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Abstract:

The Liapunov method for establishing stability has been used in a variety of fluid and plasma problems. For nondissipative systems, this stability method is related to well-known energy principles. A development of the Liapunov method for Hamiltonian systems due to Arnold uses the energy plus other conserved quantities, together with second variations and convexity estimates, to establish stability. For Hamiltonian systems, a useful class of these conserved quantities consists of the Casimir functionals, which Poisson-commute with all functionals of the given dynamical variables. Such conserved quantities, when added to the energy, help to provide convexity estimates bounding the growth of perturbations. These estimates enable one to prove nonlinear stability, whereas the commonly used second variation or spectral arguments only prove linearized stability. When combined with recent advances in the Hamiltonian structure of fluid and plasma systems, this convexity method proves to be widely and easily applicable. This paper obtains new nonlinear stability criteria for equilibria for MHD, multifluid plasmas and the Maxwell–Vlasov equations in two and three dimensions. Related systems, such as multilayer quasigeostrophic flow, adiabatic flow and the Poisson–Vlasov equation are also treated. Other related systems, such as stratified flow and reduced magnetohydrodynamic equilibria are mentioned where appropriate, but are treated in detail in other publications.

1. Introduction

The aim of this work is to establish explicit sufficient conditions for the nonlinear stability of equilibrium solutions of a variety of fluid and plasma problems in one, two, and three dimensions. As we shall explain below, many of the results in the literature only establish conditions for linearized stability or instability. The method we use is a variant of the Liapunov technique due to Arnold [1969a]. Recent advances in the Hamiltonian structure of field theories have set the stage for the new applications.

For example, the Grad Shafranov solutions of the Strauss equations for reduced MHD in the low β limit are shown in section 5 to be nonlinearly stable if the equilibrium current is a strictly monotone decreasing function of the equilibrium vector potential (see also Hazeltine, Holm, Marsden and Morrison [1984]).

Another example is three-dimensional adiabatic multifluid plasmas. Here the nonlinear stability conditions include the requirements that each fluid species be subsonic, and that the velocity, the gradients of specific entropy and generalized potential vorticity form a right-handed triad; see section 10 for details.

The classical Liapunov method finds criteria for stability of an equilibrium solution of a conservative dynamical system by seeking a constant of the motion with a local maximum or minimum at the equilibrium. In many examples, the constant of motion is the energy. An important development for the applicability of the Liapunov method to fluid dynamics is Arnold's [1965a, 1969a] nonlinear analysis of the stability of planar ideal incompressible fluid motion, providing nonlinear stability results that extend the classical linear theory of Rayleigh [1880]. Arnold adds to the energy H a conserved quantity C which corresponds to the symmetry under Lagrangian relabeling of fluid particles (in geometric language, the system in Lagrangian representation is right invariant on the cotangent bundle of the group of area-preserving diffeomorphisms). Underlying this method is the fact that the Eulerian equations of motion are Hamiltonian with respect to a certain noncanonical Poisson bracket, now called a Lie–Poisson bracket. The added constants of the motion are kinematic in the sense that they will be conserved for any system which is Hamiltonian with respect to the Lie–Poisson brackets; in fact, C Poisson commutes with all functionals; as such, it is called a Casimir. The functional C is chosen such that $H+C$ has a critical point at the stationary solution. Arnold employed convexity properties of $H+C$ to find an explicit norm and a priori estimates needed to limit the departure of finite perturbations from equilibrium. In this way, nonlinear stability was established.

In this paper, we apply the same technique to a number of other conservative systems arising in the physics of fluids and plasmas. Each example will be treated according to the general procedure alluded to above and which is detailed in section 2. The result in each case will be that if certain inequalities (the stability criteria) are satisfied for an equilibrium solution, then a priori estimates will guarantee Liapunov stability relative to an explicitly constructed norm for as long as the solutions of interest continue to exist.

Four interrelated concepts of stability, often encountered in the literature, will be of concern to us.

(1) *Neutral or spectral stability.* For a dynamical system $\dot{u} = du/dt = X(u)$, an equilibrium point u_e satisfying $X(u_e) = 0$ is called *spectrally stable*, provided the spectrum of the linearized operator $DX(u_e)$ has no strictly positive real part. A special case is *neutral stability*, for which the spectrum is purely imaginary. This corresponds to the time evolution of normal modes being purely oscillatory. For Hamiltonian systems spectral stability and neutral stability coincide.

(2) *Linearized stability.* The equilibrium solution u_e is called *linearized stable* or *linearly stable* relative to a norm $\|\delta u\|$ on infinitesimal variations δu provided for every $\varepsilon > 0$ there is a $\delta > 0$ such that if $\|\delta u\| < \delta$ at $t = 0$, then $\|\delta u\| < \varepsilon$ for $t > 0$, where δu evolves according to $(\delta u) = DX(u_e) \cdot \delta u$.

Linearized stability implies spectral stability (since, if the spectrum had a strictly positive real part, there would be an unstable eigenspace). The converse is not generally true (e.g., the equilibrium solution $(p_e, q_e) = (0, 0)$ for the dynamics generated by the Hamiltonian $H = p^2 + q^4$ is neutrally, but not linearized stable). In finite dimensions, a sufficient condition for linearized stability is that $DX(u_e)$ have distinct eigenvalues on the imaginary axis. In infinite dimensions, it is sufficient for $DX(u_e)$ to have a complete set of eigenfunctions with purely imaginary eigenvalues of multiplicity one. In the case of repeated roots on the imaginary axis, instabilities can occur with linear growth rates of a resonance type. There is an extensive theory dealing with this case going back to Krein [1950] (see also Arnold [1978]); this theory gives precise spectral conditions for linearized stability in finite dimensions. See also Levi [1977]. In finite dimensions, the spectral approach may encounter functional analytic difficulties requiring considerable effort to overcome, and the results in the literature are often only indicative of linearized stability, with no rigorous proof given. See, for example, Penrose [1960], Jackson [1960], Chandrasekhar [1961], Drazin and Reid [1981], and Friedberg [1982]. Another effective method to prove linearized stability is to look for a positive definite conserved quadratic quantity, which serves as the square of a norm. This leads to what we call "formal stability".

(3) *Formal stability.* We say that an equilibrium solution u_e of a system $\dot{u} = X(u)$ is *formally stable* if a conserved quantity is found whose first variation vanishes at the solution and whose second variation at this solution is positive (or negative) definite. Since the second variation provides a norm preserved by the linearized equations (see appendix A), formal stability implies linearized stability. Again the converse is not generally true (e.g., the equilibrium solution $(p_{1e}, q_{1e}, p_{2e}, q_{2e}) = (0, 0, 0, 0)$ for the dynamics generated by the Hamiltonian $H = (p_1^2 + q_1^2) - (p_2^2 + q_2^2)$ is linearized stable for the Euclidean norm in $\mathbb{R}^4$, but is not formally stable).

Formal stability of fluids and plasmas has been considered by a number of authors, such as Fjortoft [1946] Eliassen and Kleinschmidt [1957], Bernstein et al. [1958], Kruskal and Oberman [1958], Newcomb (see Appendix I of Bernstein [1958]), Fowler [1963], Gardner [1963], Rosenbluth [1964, p. 137ff.], Dikii [1965a], Herlitz [1967], and Davidson and Tsai [1973]. More recently, formal stability has been established by several authors who employ some aspects of Arnold's method (but not the convexity analysis). See for example Blumen [1968], Zakharov and Kuznetsov [1974], Sedenko and Iudovich [1978], Benzi et al. [1982] and Grinfeld [1984].

(4) *Nonlinear stability.* An equilibrium point u_e of a dynamical system is said to be *nonlinearly stable* if for every neighborhood U of u_e there is a neighborhood V of u_e such that trajectories $u(t)$ initially in V never leave U. This definition presupposes well-defined dynamics and a specified topology. In terms of a norm $\| \ \|$, nonlinear stability means that for every $\varepsilon > 0$ there is a $\delta > 0$ such that if $\|u(0) - u_e\| < \delta$, then $\|u(t) - u_e\| < \varepsilon$ for $t > 0$.

Many authors use the term "stability" in one of the weaker senses (1), (2) or (3) above; here we will subsequently use the term *stability* to mean nonlinear stability, in the sense of (4).

For conservative systems, it is well-known that even in finite dimensions, spectral stability is necessary for nonlinear stability, but is not sufficient (since, if the spectrum had a strictly positive real part, the nonlinear dynamics would have an unstable manifold; see, e.g., Marsden and McCracken [1976]). However, neither formal nor linearized stability is necessary for nonlinear stability. (Both counter examples above are also nonlinearly stable.) Linearized stability does not imply nonlinear stability either as shown by the following counter example discussed by Pollard [1966, p. 77] (see also, Siegel and Moser [1971, p. 109]). The dynamics generated by the Hamiltonian

$$H = \tfrac{1}{2}(q_1^2 + p_1^2) - (q_2^2 + p_2^2) + \tfrac{1}{2}p_2(p_1^2 - q_1^2) - q_1q_2p_1$$

has equilibrium $(p_{1e}, q_{1e}, p_{2e}, q_{2e}) = (0, 0, 0, 0)$ which is linearized stable in the Euclidean norm of $\mathbb{R}^4$. A one-parameter family of solutions for this system is, for any fixed value of the parameter τ,

$$p_1 = \sqrt{2}\,\frac{\sin(t-\tau)}{t-\tau}, \qquad p_2 = \frac{\sin 2(t-\tau)}{t-\tau}, \qquad q_1 = -\sqrt{2}\,\frac{\cos(t-\tau)}{t-\tau}, \qquad q_2 = \frac{\cos 2(t-\tau)}{t-\tau}.$$

The distance of time t from the equilibrium is $\sqrt{3}/(\tau - t)$, which by choosing τ, can be made as small as desired at $t = 0$ and which blows up at $t = \tau$. Thus, the equilibrium solution of this system is linearized stable, but nonlinearly unstable in the Euclidean $\mathbb{R}^4$ norm. Also, for a Hamiltonian system with at least three degrees of freedom, an equilibrium solution can be linearly stable but nonlinearly unstable, because of the phenomenon of Arnold diffusion; cf. Arnold [1978, Appendix 8], Chirikov [1979] and Lichtenberg and Liebermann [1982]. Thus, for a Hamiltonian system, spectral analysis can provide sufficient conditions for instability, but it can only give necessary conditions for stability. In this paper we provide sufficient conditions for stability.

In finite dimensions, formal stability implies stability, a classical result of Lagrange. (Indeed, if the equilibrium $X_e = 0$ is a nondegenerate minimum of the conserved quantity F, the set $\{x \mid |x| < \varepsilon, F(x) < \mu\}$, where μ is the minimum of F on the sphere of radius ε, is invariant under the flow; ε is chosen such that for all x satisfying $|x| < \varepsilon$ we have $F(0) < F(x)$; See also Siegel and Moser [1971, p. 208]). However, in the infinite dimensional case of concern to us here, formal stability need not imply stability; indeed, physically realistic examples from elasticity show that an equilibrium solution can have positive second variation of the energy and still have an infinite number of unstable directions (see Ball and Marsden [1984].) Formal stability is a step toward stability, but a further argument is needed. Arnold [1966b] provided a framework for such arguments based on convexity estimates using several quantities related to the degeneracy of the Poisson brackets describing the system (or, equivalently, to the symmetry of the system written in

Lagrangian coordinates under relabeling of fluid particles). The papers of which we are aware that actually prove nonlinear stability for conservative fluid and plasma systems are Arnold [1969a], Benjamin [1972], Bona [1975], McKean [1977], Laedke and Spatschek [1980], Holm et al. [1983], Bennet et al. [1983], Wan [1984], Wan and Pulvirente [1984], Hazeltine et al. [1984], Holm [1984], Holm et al. [1985], Abarbanel et al. [1985] and the present work.

For dissipative systems, there is a number of general results that show that linearized stability implies stability; see for example, Marsden and McCracken [1976] and references therein, and for bifurcation results see, e.g., Crawford [1983] and references therein. In the limit of zero dissipation, these methods seem to give little, or no information on the stability of the corresponding conservative system. Since conservative systems are the concern of this paper we shall not discuss dissipative systems further.

The main results of the paper give sufficient conditions for stability of equilibria for various two- and three-dimensional models of plasma physics: magnetohydrodynamics (MHD), multifluid plasmas (MFP), the Poisson–Vlasov, and Maxwell–Vlasov equations. In section 2 we start by explaining the general procedure we shall employ for proving stability. To illustrate this procedure, we present in section 3 four known examples: the free rigid body, the Lagrange top (see also Holm et al. [1984]), two-dimensional planar incompressible Euler fluid flow (Arnold [1965, 1969a]), and planar barotropic fluid flow (Holm et al. [1983b]). The first two examples are classical, and the known stability conditions are rederived using the stability algorithm presented in section 2. Next the example of planar incompressible homogeneous fluid flow is given, following Arnold's papers, which were crucial in our understanding of the general procedure and the geometric ideas underlying it. These ideas are applied to planar barotropic fluid dynamics in the fourth example.

Part I of the paper concerns various two-dimensional fluid systems. Due to its similarity to Arnold's original example and relative simplicity, we begin in section 4 with the study of multilayer quasigeostrophic two-dimensional incompressible fluid flow. We refer the reader to Abarbanel et al. [1985] for the cases of two- and three-dimensional inhomogeneous incompressible flows, including a Richardson number criterion for the stability of shear flows. Sections 5, 6 and 7 deal with magnetohydrodynamics (MHD) and multifluid plasmas (MFP) in the plane.

The second part of the paper treats three-dimensional examples. We start with adiabatic fluid flow. This is then generalized in two different manners, to MHD and MFP. Finally, the third part of the paper discusses the Poisson–Vlasov and Maxwell–Vlasov plasma equations in one, two, and three dimensions.

We have not been exhaustive in our choice of models. In the first two parts, treating fluid-like systems, we shall always present, where appropriate, an incompressible and compressible model, but emphasize the incompressible homogeneous case (i.e., constant density) over the inhomogeneous one, and we leave out completely the Boussinesq approximation. The inhomogeneous cases can be treated by introducing certain modifications, as in Abarbanel et al. [1985]. There are numerous other models to which the methods apply. For example, Hazeltine, Holm and Morrison [1985] use these methods to discuss stationary solutions for the Hasegawa–Mima–Hazeltine models. One can also use the methods here, with some additional help from Sobolev inequalities to prove the stability of a single KdV soliton (see Benjamin [1972] and Bona [1975]), or of the N-soliton (see McKean [1977].

In many of the examples, the methods of this paper are capable of establishing a priori estimates on some, but not all, of the variables. For example, in three-dimensional adiabatic flow, stability estimates are obtained only as long as solutions remain smooth, the density remains bounded: $0 < \rho_{\min} \leq \rho \leq \rho_{\max} < \infty$, and the gradient of perturbations of the entropy is bounded in terms of the perturbations of the entropy itself. This is consistent with the fact that shocks could form from initial data near a "stable" equilibrium, and that when this occurs, the density and entropy gradients can develop singularities. This kind of stability is called *conditional stability*; it requires that one "monitors" some of the dynamic variables. As long as these monitored variables remain under control, the equilibrium will be stable. As we shall see, the method enables one to identify these variables explicitly in each case and determine their stable range, in terms of equilibrium values.

2. The stability algorithm

We now present the algorithm that will be used in each of the examples. Some of the steps are facilitated and put into a larger context by the use of a Hamiltonian structure (Poisson brackets); this is explained in remarks following each step.

A. *Equations of motion and Hamiltonian*

Choose a (Banach) space P of fields u and write the equations of motion on P in first-order form as

$$\dot{u} = X(u) \tag{EM}$$

for a (nonlinear) operator X mapping a domain in P to P. Find a conserved functional H for (EM), usually representing the total energy; that is find a map $H: \mathrm{P} \to \mathbb{R}$ such that $\mathrm{d}H(u)/\mathrm{d}t = 0$ for any C^1 solution u of (EM).

Remark A. Often P is a Poisson space, i.e. a linear space (or more generally a manifold) admitting a Poisson bracket operation $\{\,,\}$ on the space of real valued functions on P which makes them into a Lie algebra, and which is a derivation in each variable. There are systematic procedures for obtaining such brackets; these procedures are not reviewed here, although we shall give references relevant to each example.* The equations (EM) can then be expressed in Hamiltonian form for such a bracket structure:

$$\dot{F} = \{F, H\}, \tag{PB}$$

where H is the Hamiltonian, F is any functional of $u \in \mathrm{P}$, and $\dot{F}$ is its time derivative through the dependence of u on t.

B. *Constants of motion*

Find a family of constants of the motion for (EM). That is, find a collection of functionals C on P such that $\mathrm{d}C(u)/\mathrm{d}t = 0$ for any C^1 solution u of (EM).

Remark B. Unless a sufficiently large family is found, the next step may not be possible. A good way to find conserved functionals is to use the Hamiltonian formalism in Remark A to find Casimir** functionals for the Poisson structure, i.e. functionals C such that $\{C, G\} = 0$ for all G. One may find additional functionals associated with symmetries of the given Hamiltonian.

* As noted in Weinstein [1982], the general notion of a Poisson manifold goes back to Sophus Lie around 1890.
** This term was used in the same context as here by Sudarshan and Mukunda [1974].

C. *First variation*

Relate an equilibrium solution u_e of (EM) to a constant of the motion C by requiring that $H_C := H + C$ have a critical point at u_e.

Note: C may or may not be uniquely determined. Keeping C as general as possible may be useful in step D. Moreover, if C retains some freedom at this stage in terms of unspecified parameters or functions, critical points of H_C will correspond to *classes* of equilibria.

Remark C. If Remarks A and B are followed, then such a C can often be expected to exist. Indeed, level sets of the constants of motion define certain "leaves" in P; if C is a Casimir, they are the "symplectic leaves" of the Poisson structure $\{,\}$. Equilibrium solutions are critical points of H restricted to such leaves. If the Casimirs are functionally independent, the Lagrange multiplier theorem implies that $H + C$ has a critical point at u_e for an appropriate Casimir function C. One cannot guarantee that such functions can be found explicitly in all cases: however, they are found in the examples we consider. These points are discussed further in appendix B.

D. *Convexity estimates*

Find quadratic forms Q_1 and Q_2 on P such that*

$$Q_1(\Delta u) \leq H(u_e + \Delta u) - H(u_e) - DH(u_e) \cdot \Delta u\,, \tag{CH}$$

$$Q_2(\Delta u) \leq C(u_e + \Delta u) - C(u_e) - DC(u_e) \cdot \Delta u\,, \tag{CC}$$

for all Δu in P. Require that

$$Q_1(\Delta u) + Q_2(\Delta u) > 0 \text{ for all } \Delta u \text{ in P, } \Delta u \neq 0\,. \tag{D}$$

Remark D. Formal stability – second variation. As a prelude to checking conditions (CH), (CC) and (D), it is often convenient to see whether the second variation $D^2H_C(u_e)$ is definite, or when feasible, whether $D^2H(u_e)$ restricted to the symplectic leaf through u_e is definite. This property, called *formal stability*, is a prerequisite for step D to work, but it is not sufficient (see Remark (2) below).

If formal stability is established, then the zero solution of the equations (EM) linearized at u_e are stable since $D^2H_C(u_e)$ provides a conserved norm under the linearized dynamics (see appendix A).

E. *A priori estimates*

If steps A through D have been carried out, then for any solution u of (EM), we have the following estimate on $\Delta u = u - u_e$:

$$Q_1(\Delta u(t)) + Q_2(\Delta u(t)) \leq H_C(u(0)) - H_C(u_e) \tag{E}$$

(this is proved below).

* Here $\Delta u - u - u_e$ denotes a *finite* variation of the solution. To avoid confusion, we shall use $\nabla^2 u$ for the Laplacian of u.

F. *(Nonlinear)stability*

Stability theorem. Suppose that steps A through D have been carried out. Set

$$\|v\|^2 = Q_1(v) + Q_2(v) > 0 \quad (\text{for } v \neq 0) \tag{N}$$

so $\|v\|$ defines a norm on P. If H_C is continuous in this norm at u_e, and solutions to (EM) exist for all time, then u_e is stable. Should solutions to (EM) not be known to exist for all time, we still have conditional stability: stability for all times during which C^1 solutions exist.

A sufficient condition for continuity of H_C is the existence of positive constants C_1 and C_2 such that

$$H(u_e + \Delta u) - H(u_e) - DH(u_e) \cdot \Delta u \leq C_1 \|\Delta u\|^2 , \tag{CH)$'$ (}$$

$$C(u_e + \Delta u) - C(u_e) - DC(u_e) \cdot \Delta u \leq C_2 \|\Delta u\|^2 . \tag{CC)$'$ (}$$

In this case there follows the stability estimate:

$$\|\Delta u(t)\|^2 = Q_1(\Delta u(t)) + Q_2(\Delta u(t)) \leq (C_1 + C_2) \|\Delta u(0)\|^2 , \tag{SE}$$

for all Δu in P (these assertions are proved below).

Proof of a priori estimate (E). Adding (CC) and (CH) gives

$$Q_1(\Delta u) + Q_2(\Delta u) \leq H_C(u_e + \Delta u) - H_C(u_e) - DH_C(u_e) \cdot \Delta u = H_C(u_e + \Delta u) - H_C(u_e) ,$$

since $DH_C(u_e) = 0$ by step C. Because H_C is a constant of the motion, $H_C(u_e + \Delta u) - H_C(u_e)$ equals its value at $t = 0$, which is (E). ■

Proof of the assertions in step F. We prove (Liapunov) stability of u_e as follows. Given $\varepsilon > 0$, find a δ such that $\|v - u_e\| < \delta$ implies $|H_C(v) - H_C(u_e)| < \varepsilon$. Thus, if $\|u(0) - u_e\| < \delta$, then (E) gives

$$\|u(t) - u_e\| \leq |H_C(u(0)) - H_C(u_e)| < \varepsilon .$$

Thus, $u(t)$ never leaves the ε-ball about u_e if it starts in the δ ball, so u_e is stable. To see that (CH)′ and (CC)′ suffice for continuity of H_C at u_e, add them to give, as in the proof of (E),

$$H_C(u_e + \Delta u) - H_C(u_e) \leq C_1 \|\Delta u\|^2 + C_2 \|\Delta u\|^2 = (C_1 + C_2) \|\Delta u\|^2 ,$$

which implies that H_C is continuous at u_e. This proves the stability estimate (SE). ■

Further remarks

(1) In some examples, Q_1 and Q_2 are each positive (so H and C are individually convex). Then (D) is automatic. However, as already noted by Arnold [1969a] (and recalled in section 3), there are some interesting examples where Q_1 is positive, Q_2 is negative and yet the sum $Q_1 + Q_2$ is positive and (D) is valid. If the sum $Q_1 + Q_2$ is shown to be negative, then one can replace H_C by $-H_C$ to obtain (D).

(2) It has been presumed that P carries a Banach space topology (although one could merely assume P is a Fréchet space) relative to which the symbols $\dot{u}$ and $\mathrm{D}H(u_e)$ are defined, and steps A, B and C are admissible. The norm $\|\cdot\|$ found in step F is usually not complete; relative to the functions H and C need not be differentiable. (This fact is related to the difficulty one encounters when trying to deduce stability from formal stability.) A sufficient condition for (CH) is that inequality

$$Q_1(v) \leq \mathrm{D}^2H(u)\cdot(v, v) \tag{CH)''$$

holds for all u and v in P. The sufficiency of (CH)″ follows from the mean value theorem. There are similar assertions for C and H_C. Note that

$$\|v\|^2 \leq \mathrm{D}^2H_C(u)(v, v) \tag{CH$_C$)''$$

is considerably stronger than formal stability: $\mathrm{D}^2H_C(u_e)(v, v)$ positive definite. Indeed, (CH$_C$)″ is a *global* convexity condition which reflects the additional hypotheses involved in step D.

(3) As already noted, in systems with a finite number of degrees of freedom, formal stability implies stability. This fact was used by Arnold [1966a] to reproduce the well-known results on stability of rigid body motion; see section 3.1. See Marsden and Weinstein [1974] for the relationship of the formal stability ideas to the stability of relative equilibria and reduction. (See also Abraham and Marsden [1978], Sections 4.3 and 4.4 and Arnold [1978], Appendices 2 and 5.)

(4) In many examples, such as compressible flow, there is no global existence of smooth solutions. This paper does not address weak solutions or solutions with shocks. The results will apply only to sufficiently smooth solutions. Moreover, one or more of the steps may require assumptions about some of the variables. For example, in two-dimensional compressible flow, (section 3.4), we obtain our estimates only under the assumption that the density satisfies $0 < \rho_{\min} \leq \rho \leq \rho_{\max} < \infty$ for constants $\rho_{\min}$ and $\rho_{\max}$. (The necessity of such assumptions is revealed by the convexity analysis; formal stability does not reveal this and would tempt one to make unjustified claims in this regard.) This type of stability, which requires one to *monitor* some of the variables will be called *conditional stability.*

(5) For Hamiltonian systems with additional symmetries, there will be additional constants of the motion besides Casimirs. These are to be incorporated into the functional C in step B. This is needed in fluid examples with a translational symmetry, for example, and in the stability analysis of a heavy top; see section 3.

Scheme: The energy-Casimir stability method

A. *Equations of motion and Hamiltonian.* Write the equations of motion (EM) on P and find the conserved energy H.
[Determine the Poisson bracket and Hamiltonian on P.]

B. *Constants of motion.* Find as many conserved quantities C as possible for (EM).
[Determine the Casimirs of P.]

C. *First variation.* Let $H_C := H + C$ and u_e be a stationary solution for (EM). Relate C and u_e by the condition $\mathrm{D}H_C(u_e) = 0$. Keep C as general as possible.

D. *Convexity estimates.* Find quadratic forms Q_1 and Q_2 on P and conditions on u_e such that (CH), (CC), and (D) hold.
[*Formal stability.* Show that $\mathrm{D}^2H_C(u_e)$ is definite and conclude linearized stability.]

E. *A priori estimates.* Write out the estimate (E).

F. *Stability.* Find sufficient conditions on u_e to guarantee that H_C is continuous in the norm (N), or prove the estimates (CH)′, (CC)′, and conclude conditional stability of u_e subject to these conditions. In the presence of a long-time existence theorem, conclude stability.

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(6) For two-dimensional incompressible flow, the appropriate Casimir function is the generalized enstrophy. This suggests, following Leith (cf. Bretherton and Haidvogel [1976]), that the Casimir functions may play a role in the "selective decay hypothesis" when dissipation is added.

For the convenience of the reader, we summarize schematically the procedure just explained, with the optional but useful steps in square brackets. In all examples, we shall follow this procedure and carry out each step explicitly.

For finite dimensional systems, formal stability implies stability. Thus, the energy-Casimir method in this case requires only steps A, B, C, and the formal stability argument in step D.

3. Background examples

In this section we discuss four examples to illustrate the stability algorithm given in section 2. These are: section 3.1 the free rigid body, section 3.2 the Lagrange top, section 3.3 ideal incompressible planar flow and section 3.4 ideal barotropic planar flow. The results for the first two examples are well-known, although they are not usually proved by this method. We follow Holm et al. [1984]. The third example follows Arnold [1969a] and the fourth is based on Holm et al. [1983b].

3.1. The free rigid body

A. *Equations of motion and Hamiltonian*

The free rigid body equations of motion are

$$\dot{\boldsymbol{m}} = \mathrm{d}\boldsymbol{m}/\mathrm{d}t = \boldsymbol{m} \times \boldsymbol{\omega}\,, \tag{3.1EM}$$

where $\boldsymbol{m}$, $\boldsymbol{\omega} \in \mathbb{R}^3$, $\boldsymbol{\omega}$ is the angular velocity and $\boldsymbol{m}$ is the angular momentum, both viewed in the body. The relation between $\boldsymbol{m}$ and $\boldsymbol{\omega}$ is given by $m_i = I_i\omega_i$, $i = 1, 2, 3$, where $I = (I_1, I_2, I_3)$ is the diagonalized moment of inertia tensor, I_1, I_2, $I_3 > 0$. A conserved quantity for (3.1EM) is the kinetic energy,

$$H(\boldsymbol{m}) = \tfrac{1}{2}\boldsymbol{m} \cdot \boldsymbol{\omega} = \tfrac{1}{2}\sum_{i=1}^{3} m_i^2/I_i\,. \tag{3.1H}$$

Remark A. This system is Hamiltonian in the Lie–Poisson structure of $\mathbb{R}^3$ considered as the dual of the Lie algebra of the rotation group SO(3). Explicitly, for F, G: $\mathbb{R}^3 \to \mathbb{R}$,

$$\{F, G\}(\boldsymbol{m}) = -\boldsymbol{m} \cdot (\nabla F(\boldsymbol{m}) \times \nabla G(\boldsymbol{m}))\,, \tag{3.1PB}$$

where $\nabla_i = \partial/\partial m_i$ in (3.1PB). With respect to this bracket, (3.1EM) is easily verified to be Hamiltonian in the sense that (3.1EM) is equivalent to $\dot{F} = \{F, H\}$ where H is given by (3.1H).*

* The first reference we know of where this is explicitly written is Sudarshan and Mukunda [1974]. The result is suggested in Arnold [1966a].

B. *Constants of motion*

For any smooth function $\phi:\mathbb{R}\to\mathbb{R}$, the function

$$C_\phi(\boldsymbol{m}) = \phi(|\boldsymbol{m}|^2/2) \tag{3.1C}$$

is a constant of motion for (3.1EM), as is easily verified.

Remark B. In fact, using (3.1PB) it is easily seen that C_ϕ are Casimir functions. These are seen to be all the Casimirs, since their level sets determine the symplectic leaves of $\mathbb{R}^3$, which are concentric spheres and the origin.

C. *First variation*

We shall find a Casimir function C_ϕ such that $H_C := H + C_\phi$ has a critical point at a given equilibrium point of (3.1EM). Such points occur when $\boldsymbol{m}$ is parallel to $\boldsymbol{\omega}$. We shall assume, without loss of generality, that $\boldsymbol{m}$ and $\boldsymbol{\omega}$ point in the x-direction. Then, after normalizing if necessary, we may even assume that the equilibrium solution is $\boldsymbol{m}_e = (1, 0, 0)$. The derivative of

$$H_C(m) := \tfrac{1}{2}\sum_{i=1}^{3} m_i^2/I_i + \phi(\tfrac{1}{2}|\boldsymbol{m}|^2)$$

is

$$\mathrm{D}H_C(\boldsymbol{m})\cdot\delta\boldsymbol{m} = (\boldsymbol{\omega} + \boldsymbol{m}\phi'(|\boldsymbol{m}|^2/2))\cdot\delta\boldsymbol{m}\,. \tag{3.1C1}$$

This equals zero at $\boldsymbol{m}_e = (1, 0, 0)$, provided that

$$\phi'(\tfrac{1}{2}) = -1/I_1\,. \tag{3.1C2}$$

Thus ϕ and $\boldsymbol{m}_e$ are related by (3.1C2).

