \to U$. Here G_μ is the coadjoint isotropy subgroup at $\mu = \mathbf{I}(\alpha_q)$, $\mathcal{O}_\mu$ is the coadjoint orbit through μ in $\mathfrak{g}^*$, and $G_\mu \cdot \alpha_q$ is the G_μ-orbit of α_q in T^*Q. The composition of the momentum map with this diffeomorphism is a $\mathfrak{g}^*$-valued function whose level set at μ is diffeomorphic to $G_\mu \times (T^*Q)_\mu$. Thus the holonomy will have G_μ-components which we call the ***Cartan angles***.

These considerations become very explicit in the case of a completely integrable system, for then $\mathcal{O}_\mu = \{\mu\}$ and $(T^*Q)_\mu$ is a point. In this case $\mathbf{I} : T^*Q \to \mathbb{R}^n$ is the momentum map of a torus action for a completely integrable system. Assume, as in §10, that $\mathbf{I}$ represent global action variables with $(\theta^1, \ldots, \theta^n)$ the conjugate angle variables. If this is not the case, the theory must be reformulated in terms of local group actions as outlined at the end of §10 and treated in detail by Montgomery [1988]. By linearity of the momentum function, we have for a given embedding m,

$$\frac{\partial}{\partial I_i} \langle \mathcal{P}(Tm^{-1} \circ u_m^T) \rangle = \langle (Tm^{-1} \circ u_m^T)^i \rangle , \tag{1}$$

where $Tm^{-1} \circ u_m^T = (Tm^{-1} \circ u_m^T)^j \, \partial/\partial\theta^j$ (sum over j) .

Now let $c(t)$, $t \in [0, 1]$ be a closed loop based at $m_0 \in M$. By the formula for the horizontal lift of the induced Hannay-Berry connection given in Proposition **10.3ii** and **i**, it follows that $(\theta(t), c(t)) \in \mathbf{I}^{-1}(\mu)$ is the horizontal lift of the loop $c(t)$ if and only if

$$\frac{d\theta^i}{dt} = \langle Tc(t)^{-1} \circ c'(t)^T \rangle ,$$

whence the holonomy of this loop is

$$\Delta\theta = \theta(1) - \theta(0) = \int_0^1 \langle Tc(t)^{-1} \circ c'(t)^T \rangle dt \ . \qquad (2)$$

This formula is analogous in spirit to the expression of the Hannay angles discussed in §**10** for integrable systems. This is why we shall call the holonomy in (2) the ***Cartan angles*** .

§12B *The Ball in the Hoop*

Let us apply formula (2) to the ball in the hoop discussed in the introduction. Consider a not necessarily circular hoop of length L in the plane, enclosing an area A. On the hoop we consider a bead sliding without friction. As in Figure **1B-1**, let s denote the arc length along the hoop measured from a reference point on the hoop and let the shape of the hoop be described by a function $\mathbf{q}(s)$ in a given inertial frame in space. No external forces are acting on the bead, so that its total energy equals its kinetic energy $\frac{1}{2} m \, \|\mathbf{q}'(s(t)) \, \dot{s}(t)\| \, r^2 = \frac{1}{2} m \, \dot{s}(t)^2$ since $\|\mathbf{q}'(s(t))\|^2 = 1$. Thus the Hamiltonian of this system $H : T^* S^1 \to \mathbb{R}$ is given by

$$H(s, p_s) \ = \ \frac{1}{2m} p_s^2 \, ;$$

the solution of the corresponding Hamiltonian system is $s(t) = \frac{1}{2m} \mu t + s_0$, where the conserved momentum is $p_s(t) = \mu \in \mathbb{R}$. The system is integrable and we choose global action angle coordinates given by $(\theta, I) = (s, p_s)$; we consider the S^1-action on the hoop being given by the flow of this system. (Strictly speaking, we should take $\theta = \frac{2\pi}{L} s$ and $I = \frac{L}{2\pi} p_s$, so that θ ranges from 0 to 2π, but we shall take θ ranging from 0 to L to simplify notation.)

Assume the hoop rotates in the plane with prescribed angular velocity $\omega(t)$. Consider the space M of embeddings of the given hoop into $\mathcal{S} = \mathbb{R}^2$ given by the possible rotated configurations of the hoop relative to a given point, say, the origin. Thus the convective velocity is $Tc(t)^{-1} \circ c'(t) = \omega(t) \, \mathbf{k} \times \mathbf{q} = \omega(-y, x)$. Therefore $Tc(t)^{-1} \circ c'(t)^T$ represents its orthogonal projection at (x, y) onto the tangent space of the hoop, *i.e.*, $Tc(t)^{-1} \circ c'(t)^T = \omega(-y, x)\cdot\mathbf{q}'(s) = \omega(-y, x)\cdot(x', y') = \omega(xy' - yx')$. Therefore, its average over the dynamics is

$$\langle Tc(t)^{-1} \circ c'(t)^T \rangle = \frac{1}{L}\int_0^L \omega(t)(x(s)y'(s) - y(s)x'(s))ds = \frac{\omega(t)}{L}\int_{\text{hoop}} (xdy - ydx) = \omega(t)\frac{2A}{L}$$

by Green's theorem. Therefore the Cartan angle equals

$$\Delta\theta = \int_0^1 \langle Tc(t)^{-1} \circ c'(t)^T \rangle dt = \frac{2A}{L}\int_0^1 \omega(t)dt = \frac{2A}{L}\int_0^1 d\alpha(t)\,dt = \frac{2A}{L}(\alpha(1) - \alpha(0))$$

where $\alpha(t)$ is the angular variable whose time derivative is $\omega(t)$. Since $\alpha(1) - \alpha(0)$ is the length of the circle of radius 1, $\alpha(1) - \alpha(0) = 2\pi$ and so

$$\Delta\theta = \frac{4\pi A}{L} \tag{3}$$

the phase shift found by the elementary methods given in the introduction.

