A Lie Group Structure for Pseudodifferential Operators

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1. Introduction

In 1979 Adler gave in [3] a Lie algebraic explanation of the complete integrability of the Korteweg-de Vries (KdV) equation. The Lie algebra used there is the algebra of formal pseudodifferential operators on the circle, \( S^1 \). (Adler was a bit more general than this but we restrict ourselves to this special case.) A formal pseudodifferential operator on \( S^1 \) is a formal series \( P(x, \xi) = \sum_{j=-\infty}^{\infty} a_j(x) \xi^j \) with \( a_j(x) \in C^\infty(S^1) \). Composition is defined by the formula

\[
P \circ Q(x, \xi) = \sum_{k \geq 0} \frac{1}{k!} \partial_x^k P(x, \xi) \partial_\xi^k Q(x, \xi).
\]

The space of formal pseudodifferential operators, \( \psi DO(S^1) \) is a topological Lie algebra with the bracket \([P, Q] = P \circ Q - Q \circ P\), and the topology given by the infinite product \( C^\infty(S^1) \times C^\infty(S^1) \times \cdots \) (see also Iacob and Sternberg [11]). With this Lie algebra Adler shows that KdV gives an infinite dimensional analog of the complete integrability of certain systems on Lie algebras described by Kostant and Symes in [12, 18, 19].

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It is natural to ask what Lie group has \(\psi DO(S^1)\) as its Lie algebra. Formally speaking it is easy to see that the group of invertible Fourier integral operators on \(S^1\) has the pseudodifferential operators as Lie algebra. To make this statement rigorous one has to give a “Lie group” structure to the invertible Fourier integral operators. The purpose of this paper along with the forthcoming paper [2] is to give a Lie group structure to the group of invertible formal Fourier integral operators on a compact manifold \(M\) and show that the Lie algebra of this group is the Lie algebra of formal pseudodifferential operators on \(M\). We describe below our strategy for doing this.

Consider the group \((\text{FIO})_\ast\) of invertible classical Fourier integral operators on \(M\), the group operation being operator multiplication. The canonical relation of a Fourier integral operator is a conic, Lagrangian submanifold of \((T^*M\setminus O)\times(T^*M\setminus O)\). If it is invertible, the canonical relation must be the graph of a canonical transformation \(\eta\) on \(T^*M\setminus O\). Since graph \(\eta\subset(T^*M\setminus O)\times(T^*M\setminus O)\) is assumed conic and Lagrangian \(\eta\) must preserve the canonical one-form \(\theta\) on \(T^*M\setminus O\), i.e. \(\eta^\ast \theta = \theta\). Let \(\mathcal{D}(T^*M\setminus O)\) be the space of smooth diffeomorphisms of \(T^*M\setminus O\) which preserve \(\theta\). Then there is a surjective map

\[
\pi: (\text{FIO})_\ast \rightarrow \mathcal{D}(T^*M\setminus O).
\]

If we denote the identity symplectomorphism by \(e\), we note that \(\pi^{-1}(e) = (\psi DO)_\ast\), the space of invertible classical pseudodifferential operators. \((\psi DO)_\ast\) is also a group under operator multiplication and we have the exact sequence of groups

\[
I \rightarrow (\psi DO)_\ast \xrightarrow{j} (\text{FIO})_\ast \xrightarrow{\pi} \mathcal{D}(T^*M\setminus O) \rightarrow e,
\]

where \(I\) is the identity operator and \(j\) is inclusion.

In Sect. 2 of this paper we describe the spaces of formal pseudodifferential operators and Fourier integral operators on a compact manifold \(M\), i.e. those which have classical symbols and are defined modulo smoothing operators. For these spaces we still get the exact sequence (1.1). The technique we use to give a Lie group structure to the group in the middle of this sequence is first to give Lie group structures to both ends of the sequence and then to produce a local section of the fibration \(\pi: (\text{FIO})_\ast \rightarrow \mathcal{D}(T^*M\setminus O)\) satisfying compatibility conditions that will allow us to define charts on \((\text{FIO})_\ast\).

The space \(\mathcal{D}(T^*M\setminus O)\) can of course be given a Fréchet manifold structure on which multiplication is smooth; but as with the diffeomorphism group \(\mathcal{D}(M)\) there is a much richer Lie group structure, namely the ILH (inverse limit Hilbert) structure described by Omori [15]. Briefly, an ILH-Lie group structure for a topological group \(G\) is a collection of topological groups \(G^t\) \(_{t \geq s_0}\) such that

1) \(G^s\) is a Hilbert manifold
2) \(G^{s+\varepsilon} \subset G^s\), densely and continuously
3) \(G = \cap G^t\) with the inverse limit topology
4) \(M_{s,t,r}: G^s \times G^t \rightarrow G^r\) defined by \((g_1, g_2) \mapsto g_1 g_2\) is \(C^{t-r}\) for \(s, t \geq r\).
5) Technical assumptions.

For \(G = \mathcal{D}(T^*M\setminus O)\) the groups \(G^s\) are the spaces \(\mathcal{D}^s(T^*M\setminus O)\), \(s > \dim M + 1\), of \(H^s\)-diffeomorphisms of \(T^*M\setminus O\) which preserve \(\theta\). \(\mathcal{D}^s(T^*M\setminus O)\) is not a closed subgroup in the group of \(H^s\)-diffeomorphisms of \(T^*M\setminus O\) but in [17] it was shown
that \( D^k_\Omega(T^\ast M \setminus O) \) is a Hilbert manifold which makes \( D^\Omega(T^\ast M \setminus O) \) into an ILH-Lie group. In Sect. 3 we review the basic facts on ILH-Lie groups and recall the construction of the ILH-Lie group structure for \( D^\Omega(T^\ast M \setminus O) \).

The rest of this paper is devoted to giving the other end of the sequence (1.1), i.e. the invertible formal pseudodifferential operators, a Lie group structure and constructing a local section of the fibration \( \pi: (\text{FIO})_\Omega \rightarrow D^\Omega(T^\ast M \setminus O) \). In the following paper [2] we describe how to patch these structures together to give a Lie group structure to the invertible formal Fourier integral operators.

To describe the Lie group structure of the invertible formal pseudodifferential operators it is convenient to restrict attention to the quotient \( (\psi DO_{0,k})_\ast \) of invertible formal pseudodifferential operators of order zero modulo those of order \(-k-1\). This gives a new exact sequence

\[
I \rightarrow (\psi DO_{0,k})_\ast \rightarrow (\text{FIO}_{0,k})_\ast \rightarrow D^\Omega(T^\ast M \setminus O) \rightarrow \mathbb{R},
\]

where \((\text{FIO}_{0,k})_\ast\) is the group of invertible Fourier integral operators of order zero modulo those of order \(-k-1\). In Sect. 5 we will show that \((\psi DO_{0,k})_\ast\) is an ILH-Lie group and then define the differentiable structure on invertible, order zero, formal pseudodifferential operators by a direct limit procedure. In order to define the ILH-Lie group structure of \((\psi DO_{0,k})_\ast\) we need a global phase function for the identity map on \( T^\ast M \setminus O \); this gives a global writing for pseudodifferential operators (modulo smoothing operators). In Sect. 4 we not only construct a global phase function for the identity map but in fact we construct a global phase function for all \( \theta \)-preserving diffeomorphisms of \( T^\ast M \setminus O \) which are \( C^1 \)-close to the identity. This gives a global writing for invertible Fourier integral operators (modulo smoothing) with canonical relation close to the identity, thus providing a local section of the fibration \( \pi: (\text{FIO}_{0,k})_\ast \rightarrow D^\Omega(T^\ast M \setminus O) \).

2. Formal Pseudodifferential and Fourier Integral Operators

For a compact manifold \( M \) let \( I^m(M, C) \) denote the space of Fourier integral operators of order \( m \) on \( M \) associated with the homogeneous canonical relation \( C \subset (T^\ast M \setminus O) \times (T^\ast M \setminus O) \). The submanifold \( C \) in \( (T^\ast M \setminus O) \times (T^\ast M \setminus O) \) is conic and Lagrangian with respect to the canonical symplectic form \( \omega \otimes \omega = p^1_\ast \omega - p^2_\ast \omega \) (here \( p_j: T^\ast M \times T^\ast M \to T^\ast M \), \( j = 1, 2 \), are projections onto the first and second factors and \( \omega = -d\theta \) where \( \theta \) is the canonical one form on \( T^\ast M \)). This means that \( ((x, \xi), (y, \eta)) \in C \) implies \( ((x, \tau \xi), (y, \tau \eta)) \in C \) for all \( \tau > 0 \) and that \( i^* (\omega \otimes \omega) \equiv 0 \), where \( i: C \to T^\ast M \times T^\ast M \) is inclusion. We consider \( A \in I^m(M, C) \) as a continuous linear operator \( A: C^\infty(M) \to C^\infty(M) \) which extends continuously to \( \mathcal{E}(M) \to \mathcal{E}(M) \) and can be represented in local charts by the standard form

\[
Au(x) = \frac{1}{(2\pi)^{n-\lambda}} \iint e^{i\varphi(x,y,\theta)} u(x, y, \theta) dy d\theta
\]

with \( x, y \in \Omega \subset \mathbb{R}^n \), \( \theta \in \mathbb{R}^n \), and \( u \in C^\infty_c(\Omega) \).

In (2.1) \( \varphi(x, y, \theta) \) is a nondegenerate phase function associated with \( C \), and the amplitude \( a(x, y, \theta) \) is an element of the space \( S^{n-\frac{1}{2} (n-\lambda)}(\Omega \times \Omega, \mathbb{R}^n) \), both in the sense of Duistermaat [5], Hörmander [10], Taylor [20] or Trèves [21]. This means explicitly
i) $\phi$ is a real valued smooth function on $\Omega \times \Omega \times \mathbb{R}^N$ homogeneous of degree one in $\theta$, i.e. $\phi(x, y, t\theta) = t\phi(x, y, \theta)$ for all $t > 0$.

ii) $d_x \phi$ and $d_y \phi$, the differentials of $\phi$ with respect to $(x, \theta)$ and $(y, \theta)$ respectively, do not vanish on the conic support of $a$.

iii) $\phi$ is nondegenerate, i.e. if $d_x \phi(x, y, \theta) = 0$ and $(x, y, \theta) \in \text{conic support of } a(x, y, \theta)$ then $d_{x,y,\theta} \frac{\partial \phi}{\partial \theta}(x, y, \theta)$ are linearly independent for $j = 1, \ldots, N$. Nondegeneracy implies that $(x, y, \theta) \mapsto (x, y, d_{x,y} \phi(x, y, \theta))$ is a diffeomorphism from the conic submanifold $C_{\phi} = \{(x, y, \theta) \in \text{conic support of } a \mid d_x \phi(x, y, \theta) = 0 \}$ onto an open cone in $C$.

iv) $a(x, y, \theta)$ is smooth on $\Omega \times \Omega \times \mathbb{R}^N$ satisfying the following property which defines $S^{-\frac{1}{2}(N-n)}(\Omega \times \Omega, \mathbb{R}^N)$. For every compact subset $K$ of $\Omega \times \Omega$ and every triplet of $n$-tuples $\alpha, \beta, \gamma$ there is a constant $C_{\alpha, \beta, \gamma}(K) > 0$ such that

$$|D^{\alpha}_{\phi}D^{\beta}_{\theta}D^{\gamma}_{\eta}(a(x, y, \theta))| \leq C_{\alpha, \beta, \gamma}(K)(1 + |\theta|)^{-\frac{1}{2}(N-n) - |\alpha|}$$

(2.2)

for all $(x, y) \in K, \theta \in \mathbb{R}^N$.

For future reference we denote $I^{-\infty}(M, C) = \bigcap_{m \in \mathbb{R}} I^m(M, C)$ and $S^{-\infty}(\Omega \times \Omega, \mathbb{R}^N) = \bigcap_{m \in \mathbb{R}} S^m(\Omega \times \Omega, \mathbb{R}^N)$.