D. *Second variation*

Since the system is finite dimensional, it suffices to check the second variation. Using (3.1C1) and (3.1C2), the second derivative at the equilibrium $\boldsymbol{m}_e = (1, 0, 0)$ is

$$\begin{aligned}\mathrm{D}^2H_C(\boldsymbol{m}_e)\cdot(\delta\boldsymbol{m})^2 &= \delta\boldsymbol{\omega}\cdot\delta\boldsymbol{m} + \phi'(|\boldsymbol{m}_e|^2/2)|\delta\boldsymbol{m}|^2 + (\boldsymbol{m}_e\cdot\delta\boldsymbol{m})^2\phi''(|\boldsymbol{m}_e|^2/2)\\ &= \sum_{i=1}^{3}(\delta m_i)^2/I_i - |\delta\boldsymbol{m}|^2/I_1 + \phi''(\tfrac{1}{2})(\delta m_1)^2\\ &= (1/I_2 - 1/I_1)(\delta m_2)^2 + (1/I_3 - 1/I_1)(\delta m_3)^2 + \phi''(\tfrac{1}{2})(\delta m_1)^2\,.\end{aligned} \tag{3.1D1}$$

This quadratic form is positive definite if and only if

$$\phi''(\tfrac{1}{2}) > 0\,, \tag{3.1D2}$$

$$I_1 > I_2\,,\qquad I_1 > I_3\,. \tag{3.1D3}$$

Consequently, $\phi(x) = (-2/I_1)x + (x - \frac{1}{2})^2$ satisfies (3.1C2) and makes the second derivative of H_C at $(1, 0, 0)$ positive definite, so stationary rotation around the longest axis is stable.

The quadratic form (3.1D1) is indefinite if

$$I_1 > I_2, \quad I_3 > I_1 \quad \text{or} \quad I_1 > I_3, \quad I_2 > I_1. \tag{3.1D4}$$

This method correctly suggests (but does not prove) that rotation around the middle axis is unstable. This may be shown by a linearized analysis. Finally, the quadratic form is negative definite, provided

$$\phi''(\tfrac{1}{2}) < 0, \tag{3.1D5}$$

and

$$I_1 < I_2, \quad I_1 < I_3. \tag{3.1D6}$$

It is obvious that we may find a function ϕ satisfying the requirements (3.1C2) and (3.1D5); e.g., $\phi(x) = (-2/I_1)x - (x - \frac{1}{2})^2$. This proves that rotation around the short axis is stable.

We summarize the results in the following well-known theorem.

Rigid body stability theorem. In the motion of a free rigid body, rotation around the long or short axis is stable.

Remark (1). It is important to keep the Casimirs as general as possible, because otherwise (3.1D2) and (3.1D5) would be contradictory. Had we chosen $\phi(x) = -(2/I_1)x + (x - \frac{1}{2})^2$ for example, (3.1D2) would be verified, but not (3.1D5). It is only the choice of *two different* Casimirs that enables us to prove the two stability results, even though the level surfaces of these Casimirs are the same.

Remark (2). The same stability theorem can also be proved by working with the second derivative along a coadjoint orbit in $\mathbb{R}^3$; i.e. a two-sphere; see Arnold [1966a]. This coadjoint orbit method has the deficiency of being inapplicable where the rank of the Poisson structure jumps (see Weinstein [1984]).

3.2. The Lagrange top

A. *Equations of motion and Hamiltonian*

The heavy top equations are

$$\mathrm{d}\boldsymbol{m}/\mathrm{d}t = \boldsymbol{m} \times \boldsymbol{\omega} + Mg\ell\boldsymbol{\gamma} \times \boldsymbol{\chi}, \tag{3.2EMa}$$

$$\mathrm{d}\boldsymbol{\gamma}/\mathrm{d}t = \boldsymbol{\gamma} \times \boldsymbol{\omega}, \tag{3.2EMb}$$

where $\boldsymbol{m}$, $\boldsymbol{\gamma}$, $\boldsymbol{\omega}$, $\boldsymbol{\chi} \in \mathbb{R}^3$. Here $\boldsymbol{m}$ and $\boldsymbol{\omega}$ are the angular momentum and angular velocity in the body, $m_i = I_i\omega_i$, $I_i > 0$, $i = 1, 2, 3$, with $\boldsymbol{I} = (I_1, I_2, I_3)$ the moment of inertia tensor. The vector $\boldsymbol{\gamma}$ represents the motion of the unit vector along the z-axis as seen from the body, and the constant vector $\boldsymbol{\chi}$ is the unit vector along the line segment of length ℓ connecting the fixed point to the center-of-mass of the body; M is the total mass of the body, and g is the strength of the gravitational acceleration, which is along Oz, pointing downward. The total energy of this system is

$$H(\boldsymbol{m}, \boldsymbol{\gamma}) = \tfrac{1}{2}\boldsymbol{m} \cdot \boldsymbol{\omega} + Mg\ell\boldsymbol{\gamma} \cdot \boldsymbol{\chi}, \tag{3.2H}$$

as can be easily verified.

Remark A. This system is Hamiltonian in the Lie–Poisson structure of $\mathbb{R}^3 \times \mathbb{R}^3$ regarded as the dual of the Lie algebra of the Euclidean group $E(3) = SO(3) \circledS \mathbb{R}^3$ ($\circledS$ denotes semidirect product). The Poisson bracket is given by

$$\{F, G\}(\boldsymbol{m}, \boldsymbol{\gamma}) = -\boldsymbol{m} \cdot (\nabla_{\boldsymbol{m}} F \times \nabla_{\boldsymbol{m}} G) - \boldsymbol{\gamma} \cdot (\nabla_{\boldsymbol{m}} F \times \nabla_{\boldsymbol{\gamma}} G + \nabla_{\boldsymbol{\gamma}} F \times \nabla_{\boldsymbol{m}} G). \tag{3.2PB}$$

The Hamiltonian is given by (3.2H) above. (See Sudarshan and Mukunda [1974], Vinogradov and Kupershmidt [1977], Ratiu and Van Moerbeke [1982] and Holmes and Marsden [1983]).

B. *Constants of motion*

It is easy to see that the functions $\boldsymbol{m} \cdot \boldsymbol{\gamma}$ and $|\boldsymbol{\gamma}|^2$ are conserved for (3.2EM). Consequently, for any smooth function Φ, the quantity

$$C(\boldsymbol{m}, \boldsymbol{\gamma}) = \Phi(\boldsymbol{m} \cdot \boldsymbol{\gamma}, |\boldsymbol{\gamma}|^2) \tag{3.2C}$$

is also conserved.

We shall be concerned here only with the Lagrange top. This is a heavy top for which $I_1 = I_2$ (i.e. it is symmetric) and the center of mass lies on the axis of symmetry in the body, i.e. $\boldsymbol{\chi} = (0, 0, 1)$. This assumption implies from the third equation of motion in (3.2EMa) that $\mathrm{d}m_3/\mathrm{d}t = 0$. Thus m_3 and hence any function $\phi(m_3)$ of m_3 is conserved.

Remark B. Using the Poisson bracket (3.2PB) it is easy to check that (3.2C) is a Casimir of the Poisson structure. In fact, the family described by (3.2C) forms all the Casimir functions, since their level sets determine the generic four-dimensional orbits $\{(\boldsymbol{m}, \boldsymbol{\gamma}) \in \mathbb{R}^3 \times \mathbb{R}^3 \mid \boldsymbol{m} \cdot \boldsymbol{\gamma} = \text{constant, and } |\boldsymbol{\gamma}|^2 = \text{constant}\}$.

C. *First variation*

We shall study the equilibrium solution $\boldsymbol{m}_e = (0, 0, \bar{m}_3)$, $\boldsymbol{\gamma}_e = (0, 0, 1)$, which represents the spinning of a symmetric top in its upright position. To begin, we look for conserved quantities of the form $H_C = H + \Phi(\boldsymbol{m} \cdot \boldsymbol{\gamma}, |\boldsymbol{\gamma}|^2) + \phi(m_3)$ which have a critical point at the equilibrium.*

The first derivative of H_C is given by

$$\begin{aligned} \mathrm{D}H_C(\boldsymbol{m}, \boldsymbol{\gamma}) \cdot (\delta\boldsymbol{m}, \delta\boldsymbol{\gamma}) = (\boldsymbol{\omega} + \dot{\Phi}(\boldsymbol{m} \cdot \boldsymbol{\gamma}, |\boldsymbol{\gamma}|^2)\boldsymbol{\gamma}) \cdot \delta\boldsymbol{m} + [MG\ell\boldsymbol{\chi} + \dot{\Phi}(\boldsymbol{m} \cdot \boldsymbol{\gamma}, |\boldsymbol{\gamma}|^2)\boldsymbol{m} \\ + 2\Phi'(\boldsymbol{m} \cdot \boldsymbol{\gamma}, |\boldsymbol{\gamma}|^2)\boldsymbol{\gamma}] \cdot \delta\boldsymbol{\gamma} + \phi'(m_3)\delta m_3, \end{aligned} \tag{3.2C1}$$

where dot and prime denote differentiation with respect to the first and second arguments of Φ. At the equilibrium solution $\boldsymbol{m}_e$, $\boldsymbol{\gamma}_e$, the first derivative H_C vanishes, provided that

$$\bar{\omega}_3 + \dot{\Phi}(\bar{m}_3, 1) + \phi'(\bar{m}_3) = 0; \qquad \bar{\omega}_3 = \bar{m}_3/I_3, \qquad Mg\ell + \dot{\Phi}(\bar{m}_3, 1)\bar{m}_3 + 2\Phi'(\bar{m}_3, 1) = 0.$$

* We could have chosen the forms $H + \Phi(\boldsymbol{m} \cdot \boldsymbol{\gamma}, |\boldsymbol{\gamma}|^2, m_3)$ or $H = \Phi_1(\boldsymbol{m} \cdot \boldsymbol{\gamma}) + \Phi_2(|\boldsymbol{\gamma}|^2) + \Phi_3(m_3)$ for H_C just as well. The form we use, however, is Casimir plus conserved quantity, consistent with the philosophy of the general energy-Casimir method.

(The remaining equations involving indices 1 and 2 are trivially verified.) Solving for $\dot{\Phi}(\bar{m}_3, 1)$ and $\Phi'(\bar{m}_3, 1)$, we get the conditions:

$$\dot{\Phi}(\bar{m}_3, 1) = -\left(\frac{1}{I_3} + \phi(\bar{m}_3)\right)\bar{m}_3 , \qquad \Phi'(\bar{m}_3, 1) = \tfrac{1}{2}\left(\frac{1}{I_3} + \phi'(\bar{m}_3)\right)\bar{m}_3^2 - \tfrac{1}{2}Mg\ell . \tag{3.2C2}$$

Thus Φ, ϕ, and the equilibrium $\boldsymbol{m}_e$, $\boldsymbol{\gamma}_e$ are related by (3.2C2).

D. *Formal stability.* Since the system is finite dimensional, it suffices to verify *formal stability.* We shall check for definiteness of the second derivative of H_C at the equilibrium point $\boldsymbol{m}_e = (0, 0, \bar{m}_3)$, $\boldsymbol{\gamma}_e = (0, 0, 1)$. To simplify notation we shall set

$$a = \Phi''(\bar{m}_3) , \qquad b = 4\Phi''(\bar{m}_3, 1) , \qquad c = \ddot{\Phi}(\bar{m}_3, 1) , \qquad d = 2\dot{\Phi}'(\bar{m}_3, 1) .$$

With this notation, (3.2C1), and (3.2C2), we find that the matrix of the second derivative of H_C at $\boldsymbol{m}_e$, $\boldsymbol{\gamma}_e$ is

$$\begin{bmatrix} 1/I_1 & 0 & 0 & \dot{\Phi}(\bar{m}_3, 1) & 0 & 0 \\ 0 & 1/I_2 & 0 & 0 & \dot{\Phi}(\bar{m}_3, 1) & 0 \\ 0 & 0 & (1/I_3) + a + c & 0 & 0 & \dot{\Phi}(\bar{m}_3, 1) + 2\bar{m}_3 c + d \\ \dot{\Phi}(\bar{m}_3, 1) & 0 & 0 & 2\Phi'(\bar{m}_3, 1) & 0 & 0 \\ 0 & \dot{\Phi}(\bar{m}_3, 1) & 0 & 0 & 2\Phi'(\bar{m}_3, 1) & 0 \\ 0 & 0 & \dot{\Phi}(\bar{m}_3, 1) + 2\bar{m}_3 c + d & 0 & 0 & 2\Phi'(\bar{m}_3, 1) + b + \bar{m}_3^2 c + 2\bar{m}_3 d \end{bmatrix} \tag{3.2D1}$$

If this form is definite, it must be positive definite, since the (1, 1) entry is positive. The six principal subdeterminants have the following values, (recall that $I_1 = I_2$):

$$1/I_1 , \qquad 1/I_1^2 , \qquad (1/I_3 + a + c)/I_1^2 ,$$

$$\frac{1}{I_1}\left(\frac{1}{I_3} + a + c\right)\left(\frac{2}{I_1}\Phi'(\bar{m}_3, 1) - \dot{\Phi}(\bar{m}_3, 1)^2\right) , \qquad \left(\frac{2}{I_1}\Phi'(\bar{m}_3, 1) - \dot{\Phi}(\bar{m}_3, 1)^2\right)^2\left(\frac{1}{I_3} + a + c\right) ,$$

$$\left(\frac{2}{I_1}\Phi'(\bar{m}_3, 1) - \dot{\Phi}(\bar{m}_3, 1)^2\right)\left[(2\Phi'(\bar{m}_3, 1) + b + \bar{m}_3^2 c + 2\bar{m}_3 d)\left(\frac{1}{I_3} + a + c\right) - (\dot{\Phi}(\bar{m}_3, 1) + 2\bar{m}_3 c + d)^2\right] .$$

Consequently, the quadratic form given by (3.2D1) is positive definite, if and only if

$$1/I_3 + a + c > 0 , \tag{3.2D2}$$

$$(2/I_1)\Phi'(\bar{m}_3, 1) - \dot{\Phi}(\bar{m}_3, 1)^2 > 0 , \tag{3.2D3}$$

$$(2\Phi'(\bar{m}_3, 1) + b + \bar{m}_3^2 c + 2\bar{m}_3 d)\left(\frac{1}{I_3} + a + c\right) - (\dot{\Phi}(\bar{m}_3, 1) + 2\bar{m}_3 c + d)^2 > 0 . \tag{3.2D4}$$

Conditions (3.2D2) and (3.2D4) can always be satisfied if we choose the numbers a, b, c, and d appropriately; e.g., $a = c = d = 0$ and b sufficiently large and positive. Thus, the determining condition for stability is (3.2D3). By (3.2C2), this becomes

$$\frac{1}{I_1}\left[\left(\frac{1}{I_3}+\phi'(\bar{m}_3)\right)\bar{m}_3^2 - Mg\ell\right] - \left(\frac{1}{I_3}+\phi'(\bar{m}_3)\right)^2\bar{m}_3^2 > 0\,. \tag{3.2D5}$$

We can choose $\phi'(\bar{m}_3)$ so that $1/I_3 + \phi'(\bar{m}_3) = e$ has any value we wish. The left side of (3.2D5) is a quadratic polynomial in e, whose leading coefficient is negative. In order for this to be positive for some e, it is necessary and sufficient for the discriminant

$$(\bar{m}_3^2/I_1)^2 - 4\bar{m}_3^2 Mg\ell/I_1$$

to be positive; that is,

$$\bar{m}_3^2 > 4Mg\ell I_1\,,$$

which is the well-known stability condition for a fast top. We have proved the following.

Heavy top stability theorem. An upright spinning Lagrange top is stable provided that the angular velocity is strictly larger than $I_3^{-1}\sqrt{4Mg\ell I_1}$.

Remarks. (1) The method suggests but does not prove that one has instability when $\bar{m}_3^2 < 4Mg\ell I_1$. In fact, an eigenvalue analysis shows that the equilibrium is linearly unstable and hence unstable in this case. (2) When $I_2 = I_1 + \varepsilon$ for small ε, the conserved quantity $\phi(m_3)$ is no longer available. In this case, a sufficiently fast top is still linearly stable, but true stability can only be established by KAM theory. (3) In Holmes and Marsden [1983] it is shown that if $I_2 = I_1 + \varepsilon$ with ε sufficiently small, the phase portrait of (3.2EM) has Poincaré–Birkhoff–Smale horseshoes (see also Ziglin [1980, 1981]).

3.3. *Two-dimensional incompressible homogeneous flow* (Arnold [1965a, 1966b, 1969a])

A. *Equations of motion and Hamiltonian*

Let D be a domain in the xy plane bounded by smooth curves $(\partial D)_i$, $i = 0, \ldots, g$. We may take $(\partial D)_0$ to be the outer boundary, so $(\partial D)_1, \ldots, (\partial D)_g$ must encircle g holes in D. Denote by $\boldsymbol{v}$ the spatial velocity of the fluid moving in D. If the fluid is incompressible and homogeneous, and $\boldsymbol{v}(\boldsymbol{x}, \boldsymbol{t})$ denotes its spatial velocity, the equations of motion are Euler's equations:

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v} = -\boldsymbol{\nabla} p\,, \qquad \operatorname{div}\boldsymbol{v} = 0\,, \qquad \boldsymbol{v}(\boldsymbol{x}, 0) = \boldsymbol{v}_0(\boldsymbol{x})\,, \tag{3.3EM}$$

where the initial condition $v_0(x)$ is a given divergence free vector field on D, and the pressure p is a real-valued function on D determined (up to a constant) by the condition that $(\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v} + \boldsymbol{\nabla} p$ be divergence free and tangent to ∂D. In fact, this condition on p is equivalent to the Neumann problem

$$\nabla^2 p = -\operatorname{div}((\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v})\,, \qquad \partial p/\partial n = -\boldsymbol{n}\cdot(\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v}\,, \tag{3.3A1}$$

where $\boldsymbol{n}$ is the outward unit normal to ∂D.

The conserved energy for (3.3EM) is

$$H(v) = \tfrac{1}{2} \int_D |v|^2 \, dx \, dy . \tag{3.3H}$$

The space $P = \mathscr{X}_{\mathrm{div}}(D)$, consisting of smooth divergence free vector fields v on D that are tangent to ∂D can be given several topologies. One choice suitable for bounded regions is H^s, $s > 1$ as in Ebin and Marsden [1970]; another is $C^{k+\alpha}$, $k \geq 0$, $0 < \alpha < 1$ as in Kato [1967]. Corresponding weighted spaces can be used if D is unbounded, as in Cantor [1975, 1979]. (The topology chosen on P must be strong enough so that the differential calculus methods employed in steps B and C are justified. This means in effect here that the vorticity ω must be continuous and vanish at infinity. In particular, vorticities that are merely in L^∞ require a modified treatment, as in Wan and Pulvirente [1984] and Tang [1984]).

Remark A. If $F, G : P \to \mathbb{R}$, define their Poisson bracket by

$$\{F, G\}(v) = -\int_D v \cdot \left[\frac{\delta F}{\delta v}, \frac{\delta G}{\delta v}\right] dx \, dy , \tag{3.3PB}$$

where the functional derivative $\delta F/\delta v \in P$ is defined by

$$DF(v) \cdot \delta v = \left\langle \frac{\delta F}{\delta v}, \delta v \right\rangle = \int \frac{\delta F}{\delta v} \cdot \delta v \, dx \, dy$$

for any $\delta v \in P$, and

$$\left[\frac{\delta F}{\delta v}, \frac{\delta F}{\delta v}\right] = \left(\frac{\delta F}{\delta v} \cdot \nabla\right) \frac{\delta G}{\delta v} - \left(\frac{\delta G}{\delta v} \cdot \nabla\right) \frac{\delta F}{\delta v}$$

is the Lie bracket of the vector fields $\delta F/\delta v$ and $\delta G/\delta v$. (The bracket (3.3PB) is the Lie–Poisson bracket for the group of volume preserving diffeomorphisms and comes from the canonical bracket in the Lagrangian representation; see Arnold [1966a] and Marsden and Weinstein [1983].) The equations of motion (3.3EM) are obtained from the Poisson bracket (3.3PB) in the following manner. First note that $\delta H/\delta v = v$. Integrating by parts and taking into account $v \cdot n|_{\partial D} = 0$ (where n is the unit vector normal to the boundary) and the L^2-orthogonality of $\delta F/\delta v$ with the vector space $\nabla \mathscr{F}(D)$ of gradient vector fields, we get

$$\begin{aligned}\{F, H\} &= -\int_D v \cdot \left[\frac{\delta F}{\delta v}, v\right] dx \, dy = -\int_D \left\{ v \cdot \left(\frac{\delta F}{\delta v} \cdot \nabla\right) v - v \cdot (v \cdot \nabla) \frac{\delta F}{\delta v} \right\} dx \, dy \\ &= -\int_D \left\{ \frac{\delta F}{\delta v} \cdot \nabla \frac{|v|^2}{2} + \frac{\delta F}{\delta v} \cdot (v \cdot \nabla v) \right\} dx \, dy = -\left\langle \frac{\delta F}{\delta v}, \mathbb{P}((v \cdot \nabla) v) \right\rangle ,\end{aligned}$$

where $\mathbb{P}$ maps $\mathscr{X}(D)$ the space of all vector fields on D to P by L^2 orthogonal projection. Thus the equations of motion defined by H via (3.3PB) are $\dot{v} + \mathbb{P}((v \cdot \nabla) v) = \mathbf{0}$. To determine $\mathbb{P}((v \cdot \nabla) v)$

explicitly, write $(\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v} = \mathbb{P}((\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v}) - \boldsymbol{\nabla}p$, take the divergence of both sides and the dot product with $\boldsymbol{n}$ to get eqs. (3.3A1). This says that p is the pressure and that $\mathbb{P}((\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v}) = (\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v} + \boldsymbol{\nabla}p$, thus yielding eqs. (3.3EM).

There is another way to describe the Hamiltonian structure of the incompressible homogeneous two-dimensional Euler equations, starting with the vorticity equation

$$\frac{\partial\omega}{\partial t} + \boldsymbol{v}\cdot\boldsymbol{\nabla}\omega = 0\,, \tag{3.3VE}$$

where $\omega = \hat{z}\cdot\operatorname{curl}\boldsymbol{v}$ is the scalar vorticity. We shall denote by $\hat{z}$ the unit vector of the z-axis, pointing upward. The vorticity equation is obtained by applying the operator $\hat{z}\cdot\operatorname{curl}$ to (3.3EM). For regions in the xy plane, any $\boldsymbol{v}$ which is divergence free and parallel to the boundary can be written uniquely as

$$\boldsymbol{v} = \operatorname{curl}(\psi\hat{z})$$

where ψ is constant on $(\partial D)_i$ and zero on $(\partial D)_0$; ψ is called the *stream function*. To show the existence of ψ, we note that the integral of $\mathrm{i}_{\boldsymbol{v}}(\mathrm{d}x\wedge\mathrm{d}y)$ around each $(\partial D)_i$ is zero since $\boldsymbol{v}$ is tangent to ∂D; since $\operatorname{div}\boldsymbol{v} = 0$, one concludes that the integral around any closed loop is zero. Hence by elementary vector calculus, $\mathrm{i}_{\boldsymbol{v}}(\mathrm{d}x\wedge\mathrm{d}y) = \mathrm{d}\psi$ for some ψ. Since $\boldsymbol{v}$ is tangent to ∂D, ψ is constant on each $(\partial D)_i$; adding a suitable constant to ψ makes it zero on $(\partial D)_0$. The following argument shows that $\boldsymbol{v}$ is uniquely determined by ω and by the circulations $\Gamma_1,\ldots,\Gamma_g$. Indeed, it suffices to show that if the stream function ϕ satisfies $\nabla^2\phi = 0$, $\phi|(\partial D)_0 = 0$, $\phi|(\partial D)_i = c_i$, a constant for $i = 1,\ldots,g$ and $\oint_{(\partial D)_i}(\partial\phi|\partial n)\,\mathrm{d}s = 0$, then $v = 0$. But this follows from Green's identity:

$$0 = \int_D \phi\nabla^2\phi\,\mathrm{d}x\,\mathrm{d}y = \sum_{i=0}^{g} c_i \oint_{(\partial D)_i} \frac{\partial\phi}{\partial n}\,\mathrm{d}s - \int_D |\boldsymbol{\nabla}\phi|^2\,\mathrm{d}x\,\mathrm{d}y\,.$$

Thus the space P can be identified with $\mathcal{F}(D)\times\mathbb{R}^g = \{\text{vorticities}\}\times\{\text{circulations}\}$. This point of view, adopted in Marsden and Weinstein [1983], is especially useful for simply connected domains. The Hamiltonian is seen to be

$$H(\omega,\Gamma_1,\ldots,\Gamma_g) = -\tfrac{1}{2}\int_D \psi\,\omega\,\mathrm{d}x\,\mathrm{d}y + \sum_{i=1}^{g} c_i\Gamma_i\,,$$

where c_i are the constant values of ψ on $(\partial D)_i$; note that if D is simply connected the last sum is omitted.

For simply connected D, the Lie–Poisson bracket in terms of vorticity equals (see Marsden and Weinstein [1983])

$$\{F,G\}(\omega) = \int_D \omega\left\{\frac{\delta F}{\delta\omega},\frac{\delta G}{\delta\omega}\right\}_{xy}\mathrm{d}x\,\mathrm{d}y\,, \tag{ωPB}$$

where $\{\,,\}_{xy}$ is the canonical (x,y)-Poisson bracket. The symbols $\delta F/\delta\omega$ in this formula must be interpreted with care, as in Marsden and Weinstein [1983]. If $\delta F/\delta\omega$ is interpreted as the usual functional derivative, (ωPB) is incorrect; to correct it a boundary term must be added as in Lewis et al. [1985].

B. *Constants of motion*

For any smooth function $\Phi:\mathbb{R}\to\mathbb{R}$ the vorticity integrals

$$C_\Phi = \int_D \Phi(\omega)\,\mathrm{d}x\,\mathrm{d}y \tag{3.3C}$$

are easily seen to be conserved using the vorticity equation (3.3VE). Here C_Φ is regarded as a function of $\boldsymbol{v}$ for (3.3PB) and of ω for (ωPB). Let ∂D consist of $g+1$ components $(\partial D)_i$, $i=0,\ldots,g$ and let

$$\Gamma_i(\boldsymbol{v}) = \oint_{(\partial D)_i} \boldsymbol{v}\cdot\mathrm{d}\boldsymbol{\ell}. \tag{3.3Γ}$$

Conservation of Γ_i is Kelvin's circulation theorem.

Remark B. The coadjoint action of $\mathrm{Diff}_{\mathrm{vol}}(D)$ on $\mathscr{X}_{\mathrm{div}}(D)$ is given by

$$\eta\cdot\boldsymbol{v} = (T\eta^{-1})^{+}\circ\boldsymbol{v}\circ\eta^{-1},$$

where $(T\eta^{-1})^{+}$ is the adjoint of $T\eta^{-1}$ pointwise on D, with respect to the Euclidean metric on $\mathbb{R}^2$. It is easily verified that both C_Φ and Γ_i are invariant under this action, i.e., are Casimir functions. If D is simply connected then the Poisson bracket (3.3PB) of C_Φ with any functional of ω vanishes. We hasten to add, however, that if D is not simply connected, the functional derivatives of C_Φ and Γ_i involve delta function distributions, so one has to interpret the bracket of C_Φ and Γ_i with any other function on $\mathscr{X}_{\mathrm{div}}(D)$ with care. Also note that velocity fields corresponding to point vortices and vortex patches are not representable as smooth elements of $\mathscr{X}_{\mathrm{div}}(D)$.

C. *First variation*

Let $\boldsymbol{a}=(a_0,\ldots,a_g)$ be a vector of constants and let

$$H_C(\boldsymbol{v}) = H(\boldsymbol{v}) + C_\Phi(\boldsymbol{v}) + \sum_{i=0}^{g} a_i\Gamma_i(\boldsymbol{v})$$

$$= \int_D [\tfrac{1}{2}|\boldsymbol{v}|^2 + \Phi(\omega)]\,\mathrm{d}x\,\mathrm{d}y + \sum_{i=0}^{g} a_i \oint_{(\partial D)_i} \boldsymbol{v}\cdot\mathrm{d}\boldsymbol{\ell}.$$

(The terms C_Φ and $\Sigma_{i=0}^{g} a_i\Gamma_i$ are not all independent, but this form proves to be convenient here and in later examples as well.) The first variation is

$$\mathrm{D}H_C(\boldsymbol{v}_e)\cdot\delta\boldsymbol{v} = \int_D [\boldsymbol{v}_e\cdot\delta\boldsymbol{v} + \Phi'(\omega_e)\hat{\boldsymbol{z}}\cdot\mathrm{curl}\,\delta\boldsymbol{v}]\,\mathrm{d}x\,\mathrm{d}y + \sum_{i=0}^{g} a_i \oint_{(\partial D)_i} \delta\boldsymbol{v}\cdot\mathrm{d}\boldsymbol{\ell}.$$

Integrating by parts the second term in the first integral gives

$$\int_D \Phi'(\omega_e)\hat{z}\cdot\operatorname{curl}\delta v \,dx\,dy = -\int_D \operatorname{div}(\Phi'(\omega_e)\hat{z}\times\delta v)\,dx\,dy + \int_D \delta v\cdot\operatorname{curl}(\Phi'(\omega_e)\hat{z})\,dx\,dy$$

$$= \sum_{i=0}^{g} \oint_{(\partial D)_i} \Phi'(\omega_e)\delta v\cdot d\ell + \int_D \delta v\cdot\operatorname{curl}(\Phi(\omega_e)\hat{z})\,dx\,dy\,.$$

Thus, since ω_e is constant on every component $(\partial D)_i$, $i = 0, 1, \ldots, g$, we get

$$DH_C(v_e)\cdot\delta v = \int_D [v_e + \operatorname{curl}(\Phi'(\omega_e)\hat{z})]\cdot\delta v\,dx\,dy + \sum_{i=0}^{g}(a_i + \Phi'(\omega_e|(\partial D)_i))\oint_{(\partial D)_i}\delta v\cdot d\ell\,.$$

Thus $DH_C(v_e) = 0$ provided

$$a_i = -\Phi'(\omega_e|(\partial D)_i)\,,\qquad i = 0,\ldots,g\,, \tag{3.3C1}$$

$$v_e + \operatorname{curl}(\Phi'(\omega_e)\hat{z}) = 0\,. \tag{3.3C2}$$

The relations (3.3C1) give the numbers a_i, once Φ is determined by (3.3C2). In order for (3.3C2) to yield a differential equation for Φ, one needs a functional relation between v_e and ω_e which can be found in the following manner. (Here we use a method a bit different from Arnold's, to facilitate the subsequent exposition.) The equations of motion (3.3EM) can also be written as

$$\frac{\partial v}{\partial t} = -\omega\times v - \nabla(p + |v|^2/2)\,, \tag{3.3C3}$$

so that applying the operator $\hat{z}\cdot\operatorname{curl}$ gives the vorticity equation

$$\partial\omega/\partial t = -v\cdot\nabla\omega\,. \tag{3.3C4}$$

For stationary flows we thus have from (3.3C4) and (3.3C3)

$$v_e\cdot\nabla\omega_e = 0\,, \tag{3.3C5}$$

$$\omega_e\hat{z}\times v_e = -\nabla(|v_e|^2/2 + p_e)\,. \tag{3.3C6}$$

Taking the dot product with v_e gives

$$v_e\cdot\nabla(|v_e|^2/2 + p_e) = 0\,. \tag{3.3C7}$$

A sufficient condition for (3.3C5) and (3.3C7) to hold is the functional relationship (Bernoulli's Law)

$$|v_e|^2/2 + p_e = K(\omega_e)\,, \tag{3.3C8}$$

where K is called the Bernoulli function. Taking the cross product of (3.3C6) with $\hat{z}$ on the left and taking into account (3.3C8) gives for $\omega_e \neq 0$

$$v_e = \frac{1}{\omega_e}\,\hat{z} \times \nabla K(\omega_e)\,. \tag{3.3C9}$$

Thus Φ is determined via (3.3C2) and (3.3C9) by

$$-\,\omega_e\,\mathrm{curl}(\Phi'(\omega_e)\hat{z}) = \hat{z} \times \nabla K(\omega_e)\,. \tag{3.3C9}$$

This holds if

$$\omega_e \Phi''(\omega_e) = K'(\omega_e)\,, \tag{3.3C10}$$

i.e.,

$$\Phi(\lambda) = \lambda\Big(\int^{\lambda} \frac{K(t)}{t^2}\,\mathrm{d}t + \mathrm{const.}\Big)\,.$$

We have proved the following.