Treating the ball in the hoop example as a family of completely integrable systems depending on a parameter and computing the associated Hannay angle yields zero. Thus, the geometric phase relative to the Hannay-Berry connection induced by the trivial connection from **§9** is zero and the entire phase is a purely dynamic phase obtained by reconstruction, as in **§2** and **§3**. As in **§1**, the Lagrangian of the rotating system with angular velocity ω is given by

$$L(s, \dot{s}; \omega) = \frac{1}{2}m\,\|\,\mathbf{q}'(s)\dot{s} + \omega \times q(s)\,\|^2 . \tag{4}$$

The Legendre transformation $p_s = \frac{\partial L}{\partial \dot{s}} = m[\dot{s} + (\omega \times \mathbf{q}(s))\cdot\mathbf{q}'(s)]$ gives the Hamiltonian

$$H(s, p_s; \omega) = \frac{1}{2m}[p_s - m(\omega \times \mathbf{q}(s))\cdot\mathbf{q}'(s)]^2 - \frac{1}{2}m\,\|\,\omega \times \mathbf{q}(s)\,\|^2 . \tag{5}$$

For each fixed ω, the Hamiltonian system on T^*S^1 given by H is completely integrable since it is one degree of freedom. Thus H defines a completely integrable system depending on the frequency parameter ω. The space of ω's form a real line $\mathbb{R}$. The holonomy of any bundle over $\mathbb{R}$ is necessarily trivial. (This is because any loop in $\mathbb{R}$ comes back on itself.) Hence the Hannay angle is zero.

§12C *The Foucault Pendulum*

As in §**10**, we cannot litterally apply the formula for the Cartan angles since, in this example, it is known that global action-angle variables do not exist (Duistermaat [1980]). This is not really a serious problem, since we can restrict to the part of the phase space corresponding to stable oscillations. The Foucault pendulum is a spherical pendulum at co-latitude α on the surface of the Earth. Denote by $\mathbf{q}$ the position of the pendulum on the sphere of radius $\ell = \|\mathbf{q}\|$, the length of the pendulum arm (see Figure **12C-1**).

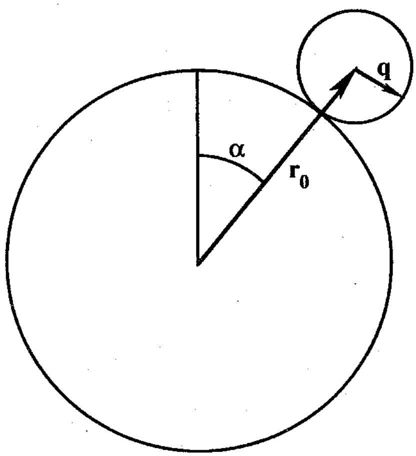


Figure 12C-1

Let $\mathbf{r}_0$ denote the vector from the center of the Earth to the point of suspension of the pendulum. The position of the tip of the pendulum in space is $R_t(\mathbf{r}_0 + \mathbf{q})$, where R_t is the rotation about the Oz-axis. Let $\omega = T/2\pi$ denote the angular velocity of the Earth's rotation. The potential energy of the pendulum is $V(\mathbf{q}) = mg\ell\mathbf{q}\cdot\hat{\mathbf{r}}_0$, where $\hat{\mathbf{r}}_0 = \mathbf{r}_0/\|\mathbf{r}_0\|$. The velocity of the pendulum's tip in space is

$$R_t\dot{\mathbf{q}} + R_t[\omega \times (\hat{\mathbf{r}}_0 + \mathbf{q})]$$

where we identify ω with the vector $\omega\mathbf{k}$. The Lagrangian is therefore given by

$$L = \frac{1}{2}m\,\|\,\omega \times (\hat{\mathbf{r}}_0 + \mathbf{q}) + \dot{\mathbf{q}}\,\|^2 - V(\mathbf{q})$$

so the Legendre transformation gives

$$\mathbf{p} = m(\dot{\mathbf{q}} + \omega \times (\hat{\mathbf{r}}_0 + \mathbf{q}))^T = m[\dot{\mathbf{q}} + (\omega \times (\hat{\mathbf{r}}_0 + \mathbf{q}))^T]$$

by identifying T^*S^2 with TS^2 via the Euclidean metric, where $(\omega \times (\hat{\mathbf{r}}_0 + \mathbf{q}))^T$ is the tangential component of $\omega \times (\hat{\mathbf{r}}_0 + \mathbf{q})$. Then

$$\dot{\mathbf{q}} = \frac{1}{m}\mathbf{p} - [\omega \times (\hat{\mathbf{r}}_0 + \mathbf{q})]^T$$

and so

$$\dot{\mathbf{q}} + \omega \times (\hat{\mathbf{r}}_0 + \mathbf{q}) = \frac{1}{m}\mathbf{p} + [\omega \times (\hat{\mathbf{r}}_0 + \mathbf{q})]^{\perp},$$

where $(\omega \times (\hat{\mathbf{r}}_0 + \mathbf{q}))^{\perp}$ denotes the normal component of $\omega \times (\hat{\mathbf{r}}_0 + \mathbf{q})$ to the sphere. Therefore the Hamiltonian is

$$\begin{aligned} H = \mathbf{p}\dot{\mathbf{q}} - L &= \frac{1}{m}\|\mathbf{p}\|^2 - \frac{1}{m}\mathbf{p}\cdot(\omega \times (\hat{\mathbf{r}}_0 + \mathbf{q}))^T \\ &\quad - \left(\frac{1}{2m}\|\mathbf{p}\|^2 + \frac{m}{2}\|(\omega \times (\hat{\mathbf{r}}_0 + \mathbf{q}))^{\perp}\|^2 - V(q)\right) \\ &= \frac{1}{2m}\|\mathbf{p}\|^2 + V(\mathbf{q}) - \mathcal{P} - \frac{m}{2}\|(\omega \times (\hat{\mathbf{r}}_0 + \mathbf{q}))^{\perp}\|^2, \end{aligned}$$

where

$$\mathcal{P} = \frac{1}{m}\mathbf{p}\cdot(\omega \times (\mathbf{r}_0 + \mathbf{q}))^T = \frac{1}{m}\mathbf{p}\cdot(\omega \times (\mathbf{r}_0 + \mathbf{q}))$$

since $\mathbf{p}$ is tangential.