When two Fourier integral operators, $A_1 \in I^m(M, C_1)$ and $A_2 \in I^m(M, C_2)$ are composed the result is again a Fourier integral operator as long as the product $C_1 \times C_2$ intersects the diagonal $\{(x, y, z) \in (T^* M \setminus \emptyset) \times (T^* M \setminus \emptyset) \times (T^* M \setminus \emptyset) \}$ transversely. In this case $C_1 \circ C_2 = \{(x, z) \in T^* M \setminus \emptyset \times T^* M \setminus \emptyset \mid \exists y \in T^* M \setminus \emptyset \text{ with } (x, y, z) \in C \}$ and $(y, z) \in C_2 \}$. More generally one can allow clean intersections of canonical relations but since we are concerned only with invertible Fourier integral operators it is no restriction to consider only transverse intersections. Indeed, if the inverse of $A \in I^m(M, C)$ is again to be a Fourier integral operator then the canonical relation $C$ must be invertible, that is, $C$ must be the graph of a diffeomorphism $\eta : T^* M \setminus \emptyset \to T^* M \setminus \emptyset$. That is canonical implies that $\eta$ is a symplectomorphism, i.e. $\eta^* \omega = \omega$, and that $C$ is conic implies that $\eta$ is homogeneous of degree one, i.e. $\eta(\tau z) = \tau \eta(z)$, $\tau > 0$. Indeed, if $C$ is conic then $(\alpha_x, \beta_z) \in C$ implies $(\tau \alpha_x, \beta_z) \in C$ so if $\eta(\alpha_x) = \beta_z$ then $\eta(\tau \alpha_x) = \tau \beta_z = \tau \eta(\alpha_x)$. It is well known (e.g. Hörmander [10], Weinstein [22]) that a diffeomorphism of $\eta : T^* M \setminus \emptyset \to T^* M \setminus \emptyset$ is symplectic and homogeneous of degree one if and only if $\eta$ preserves the canonical one form, i.e. $\eta^* \theta = \theta$. Note that for this to be true it is crucial that the zero section is deleted from $T^* M$, otherwise $\eta$ would just be a lift of a diffeomorphism $g : M \to M$.

Let $\mathcal{D}(T^* M \setminus \emptyset) = \{ \eta : T^* M \setminus \emptyset \to T^* M \setminus \emptyset | \eta \text{ is a diffeomorphism and } \eta^* \theta = \theta \}$ be the group of homogeneous canonical diffeomorphisms of $T^* M \setminus \emptyset$. Denote the class of Fourier integral operators of order $m$ with $C = \text{graph}(\eta)$ by $I^m(M, \eta)$. If $A_1 \in I^m(M, \eta_1)$ and $A_2 \in I^m(M, \eta_2)$ then $A_1 \circ A_2 \in I^{m_1 + m_2}(M, \eta_1 \circ \eta_2)$ since $\text{graph}(\eta_1 \circ \eta_2) = \text{graph}(\eta_2) \setminus \text{graph}(\eta_1 \circ \eta_3)$.

The class of pseudodifferential operators of order $m$ on $M$ is defined as $L^m(M) = I^m(M, e)$ where $e$ is the identity $e : T^* M \setminus \emptyset \to T^* M \setminus \emptyset$. A pseudodifferential
operator $P$ can be locally represented in the form

$$Pu(x) = (2\pi)^{-n} \iint e^{i(x-y)\cdot\xi} a(x, y, \xi) u(y) dyd\xi,$$

where $x, y \in \Omega \subset \mathbb{R}^n$, $\xi \in \mathbb{R}^n$ and $a(x, y, \xi) \in S^m(\Omega \times \Omega \times \mathbb{R}^n)$.

The Schwartz kernel theorem states that for any continuous map $K : C^\infty(M) \to \mathcal{D}'(M)$, there exists a unique distribution $\hat{K} \in \mathcal{D}'(M \times M)$ called the Schwartz Kernel such that for all $\varphi, \psi \in C^\infty(M)$, $\langle K\varphi, \psi \rangle = \langle \hat{K}, \varphi \otimes \psi \rangle$. Here and in what follows $C^\infty(M)$ and $\mathcal{D}'(M)$ are equipped with their standard seminorms with respect to which they are Fréchet spaces. A continuous linear operator from $C^\infty(M)$ to $\mathcal{D}'(M)$ is called regularizing (or smoothing), if it extends to a continuous linear map of $\mathcal{D}'(M)$ into $C^\infty(M)$. This happens if and only if the associated Schwartz Kernel is $C^\infty$ on $M \times M$. The spaces $L^{-\infty}(M) = \bigcap_{m\in\mathbb{R}} L^m(M)$ and $I^{-\infty}(M, \eta) = \bigcap_{m\in\mathbb{R}} I^m(M, \eta)$ both coincide with the space of regularizing operators on $M$ (see e.g. Trèves [21]).

In this paper we shall consider only formal pseudodifferential and Fourier integral operators: These are defined as follows. Let $S^m(\Omega)$ be the subspace of $S^m(\Omega \times \Omega \times \mathbb{R}^n)$ consisting of amplitudes $a(x, \xi)$ independent of $y$. The map associating to $p(x, \xi) \in S^m(\Omega)$ the operator $P \in L^m(\Omega)$ given by

$$Pu(x) = (2\pi)^{-n} \iint e^{i(x-y)\cdot\xi} p(x, \xi) u(y) dyd\xi$$

induces an isomorphism of $S^m(\Omega)/S^{-\infty}(\Omega)$ with $L^m(\Omega)/L^{-\infty}(\Omega)$ [26]. Elements of $S^m(\Omega)$ are called symbols of order $m$. A symbol is called classical if it has an asymptotic expansion $\sum_{j=0}^{\infty} a_{m-j}(x, \xi)$ where each term $a_{m-j}(x, \xi)$ is smooth off $\xi = 0$ and is positive homogeneous of degree $m-j$ in $\xi$, i.e. $a_{m-j}(x, \tau\xi) = \tau^{-j} a_{m-j}(x, \xi)$ for all $\tau > 0$. (To say that $a(x, \xi)$ has the asymptotic expansion $\sum_{j=0}^{\infty} a_{m-j}(x, \xi)$ means that

$$(1 - \chi(x, \xi)) \left( a(x, \xi) - \sum_{j=0}^{k-1} a_{m-j}(x, \xi) \right) \in S^{m-k}(\Omega)$$

for all $k$ where $\chi(x, \xi)$ is a bump function on $\Omega \times \mathbb{R}^n$ which has the value one on a neighborhood of $\Omega \times \{0\}$ and is compactly supported in the $\xi$ direction.) Since the asymptotic expansion is independent of changes in $a(x, \xi)$ by elements of $S^{-\infty}(\Omega)$ it makes sense to talk about classical pseudodifferential operators on $\Omega$ as pseudodifferential operators on $\Omega$ which can be written in the form (2.4) modulo a smoothing operator where $p(x, \xi)$ is a classical symbol. For formal manipulations it is convenient to ignore the smoothing part of $P$. In particular the equivalence classes of classical pseudodifferential operators in $L^m(\Omega)/L^{-\infty}(\Omega)$ are in one to one correspondence with the space of asymptotic expansions $\sum_{j=0}^{\infty} a_{m-j}(x, \xi)$. This leads to the definition: A formal pseudodifferential operator on $M$ of order $m$ is an element of $L^m(M)/L^{-\infty}(M)$ which has a representative in any local chart $\Omega$ of the form (2.4) with $p(x, \xi)$ defining a classical symbol. We denote the space of formal pseudodifferential operators of order $m$ by $\psi DO^m$. The principal symbol $\sigma(P)(x, \xi)$
of \( P \in \psi DO_m \) equals \( a_m(x, \xi) \) in any local chart and is globally defined as a smooth homogeneous function of degree \( m \) on \( T^*M \setminus \Omega \).

A formal Fourier integral operator of order \( m \), with canonical relation graph(\( \eta \)), is an element of the quotient space \( \Gamma^m(M, \eta)/\Gamma^{-\infty}(M, \eta) \) which has a representative in any local chart \( \Omega \) of the form (2.1) with \( a(x, y, \xi) \) defining a classical symbol. We call this space \( \text{FIO}_m(\eta) \) and denote

\[
\text{FIO}_m = \bigcup_{\eta \in \mathscr{D}(T^*M, \Omega)} \text{FIO}_m(\eta).
\]

Notice that for \( -k \leq m \) we have \( \psi DO_{-m} \circ \psi DO_m \). We let

\[
\psi DO_{m,k} = \psi DO_m/\psi DO_{-k-1},
\]

these are the formal pseudodifferential operators of order \( m \), modulo those of order \( -k-1 \). Note that on \( \Omega \) there is an isomorphism between \( \psi DO_{m,k} \) and the classical symbols in \( S^m(\Omega)/S^{-k-1}(\Omega) \), which in turn can be thought of as finite expansions

\[
\sum_{j=1}^{m-k} a_{m,j}(x, \xi)
\]

where \( \sum_{j=1}^{m-k} a_{m,j}(x, \xi) \) is homogeneous of degree \( m-j \). Likewise we define \( \text{FIO}_{m,k}(\eta) = \text{FIO}_m(\eta)/\text{FIO}_{-k-1}(\eta) \) the formal Fourier integral operators of order \( m \) with canonical relation graph(\( \eta \)) modulo those of order \( -k-1 \). In any local chart \( \Omega \) we can think of an element of \( \text{FIO}_{m,k}(\eta) \) as the equivalence class of an operator of the form (2.1) where \( a(x, y, \theta) \) has a finite expansion \( \sum_{j=1}^{m-k} a_{m,j}(x, y, \theta) \) where \( m_0 = m - \frac{1}{2}(N-n) \). Of course \( \text{FIO}_{m,k}(e) = \psi DO_{m,k} \), where \( e \) is the identity. Finally, we let

\[
\text{FIO}_{m,k} = \bigcup_{\eta \in \mathscr{D}(T^*M, \Omega)} \text{FIO}_{m,k}(\eta).
\]

Notice that if \( m=0 \) then composition is well defined. Indeed, let \([A], [B] \in \text{FIO}_{0,k}\) and let \( A \) and \( B \) be representatives of these classes, then \([A \circ B] \) is independent of the choice of representatives since if \( R_1, R_2 \in \text{FIO}_{-k-1} \) then \((A + R_1) \circ (B + R_2) = A \circ B + R_1 \circ B + A \circ R_2 + R_1 \circ R_2 \) and \( R_1 \circ B + A \circ R_2 + R_1 \circ R_2 \in \text{FIO}_{-k-1} \). We write \([A] \circ [B] = [A \circ B] \). Of course if \([A], [B] \in \psi DO_{0,k} \) then \([A] \circ [B] \in \psi DO_{0,k} \) as well since in general if \([A] \in \text{FIO}_{0,k}(\eta_1) \) and \([B] \in \text{FIO}_{0,k}(\eta_2) \) then \([A] \circ [B] \in \text{FIO}_{0,k}(\eta_1 \circ \eta_2) \). For \([A], [B] \in \psi DO_{0,k} \) the asymptotic formula for the symbol of \( A \circ B \) gives that the symbol for \([A] \circ [B] \) is

\[
\sum_{j=0}^{k} c_{-j}(x, \xi),
\]

where

\[
c_{-j}(x, \xi) = \sum_{r+s=|a|=j} D_x^r a_{0}(x, \xi) \xi^s b_{0}(x, \xi).
\]

Let \( (\text{FIO}_{0,k})_* \) and \( (\psi DO_{0,k})_* \) denote the groups of invertible elements of \( \text{FIO}_{0,k} \) and \( \psi DO_{0,k} \) respectively.

**Proposition 2.1.** The following sequence of groups is exact

\[
I \rightarrow (\psi DO_{0,k})_* \xrightarrow{i} (\text{FIO}_{0,k})_* \xrightarrow{\pi} \mathscr{D}(T^*M, \Omega) \rightarrow e.
\]
Proof. The map \( j \) is the inclusion. The map \( \pi \) which associates to any \([A] \in (\text{FIO}_{0,k})_*\) its canonical relation is by previous remarks a surjective group homomorphism, i.e., \( \pi([A]) = \eta \in \mathcal{D}(T^*M \setminus \partial O) \) if and only if \([A] \in (\text{FIO}_{0,k}\eta)\). The kernel of \( \pi \) consists of elements in \((\text{FIO}_{0,k})_*\) with canonical relation given by the identity, i.e., \( \ker(\pi) = (\psi \text{DO}_{0,k})_* \). \( \square \)

3. ILB-Lie Groups and \( \mathcal{D}(T^*M \setminus O) \)

In this section we describe what is meant by the “Lie group” structure of diffeomorphism groups. This will also be the setting in which the groups of invertible, zero order, pseudodifferential operators and Fourier integral operators are Lie groups.

Definition 3.1. A collection of topological vector spaces \( \{E^s, E^s | s \geq s_0\} \) is called a nested Banach space if

(i) each \( E^s, s \geq s_0, s \neq \infty \) is a Banach space;

(ii) for each \( s \geq s_0 \) there are linear continuous dense inclusions \( E^{s+1} \subset E^s \);

(iii) \( E^s = \bigcap_{s \geq s_0} E^s \) with the inverse limit topology.

Note that \( E^s \) is a graded Fréchet space [9, p. 133]. Since every Fréchet space is an inverse limit of Banach spaces, one can always consider the nested Banach space associated to a given Fréchet space. Nested Banach spaces serve as models for nested manifolds.

Definition 3.2. A collection of topological spaces \( \{M^s, M^s | s \geq s_0\} \) is called a nested Banach manifold modeled on the nested Banach space \( \{E^s, E^s | s \geq s_0\} \) if

(i) each \( M^s, s \geq s_0, s \neq \infty \), is a Banach manifold of class \( C^{\infty} \) modeled on \( E^s \), where the order of differentiability \( k(s) \) is strictly increasing as \( s \to \infty \);

(ii) for each \( s \geq s_0, s \neq \infty \), there exist dense inclusions \( M^{s+1} \subset M^s \) of class \( C^{\infty} \);

(iii) \( M^s = \bigcap_{s \geq s_0} M^s \) with the inverse limit topology;

(iv) if \( (U^s, \varphi^s) \) is a chart on \( M^s, s \geq s_0, s \neq \infty \), then \( (U^s \cap M^t, \varphi^s|_{U^s \cap M^t}) \) is a chart on \( M^t \) for all \( t \geq s \).