Proposition. *Stationary solutions v_e of the two-dimensional, homogeneous, incompressible, Euler flow with $\omega_e \neq 0$ are critical points of $H + C\Phi + \Sigma_{i=0}^{g} a_i\Gamma_i$, where*

$$\Phi(\lambda) = \lambda\Big(\int^{\lambda} \frac{K(t)}{t^2}\,\mathrm{d}t + \mathrm{const.}\Big)\,,$$

K is the Bernoulli function for the stationary solution v_e, and

$$a_i = -\Phi'(\omega_e|(\partial D)_i)\,.$$

If ψ denotes the stream function for v, i.e. $v = (\psi_y, -\psi_x)$, then proceeding as before, the condition $v_e \cdot \nabla\omega_e = 0$ becomes $\{\psi_e, \omega_e\} = 0$ which holds if ψ_e and ω_e are functionally related. Thus, if $\omega_e \neq 0$, there exists a function Ψ such that $\psi_e = \Psi(\omega_e)$. On the other hand, v_e is a critical point of H_C($g = 0$ in this case) if $\psi_e + \Phi'(\omega_e) = 0$, i.e., if $\Phi' = -\Psi$, and one could now state the above proposition in this case, in the form given by Arnold [1965a, 1969a].

Remark D. Formal stability. The second variation of $H_C = H + C_\Phi + \Sigma_{i=0}^{g} a_i\Gamma_i$ is

$$\mathrm{D}^2 H_C(v_e)\cdot(\delta v, \delta v) = \int_D (|\delta v|^2 + \Phi''(\omega_e)(\delta\omega)^2)\,\mathrm{d}x\,\mathrm{d}y\,.$$

If the domain is simply connected, this expression equals

$$\int_D [-\delta\omega(\nabla^2)^{-1}\delta\omega + \Phi''(\omega_e)(\delta\omega)^2]\,\mathrm{d}x\,\mathrm{d}y\,,$$

where $-\delta\psi = (\nabla^2)^{-1}\delta\omega$ denotes the unique solution of the problem $-\nabla^2\delta\psi = \delta\omega$, $\delta\psi|(\partial D)_0 = 0$. This quadratic form is positive definite if $\Phi''(\omega_e) > 0$. If $\Phi''(\omega_e)$ is sufficiently negative (as determined from the Poincaré inequality for the domain D), this form is negative definite. (In the latter case, the conditions for formal stability are weaker than those given by the convexity analysis, as noted in the final remark of Arnold [1969a].) Linearized stability follows now from definiteness of $\mathrm{D}^2H_C(\boldsymbol{v}_e)\cdot(\delta\boldsymbol{v}, \delta\boldsymbol{v})$ when either $\Phi''(\omega_e) > 0$, or $\Phi''(\omega_e)$ is sufficiently negative. As will be clear below, this linearized stability condition slightly generalizes Rayleigh's result [1880] that a plane-parallel incompressible shear flow requires an inflection point in its velocity profile in order to be linearly unstable.

D. *Convexity estimates*

Since H is quadratic, condition (CH) from section 2 is trivially satisfied with $Q_1 = H$. For (CC), we require

$$Q_2(\Delta\omega) \le \int_D [\Phi(\omega_e + \Delta\omega) - \Phi(\omega_e) - \Phi'(\omega_e)\cdot\Delta\omega]\,\mathrm{d}x\,\mathrm{d}y\,.$$

This holds with $Q_2(\Delta\omega) - c_2\int_D (\Delta\omega)^2\,\mathrm{d}x\,\mathrm{d}y$, where c_2 is a constant, provided $c_2 \le \Phi''(\lambda)$ for all λ.

Condition (D) requires

$$\int_D |\Delta\boldsymbol{v}|^2\,\mathrm{d}x\,\mathrm{d}y + c_2\int_D (\Delta\omega)^2\,\mathrm{d}x\,\mathrm{d}y > 0$$

for all $\Delta\boldsymbol{v} \neq 0$. This holds, for example, if $c_2 > 0$. This quadratic form can be negative definite in certain cases where $c_2 < 0$ because of the Poincaré inequality, as shown by Arnold [1965a]. Thus, there are two cases to consider for stability: $\Phi''(\lambda) \ge c_2 > 0$ and $-\Phi''(\lambda) \ge -c_2 > 0$. By (3.3C9) and (3.3C10) these conditions translate into conditions on the flow velocity profile at equilibrium, since

$$\Phi''(\omega_e) = K'(\omega_e)/\omega_e = \boldsymbol{v}_e\cdot\hat{\boldsymbol{z}}\times\nabla\omega_e/|\nabla\omega_e|^2\,.$$

For example, plane-parallel incompressible flows along the x-axis in the strip $0 \le y \le Y$ have

$$\boldsymbol{v}_e = \hat{\boldsymbol{x}}u(y)\,,\qquad \omega_e = -u'(y)\,,\qquad \boldsymbol{v}_e\cdot\hat{\boldsymbol{z}}\times\nabla\omega_e/|\nabla\omega_e|^2 = u(y)/u''(y)\,.$$

Consequently, for such flows the requirement for stability in the first case above becomes $\Phi''(\omega_e(y)) = u(y)/u''(y) \ge c_2 > 0$. Thus, when the sign of u is everywhere the same as the sign of u'', all flows having no inflection points will be stable. Existence of an inflection point, however, does not necessarily imply instability. Consider stationary plane-parallel flows in the second case, with $-u(y)/u''(y) \ge -c_2 > 0$.

Then one bounds $-H_C$ to find

$$-(Q_1+Q_2) \geq \int_D [(\Delta\omega)(\nabla^2)^{-1}(\Delta\omega) - c_2(\Delta\omega)^2]\,dx\,dy \geq \int_D (-k_{\min}^{-2} - c_2)(\Delta\omega)^2\,dx\,dy\,,$$

where $k_{\min}^2$ is the minimum eigenvalue of minus the Laplacian $(-\nabla^2)$ in the domain D. Consequently, stationary flows with $\min|\Phi''(\omega_e)| > k_{\min}^{-2}$ and thus, (Q_1+Q_2) negative definite, will be stable. For example, sinusoidal plane-parallel flows $u(y) = \sin(ky)$ with $k^{-2} > k_{\min}^{-2}$ are stable. (This statement is a bit imprecise: if the region is a strip $0 \leq y \leq d$ and periodic in x, then one must confine oneself to perturbations which preserve the circulations and flow rates in the x-direction. The reason is that it is only for such perturbations that the kinetic energy has the form $-\int_D \omega(\nabla^2)^{-1}\omega\,dx\,dy$; see Holm, Marsden and Ratiu [1985] for details).

E. *A priori estimates*

For $c_2 > 0$, the estimate (E) from section 2 gives the following estimate on the growth of perturbations:

$$\int_D |\Delta \boldsymbol{v}|^2\,dx\,dy + c_2 \int_D (\Delta\omega)^2\,dx\,dy \leq \tfrac{1}{2}\int_D |\Delta \boldsymbol{v}_0|^2\,dx\,dy + \int_D \Phi(\omega_0)\,dx\,dy - \tfrac{1}{2}\int_D |\Delta \boldsymbol{v}_e|^2\,dx\,dy - \int_D \Phi(\omega_e)\,dx\,dy\,, \tag{3.3E}$$

where $\omega_0 = \omega|_{t=0}$ and $\Delta\omega = \omega - \omega_e$ depends on time.

F. *Nonlinear stability*

For $c_2 > 0$, we set

$$\|\Delta \boldsymbol{v}\|^2 = \int_D |\Delta \boldsymbol{v}|^2\,dx\,dy + c_2 \int_D (\Delta\omega)^2\,dx\,dy\,. \tag{3.3N}$$

This norm is equivalent to the H^1 norm on $\Delta \boldsymbol{v}$, so we get stability estimates from (3.3E) that are H^1 in $\boldsymbol{v}$. [If $c_2 < 0$, the estimates are only L^2 in $\boldsymbol{v}$, as noted by Arnold [1969a].] With $c_2 > 0$, (CH)′ holds, and (CC)′ holds provided $K'(\lambda)/\lambda = \Phi''(\lambda) \leq C_2$ for some $C_2 < \infty$. If one works in terms of a stream function for the velocity field, this condition becomes

$$\Psi' \geq -C_2\,.$$

Results of Wolibner [1933], Iudovich [1963] and Kato [1967] show that global solutions exist in the space P. Thus we can state the following result of Arnold [1969a]:

Rayleigh–Arnold stability theorem. Stationary solutions v_e of the two-dimensional homogeneous incompressible Euler flow with $\omega_e \neq 0$ are (nonlinearly, Liapunov) stable in the norm (3.3N) provided the equilibrium solution satisfies

$$0 < c_2 \le K'(\omega_e)/\omega_e \le C_2 < \infty ,$$

where K is the Bernoulli function. Equivalently, this condition can be replaced by

$$0 < c_2 \le \frac{v_e \cdot \hat{z} \times \nabla\omega_e}{|\nabla\omega_e|^2} \le C_2 < \infty .$$

Example (Kelvin–Stuart cat's eyes).* In addition to the shear flow example already discussed, we show now that the methods can be applied to a stationary flow due to Kelvin [1880] in the linearized case and Stuart [1967] in the nonlinear case. The linear stability analysis for this example and an analysis of nonlinear terms were given by Stuart [1971].

The stationary solution of the two-dimensional Euler equations we consider is given in the xy-plane by

$$\omega_e = -\exp(-2\psi_e) = -[a \cosh y + (a^2 - 1)^{1/2} \cos x]^{-2} , \qquad a \ge 1 .$$

The streamlines are the familiar pattern in fig. 1. In this case $\omega_e < 0$ and $\Phi'(\omega_e) = (2\omega_e)^{-1} < 0$, so Q_1 and Q_2 have opposite sign. To get stability we use the Poincaré inequality and require $\min|\Phi''(\omega_e)| > k_{\min}^{-2}$ (see the discussion in remark D above). This requires a bounded region, so we limit our flows to be 2π periodic in x and bounded by streamlines in y, as in fig. 1. One finds that below a critical value of $a < 1.175\ldots$, the region can be chosen to contain the separatrices in fig. 1 and so produces nonlinear stability for the cat's eyes, as long as perturbations are initially chosen to have the same circulation as the cats eyes, and zero net flow rate in the x-direction. See Holm, Marsden and Ratiu [1985] for further details.

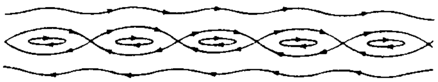

Fig. 1.

3.4. *Two-dimensional barotropic flow* (*Holm et al.* [1983*b*], *Grinfeld* [1984])

A. *Equations of motion and Hamiltonian*

Let D be a domain in $\mathbb{R}^2$ with smooth boundary. The evolution equations for the velocity field $v(x, y, t)$ and density $\rho(x, y, t)$ are

$$\frac{\partial v}{\partial t} + (v \cdot \nabla)v = -\nabla h(\rho) , \qquad \frac{\partial \rho}{\partial t} + \operatorname{div}(\rho v) = 0 , \tag{3.4EM}$$

* We thank John Gibbon for pointing out this example.

where $\boldsymbol{v}$ is parallel to ∂D and $h(\rho)$ is the specific enthalpy, a given function of $\rho > 0$, satisfying $p'(\rho) = \rho h'(\rho)$, where p is the pressure.

We choose P to be a space of $\boldsymbol{v}$ and ρ that are C^1 (say H^s, $s > 2$) and tending to a fixed vector field and density at ∞ if D is unbounded (in the weighted spaces as in example 3.3 say), or with $\boldsymbol{v}$ parallel to ∂D. We shall also need to exclude from the beginning of the discussion certain important features over which the present methods have no control. These are as follows, taken as part of our definition of P;

(a) shocks; solutions considered are C^1;

(b) cavitation and extreme compression: the density satisfies $0 < \rho_{\min} \le \rho \le \rho_{\max} < \infty$, where $\rho_{\min}$ and $\rho_{\max}$ are constants (that will shortly be required to satisfy certain inequalities involving other constants in the problem).

The conserved energy is

$$H(\boldsymbol{v}, \rho) = \int_D [\tfrac{1}{2}\rho|\boldsymbol{v}|^2 + \varepsilon(\rho)]\, dx\, dy\,,$$

where $\varepsilon(\rho)$ is the internal energy per unit area, related to the specific enthalpy by $\varepsilon'(\rho) = h(\rho)$.

Remark A. The equations of motion (3.4EM) are Hamiltonian. The configuration space of compressible fluid motion is the group of diffeomorphisms of D whose Lie algebra consists of the space $\mathscr{X}(D)$ of all vector fields on D. $\mathscr{X}(D)$ is represented on the vector space $\mathscr{F}(D)$ of functions on D by minus the Lie derivative, i.e.,

$$\boldsymbol{X} \cdot f := -\boldsymbol{X}[f] = -df(\boldsymbol{X})\,, \quad \text{for} \quad \boldsymbol{X} \in \mathscr{X}(D)\,,$$

$f \in \mathscr{F}(D)$. On the dual of the semidirect product $\mathscr{X}(D) \circledS \mathscr{F}(D)$ with variables $\boldsymbol{M} = \rho\boldsymbol{v}$ and ρ, the eqs. (3.4EM) are Hamiltonian (i.e. (PB) section 2 holds) relative to the Lie–Poisson bracket

$$\{F, G\} = \int_D \boldsymbol{M} \cdot \left[\left(\frac{\delta G}{\delta \boldsymbol{M}} \cdot \nabla\right)\frac{\delta F}{\delta \boldsymbol{M}} - \left(\frac{\delta F}{\delta \boldsymbol{M}} \cdot \nabla\right)\frac{\delta G}{\delta \boldsymbol{M}}\right] dx\, dy$$

$$+ \int_D \rho\left[\frac{\delta G}{\delta \boldsymbol{M}} \cdot \left(\nabla \frac{\delta F}{\delta \rho}\right) - \frac{\delta F}{\delta \boldsymbol{M}} \cdot \left(\nabla \frac{\delta G}{\delta \rho}\right)\right] dx\, dy\,.$$

This bracket is found in Iwinski and Turski [1976], Morrison and Greene [1980] and Dzyaloshinsky and Volovick [1980]; see also Dashen and Sharp [1965] and Bialynicki-Birula and Iwinski [1973]. The bracket was derived from Clebsch variables by Enz and Turski [1979], Greene, Holm and Morrison [1980], by Morrison [1982] and Holm and Kupershmidt [1983]. This bracket is the Lie–Poisson bracket for a semi-direct product. This is noted in Marsden [1982], where it is also pointed out that the bracket could be obtained as an instance of the abstract results concerning the Lagrange to Euler map of Ratiu [1980] and Guillemin and Sternberg [1980]. Holm and Kupershmidt [1983] also showed that other interesting systems, such as MHD are Lie–Poisson for semi-direct products. These and related brackets are derived from canonical brackets in Lagrangian representation in Marsden, Weinstein et al. [1983], Holm, Kupershmidt and Levermore [1983a] and Marsden, Ratiu and Weinstein [1984a,b].

B. *Constants of motion*

From (3.4EM) one finds that ω/ρ is advected by the flow, i.e., $\partial(\omega/\rho)/\partial t + \boldsymbol{v}\cdot\nabla(\omega/\rho) = 0$. Thus, for any function $\Phi:\mathbb{R}\to\mathbb{R}$, the quantity

$$C_\Phi(\boldsymbol{v},\rho) = \int_D \rho\Phi(\omega/\rho)\,\mathrm{d}x\,\mathrm{d}y$$

is a constant of the motion, where $\omega = \hat{z}\cdot(\nabla\times\boldsymbol{v})$ is the scalar vorticity. Similarly, by Kelvin's circulation theorem the quantities

$$\Gamma_i(\boldsymbol{v},\rho) = \oint_{(\partial D)_i} \boldsymbol{v}\cdot\mathrm{d}\boldsymbol{\ell}, \qquad i = 0,\dots,g$$

are conserved, where $(\partial D)_i$ are the connected components of the boundary.

Remark B. The functions C_Φ are Casimirs for the Poisson structure in remark A. This can be checked directly, or it can be proved by noting that C_Φ, as a function of $(\boldsymbol{M},\rho)$, is invariant under the coadjoint action of $\mathrm{Diff}(D)\,Ⓢ\,\mathscr{F}(D)$ (semidirect product of the group of diffeomorphisms and functions) on P. For $(\eta, f)\in\mathrm{Diff}(D)\,Ⓢ\,\mathscr{F}(D)$, this action is

$$(\eta, f)\cdot(M,\rho) = (\eta_* M - \mathrm{d}f\otimes\eta_*\rho,\ \eta_*\rho)$$

where ρ is regarded as a density. Similarly, all Γ_i are invariant under the coadjoint action, but they do not have functional derivatives in the usual sense of the formal calculus of variations. Thus, their brackets with arbitrary functionals require care in interpretation; see Lewis et al. [1985].

C. *First variation*

Let $(\boldsymbol{v}_e,\rho_e)$ be an equilibrium solution of (3.4EM). Then $H_C(\boldsymbol{v},\rho) = H(\boldsymbol{v},\rho) + C_\Phi(\boldsymbol{v},\rho) + \sum_{i=0}^{g} a_i\Gamma_i(\boldsymbol{v},\rho)$ has a critical point at $(\boldsymbol{v}_e,\rho_e)$, provided the following holds for all $\delta\boldsymbol{v}$, $\delta\rho$ (such that $(\boldsymbol{v}_e+\delta\boldsymbol{v},\ \rho_e+\delta\rho)$ lies in P):

$$0 = \mathrm{D}H_C(\boldsymbol{v}_e,\rho_e)\cdot(\delta\boldsymbol{v},\delta\rho)$$

$$= \int_D [\rho_e\boldsymbol{v}_e\cdot\delta\boldsymbol{v} + \Phi'(\omega_e/\rho_e)\hat{z}\cdot(\nabla\times\delta\boldsymbol{v})]\,\mathrm{d}x\,\mathrm{d}y + \sum_{i=0}^{g} a_i \oint_{(\partial D)_i} \delta\boldsymbol{v}\cdot\mathrm{d}\boldsymbol{\ell}$$

$$+ \int_D \left[\frac{|\boldsymbol{v}_e|^2}{2} + h(\rho_e) + \Phi\left(\frac{\omega_e}{\rho_e}\right) - \frac{\omega_e}{\rho_e}\Phi'\left(\frac{\omega_e}{\rho_e}\right)\right]\delta\rho\,\mathrm{d}x\,\mathrm{d}y\,.$$

Integrating the second term in the first integral by parts gives

$$0 = \int_D \left\{ \left[\frac{|v_e|^2}{2} + h(\rho_e) + \Phi\left(\frac{\omega_e}{\rho_e}\right) - \frac{\omega_e}{\rho_e} \Phi'\left(\frac{\omega_e}{\rho_e}\right) \right] \delta\rho + \left[\rho_e v_e - \hat{z} \times \nabla \Phi'\left(\frac{\omega_e}{\rho_e}\right) \right] \cdot \delta v \right\} dx\, dy$$

$$+ \sum_{i=0}^{g} \oint_{(\partial D)_i} \Phi'\left(\frac{\omega_e}{\rho_e}\right) \delta v \cdot d\ell + \sum_{i=0}^{g} a_i \oint_{(\partial D)_i} \delta v \cdot \delta \ell . \tag{3.4FV}$$

For stationary solutions, ω_e/ρ_e is constant along streamlines, so the $2(g+1)$ boundary terms cancel, provided $a_i = -\Phi'((\omega_e/\rho_e)|(\partial D)_i)$. From (3.4EM), stationary flows satisfy

$$v_e \cdot \nabla(|v_e|^2/2 + h(\rho_e)) = 0 , \qquad v_e \cdot \nabla(\omega_e/\rho_e) = 0 . \tag{3.4C1}$$

This is consistent with assuming a Bernoulli Law

$$|v_e|^2/2 + h(\rho_e) = K\left(\frac{\omega_e}{\rho_e}\right) , \tag{3.4C2}$$

for K a smooth function of a real variable. The condition $0 = DH_C(v_e, \rho_e) \cdot (\delta v, \delta \rho)$ holds if the coefficients of $\delta\rho$ and δv vanish. For $\delta\rho$ this is

$$K(\zeta) + \Phi(\zeta) - \zeta \Phi'(\zeta) = 0 ,$$

which uniquely determines Φ (up to a constant):

$$\Phi(\zeta) = \zeta \left(\int^{\zeta} \frac{K(t)}{t^2} dt + \text{const.} \right) .$$

An important point is that the coefficient of δv in (3.4FV) also vanishes by virtue of the expression for Φ. Indeed, from Bernoulli's Law,

$$\nabla(|v_e|^2/2 + h(\rho_e)) = \nabla K(\omega_e/\rho_e) ,$$

so that for stationary solutions, (3.4EM) gives

$$0 = \partial v_e / \partial t = -\nabla(|v_e|^2/2 + h(\rho_e)) + v_e \times \omega_e \hat{z} ,$$

and hence

$$v_e \times \omega_e \hat{z} = \nabla K\left(\frac{\omega_e}{\rho_e}\right) \quad \text{or} \quad \rho_e v_e = \frac{\rho_e}{\omega_e} \hat{z} \times \nabla K\left(\frac{\omega_e}{\rho_e}\right) = \hat{z} \times \nabla \Phi'\left(\frac{\omega_e}{\rho_e}\right) ,$$

using the relation $K'(\zeta) - \zeta \Phi''(\zeta)$ between K and Φ. Consequently, we have the following.

Proposition. Stationary solutions $(\boldsymbol{v}_e, \rho_e)$ of two-dimensional barotropic Euler flow with $\rho_e > 0$ are critical points of $H + C_\Phi + \Sigma_{i=0}^{g} a_i \Gamma_i$, where Φ is given in terms of the Bernoulli function K for the stationary solution by

$$\Phi(\zeta) = \zeta\left(\int^{\zeta} \frac{K(t)}{t^2}\,\mathrm{d}t + \text{const.}\right), \qquad a_i = -\Phi'(\omega_e/\rho_e)|(\partial D)_i\,.$$

Remark D. The second variation of $H_C(\boldsymbol{v}_e, \rho_e)$ is computed to be

$$\mathrm{D}^2 H_C(\boldsymbol{v}_e, \rho_e)(\delta \boldsymbol{v}, \delta\rho) = \int_D \left\{\frac{|\delta(\rho \boldsymbol{v})|^2}{\rho_e} + \left[\varepsilon''(\rho_e) - \frac{|\boldsymbol{v}_e|^2}{\rho_e}\right](\delta\rho)^2 + \frac{1}{\omega_e} K'\left(\frac{\omega_e}{\rho_e}\right)\left[\delta\left(\frac{\omega}{\rho}\right)\right]^2\right\} \mathrm{d}x\,\mathrm{d}y, \qquad \text{(3.4SV)}$$

where $\delta(\rho \boldsymbol{v}) := \boldsymbol{v}_e \delta\rho + \rho_e \delta \boldsymbol{v}$ and $\delta(\omega/\rho) := (\rho_e \delta\omega - \omega_e \delta\rho)/\rho_e^2$.

Expression (3.4SV), suggests that conditions for stability are $\rho_e > 0$ and $\varepsilon''(\rho_e)\rho_e > |\boldsymbol{v}_e|^2$ (the latter meaning that the stationary flow is subsonic), and $(1/\omega_e)K'(\omega_e/\rho_e) > 0$. This is the condition for linearized stability, but the nonlinear theory requires more stringent conditions. (The second variation calculation has also recently been done by Grinfeld [1984].)

D. *Convexity estimates*

We have, after a short computation,

$$H(\boldsymbol{v}_e + \Delta \boldsymbol{v}, \rho_e + \Delta\rho) - H(\boldsymbol{v}_e, \rho_e) - \mathrm{D}H(\boldsymbol{v}_e, \rho_e)\cdot(\Delta \boldsymbol{v}, \Delta\rho)$$

$$= \int_D \left\{\frac{|\Delta(\rho \boldsymbol{v})|^2}{2\rho} - \frac{|\boldsymbol{v}_e|^2}{2}\frac{(\Delta\rho)^2}{\rho} + [\varepsilon(\rho_e + \Delta\rho) - \varepsilon(\rho_e) - \varepsilon'(\rho_e)\Delta\rho]\right\} \mathrm{d}x\,\mathrm{d}y,$$

where $\Delta(\rho \boldsymbol{v}) := (\rho_e + \Delta\rho)(\boldsymbol{v}_e + \Delta \boldsymbol{v}) - \rho_e \boldsymbol{v}_e$. Assume $\varepsilon''(\tau) \geq c_{\min}^2/\tau$ for all τ and a constant $c_{\min}$ (the minimum sound speed). Then we get (CH) with

$$Q_1(\Delta(\rho \boldsymbol{v}), \Delta\rho) = \tfrac{1}{2}\int_D \left\{\frac{|\Delta(\rho \boldsymbol{v})|^2}{\rho_{\max}} + \left[\frac{c_{\min}^2}{\rho_{\max}} - \frac{|\boldsymbol{v}_e|^2}{\rho_{\min}}\right](\Delta\rho)^2\right\} \mathrm{d}x\,\mathrm{d}y,$$

where $0 < \rho_{\min} \leq \rho \leq \rho_{\max} < \infty$. Note that Q_1 is a quadratic form in the variables $(\rho \boldsymbol{v}, \rho)$ rather than $(\boldsymbol{v}, \rho)$.

If the Bernoulli function K satisfies

$$a \leq \frac{1}{\zeta} K'(\zeta) = \Phi''(\zeta),$$

then one finds (CC) with a quadratic form in $\Delta(\omega/\rho)$:

$$Q_2(\Delta(\rho \boldsymbol{v}), \Delta\rho) = \tfrac{1}{2} a \rho_{\min} \int_D [\Delta(\omega/\rho)]^2 \, dx \, dy, \qquad \text{(CC)}$$

where $\Delta(\omega/\rho) := [\omega_e + \Delta\omega)/(\rho_e + \Delta\rho) - \omega_e/\rho_e]$. Thus, (D) holds provided

$$a > 0 \quad \text{and} \quad c^2_{\min}/\rho_{\max} > |\boldsymbol{v}_e|^2/\rho_{\min}.$$

E. *A priori estimates*
The estimate (E) of section 2 holds where Q_1 and Q_2 are as above.

F. *Nonlinear stability*
If we have

$$\varepsilon''(\tau) \le c^2_{\max}/\rho_{\min} \quad \text{for all } \tau, \quad \rho_{\min} \le \tau \le \rho_{\max}$$

and

$$\frac{1}{\zeta} K'(\zeta) \le A < \infty,$$

then (CH)′ and (CC)′ hold for arguments similar to those given in step D. Thus, with this hypothesis, and for solutions in P satisfying $\rho_{\min} \le \rho \le \rho_{\max}$, we have Liapunov stability in the norm $\| \ \|^2 = Q_1 + Q_2$ as long as solutions remain in P. (The existence theory for solutions to these equations is not well established, except for short-time solutions – see Courant and Hilbert [1962, Vol. II] – so there is little more one can expect in the present circumstances.)

We summarize our results as follows.

Stability theorem. Stationary solutions $(\boldsymbol{v}_e, \rho_e)$ of the two-dimensional barotropic Euler flow which satisfy the conditions

$$0 < \rho_{\min} \le \rho_e \le \rho_{\max} < \infty, \qquad \text{(3.4SC1)}$$

$$0 < a \le \frac{1}{\zeta} K'(\zeta) \le A < \infty, \qquad \text{(3.4SC2)}$$

$$c^2_{\min}/\rho_{\max} \le \varepsilon''(\tau) \le c^2_{\max}/\rho_{\min}, \qquad \text{(3.4SC3)}$$

where K is the Bernoulli function for $(\boldsymbol{v}_e, \rho_e)$ are conditionally stable in the norm on $(\rho \boldsymbol{v}, \rho)$ given by $Q_1 + Q_2$, that is, perturbations from equilibrium are a priori bounded in time in the norm determined by $Q_1 + Q_2$ as long as the solutions satisfy $\rho_{\min} \le \rho \le \rho_{\max}$.

Example A. *Shear flow.* A stationary solution of (3.4EM) in the strip $\{(x, y) \in \mathbb{R}^2 | Y_1 \le y \le Y_2\}$, is given by the plane parallel flows with arbitrary velocity profile $\boldsymbol{v}_e(x, y) = (u(y), 0)$ and constant density $\rho_e = 1$. We can allow x to be unrestricted in $\mathbb{R}$ or to be periodic. In the former case, we require that the perturbations allowed be initially square integrable. Note that $(\omega_e/\rho_e)(x, y) = -u'(y)$. Let c_e denote the sound speed of this stationary solution. By our earlier analysis this flow is formally hence linearized, stable if and only if $c_e^2 - u(y)^2 > 0$ and $u(y)/u''(y) > 0$.

The hypothesis on the existence of the Bernoulli function K is in this case $u''(y) \neq 0$. In other words, plane parallel flows with constant density and velocity profile with no inflection point are formally, hence linearly, stable. This is analogous to Rayleigh's theorem for the incompressible problem.

We turn now to the study of our a priori estimates for this shear flow. For this, we must compute the Bernoulli function K from its defining relation (3.4C2) under the hypothesis $\nabla(\omega_e/\rho_e) = u''(y)\hat{y} \neq 0$. Denote by ϕ the inverse of u; we get $K(\zeta) = u[\phi(\zeta)]^2/2 + h(1)$ and thus $K'(\zeta) = -u(\phi(\zeta))u'(\phi(\zeta))/u''(\zeta)) = \zeta u(\phi(\zeta))/u''(\phi(\zeta))$, so that condition (3.4SC2) becomes $0 < a \leq u(y)/u''(y) \leq A < \infty$. To get the a priori estimate (E), one imposes condition (3.4SC3), which bounds $\varepsilon''(\tau)$. Condition (3.4SC3), for example, is satisfied for an ideal gas with $\gamma = 2$, i.e., a monatomic gas in two dimensions. The a priori estimate (E) then results, with $\rho_e = 1$ and velocity profile $u(y)$, satisfying (3.4SC2) but arbitrary otherwise.