Let M be the space of embeddings of the sphere of radius ℓ in $\mathbb{R}^3$ tangent to the Earth at co-latitude α, as in Figure **12C-1**. We recognize in $\mathcal{P}$ the Hamiltonian defining the induced Cartan connection on $T^*S^2 \times S^2$ (formula (4) of §**11**).

Remark Since $(\omega \times \mathbf{q})^{\perp} = 0$, we have $[\omega \times (\mathbf{r}_0 \times \mathbf{q})]^{\perp} = (\omega \times \mathbf{r}_0)^{\perp}$ = constant. In the equations of motion one can drop this constant. Had we considered a Foucault pendulum on an ellipsoid, this term would not be constant, but it would be of order ω^2, which is the general case. As explained in the introduction, applying the averaging principle, one would ignore this term.

Let S^1 act on the phase space of the pendulum by rotation about the $\mathbf{r}_0$ axis. We want to compute the part of the holonomy corresponding to this S^1-action. According to Proposition **11.4,** the horizontal lift of the induced Hannay-Berry connection is given by the Hamiltonian vector field of $\mathcal{P}$, *i.e.*, by $\langle p\cdot[\omega \times (\mathbf{r}_0 \times \mathbf{q})]\rangle$. If $\mathbf{v}$ is any constant vector,

$$\langle \mathbf{p}\cdot\mathbf{v}\rangle = \langle \mathbf{p}\cdot\hat{\mathbf{r}}_0\rangle(\mathbf{v}\cdot\hat{\mathbf{r}}) \tag{8}$$

since the S^1-action over which we average has $\mathbf{r}_0$ as its axis of rotation. Setting $\mathbf{v} = \omega \times \mathbf{r}_0$ this implies that

$$\langle \mathbf{p}\cdot(\omega \times \mathbf{r}_0)\rangle = 0 ,$$

and hence

$$\langle \mathbf{p}\cdot[\omega \times (\mathbf{r}_0 + \mathbf{q})]\rangle = \langle \mathbf{p}\cdot(\omega \times \mathbf{q})\rangle .$$

Let $I = \mathbf{p}\cdot(\mathbf{r}_0 \times \mathbf{q})$ be the momentum map of the S^1-action (rotation about $\mathbf{r}_0$). By (8)

$$\langle \mathbf{p}\cdot(\omega \times \mathbf{q})\rangle = \langle \omega\cdot(\mathbf{q} \times \mathbf{p})\rangle = [(\mathbf{q} \times \mathbf{p})\cdot\mathbf{r}_0]\,\langle \omega\cdot\hat{\mathbf{r}}_0\rangle = I\omega\cos\alpha$$

since $\omega\cdot\hat{\mathbf{r}}_0$ is constant. Thus,

$$\langle \mathcal{P} \rangle = I\,\omega\cos\alpha$$

and so the horizontal lift of ω is given by $(-X_{\langle \mathcal{P}\rangle}, \omega) = \left(-\omega\cos\alpha\frac{\partial}{\partial\theta}, \omega\right)$, where $\frac{\partial}{\partial\theta}$ is the infinitesimal generator of the S^1-action corresponding to I. Therefore the horizontal lift of a curve in θ is given by the differential equation

$$\dot{\theta} = -\omega\cos\alpha$$

so that if this curve is a loop parametrized on $[0, T/2\pi]$ we get

$$\boxed{\theta(T) - \theta(0) = -\int_0^T \omega\cos\alpha\, dt = -\omega T\cos\alpha = -2\pi\cos\alpha}$$

which is the deviation of the plane of oscillation in the laboratory frame (*i.e.*, a frame fixed on the Earth) of the Foucault pendulum during 24 hours.

For example, if we are at the equator, where $\alpha = \pi/2$, there is no deviation. If we are at the North Pole, where $\alpha = 0$, the plane rotates in the opposite direction to that of the Earth's rotation, performing a full circular motion. This is the usual Foucault result for a lab frame attached to the Earth. The laboratory and inertial frames are related by $\mathbf{q}_{\text{inertial}} = R_t(\mathbf{r}_0 + \mathbf{q}_{\text{lab}})$. As long as we regard angles as taking values in the circle, where 0 is identified with 2π, then the answer is the same in the inertial frame as in the lab frame. (It seems that there is no way of keeping track of the full angular change in the inertial frame, since this would require a reference line in each tangent space to the Earth relative to which we measure the angle. Since there are no nonvanishing vector fields on S^2, this is not possible.)

§13 *Induced Connections*

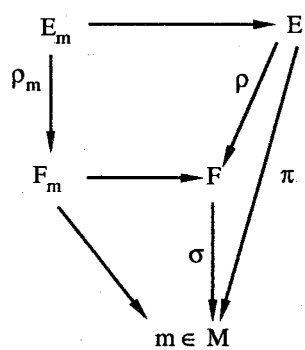
Assume that $\pi : E \to M$, $\rho : E \to F$, and $\sigma : F \to M$ are surjective submersions such that $\pi = \sigma \circ \rho$ and $\rho_m : E_m \to F_m$ defined by $\rho_m = \rho \mid E_m$ is a surjective submersion for all $m \in M$. Let $E_m = \pi^{-1}(m)$ and $F_m = \sigma^{-1}(m)$ denote the fibers of π and σ respectively. Assume there is a connection $\gamma^M \in \Omega^1(E; V^M)$ with horizontal lift $\mathrm{hor}^M : TM \to \mathfrak{X}_{\mathrm{hor}}(E; M)$ for $\pi : E \to M$, where $V^M = \ker T\pi$ is the vertical subbundle given by π. Denote by $H^M = \ker \gamma^M$ the horizontal subbundle of TE defined by γ^M. In addition, assume that for each $m \in M$ the bundles $\rho_m : E_m \to F_m$ have connections $\gamma^m \in \Omega^1(E_m; V^m)$ with horizontal lifts $\mathrm{hor}^m : TF_m \to \mathfrak{X}_{\mathrm{hor}}(E_m; V^m)$, where $V^m = \ker T\rho_m$ is the vertical subbundle of TE_m given by ρ_m. Let $H^m = \ker \gamma^m$ be the horizontal subbundle of TE_m defined by γ^m and assume that the connections γ^m depend smoothly on m in the sense that if $Z : M \times F \to TF$ is a smooth map satisfying $Z(m, \cdot) \in \mathfrak{X}(F_m)$ for all $m \in M$, then the mapping