Note that condition (iv) is stronger than (ii). It says that in spite of the fact that \( M^t \) is not a submanifold of \( M^s \) for \( t > s \), charts on \( M^t \) do induce charts on \( M^s \). In applications, this hypothesis becomes essential in proofs of regularity theorems. An important consequence of the definition is that \( M^s \) is a Fréchet manifold, modeled on \( E^s \); the charts are \( (U^s \cap M^s, \varphi^s|_{U^s \cap M^s}) \) for any \( s \geq s_0 \). (The definition we adopt for \( C^k \) maps between Fréchet spaces is the following: \( D^k f(x) (h_1, ..., h_k) \) is jointly continuous in \( x, h_1, ..., h_k \); see [9] for details.)

Nested Banach spaces and manifolds are also called ILB (inverse limit of Banach) spaces and manifolds; this terminology is due to Omori [15] and emphasizes \( E^s \) and \( M^s \) respectively. In our point of view, however, the study of \( M^s \) is emphasized, the properties of \( M^s \) being corollaries. To prove the theorems about \( M^s \), one can take different points of view. One point of view is to attempt to systematically use the powerful Nash-Moser implicit function theorem in the tame category [9]. For many purposes, however, a second point of view is often useful. One applies the ordinary implicit function theorem on \( M^s \) and proves in addition a
regularity theorem that allows the passage \( s \to \infty \), without shrinking the interval of
time existence, or the neighborhood in question, to a point. This approach enables
one to obtain results just as good as with the Nash-Moser theorem; [7, 13]. Of
course the applications of the two points overlap but are by no means equal.

**Definition 3.3.** Let \( \{ M^t, M^s \}_{s \geq s_0} \) and \( \{ N^t, N^s \}_{t \geq t_0} \) be two nested manifolds. A
map \( f : M^s \to N^s \) is a \( C^k \) nested map, \( k \geq 0 \), if there is a \( t_f \geq t_0 \) such that for every \( t \geq t_f \) there exists \( s(t) \geq s_0 \) and a \( C^k \)-extension \( f' : M^{00} \to N^t \) of \( f \). A map \( f : M^s \to N^s \) is smooth, or \( C^\infty \) nested, if it is \( C^k \) nested for every \( k \geq 0 \). (Here, \( t_f \) is allowed to
depend on \( k \).) A map \( f : M^s \to N^s \) is a \( C^k \) nested diffeomorphism, \( 0 \leq k \leq \infty \), if it has a
\( C^k \) nested inverse \( g : N^s \to M^s \).

In this definition note that \( k \leq k(t_f) \), \( k \leq k(s(t_f)) \), since the order of smoothness
of the manifold must exceed or be equal to \( k \). Also note that in spite of the fact that
\( M^t \) and \( N^t \) are in general not \( C^\infty \) manifolds, we can define \( C^\infty \) maps between the
nested manifold \( \{ M^s, M^t \}_{s \geq s_0} \), \( \{ N^t, N^s \}_{t \geq t_0} \).

Definitions 3.2 and 3.3 define the category of nested manifolds. We shall be
concerned in this paper exclusively with infinite dimensional nested manifolds. Let
us recall the standard example. Let \( M \) and \( N \) be smooth finite dimensional
manifolds with \( M \) compact, possibly with boundary and \( N \) boundaryless. Let
\( C^\infty(M, N) \), \( H^s(M, N) \) denote the \( C^\infty \) and Sobolev class \( H^s \) maps from \( M \) into \( N \),
\( s > (\dim M)/2 \). \( H^s(M, N) \) is a \( C^\infty \) Hilbert manifold whose tangent space at \( f \) equals

\[
T_f(H^s(M, N)) = \{ X \in H^s(M, TN) | \tau_N \circ X = f \}
\]

where \( \tau_N : TN \to N \) is the canonical tangent bundle projection. Thus \( T_f(H^s(M, N)) \)
consists of \( H^s \)-vector fields over \( f \in H^s(M, N) \). A chart at \( f \) in \( H^s(M, N) \) is given by
the exponential map of a Riemannian metric on \( N \) [6, 16, 14]. In this way
\( \{ C^\infty(M, N), H^s(M, N) | s > (\dim M)/2 \} \) is a nested Hilbert manifold.

An example of a smooth nested map often encountered is the following. Let
\( H^s(\Lambda^k(M)) \) denote the Hilbertizable space of all \( H^s \) \( k \)-forms on the compact
manifold \( M \) and let \( C^\infty(\Lambda^k(M)) = \Omega^k(M) \) be the smooth sections of the bundle of
\( k \)-forms \( \Lambda^k(M) \). Then for \( \alpha \in \Omega^k(M) \) define the map \( \psi_{\alpha} : C^\infty(M, N) \to \Omega^k(M) \) by

\[
\psi_{\alpha}(f) = f^* \alpha.
\]

This map is a \( C^\infty \) nested map between
\( \{ C^\infty(M, N), H^s(M, N) | s > \dim M/2 + 1 \} \) and \( \{ \Omega^k(M), H^s(\Lambda^k(M)) | s > (\dim M)/2 \} \)
since \( \psi_{\alpha} : H^{\ast + 1}(M, N) \to H^s(\Lambda^k(M)) \) is \( C^\infty \); see [6].

In these examples the manifolds \( M^s, N^t \) were \( C^\infty \) Banach manifolds. This is not
the case for the manifold structures of the groups of invertible pseudodifferential
and Fourier integral operators, as we shall see in the course of this paper and the
forthcoming paper [2]. The definition of the topology will impose restrictions on the
order of differentiability.

With the concept of nested manifolds and nested maps one can define tangent
bundles, bundles of \( k \)-forms, nested Lie groups and nested Lie algebras, just by
following the standard definitions. But as is well-known, diffeomorphism groups
have a much finer structure than that of a nested Lie group. The following
definition weakens Omori's concept of an ILB-Lie group which is adapted to the
only known examples, the diffeomorphism groups. In addition, as we shall see, this
definition encompasses also the Lie group structures of invertible pseudodifferential
and Fourier integral operators.
Definition 3.4. An ILB-Lie group \( \{G^\alpha, G^\beta|s \geq s_0\} \) is a nested manifold with \( G^\alpha, G^\beta \) topological groups (in the manifold topology), satisfying the following additional conditions:

(i) the group multiplication \( G^\alpha \times G^\alpha \to G^\alpha \) can be extended to a \( C^k \)-map \( G^{s+k} \times G^s \to G^s \) for every \( s \) satisfying \( k \leq k(s) \);

(ii) the inversion mapping \( G^\alpha \to G^\alpha \) can be extended to a \( C^k \)-map \( G^{s+k} \to G^s \) for every \( s \) satisfying \( k \leq k(s) \);

(iii) right translation \( R_g \) by \( g \in G^s \), \( R_g(h) = hg \), is a \( C^{(k)} \)-map, \( R_g: G^s \to G^s \).

If \( G^s \) is the group \( \mathcal{D}(M) \) of \( H^s \) diffeomorphisms of a compact manifold, then \( k(s) = \infty \) and the nested Lie group \( \{ \mathcal{D}^\alpha(M), \mathcal{D}^\beta(M)|s \geq (\dim M)/2 + 1\} \) satisfies the following additional property:

The tangent map of right translation

\[
TR: g^{s+k} \times G^s \to TG^s, \quad TR(g, \xi) = T_gR_g(\xi),
\]

where

\[
g^{s+k} = T_gG^{s+k}, \quad \text{is a } C^k \text{ map.}
\]

We shall point out below what consequences such a property has in the definition of Lie algebras. In the general case, when \( G^s \) is only of class \( k(s) \), the tangent bundle is of class \( C^{(k-1)} \) and one expects the loss of a derivative. Indeed, from property (i) it follows easily that \( TR \) is of class \( C^{k-1} \).

Note that condition (iv) in Definition 3.2 is equivalent in the case of ILB-Lie groups to the following: there are open neighborhoods \( U \) of \( e \) in \( G^\alpha \) and \( V \) of \( 0 \) in \( g^\alpha \) and a diffeomorphism \( \varphi: U \to V \) such that \( \varphi|_{U \cap G^\alpha} \) is a diffeomorphism of \( U \cap G^\alpha \) in the manifold structure of \( G^\alpha \) with the open subset \( V \cap g^\alpha \) in \( g^\alpha \) for all \( s \geq s_0 \).

Proposition 3.5. Let \( \{G^\alpha, G^\beta|s \geq s_0\} \) be an ILB-Lie group and let \( g^\alpha = T_gG^\alpha \). For \( \xi \in g^\alpha \), let \( X_\xi \) denote the right invariant vector field \( X_\xi(g) = T_gR_g(\xi(\xi)) \) on \( G^\alpha \). The operation

\[
[\xi, \eta] = -[X_\xi, X_\eta](e)
\]

for \( \xi, \eta \in g^{s+2}, \xi, \eta \in g^{s+2} \) is a bilinear, continuous, antisymmetric map \( g^{s+2} \times g^{s+2} \to g^{\min(s,t)} \), it satisfies the Jacobi identity in \( g^{\min(s,t,r)} \) for elements in \( g^{s+4}, g^{s+4}, g^{s+4} \).

Proof. Since \( TR: g^{s+k} \times G^s \to TG^s \) is \( C^1 \), \( X_\xi \) is a \( C^1 \) vector field on \( G^s \). Thus, on \( G^{\min(s,t)} \), \([X_\xi, X_\eta]\) is a well-defined vector field on the densely embedded manifold \( G^{\min(s,t)} \); this vector field is of class \( C^0 \). All other properties are easily verified. The minus sign in the definition of the bracket appears since the Lie algebra bracket is usually thought of as being defined by left invariant vector fields, an operation that does not have enough smoothness in the present context.

Motivated by this proposition, we state the following

Definition 3.6. An ILB-Lie algebra is a nested Banach space \( \{g^\alpha, g^\beta|s \geq s_0\} \) with bilinear, continuous antisymmetric maps \( g^{s+2} \times g^{t+2} \to g^{\min(s,t,r)} \) for all \( s, t \geq s_0 \) which satisfy the Jacobi identity on \( g^{\min(s,t,r)} \) for elements in \( g^{s+4} \times g^{s+4} \times g^{s+4} \).

The classical examples of ILB-Lie groups are the diffeomorphism groups. The ILB-Lie algebra of the ILB-Lie groups \( \{\mathcal{D}^\alpha(M), \mathcal{D}^\beta(M)|s \geq (\dim M)/2 + 1\} \) is \( \{\mathcal{X}(M), H^s(TM)|s \geq (\dim M)/2 + 1\} \), where \( \mathcal{X}(M) \) denotes the \( C^\infty \) vector fields on
$M$ and $H'(T M)$ the $H'$ vector fields on $M$. In this case, as already mentioned, each $D^k (M)$ is a $C^\infty$ Hilbert manifold. In Proposition 3.5, all orders of differentiability drop by one, i.e. we get maps $g^{s+1} \times g^{s+1} \to g^{s+1}$, etc; this is due to the fact that $TR: g^{s+1} \times g^{s+1} \to T G'$ is in this particular case $C^k$ and not merely $C^{k-1}$. The Lie algebra bracket is minus the usual Lie bracket operation of vector fields [7, 15]. If the models of all $G'$ in Definition 3.4 are Hilbert spaces, $\{G'^s, G'^s, s \geq s_0\}$ is called an ILH-(inverse limit of Hilbert) Lie group. In particular, all diffeomorphism groups are ILH-Lie groups. In the course of this paper we shall often refer to $G'^s$ or $G'$ as Lie groups and understand that we mean $\{G'^s, G'^s, s \geq s_0\}$ as ILB or ILH-Lie groups.

3.7. The Lie group $D^k (T^* M \setminus O)$. The group $D^k (T^* M \setminus O)$ of diffeomorphisms of the cotangent bundle $T^* M$ minus the zero section, preserving the canonical one-form $\theta$ of $T^* M$, will play a crucial role in the definition of a Lie group structure for $(FIO)_\omega$. Referring to the prior discussion of manifolds of maps, it should be noted that this group has an added difficulty in that $T^* M \setminus O$ is not compact, so the standard results on diffeomorphism groups do not apply in a straightforward manner. We shall review here, following [17], how $D^k (T^* M \setminus O)$ is made into a Lie group with Lie algebra $\mathfrak{d}^k (T^* M \setminus O) = \{ H \in \mathfrak{h}^{k-1} (T^* M \setminus O, \mathbb{R}) | H \text{ homogeneous of degree one} \}$ with the Poisson bracket as Lie algebra bracket, $s > \dim M + 1/2$. The main idea is that $\mathfrak{d}^k (T^* M \setminus O)$ is algebraically isomorphic to the Lie group of all $H^{k+1}$ contact transformations of the cosphere bundle of $M$, which is a compact manifold. We start by recalling the relevant facts.