For the Mie–Grüneisen equation of state $\varepsilon(\tau) = A\tau + B/\tau + C$, with constants $A = \alpha\rho_e^3/2$, $B = \varepsilon'(\rho_e) + \alpha\rho_e/2$, $c = \varepsilon(\rho_e) - \rho_e\varepsilon'(\rho_e) - \alpha\rho_e^2$, where the constant α satisfies $c^2_{\min}/\rho_{\max} \leq \alpha \leq c^2_{\max}/\rho_{\min}$, condition (3.4SC3) is sufficient for the a priori estimate for the "elastic fluid", again with $\rho_e = 1$.

Parallel shear flows with one inflection point taking place at $y = 0$ $[u''(0) = 0]$ can also be considered, under the assumption that the equilibrium velocity profile is antisymmetric about the inflection point: $u(-y) = -u(y)$. For the case in which the ratio $u(y)/u''(y)$ is positive and bounded, as in (3.4SC3), one again obtains a priori bounds. For example, one may take $u(y) = \operatorname{arc\,tanh} y$, $|y| \leq 1$.

Compressible shear flow in the plane can also be stationary if $\boldsymbol{v}_e(x, y) = (u(y), 0)$ and $\rho_e(x, y) = f(y)$, for arbitrary functions $u(y)$, $f(y)$. In this case, $\omega_e(x, y)/\rho_e = u'(y)/f(y)$ and the assumption on the existence of the Bernoulli function K is $[u'(y)/f(y)]' \neq 0$. This flow is formally stable provided $c_e^2(y) - u^2(y) > 0$ and $\zeta^{-1}K'(\zeta) > 0$, where $c_e(y)$ is the sound speed. Thus, the stationary flow must be subsonic everywhere, and $K(\zeta)$ must be increasing as a function of $\zeta^2/2$. The a priori estimate (E) holds, if ε and K satisfy the inequalities in the theorem.

Example B. *Circular flows.* To illustrate the effect of barotropic compressibility on stability, we consider circular flow in an annular domain, so in polar coordinates (r, θ), $\boldsymbol{v}_e = \boldsymbol{\theta} v_e(r)$ where $\boldsymbol{\theta}$ is a unit vector in the azimuthal direction, and $\rho_e = \rho_e(r)$. Because of circular symmetry, there are additional conserved quantities: namely, the angular momentum $\int_D (\rho\boldsymbol{v} \times \boldsymbol{r}) \cdot \hat{z}\, \mathrm{d}x\, \mathrm{d}y$ and moment of inertia $\int_D \rho r^2\, \mathrm{d}x\, \mathrm{d}y$. Hence, we take

$$H_C = \int_D \mathrm{d}x\, \mathrm{d}y [\tfrac{1}{2}\rho|\boldsymbol{v}|^2 + \varepsilon(\rho) + \rho\Phi(\omega/\rho) + \tfrac{1}{2}\Omega\rho\boldsymbol{v} \times \boldsymbol{r} \cdot \hat{z} + \tfrac{1}{8}\Omega^2\rho r^2 + \Omega] + \oint_{(\partial D)_i} \boldsymbol{v} \cdot \mathrm{d}\boldsymbol{\ell},$$

where Ω = constant and $(\partial D)_i$ are circularly symmetric. This can be rewritten as

$$H_C = \int_D \mathrm{d}x\, \mathrm{d}y [\tfrac{1}{2}\rho|\tilde{\boldsymbol{v}}|^2 + \varepsilon(\rho) + \rho\Phi((\tilde{\omega} + \Omega)/\rho)] + \oint_{(\partial D)_i} \tilde{\boldsymbol{v}} \cdot \mathrm{d}\boldsymbol{\ell}$$

where $v = v + \frac{1}{2}\Omega r \times z$ is the fluid velocity relative to a frame rotating with angular velocity $\Omega/2$, and $\tilde{\omega} = \hat{z} \cdot \text{curl}\, \tilde{v} = \omega - \Omega$. Since H_C in these variables retains its previous form, the stability condition $\Phi''(\omega_e/\rho_e) > 0$ can be written as either

$$\tilde{v}_e \Big/ \frac{d}{dr}[\rho_e^{-1}(\tilde{\omega}_e + \Omega)] > 0\,,$$

where $\tilde{\omega}_e = r^{-1}d(r\tilde{v}_e)/dr$, or, equivalently, using the equilibrium condition $d\rho_e/dr = \rho_e\tilde{v}_e(\tilde{v}_e + \Omega r)/rc_e^2$ where $c_e^2 = \rho_e h'(\rho_e)$, as

$$\rho_e\tilde{v}_e \Big/ \left[\frac{d}{dr}(\tilde{\omega}_e + \Omega) - (\tilde{v}_e + \Omega r)(\tilde{\omega}_e + \Omega)\tilde{v}_e/rc_e^2\right] > 0 \tag{3.4BSC}$$

for stability.* Thus, compressibility can be either stabilizing or not, depending on the relative signs and magnitudes of $\tilde{v}_e/r$, $\tilde{\omega}_e$, and Ω, and the magnitude of c_e^2. In the limit that c_e^{-2} tends to zero, the second term in the denominator vanishes in (3.4BSC) and it becomes the counterpart for circular incompressible flow of Rayleigh's inflection point criterion. Of course, this incompressible case could also be done directly in the context of section 3.3.

For rigidly rotating flows, $\tilde{v}_e = \tilde{\omega}_e r/2$, $\tilde{\omega}_e = \text{const}$, and condition (3.4BSC) becomes

$$(2\Omega + \tilde{\omega}_e)(\tilde{\omega}_e + \Omega) < 0$$

for stability, which is satisfied when $\tilde{\omega}_e\Omega < 0$ and $2|\Omega| > |\tilde{\omega}_e| > |\Omega|$, independently of the domain considered. For $\Omega = 0$, homogeneous flows with $v_e(r) = v_0(r/r_0)^n$ for constants n, v_0, r_0, and $v_e^2/c_e^2 = d\log\rho_e/d\log r = m^2 = \text{const}$, are stable according to (3.4BSC) for either $n > 1 + m^2$, or $n < -1$, also independently of the domain.

Remark. The barotropic equations in a rotating frame are

$$\partial\tilde{v}/\partial t = -(\tilde{v}\cdot\nabla)\tilde{v} - \nabla h(\rho) + \Omega\tilde{v}\times\hat{z}\,, \qquad \partial\rho/\partial t = -\text{div}\,\rho\tilde{v}\,,$$

which imply

$$(\partial/\partial t + \tilde{v}\cdot\nabla)[(\tilde{\omega} + \Omega)/\rho] = 0\,,$$

and their corresponding stability criteria discussed here have certain meteorological applications and also apply to large-scale topographical planetary waves in the ocean when $h(\rho) = g\rho^2/2$ and ρ is identified with the height of a water surface over a flat bottom (see, e.g., LeBlond and Mysak [1978]). In the absence of circular symmetry, the nonlinear stability analysis for the barotropic equations in a rotating frame is an obvious modification of what is presented earlier in this section for the case without rotation.

* For the homogeneous case, nonlinear stability of circular, elliptical and annular patches of vorticity is studied theoretically by Wan [1984], Wan and Pulvirente [1984] and Tang [1984], and numerically by Dritschel [1984].

Part I. Two-Dimensional Fluid Systems

The examples presented in the introductory sections included two-dimensional incompressible and compressible flow. In this first part we study quasigeostrophic flow, planar MHD, and planar multifluid plasmas. The second part will deal with analogous three-dimensional problems.

The examples presented can be read independently. Because of the different nature of the Casimirs for different problems, approximate equations and two-dimensional equations are not necessarily more tractable than more exact equations or three-dimensional ones.

4. Multilayer quasigeostrophic flow

In this section we shall apply Arnold's method to the study of a widely used model in physical oceanography and astrophysics: multilayer quasigeostrophic flow. This a straightforward extension of the examples in sections 3.3 and 3.4. (See especially the discussion in example B of section 3.4 of planetary topographic waves at the end of section 3). This type of example has been of considerable interest in the literature (see for example, Dikii [1965a,b], Blumen [1968], Pierini and Vulpiani [1981], Benzi et al. [1982] and Andrews [1983]). However, the stability analyses of stationary flows so far have only been for formal stability. Here we complete the proofs by providing convexity estimates. This section also provides a transition between the easier examples in section 3 and the more complicated case of planar MHD considered in the ensuing two sections.

A. *Equations of motion and Hamiltonian*

Consider a stratified fluid of N superimposed layers of constant densities $\rho_1 < \cdots < \rho_N$, the layers being stacked according to increasing density, such that the density of the upper layer is ρ_1. The quasigeostrophic approximation assumes that the velocity field is constant in the vertical direction and that in the horizontal direction the motion obeys a system of coupled incompressible shallow water equations. We shall denote by $\boldsymbol{v}_i = (-\partial\psi_i/\partial y, \partial\psi_i/\partial x)$ the velocity field of the ith layer, where ψ_i is its stream function. Let

$$\omega_i = \nabla^2\psi_i + \alpha_i \sum_{j=1}^{N} T_{ij}\psi_j + f_i\,, \qquad i = 1, \ldots, N \tag{4A1}$$

be the generalized vorticity of the ith layer, where

$$\alpha_i = f_0^2/g[(\rho_{i+1} - \rho_i)/\rho_0]D_i\,, \qquad i = 1, \ldots, N$$

$$f_i = f_0 + \beta y\,, \qquad i = 1, \ldots, N-1$$

$$f_N = f_0 + \beta y + f_0 d(y)/D_N\,,$$

$$f_0 = 2\Omega \sin\phi_0\,, \qquad \beta = (2\Omega\cos\phi_0)/R\,,$$

and

$$[T_{ij}] = \begin{bmatrix} -1 & 1 & 0 & & & 0 \\ 1 & -2 & 1 & & & \\ & \ddots & \ddots & \ddots & & \\ & & & 1 & -2 & 1 \\ 0 & & & 0 & 1 & -1 \end{bmatrix}, \quad i, j = 1, \ldots, N.$$

The $N \times N$ tensor T_{ij} is the second-order difference operator: $T_{ij}\psi_j = (\psi_{i-1} - \psi_i) - (\psi_i - \psi_{i+1})$, g is the gravitational acceleration, $\rho_0 = (1/N)(\rho_1 + \cdots + \rho_N)$ is the mean density, D_i is the mean thickness of the i-th layer, R is the Earth's radius, Ω is the Earth's angular velocity, ϕ_0 is the reference latitude, and $d(y)$ is the shape of the bottom. With these notations, the motion of the multilayered fluid is given by (see Pedlosky [1979]):

$$\partial\omega_i/\partial t + \{\psi_i, \omega_i\}_{xy} = 0\,, \qquad i = 1, \ldots, N \tag{4EM}$$

where $\{\,,\}_{xy}$ denotes the usual xy-Poisson bracket in $\mathbb{R}^2$. The boundary conditions in a compact domain D with smooth boundary

$$\bigcup_{j=0}^{g} (\partial D)_j$$

are $\psi_i|_{(\partial D)_i}$ = constant, whereas in $\mathbb{R}^2$ they are

$$\lim_{(x,y)\mapsto\pm\infty} \nabla\psi_i = 0\,.$$

The space P consists of N-tuples $(\omega_1, \ldots, \omega_N)$ of real-valued functions on D (the "generalized vorticities") with the above boundary conditions and certain smoothness properties that guarantee that solutions are at least of class C^1.

The Hamiltonian for (4EM) is the total energy

$$H(\omega_1, \ldots, \omega_N) = \tfrac{1}{2} \int_D \Big[\sum_{i=1}^{N} \frac{1}{\alpha_i} |\nabla\psi_i|^2 + \sum_{i=1}^{N-1} (\psi_i - \psi_{i+1})^2\Big]\, dx\, dy\,, \tag{4H}$$

with ψ_i determined from ω_i by the elliptic equation (4A1) with the boundary conditions discussed above.

Remark A. The eqs. (4EM) are Hamiltonian with respect to the Lie–Poisson bracket on the dual of $\bigoplus_{i=1}^{N} \mathcal{F}(D)$ given by

$$\{F, G\}(\omega_1, \ldots, \omega_N) = \sum_{i=1}^{N} \int_D \omega_i \left\{\frac{\delta F}{\delta\omega_i}, \frac{\delta G}{\delta\omega_i}\right\}_{xy} dx\, dy\,, \tag{4PB}$$

if D is simply connected. If D is not simply connected, one can proceed as in example 3.3, considering $\boldsymbol{v}_i$ as the basic dynamic variables and $\delta F/\delta\omega_i$ is interpreted with care (Marsden and Weinstein [1983], Lewis et al. [1985]).

B. *Constants of motion*

It is easy to see that the material time derivative of $\omega_i(t, x, y)$ along the flow of (4EM) is zero. Consequently, for every function $\phi_i:\mathbb{R}\to\mathbb{R}$ the functional

$$C_i(\omega_i) = \frac{1}{\alpha_i}\int_D \Phi_i(\omega_i)\,\mathrm{d}x\,\mathrm{d}y \tag{4C1}$$

is a conserved quantity for the system (4EM), provided the integrals exist. (The constant α_i is inserted for later convenience). By Kelvin's circulation theorem, the following integrals are conserved

$$\Gamma_{ji}(\omega_i) = -\oint_{(\partial D)_j} \nabla\psi_i\cdot\boldsymbol{n}\,\mathrm{d}s\,, \tag{4C2}$$

where $\boldsymbol{n}$ is the outward unit normal.

Remark B. As in example 3.3, the functionals C_i and Γ_{ji} are preserved by the push-forward of functions by area-preserving diffeomorphisms. Thus C_i and Γ_{ji} are Casimirs.

C. *First variation*

Stationary solutions will be sought as conditional extrema of the energy H at fixed $C := \Sigma_{i=1}^N C_i$ and Γ_{ji} by means of a constrained variational principle. Let $H_C := H + C + \Sigma_{i,j}\lambda_{ji}\Gamma_{ji}$ where all the Φ_i and λ_{ji} will eventually be chosen such that the first variation $\mathrm{D}H_C$ vanishes at a stationary solution $\boldsymbol{\omega}_\mathrm{e} := (\omega_1^\mathrm{e}, \ldots, \omega_N^\mathrm{e})$. After integration by parts, one finds

$$\begin{aligned}
\mathrm{D}H(\boldsymbol{\omega}_\mathrm{e})\cdot\delta\boldsymbol{\omega} &= \int_D \Big[\sum_{i=1}^N \frac{1}{\alpha_i}\nabla\psi_i^\mathrm{e}\cdot\nabla\delta\psi_i + \sum_{i=1}^{N-1}(\psi_i^\mathrm{e}-\psi_{i+1}^\mathrm{e})(\delta\psi_i-\delta\psi_{i+1}) + \sum_{i=1}^N\frac{1}{\alpha_i}\Phi_i'(\omega_i^\mathrm{e})\delta\omega_i\Big]\mathrm{d}x\,\mathrm{d}y\\
&\quad - \sum_{i,j}\lambda_{ji}\oint_{(\partial D)_j}\nabla\delta\psi_i\cdot\boldsymbol{n}\,\mathrm{d}s\\
&= \int_D \Big[\sum_{i=1}^N\frac{1}{\alpha_i}\nabla^2\delta\psi_i(\Phi_i'(\omega_i^\mathrm{e})-\psi_i^\mathrm{e}) + \sum_{i=1}^{N-1}(\delta\psi_i^\mathrm{e}-\delta\psi_{i+1}^\mathrm{e})(\psi_i - \Phi_i'(\omega_i) - \psi_{i+1}^\mathrm{e} + \Phi_{i+1}'(\omega_{i+1}^\mathrm{e}))\Big]\mathrm{d}x\,\mathrm{d}y\\
&\quad + \sum_{i,j}^N\Big[\frac{(\psi_i^\mathrm{e})|(\partial D)_j}{\alpha_i} - \lambda_{ji}\oint_{(\partial D)_j}\nabla\delta\psi_i\cdot\boldsymbol{n}\,\mathrm{d}s\Big]\,.
\end{aligned} \tag{4FV}$$

In these calculations, we use the relations between ω_i and ψ_i, the form of the matrix T_{ij} and the fact that ψ_i^e is constant on each $(\partial D)_i$.

The equilibrium solutions of (4EM) satisfy

$$\{\psi_i^e, \omega_i^e\} = 0\,, \qquad i = 1, \ldots, N, \tag{4S1}$$

i.e. $\nabla\psi_i^e$ and $\nabla\omega_i^e$ are collinear. Sufficient conditions for this to happen are the functional relationships

$$\psi_i^e = \Psi_i(\omega_i^e)\,, \qquad i = 1, \ldots, N, \tag{4S2}$$

for some real-valued functions Ψ_i. (Note that this is possible since ω_i^e is constant on the boundary $(\partial D)_i$.) From (4FV) and (4S2) it follows that at a stationary solution $\boldsymbol{\psi}_e$, $DH_C(\boldsymbol{\psi}_e) = 0$ if and only if $\psi_i^e = \Phi_i'(\omega_i^e)$, $i = 1, \ldots, N$ and hence $\Phi_i' = \Psi_i$, $i = 1, \ldots, N$. We have proved the following.

Proposition. A stationary solution of (4EM) is a critical point of $H_C = H + C + \Sigma_{i,j}\,\lambda_{ji}\Gamma_{ji}$ where

$$C = \sum_{i=1}^{N} C_i\,, \qquad C_i = (1/\alpha_i)\int_D \Phi_i(\omega_i)\,dx\,dy\,, \qquad \Phi_i' = \Psi_i\,, \qquad i = 1, \ldots, N, \qquad \lambda_{ji} = \psi_i^e|(\partial D)_j/\alpha_i\,.$$

Remark D. Formal stability. The second derivative of H_C at $\boldsymbol{\omega}_e$ equals

$$D^2H(\boldsymbol{\omega}_e)(\delta\boldsymbol{\omega})^2 = \int_D \left[\sum_{i=1}^{N} \frac{1}{\alpha_i}(|\nabla\delta\psi_i|^2 + \Phi_i''(\omega_i^e)(\delta\omega_i)^2) + \sum_{i=1}^{N-1}(\delta\psi_i - \delta\psi_{i+1})^2\right] dx\,dy\,. \tag{4SV}$$

Thus, if $\Phi''(\omega_i^e) = \Psi_i'(\omega_i^e) > 0$, the second derivative is positive for any perturbation $\delta\omega_i$. This proves the following result of Pierini and Vulpiani [1981]: *The stationary solutions of (EM) are formally stable, if* $\Psi_i'(\zeta) > 0$, *for all* $i = 1, \ldots, N$. Similar results for special classes of flows can be found in Blumen [1968] and Andrews [1983]. In particular, the conditions $\Psi_i'(\zeta) > 0$ for all $i = 1, \ldots, N$ imply linearized stability.

D. *Convexity estimates*

Since H is quadratic, condition (CH) from section 2 is trivially satisfied with $Q_1 = H$. For (CC) we require

$$Q_2(\Delta\boldsymbol{\omega}) \le \sum_{i=1}^{N} \int_D \Phi_i(\omega_i^e + \Delta\omega_i) - \Phi_i(\omega_i^e) - \Phi_i'(\omega_i^e)\alpha\omega]\,dx\,dy\,.$$

This holds with

$$Q_2(\Delta\omega) = \tfrac{1}{2}c_2 \sum_{i=1}^{N} \int_D (\Delta\omega)^2\,dx\,dy, \text{ where } c_2 \text{ is a constant, provided } c_2 \le \Phi_i''(\zeta) \quad \text{for all} \quad \zeta.$$

Condition (D) requires

$$\int_D \left[\sum_{i=1}^{N} \frac{1}{\alpha_i} |\nabla \delta\psi_i|^2 + \sum_{i=1}^{N} (\Delta\psi_{i+1})^2 + c_2 \sum_{i=1}^{N} (\Delta\psi_i)^2\right] \mathrm{d}x\,\mathrm{d}y > 0\,,$$

for all $\Delta\omega_i \neq 0$. If $c_2 > 0$ this condition is satisfied. This means that for all $i = 1, \ldots, N$ and ζ we have

$$\Phi_i''(\zeta) \geq c_2 > 0\,, \quad \text{i.e.,} \quad \Psi_i'(\zeta) \geq c_2 > 0\,.$$

From $\psi_i^{\mathrm{e}} = \Psi_i(\omega_i^{\mathrm{e}})$, it follows that $\nabla\psi_i^{\mathrm{e}} = \Psi_i'(\omega_i^{\mathrm{e}})\nabla\omega_i^{\mathrm{e}}$ so that

$$\Psi_i'(\omega_i^{\mathrm{e}}) = \nabla\psi_i^{\mathrm{e}} / \nabla\omega_i^{\mathrm{e}}\,.$$

Thus condition (D) holds if

$$\nabla\psi_i^{\mathrm{e}} / \nabla\omega_i^{\mathrm{e}} \geq c_2 > 0$$

for all $i = 1, \ldots, N$.

E. *A priori estimate*

Let $\omega_{i,0} = \omega_i|_{t=0}$, $\psi_{i,0} = \psi_i|_{t=0}$, $\Delta\omega_i = \omega_i - \omega_i^{\mathrm{e}}$, and $\Delta\psi_i = \psi_i - \psi_i^{\mathrm{e}}$. Then for $\Psi_i'(\zeta) \geq c_2 > 0$ we have the following estimate:

$$\frac{1}{2}\int_D \left[\sum_{i=1}^{N} \frac{1}{\alpha_i} |\nabla\Delta\psi_i|^2 + \sum_{i=1}^{N-1} (\Delta\psi_i - \Delta\psi_{i+1})^2 + c_2 \sum_{i=1}^{N} \frac{1}{\alpha_i} (\Delta\omega_i)^2\right] \mathrm{d}x\,\mathrm{d}y$$

$$\leq \frac{1}{2}\int_D \left[\sum_{i=1}^{N} \frac{1}{\alpha_i} |\nabla\Delta\psi_{i,0}|^2 + \sum_{i=1}^{N-1} (\Delta\psi_{i,0} - \Delta\psi_{i+1,0})^2\right] \mathrm{d}x\,\mathrm{d}y$$

$$+ \int_D \sum_{i=1}^{N} \frac{1}{\alpha_i} [\Phi_i(\omega_i^{\mathrm{e}} + \Delta\omega_{i,0}) - \Phi_i(\omega_i^{\mathrm{e}}) - \Phi_i'(\omega_i^{\mathrm{e}})\Delta\omega_{i,0}]\, \mathrm{d}x\,\mathrm{d}y\,. \tag{4E}$$

F. *Nonlinear stability*

For $c_2 > 0$ we set

$$\|\nabla\psi\|^2 = \int_D \left[\sum_{i=1}^{N} \frac{1}{\alpha_i} |\nabla\Delta\psi_i|^2 + \sum_{i=1}^{N-1} (\Delta\psi_i - \Delta\psi_{i+1})^2 + c_2 \sum_{i=1}^{N} \frac{1}{\alpha_i} (\Delta\omega_i)^2\right] \mathrm{d}x\,\mathrm{d}y\,. \tag{4N}$$

Then (CH)′ and (CC)′ hold provided

$$\Psi_i'(\zeta) \leq C_2 < +\infty$$

for all ζ. Then we get the following.

Stability theorem. Assume that

$$\infty > C_2 \geq \Psi_i'(\zeta) \geq c_2 > 0\,, \qquad i = 1, \ldots, N\,,$$

for all values of ζ. Then the stationary solution ψ_e of (EM) is Liapunov stable (as long as solutions remain C^1).

Example. Shear flow in a two-layer system. Consider $N = 2$ in (4EM) and the steady solution

$$\psi_i^e = U_i y\,, \qquad i = 1, 2\,, \qquad \text{where } U_i > 0 \text{ is a constant}\,,$$

studied by Pierini and Vulpiani [1981] with periodic boundary conditions in a finite x and y interval. The derivatives ψ_1', ψ_2' are easily seen to equal

$$\psi_1' = U_1/[\alpha_1(U_2 - U_1) + \beta]\,, \qquad \psi_2' = U_2/[\alpha_2(U_1 - U_2) + \beta + f_0 d'(y)/D_2]\,.$$

Hence, the hypotheses of the stability theorem are satisfied and the two-layer flow is stable if

$$U_2 - U_1 > -\beta/\alpha_1\,,$$

and the shape of the bottom is such that

$$C_2 > \alpha_2(U_1 - U_2) + \beta + f_0 d'(y)/D_2 > c_2 > 0\,,$$

for some constants C_2 and $c_2 > 0$.

The case when both U_1, $U_2 < 0$ can be treated similarly, by passing to a moving reference frame $x = x' + kt$, $y = y'$, $t = t$; for $k > -U_1, -U_2$.

Remark. The same considerations apply for the Liapunov stability of a uniform one-layer quasi-geostrophic motion considered by Benzi et al. [1982]. The rigorous convexity argument alters their sufficient condition by placing a positive lower bound on the derivative of ψ. With this modification, the applications given in the aforementioned paper have obvious changes.

5. Planar MHD with B in the plane

In the barotropic (resp. incompressible) magnetohydrodynamics (MHD) approximation, plasma motion in three dimensions is governed by the following system of equations:

$$\frac{\partial \rho}{\partial t} = -\mathrm{div}(\rho \boldsymbol{v}), \qquad \rho \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = -\boldsymbol{\nabla} p + \boldsymbol{J} \times \boldsymbol{B}, \qquad \frac{\partial \boldsymbol{B}}{\partial t} = -\mathrm{curl}\, \boldsymbol{E}, \quad \text{and} \quad \mathrm{div}\, \boldsymbol{B} = 0, \tag{3dMHD}$$

where

$$p = p(\rho) \quad [\text{resp. div}\, \boldsymbol{v} = 0], \qquad \boldsymbol{J} = \mathrm{curl}\, \boldsymbol{B} \quad \text{and} \quad \boldsymbol{E} = -\boldsymbol{v} \times \boldsymbol{B}.$$

In the barotropic case, the pressure p is a given function of the mass density ρ: $p = p(\rho)$. In the incompressible case p is determined, as usual, by the condition div $\boldsymbol{v} = 0$. In (3dMHD), $\boldsymbol{E}$ is the electric field, $\boldsymbol{v}$ the fluid velocity, $\boldsymbol{B}$ the magnetic field, $\boldsymbol{J}$ the electric current density, $\mathrm{d}/\mathrm{d}t := \partial/\partial t + \boldsymbol{v} \cdot \boldsymbol{\nabla}$ is the material derivative, and the equations are written in rationalized Gaussian units. The conditions under which these equations are an appropriate physical model are discussed by Bernstein et al. [1958] and Freidberg [1982]. The boundary conditions we assume are those of a fixed, ideally conducting interface; i.e. the velocity $\boldsymbol{v}$, and magnetic field $\boldsymbol{B}$, are tangential to the boundary.

In this section we consider two-dimensional incompressible and barotropic MHD taking place in a domain D in the xy plane with $\boldsymbol{B}$ parallel to the plane. We shall begin with the incompressible case. This case is of interest since it corresponds to the equations of reduced magnetohydrodynamics (RMHD) in the low β limit and with a helical symmetry imposed. In particular, we determine the stability of Alfvén solutions and Grad–Shafranov equilibria.

5.1. Homogeneous incompressible case

Some of the key features of this example are discussed in the context of RMHD in Hazeltine et al. [1984]. Stability analyses for the more complex models occurring in Hazeltine and Morrison [1983] are found in Hazeltine, Holm and Morrison [1984].

A. *Equations of motion and Hamiltonian*

We shall assume that the domain D containing the fluid has a smooth boundary and lies in the xy plane. Since the Eulerian velocity field $\boldsymbol{v}$ and magnetic field $\boldsymbol{B}$ are in the xy plane and satisfy $\mathrm{div}(\boldsymbol{B}) = 0$, $\boldsymbol{B} \cdot \boldsymbol{n} = 0$, $\mathrm{div}(\boldsymbol{v}) = 0$ and $\boldsymbol{v} \cdot \boldsymbol{n} = 0$ ($\boldsymbol{n}$ is the unit outward normal to ∂D), there exist functions A and ψ on D (the scalar magnetic potential and stream function) such that $\boldsymbol{B} = \mathrm{curl}(A\hat{\boldsymbol{z}})$, $\boldsymbol{v} = \mathrm{curl}(\psi\hat{\boldsymbol{z}})$ and A and ψ are constant on connected components of ∂D. Thus, the current has the expression

$$J = \hat{\boldsymbol{z}} \cdot (\boldsymbol{\nabla} \times \boldsymbol{B}) = -\nabla^2 A.$$

To simplify matters, we shall assume that D is simply connected, so we can take A and ψ zero on ∂D. The equations of motion (3dMHD) become

$$\dot{\omega} = \{\psi, \omega\}_{xy} + \{J, A\}_{xy}, \qquad \dot{A} = \{\psi, A\}_{xy}, \tag{5.1EM}$$

where $\{\ ,\ \}_{xy}$ is the canonical Poisson bracket in the xy plane. The space P consists of pairs of functions (ω, A) on D with appropriate smoothness properties. The total energy

$$\tfrac{1}{2} \int_D (|\boldsymbol{v}|^2 + |\boldsymbol{B}|^2)\, \mathrm{d}x\, \mathrm{d}y$$

has the expression

$$H(\omega, A) = \tfrac{1}{2} \int_D [\omega(-\nabla^2)^{-1}\omega + A(-\nabla^2)A] \, dx \, dy, \tag{5.1H}$$

and represents the conserved Hamiltonian for the eqs. (5.1EM).

The eqs. (5.1EM) coincide with the RMHD equations in the low β limit with a helical symmetry; see, for example, Morrison and Hazeltine [1984].

Remark A. Poisson bracket. The eqs. (5.1EM) are Hamiltonian with respect to the following semidirect product Lie–Poisson bracket

$$\{F, G\} = \int_D \left[\omega\left\{\frac{\delta F}{\delta\omega}, \frac{\delta G}{\delta\omega}\right\}_{xy} + A\left(\left\{\frac{\delta F}{\delta\omega}, \frac{\delta G}{\delta A}\right\}_{xy} - \left\{\frac{\delta G}{\delta\omega}, \frac{\delta F}{\delta A}\right\}_{xy}\right)\right] dx \, dy\,; \tag{5.1PB}$$

the functional derivatives $\delta F/\delta\omega$ and $\delta F/\delta A$ must be interpreted carefully as in Marsden and Weinstein [1983] and Lewis et al. [1985]. The verification that $\dot{F} = \{F, H\}$ for any F with $\{\,,\}$ given in (5.1PB) uses the following integration by parts formula:

$$\int_D \{f, g\} h \, dx \, dy = \int_D f\{g, h\} \, dx \, dy + \oint_{\partial D} fh\boldsymbol{X}_g \cdot \boldsymbol{n} \, ds\,,$$

where $\boldsymbol{X}_g = \nabla g \times \hat{z}$ is the divergenceless vector field with stream function g. The Poisson bracket (5.1PB) coincides with the Lie–Poisson bracket on the dual of the semidirect product Lie algebra of the Lie group $\mathrm{Diff}_{\mathrm{vol}}(D) \circledS \mathscr{F}(D)$. The Poisson bracket (5.1PB) is obtained from canonical brackets in Lagrangian coordinates under reduction and an assumption of helical symmetry (see Marsden and Morrison [1984]).