$$e \in E \mapsto (\mathrm{hor}^m Z(m, \rho_m(e)))(e) \in TE \quad , \text{ for } m = \pi(e) ,$$

is a smooth vector field on E. Equivalently, in terms of the connection forms γ^m, smoothness means that

$$e \in E \mapsto \gamma^m(Z(m, \rho_m(e)) \in V^m, \quad \text{for } m = \pi(e),$$

is a smooth vector field on E; its value at each e is vertical relative to ρ_m. Let $V_F = \ker T\rho$ be the vertical subbundle of TE relative to ρ. The maps involved are summarized in the following diagram



whose horizontal arrows are inclusions.

13.1 Theorem *The map* $\gamma^F(e) = \gamma^m(e) \circ \gamma^M(e)$ *defines a connection form* $\gamma^F \in \Omega^1(E; V^F)$ *for the bundle* $\rho : E \to F$. *The pull-back of* γ^F *to each* E_m *coincides with* γ^m. *If* $H^F = \ker \gamma^F$ *denotes the horizontal subbundle of* TE *defined by* γ^F, *then*

$$H^F_e = H^m_e \oplus H^M_e, \quad V^F_e = V^m_e \subset V^M_e, \quad \textit{and} \quad H^F_e \cap V^M_e = H^m_e .$$