The multiplicative group of strictly positive reals $\mathbb{R}^+$ acts smoothly on $T^* M \setminus O$ by $\alpha_x \cdot \tau \cdot x = \tau \cdot x$, $\tau > 0$, $\alpha_x \in T^*_x M$, $\alpha_x \neq 0$. This action is free and proper and therefore $\pi: T^* M \setminus O \to Q = \pi (T^* M \setminus O) / \mathbb{R}^+$ is a smooth principal fiber bundle over $Q$, the cosphere bundle of $M$. Note that $Q$ is compact and odd-dimensional. Also $Q$ carries a canonical contact structure given by projecting the sub-bundle $\text{Ker} \theta \subset T (T^* M)$ down to a sub-bundle $E \subset T Q$ by the differential of the projection $\pi: T^* M \to Q$. A contact one-form on $Q$ is a one-form $\alpha$ on $Q$ with $\text{Ker} \alpha = E$. $Q$ carries no canonical contact one-form but for each global section $\sigma: Q \to T^* M \setminus O$ we can define an exact contact one-form $\theta_\sigma$ on $Q$ by $\theta_\sigma = \sigma^* \theta$. Such global sections exist in abundance; for example, any Riemannian metric on $M$ identifies $T^* M$ with $T M$ and $Q$ with the unit sphere bundle. Then the usual inclusion of the sphere bundle into $T M$ gives a section $\sigma$. The section $\sigma$ is uniquely determined by a smooth function $f_\sigma: T^* M \setminus O \to \mathbb{R}$, defined by $\sigma (\pi (x, \xi)) = f_\sigma (\alpha_x, \xi)$. In other words, $f_\sigma$ measures how far from the section $\sigma$ an element $x \in T^* M \setminus O$ lies. The function $f_\sigma$ is homogeneous of degree $-1$ and $\pi^* \theta_\sigma = f_\sigma \theta$.

An $H^{k+1}$ contact transformation on $Q$ is a diffeomorphism $\varphi \in \mathcal{D}^{k+1} (Q)$ such that for any two sections $\alpha, \zeta: Q \to T^* M \setminus O$, there exists an $H^{k+1}$ function $h_\varphi: Q \to \mathbb{R}$ satisfying $\varphi^* \theta_\sigma = h_\varphi \theta_\sigma$. Equivalently, $\varphi \in \mathcal{D}^{k+1} (Q)$ is a $H^{k+1}$ contact transformation if and only if for each global section $\sigma$ there exists an $H^{k+1}$ function $h_\sigma: Q \to \mathbb{R}$ such that $\varphi^* \theta_\sigma = h_\sigma \theta_\sigma$. The function $h_\sigma$ is uniquely determined by $\sigma$, namely $h_\sigma = \langle \varphi^* \theta_\sigma, E_\sigma \rangle$, where $E_\sigma$ is the Reeb vector field on $Q$ determined by the contact structure $\theta_\sigma$ and $\langle \cdot, \cdot \rangle$ denotes the pairing between vector fields and one forms. ($E_\sigma$ is the unique vector field satisfying $\langle \theta_\sigma, E_\sigma \rangle = 1$ and $i_{E_\sigma} (d \theta_\sigma) = 0$, where $i_{E_\sigma} (d \theta_\sigma)$ denotes the interior product of $E_\sigma$ with $d \theta_\sigma$, in local coordinates $(x^1, \ldots, x^{n-1}, y^1, \ldots, y^{n-1}, t)$ on $Q$, where $\theta_\sigma = \sum_{i=1}^{n-1} y^i dx^i + dt$ we have $E_\sigma = \frac{\partial}{\partial t})$. 

\[\text{...} \]
Therefore the group of $H^{s+1}$-contact transformations on $Q$ is isomorphic to the group
\[ \text{Con}^{s+1}_+(Q) = \{(\varphi, h) \in D^{s+1}_0(Q) \times H^{s+1}(Q, \mathbb{R}; \mathcal{O}) | \varphi^* \theta_e = h \theta_e \} \]
for any fixed but arbitrary global section $\sigma$, where $D^{s+1}_0(Q) \times H^{s+1}(Q, \mathbb{R}; \mathcal{O})$ is the semidirect product of the Lie groups $D^{s+1}_0(Q)$ and $H^{s+1}(Q, \mathbb{R}; \mathcal{O})$ as a multiplicative group) with composition law $(\varphi_1, h_1) \cdot (\varphi_2, h_2) = (\varphi_1 \circ \varphi_2, h_2(h_1 \circ \varphi_2))$. Omori [15] has shown that $\text{Con}^{s+1}_+(Q)$ is a closed Lie subgroup of the Lie group $D^{s+1}_0(Q) \times H^{s+1}(Q, \mathbb{R}; \mathcal{O})$. The Lie algebra of $D^{s+1}_0(Q) \times H^{s+1}(Q, \mathbb{R}; \mathcal{O})$ is the semidirect product $\mathfrak{d}^{s+1}_0(Q) \times \mathfrak{h}^{s+1}(Q, \mathbb{R}; \mathcal{O})$ of $H^{s+1}$-vector fields and $H^{s+1}$ functions, with bracket $(X, f), (Y, g) = ([X, Y], X(g) - Y(f))$. The Lie algebra of $\text{Con}^{s+1}_+(Q)$ is $\mathfrak{d}^{s+1}_0(Q) \times \mathfrak{h}^{s+1}(Q, \mathbb{R}; \mathcal{O}) | L_q \theta_e = g \theta_e \}$ (denoting the Lie derivative along the vector field $Y$).

In [17, Theorem 4.1] it is shown that the group $D^{s+1}_0(T^*M; \mathcal{O})$ is isomorphic (as a group) to the Lie group $\text{Con}^{s+1}_+(Q)$. The isomorphism is given by $\Phi: D^{s+1}_0(T^*M; \mathcal{O}) \rightarrow \text{Con}^{s+1}_+(Q)$, $\Phi(\eta) = (\varphi, h)$ where $\varphi$ is defined by $\circ \pi = \pi \circ \eta$ and $h$ by $h \pi = (f_\eta \circ \pi) f_\eta$; $\sigma(\pi(x)) = f_\eta(x_\eta) x_\eta$. The inverse of $\Phi$ is given by $\Phi^{-1}(\varphi, h) = (\varphi \circ \pi, (h \circ \pi) \cdot f_\eta)$.

Since $\text{Con}^{s+1}_+(Q)$ and $\text{Con}^{s+1}_+(Q)$ are isomorphic as ILH-Lie groups for any two global sections $\sigma$ and $\zeta$, the isomorphism $\Phi$ determines an ILH-Lie group structure on $D^{s+1}_0(T^*M; \mathcal{O})$ which is independent of $\sigma$, (or independent of the Riemannian metric if $\sigma$ is induced from such). Furthermore, the Lie algebra of $D^{s+1}_0(T^*M; \mathcal{O})$, $\mathfrak{d}^{s+1}_0(T^*M; \mathcal{O}) = \{ Y \in \mathfrak{d}^{s+1}_0(T^*M; \mathcal{O}) | L_q = 0 \}$ is isomorphic to $\mathfrak{d}^{s+1}_0(T^*M; \mathcal{O}) \times \mathfrak{h}^{s+1}(T^*M; \mathcal{O})$ $\{ H \in H^{s+1}(T^*M; \mathcal{O}) | h \text{ homogeneous of degree one} \}$. This is because $L_q \theta = 0$ if and only if $Y$ is globally Hamiltonian, homogeneous of degree zero, with Hamiltonian function $H = \theta(Y)$, homogeneous of degree one. Moreover, the Lie algebras $\mathfrak{d}^{s+1}_0(T^*M; \mathcal{O})$ and $\mathfrak{d}^{s+1}_0(T^*M; \mathcal{O})$ are isomorphic via $T_q \Phi: \mathfrak{d}^{s+1}_0(T^*M; \mathcal{O}) \rightarrow \mathfrak{d}^{s+1}_0(T^*M; \mathcal{O})$. Explicitly, $T_q \Phi(X_q) = (X, k)$, where $X$ is uniquely defined by $T_q \circ X_q = X \circ \pi$ and $k$ by $k \pi = \{ f_\eta, H \} / f_\eta$; $(\cdot, \cdot)$ is the canonical Poisson bracket on $T^*M$. The map $H \rightarrow H \circ \sigma$ is an isomorphism from $\mathfrak{d}^{s+1}_0(T^*M; \mathcal{O})$ onto $H^{s+1}(Q, \mathbb{R})$, with inverse $j$: $H^{s+1}(Q, \mathbb{R}) \rightarrow \mathfrak{d}^{s+1}_0(T^*M; \mathcal{O})$, where $j(f)$ is the extension to $T^*M; \mathcal{O}$ by homogeneity of degree one of $f \circ \pi^{-1}: \sigma(Q) \subset (T^*M; \mathcal{O}) \rightarrow \mathbb{R}$, for $f \in H^{s+1}(Q, \mathbb{R})$. The composition of these two isomorphisms with $T_q \Phi^{-1}$ gives an isomorphism

\[ F: \mathfrak{d}^{s+1}_0(Q) \xrightarrow{T_q \Phi^{-1}} \mathfrak{d}^{s+1}_0(T^*M; \mathcal{O}) \xrightarrow{j^{-1}} H^{s+1}(Q, \mathbb{R}) \]

$F(X, k) = \theta_e(X)$. In the condition $L_q \theta_e = k \theta_e$, the function $k$ is uniquely determined by $X$, namely, $k = E_q(\theta_e(X))$. From this it follows easily that $F$ is continuous and hence an isomorphism between $\mathfrak{d}^{s+1}_0(Q)$ and $H^{s+1}(Q, \mathbb{R})$ (note the gain of one derivative). We see thus once again that $\text{Con}^{s+1}_+(Q)$ and $\text{Con}^{s+1}_+(Q)$ are isomorphic as ILH-Lie groups for any two global sections $\sigma, \zeta: Q \rightarrow T^*M; \mathcal{O}$, since both are modelled on $H^{s+1}(Q, \mathbb{R})$.

Defining the Hilbert space structure of $\mathfrak{d}^{s+2}(T^*M; \mathcal{O})$ as the one induced by the isomorphism $j$: $H^{s+2}(Q, \mathbb{R}) \rightarrow \mathfrak{d}^{s+2}(T^*M; \mathcal{O})$, it follows that $H^{s+2}(Q, \mathbb{R})$, $\mathfrak{d}^{s+2}(T^*M; \mathcal{O})$, $\mathfrak{d}^{s+1}_0(T^*M; \mathcal{O})$ and $\mathfrak{d}^{s+1}_0(Q)$ are all isomorphic as Hilbert spaces. It is desirable to compare the topology of $\mathfrak{d}^{s+2}(T^*M; \mathcal{O})$ with the strong $C^1$-Whitney topology. Since all elements of $\mathfrak{d}^{s+2}(T^*M; \mathcal{O})$ are $C^1$ by the Sobolev
imbedding theorem, we can define a new topology on \( S^{s+2}(T^*M\setminus O) \) in the following way: a neighborhood of zero consists of all those functions \( H \in S^{s+2}(T^*M\setminus O) \) for which \( dH : (T^*M\setminus O) \rightarrow T^*(T^*M\setminus O) \) is \( C^1 \)-close to zero in the strong \( C^1 \)-Whitney topology. Denote by \( S^{s+2}_w(T^*M\setminus O) \) the space \( S^{s+2}(T^*M\setminus O) \) equipped with this new topology. It is then a routine matter to check that \( j : H^{s+2}(\mathbb{Q}, \mathbb{R}) \rightarrow S^{s+2}_w(T^*M\setminus O) \), or equivalently the identity \( S^{s+2}(T^*M\setminus O) \rightarrow S^{s+2}_w(T^*M\setminus O) \), are continuous with discontinuous inverses, i.e. the new topology is strictly coarser than the original one on \( S^{s+2}(T^*M\setminus O) \). This remark will be useful in the construction of an explicit chart at \( e \) in \( S^{s+1}_w(T^*M\setminus O) \), which we will give in Sect. 4.