B. *Constants of the motion*

For arbitrary real valued functions Φ and Ψ of one variable, the functional

$$C_{\Phi,\Psi}(\omega, A) = \int_D (\omega\Phi(A) + \Psi(A)) \, dx \, dy$$

is preserved by the equations of motion.

Remark B. These functions are Casimirs for the Poisson bracket (5.1PB). This may be checked either by direct verification, or by checking that $C_{\Phi,\Psi}$ is invariant under the coadjoint action of $\mathrm{Diff}_{\mathrm{vol}}(D) \circledS \mathscr{F}(D)$:

$$(\eta, f) \cdot (\omega, A) = (\eta_* \omega + \{\eta_* A, f\}, \eta_* A)\,.$$

C. *First variation*

The derivative of $H_C := H + C_{\Phi,\Psi}$ at an equilibrium (ω_e, A_e) is given by

$$DH_C(\omega_e, A_e)(\delta\omega, \delta A) = \int_D\int [\psi_e + \Phi(A_e)]\delta\omega + \int_D\int [J_e + \omega_e\Phi'(A_e) + \Psi'(A_e)]\,\delta A$$

where $\psi_e = (-\nabla^2)^{-1}\omega_e$ and $J_e = -\nabla^2 A_e$. Thus (ω_e, A_e) is a critical point when

$$\psi_e + \Phi(A_e) = 0\,, \qquad (5.1\text{FV})_1$$

and

$$J_e + \omega_e\Phi'(A_e) + \Psi'(A_e) = 0\,. \qquad (5.1\text{FV})_2$$

From the second equation of (5.1EM), it follows that $\nabla\psi_e$ and ∇A_e are collinear in the plane. A sufficient condition for this to hold is the functional relationship $\psi_e = \bar{\psi}(A_e)$ which, in turn, determines $\Phi = -\bar{\psi}$ from $(5.1\text{FV})_1$. From the first equation of (5.1EM) we have

$$\{\psi_e, \omega_e\}_{xy} + \{J_e, A_e\}_{xy} = 0\,,$$

so using (5.1FV), we get

$$\{\omega_e\Phi'(A_e), A_e\}_{xy} + \{J_e, A_e\}_{xy} = 0\,,$$

so the vectors $\nabla(J_e + \omega_e\Phi'(A_e))$ and ∇A_e must be collinear in the plane. A sufficient condition for this collinearity is the functional relationship

$$J_e + \omega_e\Phi'(A_e) = G(A_e) \qquad (5.1\text{G})$$

for some function G; thus Ψ is determined by $(5.1\text{FV})_2$ and (5.1G) to be

$$\Psi(a) = -\int^a G(s)\,ds\,.$$

Proposition. Stationary solutions (ω_e, A_e) of the planar homogeneous incompressible MHD equations with $\boldsymbol{B}$ in the plane in a simply connected domain D satisfying the functional relationships

$$\psi_e = \bar{\psi}(A_e)\,, \qquad J_e - \omega_e\bar{\psi}'(A_e) = G(A_e)$$

for some real valued functions of a real variable $\bar{\psi}$ and G, are critical points of $H + C_{\Phi,\Psi}$, where

$$\Phi = -\bar{\psi} \quad \text{and} \quad \Psi(a) = -\int^a G(s)\,ds\,.$$

(Conversely, if (ψ_e, A_e) is a critical point of $H + C_{\Phi,\Psi}$, then it is an equilibrium solution of 5.1EM).

Remarks:

(i) For any ψ_e, the choice $A_e = c\psi_e$ gives an equilibrium solution; these are *Alfvén solutions* (see Chandrasekhar [1961, §113]). Here $\Phi(a) = -a/c$. From $(5.1\text{FV})_2$ we see $\Psi'(a) = 0$, so Ψ is constant.

(ii) If $\psi_e = 0$ (so $\Phi = 0$) then we have a static equilibrium and (5.1G) reduces to

$$J_e = G(A_e)$$

for a function G. These are *Grad–Shafranov equilibria* (cf. Chandrasekhar [1961, §115]).

(iii) A particular solution of $J_e = G(A_e)$, $-\nabla^2 A_e = J_e$, is given by the Kelvin–Stuart cat's eye formula

$$J_e = -[a \cosh y + \sqrt{a^2 - 1} \cos x]^{-2}$$

(see example 3.3). Using a Poincaré-type inequality, $\int |\nabla \delta A|^2 \, dx \, dy \geq k_{\min} \int |\delta A|^2 \, dx \, dy$, one gets a stability estimate similar to the analysis of the cat's eye solution in fluid mechanics if $a < 1.175\ldots$ (Holm, Marsden and Ratiu [1984]). We note that the magnetic field lines in this case are such that they are confining for the plasma. In the literature (Finn and Kaw [1977], Pritchett and Wu [1979], and Bondeson [1983]) these magnetic island solutions are shown to be unstable; this can happen only if one allows arbitrary distrubances in the y direction – transverse to the eyes. Our approach gives stability since our disturbances are confined to a finite extent in that direction.

Remark D. Formal stability. As a prelude to the convexity estimates, we determine conditions under which the second variation of $H_C = H + C_{\Phi,\Psi}$ is definite. Integrating by parts, using the boundary conditions $\psi|\partial D = 0$, $A|\partial D = 0$, regrouping, and using $\psi_e = -\Phi(A_e)$, we get

$$\begin{aligned}
D^2 H_C(\omega_e, A_e) \cdot (\delta\omega, \delta A)^2 &= \int_D [\delta\omega(-\nabla^2)^{-1}\delta\omega + \delta A(-\nabla^2)\delta A + 2\Phi'(A_e)\delta\omega\, \delta A \\
&\quad + (\omega_e \Phi''(A_e) + \Psi''(A_e))(\delta A)^2] \, dx \, dy \\
&= \int_D [|\nabla\delta\psi - \nabla(\Phi'(A_e)\delta A)|^2 + (1 - \Phi'(A_e)^2)|\nabla\delta A|^2 + (\omega_e \Phi''(A_e) \\
&\quad + \Psi''(A_e) + \Phi'(A_e)\nabla^2\Phi'(A_e))(\delta A)^2] \, dx \, dy .
\end{aligned}$$

Thus, sufficient conditions for formal stability are

(i) $|\Phi'(A_e)| \leq 1$,

(ii) $\omega_e \Phi''(A_e) + \Psi''(A_e) + \Phi'(A_e)\nabla^2\Phi'(A_e) \geq 0$,

where Φ and Ψ are determined in the proposition. Notice that from $\psi_e = -\Phi(A_e)$, we get $\nabla\psi_e = -\Phi'(A_e)\nabla A_e$ and so $\boldsymbol{v}_e = -\Phi'(A_e)\boldsymbol{B}_e$. Thus, condition (i) may be phrased: *$\boldsymbol{v}_e$ and $\boldsymbol{B}_e$ are collinear with $\|\boldsymbol{v}_e\| \leq \|\boldsymbol{B}_e\|$*. Likewise, (ii) says ∇J_e and ∇A_e are parallel with ∇J_e a negative multiple of ∇A_e; in other words, (ii) says that *∇J_e and $\boldsymbol{B}_e$ if not zero, form an orthogonal basis in the xy plane with the same orientation as the x and y axis*; i.e. ∇J_e and $\boldsymbol{B}_e$ form a right-handed system (∇J_e being zero is allowed).

D. Convexity estimates

Since H is quadratic, we choose $Q_1 = H$. Next, consider

$$\hat{C} = C_{\Phi,\Psi}(\omega_e + \Delta\omega, A_e + \Delta A) - C_{\Phi,\Psi}(\omega_e, A_e) - DC_{\Phi,\Psi}(\omega_e, A_e)(\Delta\omega, \Delta A)$$

$$= \int_D \{(\omega_e + \Delta\omega)\Phi(A_e + \Delta A) + \Psi(A_e + \Delta A) - \omega_e\Phi(A_e) - \Psi(A_e) - \Phi(A_e)\Delta\omega - \omega_e\Phi'(A_e)\Delta A$$

$$- \Psi'(A_e)\Delta A\}\,dx\,dy$$

$$= \int_D \{\omega_e(\Phi(A_e + \Delta A) - \Phi(A_e) - \Phi'(A_e)\Delta A) + \Delta\omega(\Phi(A_e + \Delta A) - \Phi(A_e))$$

$$+ \Psi(A_e + \Delta A) - \Psi(A_e) - \Psi'(A_e)\Delta A\}\,dx\,dy\,.$$

Suppose that

$$\Phi'(a) \geq q\,, \qquad 2\Phi''(a) \geq r\,,$$

and

$$2\Psi''(a) \geq s\,, \quad \text{i.e.,} \quad G'(a) \leq -s/2$$

for constants q, r and s. Then

$$2\hat{C} \geq \iint_D [r\omega_e(\Delta A)^2 + q\Delta\omega\Delta A + s(\Delta A)^2]\,dx\,dy := Q_2(\Delta\omega, \Delta A)\,.$$

Now we consider $Q_1 + Q_2$:

$$(Q_1 + Q_2)(\Delta\omega, \Delta A) = \tfrac{1}{2}\int_D \{(\Delta\omega)(-\nabla^2)^{-1}(\Delta\omega) + (\Delta A)(-\nabla^2)(\Delta A)\}\,dx\,dy$$

$$+ \int_D \{(r\omega_e + s)(\Delta A)^2 + q\Delta\omega\Delta A\}\,dx\,dy$$

$$= \tfrac{1}{2}\int_D |\nabla(\Delta\psi) - q\nabla(\Delta A)|^2\,dx\,dy + \int_D (1 - q^2)|\nabla(\Delta A)|^2\,dx\,dy$$

$$+ \int_D (r\omega_e + s)(\Delta A)^2\,dx\,dy\,.$$

This is positive if:

(a) $|q| \le 1$.

and

(b) $r\omega_e + s \ge 0$,

and is definite if at least one inequality is strict.

The two special cases of Alfvén solutions and Grad–Shafranov equilibria deserve special note:

(i) If Ψ is constant and Φ is linear ($\Phi(a) = -a/c$), then $r = s = 0$ and $q = -1/c$. In this case,

$$\hat{C} = \int_D \Delta\omega \cdot \left(-\frac{1}{c}\right)(\Delta A)\, dx\, dy ,$$

and so

$$Q_1 + \hat{C} = \int_D [(\Delta\omega)(-\nabla^2)^{-1}(\Delta\omega) + \Delta A(-\nabla^2)\Delta A - \frac{1}{c}\Delta\omega\Delta A]\, dx\, dy$$

$$= \int_D \left\{\left| \nabla(\Delta\psi) - \frac{1}{c}\Delta\nabla A\right|^2 + \left(1 - \frac{1}{c^2}\right)|\Delta A|^2\right\} dx\, dy$$

which is conserved. Thus, one has Liapunov stability in the above norm if $c > 1$. If $c = 1$ this quadratic form simplifies to

$$Q_1 + \hat{C} = \int_D |\nabla\Delta\psi - \nabla\Delta A|^2\, dx\, dy$$

which is a "degenerate" norm (a semi-norm). In this case one has an a priori bound on the difference $\Delta\psi - \Delta A$ (i.e. ψ must remain close to A) but of the perturbations each may grow.

(ii) In the Grad–Shafranov case, $\Phi = 0$ so we take $q = r = 0$ and the condition (b) above becomes $s > 0$, i.e., J_e is a decreasing function of A_e.

E. *A priori estimates*

If we set

$$\|(\Delta\omega, \Delta A)\|^2 = (Q_1 + Q_2)(\Delta\omega, \Delta A)$$

and (a) and (b) above hold, then estimate (E) from section 2 gives

$$\|(\Delta\omega, \Delta A)\|^2 \le H_C(\omega_e + \Delta\omega|_{t=0}, A_e + \Delta A|_{t=0}) - H_C(\omega_e, A_e) .$$

F. *Nonlinear stability*

Sufficient conditions for stability in the norm $\|\cdot\|$ are obtained by bounding $Q_1 + \hat{C}$ above, in addition to (a) and (b). One gets:

$$q \le \Phi'(a) \le Q\,, \qquad r \le 2\Phi''(a) \le R\,, \qquad s \le 2\Psi''(a) \le S\,,$$

where

$$|q| < 1 \quad \text{and} \quad r\omega_e + s > 0$$

(if one inequality becomes an equality, then one uses the earlier arguments special to the Alfvén and Grad–Shafranov cases).

Remark. The above analysis does not reduce to the Arnold case in the sense that an equilibrium of H_C for the 2dMHD equations (5.1EM) with $A_e = 0$ does not give the same stability conditions as in example 3 in section 3. The reason is that the Casimirs for the two problems are rather different. In particular, a 2dMHD equilibrium of H_C is static, and the functional relations assumed above become trivial in the case $A_e = 0$.

We summarize our findings:

Stability theorem. Let (ω_e, A_e) be an equilibrium solution of (5.1EM). Assume that

$$\psi_e = -\Phi(A_e)\,, \qquad J_e + \omega_e \Phi'(A_e) = G(A_e)\,,$$

for functions Φ and G. Moreover, assume Φ and G satisfy

$$-\infty < q \le \Phi'(a) \le Q < \infty\,, \qquad -\infty < r \le 2\Phi''(a) \le R < \infty\,, \qquad -\infty < s \le -2G'(a) \le S < \infty\,,$$

for all a, where q, r, s, Q, R, S are constants satisfying

$$|q| < 1 \quad \text{and} \quad r\omega_e + s > 0\,.$$

Then (ω_e, A_e) is (nonlinearly) stable relative to the norm

$$\|\Delta\psi, \Delta A)\|^2 = \tfrac{1}{2} \int_D |\nabla(\Delta\psi) - q\nabla(\Delta A)|^2 \,\mathrm{d}x\,\mathrm{d}y + \int_D (1-q^2)|\nabla \Delta A)|^2 \,\mathrm{d}x\,\mathrm{d}y + \int_D (r\omega_e + s)|\Delta A|^2 \,\mathrm{d}x\,\mathrm{d}y$$

as long as solutions exist and are C^1.

The Alfvén solutions, with $\psi_e = A_e$ and $G = 0$, are stable as a family,* relative to the semi-norm

$$\|(\Delta\psi, \Delta A)\|^2 = \int_D |\nabla(\Delta\psi - \Delta A)|^2 \,\mathrm{d}x\,\mathrm{d}y\,.$$

Grad-Shafranov solutions, with $\Phi = 0$ and $J_e = G(A_e)$ which satisfy

* That is, with stability defined in terms of neighborhoods of sets of equilibrium solutions, rather than individual equilibria.

$$0 < s \leq -2G'(a) \leq S < \infty$$

are stable relative to the norm

$$\|(\Delta\psi, \Delta A)\|^2 = \tfrac{1}{2} \int_D (|\nabla(\Delta\psi)|^2 + |\nabla(\Delta A)|^2 + s|\Delta A|^2)\, dx\, dy .$$

Finally, we remark that the global existence of smooth solutions is not known (to us) for the system (5.1EM), so the stability has to be conditional: it is valid for times as long as smooth (C^1) solutions exist.

5.2. Compressible case

A. *Equations of motion and Hamiltonian*

Just as in incompressible planar MHD with $\boldsymbol{B}$ in the xy plane, the relations

$$\nabla \cdot \boldsymbol{B} = 0 , \qquad \boldsymbol{B} \cdot \hat{z} = 0 , \quad \text{and} \quad \boldsymbol{B} \cdot \boldsymbol{n} = 0$$

in a simply connected domain $D \subset \mathbb{R}^2$ imply the existence of a function A such that

$$\boldsymbol{B} = \operatorname{curl}(A\hat{z}) = \nabla A \times \hat{z} , \quad \text{and} \quad A|\partial D = 0 .$$

As in section 5.1., let the current be given by

$$J := \hat{z} \cdot \operatorname{curl} \boldsymbol{B} = -\nabla^2 A .$$

The compressible MHD equations for this situation are, with $\omega = \hat{z} \cdot \operatorname{curl} \boldsymbol{v}$,

$$\dot{\rho} = -\operatorname{div} \rho \boldsymbol{v} , \qquad \dot{\boldsymbol{v}} = -\omega \hat{z} \times \boldsymbol{v} - \nabla(\tfrac{1}{2}|\boldsymbol{v}|^2 + h(\rho)) + \frac{J}{\rho} \nabla A , \qquad \dot{A} = -\boldsymbol{v} \cdot \nabla A , \tag{5.2EM}$$

where $h(\rho)$ is specific enthalpy, obeying $h'(\rho) = \rho^{-1} p'(\rho)$ where $p(\rho)$, the pressure, is a function of density ρ. The space P consists of triplets $(\boldsymbol{v}, \rho, A)$, lying in appropriate function spaces, and sufficiently smooth in the domain $D \subset \mathbb{R}^2$. The conserved Hamiltonian is

$$H = \int_D (\tfrac{1}{2}\rho|\boldsymbol{v}|^2 + \varepsilon(\rho) + \tfrac{1}{2}|\nabla A|^2)\, dx\, dy , \tag{5.2H}$$

where $\varepsilon(\rho)$ is the internal energy density, satisfying $\varepsilon'(\rho) - h(\rho)$.

Remark A. The equations of motion (5.2EM) are Hamiltonian with respect to the Lie–Poisson bracket on the dual of the semidirect product Lie algebra $\mathscr{X}(D) \circledS (\mathscr{F}(D) \times \Lambda^2(D))$, where the action of the vectorfields $\mathscr{X}(D)$ on the functions $\mathscr{F}(D)$ and two-forms $\Lambda^2(D)$ is by minus the Lie derivative. The dual spaces of $\mathscr{F}(D)$ and $\mathscr{X}(D)$ are identified with themselves by the L^2-pairing, whereas the dual of

$\Lambda^2(D)$ consists of functions on D. The dynamic variables in $(\mathscr{X}(D) \circledS (\mathscr{F}(D) \times \Lambda^2(D))^* = \mathscr{X}(D) \times \mathscr{F}(D) \times \mathscr{F}(D)$ are $(\boldsymbol{M} = \rho\boldsymbol{v}, \rho, A)$, with $\boldsymbol{M} = \rho\boldsymbol{v}$ the Eulerian momentum density of the fluid. With these notations, the Poisson bracket of two functionals F and G of $(\boldsymbol{M}, \rho, A)$ is given by

$$\{F, G\}(\boldsymbol{M}, \rho, A) = \int_D \left\{\boldsymbol{M}\left[\left(\frac{\delta G}{\delta \boldsymbol{M}} \cdot \nabla\right)\frac{\delta F}{\delta \boldsymbol{M}} - \left(\frac{\delta F}{\delta \boldsymbol{M}} \cdot \nabla\right)\frac{\delta G}{\delta \boldsymbol{M}}\right] + \rho\left[\frac{\delta G}{\delta \boldsymbol{M}} \cdot \left(\nabla \frac{\delta F}{\delta \rho}\right) - \frac{\delta F}{\delta \boldsymbol{M}} \cdot \left(\nabla \frac{\delta G}{\delta \rho}\right)\right]\right.$$

$$\left. + A\left[\frac{\delta G}{\delta \boldsymbol{M}} \cdot \left(\nabla \frac{\delta F}{\delta A}\right) - \frac{\delta F}{\delta \boldsymbol{M}} \cdot \left(\nabla \frac{\delta G}{\delta A}\right) + \frac{\delta F}{\delta A} \operatorname{div} \frac{\delta G}{\delta \boldsymbol{M}} - \frac{\delta G}{\delta A} \operatorname{div} \frac{\delta F}{\delta \boldsymbol{M}}\right]\right\} \mathrm{d}x\, \mathrm{d}y . \tag{5.2PB}$$

This bracket is related to the MHD bracket in Morrison and Greene [1980], is derived from Clebsch variables in Holm and Kupershmidt [1983], and is obtained from a canonical bracket via the Lagrangian to Eulerian map in Holm, Kupershmidt and Levermore [1983], Marsden et al. [1983], and Marsden, Ratiu and Weinstein [1984a, b].

B. *Constants of motion*

Applying the operator $\hat{z} \cdot \operatorname{curl}$ to the motion equations (5.2EM) and using the identity

$$\operatorname{div}(g\hat{z} \times \nabla f) = (\nabla f \times \nabla g) \cdot \hat{z} = -\operatorname{div}(f\hat{z} \times \nabla g) \tag{5.2VI}$$

for any functions f, g depending only on x and y we get the vorticity equation

$$\dot{\omega} = -\operatorname{div}(\omega \boldsymbol{v} - A\hat{z} \times \nabla(J/\rho)) . \tag{5.2ω}$$

Consider the quantity

$$C_{\Phi, \Psi}(\boldsymbol{v}, \rho, A) = \int_D [\omega\Phi(A) + \rho\Psi(A)]\, \mathrm{d}x\, \mathrm{d}y \tag{5.2C}$$

for arbitrary smooth real valued functions of a real variable Φ and Ψ. Taking into account (5.2ω) and the first and third equations in (5.2EM), the time derivative of (5.2C) equals

$$\frac{\mathrm{d}}{\mathrm{d}t} C_{\Phi, \Psi}(\boldsymbol{v}, \rho, A) = -\int_D \operatorname{div}[(\Phi(A)\omega + \Psi(A)\rho)\boldsymbol{v}]\, \mathrm{d}x\, \mathrm{d}y + \int_{\partial D} \operatorname{div}(A\hat{z} \times \nabla(J/\rho))\Phi(A)\, \mathrm{d}x\, \mathrm{d}y$$

$$= -\oint_{\partial D} (\Phi(A)\omega + \Psi(A)\rho)\boldsymbol{v} \cdot \boldsymbol{n}\, \mathrm{d}s + \oint_{\partial D} \frac{J}{\rho} \Phi(A)(\nabla A \times \hat{z}) \cdot \boldsymbol{n}\, \mathrm{d}s$$

upon applying the identity (5.2VI). Both terms vanish provided $\boldsymbol{v} \cdot \boldsymbol{n} = 0$ and $\boldsymbol{B} \cdot \boldsymbol{n} = 0$, where $\boldsymbol{B} = \nabla A \times \hat{z}$. So $C_{\Phi, \Psi}$ is conserved.

Remark B. The coadjoint action of the semidirect product $\mathrm{Diff}(D) \circledS (\mathcal{F}(D) \times \Lambda^2(D))$ on the dual of its Lie algebra is given by

$$(\eta, f, \alpha\, \mathrm{d}x \wedge \mathrm{d}y) \cdot (\boldsymbol{M}, \rho, A) = ((\rho\circ\eta^{-1})J\eta_{*}\boldsymbol{v} + \alpha\nabla(A\circ\eta^{-1}) - (\rho\circ\eta^{-1})J\nabla f, (\rho\circ\eta^{-1})\, f, A\circ\eta^{-1}),$$

where J is the Jacobian of η and $\boldsymbol{M} = \rho\boldsymbol{v}$. A straightforward computation shows that if $\boldsymbol{v}\cdot\boldsymbol{n} = \boldsymbol{B}\cdot\boldsymbol{n} = 0$, the quantity $C_{\Phi,\Psi}$ is invariant under the coadjoint action and consequently it is a Casimir for the Poisson bracket (5.2PB).

C. *First variation*

The stationary solutions $(\boldsymbol{v}_e, \rho_e, A_e)$ of (5.2EM) and (5.2ω) obey

$$\operatorname{div} \rho_e\boldsymbol{v}_e = 0\,, \tag{5.2S1}$$

$$-\omega_e\hat{\boldsymbol{z}} \times \boldsymbol{v}_e - \nabla(\tfrac{1}{2}|\boldsymbol{v}_e|^2 + h(\rho_e)) + (J_e/\rho_e)\nabla A_e = 0\,, \tag{5.2S2}$$

$$\boldsymbol{v}_e \cdot \nabla A_e = 0\,, \tag{5.2S3}$$

$$\operatorname{div}(\omega_e\boldsymbol{v}_e - A_e\hat{\boldsymbol{z}} \times \nabla(J_e/\rho_e)) = 0\,. \tag{5.2S4}$$

Taking the scalar product of $\boldsymbol{v}_e$ with (5.2S2) and using (5.2S3) gives

$$\boldsymbol{v}_e \cdot \nabla(\tfrac{1}{2}|\boldsymbol{v}_e|^2 + h(\rho_e)) = 0\,, \tag{5.2B1}$$

so that the gradient vectors $\nabla(\frac{1}{2}|\boldsymbol{v}_e|^2 + h(\rho_e))$ and ∇A_e are collinear in the plane. A sufficient condition for this collinearity is the functional relationship

$$\tfrac{1}{2}|\boldsymbol{v}_e|^2 + h(\rho_e) = K(A_e)\,, \tag{5.2B2}$$

for a function K, called the *Bernoulli function.* Taking the cross product of the unit vector $\hat{\boldsymbol{z}}$ with (5.2S2), applying (5.2B2) and assuming that $\omega_e \neq 0$, we get

$$\boldsymbol{v}_e = \frac{1}{\omega_e}\Big(\frac{J_e}{\rho_e} - K'(A_e)\Big)\boldsymbol{B}_e\,, \tag{5.2B3}$$

where $\boldsymbol{B}_e = \nabla A_e \times \hat{\boldsymbol{z}}$. Thus for stationary solutions, the magnetic field $\boldsymbol{B}_e$ is collinear with the velocity $\boldsymbol{v}_e$ in the xy plane with coefficient given in (5.2B3). Equation (5.2B3) agrees with (5.2S4) and together with (5.2S1) implies that $\boldsymbol{B}_e \cdot \nabla[(J_e - \rho_e K'(A_e))/\omega_e] = 0$. Thus by (5.2S3) and (5.2B3) the vectors ∇A_e and

$$\nabla[(\rho_e/\omega_e)(K'(A_e) - J_e/\rho_e)]$$

are collinear in the plane. A sufficient condition for this to hold is the functional relationship

$$(\rho_e/\omega_e)[K'(A_e) - J_e/\rho_e] = L(A_e)\,. \tag{5.2IL}$$

This relation is analogous to *Long's equation* in stratified fluid flow (see Drazin and Reid [1981] and Abarbanel et al. [1985]).

Let $H_C(\boldsymbol{v}, \rho, A) := H(\boldsymbol{v}, \rho, A) + C_{\Phi,\Psi}(\boldsymbol{v}, \rho, A) + \lambda \int_D \omega \, dx \, dy$. The first variation vanishes at an equilibrium $(\boldsymbol{v}_e, \rho_e, A_e)$ when the functions Φ, Ψ and the constant λ satisfy certain conditions, to be determined now. Integrating by parts gives

$$
\begin{aligned}
DH_C(\boldsymbol{v}_e, \rho_e, A_e) \cdot (\delta\boldsymbol{v}, \delta\rho, \delta A) &= \int_D dx\, dy\, \{[\rho_e \boldsymbol{v}_e \cdot \delta\boldsymbol{v} + \Phi(A_e)\delta\omega] + [\tfrac{1}{2}|\boldsymbol{v}_e|^2 + h(\rho_e) + \Psi(A_e)]\delta\rho \\
&\quad + [J_e + \omega_e \Phi'(A_e) + \rho_e \Psi'(A_e)]\delta A\} + \lambda \int_D \delta\omega \, dx\, dy \\
&= \int_D dx\, dy \{[\rho_e \boldsymbol{v}_e + \Phi'(A_e)\boldsymbol{B}_e] \cdot \delta\boldsymbol{v} + [\tfrac{1}{2}|\boldsymbol{v}_e|^2 + h(\rho_e) + \Psi(A_e)]\delta\rho \\
&\quad + [J_e + \omega_e \Phi'(A_e) + \rho_e \Psi'(A_e)]\delta A\} + \oint_{\partial D} \delta A \, \nabla A_e \cdot \boldsymbol{n} \, ds \\
&\quad + \Phi(A_e)|_{\partial D} \oint_{\partial D} \delta\boldsymbol{v} \cdot d\boldsymbol{\ell} + \lambda \oint_{\partial D} \delta\boldsymbol{v} \cdot d\boldsymbol{\ell} .
\end{aligned}
$$

Using $A_e|\partial D = 0$, the first derivative of H_C will vanish at the stationary solution, provided

$$\rho_e \boldsymbol{v}_e + \Phi'(A_e)\boldsymbol{B}_e = 0 , \tag{5.2FV1}$$

$$\tfrac{1}{2}|\boldsymbol{v}_e|^2 + h(\rho_e) + \Psi(A_e) = 0 , \tag{5.2FV2}$$

$$J_e + \omega_e \Phi'(A_e) + \rho_e \Psi'(A_e) = 0 , \tag{5.2FV3}$$

$$\lambda + \Phi(A_e)|_{\partial D} = \lambda + \Phi(0) = 0 . \tag{5.2FV4}$$

Relation (5.2FV4) determines λ once Φ is known. From (5.2B2) and (5.2FV2), it follows that

$$\Psi(A_e) = -K(A_e) . \tag{5.2FV5}$$

Substituting (5.2B3) in (5.2FV1) and taking into account (5.2L) yields for $\boldsymbol{B}_e \neq \boldsymbol{0}$

$$\Phi'(A_e) = L(A_e) , \tag{5.2FV6}$$

which in turn, together with (5.2FV5) and (5.2L) makes (5.2FV3) into an identity. We have proved the following.

Proposition. Stationary solutions $(\boldsymbol{v}_e, \rho_e, A_e)$ of the planar barotropic MHD equations with $\boldsymbol{B}$ in the plane and $A_e|\partial D = 0$ satisfying $\omega_e \neq 0$, $\boldsymbol{B}_e \neq 0$ are critical points of $H(\boldsymbol{v}, \rho, A) + C_{\Phi,\Psi}(\boldsymbol{v}, \rho, A) + \lambda \int_D \omega \, dx \, dy$, provided

$$\Psi(s) = -K(s), \qquad \Phi(s) = \int^s L(u)\, du, \qquad \lambda = -\Phi(0),$$

where K and L are the Bernoulli and Long functions respectively, given by (5.2B2) and (5.2L). Conversely, a critical point of H_C is a stationary solution.

Remarks:

(i) Equation (5.2FV3) can also be written as

$$J_e + \omega_e L(A_e) = \rho_e K'(A_e). \tag{5.2FV7}$$

Suppose that $-L(A_e) = c = \text{constant}$, $c \neq 0$. Then by (5.2FV1) and (5.2FV6), $\rho_e \boldsymbol{v}_e = c\boldsymbol{B}_e$, $J_e = (\rho_e \omega_e + \hat{\boldsymbol{z}} \cdot \nabla\rho_e \times \boldsymbol{v}_e)/c$ and by (5.2FV7) we have

$$\left(1 - \frac{\rho_e}{c^2}\right)\omega_e + \frac{\rho_e}{c} K'(A_e) + \frac{1}{c^2} \hat{\boldsymbol{z}} \cdot \boldsymbol{v}_e \times \nabla\rho_e = 0. \tag{5.2FV8}$$

If in addition $\rho_e = c^2$, this class of solutions reduces to the Alfvén or equipartition solution (see Chandrasekhar [1961, §113]).