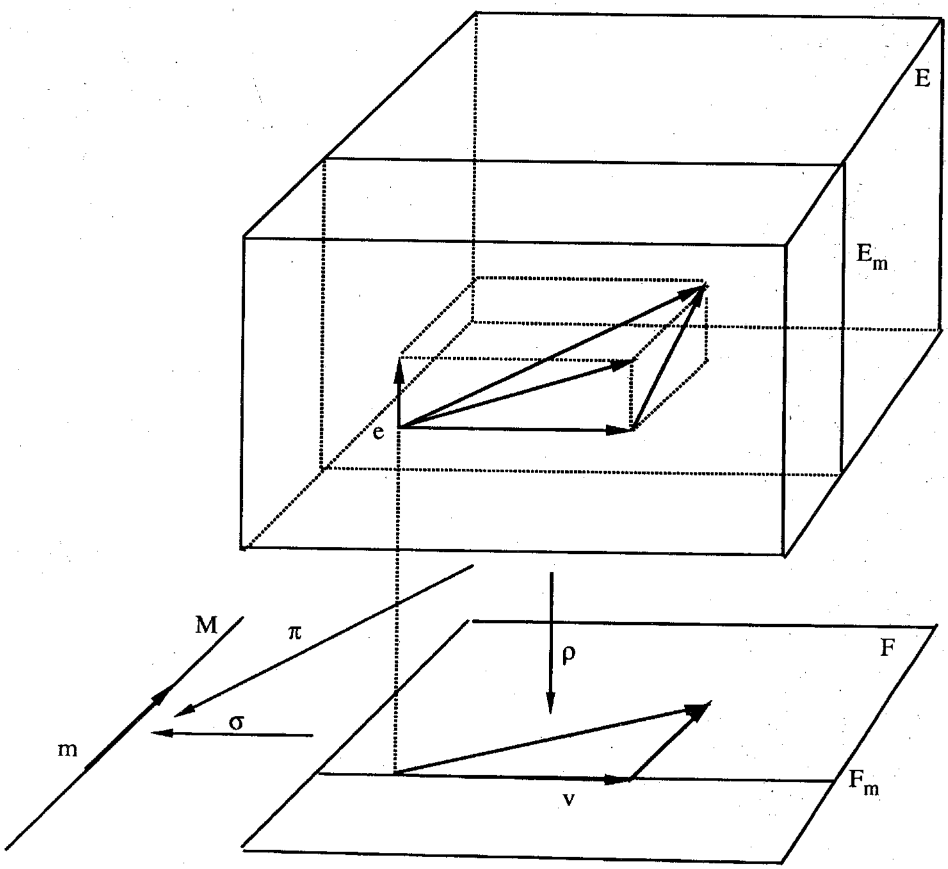


Figure 13-1

Proof By definition, $\gamma^M(e)(v) \in V^M_e = \ker T_e\pi = T_eE_m$ for $m = \pi(e)$. Thus $\gamma^m(e) \circ \gamma^M(e)$ defines a map $\gamma^F(e) : T_eE \to V^m_e$. Since $E_m \subset E$ and $\rho_m = \rho \mid E_m$, we obtain $V^m_e = \ker T_e\rho_m \subset \ker T_e\rho = V^F_e$. Conversely, if $v \in V^F_e$, then $T_e\rho(v) = 0$, so that $0 = (T_{\rho(e)}\sigma \circ T_e\rho)(v) = T_e\pi(v)$, *i.e.*, $v \in \ker T_e\pi = TE_m$. On E_m, ρ and ρ_m coincide, so $T_e\rho_m(v) = 0$; *i.e.*, $v \in V^m_e$. This shows that $V^m_e = V^F_e$ and hence $\gamma^F(e) : T_eE \to V^F_e$ is the identity on V^F_e. Since $\sigma \circ \rho = \pi$, we see that $V^F_e \subset V^M_e$ and hence if Z is any smooth vector field on E defined on a neighborhood of e, then $\gamma(Z)$ is a smooth vertical vector field (on this neighborhood) relative to π and $e \mapsto \gamma^F(Z)(e) = \gamma^{\pi(e)}(\gamma(Z)(e))$ is smooth by the smoothness condition on the γ^m. Thus $\gamma^F \in \Omega^1(E; V^F)$ is a connection one-form. Note that $T_eE = H^M_e \oplus V^M_e = H^M_e \oplus T_eE_m = H^M_e \oplus H^m_e \oplus V^F_e$, whence $H^F_e = H^M_e \oplus H^m_e$. Intersect this relation with V^M_e to get $H^F_e \cap V^M_e = H^m_e$ since $H^m_e \subset V^M_e = T_eE_m$. Finally, if $v \in T_eE_m = V^M_e$, then $\gamma^M(e)(v) = v$ and thus $\gamma^F(e)(v) = \gamma^m(e)(v)$, *i.e.*, the pull-back of γ^F to E_m equals γ^m. ■

We compute the curvature Ω^F of $\gamma^F \in \Omega^1(E; V^F)$ in terms of the curvatures Ω^M of $\gamma^M \in \Omega^1(E; V^M)$ and Ω^m of $\gamma^m \in \Omega^1(E_m; V^m)$. If $X \in \mathfrak{X}(E)$, denote by X^F and X^M the horizontal projections of X given by γ^F and γ^M. By the previous theorem, X^F splits into two components: one in H^m, the other in H^M. Note that ${}_mX := \langle \gamma^M, X \rangle \mid E_m$ defines a vector field on E_m. Writing

$$X^F(e) = X^1(e) + X^2(e) \in H^m_e \oplus H^M_e , \tag{1}$$

we get $T_e\pi(X^F(e)) = T_e\pi(X^2(e))$ since $T_e\pi(X^1(e)) = 0$ *i.e.*, $X^1(e) \in H^m_e \subset V^M_e = \ker T_e\pi$. Since $T_e\rho(X(e)) = T_e\rho(X^F(e))$ and $\pi = \sigma \circ \rho$, we conclude that

$$T_e\pi(X^F(e)) = (T_{\rho(e)}\sigma \circ T_e\rho)(X^F(e)) = (T_{\rho(e)}\sigma \circ T_e\rho)(X(e)) = T_e\pi(X(e)) = T_e\pi(X^2(e)),$$

i.e., $X(e)$ and $X^2(e)$ have the same horizontal component in the splitting $T_eE = H^M_e \oplus V^M_e$. Since $X^2(e) \in H^m_e$, we conclude that $X^2(e) = X^M(e)$. Applying γ^M to (1) we get

$$\langle \gamma^M, X^F \rangle(e) = \langle \gamma^M, X^1 \rangle(e) \tag{2}$$

since $\gamma^M(e)(X^M(e)) = 0$. However, $\langle \gamma^M, X^F \rangle = \langle \gamma^M, X \rangle$ since γ^M is the identity on V^M and $V^F \subset V^M$. On the other hand, since $X^1(e) \in H^m_e \subset V^M_e$, we have $\langle \gamma^M, X^1 \rangle(e) = X^1(e)$ and so by (2), $X^1(e) = \langle \gamma^M, X \rangle(e)$. We conclude that

$$X^F = \langle \gamma^M, X \rangle + X^M , \tag{3}$$

or if $m = \pi(e)$,

$$X^F(e) = {}_mX(e) + X^M(e) . \tag{4}$$

By formula (6) of §**4**, the curvature of γ^F equals

$$\begin{aligned} \Omega^F(X, Y)(e) &= -\langle \gamma^F, [X^F, Y^F] \rangle(e) = -\langle \gamma^F, [\langle \gamma^M, X \rangle + X^M, \langle \gamma^M, Y \rangle + Y^M] \rangle(e) \\ &= -(\gamma^m(e) \circ \gamma^M(e))([\langle \gamma^M, X \rangle, \langle \gamma^M, Y \rangle](e) \end{aligned}$$

$$+ [\langle\gamma^M, X\rangle, Y^M](e) + [X^M, \langle\gamma^M, Y\rangle](e) + [X^M, Y^M](e))$$

$$= -(\gamma^m(e) \circ \gamma^M(e))([\langle\gamma^M, X\rangle, \langle\gamma^M, Y\rangle](e))$$

$$-(\gamma^m(e) \circ \gamma^M(e))([\langle\gamma^M, X\rangle, Y^M](e) + [X^M, \langle\gamma^M, Y\rangle](e))$$

$$-(\gamma^m(e) \circ \gamma^M(e))([X^M, Y^M](e)) \,. \tag{5}$$

where $X, Y \in \mathfrak{X}(E)$. Since $\langle \gamma^M, X\rangle$ and $\langle \gamma^M, Y\rangle$ restricted to each E_m are tangent to E_m, it follows that their bracket is again tangent to each E_m, *i.e.*, $[\langle\gamma^M, X\rangle, \langle\gamma^M, Y\rangle](e) \in T_eE_m = V^M_e$, the first term in (5) becomes

$$\Omega^m(\langle\gamma^M, X\rangle, \langle\gamma^M, Y\rangle)(e)\,, \text{ for } m = \pi(e) \,. \tag{6}$$

The last term in (5) is

$$\gamma^m(e)(\Omega^M(X, Y)(e)) \,. \tag{7}$$

The second term in (5) can be simplified if there is an additional compatibility condition between γ^M and the family $\{\gamma^m\}$. Note that $\langle\gamma^M, X\rangle$ is a vector field on E whose value at e is in H^m_e. There are two obvious bracket conditions one can request: for any vector field X on E whose value at e is in H^m_e and any $Y \in \mathfrak{X}_{hor}(E; M)$, the bracket $[X, Y]$ is

i either in $\mathfrak{X}_{hor}(E; M)$, or

ii its value at e is in H^m_e.