Remarks. (1) The gain of one derivative at the Lie algebra level has a corresponding statement in \( S^{s+1}_w(T^*M\setminus O) \). For every \( \eta \in \mathcal{D}_w^{s+1}(T^*M\setminus O) \), \( \tau \circ \eta : (T^*M\setminus O) \rightarrow M \) is of class \( H^{s+1} \), where \( \tau : T^*M \rightarrow M \) is the cotangent bundle projection. Locally, this means that if \( \eta(x, \alpha) = \langle y(x, \alpha) \rangle \), then \( y \) is \( H^{s+2} \) jointly in \( x \) and \( \alpha \). To prove this, note that \( \eta^* \theta = \theta \) is equivalent locally to

\[
\sum_{i=1}^{n} \beta_i \frac{\partial y^i}{\partial \alpha_k} = \alpha_k, \quad \sum_{i=1}^{n} \beta_i \frac{\partial y^i}{\partial x_k} = 0, \quad k = 1, \ldots, n.
\]

Since \( \eta \) is a diffeomorphism of class \( H^{s+1} \), for fixed \( x \), there exists a unique \( \alpha \) such that \( \beta = (0, \ldots, 0, 1, \ldots, 0) \), the \( i \)-th basis vector. For this choice of \( \alpha \), the first relation shows that the \( i \)-th column of the matrix \( \left( \frac{\partial y^i}{\partial x} \right) \) is \( H^{s+1} \). This says that \( y(x, \alpha) \) has all derivatives of order at most \( s+2 \) square integrable except the derivatives involving only \( \alpha \)'s. The second relation is an elliptic equation with \( H^{s+1} \) coefficients of first order in \( y \) regarded as a function of \( \alpha \) only (its symbol maps \( \beta \) \in \( \mathbb{R}^n \), to \( \beta_1 + \cdots + \beta_1 \beta_2 + \cdots + \beta_n \) \in \( \mathbb{R}^n \)) and thus its solution is of class \( H^{s+2} \), i.e. the \( (s+2)nd \) derivative of \( y \) with respect to \( \alpha \) is square integrable and thus \( y \) is of class \( H^{s+2} \).

(2) Let \( \eta \in \mathcal{D}_w^{s+1}(T^*M\setminus O) \) be fiber preserving, i.e. \( \tau^* \eta(\alpha) = \tau^* \eta(\alpha') \) for all \( \alpha, \alpha' \in T^*M\setminus O \). Then \( \eta \) can be extended \( H^{s+1} \)-smoothly to the zero section by \( \eta(0) \circ \eta(\alpha) \), \( y = \tau^* \eta(\alpha) \), \( \alpha \in T^*M\setminus O \). So \( \eta : T^*M \rightarrow T^*M \), \( \eta^* \theta = \theta \) and hence \( \eta = T^*g \) for an \( H^{s+2} \) diffeomorphism \( g : M \rightarrow M \). In particular if \( \tau^* \eta(\alpha) = \alpha \) for all \( \alpha \in T^*M\setminus O \), then \( \eta = e \). From this it follows that the effect of \( \eta \) on base points uniquely determines \( \eta \), i.e. if \( \eta, \eta \in \mathcal{D}_w^{s+1}(T^*M\setminus O) \) satisfy \( \tau^* \eta = \tau^* \eta \), then \( \eta = \eta \) (since \( \tau^* \eta = \tau^* \eta^{-1}(\alpha) = \alpha \)).

(3) For all the function spaces introduced in this section when the index \( s+1 \) or \( s+2 \) is missing we always mean \( C^\infty \) mappings.

4. Global Writing of Pseudodifferential Operators and a Local Section from \( \mathcal{D}(T^*M\setminus O) \) into \( \text{FIo}_{\theta}(\pi) \)

We start by recalling a few facts about phase functions of Lagrangian submanifolds as they are presented in Weinstein [22].

Let \( \pi : B \rightarrow M \) be a submersion and \( N = \{ \alpha \in T^*B | \alpha(v) = 0 \} \) for all \( v \in \text{Ker}(T\pi) \subset TB \subset T^*B \), the conormal bundle to the fibers. Locally, if \( q, (q, q') \), \( (q', q, q') \), \( (q, q', p, p') \) are coordinates on \( M, B, TB, T^*B \) respectively, and \( \pi(q, q') = q \), then
(q, q', O, q') represents $\text{Ker}(\mathcal{T}_0)$, and $(q, q', p, O)$ is $N_n$. Thus the $\omega$-orthogonal bundle $(TN_z)^\omega \subset TT'B$ is generated by the vector fields $\partial/\partial q'$ and it follows that $(TN_z)^\omega \subset TN_z$, i.e., $N_z$ is coisotropic, thus defining by the Frobenius theorem an integrable foliation, denoted by $\mathcal{N}_z^\omega$ of $N_z$ by isotropic submanifolds. If $\dim B = n + k$, $\dim M = n$, the leaf of $\mathcal{N}_z^\omega$ through $z = (q, q', p, O) \in N_z \subset T'B$ is given locally by $(q, \mathbb{R}^4, p, O)$ and globally by $(\beta_z, N_q \pi(y) = \pi(y))$, and if $T_z \pi(v_z) = T_z \pi(w_z)$, then $z, (v_z) = (\beta_z(w_z))$. The quotient space $N_z/\mathcal{N}_z^\omega$, the reduction of $T'B$ by $N_z$, is locally a symplectic manifold with coordinates $(q, p, O)$ and symplectic structure canonically induced from $T'B$.

There exists a canonical map $\zeta: N_z/\mathcal{N}_z^\omega \to T'M$, $\zeta(q, O, p, O) = (q, p)$ which is a local symplectic diffeomorphism. $\zeta$ is injective if and only if the fibers of $\pi$ are connected and is surjective if and only if $\pi$ is surjective. Thus if these conditions on $\pi$ are satisfied $N_z/\mathcal{N}_z^\omega$ is a symplectic manifold symplectically diffeomorphic by $\zeta$ to $T'B$. We assume this to be the case.

A $C^2$ map $\varphi: B \to \mathbb{R}$ is called a global phase function (or Morse family) on $B$, if the submanifold $d\varphi(B) \subset T'B$ is transversal to $\mathcal{N}_z$. In the above local coordinates, if $d\varphi(q, q') = (q, q', d_1 \varphi(q, q'), d_2 \varphi(q, q'))$ this means that the $k \times (n + k)$ matrix $(d_1 d_2 \varphi(q, q'), d_2 d_2 \varphi(q, q'))$ has rank $k$. From general theorems on reduction [22, 1, p. 417] it follows then that the reduced manifold $d\varphi(B)_z = (d\varphi(B) \cap N_z)/\mathcal{N}_z^\omega$ is Lagrangian in $N_z/\mathcal{N}_z^\omega \cong T'B$. In the terminology of Fourier integral operators $\varphi$ is indeed a global phase function for $d\varphi(B)_z$ as defined in Sect. 2. It is well known that every Lagrangian submanifold in $T'B$ admits local phase functions [8, 10, 22].

For the chart construction that follows one more ingredient is needed.

A linearization (Bokobza [4]) on a manifold $M$ is a smooth map $v: \Omega \subset M \times M \to T'M$, where $\Omega$ is an open neighborhood of the diagonal in $M \times M$, such that

1) $v(x, y) \in T_zM$, for all $(x, y) \in \Omega$
2) $v(x, x) = \mathcal{O}_x \subset T_xM$, for all $x \in M$
3) for all $x \in M$ the tangent map of $y \mapsto v(x, y)$, which is a linear map from $T_xM$ to $T_zM$, is the identity for $y = x$.

Note that a linearization always exists by choosing a Riemannian metric and putting $v = \text{Exp}^{-1}$, where $\text{Exp} = (\tau, \exp): TM \to M \times M$ is a diffeomorphism from a neighborhood $U$ of the zero section of $TM$ onto a neighborhood $\Omega$ of the diagonal in $M \times M$.

Now let in the prior construction $B = T'M \times M$, and $\pi = \tau \times e: T'M \times M \to M \times M$, i.e., $\pi(x, y) = (x, y)$ and let $v: M \times M \to T'M$ be a linearization of $M$.

**Proposition 4.1.** The function $\varphi_0: T'M \times M \to \mathbb{R}$ defined by

$$\varphi_0(x, y) = \mathcal{O}_x \cdot (v(x, y))$$

is a smooth global phase function for graph($e$) in $T'M \times T'M$.

**Proof.** $B$ is isomorphic to the pullback bundle $p_1^*(T'M)$, where $p_1: M \times M \to M$ is the first projection, and $\mathcal{T}_0 = \mathcal{T}_0 \times \mathcal{I}$. Hence $\ker(\mathcal{T}_0) = (\ker(\mathcal{T}_0^*) \times O_{TM} = V_{TT'M}$ $\times O_{TM} \subset TT'M \times T'M$, where $O_{TM}$ is the zero section in $TM$ and $V_{TT'M}$ is the vertical subbundle of $TT'M$. Furthermore, $N_z = H_{TT'M} \times T'M \subset T'M \times T'M$, where the horizontal subbundle $H_{TT'M} = \{I_{\mathcal{O}_z} \in T'M | I_{\mathcal{O}_z}(V_{TT'M}) = 0\}$
is the annihilator of $V_{TT^*M}$. In local coordinates:
\[ \ker(T\nu) = \{(x, \alpha, O; A; y, O) \} \quad \text{and} \quad N_\alpha = \{(x, \alpha, O; y, \eta) \}. \]
Recall that $v(x, y)$ is a diffeomorphism from a neighborhood $\Omega$ of $d$ in $M \times M$ onto a neighborhood $U$ of $O_{TM}$ in $TM$. So locally $v(x, y) = (x, v(x, y))$ where $v(x, y) = 0$ if and only if $x = y$. We are working in the open submanifold $\pi^{-1}(\Omega)$ in $T^*M \times M$.
Note that this submanifold contains whole fibers of $T^*M$.

To show that $\varphi_0$ is a Morse family we work in local coordinates where $\varphi_0(x, \alpha; y) = (v(x, y))$ and straightforward calculations show that
\[ d\varphi_0(x_\alpha, y) = \{(x, \alpha, \alpha \circ d_1 v'(x, y), v(x, y)), (y, \alpha \circ d_2 v'(x, y)) \} \in T^*_{x_\alpha,y}(T^*M \times M) \]
and that $T_{\varphi_0(x, y)}(N_\alpha)$ is of the form
\[ T_{\varphi_0(x_\alpha, y)}(N_\alpha) = \{(X, A, W, O; Y, b) \} \subset T_{\varphi_0(x_\alpha, y)}(T^*T^*M \times T^*M) \].
Furthermore $\text{Im}(T_{x_\alpha,y}, d\varphi_0)$ is locally of the form (put $V_x = (x, \alpha, X, A)$, $Y_y = (y, Y)$)
\[ (T_{x_\alpha,y}, d\varphi_0)(V_x, Y_y) = \{(X, A, \alpha \circ d_1 v'(x, y) \cdot X + d_2 \alpha \cdot d_1 v'(x, y) \cdot Y), d_1 v'(x, y) \cdot X + d_2 \alpha \cdot d_1 v'(x, y) \cdot Y \} \].

By property 3) of $v$, this shows that for all $(x_\alpha, y) \in T^*M \times M$ such that $d\varphi_0(x_\alpha, y) \in N_\alpha$ we have
\[ \text{Im}(T_{x_\alpha,y}, d\varphi_0)(V_x, Y_y) + T_{\varphi_0(x_\alpha, y)}(N_\alpha) = T_{\varphi_0(x_\alpha, y)}(T^*T^*M \times T^*M) \]
\[ \text{i.e. } d\varphi_0(T^*T^*M \times M) \text{ is transversal to } N_\alpha. \]

The induced foliation $\mathcal{N}_\alpha^\perp$ is of the form $\mathcal{N}_\alpha^\perp = \{(x, \Omega, \xi, O; y, \eta) \}$ and hence $\mathcal{N}_\alpha^\perp \subset \mathcal{N}_\alpha^\perp \subset \mathcal{N}_\alpha^\perp$. So $d\varphi_0(T^*M \times M)|_{N_\alpha} = (d\varphi_0(T^*T^*M \times M) \cap N_\alpha)|_{\mathcal{N}_\alpha^\perp} = \{(x, x, \alpha \circ d_2 v'(x, x), \alpha) \} \subset \mathcal{N}_\alpha^\perp \subset \mathcal{N}_\alpha^\perp$ since $v'(x, y) = 0 \iff x = y$ and $d_1 v'(x, x) = -\text{id}$. \[ \square \]

**Remarks** 1). Take $B = (T^*M \times O) \times M$ and $\varphi_0$ as in Proposition 4.1. The proof shows that
\[ d\varphi_0((T^*M \times O) \times M)|_{N_\alpha} = \{(x, x, -\alpha, \alpha) \}, \]
i.e. $d\varphi_0((T^*M \times O) \times M)|_{N_\alpha} = \text{graph}(\alpha) \subset (T^*M \times O) \times (T^*M \times O), \alpha \ominus \omega)$. \[ 2) \pi : (T^*M \times O) \times M \to M \times M \text{ has not connected fibers if and only if } \dim M = 1. \]

But in this case we have $M = S^1$ and $T^*M \times O = (\mathbb{R} \times O) \times S^1$ and the reduced manifold is $d\varphi_0(T^*M \times O) |_{N_\alpha} = \{(x, x, -\alpha, \alpha) : x > 0 \} \cup \{(x, x, -\alpha, \alpha) : x < 0 \}$ which is a Lagrangian submanifold in
\[ N_\alpha / \mathcal{N}_\alpha^\perp = (S^1 \times \mathbb{R} \times S^1 \times \mathbb{R}) \cup (S^1 \times \mathbb{R} \times S^1 \times \mathbb{R}) \].