(ii) Suppose that $L(A_e) = 0$, then by (5.2FV1) and (5.2FV6), $\boldsymbol{v}_e = \boldsymbol{0}$, i.e. we have a static equilibrium. Relation (5.2FV7) then reduces to

$$J_e = \rho_e K'(A_e), \tag{5.2G}$$

which is the *compressible Grad–Shafranov equation.*

Remark D. Second variation. After integrating by parts and using the boundary condition $\delta A|\partial D = 0$, we find that the second variation of H_C at a stationary solution $(\boldsymbol{v}_e, \rho_e, A_e)$ is given by $\delta^2 H_C := D^2 H_C(\boldsymbol{v}_e, \rho_e, A_e) \cdot (\delta \boldsymbol{v}, \delta\rho, \delta A)^2$

$$= \int_D [\rho_e |\delta \boldsymbol{v}|^2 + 2\boldsymbol{v}_e \cdot \delta \boldsymbol{v} \delta\rho + \varepsilon''(\rho_e)(\delta\rho)^2 + |\nabla \delta A|^2 + 2\Psi'(A_e)\, \delta\rho\, \delta A$$
$$+ (\omega_e \Phi''(A_e) + \rho_e \Psi''(A_e))(\delta A)^2 + 2(\hat{\boldsymbol{z}} \times \delta \boldsymbol{v}) \cdot (\nabla\Phi'(A_e))\delta A + 2\Phi'(A_e)(\hat{\boldsymbol{z}} \times \delta \boldsymbol{v}) \cdot \nabla\delta A]\, dx\, dy. \tag{5.2SV}$$

Taking as variables the two components of $\delta \boldsymbol{v}$, $\delta\rho$, δA and the two components of $\nabla\delta A$, the quadratic form under the integral sign has the 6×6 matrix

$$\begin{bmatrix} \rho_e & 0 & v_e^1 & \partial_y\Phi'(A_e) & 0 & \Phi'(A_e) \\ 0 & \rho_e & v_e^2 & -\partial_x\Phi'(A_e) & -\Phi'(A_e) & 0 \\ v_e^1 & v_e^2 & \varepsilon''(\rho_e) & \Psi'(A_e) & 0 & 0 \\ \partial_y\Phi'(A_e) & -\partial_x\Phi'(A_e) & \Psi'(A_e) & \alpha & 0 & 0 \\ 0 & -\Phi'(A_e) & 0 & 0 & 1 & 0 \\ \Phi'(A_e) & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

where $\alpha = \omega_e\Phi''(A_e) + \rho_e\Psi''(A_e)$. The six principal minors of this matrix are:

$$\mu_1 = \rho_e\,, \qquad \mu_2 = \rho_e^2\,, \qquad \mu_3 = \rho_e^2\varepsilon''(\rho_e) - |v_e|^2\rho_e\,,$$

$$\mu_4 = \rho_e^2\alpha\varepsilon''(\rho_e) - \rho_e^2\Psi'(A_e)^2 + (v_e\cdot\nabla\Phi'(A_e))^2 - \rho_e\varepsilon''(\rho_e)|\nabla\Phi'(A_e)|^2$$
$$-\rho_e\alpha|v_e|^2 - 2\rho_e\Psi'(A_e)\hat{z}\cdot(v_e\times\nabla\Phi'(A_e))\,,$$

$$\mu_5 = \mu_4 - \Phi'(A_e)^2(\rho_e\varepsilon''(\rho_e)\alpha - \varepsilon''(\rho_e)(\partial_y\Phi'(A_e))^2 - \Psi'(A_e)\rho_e - (v_e^1)^2\alpha + 2\Psi'(A_e)v_e^1\partial_y\Phi'(A_e))\,,$$

$$\mu_6 = \mu_5 + \Phi'(A_e)^3(\alpha\varepsilon''(\rho_e)\Phi'(A_e) - \Phi'(A_e)^2\Psi'(A_e)) + \rho_e\varepsilon''(\rho_e)\alpha - 2\Psi'(A_e)v_e^2\partial_x\Phi'(A_e)$$
$$-\varepsilon''(\rho_e)(\partial_x\Phi'(A_e))^2 - (v_e^2)^2\alpha - \Psi'(A_e)^2\rho_e\,.$$

The conditions for formal (and hence linearized) stability are

$$\rho_e > 0\,, \qquad \rho_e\varepsilon''(\rho_e) - |v_e|^2 > 0\,, \qquad \mu_4 > 0\,, \qquad \mu_5 > 0\,, \qquad \mu_6 > 0\,. \tag{5.2F5}$$

We examine these conditions of formal stability for the cases of the Grad–Shafranov and Alfvén solutions.

(i) The Grad–Shafranov solutions are static, i.e., $v_e = \mathbf{0}$ and $\Phi' = 0$. In this case $-K(A_e) = -h(\rho_e) = -\varepsilon'(\rho_e) = \Psi(A_e)$, and $-K'(A_e) = -J_e/\rho_e = \Psi'(A_e)$. The quadratic form (5.2SV) in this case simplifies to

$$\int_D [\rho_e|\delta v|^2 + \varepsilon''(\rho_e)(\delta\rho)^2 + |\nabla\delta A|^2 + 2\Psi'(A_e)\delta\rho\delta A + \rho_e\Psi''(A_e)(\delta A)^2]\,\mathrm{d}x\,\mathrm{d}y\,,$$

and so the conditions for formal stability in this case become

$$\rho_e > 0\,, \qquad \rho_e\varepsilon''(\rho_e) = c_e^2 > 0\,, \qquad \rho_e^2c_e^2K''(A_e) > J_e^2\,, \tag{5.2GS}$$

where $c_e^2 = \rho_e\varepsilon''(\rho_e)$ is the sound speed. These conditions are equivalent to

$$(J_e/\rho_e)^2 + c_e^2\nabla(J_e/\rho_e)/\nabla A_e > 0\,,$$

if $\nabla A_e \neq 0$.

(ii) The Alfvén solutions are characterized by $\rho_e = c^2$, $v_e = B_e/c$, and $-\Phi' = c$, $c = \text{constant}$. Then from (5.2FV8) with $\rho_e = c^2$, it follows that $-\Psi' = K' = 0$. The quadratic form (5.2SV) simplifies in this case to

$$\int_D [c^2|\delta \boldsymbol{v}|^2 + 2\boldsymbol{v}_e \cdot \delta \boldsymbol{v} \delta\rho + \varepsilon''(c^2)(\delta\rho)^2 + |\nabla\delta A|^2 + 2c(\hat{z} \times \delta \boldsymbol{v}) \cdot \nabla\delta A] \,dx\,dy\,. \tag{5.2ALF}$$

The quadratic form in $\delta \boldsymbol{v}$, $\delta\rho$, $\nabla\delta A$ under the integral sign in (5.2ALF) has the 5×5 matrix

$$\begin{bmatrix} c^2 & 0 & v_e^1 & 0 & c \\ 0 & c^2 & v_e^2 & -c & 0 \\ v_e^1 & v_e^2 & \varepsilon''(c^2) & 0 & 0 \\ 0 & -c & 0 & 1 & 0 \\ c & 0 & 0 & 0 & 1 \end{bmatrix}$$

whose principal subdeterminants are c^2, c^4, $c^2(c_e^2 - |\boldsymbol{v}_e|^2)$, $-c^2(v_e^2)^2$, 0. Thus, the second variation is indefinite in the case of Alfvén solutions. In fact, grouping together terms in (5ALF) that involve $\delta \boldsymbol{v}$, and completing squares leads to

$$\delta^2 H_C = \int_D \{c^2|\delta \boldsymbol{v} + c^{-1}\nabla\delta A \times \hat{z}|^2 + c^2|\delta \boldsymbol{v} + c^{-2}\boldsymbol{v}_e\delta\rho|^2 - c^2|\delta \boldsymbol{v}|^2 + [\varepsilon''(c^2) - |\boldsymbol{v}_e|^2/c^2](\delta\rho)^2\} \,dx\,dy\,.$$

One can check that, provided $\varepsilon''(c^2) - 2|\boldsymbol{v}_e|^2/c^2 > 0$, the minimum possible value of $\delta^2 H_C$ here is zero, which occurs for $\delta\rho = 0$ and $\delta \boldsymbol{v} = -c^{-1}\nabla\delta A \times \hat{z}$, i.e., precisely for those variations in the class of incompressible Alfvén solutions. Thus, the incompressible Alfvén solutions are minimum energy solutions amongst the compressible solutions. Thus, if $\varepsilon''(c^2) - 2|\boldsymbol{v}_e|^2/c^2 > 0$, the incompressible Alfvén solutions are formally stable *as a class* within the compressible solutions. This extends a result of Hazeltine, Holm, Marsden and Morrison [1984] given above in example 5.1, where stability of this class among incompressible solutions is shown. (The convexity analysis given below can also be extended to cover this case.) See also Chandrasekhar [1961, §115].

D. *Convexity analysis*

Because of the complexity of the general case, we shall confine our convexity analysis to Grad–Shafranov solutions. With H given by (5.2H) and $\boldsymbol{v}_e = \mathbf{0}$, let

$$\hat{H}(\Delta \boldsymbol{v}, \Delta\rho, \Delta A) = H(\Delta \boldsymbol{v}, \rho_e + \Delta\rho, A_e + \Delta A) - H(\mathbf{0}, \rho_e, A_e) - \mathrm{D}H(\mathbf{0}, \rho_e, A_e) \cdot (\Delta \boldsymbol{v}, \Delta\rho, \Delta A)$$

$$= \int_D [\tfrac{1}{2}(\rho_e + \Delta\rho)|\Delta \boldsymbol{v}|^2 + [\varepsilon(\rho_e + \Delta\rho) - \varepsilon(\rho_e) - \varepsilon'(\rho_e)\Delta\rho] + \tfrac{1}{2}|\nabla(\Delta A)|^2] \,dx\,dy\,.$$

Thus, if we assume that ε satisfies the stability criterion

$$\varepsilon''(\rho_e) \geq a\,, \qquad a \text{ constant}\,, \tag{5.2SC1}$$

and we confine our attention to solutions satisfying

$$\rho = \rho_e + \Delta\rho \geq \rho_{\min} > 0\,, \tag{5.2SC2}$$

then condition (CH) of section 2 holds with

$$Q_1(\Delta \boldsymbol{v}, \Delta\rho, \Delta A) = \tfrac{1}{2}\int_D [\rho_{\min}|\Delta \boldsymbol{v}|^2 + a(\Delta\rho)^2 + |\nabla(\Delta A)|^2]\,\mathrm{d}x\,\mathrm{d}y\,.$$

The conserved quantity used is

$$C_\Psi(\boldsymbol{v}, \rho, A) = \int_D \rho\Psi(A)\,\mathrm{d}x\,\mathrm{d}y\,.$$

Let $\hat{C}(\Delta \boldsymbol{v}, \Delta\rho, \Delta A) = C_\Psi(\boldsymbol{v}_e + \Delta \boldsymbol{v}, \rho_e + \Delta\rho, A_e + \Delta A) - C_\Psi(\boldsymbol{v}_e, \rho_e, A_e) - \mathrm{D}C_\Psi(\boldsymbol{v}_e, \rho_e, A_e)\cdot(\Delta \boldsymbol{v}, \Delta\rho, \Delta A)$

$$= \int_D [(\rho_e + \Delta\rho)\Psi(A_e + \Delta A) - \rho_e\Psi(A_e) - \Delta\rho\Psi(A_e) - \rho_e\Psi'(A_e)\Delta A]\,\mathrm{d}x\,\mathrm{d}y$$

$$= \int_D \{\rho_e[\Psi(A_e + \Delta A) - \Psi(A_e) - \Psi'(A_e)\Delta A] + [\Psi(A_e + \Delta A) - \Psi(A_e)]\Delta\rho\}\,\mathrm{d}x\,\mathrm{d}y\,.$$

Thus, if we assume

$$\Psi''(A_e) \geq r \quad \text{and} \quad \Psi'(A_e) \geq s\,, \tag{5.2SC3}$$

then condition (CC) of section 2 holds with

$$Q_2(\Delta v, \Delta\rho, \Delta A) = \tfrac{1}{2}\int_D (\rho_e r|\Delta A|^2 + 2s\Delta A\Delta\rho)\,\mathrm{d}x\,\mathrm{d}y\,.$$

Condition (D) of section 2 holds when $Q_1 + Q_2$ is positive; this holds if $\rho_e > 0$, (5.2SC1–3) holds and

$$a > 0\,, \qquad \rho_e a r - s^2 > 0\,. \tag{5.2SC4}$$

Thus, in the norm

$$\|(\Delta \boldsymbol{v}, \Delta\rho, \Delta A)\|^2 = \tfrac{1}{2}\int_D \{\rho_{\min}|\Delta \boldsymbol{v}|^2 + |\nabla(\Delta A)|^2 + a|\Delta\rho|^2 + 2s\Delta\rho\Delta A + \rho_e r|\Delta A|^2\}\,\mathrm{d}x\,\mathrm{d}y \tag{5.2N}$$

we get the a priori estimate (E) of section 2.

Stability theorem – compressible Grad–Shafranov case. Let $(\mathbf{0}, \rho_e, A_e)$ be an equilibrium solution of (5.2EM) and suppose the current $J_e = -\nabla^2 A_e$ satisfies

$$J_e + \rho_e \Psi'(A_e) = 0$$

for a real valued function Ψ of one variable. Assume the internal energy satisfies

$$0 < a \leq \varepsilon''(\rho_e) \leq \bar{a} < \infty$$

for constants a, $\bar{a}$, and Ψ satisfies

$$r \leq \Psi''(A_e) \leq R\,, \qquad s \leq \Psi'(A_e) \leq S\,,$$

where

$$\rho_e a r - s^2 > 0 \quad \text{and} \quad \rho_e \bar{a} R - S^2 > 0\,.$$

Then for smooth solutions satisfying $\infty > \rho_{max} \geq \rho \geq \rho_{min} > 0$, we have stability of the equilibrium in the norm (5.2N).

Proof. All that remains is to show that (CH)′ and (CC)′ of section 2 hold. But this follows from the upper estimates on ε'', Ψ'' and Ψ'. ∎

6. Planar MHD with $\boldsymbol{B}$ perpendicular to the plane

In this section, we consider the two-dimensional cases of incompressible and barotropic MHD flow taking place in a simply connected domain D of the x, y plane with $\boldsymbol{B}$ normal to the plane. We shall begin with the homogeneous incompressible case. Here the results are essentially the same as in Arnold's case (see section 3.3) for the simple reason that the total energy H is convex in $\boldsymbol{B}$, $\boldsymbol{B}$ is advected by the flow, and $\boldsymbol{B}$ enters the evolution equation for the vorticity only in terms of the gradient of the energy of the magnetic field. Nevertheless, this case is quite instructive to do by the stability algorithm and will give insight for the compressible case.

6.1. *Homogeneous incompressible case*

A. *Equations of motion and Hamiltonian*

The MHD equations for this case are simply

$$\partial \boldsymbol{v}/\partial t + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} = -\nabla p - \nabla B^2/2\,; \qquad \operatorname{div} \boldsymbol{v} = 0\,, \tag{6.1EM$\boldsymbol{v}$}$$

$$\partial B/\partial t + \boldsymbol{v} \cdot (\nabla B) = 0\,, \tag{6.1EMB}$$

where the velocity $\boldsymbol{v}$ and the scalar magnetic field B depend only on x and y. The velocity $\boldsymbol{v}$ has no z-component and the magnetic field $\boldsymbol{B} := B\hat{\boldsymbol{z}}$ is perpendicular to the x, y plane.

The space P consists of pairs $(\boldsymbol{v}, B)$, with the velocity $\boldsymbol{v}$ divergence free, parallel to the boundary ∂D of the domain D (and tends to a constant at infinity, if D is unbounded).

The total energy of the system is

$$H(\boldsymbol{v}, B) = \tfrac{1}{2} \int_D |\boldsymbol{v}|^2 \, dx\, dy + \tfrac{1}{2} \int_D B^2 \, dx\, dy \,, \tag{6.1H}$$

which is easily seen to be conserved by the system (6.1EM).

Remark A. Poisson structure. As in section 3.3, we identify the vector space $\mathscr{X}_{\mathrm{div}}(D)$ of divergence free vector fields on D with itself by the L^2-pairing of vector fields. The Poisson bracket is the Lie–Poisson bracket associated with the semidirect product of $\mathscr{X}_{\mathrm{div}}(D)$ acting on the vector space of functions $\mathscr{F}(D)$ by minus the Lie derivative, where $\mathscr{X}_{\mathrm{div}}(D)$ has as Lie algebra bracket *minus* the Lie bracket of vector fields. Thus, if F, G: $(\mathscr{X}_{\mathrm{div}}(D) \circledS \mathscr{F}(D))^* = \mathscr{X}_{\mathrm{div}}(D) \times \mathscr{F}(D) \to \mathbb{R}$, their Lie–Poisson bracket is given by

$$\{F, G\}(\boldsymbol{v}, B) = \int_D \left[\boldsymbol{v} \cdot \left(\left(\frac{\delta G}{\delta \boldsymbol{v}} \cdot \nabla \right) \frac{\delta F}{\delta \boldsymbol{v}} - \left(\frac{\delta F}{\delta \boldsymbol{v}} \cdot \nabla \right) \frac{\delta G}{\delta \boldsymbol{v}} \right) + B \left(\frac{\delta G}{\delta \boldsymbol{v}} \cdot \left(\nabla \frac{\delta F}{\delta B} \right) - \frac{\delta F}{\delta \boldsymbol{v}} \cdot \left(\nabla \frac{\delta G}{\delta B} \right) \right) \right] dx\, dy \,,$$

where $\delta F/\delta \boldsymbol{v}$, $\delta G/\delta \boldsymbol{v}$ are divergence-free vector fields in the plane and $\delta F/\delta B$, $\delta G/\delta B$ are functions on D. As usual, we have identified $\mathscr{F}(D)$ with itself by the L^2-pairing of functions. As in Marsden and Weinstein [1983] and Marsden, Ratiu and Weinstein [1984], this bracket may be derived from the canonical bracket in the Lagrangian representation.

B. *Constants of motion*

Taking the curl of (6.1EM$\boldsymbol{v}$) yields the vorticity equation

$$\partial \omega / \partial t + \boldsymbol{v} \cdot (\nabla \omega) = 0 \,, \tag{6.1EM}$$

where $\omega = \hat{\boldsymbol{z}} \cdot \operatorname{curl} \boldsymbol{v}$ is the scalar vorticity. Thus ω and B are advected by the flow and hence

$$C_\Phi(\boldsymbol{v}, B) = \int_D \Phi(\omega, B) \, dx\, dy \tag{6.1C}$$

is a constant of motion for any function Φ of two variables.

Remark B. Casimirs. Although C_Φ is preserved by (6.1EM) for any Φ, it is a Casimir only when Φ is linear in ω: i.e., $\Phi(\boldsymbol{v}, B) = \omega \Phi_1(B) + \Phi_2(B)$. This may be verified directly using the Poisson bracket or by checking for invariance of C_Φ under the coadjoint action of $\mathrm{Diff}_{\mathrm{vol}}(D) \circledS \mathscr{F}(D)$ on the dual of its Lie algebra; this action is $(\eta, f) \cdot (\boldsymbol{v}, B) = (\eta_* \boldsymbol{v} - \mathbb{P}(\eta_* B \nabla f), \eta_* B)$ where $\mathbb{P}$ projects a vector field onto its divergence free part which is also parallel to the boundary by the Weyl–Helmholtz–Hodge decomposition (i.e. $\mathbb{P}\boldsymbol{u} = \boldsymbol{u} + \nabla g$, with $\operatorname{div} \boldsymbol{u} = 0$, $\boldsymbol{u} \| \partial D$ and the sum is orthogonal). The larger family of conserved quantities is due to the special form of the equations. [If one enlarged the variables from $(\boldsymbol{v}, B)$ to $(\boldsymbol{v}, \omega, B)$ and used the larger semidirect product bracket, the equations would still be Hamiltonian (this is what is special about the equations) and (6.1C) would appear as a Casimir.]

C. *First variation*

As in section 3.3, a stationary solution $\boldsymbol{v}_e$, B_e with stream function ψ_e, (constant on ∂D) satisfies

$$\psi_e = \bar{\psi}(B_e) \quad \text{and} \quad \omega_e = \bar{\omega}(B_e)\,.$$

Consider the conserved quantity

$$H_C(v, B) := H(\boldsymbol{v}, B) + C_\Phi(\boldsymbol{v}, B) + \lambda \int_D \omega \,\mathrm{d}x\,\mathrm{d}y\,,$$

where λ is a constant. (Use the circulations around each hole as in section 3.3 if D is not simply connected.) Then H_C has a critical point at $(\boldsymbol{v}_e, B_e)$, provided

$$0 = \mathrm{D}H_C(\boldsymbol{v}_e, B_e)\cdot(\delta\boldsymbol{v}, \delta B) = \int_D [\boldsymbol{v}_e\cdot\delta\boldsymbol{v} + B_e\delta B + \dot{\Phi}(\omega_e, B_e)\delta\omega + \Phi'(\omega_e, B_e)\delta B]\,\mathrm{d}x\,\mathrm{d}y$$

$$+ \lambda \int_D \delta\omega \,\mathrm{d}x\,\mathrm{d}y$$

$$= \int_D [(\psi_e + \dot{\Phi}(\omega_e, B_e))\delta\omega + (B_e + \Phi'(\omega_e, B_e)]\delta B)\,\mathrm{d}x\,\mathrm{d}y + \oint_{\partial D} \psi_e\delta\boldsymbol{v}\cdot\mathrm{d}\boldsymbol{\ell} + \lambda \oint_{\partial D} \delta\boldsymbol{v}\cdot\mathrm{d}\boldsymbol{\ell}\,,$$

where $\dot{\Phi}$ and Φ' denote the derivatives of Φ with respect of its first and second arguments, respectively. Since ψ_e is constant on the boundary, the last two boundary integrals will cancel, provided

$$\lambda + \psi_e|\partial D = 0\,, \tag{λ}$$

and thus $\mathrm{D}H_C(\boldsymbol{v}_e, B_e) = 0$ if

$$\psi_e + \dot{\Phi}(\omega_e, B_e) = 0\,, \qquad B_e + \Phi'(\omega_e, B_e) = 0\,. \tag{6.1FV}$$

There is some functional freedom remaining in $\dot{\Phi}$ and Φ', since (6.1FV) specifies the partial derivatives of Φ only along the curve where ω_e and B_e are related by the equilibrium conditions. In particular, this imposes two conditions on the second derivatives obtained by implicit differentiation:

$$\frac{\mathrm{d}\bar{\psi}}{\mathrm{d}B_e} + \ddot{\Phi}(\omega_e, B_e)\frac{\mathrm{d}\bar{\omega}}{\mathrm{d}B_e} + \dot{\Phi}'(\omega_e, B_e) = 0\,; \tag{6.1C1}$$

$$1 + \Phi''(\omega_e, B_e) + \dot{\Phi}'(\omega_e, B_e)\frac{\mathrm{d}\bar{\omega}}{\mathrm{d}B_e} = 0\,; \tag{6.1C2}$$

these relations will be used in the next step.

Remark D. Second variation. As a guide to the convexity estimates that follow, we shall find conditions under which the second variation of H_C is definite. One has

$$D^2H_C(\boldsymbol{v}_e, B_e)\cdot(\delta\boldsymbol{v}, \delta B)^2 = \int_D \left[|\delta\boldsymbol{v}|^2 + (\delta\omega, \delta B)\begin{bmatrix} \ddot{\Phi} & \dot{\Phi}' \\ \dot{\Phi}' & \Phi''+1 \end{bmatrix}\begin{pmatrix}\delta\omega \\ \delta B\end{pmatrix}\right] \mathrm{d}x\,\mathrm{d}y,$$

where $\ddot{\Phi}$, $\dot{\Phi}'$, and Φ'' are to be evaluated at (ω_e, B_e). Sufficient conditions for this quadratic form to be positive definite are

$$\ddot{\Phi}(\omega_e, B_e) > 0 \quad \text{and} \quad \ddot{\Phi}(\omega_e, B_e)(\Phi''(\omega_e, B_e)+1) - \dot{\Phi}'(\omega_e, B_e)^2 > 0\,.$$

Eliminating $\Phi''(\omega_e, B_e)$ from the second condition by the use of (6.1C2) gives

$$-\frac{\mathrm{d}\bar{\omega}}{\mathrm{d}B_e}\frac{\ddot{\Phi}(\omega_e, B_e)}{\dot{\Phi}'(\omega_e, B_e)} > 1\,,$$

provided $\dot{\Phi}'(\omega_e, B_e) \neq 0$. Using (6.1C1) produces

$$\frac{\ddot{\Phi}(\omega_e, B_e)\,\mathrm{d}\bar{\omega}/\mathrm{d}B_e}{\ddot{\Phi}(\omega_e, B_e)\,\mathrm{d}\bar{\omega}/\mathrm{d}B_e + \mathrm{d}\bar{\psi}/\mathrm{d}B_e} > 1\,,$$

i.e.,

$$\frac{\mathrm{d}\bar{\psi}/\mathrm{d}B_e}{\ddot{\Phi}(\omega_e, B_e)\,\mathrm{d}\bar{\omega}/\mathrm{d}B_e} < 0\,.$$

Now since $\ddot{\Phi}(\omega_e, B_e) > 0$ and $\omega_e = -\nabla^2\psi_e$, this becomes

$$\nabla\psi_e / \nabla\nabla^2\psi_e > 0\,,$$

which is identical to the condition for stability of stationary solutions for planar, incompressible, homogeneous Euler flow, as found by Arnold [1965, 1969a] (see section 3.3). Thus, in this case the magnetic field does not affect the stability condition. This stability condition implies linearized stability of $(\boldsymbol{v}_e, B_e)$.

D. *Convexity estimates*

The choice

$$\Phi(\omega, B) = -\tfrac{1}{2}B^2 + \Phi_1(\omega)$$

effectively eliminates the magnetic field in the expression for H_C. Then the only remaining condition for formal stability is $\ddot{\Phi}(\omega_e, B_e) = \ddot{\Phi}_1(\omega_e) > 0$. With this choice of Φ the ensuing steps D, E, F in the stability algorithm repeat the corresponding steps in example 3, section 3.

6.2. Compressible case

In this case, the MHD equations simplify to

$$\partial \boldsymbol{v}/\partial t = -(\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v} - \boldsymbol{\nabla} h(\rho) - (1/\rho)\boldsymbol{\nabla} B^2/2\,, \qquad (6.2\text{EM}\boldsymbol{v})$$

$$\partial\rho/\partial t = -\operatorname{div}\rho\boldsymbol{v}\,, \qquad (6.2\text{EM}\rho)$$

$$\partial B/\partial t = -\operatorname{div} B\boldsymbol{v}\,, \qquad (6.2\text{EMB})$$

which is a dynamical system for the fluid velocity $\boldsymbol{v}(x, y, t)$, the mass density $\rho(x, y, t)$, and scalar magnetic field $B(x, y, t)$. The single component of B is normal to the plane, along the unit vector, $\hat{z}$. Here $h(\rho)$, the specific enthalpy, is a given function related to the barotropic pressure, $p(\rho)$, by $h'(\rho) = \rho^{-1}p'(\rho)$. Again, the velocity $\boldsymbol{v}$ must be tangential at the boundary ∂D of the domain D. We choose P, as in example 4 in section 3, to be an appropriate Sobolev space (weighted if D is unbounded) of triples $(\boldsymbol{v}, \rho, B)$ which satisfy the given boundary conditions and have specified asymptotic behavior. The eqs. (6.1EM) define a dynamical system in P, at least for a short time. As in section 3.4, shocks are excluded by confining our attention to C^1 solutions, and cavitation and extreme compression are avoided by confining our attention to solutions with a density satisfying $0 < \rho_{\min} \le \rho \le \rho_{\max} < \infty$.

The conserved energy is

$$H(\boldsymbol{v}, \rho, B) = \int_D [\tfrac{1}{2}\rho v^2 + \varepsilon(\rho) + B^2/2]\,\mathrm{d}x\,\mathrm{d}y\,, \qquad (6.2H)$$

where the internal energy $\varepsilon(\rho)$ is as in section 3.4. As before, we shall sometimes find it convenient to work with the variables $\boldsymbol{M} = \rho\boldsymbol{v}$, ρ and B.

Remark A. Poisson structure. The equations (6.1EM) are Hamiltonian relative to the Poisson structure on P given by the same expression as in remark A in example 3.4, plus the terms

$$\int_D B\left[\frac{\delta G}{\delta \boldsymbol{M}}\cdot\boldsymbol{\nabla}\left(\frac{\delta F}{\delta B}\right) - \frac{\delta F}{\delta \boldsymbol{M}}\cdot\boldsymbol{\nabla}\left(\frac{\delta G}{\delta B}\right)\right]\mathrm{d}x\,\mathrm{d}y$$

(see Morrison and Greene [1980], Holm and Kupershmidt [1983], Marsden et al. [1983] and Marsden, Ratiu and Weinstein [1984]).

B. Constants of motion

Equations (6.2EM) imply that B/ρ is advected by the flow:

$$(\mathrm{d}/\mathrm{d}t)(B/\rho) = 0\,,$$

where $\mathrm{d}/\mathrm{d}t = \partial/\partial t + \boldsymbol{v}\cdot\boldsymbol{\nabla}$ is the material derivative. Thus, the following functional is conserved:

$$C_\Phi(\rho, B) = \int_D \rho\Phi(B/\rho)\,\mathrm{d}x\,\mathrm{d}y\,, \tag{6.2C$_\Phi$}$$

for an arbitrary real valued function Φ. Taking $\hat{z}\cdot$curl of the motion equation (6.2EM$\boldsymbol{v}$) leads to

$$\frac{\partial\omega}{\partial t} = -\mathrm{div}\left(\omega\boldsymbol{v} + \frac{B}{\rho}\,\mathrm{curl}\;\hat{z}B\right). \tag{6.2ω}$$

The circulation $C_\omega = \int_D \omega\,\mathrm{d}x\,\mathrm{d}y$ will be preserved provided

$$\boldsymbol{v}\cdot\boldsymbol{n} = 0 \quad \text{and} \quad \boldsymbol{J}\cdot\boldsymbol{n} = 0$$

on the boundary ∂D, where $\boldsymbol{J} = \boldsymbol{\nabla}\times\boldsymbol{B}$ is the current. With these boundary conditions, we shall prove that there is another constant of the motion, namely

$$C_\Lambda(\boldsymbol{v}, \rho, B) = \int_D \omega\Lambda(B/\rho)\,\mathrm{d}x\,\mathrm{d}y\,. \tag{6.2C_Λ}$$

Now using (6.2ω) and (6.2B/ρ) yields

$$\frac{\partial}{\partial t} C_\Lambda(\boldsymbol{v}, \rho, B) = -\int_D [\mathrm{div}(\omega\Lambda(B/\rho)\boldsymbol{v}) + \Lambda(B/\rho)\,\mathrm{div}(B\boldsymbol{J}/\rho)]\,\mathrm{d}x\,\mathrm{d}y$$

$$= -\int_D \mathrm{div}[\omega\Lambda(B/\rho)\boldsymbol{v} + \boldsymbol{J}\cdot\boldsymbol{\nabla}N(B/\rho)]\,\mathrm{d}x\,\mathrm{d}y\,,$$

where $N'(B/\rho) = \Lambda(B/\rho)$. Thus

$$\frac{\partial}{\partial t} C_\Lambda(\boldsymbol{v}, \rho, B) = -\oint_{\partial D} [\omega\Lambda(B/\rho)\boldsymbol{v} + N(B/\rho)\boldsymbol{J}]\cdot\boldsymbol{n}\,\mathrm{d}x\,,$$

which vanishes since we assume $\boldsymbol{v}\cdot\boldsymbol{n} = 0$, and $\boldsymbol{J}\cdot\boldsymbol{n} = 0$ on the boundary. The condition $\boldsymbol{J}\cdot\boldsymbol{n} = 0$ on ∂D means that ∂D is a fixed insulating boundary (no current crosses ∂D).