In case **i** the second term vanishes since $\gamma^M(e)$ has kernel H^M_e. In case **ii**, since $H^m_e \subset V^M_e$, $\gamma^M(e)$ is the identity on H^m_e, and then $\gamma^m(e)$ kills the second term, so in both cases the second term in (5) vanishes. We have proved the following:

13.2 Proposition *Assume either condition* **i** *or* **ii** *holds. Then the curvature* $\Omega^F \in \Omega^2(E; V^F)$ *is given by*

$$\Omega^F(X, Y)(e) = \Omega^m(\langle\gamma^M, X\rangle, \langle\gamma^M, Y\rangle)(e) + \gamma^m(e)(\Omega^M(X, Y)(e)) \tag{8}$$

for any $X, Y \in \mathfrak{X}(E)$.

To compute the induced map $\bar{\Omega}^F\colon \mathfrak{X}(F) \times \mathfrak{X}(F) \to \mathfrak{X}_{vert}(E; F)$ on the base, we use (8) and the horizontal lift. Let $hor^M : \mathfrak{X}(M) \to \mathfrak{X}_{hor}(E; M)$, $hor^m : \mathfrak{X}(F_m) \to \mathfrak{X}_{hor}(E_m; F_m)$, and $hor^F\colon \mathfrak{X}(F) \to \mathfrak{X}_{hor}(E; F)$ be the horizontal lifts given by γ^M, γ^m, and γ^F respectively. If $U \in \mathfrak{X}(F)$, then $hor^F U \in \mathfrak{X}_{hor}(E; F)$ so that by (3)

$$hor^F U = \langle\gamma^M, hor^F U\rangle + (hor^F U)^M \,,$$

with $(hor^F U)^M(e) \in H^M_e$ and $\langle\gamma^M, hor^F U\rangle(e) \in H^{\pi(e)}_e \subset V^M_e$. Thus, if $W \in \mathfrak{X}(F)$, we get

$$\bar{\Omega}^F(U, W)(e) = \Omega^F(\mathrm{hor}^F U, \mathrm{hor}^F W)(e)$$
$$= \Omega^m(e)(\langle\gamma^M, \mathrm{hor}^F U\rangle, \langle\gamma^M, \mathrm{hor}^F W\rangle)(e) + \gamma^m(e)(\Omega^M((\mathrm{hor}^F U)^M, (\mathrm{hor}^F W)^M)(e)). \quad (9)$$

This expression can be simplified in the following way. We show that

$$(\mathrm{hor}^F U)^M(e) = \mathrm{hor}^M_e(T_{\rho(e)}\sigma \cdot U(e)) . \quad (10)$$

Since both sides are in H^M_e, it suffices to show that $T_e\pi((\mathrm{hor}^F U)^M(e)) = T_{\rho(e)}\sigma \cdot U(e)$, which is proved as follows. Since $\pi = \sigma \circ \rho$ and $T_e\rho((\mathrm{hor}^F U(e)) = U(e)$, we get $T_e\pi((\mathrm{hor}^F U)^M(e)) = (T_{\rho(e)}\sigma \circ T_e\rho)((\mathrm{hor}^F U)(e)) = T_{\rho(e)}\sigma \cdot U(e)$. By (10), the last term in (9) equals

$$\gamma^m(e)(\bar{\Omega}^M(e)(T_{\rho(e)}\sigma \cdot U(e), T_{\rho(e)}\sigma \cdot W(e)) . \quad (11)$$

For the first term in (9), write $\langle\gamma^M, \mathrm{hor}^F U\rangle(e)$ as hor^m_e of the vector $T_e\rho(\langle\gamma^M, \mathrm{hor}^F U\rangle(e))$ in $F_m = \sigma^{-1}(m)$. But

$$T_e\rho(T_eE) = T_e\rho(V^F_e \oplus H^F_e) = T_e\rho(H^F_e) = T_e\rho(H^m_e \oplus H^M_e) = T_e\rho(H^m_e) \oplus T_e\rho(H^M_e) = T_{\rho(e)}F.$$

Note that $T_e\rho(H^m_e) = \ker T_{\rho(e)}\sigma$. Indeed $(T_{\rho(e)}\sigma \circ T_e\rho)(H^m_e) = T_e\pi(H^m_e) = 0$ since $V^M_e \supset H^m_e$, so that $T_e\rho(H^m_e) \subset \ker T_{\rho(e)}\sigma$. Also $\ker T_{\rho(e)}\sigma \cap T_e\rho(H^M_e) = \{0\}$, for if $v \in H^M_e$ and $(T_{\rho(e)}\sigma \circ T_e\rho)(v) = 0$, then $T_e\pi(v) = 0$, whence $v \in V^M_e \cap H^M_e = \{0\}$.

Since $T\rho(H^M)$ is a smooth subbundle of TF, being the image of a subbundle by a vector bundle map, it follows that

$$TF = \ker T\sigma \oplus T\rho(H^M) , \quad (12)$$

i.e., there is an induced connection $\gamma^{M,F} \in \Omega^1(F; \ker T\sigma)$. In addition, since $T_e\rho(H^m_e) = \ker T_e\sigma$, we get

$$T_e\rho(\langle\gamma^M, \mathrm{hor}^F U\rangle(e)) = \gamma^{M,F}(e)(U(e)) . \quad (13)$$

Thus the first term in (9) equals

$$\bar{\Omega}^m(e)(\gamma^{M,F}(e) \cdot U(e), \gamma^{M,F}(e) \cdot W(e)) \quad (14)$$

and so by (11) and (13),

$$\bar{\Omega}^F(U, W)(e) = \bar{\Omega}^m(e)(\gamma^{M,F}(e) \cdot U(e), \gamma^{M,F}(e) \cdot W(e))$$
$$+ \gamma^m(\bar{\Omega}^M(e)(T_{\rho(e)}\sigma \cdot U(e), T_{\rho(e)}\sigma \cdot W(e))). \quad (15)$$

We have proved the following.