In terms of a linearization on $M$ we are able to write a global formula for pseudodifferential operators on a manifold, [4].

Let $v : \Omega \subset M \times M \to TM$ be a linearization of $M$ and let $U \subset TM$ be the neighborhood of the zero section in $TM$ where $v : \Omega \to U$ is a diffeomorphism. For any $x \in M$ let $\Omega_x = \{ y \in M | (x, y) \in \Omega \}$ and $U_x = U \cap T_x M$, then $v_x = v(x, \cdot) : \Omega_x \to U_x$ is a diffeomorphism. The tangent map $T_x v_x : T_x M \to T_x U_x$ is the identity for $y = x$. Its dual $T_x^* v_x : T_x^* M \to T_x^* U_x$ induces a linear map of densities $|\Lambda^*(T_x v_x)| : |\Lambda^*(T_x^* M)| \to |\Lambda^*(T_x^* U_x)|$ for $y \in \Omega_x$.\]
Choose a density $\zeta_x$ on $T_x^*M$, i.e. $\zeta_x \in |A^*T_x^*M|$ and pull it back to $\Omega_x$ by means of $|A^*T_x^*v_x|$, i.e.

$$(v_x^*\zeta_x)(y) = |A^*T_x^*v_x|^{-1}(\zeta_x) \in |A^*T_x^*\Omega_x| = |A^*T_x^*M|$$

for $y \in \Omega_x$. In local coordinates $\partial_{x_1}, \ldots, \partial_{x_n}$ on $T_x^*M$, and $y_1, \ldots, y_n$ on $\Omega_x$ we can choose $\zeta_x = dx = dx_1 \wedge \ldots \wedge dx_n$ and $v_x^*\zeta_x = v_x^*dx$.

$$= |\det(dx, dy) v_x| dy_1 \wedge \ldots \wedge dy_n = \det \left| \frac{\partial u(x, y)}{\partial y} \right| dy.$$

Now, let $d\xi \in |A^*T^*_xM|$ be the dual basis of $\zeta_x \in |A^*T_x^*M|$. Considering $d\xi$ as a density on $T^*_xM$ we have $d\xi \otimes v_x^*d\xi$ is a density on $T^*_xM \times \Omega_x$.

Let $f: T^*M \times M \to \mathbb{C}$, $f(x, y)$ be such that $\text{supp}_x f \subset \Omega_x$ for all $x \in T^*M$. Then the integral

$$\int_{\Omega_x} f(x, y) |\det(dx, dy) v_x| dy$$

is a function on $T^*_xM$ which we can integrate with respect to $d\xi$,

$$\int_{T^*_xM} d\xi \int_{\Omega_x} f(x, y) |\det(dx, dy) v_x| dy \in \mathbb{C}.$$

Let $\chi(x, y)$ be a bump function on $M \times M$ such that $\text{supp} \chi \subset \Omega$ and $\chi \equiv 1$ on a neighborhood of the diagonal. Let $a(x, \xi) \in S^m(\Omega)$ be a classical symbol of order $m$. We define an operator $P$ on $M$ by

$$Pu(x) = (2\pi)^{-n} \int_{T^*_xM} a(x, \xi) d\xi \int_{\Omega_x} \chi(x, y) e^{ia(\xi, y)} u(y) |\det(dx, dy) v_x| dy,$$  \hspace{1cm} (4.1)

where $u \in C^\infty(M)$ and $\varphi_0$ is the global phase function of the diagonal in $T^*M \times T^*M$ defined in Proposition 4.1. The integral (4.1) converges in the oscillatory sense.

Bokobza [4] showed that $P$ is a continuous linear operator from $C^\infty(M)$ to $C^\infty(M)$, and from $\mathcal{D}(M)$ into $\mathcal{D}(M)$, i.e. $P$ is a classical pseudodifferential operator of order $m$ on $M$ in our previous sense.

**Remark.** If $M = \mathbb{R}^n$, a linearization $\nu: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is given by $\nu(x, y) = x - y$, and the phase function $\varphi_0: (\mathbb{R}^n)^* \times \mathbb{R}^n \to \mathbb{R}$ of graph(e), becomes $\varphi_0(x, \xi) = \xi \cdot (x - y)$. So a pseudodifferential operator $P$ on $\mathbb{R}^n$, according to (4.1) can be written as

$$Pu(x) = (2\pi)^{-n} \int \chi^{(x, y)} \varphi_0(x, \xi) u(y) dy d\xi$$  \hspace{1cm} (4.2)

which is indeed the standard form (2.3).

Now by perturbing $\varphi_0$ by “small” functions $H \in \mathcal{S}^{s+2}(T^*M\setminus O)$ we obtain global phase functions for Lagrangian submanifolds close to graph(e) in $((T^*M\setminus O) \times (T^*M\setminus O), \omega \otimes \omega)$.

**Lemma 4.2.** Let $H \in \mathcal{S}^{s+2}(T^*M\setminus O)$ be close to zero and define $\varphi_H: (T^*M\setminus O) \times M \to \mathbb{R}$ by

$$\varphi_H(x, y) = \varphi_0(x, y) + H(x, y).$$

Then there exists an $\eta \in \mathcal{S}^{s+1}(T^*M\setminus O)$ such that $\varphi_H$ is a global phase function for graph($\eta$), i.e. $d\varphi_H((T^*M\setminus O) \times M)_{\eta_x} = \text{graph}(\eta)$.
Proof. The Lagrangian submanifold in \((T^*M \setminus O) \times (T^*M \setminus O)\) generated by \(\varphi_H\) is locally
\[
d\varphi_H((T^*M \setminus O) \times M)_{N_x} = \{(x, y, x \cdot d_1 v'(x, y) + d_1 H(x, x), x \cdot d_2 v'(x, y))v'(x, y) + d_2 H(x, x) = 0\}.
\]
So we have to solve the system of equations for \((x, x)\) when \((y, \beta)\) are given
\[
x \cdot d_2 v'(x, y) = -\beta, \tag{4.3}
\]
\[
v'(x, y) + d_2 H(x, x) = 0. \tag{4.4}
\]
Since we are looking for solutions \((x, x)\) only in the neighborhood \(\Omega\) of \(d\) in \(M \times M\) where \(v\) is a diffeomorphism, we can solve (4.3) for \(x, x = -\beta \cdot d_2 v'(x, y)^{-1}\). The map \(\phi_x : M \rightarrow TM\) defined for every \((x, y) \in \Omega\) by \(\phi_x(x) = v'(x, y) + d_2 H(x, x, -\beta \cdot d_2 v'(x, y)^{-1})\) is for \(H \equiv 0\) transversal to \(O_{TM}\) in \(TM\) and \(y \in \phi_x^{-1}(O_{TM})\). Since \(H \in \mathcal{D}^{s+2}_{\ast}(T^*M \setminus O)\) is close to zero, i.e. \(dH\) is \(C^1\)-close to zero, \(\phi_x\) is still transversal to \(O_{TM}\) and \(\phi_x^{-1}(O_{TM}) \neq 0\). By the implicit function theorem there is a unique \(x \in \Omega\) such that \(\phi_x(x) = 0\). Denote this \(x(y, \beta)\) and let \(v'(x(y, \beta), y) = d_1 v'(x(y, \beta), y) + d_1 H(x(y, \beta), x(y, \beta), x(y, \beta))\). This defines a diffeomorphism \(\eta : (T^*M \setminus O) \rightarrow (T^*M \setminus O)\) such that \(\eta^{-1}(y, \beta) = (x(y, \beta), \xi(y, \beta))\) and (4.4) is satisfied. So \(d\varphi_H((T^*M \setminus O) \times M)_{N_x} = \text{graph}(\eta)\). Since \(d\varphi_H((T^*M \setminus O) \times M)_{N_x}\) is a conic Lagrangian submanifold it follows that \(\eta^\ast \theta = \theta\) i.e. \(\eta \in \mathcal{D}^{s+1}_{\ast}(T^*M \setminus O)\). \(\square\)

Lemma 4.3. The map \(H \rightarrow \eta\) given by Lemma 4.2 is a bijection from a neighborhood \(\mathcal{V}\) of zero in \(\mathcal{S}^{s+2}_{\ast}(T^*M \setminus O)\) onto a neighborhood \(\mathcal{U}\) of the identity in \(\mathcal{D}^{s+2}_{\ast}(T^*M \setminus O)\). The inverse mapping is given as follows: for \(\eta \in \mathcal{D}^{s+1}_{\ast}(T^*M \setminus O)\) define \(H \in \mathcal{S}^{s+2}_{\ast}(T^*M \setminus O)\) by \(H(x) = -x \cdot (v(x, \tau^\ast \eta^{-1}(\beta_x)))\).

Proof. Given \(\eta \in \mathcal{D}^{s+1}_{\ast}(T^*M \setminus O)\), associate to it \(H \in \mathcal{S}^{s+2}_{\ast}(T^*M \setminus O)\) by
\[
H(\beta_z) = -\beta_z \cdot v(z, \tau^\ast \eta^{-1}(\beta_z)). \tag{4.5}
\]
\(H\) is \(H^{s+2}\) because \(\tau^\ast \eta^{-1}\) is \(H^{s+2}\) (Remark 1 at the end of Sect. 3) and it is homogeneous of degree one. With this we build the phase function \(\varphi_H\) by Lemma 4.2
\[
\varphi_H(\beta_z, y) = \varphi_0(\beta_z, y) + H(\beta_z) = \beta_z \cdot v(z, y) - \beta_z \cdot v(z, \tau^\ast \eta^{-1}(\beta_z)).
\]
The Langrangian submanifold in \(((T^*M \setminus O) \times (T^*M \setminus O), \omega \otimes \omega)\) generated by \(\varphi_H\) is
\[
d\varphi_H((T^*M \setminus O) \times M)_{N_x} = \{(x, y, x \cdot d_1 v'(x, z) + d_1 H(x, x), x \cdot d_2 v'(x, z))d_2 H(x, x) = -v'(x, z)\}.
\]
Define \(h : (T^*M \setminus O) \rightarrow (T^*M \setminus O)\) by \(h(\beta_z) = -\beta_z \cdot T_z v(\tau^\ast \eta(\beta_z), z)^{-1}\), where \(T_z v(x, y) = T_z v(x, x) : T_z M \rightarrow T_x M\). If \(\eta\) is close to \(e \in \mathcal{D}^{s+1}_{\ast}(T^*M \setminus O)\) we get \((\tau^\ast \eta(\beta_z), z) \in \Omega\) where \(\Omega\) is the neighborhood of \(\Delta \subset M \times M\) where \(v\) is a diffeomorphism. It is easy to check that \(h\) is an \(H^{s+2}\) diffeomorphism homogeneous of degree one with inverse \(h^{-1}(x_z) = x_z \cdot T_z v(x_z, x_z)^{-1}(x_z)\).

Take \(H \in \mathcal{V} \subset \mathcal{S}^{s+2}_{\ast}(T^*M \setminus O)\) and associate to it \(\eta \in \mathcal{U} \subset \mathcal{D}^{s+1}_{\ast}(T^*M \setminus O)\) according to Lemma 4.2. This defines an \(\tilde{H} \in \mathcal{S}^{s+2}_{\ast}(T^*M \setminus O)\) by (4.5). To show that \(H = \tilde{H}\) it is sufficient to prove \(H + h = \tilde{H} + h\). Using the fact that \(H\) is homogeneous of degree
one and (4.4) we get
\[
(H \circ h)(z, \beta) = H(x, -\beta \circ d_2\nu'(x, z)^{-1}) = \beta \circ d_2\nu'(x, z)^{-1} \cdot \nu'(x, z)
\]
\[
= -\beta \circ d_2\nu'(x, z)^{-1} \cdot d_2H(x, -\beta \circ d_2\nu'(x, z)^{-1})
\]
\[
= H(x, -\beta \circ d_2\nu'(x, z)^{-1})
\]
\[
= (H \circ h)(z, \beta).
\]

Given \( \eta \in \Psi \subset \mathcal{D}'(T^*M) \) define \( H \) by (4.5) and associate to it \( \tilde{\eta} \) according to Lemma 4.2, i.e. \( \tilde{\eta} \) is defined from \( d\omega_H((T^*M) \times M)_{\tilde{\eta}} \). Given \( (z, \beta) \) with 
\[
\beta = \gamma \circ d_2\nu'(x', z) \quad \text{where} \quad x' = \tau^*\tilde{\eta}(z, \beta) \quad \text{is the unique solution of}
\]
\[
d_2H(x', \gamma) = -\nu'(x', z).
\] (4.6)

Let \( \eta^{-1}(z, \beta) = (a(z, \beta), b(z, \beta)). \) Then \( H(z, \beta) = -\beta \cdot \nu'(z, a(z, \beta)) \) and \( d_2H(z, \beta) = -\nu'(z, a(z, \beta)) - \beta \circ d_2\nu'(z, a(z, \beta)) \circ d_2a(z, \beta). \)