Remark B. Casimirs. The functions C_Φ are Casimirs for the Poisson structure in remark A. This may be checked by direct computation, or by an invariance argument similar to that given in remark B, section 3.4 and in part I. The coadjoint action is as follows: for

$$(\eta, f, g) \in \mathrm{Diff}(D) \circledS (\mathscr{F}(D) \oplus \mathscr{F}(D)), \quad (\eta, f, g) \cdot (\boldsymbol{M}, \rho, B) = (\eta_* \boldsymbol{M} - \eta_* \rho \nabla f - \eta_* B \nabla g, \eta_* \rho, \eta_* B).$$

The action defining the semidirect product Lie algebra is minus the Lie derivative.

The functions C_Λ are not Casimirs; their conservation relies on the special fact that $\partial\omega/\partial t$ is a divergence (see (6.1ω)). One could make C_Λ appear as a Casimir by introducing $\zeta = B/\rho$ as an extra variable, and adding on a corresponding advected term in the Poisson bracket.

C. *First variation*

We shall relate the equilibrium solutions $\boldsymbol{v}_e$, ρ_e, B_e of (6.1EM) to critical points of the conserved functional

$$H_C(\boldsymbol{v}, \rho, B) := H(\boldsymbol{v}, \rho, B) + C_\Phi(\boldsymbol{v}, \rho, B) + C_\Lambda(\boldsymbol{v}, \rho, B) + \lambda \int_D \omega \, \mathrm{d}x \, \mathrm{d}y .$$

The first variation of H_C becomes, after an integration by parts:

$$\begin{aligned} \mathrm{D}H_C(\boldsymbol{v}_e, \rho_e, B_e) = \int_D \Big\{ & \Big[\tfrac{1}{2}|\boldsymbol{v}_e|^2 + \varepsilon'(\rho_e) + \Phi(B_e/\rho_e) - \frac{B_e}{\rho_e} [\Phi'(B_e/\rho_e) + \frac{\omega_e}{\rho_e} \Lambda'(B_e/\rho_e)] \Big] \delta\rho \\ & + \Big[B_e + \Phi'(B_e/\rho_e) + \frac{\omega_e}{\rho_e} \Lambda'(B_e/\rho_e) \Big] \delta B + [\rho_e \boldsymbol{v}_e - \hat{\boldsymbol{z}} \times \nabla \Lambda(B_e/\rho_e)] \cdot \delta \boldsymbol{v} \Big\} \\ & + \oint_{\partial D} \Lambda(B_e/\rho_e)(\delta \boldsymbol{v} \times \hat{\boldsymbol{z}}) \cdot \hat{\boldsymbol{n}} \, \mathrm{d}s + \lambda \oint_{\partial D} \delta \boldsymbol{v} \cdot \mathrm{d}\boldsymbol{\ell} . \end{aligned}$$

Since B_e and ρ_e are necessarily constant on the boundary, the two boundary integrals cancel if

$$\Lambda(B_e/\rho_e)|\partial D + \lambda = 0 . \tag{6.2λ}$$

With this choice of λ, the first variation vanishes at $(\boldsymbol{v}_e, \rho_e, B_e)$ provided

$$\tfrac{1}{2}|\boldsymbol{v}_e|^2 + \varepsilon'(\rho_e) + \Phi(B_e/\rho_e) - \frac{B_e}{\rho_e} \Big[\Phi'(B_e/\rho_e) + \frac{\omega_e}{\rho_e} \Lambda'(B_e/\rho_e) \Big] = 0 , \tag{6.2FV1}$$

$$\frac{\omega_e}{\rho_e} \Lambda'(B_e/\rho_e) + \Phi'(B_e/\rho_e) + B_e = 0 , \tag{6.2FV2}$$

and

$$\rho_e \boldsymbol{v}_e - \hat{\boldsymbol{z}} \times \nabla \Lambda(B_e/\rho_e) = 0 . \tag{6.2FV3}$$

To relate Φ, Λ with conditions satisfied by stationary solutions, a closer look at the stationary solutions is in order.

The stationary equations are

$$\operatorname{div}(\rho_e \boldsymbol{v}_e) = 0\,. \tag{6.2S1}$$

$$\boldsymbol{v}_e \cdot \boldsymbol{\nabla}(B_e/p_e) = 0\,, \tag{6.2S2}$$

$$\boldsymbol{v}_e \times \hat{z}\omega_e - \boldsymbol{\nabla}(\tfrac{1}{2}|\boldsymbol{v}_e|^2 + h(\rho_e) + B_e^2/\rho_e) + B_e \boldsymbol{\nabla}(B_e/\rho_e) = 0\,. \tag{6.2S3}$$

Taking the dot product of $\boldsymbol{v}_e$ with the last equation, we get

$$\boldsymbol{v}_e \cdot \boldsymbol{\nabla}(\tfrac{1}{2}|\boldsymbol{v}_e|^2 + h(\rho_e) + B_e^2/\rho_e) = 0\,.$$

This relation and (6.2S2) are satisfied if the following functional relationship, called *Bernoulli's Law*, holds

$$\tfrac{1}{2}|\boldsymbol{v}_e|^2 + h(\rho_e) + B_e^2/\rho_e = K(B_e/\rho_e)\,; \tag{6.2B}$$

the function K is called *Bernoulli's function*. Taking the cross product of $\hat{z}$ with (6.2S3) and applying (6.2B) leads to

$$0 = \operatorname{div}(\rho_e \boldsymbol{v}_e) = (\hat{z} \times \boldsymbol{\nabla}(B_e/\rho_e)) \cdot \boldsymbol{\nabla}\left[\frac{\rho_e}{\omega_e}(K'(B_e/\rho_e) - B_e)\right],$$

and hence

$$\boldsymbol{v}_e \cdot \boldsymbol{\nabla}\left[\frac{\rho_e}{\omega_e}(K'(B_e/\rho_e) - B_e)\right] = 0\,.$$

For this to hold, another functional relationship suffices,

$$\frac{\rho_e}{\omega_e}[K'(B_e/\rho_e) - B_e] = L(B_e/\rho_e)\,, \tag{6.2S5}$$

for a function $L(B_e/\rho_e)$. (This is analogous to *Long's equation* in stratified flow; see Abarbanel et al. [1985].)

Returning to the conditions for the vanishing of the first variation of H_C and comparing (6.2C1), (6.2C2), (6.2C3) with (6.2B), (6.2S4), and (6.2S5), we identify

$$\Phi(B_e/\rho_e) = -K(B_e/\rho_e) \tag{6.2Φ}$$

$$\Lambda'(B_e/\rho_e) = L(B_e/\rho_e)\,. \tag{6.2Λ}$$

Summarizing, we have proved:

Proposition. Stationary solutions of the barotropic planar MHD equations with $\boldsymbol{B}$ perpendicular to the plane and boundary conditions $\boldsymbol{v}\cdot\boldsymbol{n}=0$, $\boldsymbol{J}\cdot\boldsymbol{n}=0$ are critical points of the total energy H constrained by C_Φ, C_Λ and $\lambda\int_D \omega\,\mathrm{d}x\,\mathrm{d}y$, i.e., are critical points of H_C, provided Φ, Λ and λ satisfy eqs. (6.2Φ), (6.2Λ) and (6.2λ).

Remark D. Formal stability. The criterion of formal stability is calculated from the second variation of H_C, evaluated at the equilibrium point. Letting

$$\delta^2 H_C := \mathrm{D}^2 H_C(\boldsymbol{v}_e,\rho_e,B_e)\cdot(\delta\boldsymbol{v},\delta\rho,\delta B)^2\,,$$

we obtain, after some computation:

$$\delta^2 H_C=\int_D \mathrm{d}x\,\mathrm{d}y\left\{\frac{1}{\rho_e}|\rho_e\delta\boldsymbol{v}+\boldsymbol{v}_e\delta\rho|^2+(\varepsilon''(\rho_e)-|\boldsymbol{v}_e|^2/\rho_e)(\delta\rho)^2\right.$$
$$\left.+(\delta B)^2\rho_e A\left[\delta(B/\rho)+\frac{\Lambda'(B_e/\rho_e)}{A}\delta\left(\frac{\omega}{\rho}\right)\right]^2-\rho_e\frac{(\Lambda'(B_e/\rho_e))^2}{A}\left(\delta\left(\frac{\omega}{\rho}\right)\right)^2\right\}. \tag{6.2SV}$$

In (6.2SV), A is defined by

$$A=\frac{\omega_e}{\rho_e}\Lambda''(B_e/\rho_e)+\Phi''(B_e/\rho_e)\,,$$

and the variation of ω/ρ is given by

$$\delta(\omega/\rho)=\frac{1}{\rho_e}\delta\omega-\frac{\omega_e}{\rho_e^2}\delta\rho\,,$$

with a similar expression for $\delta(B/\rho)$. The quantity $\delta^2 H_C$ is positive definite, provided the following three conditions are satisfied

(a) $\varepsilon''(\rho_e)-|\boldsymbol{v}_e|^2/\rho_e>0\,,$

(b) $\Lambda'(B_e/\rho_e)=0\,,$

(c) $\Phi''(B_e/\rho_e)>0\,.$

Since $\rho_e\varepsilon''(\rho_e)=c_e^2$, where c_e is the sound speed of the equilibrium solution, condition (a) requires the equilibrium flow to be subsonic everywhere. Condition (b), via (6.2Λ), (6.2B) and (6.2S5), requires the equilibrium to be static ($\boldsymbol{v}_e=\boldsymbol{0}$), as well as to satisfy

$$K'(B_e/\rho_e)-B_e=0\,. \tag{b'}$$

Condition (c) implies

$$K''(B_e/\rho_e) < 0\,,$$

or, via (b′)

$$\frac{\mathrm{d}\log B_e}{\mathrm{d}\log \rho_e} < 1\,,$$

for formal and hence linearized stability of a static, planar, barotropic, MHD equilibrium, with $\boldsymbol{B}$ perpendicular to the plane.

Another useful way of stating this, which will facilitate comparison with the three-dimensional case is as follows. Taking the gradient of (b′), we get

$$K''(B_e/\rho_e)\,\nabla(B_e/\rho_e) = \nabla B_e\,.$$

Thus, condition (c) becomes:

$$\Phi''(B_e/\rho_e) = \frac{\nabla B_e\cdot[\hat{z}\times\nabla(B_e/\rho_e)]}{|\nabla(b_e/\rho_e)|^2} = -\frac{\nabla B_e}{\nabla(B_e/\rho_e)} > 0 \tag{c′}$$

(note that this is impossible for constant density solutions).

Additional remarks.

(i) Substitution of the critical point relations (6.2Φ), (6.2Λ), and stationary flow relations (6.2S1, 3) into H_C with $\Lambda'(B_e/\rho_e) = 0$ gives

$$-H_C(\boldsymbol{v}_e, \rho_e, B_e) = \int_D [p(\rho_e) + B_e^2/2]\,\mathrm{d}x\,\mathrm{d}y$$

upon using the thermodynamic relation for the pressure $p(\rho) = \rho\varepsilon'(\rho) - \varepsilon(\rho)$. In fact, the time integral $-\int H_C(\boldsymbol{v}_e, \rho_e, B_e)\,\mathrm{d}t$ is the Lagrangian for the barotropic MHD equations, see Seliger and Whitham [1968]. Taking variational derivatives gives

$$-2\delta^2[H_C|_e] = \int_D [p''(\rho_e)(\delta\rho)^2 + (\delta B)^2]\,\mathrm{d}x\,\mathrm{d}y\,,$$

which is negative definite if $p''(\rho_e) > 0$. Note, however, that this is not equivalent to (6.2SV), since the order is opposite: the latter is obtained by taking variations and then imposing equilibrium relations, while the expression here is obtained by imposing equilibrium relations and then taking variations. Definiteness of (6.2SV) gives conditions for formal stability. (However, $p''(\rho_e) > 0$ is the condition for "thermodynamic stability", as in Courant and Friedrichs [1948], which is related to well-posedness: continuous dependence on initial data, not to dynamic stability.)

(ii) Indefiniteness of the second variation in (6.2SV) does not prove, but strongly suggests instability, caused by the combined presence of perpendicular magnetic field and heterogeneity of the density ($\rho \neq \text{const}$). Density heterogeneity rather than compressibility is the cause, since heterogeneous, incompressible, ($\operatorname{div} \boldsymbol{v} = 0$), MHD flows with perpendicular magnetic field have the same type of second-variational features, constants of the motion and stability conditions as in the present case.

D. *Convexity estimates*

For a stationary solution $(\boldsymbol{v}_e, \rho_e, B_e)$ we have

$$\hat{H} := H(\boldsymbol{v}_e + \Delta\boldsymbol{v}, \rho_e + \Delta\rho, B_e + \Delta B) - H(\boldsymbol{v}_e, \rho_e, B_e) - \mathrm{D}H(\boldsymbol{v}_e, \rho_e, B_e)\cdot(\Delta\boldsymbol{v}, \Delta\rho, \Delta B)$$

$$= \int_D \left[\frac{1}{2}\frac{|\Delta(\rho\boldsymbol{v})|^2}{\rho_e + \Delta\rho} - \tfrac{1}{2}|\boldsymbol{v}_e|^2 \frac{(\Delta\rho)^2}{\rho_e + \Delta\rho} + \tfrac{1}{2}(\Delta B)^2 + \varepsilon(\rho_e + \Delta\rho) - \varepsilon(\rho_e) - \varepsilon'(\rho_e)\Delta\rho\right] \mathrm{d}x\,\mathrm{d}y\,,$$

where $\Delta(\rho\boldsymbol{v}) = (\rho_e + \Delta\rho)(\boldsymbol{v}_e + \Delta\boldsymbol{v}) - \rho_e\boldsymbol{v}_e$. Assume that

$$0 < e \leq \varepsilon''(\tau)$$

for all values of the argument τ. Since

$$0 < \rho_{\min} \leq \rho_e + \Delta\rho \leq \rho_{\max} < \infty\,,$$

we have

$$\hat{H} \geq Q_1(\Delta(\rho\boldsymbol{v}), \Delta\rho, \Delta B) := \int_D \left[\frac{1}{2}\frac{|\Delta(\rho\boldsymbol{v})|^2}{\rho_{\max}} + \tfrac{1}{2}(\Delta B)^2 - \tfrac{1}{2}|\boldsymbol{v}_e|^2\frac{(\Delta\rho)^2}{\rho_{\min}} + \tfrac{1}{2}(\Delta\rho)^2\right] \mathrm{d}x\,\mathrm{d}y\,. \tag{6.2CH}$$

So we have the inequality (CH) of section 2.

Similarly, we get

$$\hat{C}_\Phi := C_\Phi(\boldsymbol{v}_e + \Delta\boldsymbol{v}, \rho_e + \Delta\rho, B_e + \Delta B) - C_\Phi(\boldsymbol{v}_e, \rho_e, B_e)$$
$$- \mathrm{D}C_\Phi(\boldsymbol{v}_e, \rho_e, B_e)\cdot(\Delta\boldsymbol{y}, \Delta\rho, \Delta B)$$

$$= \int_D (\rho_e + \Delta\rho)\left[\Phi\left(\frac{B_e + \Delta B}{\rho_e + \Delta\rho}\right) - \Phi\left(\frac{B_e}{\rho_e}\right) - \Phi'\left(\frac{B_e}{\rho_e}\right)\Delta\left(\frac{B}{\rho}\right)\right] \mathrm{d}x\,\mathrm{d}y\,,$$

where $\Delta(B/\rho) = (B_e + \Delta B)/(\rho_e + \Delta\rho) - B_e/\rho_e$. Now assuming that

$$0 < a \leq \Phi''(\zeta)$$

for all values of the argument, we get

$$\hat{C}_\Phi \geq Q_2(\Delta(B/\rho)) := \oint_D \tfrac{1}{2} a \rho_{\min} (\Delta(B/\rho))^2 \, dx \, dy . \tag{6.2CC}$$

The right-hand side of this inequality defines a quadratic form $Q_2(\Delta(B/\rho))$.

Since the second variation argument suggests we can get stability only for static equilibria, we confine our attention to this case, taking $\Lambda' = 0$:

Stability theorem (static equilibria). Suppose $(\mathbf{0}, \rho_e, B_e)$ is a stationary solution of (6.2EM) satisfying

$$B_e = K'(B_e/\rho_e)$$

for a function K. Suppose

$$0 \leq e < \varepsilon''(\rho_e) \leq E < \infty , \qquad 0 < a \leq -\frac{\nabla B_e}{\nabla(B_e/\rho_e)} \leq A < \infty .$$

For solutions obeying $0 < \rho_{\min} \leq \rho_e + \Delta\rho \leq \rho_{\max} < \infty$, $(\mathbf{0}, \rho_e, B_e)$ is stable in the norm given by

$$\|(\Delta \boldsymbol{v}, \Delta\rho, \Delta B, \Delta(B/\rho))\|^2 = Q_1(\Delta(\rho \boldsymbol{v}), \Delta\rho, \Delta B) + Q_2(\Delta(B/\rho)) ,$$

where Q_1 and Q_2 are given by (6.2CH) and (6.2CC) above.

7. Multifluid plasmas

Here we consider the stability of equilibria for two-dimensional charged fluids, following Holm [1984]. The results will be generalized to three dimensions in section 10.

A. *Equations of motion and Hamiltonian*

The multifluid plasma equations (MFP) describe the motion of a system of ideal charged fluids, interacting via self-consistent electromagnetic forces. The fluid species will be labelled by superscript s; no summation on repeated indices s is imposed in this section. Each species is composed of particles of constant mass m^s and charge q^s, with charge to mass ratio $a^s = q^s/m^s$. The dynamical fluid variables in the Eulerian picture are: fluid velocity $\boldsymbol{v}^s$, mass density ρ^s (with barotropic partial pressure $p^s = p^s(\rho^s)$ and internal energy density $\varepsilon^s = \varepsilon^s(\rho^s)$, each depending *only* on ρ^s), electric field $\boldsymbol{E}$, and magnetic field $\boldsymbol{B}$. In this section we consider planar MFP motion in some domain $D \subset \mathbb{R}^2$ in the xy plane (simply connected for simplicity). In order that such motion remain planar, each of the dependent variables $\boldsymbol{v}^s$, ρ^s, $\boldsymbol{E}$, $\boldsymbol{B}$ must be functions only of (x, y, t); $\boldsymbol{v}^s$ and $\boldsymbol{E}$ must lie in the xy plane, and the vorticity ω^s of the species s and magnetic field $\boldsymbol{B}$ must be directed normal to the xy plane, along $\hat{z}$. We define the scalar vorticity for the species s and the scalar magnetic field by $\boldsymbol{\omega}^s = \omega^s \hat{z}$ and $\boldsymbol{B} = B\hat{z}$. The planar (MFP) equations consist of Euler's equations for charged barotropic fluids in the xy plane interacting self-consistently via Maxwell's field equations, i.e.,

$$\partial_t \boldsymbol{v}^s = -(\omega^s + a^s B)\hat{\boldsymbol{z}} \times \boldsymbol{v}^s - \boldsymbol{\nabla}(\tfrac{1}{2}|\boldsymbol{v}^s|^2 + h^s(\rho^s)) + a^s \boldsymbol{E}, \qquad \partial_t \rho^s = -\operatorname{div} \rho^s \boldsymbol{v}^s,$$

$$\partial_t B = -\hat{\boldsymbol{z}} \cdot \operatorname{curl} \boldsymbol{E} = E_{1,2} - E_{2,1}, \qquad \partial_t \boldsymbol{E} = \boldsymbol{\nabla} B \times \hat{\boldsymbol{z}} - \sum_s a^s \rho^s \boldsymbol{v}^s, \qquad \operatorname{div} \boldsymbol{E} = \sum_s a^s \rho^s, \qquad \operatorname{div} B\hat{\boldsymbol{z}} = 0, \tag{7EM}$$

where $h^s(\rho^s)$ is specific enthalpy, related to pressure p^s and internal energy density $\varepsilon^s(\rho^s)$ by

$$\mathrm{d}\varepsilon^s = h^s \,\mathrm{d}\rho^s, \tag{7A1}$$

$$\mathrm{d}h^s = (\rho^s)^{-1} \,\mathrm{d}p^s. \tag{7A2}$$

The boundary conditions are $\boldsymbol{v} \cdot \boldsymbol{n} = 0$, and $\boldsymbol{E} \times \boldsymbol{n} = 0$ where $\boldsymbol{n}$ is the outward unit normal to ∂D.

For a single fluid species and when $\boldsymbol{E}$ and B are absent, these equations reduce to the equations for planar motion of a barotropic fluid; the stability criteria for stationary solutions of these equations was derived in section 3.4.

The Hamiltonian for (7EM) is the total energy:

$$H(\boldsymbol{v}^s, \rho^s, \boldsymbol{E}, B) := \int_D \left\{ \sum_s \tfrac{1}{2}\rho^s |\boldsymbol{v}|^s + \varepsilon^s(\rho^s) + \tfrac{1}{2}|\boldsymbol{E}|^2 + \tfrac{1}{2}B^2 \right\} \mathrm{d}x\,\mathrm{d}y. \tag{7H}$$

The space P consists of the set of pairs $(\boldsymbol{v}^s, \rho^s)$ for each species s and $(\boldsymbol{E}, B)$ satisfying $\operatorname{div} \boldsymbol{E} = \Sigma_s a^s \rho^s$, which is preserved by the dynamics of (7EM).

Remark A. The dynamic equations (7EM) are Hamiltonian with respect to the following Poisson bracket due to Iwinski and Turski [1976], Spencer and Kaufman [1982] and Spencer [1982]; see also Holm and Kupershmidt [1983], Marsden et al. [1983], Holm, Kupershmidt and Levermore [1983], Marsden, Ratiu and Weinstein [1984a] and Montgomery, Marsden and Ratiu [1984]. Let $\boldsymbol{M}^s = \rho^s \boldsymbol{v}^s$ denote the momentum density for the species s. Then

$$\begin{aligned}
\{F, G\}(\boldsymbol{M}^s, \rho^s, \boldsymbol{E}, B) = \sum_s \int_D \Bigg[& M_i^s \left(\left(\frac{\delta G}{\delta \boldsymbol{M}^s} \cdot \boldsymbol{\nabla} \right) \frac{\delta F}{\delta M_i^s} - \left(\frac{\delta F}{\delta \boldsymbol{M}^s} \cdot \boldsymbol{\nabla} \right) \frac{\delta G}{\delta M_i^s} \right) \\
& + \rho^s \left(\frac{\delta G}{\delta \boldsymbol{M}^s} \cdot \boldsymbol{\nabla} \frac{\delta F}{\delta \rho^s} - \frac{\delta F}{\delta \boldsymbol{M}^s} \cdot \boldsymbol{\nabla} \frac{\delta G}{\delta \rho^s} \right) + a^s \rho^s \left(\frac{\delta F}{\delta \boldsymbol{M}^s} \cdot \frac{\delta G}{\delta \boldsymbol{E}} - \frac{\delta G}{\delta \boldsymbol{M}^s} \cdot \frac{\delta F}{\delta \boldsymbol{E}} \right) \\
& + B \left(\frac{\delta F}{\delta M_1^s} \frac{\delta G}{\delta M_2^s} - \frac{\delta F}{\delta M_2^s} \frac{\delta G}{\delta M_1^s} \right) \Bigg] \mathrm{d}x\,\mathrm{d}y \\
& + \int_D \hat{\boldsymbol{z}} \cdot \left(\frac{\delta F}{\delta \boldsymbol{E}} \times \boldsymbol{\nabla} \frac{\delta G}{\delta B} - \frac{\delta G}{\delta \boldsymbol{E}} \times \boldsymbol{\nabla} \frac{\delta F}{\delta B} \right) \mathrm{d}x\,\mathrm{d}y.
\end{aligned} \tag{7PB}$$

As identified in Holm and Kupershmidt [1983], the first group of terms is a Lie–Poisson bracket in a semidirect product. The last four references show how to derive this bracket by reduction from Lagrangian coordinates and investigate the accompanying geometry.

B. *Constants of the motion*

Define the "modified vorticities" Ω^s by

$$\Omega^s := (\omega^s + a^s B)/\rho^s . \tag{7Ω^s}$$

Taking the curl of the first (7EM) equation and using the second leads to the advection of Ω^s, i.e.,

$$\partial_t \Omega^s + \boldsymbol{v}^s \cdot \nabla \Omega^s = 0 .$$

This and the continuity equation for ρ^s imply that for every real valued function of a real variable $\Phi^s(\zeta)$, the functional

$$C_{\Phi^s}(\Omega^s) =: \int_D \rho^s \Phi^s(\Omega^s)\, \mathrm{d}x\, \mathrm{d}y \tag{7C}$$

is conserved by the planar MFP equations (provided the integral exists and the solutions are smooth; as in barotropic flow, we presume Ω^s can be created at a shock discontinuity).

Remark B. By direct computation one may show that the functionals (7C) are Casimirs for the Poisson bracket (7PB). This can also be shown by proving that the functionals (7C) are invariant under the coadjoint action of the semidirect product group underlying the first three terms of (7PB) (the last term plays no role in these Casimirs – it in fact arises from the canonical $(\boldsymbol{E}, \boldsymbol{A})$ bracket by reduction by the electromagnetic gauge group.)

C. *First variation*

The equilibrium states ρ^s_e, $\boldsymbol{v}^s_e$, $\boldsymbol{E}_e$, B_e, of the system (7EM) are the stationary, two-dimensional, barotropic MFP flows. For such stationary flows, one has the relations

$$\begin{aligned} &\operatorname{div} \boldsymbol{E}_e = \sum_s a^s \rho^s_e , \qquad \boldsymbol{E}_e = -\nabla\phi_e , \qquad \nabla B_e \times \hat{\boldsymbol{z}} = \sum_s a^s \rho^s_e \boldsymbol{v}^s_e , \\ &\operatorname{div} \rho^s_e \boldsymbol{v}^s_e = 0 , \qquad \boldsymbol{v}^s_e \cdot \nabla[\tfrac{1}{2}|\boldsymbol{v}^s_e|^2 + h^s(\rho^s_e) + a^s\phi_e] = 0 , \qquad \boldsymbol{v}^s_e \cdot \nabla\Omega^s_e = 0 . \end{aligned} \tag{7S1}$$

According to the last two equations in (7S1), the gradient vectors $\nabla\Omega^s_e$ and $\nabla(\frac{1}{2}|\boldsymbol{v}^s_e|^2 + h^s(\rho^s_e) + a^s\phi_e)$ are orthogonal to the equilibrium species velocity $\boldsymbol{v}^s_e$. Consequently, these two gradient vectors are collinear, provided the velocity does not vanish. A sufficient condition for such collinearity in the plane is the functional relationship

$$\tfrac{1}{2}|\boldsymbol{v}^s_e|^2 + h^s(\rho^s_e) + a^s\phi_e = K^s(\Omega^s_e) , \tag{7K}$$

for certain functions $K^s(\zeta)$ of the real variable ζ; K^s are called the *Bernoulli functions* and (7K) represents *Bernoulli's Law* for each species. Either applying the operator $(\Omega^s_e)^{-1}\hat{\boldsymbol{z}} \times \nabla$ to (7K), or vector multiplying by $\hat{\boldsymbol{z}}$ the stationary motion equation

$$(\omega^s_e + a^s B_e)\hat{\boldsymbol{z}} \times \boldsymbol{v}^s_e = -\nabla(\tfrac{1}{2}|\boldsymbol{v}^s_e|^2 + h^s(\rho^s_e) + a^s\phi_e)$$

we get

$$\rho^s_e v^s_e = \frac{K^{s\prime}(\Omega^s_e)}{\Omega^s_e}\hat{z}\times\nabla\Omega^s_e = \frac{1}{\Omega^s_e}\hat{z}\times\nabla K^s(\Omega^s_e)\,, \tag{7S2}$$

where prime ′ denotes derivative of a function with respect to its stated argument. Substitution of (7S2) into Ampère's Law (the second equation in (7S1)) leads to another relation for stationary flows, namely,

$$\nabla B_e = -\sum_s \frac{a^s}{\Omega^s_e}\nabla K^s(\Omega^s_e)\,. \tag{7S3}$$

Let

$$H_C(v^s,\rho^s,E,B) = \int_D \Big\{\sum_s\big[\tfrac{1}{2}\rho^s|v^s|^2 + \varepsilon^s(\rho^s) + \rho^s\Phi^s(\Omega^s) + \lambda^s\rho^s\Omega^s + \mu a^s\rho^s\big] + \tfrac{1}{2}|E|^2 + \tfrac{1}{2}B^2\Big\}\,dx\,dy\,, \tag{7H$_C$}$$

where λ^s and μ are constants multiplying Casimirs that are separated out for later convenience. After integration by parts, the derivative DH_C becomes

$$\begin{aligned} DH_C(v^s_e,\rho^s_e,E_e,B_e)\cdot(\delta v^s,\delta\rho^s,\delta E,\delta B) = \int_D &\Big\{\sum_s\big[\tfrac{1}{2}|v^s_e|^2 + h^s(\rho^s_e) + \mu a^s + \Phi^s(\Omega^s_e) - \Omega^s_e\Phi^{s\prime}(\Omega^s_e)\big]\delta\rho^s \\ &+ \sum_s\big[\rho^s_e v^s_e - \hat{z}\times\nabla\Phi^{s\prime}(\Omega^s_e)\big]\cdot\delta v^s + \Big[B_e + \sum_s a^s\Phi^{s\prime}(\Omega^s_e)\Big]\delta B \\ &+ E_e\cdot\delta E\Big\}\,dx\,dy + \oint_{\partial D}\sum_s \Phi^{s\prime}(\Omega^s)\delta v^s\cdot d\ell \end{aligned} \tag{7FV1}$$

where $d\ell$ is the line element along the boundary ∂D. For a stationary solution, the connected components of the boundary ∂D are both streamlines and equipotential lines. Thus, Ω^s_e and ϕ_e are constants on ∂D and the last two boundary integrals combine into

$$\sum_s\big[\lambda^s + \Phi^{s\prime}(\Omega^s_e)|_{\partial D}\big]\int_{\partial D}\delta v^s\cdot d\ell\,,$$

which vanishes when λ^s is chosen to be $\lambda^s = -\Phi^{s\prime}(\Omega^s_e)|_{\partial D}$. The terms involving $\mu a^s\delta\rho^s$ and $E_e\cdot\delta E$ in (7FV1) combine into

$$\sum_s\int_D \mu a^s\delta\rho^s\,dx\,dy + \int_D \phi_e\,\mathrm{div}\,\delta E\,dx\,dy - \int_{\partial D}\phi_e\,\delta E\cdot n\,dx\,dy$$

$$= -\phi_e|_{\partial D}\int_{\partial D}\mathrm{div}\,\delta E\,dx\,dy + \mu\int_D\sum_s a^s\delta\rho^s dx\,dy + \int_D \phi_e\,\mathrm{div}\,\delta E\,dx\,dy\,.$$

For variations that preserve Gauss' Law, these terms further combine into

$$(-\phi_e|_{\partial D} + \mu) \int_D \operatorname{div} \delta \boldsymbol{E}\, dx\, dy + \sum_s \int_D \phi_e a^s \delta\rho^s ,$$

and the first term vanishes when μ is chosen to satisfy $\mu = \phi_e|_{\partial D}$. Thus, (7FV1) becomes

$$\begin{aligned} \delta H_C(\boldsymbol{v}_e^s, \rho_e^s, \boldsymbol{E}_e, B_e) \cdot (\delta \boldsymbol{v}^s, \delta\rho^s, \delta \boldsymbol{E}, \delta B) = \int_D \Big\{ & \Big[\sum_s \tfrac{1}{2}|\boldsymbol{v}_e^s|^2 + h^s(\rho_e) + a^s\phi_e + \Omega_e^s \Phi^{s\prime}(\Omega_e^s)\Big]\delta\rho^s \\ & + \Big[\sum_s \rho_e^s \boldsymbol{v}_e^s - \hat{\boldsymbol{z}} \times \nabla\Phi^{s\prime}(\Omega_e^s)\Big] \cdot \delta \boldsymbol{v}^s \\ & + \Big[B_e + \sum_s a^s \Phi^{s\prime}(\Omega_e^s)\Big]\delta B \Big\}\, dx\, dy . \end{aligned} \tag{7FV2}$$

In this expression, the $\delta\rho^s$ coefficient vanishes for a stationary flow obeying (7K), provided that Φ^s is related to the Bernoulli function K^s by

$$K^s(\zeta) + \Phi^s(\zeta) - \zeta\Phi^{s\prime}(\zeta) = 0 ; \quad \text{i.e.,} \quad \Phi^s(\zeta) = \zeta\Big(\int^{\zeta} \frac{K^s(t)}{t^2}\, dt + \text{const}\Big) .$$

Differentiating this relation with respect to ζ implies $\zeta^{-1}K^{s\prime}(\zeta) - \Phi^{s\prime\prime}(\zeta) = 0$. Then the $\delta\boldsymbol{v}^s$ coefficient in (7FV2) vanishes by (7S2), since $\nabla\Phi^{s\prime}(\Omega_e^s) = (\Omega_e^s)^{-1}\nabla K^s(\Omega_e^s)$. Writing $\delta B = \hat{\boldsymbol{z}} \cdot \operatorname{curl} \delta \boldsymbol{A}$ and integrating by parts, the δB term in (7FV2) becomes

$$- \int_D \delta\boldsymbol{A} \cdot \hat{\boldsymbol{z}} \times \Big(\nabla B_e + \sum_s a^s \nabla\Phi^{s\prime}(\Omega_e^s)\Big)\, dx\, dy ,$$

which vanishes for stationary flows by (7S3) and (7FV3).