13.3 Theorem *Assuming the hypotheses of Theorem* **13.1**, *there is an induced connection* $\gamma^{M,F} \in \Omega^1(F; \ker T\sigma)$ *on* F *whose horizontal subbundle is given by* $T\rho(H^M)$. *Under the hypotheses of* **13.2**, *the curvature of the connection* γ^F *induces a map* $\overline{\Omega}^F$ *on the base* F *of the surjective submersion* $\rho : E \to F$ *given by* (15) *for any* $U, V \in \mathfrak{X}(F)$.

13.4 Corollary *Denote by* $\mathrm{hor}^{M,F} : \mathfrak{X}(M) \to \mathfrak{X}_{\mathrm{hor}}(F; \ker T\sigma)$ *the horizontal lift of the connection* $\gamma^{M,F} \in \Omega^1(F; \ker \sigma)$. *Then* $\mathrm{hor}^M = \mathrm{hor}^F \circ \mathrm{hor}^{M,F}$.

Proof Fix $e \in E$ and $v \in T_{\pi(e)}M$. By the proof of **13.3**, the $\gamma^{M,F}$-horizontal lift of v at $\rho(e)$ equals $\mathrm{hor}^{M,F}_{\rho(e)}(v) = T_e\rho \cdot u$ for some $u \in H^M_e = \ker \gamma^M(e)$. But then its γ^F-hrizontal lift at e is $\mathrm{hor}^F_e(\mathrm{hor}^{M,F}_{\rho(e)}(v)) = (\mathrm{hor}^F_e \circ T_e\rho)(u) = u$ since $u \in H^M_e \subset H^F_e$, by Theorem **13.1**. Therefore $\langle \gamma^M(e), (\mathrm{hor}^F \circ \mathrm{hor}^{M,F})_e(v) \rangle = \langle \gamma^M(e), u \rangle = 0$, *i.e.*, $(\mathrm{hor}^F \circ \mathrm{hor}^{M,F})_e(v) \in H^M_e$. From $\pi = \sigma \circ \rho$, one gets $T_e\pi \cdot (\mathrm{hor}^F \circ \mathrm{hor}^{M,F})_e(v) = v$ which proves the corollary. ■

§14 *The Hannay-Berry Connection for General Systems*

To start with, let $\pi : P \times M \to M$ be a trivial bundle where P is a symplectic manifold. Endow this bundle with the trivial connection. Let H be a Hamiltonian on $P \times M$ defining a completely integrable system for which global action variables exist; thus there is a parametrized torus action on $P \times M$. Let c be a loop at m_0 in M. The holonomy of c relative to the Hannay-Berry connection is called the ***geometric phase***. We will compare the angular variables in the torus over m_0, once a complete circuit around the loop c has been performed. Since the dynamics in the fiber varies as we move in c, we call the ***dynamic phase*** the total angular shift due to the frequencies $\omega^i = \partial H/\partial I^i$ of the integrable system, namely

$$\text{dynamic phase} = \int_0^1 \omega^i(I(c(t)), c(t))dt \ . \tag{1}$$

In writing this, we either assume that the loop is contained in a neighborhood whose standard action coordinates are defined, or we postulate the existence of global action variables, which is not always possible due to monodromy (Duistermaat [1980]). In any of these two cases, in completing the circuit c, we return to the same torus, so a comparison between the angles makes sense. Since the geometric phase is given in such a situation by Hannay's angles as we saw in §**10**, the total angular shift going once around c is given by

$$\Delta\theta = \text{dynamic phases} + \text{Hannay's angles} \ . \tag{2}$$

If we are dealing with a non-integrable system or with a moving system, the fiber dynamics is not so easily accounted for. The general set-up is a Poisson fiber bundle $\pi : E \to M$ with a Poisson Ehresmann connection with horizontal lift hor_0 and a family of Hamiltonian group actions given by a compact Lie group G defining a parametrized momentum map $\mathbf{I} : E \to \mathfrak{g}^*$. We form, for regular values $\mu \in \mathfrak{g}$, the parametrized reduced spaces $\mathbf{I}^{-1}(\mu)/G_\mu \to M$ whose fibers are the reduced spaces (symplectic if the fiber of $\pi : E \to M$ were symplectic--Marsden and Weinstein [1974], or Poisson in general--Marsden and Ratiu [1986]). The dynamics on these fibers will, in general, be non-integrable. In the integrable case, these fibers are points and the action of the torus is given by the frequencies $\omega_1, \ldots, \omega_n$ for each frozen value of the parameter. Our goal, as stated at the beginning of the paper, is to determine the geometric (or kinematic) part of the phase shift in the level sets $\mathbf{I}^{-1}(\mu)$. To do so, consider the tower of bundles

$$\pi_\mu : \mathbf{I}^{-1}(\mu) \to E_\mu := \mathbf{I}^{-1}(\mu)/G_\mu \ , \quad \sigma_\mu : E_\mu \to M \ . \tag{3}$$

Assume that $\langle \mathbf{D}_0 \mathbf{I} \rangle = 0$, which guarantees, by the theorems in Sections **6** and **7**, the existence and uniqueness of the Hannay-Berry connection on the bundle $\pi : E \to M$ which preserves the level sets of $\mathbf{I}$. Thus we get an induced Hannay-Berry connection with horizontal lift hor on the bundle $\sigma_\mu \circ \pi_\mu = \pi \mid \mathbf{I}^{-1}(\mu) : \mathbf{I}^{-1}(\mu) \to M$. This, however, is not enough to study the original dynamics on $\mathbf{I}^{-1}(\mu)$ since it ignores the family of reduced dynamics. What we need is a consistent way of lifting curves from the family of reduced spaces E_μ to $\mathbf{I}^{-1}(\mu)$. If there was no parameter involved here, this would be exactly the situation studied in Sections **2** and **3**. To put ourselves in this case, we use Sections **2** and **3** to construct on the fiberwise bundles $\pi_\mu^m : \mathbf{I}^{-1}(\mu)^m \to E_\mu^m$ a connection. We assume this family of connections depends smoothly on $m \in M$. In this way, using §**2** and §**3** we know how to reconstruct the dynamics for each frozen value of m. Of course, we want to reconstruct it, as m varies.