Therefore (4.6) becomes
\[
-\nu'(x', a(x', \gamma)) - \gamma \circ d_2\nu'(x', a(x', \gamma)) \circ d_2a(x', \gamma) = -\nu'(x', z). \quad (4.6')
\]

Now let \( x = \tau^*\eta(\beta) \), and replace in (4.6') everywhere \( x' \) by \( x \). Since \( a(x, \gamma) = a(h(z, \beta)) = z \) and \( \gamma \circ d_2\nu'(x, a(x, \gamma)) = \beta \circ d_2\nu'(x, z)^{-1} \circ d_2\nu'(x, z) = \beta \), (4.6) becomes
\[
\beta \circ d_2a(x, \beta \circ d_2\nu'(x, z)^{-1}) = 0 \ . \quad (4.6'')
\]

Since \( h(z, \beta) = (x, \gamma) \), calling \( h^{-1}(x, \gamma) = (a(x, \gamma), c(x, \gamma)) \) it follows \( c(x, \gamma) = \beta \) and (4.6'') becomes \( c(x, \gamma) \circ d_2a(x, \gamma) = 0 \) which is verified since \( h^{-1} \) is homogeneous of degree one and \( (h^{-1})^*\theta(x, \gamma)(A, B) = (c \circ d_1a) \cdot A + (c \circ d_2a) \cdot B = \theta(x, \gamma)(A, B) = \gamma \circ A \) for any vector \((A, B)\). Hence \( x = \tau^*\eta(\beta) \) solves (4.6) follows \( \tau^* \circ \tilde{\eta} = \tau^* \circ \eta \) and by Remark 2 at the end of Sect. 3, we get \( \tilde{\eta} = \eta \). \( \square \)

Twice application of the \( \Omega \)-lemma [16] proves that \( \eta^{-1} \lor H \) is a \( C^\infty \) map. Since the map \( H \lor \eta^{-1} \) is continuous, bijective and has a \( C^\infty \) inverse, it is a \( C^\infty \) diffeomorphism.

Since \( \mathcal{D}'(T^*M) \) is continuously imbedded in \( \mathcal{D}'(T^*M) \) the prior lemmas yield the following theorem.

**Theorem 4.4.** (\( \Psi, \Phi, \mathcal{D}'(T^*M) \)) is a smooth chart at the identity of \( \mathcal{D}'(T^*M) \), where \( \Phi : \Psi \subset \mathcal{D}'(T^*M) \to \mathcal{D}'(T^*M) \) is defined by
\[
\Phi(\eta)(x) = -\alpha_x \cdot v(x, \tau^*\eta^{-1}(x))
\]
for all \( x \in T^*M \).

**Remarks.** (1) This chart depends upon the linearization \( v \). But two different linearizations \( v \) and \( \tilde{v} \) define a diffeomorphism
\[
g : (T^*M) \times M \to (T^*M) \times M, \pi \circ g = \pi, \text{ satisfying } \tilde{\eta}_0 = \varphi_0 \circ g
\]
and therefore they generate the same Lagrange submanifold. This is crucial for our application to Fourier integral operators because these are invariant under change of phase functions as long as they define the same Lagrangian submanifold.

(2) Note that \( \eta \in \Psi \) is \( C^\infty \) if and only if \( \Phi(\eta) \) is \( C^\infty \).
We are now equipped to generalize the global formula (4.1) for pseudodifferential operators on a manifold to give a global formula for Fourier integral operators near the identity. Let \( A \in \text{FIO}_0(\eta) \) with \( \eta \in \mathcal{U} \cap \mathcal{D}(T^*M \setminus O) \) and let \( a(x, \xi) \) be a representative of the classical symbol of \( A \). Let \( \chi(x, y) \) be the bump function used in (4.1), then, modulo smoothing,

\[
Au(x) = (2\pi)^{-n} \int_{T^*M} a(x, \xi) d^\xi \int_{\Omega_x} \chi(x, y) e^{i\gamma(x, \xi) y} \left| \det \frac{\partial^m v}{\partial y^m} \right| dy,
\]

(4.7)

where \( \varphi_H \) is the global phase function of \( \gamma(x, \xi) \), given by Lemmas 4.2 and 4.3.

**Remark.** Let \( M = \mathbb{R}^n \) and take the linearization \( v(x, y) = x - y \). Then for \( H \in \Phi(\mathcal{U}) \cap \mathcal{D}(\mathbb{R}^{2n} \setminus O) \) we get \( \varphi_H(x, y) = (x - y) \cdot \xi + H(\xi) \). Thus a representative of a formal Fourier integral operator \( A \) on \( \mathbb{R}^n \) can be written, according to (4.7), as

\[
Au(x) = (2\pi)^{-n} \int_{T^*M} e^{i(x(x - y) \cdot \xi + H(\xi))} a(x, \xi) u(y) d^y d\xi
\]

(4.8)

which corresponds to the standard form \([5, 10]\).

Finally, we can define a local section \( \sigma \) from \( \mathcal{D}(T^*M \setminus O) \) into \( \text{FIO}_{0, k}\) as follows:

For \( \eta \in \mathcal{U} \cap \mathcal{D}(T^*M \setminus O) \), define \( \sigma(\eta) \in \text{FIO}_{0, k} \) by

\[
\sigma(\eta) u(x) = (2\pi)^{-n} \int_{T^*M} a(x, \xi) d^\xi \int_{\Omega_x} \chi(x, y) e^{i\gamma(x, \xi) y} \left| \det \frac{\partial^m v}{\partial y^m} \right| dy.
\]

(4.9)

where \( \varphi_M(\alpha, y) = \varphi_0(\alpha, y) + H(\alpha) \) and \( H = \Phi(\eta) \), so \( \varphi_M(\alpha, y) = \alpha_y \cdot (v(x, y)) \). Therefore

\[
\sigma(\eta) u(x) = (2\pi)^{-n} \int_{T^*M} a(x, \xi) d^\xi \int_{\Omega_x} \chi(x, y) e^{i\gamma(x, \xi) y} \left| \det \frac{\partial^m v}{\partial y^m} \right| dy.
\]

(4.10)

In other words, \( \sigma(\eta) \) is defined as the Fourier integral operator whose phase function is \( \varphi_{\phi_M} \), the global generating function of \( \gamma(x, \xi) \), and whose amplitude is 1. Recall that \( \Phi(\eta) \) is smooth if \( \eta \) is smooth, in which case \( \sigma(\eta) \) is a well defined Fourier integral operator of order zero. Its principal symbol is 1 (in the trivializations given by the global phase function \( \varphi_{\phi_M} \), hence \( \sigma(\eta) \) is invertible modulo smoothing operators, in particular, \( \sigma(\eta) \in \text{FIO}_{0, k} \) for any \( k \). Notice \( \pi(\sigma(\eta)) = \eta \) for all \( \eta \in \mathcal{U} \cap \mathcal{D}(T^*M \setminus O) \), i.e. \( \sigma \) is a local section.

5. The Lie Group Structure of \( \psi DO_{0, k} \)

The space \( \psi DO_{0, k} \) can be given the structure of a topological group as follows. Let \( v \) be a linearization of \( M \) and \( \varphi_0 : T^*M \times M \to \mathbb{R} \) the global phase function given by \( v \) as in Proposition 4.1. Using \( \varphi_0 \) we can write a representative of \( [P] \in \psi DO_m \) in the form (4.1) where \( a(x, \xi) \sim \sum_{j=0}^{\infty} a_{m-j}(x, \xi) a_{m-j}(x, \xi) \) being a smooth function on \( T^*M \setminus O \), homogeneous of degree \( m-j \). To do this let \( a_{m-j}(x, \xi) \) be the principal symbol of \( [P] \in \psi DO_m \) and let \( P_{m-j} \) be the operator given by (4.1) with total symbol \( a_{m-j}(x, \xi) \), i.e. replace \( a(x, \xi) \) in (4.1) by \( a_{m-j}(x, \xi) \). Then \( [P - P_{m-j}] \in \psi DO_{m-1} \). Let \( a_{m-j}(x, \xi) \) be the principal symbol of \( [P - P_{m-j}] \) and let \( P_{m-j} \) be given by (4.1) with total symbol \( a_{m-j}(x, \xi) \), then \( [P - P_{m-j}] \in \psi DO_{m-2} \). Continue in this fashion.
to get \( a_{m_{-}j}(x, \xi) \) and \( P_{m_{-}j} \) so that \([P - P_{m_{-}j - 1} - \ldots - P_{m_{-}j}] \in \psi \text{DO}_{m_{-}j-1} \) for all \( j \geq 0 \). The sequence \((a_{m}(x, \xi), a_{m-1}(x, \xi), \ldots)\) will be called the symbol of \([P] \) with respect to the phase function \( \varphi_{0} \). Clearly, in this way, we have constructed an isomorphism between \( \psi \text{DO}_{m} \) and the space of sequences \((a_{m}(x, \xi), a_{m-1}(x, \xi), \ldots)\) of functions \( C^{\infty}(T^{*}M\setminus O) \) and is homogeneous of degree \( m_{-}j \). This can be identified to a somewhat simpler space by restricting to a cosphere bundle. Namely, as in Sect. 3 let \( Q = T^{*}M\setminus O/\mathbb{R}^{+} \) be the cosphere bundle and choose \( \sigma : Q \to T^{*}M\setminus O \) a section of the \( \mathbb{R}^{+} \) bundle \( \pi : T^{*}M\setminus O \to Q \). Then by pulling back functions on \( T^{*}M\setminus O \) to \( Q \) by \( \sigma \) we can identify \( \psi \text{DO}_{m} \) with the infinite product \( C^{\infty}(Q) \times C^{\infty}(Q) \times \ldots \) by \([P] \mapsto (a_{m_{-}j} \circ \sigma, a_{m_{-}j-1} \circ \sigma, \ldots)\) where \((a_{m}, a_{m-1}, \ldots)\) is the symbol of \([P] \). Since the product \( C^{\infty}(Q) \times C^{\infty}(Q) \times \ldots \) is a topological space we can endow \( \psi \text{DO}_{m} \) with the topology induced by this identification. It can be seen that the topology is independent of the choices of \( v \) and \( \sigma \) and that composition on \( \psi \text{DO}_{m} \) is continuous in this topology (c.f. Propositions 5.3, 5.4, and 5.5). From this last remark it follows that \( (\psi \text{DO}_{0})_{\psi} \) is a topological group. The problem with this topological group arises when we try to find the spaces \( \psi \text{DO}_{l} \) to make \( (\psi \text{DO}_{0})_{\psi} \) an ILH group. One would want to take an infinite product of various \( H^{s} \) spaces on \( Q \). Unfortunately the infinite product of Hilbert spaces is no longer a Hilbert space. For this reason we are forced to use the spaces \( \psi \text{DO}_{m,k} \) and then consider \( \psi \text{DO}_{m} \) as a direct limit of the ILH spaces \( \psi \text{DO}_{m,k} \) as \( k \to \infty \).

Following the construction above we choose a global phase function \( \varphi_{0} \), coming from the linearization \( v \), and a section \( \sigma : Q \to T^{*}M\setminus O \). Then \( \psi \text{DO}_{m,k} \) is isomorphic to the \((m + k + 1)\)-fold product \( C^{\infty}(Q) \times \ldots \times C^{\infty}(Q) \) by extending the \( j \)th function to \( T^{*}M\setminus O \) by homogeneity of degree \( m_{-}j \), \( 0 \leq j \leq m + k \), and then using \( \varphi_{0} \) to give an identification of \( \psi \text{DO}_{m,k} \) with the finite sequences \((a_{m}(x, \xi), a_{m-1}(x, \xi), \ldots, a_{m-k}(x, \xi))\). We will see momentarily that the induced topology on \( \psi \text{DO}_{m,k} \) is independent of the choices \( v \) and \( \sigma \) and that \( (\psi \text{DO}_{0})_{\psi} \) is a topological group with this topology. First we describe the spaces \( \psi \text{DO}_{m,k} \).

We use \( H^{s} \) structures on the symbol spaces to define \( \psi \text{DO}_{m,k} \). Let \( a_{l}(x, \xi) \) be a smooth function on \( T^{*}M\setminus O \) which is homogeneous of degree \( l \) in \( \xi \). If \( l \geq -n \), \( a_{l}(x, \xi) \) won't be in any of the spaces \( H^{s}(T^{*}M\setminus O) \) because of its growth at infinity. To get around this we restrict to the cosphere bundle \( Q \) by means of \( \sigma \). Namely, define (for \( s > \text{dim}M \))

\[
\|a_{l}(x, \xi)\|_{s} = \|a_{l} \circ \sigma\|_{s},
\]  

(5.1)

where the norm \( \| \cdot \|_{s} \) on the right is the \( H^{s} \) norm of \( a_{l} \circ \sigma \) as a function on the compact manifold \( Q \). Let \( \mathcal{S}_{l}^{\sigma}(T^{*}M\setminus O) \) be the completion of the set of smooth functions on \( T^{*}M\setminus O \), homogeneous of degree \( l \), with respect to this topology. This space can of course be thought of as the space of \( H^{s} \) functions on \( S^{*}M = \sigma(Q) \), extended to \( T^{*}M\setminus O \) by homogeneity of degree \( l \).