The situation described in the previous section deals with such a case: from a connection on a big bundle and fiberwise connections, it gives two connections on the remaining bundles. Theorem **13.1** gives a ***generalized Hannay-Berry connection*** on the bundle $\pi_\mu : \mathbf{I}^{-1}(\mu) \to E_\mu$; we will denote its horizontal lift by hor^μ. We also get a ***base Hannay-Berry connection*** on the fiber bundle $\sigma_\mu : E_\mu \to M$ by Theorem **13.3**; its horizontal lift will be denoted by hor^M. If the G_μ-actions restricted to each fiber of $\sigma_\mu \circ \pi_\mu : \mathbf{I}^{-1}(\mu) \to M$ define principal bundles, we conclude that $\pi_\mu : \mathbf{I}^{-1}(\mu) \to E_\mu$ is a principal G_μ-bundle and therefore the generalized Hannay-Berry connection is a principal connection. Consequently, its holonomy can be computed by the method of Proposition **4.1**, or if G_μ is abelian, one can give an explicit formula in terms of its curvature, as in Corollary **4.2** (see equation (11)).

For connections induced by Cartan connections, the generalized Hannay-Berry connection can be explicitly computed as follows. Let $E = T^*Q \times M$, G be a Lie group acting on Q, whose lifted action defines an equivariant momentum map regarded as a parametrized momentum map $\mathbf{I} : T^*Q \times M \to \mathfrak{g}^*$, depending trivially on M. Then we get the bundles

$$\pi_\mu : \mathbf{I}^{-1}(\mu) \times M \to \mathbf{I}^{-1}(\mu)/G_\mu \times M \quad \text{and} \quad \sigma_\mu : \mathbf{I}^{-1}(\mu)/G_\mu \times M \to M,$$

the first one being a principal G_μ-bundle and the second a trivial bundle. Let γ denote the connection one-form of the induced Hannay-Berry connection given by the Cartan connection γ_0 on $\pi : T^*Q \times M \to M$ (see Corollary **6.5**). We have

$$\begin{aligned} \gamma(\alpha_q, m)\cdot(U_{\alpha_q}, u_m) &= \gamma_0(\alpha_q, m)\cdot(U_{\alpha_q}, u_m) - X_{K\cdot T\pi(U_{\alpha_q}, u_m)}(\alpha_q, m) \\ &= (U_{\alpha_q} + X_{\mathcal{P}\cdot u_m}(\alpha_q), 0) - (X_{K\cdot u_m)}(\alpha_q), 0) \\ &= (U_{\alpha_q} + X_{\langle \mathcal{P}\cdot u_m \rangle}(\alpha_q), 0) \end{aligned} \tag{4}$$

by (4) of §**11** and Proposition **11.4 ii**.

Now let $\kappa \in \Omega^1(Q: \mathfrak{g}_\mu)$ be a principal connection on the bundle $Q \to Q/G_\mu$ and let κ^μ be the induced connection on $I^{-1}(\mu) \to I^{-1}(\mu)/G_\mu$ (Corollary **2.3**). Denote by $\tau : T^*Q \to Q$ the cotangent bundle projection and extend κ^μ trivially to the bundle $I^{-1}(\mu) \times M \to I^{-1}(\mu)/G_\mu \times M$. Think of κ^μ as a family of connection parametrized (in a trivial way) by M. Then Theorem **13.1** defines the connection one-form of the generalized Hannay-Berry connection : $\chi(\alpha_q, m) = \kappa^\mu(\alpha_q, m) \circ \gamma(\alpha_q, m)$, *i.e.*, by (4)

$$\begin{aligned}\chi(\alpha_q, m)\cdot(U_{\alpha_q}, u_m) &= \kappa^\mu(\alpha_q, m)\cdot(\gamma(\alpha_q, m)\cdot(U_{\alpha_q}, u_m)) \\ &= \kappa^\mu(\alpha_q, m)\cdot(U_{\alpha_q} + X_{\langle \mathcal{P}\cdot u_m\rangle}(\alpha_q), 0) \\ &= \kappa(q)\cdot T_{\alpha_q}\tau(U_{\alpha_q} + X_{\langle \mathcal{P}\cdot u_m\rangle}(\alpha_q)) \,, \end{aligned} \tag{5}$$

where $\tau : T^*Q \to Q$ is the projection. We proved the following

14.1 Theorem *The generalized HB-connection on* $\pi_\mu : I^{-1}(\mu) \times M \to I^{-1}(\mu)/G_\mu \times M$ *induced by a principal connection* $\kappa \in \Omega^1(Q; \mathfrak{g}_u)$ *and the Cartan connection on* $T^*Q \times M \to M$ *is a principal connection* $\chi \in \Omega^1(I^{-1}(\mu) \times M; \mathfrak{g}_\mu)$ *given by* (5).

In summary, we have connections on all levels of the tower of bundles

$$\begin{array}{c} I^{-1}(\mu) \times M \\ \downarrow \\ I^{-1}(\mu)/G_\mu \times M \\ \downarrow \\ M \ . \end{array}$$

We expect this general context is important for systems like coupled rigid bodies that have joint controls and are subject to overall motions, such as the space telescope. The joint controls can prescribe a motion in $I^{-1}(\mu)/G_\mu$, while the overall motion might be the system in orbit about the Earth. The reconstruction of the dynamics on $I^{-1}(\mu) \times M$ then involves both the cotangent bundle connection on $I^{-1}(\mu) \to I^{-1}(\mu)/G_\mu$ (either by Section **2B** or **2C**) and on $I^{-1}(\mu) \times M \to M$ by the Cartan-Hannay-Berry connection.

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