Before proceeding, we gather some of the properties of these norms.

**Proposition 5.1.** The Hilbert manifold structure of \( \mathcal{S}_{l}^{\sigma}(T^{*}M\setminus O) \) is independent of the choice of section \( \sigma : Q \to T^{*}M\setminus O \).

**Proof.** Let \( \bar{\sigma} : Q \to T^{*}M\setminus O \) be another section. As in Sect. 3 there is a function \( f_{\sigma} \) such that

\[
\sigma(\pi(\alpha_{x})) = f_{\sigma}(\alpha_{x}) \alpha_{x}, \quad \forall \alpha_{x} \in T^{*}M\setminus O.
\]
Let $h_{a\sigma} : Q \to \mathbb{R}$ be given by
\[ h_{a\sigma} = f_{a\sigma} \circ \sigma. \]

$h_{a\sigma}$ tells us how far (multiplicatively) $\sigma(q)$ is from $\sigma(q)$, namely $\sigma(q) = h_{a\sigma}(q) \tilde{\sigma}(q)$. Hence if $a \in \mathcal{P}^m(T^*M;O)$ then
\[ (a \circ \sigma)(q) = (h_{a\sigma}(q))^{\beta}(a \circ \tilde{\sigma})(q). \]

But $h_{a\sigma}$ is a fixed smooth, nonvanishing function on $Q$ so that multiplication by $(h_{a\sigma})^{\beta}$ is a diffeomorphism on $H^p(Q)$.

**Proposition 5.2.** Let $\Omega \subset \mathbb{R}^n$ be open and let $a(x, \xi) \in \mathcal{S}^o(T^*\Omega \setminus \Omega)$. Then
\[ D_x^2 D_{\xi}^2 a(x, \xi) \in \mathcal{S}^{-|\alpha|-|\beta|}_0(T^*\Omega \setminus \Omega). \]

**Proof.** By Proposition 5.1 we are free to choose a convenient section $\sigma : Q \to T^*\Omega \setminus \Omega$. We choose $\sigma$ to give the unit sphere bundle in the standard metric on $\mathbb{R}^n$, i.e. $\sigma(Q) = \{(x, \xi) | x \in \Omega, |\xi|| = 1\}$.

We need to bound the norm of $D_x^2 D_{\xi}^2 (a \circ \sigma)$ in $H^p \circ \sigma^{-1} \beta (Q)$ by the norm of $a \circ \sigma$ in $H^p(Q)$. The $x$-derivatives are no problem so it is enough to prove the statement for $|\alpha|| = 0, |\beta|| = 1$ (the argument can be repeated for longer $\beta$). To do this write the operator $\frac{\partial}{\partial \xi_i}$ as the sum of a radial part and a tangential part, $\frac{\partial}{\partial \xi_i} = D_r + D_{R_i}$. In this case $D_{R_i} = \xi_i \frac{\partial}{|\xi||}$. By the homogeneity of $a_i(x, \xi)$ we have that
\[ |\xi|| \frac{\partial}{|\xi||} a_i(x, \xi) = (l-1) a_i(x, \xi) \]
so that
\[ D_{R_i} (a \circ \sigma) = (\xi_i (l-1) a) \circ \sigma. \]

Now $\xi_i$ is a smooth function on $Q$, bounded by 1, so
\[ \| (\xi_i (l-1) a) \circ \sigma \|_0 \leq (l-1) \| a \circ \sigma \|_0. \]

It follows that
\[ \| D_{\xi_i} (a \circ \sigma) \|_{l_x \sigma^{-1}} \leq C \| a \circ \sigma \|_{l_x} \quad \text{for some } C. \]

We are now ready to define the $s$-norm of an element $[P] \in \mathcal{D}O_{m,k}$. Fix a linearization $v$ on $M$ and let $\sum_{j=0}^{m+k} a_{m-j}$ be the symbol of $P$ with respect to this linearization. Define
\[ \| P \|_{m,k;x} = \| a_m \|_{l_x} + \| a_{m-1} \|_{l_x} + \cdots + \| a_{-k} \|_{l_x}^2, \quad (5.2) \]
where $\| a_{m-j} \|_{l_x+m+k-j}$ is the norm of $a_{m-j}$ in $\mathcal{P}^{m+k+m-j}(T^*M;O)$ given by some choice of section $\sigma : Q \to T^*M;O$. Let $\mathcal{D}O_{m,k}$ denote the completion of $\mathcal{D}O_{m,k}$ with respect to this norm. This makes $\mathcal{D}O_{m,k}$ a Hilbert space. As an immediate consequence of Proposition 5.1 we have
**Proposition 5.3.** The Hilbert manifold structure of \( \psi DO_{m,k}^s \) is independent of the choice of section \( \sigma : Q \to T^*M \setminus O \).

The norm also depends on the linearization \( v \).

**Proposition 5.4.** The Hilbert manifold structure on \( \psi DO_{m,k}^s \) is independent of the choice of linearization.

**Proof.** We work with a fixed section \( \sigma : Q \to T^*M \setminus O \) and suppose \( \tilde{v} \) is another linearization. Let \( \phi_0 \) and \( \tilde{\phi}_0 \) be the phase functions induced by \( v \) and \( \tilde{v} \). Then as in (4.1) for \( [P] \in \psi DO_{m,k}^s \) we can write

\[
P_u(x) = (2\pi)^{-n} \int_{\Omega_0} a(x, \xi) d\xi \left( \frac{\det \nu_x}{\det \nu_{\tilde{v}_x}} \right) \left[ \frac{\partial}{\partial y_j} a(x, \xi) \right] \frac{\partial}{\partial y_j} \chi(x,y) e^{i\tilde{\phi}_0(x,y)} \nu_0(y) dy
\]

where \( a(x, \xi) \) is given locally by the asymptotic formula

\[
a(x, \xi) = (2\pi)^s \sum_{k \geq 0} \frac{1}{2^k k!} (iD_x)^k D_{\xi}^k a(x, \xi) \bigg|_{y = x} \quad (5.3)
\]

with

\[
a(x, y, \xi) = a(x, \xi) e^{i(\phi_0(x,y) - \tilde{\phi}_0(x,y))} \left( \frac{\det \nu_x}{\det \nu_{\tilde{v}_x}} \right)(y)
\]

(see e.g. [10]). From this we get that

\[
a_m(x, \xi) = a_m(x, \xi),
\]

\[
a_m - 1(x, \xi) = a_m - 1(x, \xi) + \frac{1}{(n-1)!} \sum_{j=1}^{n} \frac{\partial}{\partial \xi_j} a_m(x, \xi) \frac{\partial}{\partial y_j} \left( \frac{\det \nu_x}{\det \nu_{\tilde{v}_x}} \right)(y) \bigg|_{y = x}
\]

etc.

By Proposition 5.2 the derivatives \( D_{\xi}^k a_m - j(x, \xi) \) are bounded in the \( H^{s+m+k-j-|a|} \) norm by the \( H^{s+m+k-j} \) norm of \( a_m - j(x, \xi) \). Since also \( v \) and \( \tilde{v} \) are smooth the map

\[
\sum_{j=0}^{m+k-1} a_m - j(x, \xi) \mapsto \sum_{j=0}^{m+k-1} a_{m-j}(x, \xi)
\]

is a diffeomorphism. □

By choosing a linearization \( v \) and a section \( \sigma : Q \to T^*M \setminus O \) we can think of \([P] \in \psi DO_{m,k}^s \) as represented by a pseudodifferential operator with symbol \( a(x, \xi) \)

\[
a_m(x, \xi) \in \sum_{j=0}^{m+k} a_{m-j}(x, \xi)
\]

with \( a_{m-j} \in H^{s+m+k-j}(Q) \). In fact this description can be given rigorous sense if \( s \) is large enough. Indeed, the operator defined by (4.1) still makes sense as an oscillatory integral if we can differentiate \( a(x, \xi) \) enough times so that it becomes integrable on \( T^*M \setminus O \). If \( s > 2n + m \) the first \( n + m \) derivatives of \( a(x, \xi) \) are continuous so integration by parts \( n + m \) times gives a convergent expression.
Recall that if $m=0$ composition in $\psi DO_{0,k}$ is well defined.

**Proposition 5.5.** Composition extends continuously to $\psi DO_{0,k}$.

**Proof.** If $P_1$ has symbol $a = a_0(x, \xi) + a_{-1}(x, \xi) + \ldots + a_{-K}(x, \xi)$ and $P_2$ has symbol $b = b_0(x, \xi) + b_{-1}(x, \xi) + \ldots + b_{-K}(x, \xi)$ then $P_1 \circ P_2$ has symbol $c = c_0(x, \xi) + c_{-1}(x, \xi) + \ldots + c_{-K}(x, \xi)$ where

$$
c_0 = a_0 b_0$$

$$c_{-1} = a_0 b_{-1} + a_{-1} b_0 + \sum_{i=1}^n \partial_x a_0 \hat{\partial}_x b_i b_0$$

$$c_{-2} = a_0 b_{-1} + a_0 b_{-2} + a_{-2} b_0 + \sum_{i,j} \partial_x \partial_x a_0 \hat{\partial}_x b_i b_j b_0 + \sum_{i=1}^n \partial_x a_{-1} \hat{\partial}_x b_{-1} b_0$$

$$+ \sum_{i=1}^n \partial_x a_{-1} \hat{\partial}_x b_{-1} + \sum_{i=1}^n \partial_x a_{-1} \hat{\partial}_x b_0$$

etc.

By Proposition 5.2 it is clear that the $H^{s+k-j}$ norm of $c_{-j}$ is bounded by $C \|P_1\|_s \|P_2\|_s$, for example

$$\|c_0\|_{s+k} \leq \|a_0\|_{s+k} \|b_0\|_{s+k}$$

$$\|c_{-1}\|_{s+k-1} \leq \|a_0\|_{s+k-1} \|b_{-1}\|_{s+k-1} + \|a_{-1}\|_{s+k-1} \|b_0\|_{s+k-1}$$

$$+ n \|a_0\|_{s+k} \|b_0\|_{s+k}$$

etc. □

Thus $\psi DO_{0,k}^*$ is a Hilbert algebra. It follows that the subset $(\psi DO_{0,k}^*)_*$ of invertible elements in $\psi DO_{0,k}^*$ is open and that inversion, $P \mapsto P^{-1}$, is a homeomorphism of $(\psi DO_{0,k}^*)_*$. If $P$ has symbol $a = a_0(x, \xi) + a_{-1}(x, \xi) + \ldots + a_{-K}(x, \xi)$ then $P^{-1}$ has symbol $\tilde{a} = \tilde{a}_0(x, \xi) + \ldots + \tilde{a}_{-K}(x, \xi)$ where $\tilde{a}_0(x, \xi) = \frac{1}{a_0(x, \xi)}$ and $\tilde{a}_{-j}(x, \xi)$ depends only on $a_{-j}(x, \xi)$, the first derivatives of $a_{-j+1}(x, \xi)$, the second derivatives of $a_{-j+2}(x, \xi)$, and so on. Composition in $(\psi DO_{0,k}^*)_*$ is in fact smooth since it is the restriction of the composition map in $\psi DO_{0,k}^*$ which is bilinear and continuous, hence smooth. Therefore $(\psi DO_{0,k}^*)_*$ is a Hilbert Lie group.

Summarizing we have the following

**Theorem 5.6.** $(\psi DO_{0,k}^*)_* = \{\psi DO_{0,k}^*: s > \dim M\}$ is an ILH-Lie group where each $(\psi DO_{0,k}^*)_*$ is a smooth Hilbert Lie group. Its ILH-Lie algebra is $(\psi DO_{0,k}^*)_*$.

**Remark.** Now that we have given $(\psi DO_{0,k}^*)_*$ an ILH-Lie group structure we can take limits to describe a Lie group structure for $(\psi DO)_*$. First, consider $(\psi DO_0)_*$:

**Corollary 5.7.** $(\psi DO_0)_*$ has the Lie group structure of a direct limit of ILH-Lie groups, i.e. $(\psi DO_0)_* = \lim_{\searrow} (\psi DO_{0,k})_*$. 
This situation for the full group $\psi DO_a$ is somewhat more complicated. For any $m$ we can give $\psi DO_m \cap \psi DO_a$ the structure of a direct limit of ILH-manifolds by using an elliptic operator to identify $\psi DO_m \cap \psi DO_a$ with $(\psi DO)_{\cap}$. Multiplication will be smooth in the appropriate sense between the appropriate spaces. Piecing this together for all $m$ makes $\psi DO_a$ a graded direct limit of ILH-Lie groups. We omit precise formulations of these concepts since they are direct generalizations of Sect. 3.

References


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