We summarize our findings as follows:

Proposition. For smooth solutions of (7EM) with boundary conditions $\boldsymbol{v}^s \cdot \boldsymbol{n} = 0$ and $\boldsymbol{E} \times \boldsymbol{n} = 0$, a stationary solution $(\boldsymbol{v}_e^s, \rho_e^s, \boldsymbol{E}_e, B_e)$ is a critical point of H_C given by eq. (7H_C) provided Φ^s satisfies (7FV3), where K^s is the Bernoulli function, determined by eq. (7K). Conversely, a critical point for H_C for any Φ^s gives a stationary solution satisfying (7K) where K is given by (7FV3).

Remark D. Second variation. The quadratic form defined by the second derivative of H_C at the stationary solution is

$$D^2H_C(\boldsymbol{v}_e^s, \rho_e^s, \boldsymbol{E}_e, B_e)\cdot(\delta\boldsymbol{v}^s, \delta\rho^s, \delta\boldsymbol{E}, \delta B)^2 = \int_D \Big\{\sum_s [\rho_e^s|\delta\boldsymbol{v}^s + \boldsymbol{v}_e^s\delta\rho^s/\rho_e^s|^2$$
$$+ (h^{s\prime}(\rho_e^s) - |\boldsymbol{v}_e^s|^2/\rho_e^s)(\delta\rho^s)^2 + \rho_e^s\Phi^{s\prime\prime}(\Omega_e^s)(\delta\Omega^s)^2]$$
$$+ (\delta B)^2 + |\delta\boldsymbol{E}|^2\Big\}\,\mathrm{d}x\,\mathrm{d}y\,. \tag{7SV}$$

Sufficient conditions for this quadratic form to be positive definite are:

$$h^{s\prime}(\rho_e^s) - |\boldsymbol{v}_e^s|^2/\rho_e^s = ((c_e^s)^2 - |\boldsymbol{v}_e^s|^2)/\rho_e^s > 0\,, \tag{7FS1}$$

where c_e^s is the sound speed of species s for the stationary solution, defined by $\rho_e^s h^{s\prime}(\rho_e^s) = (c_e^s)^2$, i.e., the stationary flow is everywhere subsonic; and

$$(\Omega_e^s)^{-1}K^{s\prime}(\Omega_e^s) = \Phi^{s\prime\prime}(\Omega_e^s) > 0\,, \tag{7FS2}$$

i.e., by (7S2), $\boldsymbol{v}_e^s\cdot\hat{\boldsymbol{z}}\times\nabla\Omega_e^s > 0$ throughout the flow. For a single, incompressible fluid without charge ($s \equiv 1$, $\rho_e^s = 1$, $\delta\rho^s = 0$, $\delta B = 0$, $\delta\boldsymbol{E} = 0$), formula (7SV) reduces to the second variation formula in example 3.3; for a single compressible fluid (7SV) reduces to the corresponding formula in example 3.4.

D. *Convexity estimates*

We have, after a short computation

$$\hat{H}(\Delta\boldsymbol{v}^s, \Delta\rho^s, \Delta\boldsymbol{E}, \Delta B) := H(\boldsymbol{v}_e + \Delta\boldsymbol{v}, \rho_e + \Delta\rho, \boldsymbol{E}_e + \Delta\boldsymbol{E}, B_e + \Delta B) - H(\boldsymbol{v}_e, \rho_e, \boldsymbol{E}_e, B_e)$$
$$- DH(\boldsymbol{v}_e, \rho_e, \boldsymbol{E}_e, B_e)\cdot(\Delta\boldsymbol{v}, \Delta\rho, \Delta\boldsymbol{E}, \Delta B)$$
$$= \sum_s \int_D \Big\{\frac{|\Delta(\rho^s\boldsymbol{v}^s)|^2}{2\rho_e^s} - \frac{|\boldsymbol{v}_e^s|^2(\Delta\rho^s)^2}{2\rho_e^s} + [\varepsilon^s(\rho_e + \Delta\rho) - \varepsilon^s(\rho_e) - \varepsilon^{s\prime}(\rho_e)\Delta\rho]\Big\}\,\mathrm{d}x\,\mathrm{d}y$$
$$+ \int_D (|\Delta\boldsymbol{E}|^2 + (\Delta B)^2)\,\mathrm{d}x\,\mathrm{d}y\,,$$

where $\Delta(\rho^s\boldsymbol{v}^s) = (\rho_e^s + \Delta\rho^s)(\boldsymbol{v}_e^s + \Delta\boldsymbol{v}^s) - \rho_e^s\boldsymbol{v}_e^s$. Assume $\Sigma_s\,\varepsilon^{s\prime\prime}(\tau) \geq c_{\min}^2/\tau$ for all τ and a constant $c_{\min}$. Then we get

$$Q_1(\Delta(\rho^s\boldsymbol{v}^s), \Delta\rho^s, \Delta\boldsymbol{E}, \Delta B) \leq \hat{H}(\Delta\boldsymbol{v}^s, \Delta\rho^s, \Delta\boldsymbol{E}, \Delta B) \tag{7CH}$$

with

$$Q_1(\Delta(\rho^s\boldsymbol{v}^s), \Delta\rho^s, \Delta\boldsymbol{E}, \Delta B) = \tfrac{1}{2}\int_D \Big\{\sum_s |\Delta(\rho^s\boldsymbol{v}^s)|^2/\rho_{\max}^s + \Big(c_{\min}^2 - \sum_s |\boldsymbol{v}_e^s|^2/\rho_{\min}^s\Big)(\Delta\rho)^2$$
$$+ |\Delta\boldsymbol{E}|^2 + (\Delta B)^2\Big\}\,\mathrm{d}x\,\mathrm{d}y\,,$$

for solutions obeying (as in section 3.4), $0 < \rho_{\min}^s \leq \rho^s \leq \rho_{\max}^s < \infty$.

If the sum of the Bernoulli functions K^s satisfies

$$a \le \frac{1}{\zeta} \sum_s K^{s\prime}(\zeta) = \sum_s \Phi^{s\prime\prime}(\zeta) ,$$

then one finds

$$Q_2(\Delta\Omega^s) \le \sum_s \hat{C}_{\Phi^s}(\Delta v^s, \Delta\rho^s, \Delta E, \Delta B) , \tag{7CC}$$

where

$$Q_2(\Delta\Omega^s) = \tfrac{1}{2} a \sum_s \rho^s_{\min} \int_D (\Delta\Omega^s)^2 \, dx \, dy ,$$

and

$$\Delta\Omega^s = (\omega^s_e + a^s B_e + \Delta\omega^s + a^s \Delta B)/(\rho^s_e + \Delta\rho^s) - (\omega^s_e + a^s B_e)/\rho^s_e . \tag{7D}$$

Thus

$$Q_1 + Q_2 > 0$$

holds, provided

$$a > 0 \quad \text{and} \quad c^2_{\min} > \sum_s |v^s_e|^2/\rho^s_{\min} .$$

E. *A priori estimates*
The following estimate holds:

$$\hat{H}_C(\Delta v^s(0), \Delta\rho^s(0), \Delta E(0), \Delta B(0)) \ge Q_1(\Delta(\rho^s(t) v^s(t)), \Delta\rho^s(t), \Delta E(t), \Delta B(t)) + Q_2(\Delta\Omega^s(t)) . \tag{7E}$$

F. *(Nonlinear) stability*
If we have

$$\sum_s \varepsilon^{s\prime\prime}(\tau) \le c^2_{\max}/\tau ,$$

for all τ, and

$$\frac{1}{\zeta} \sum_s K^{s\prime}(\zeta) \le A < \infty ,$$

for constants $c_{\max}$ and A, then as above,

$$C_1 Q_1(\Delta(\rho^s \boldsymbol{v}^s), \Delta\rho^s, \Delta\boldsymbol{E}, \Delta B) \geq \hat{H}(\Delta\boldsymbol{v}^s, \Delta\rho^s, \Delta\boldsymbol{E}, \Delta B)\,, \tag{CH'}$$

and

$$C_2 Q_2(\Delta\Omega^s) \geq \sum \hat{C}_{\Phi^s}(\Delta\boldsymbol{v}^s, \Delta\rho^s, \Delta\boldsymbol{E}, \Delta B)\,, \tag{CC'}$$

for constants C_1 and C_2. We summarize these results as follows.

Stability theorem. Stationary solutions $(\boldsymbol{v}_e^s, \rho_e^s, \boldsymbol{E}_e, B)$ of the planar compressible MFP equations (7EM) with velocity field tangent to the boundary and electric field normal to it satisfying

$$0 < \rho_{\min} \leq \rho_e^s \leq \rho_{\max} < +\infty, \quad \text{for all s}\,,$$

$$0 < c_{\min}^2/\tau \leq \sum_s \varepsilon^{s\prime\prime}(\tau) \leq c_{\max}^2/\tau < +\infty\,, \quad \text{for all } \tau\,,$$

and

$$0 < a \leq \frac{1}{\zeta} \sum K^{s\prime}(\zeta) \leq A < +\infty\,, \quad \text{for all s and } \zeta\,,$$

where ε^s and K^s are the internal energy density and Bernoulli function for the species s, are stable in the norm on $(\Delta(\rho^s\boldsymbol{v}^s), \Delta\rho^s, \Delta\boldsymbol{E}, \Delta B, \Delta\Omega^s)$ determined by $Q_1 + Q_2$ for smooth solutions satisfying $\rho_{\min} \leq \rho^s \leq \rho_{\max}$.

When there is only a single fluid species and electromagnetic fields are absent, the result of the stability theorem reduces to the estimate in example 3.4 for planar barotropic flow. These estimates can break down when smooth solutions cease to exist; for example, upon occurrence of cavitation, and/or the formation of shocks from an initially smooth, steady flow. When these phenomena occur, however, it is questionable whether the barotropic approximation should still be used. One could exclude cavitation by replacing (CH), (CC), (CH)′ and (CC)′ by an estimate as in Holm et al. [1983], modeling an elastic fluid.

One can treat the case of incompressible homogeneous multifluid plasmas by the same methods. In this case the criterion reduces to the same one as in Example 3.3 with the vorticity ω replaced by Ω defined by ($7\Omega^s$). One can also treat incompressible inhomogeneous MFP by combining the techniques here with those of Abarbanel et al. [1985].

Example. Subsonic shear flows. A stationary solution in the strip $\{(x, y) \subset \mathbb{R}^2 | Y_1 \leq y \leq Y_2\}$ is a plane parallel flow along x, admitting arbitrary velocity profile $\boldsymbol{v}_e^s(x, y) = (\bar{v}^s(y), 0)$, electrostatic potential $\phi_e(x, y) = \bar{\phi}(y)$, and density $\rho_e^s(x, y) = \bar{\rho}^s(y)$. The density profile is subject only to the subsonic conditions (7FS1), expressible as

$$\frac{\mathrm{d}\bar{p}^s}{\mathrm{d}\bar{\rho}^s}(y) - (\bar{v}^s(y))^2 > 0\,, \tag{7E1}$$

and depending on the barotropic relation $\bar{p}^s = p^s(\bar{\rho}^s)$. In this domain, the independent variable x can be either unrestricted on the entire real line, or periodic. The former case requires that initial perturbations be sufficiently integrable for $H_C(\Delta \boldsymbol{v}^s(0), \Delta\rho^s(0), \Delta\boldsymbol{E}(0), \Delta B(0))$ to be bounded above.

To determine the limits of stability for subsonic stationary planar MFP flows, we proceed as follows. (i) Choose profiles $\bar{v}^s(y)$, $\phi(y)$, and $\bar{\rho}^s(y)$, satisfying the subsonic condition (7E1). Relations (7S2) and (7S3) then imply y-dependence only, for the magnetic field and modified vorticity: $B_e(x, y) = \bar{B}(y)$, $\Omega^s_e(x, y) = \bar{\Omega}^s(y)$. (ii) Use Ampère's Law in the form (7S3) to determine $B(y)$ from $\bar{\rho}^s(y)$ and $\bar{v}^s(y)$, then compute $\bar{\rho}^s(y)$ from its definition in terms of $\bar{\rho}^s$, $\bar{v}^s$, $\bar{B}$. (iii) Solve for an expression for the quantity $(\bar{\Omega}^s)^{-1}K^s\,(\bar{\Omega}^s)$ appearing in the stability theorem and consider its sign, thereby determining the limits of stability in terms of the profiles $\bar{\rho}^s(y)$, $\bar{v}^s(y)$, $\bar{B}(y)$.

Given the profiles $\bar{v}^s(y)$, $\bar{\rho}^s(y)$, and $\bar{\phi}(y)$, one finds $\bar{\omega}^s(y)$ and $\bar{\Omega}^s(y)$ from their definitions

$$\omega^s_e = \hat{z}\cdot \text{curl}\, \boldsymbol{v}^s_e = -\bar{v}^{s\prime}(y) =: \bar{\omega}^s(y)\,, \tag{7E2}$$

and

$$\Omega^s_e = (\rho^s_e)^{-1}(\omega^s_e + a^s B_e) = (\bar{\rho}^s(y))^{-1}(-\bar{v}^{s\prime}(y) + a^s\bar{B}(y)) =: \bar{\Omega}^s(y)\,. \tag{7E3}$$

Equations (7S2) and (7S3) give the relations

$$\bar{\rho}^s(y)\bar{v}^s(y) = -\frac{1}{\bar{\Omega}^s}K^{s\prime}(\bar{\Omega}^s)\bar{\Omega}^{s\prime}(y)\,, \tag{7E4}$$

and

$$\bar{B}'(y) = \sum_s a^s\bar{\rho}^s(y)\bar{v}^s(y)\,, \tag{7E5}$$

which determine $\bar{B}(y)$ and $(\bar{\Omega}^s)^{-1}K^{s\prime}(\bar{\Omega}^s)$. Solving (7E4) gives the formula

$$(\bar{\Omega}^s)^{-1}\frac{\mathrm{d}K^s(\bar{\Omega}^s)}{\mathrm{d}\bar{\Omega}^s} = -\frac{\bar{\rho}^s(y)\bar{v}^s(y)}{\mathrm{d}\bar{\Omega}^s/\mathrm{d}y}$$
$$= \frac{(\bar{\rho}^s)^2\bar{v}^s}{\bar{v}^{s\prime\prime} - (\bar{\rho}^{s\prime}/\bar{\rho}^s)\bar{v}^{s\prime} + a^s\bar{B}(\bar{\rho}^{s\prime}/\bar{\rho}^s - \bar{B}'/\bar{B})}\,, \tag{7E6}$$

where, e.g., $\bar{v}^{s\prime\prime} = \mathrm{d}^2\bar{v}^s(y)/\mathrm{d}y^2$, $\bar{B}' = \mathrm{d}\bar{B}(y)/\mathrm{d}y$, etc. Thus, control of positivity of $(\bar{\Omega}^s)^{-1}K^{s\prime}(\bar{\Omega}^s)$ in the stability theorem and, hence, of stability for MFP involves an interplay among velocity, density and magnetic field profiles, through the positivity condition,

$$-\bar{v}^s(y)/\bar{\Omega}^{s\prime}(y) > 0\,. \tag{7E7}$$

Given that condition (7E7) holds, planar MFP flows will be stable, provided

$$\bar{\Omega}^{s\prime}(y) \neq 0\,. \tag{7E8}$$

We consider several cases.

Case A. In the case of neutral fluids ($a^s = 0$) and stationary flows with constant density ($\bar{\rho}^{s\prime}(y) = 0$), positivity of $(\bar{\Omega}^s)^{-1} K^{s\prime}(\bar{\Omega}^s)$ in (7E7) reduces to

$$\bar{v}^s(y)/\bar{v}^{s\prime\prime}(y) > 0 . \tag{7E9}$$

Provided (7E9) holds throughout the domain D, one recovers Rayleigh's criterion (see example 3.3) for stability of shear flows: all flows in this case with no inflection points in their velocity profile are stable.

Case B. For the case of charged fluids ($a^s \neq 0$) at constant density ($\bar{\rho}^{s\prime}(y) = 0$), positivity in (7E7) reduces to

$$(\bar{\rho}^s)^2 \bar{v}^s / [\bar{v}^{s\prime\prime} - a^s \bar{B}'] > 0 . \tag{7E10}$$

Provided (7E10) holds throughout D, one obtains the following criterion for stability in this case,

$$\bar{v}^{s\prime\prime}(y) \neq a^s \bar{B}'(y) . \tag{7E11}$$

Case C. In the general MFP case, with charged, compressible fluids, ($a^s \neq 0$, $\bar{\rho}^{s\prime}(y) \neq 0$), (7E7) holds, the stability condition (7E8) becomes

$$\bar{v}^{s\prime\prime} \neq (\bar{\rho}^{s\prime}/\bar{\rho}^s)(\bar{v}^{s\prime} - a^s \bar{B}) + a^s \bar{B}' , \tag{7E12}$$

which involves all three stationary profiles.

Note that the conditions obtained in these examples by Arnold's method are sufficient for stability. Thus, violation of these conditions would be necessary for the onset of instability but not necessary and sufficient, except in the fortunate event where they coincide with instability conditions found by linear analysis.

Part II. Three-Dimensional Fluid Systems

The results of Arnold [1965b] and Dikii [1965b] suggest that it may be difficult to extend the two-dimensional results to three dimensions. For incompressible homogeneous flows in the Eulerian representation this indeed seems to be the case because of a lack of Casimirs. However, for three-dimensional systems with sufficiently rich Casimir structures, we will show in this part that the stability methods are indeed applicable and give conditional stability results. As we shall remark in the MHD section (section 10), the methods are compatible with the δW method of Bernstein [1958, 1983]. Here we shall treat only the compressible case; for the inhomogeneous incompressible case of interest in stratified flows, see Abarbanel et al. [1985].

We shall begin with three-dimensional adiabatic flow. This will be crucial to the following two sections for, as we shall see, the computations needed for MFP and MHD essentially reduce to the adiabatic case.

8. Three-dimensional adiabatic flow

The stability algorithm we are using depends on a good supply of Casimir functions for its application. In this regard, the three-dimensional situation is rather different from the two-dimensional one. In particular, for ideal, homogeneous, incompressible flow in the Eulerian representation, we know of only a relatively trivial three-dimensional version of example 3.3. The difficulty is that the only known Casimir is the helicity, $\int \boldsymbol{v}\cdot \operatorname{curl} \boldsymbol{v}\, \mathrm{d}^3x$. The corresponding equilibrium states are the Beltrami flows (see, e.g., Arnold [1966b] and Holm [1984]). However, if density, or entropy variations are also allowed, we recover abundant Casimirs and the method again applies. The results of this section will be generalized in the next two sections to MHD and multifluid plasmas.

Three-dimensional adiabatic flow was studied using Arnold's ideas in Dikii [1965b]. There, an expression for D^2H_C and hence an implicit condition for formal stability was given; however, no workable hypotheses for stability were found. We shall see that if we limit the range of the density ρ (as in section 3.4) and the size of $\nabla\eta$ compared to η, then we can get explicit expressions for definiteness. In addition, we obtain rigorous *a priori* stability estimates using the convexity method and obtain thereby, a (conditional) stability result.

As usual, we follow the algorithm in section 2.

A. *Equations of motion and Hamiltonian*

Let D be a fixed domain in three dimensions and $\boldsymbol{x}=(x, y, z)\in D\subset\mathbb{R}^3$. The adiabatic fluid equations define a dynamical system in terms of the spatial fluid velocity $\boldsymbol{v}(\boldsymbol{x}, t)$, mass density $\rho(\boldsymbol{x}, t)$, and specific entropy $\eta(\boldsymbol{x}, t)$, with $\boldsymbol{v}$ tangent to the boundary ∂D:

$$\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = -\frac{1}{\rho}\nabla p(\rho,\eta)\,,\qquad \frac{\mathrm{d}\rho}{\mathrm{d}t} = -\rho \operatorname{div}\boldsymbol{v}\,,\qquad \frac{\mathrm{d}\eta}{\mathrm{d}t}=0\,, \tag{8EM}$$

where $\mathrm{d}/\mathrm{d}t = \partial/\partial t + \boldsymbol{v}\cdot\nabla$ is the material derivative. The adiabatic condition, $\mathrm{d}\eta/\mathrm{d}t=0$, means that no heat is exchanged across flow lines. The pressure $p=p(\rho,\eta)$ is assumed to be given in terms of a thermodynamic equation of state for the internal energy density $\varepsilon(\rho,\eta)$ as follows. Given ε, define the temperature T and specific enthalpy h by

$$\rho T = \partial\varepsilon/\partial\eta \quad\text{and}\quad h=\partial\varepsilon/\partial\rho\,,$$

i.e.,

$$\mathrm{d}\varepsilon = \rho T\,\mathrm{d}\eta + h\,\mathrm{d}\rho\,. \tag{8T1}$$

The pressure p, defined by $p=\rho^2\partial(\varepsilon/\rho)/\partial\rho$, satisfies $\rho h = \varepsilon + p$ and

$$\mathrm{d}h = T\,\mathrm{d}\eta + \frac{1}{\rho}\,\mathrm{d}p = \left[T + \rho\left.\frac{\partial T}{\partial\rho}\right|_{\eta}\right]\mathrm{d}\eta + \frac{1}{\rho}c^2\,\mathrm{d}\rho\,, \tag{8T2}$$

where c is the adiabatic sound speed, defined by

$$c^2 = \partial p(\rho,\eta)/\partial\rho = \rho\,\partial h(\rho,\eta)/\partial\rho\,. \tag{8T3}$$

The space P consists of triples (v, ρ, η) in a suitable Sobolev space (see section 2) such that these quantities are C^1; as usual, this condition precludes shocks. The Hamiltonian is

$$H(v, \rho, \eta) = \int_D [\tfrac{1}{2}\rho |v|^2 + \varepsilon(\rho, \eta)] \, dx \, dy \, dz . \tag{8H}$$

Remark A. Poisson structure. The equations (EM) are Hamiltonian using the variables (M, ρ, η) with $M = \rho v$ and the semidirect product Lie–Poisson structure

$$\{F, G\} = \int_D M \cdot \left[\left(\frac{\delta G}{\delta M} \cdot \nabla \right) \frac{\delta F}{\delta M} - \left(\frac{\delta F}{\delta M} \cdot \nabla \right) \frac{\delta G}{\delta M} \right] dx \, dy \, dz$$
$$+ \int_D \rho \left[\left(\frac{\delta G}{\delta M} \cdot \nabla \right) \frac{\delta F}{\delta \rho} - \left(\frac{\delta F}{\delta M} \cdot \nabla \right) \frac{\delta G}{\delta \rho} \right] dx \, dy \, dz + \int_D \eta \nabla \cdot \left(\frac{\delta G}{\delta M} \frac{\delta F}{\delta \eta} - \frac{\delta F}{\delta M} \frac{\delta G}{\delta \eta} \right) dx \, dy \, dz . \tag{8PB}$$

This may be derived, for example by consulting references given in remark A, section 3.4.

B. *Constants of the motion*

The *potential vorticity* is defined by

$$\Omega := \frac{1}{\rho} \omega \cdot \nabla \eta = \frac{1}{\rho} \operatorname{div}(\eta \omega) , \tag{8Ω}$$

where $\omega := \operatorname{curl} v$ is the vorticity. Using (8EM) one finds that Ω is a flow line invariant, i.e.,

$$d\Omega/dt = 0 .$$

Thus, the following functional is conserved for solutions of system (8EM),

$$C_\Phi(\eta, \Omega) := \int_D \rho \Phi(\eta, \Omega) \, dx \, dy \, dz , \tag{8C}$$

for an arbitrary function Φ. In particular, the functional

$$C_\mu(\eta, \Omega) = \mu \int_D \rho \Omega \, dx \, dy \, dz$$

is also conserved for any constant μ.

Remark B. Casimirs. The functionals C_Φ are Casimirs for the Poisson structure (8PB). This can be checked by direct computation, or by noting that the group underlying the Poisson structure leaves C_Φ

invariant. This group is the semidirect product Diff Ⓢ (Functions × Densities) and acts on $(\boldsymbol{M}, \rho, \eta)$ by

$$(\phi, f, \nu)\cdot(\boldsymbol{M}, \rho, \eta) = (\phi_* \boldsymbol{M} - \mathrm{d}f \otimes \phi_* \rho - \nu \otimes \mathrm{d}(\phi_* \eta), \phi_* \rho, \phi_* \eta)\,,$$

for $(\phi, f, \nu) \in$ Diff × Functions × Densities, where $\boldsymbol{M}$ is regarded here as a one-form density.

C. *First variation*

The equilibrium states of (8EM) are stationary adiabatic flows (in three dimensions). As usual, we let the subscript e denote quantities associated with such flows. Since Ω_e and η_e are constant along streamlines, their gradients are perpendicular to $\boldsymbol{v}_\mathrm{e}$. On the other hand, for stationary flows, the conservation of mass and energy imply that the quantity $\frac{1}{2}|\boldsymbol{v}_\mathrm{e}|^2 + h(\rho_\mathrm{e}, \eta_\mathrm{e})$ is also constant along streamlines. Thus we have:

$$\boldsymbol{v}_\mathrm{e}\cdot\nabla\eta_\mathrm{e} = \boldsymbol{v}_\mathrm{e}\cdot\nabla\Omega_\mathrm{e} = \boldsymbol{v}_\mathrm{e}\cdot\nabla(\tfrac{1}{2}|\boldsymbol{v}_\mathrm{e}|^2 + h(\rho_\mathrm{e}, \eta_\mathrm{e})) = 0\,.$$

Sufficient conditions for these relationships are the existence of functions $K(\eta_\mathrm{e}, \Omega_\mathrm{e})$ and $\lambda(x, y, z)$ such that

$$\tfrac{1}{2}|\boldsymbol{v}_\mathrm{e}|^2 + h(\rho_\mathrm{e}, \eta_\mathrm{e}) = K(\eta_\mathrm{e}, \Omega_\mathrm{e})\,, \tag{8S1}$$

$$\rho_\mathrm{e}\boldsymbol{v}_\mathrm{e} = (\nabla\eta_\mathrm{e}\times\nabla\Omega_\mathrm{e})\lambda\,. \tag{8S2}$$

As usual, K is called the *Bernoulli function.* The gradient of (8S1) together with the relation $\frac{1}{2}\nabla|\boldsymbol{v}_\mathrm{e}|^2 + (1/\rho_\mathrm{e})\nabla p_\mathrm{e} + \boldsymbol{\omega}_\mathrm{e}\times\boldsymbol{v}_\mathrm{e} = \boldsymbol{0}$ for stationary solutions and the relation (8T2) yields the formula

$$\boldsymbol{v}_\mathrm{e}\times\boldsymbol{\omega}_\mathrm{e} = \nabla K(\eta_\mathrm{e}, \Omega_\mathrm{e}) - T_\mathrm{e}\nabla\eta_\mathrm{e}\,. \tag{8S3}$$

After integration by parts, one finds that the first variation of $H_C := H + C_\Phi + C_\mu$ is given by

$$\begin{aligned}\mathrm{D}H_C(\boldsymbol{v}, \rho, \eta)\cdot(\delta\boldsymbol{v}, \delta\rho, \delta\eta) = \int_D \Big\{ &[\tfrac{1}{2}|\boldsymbol{v}|^2 + h(\rho, \eta) + \Phi(\eta, \Omega) - \Omega\Phi'(\eta, \Omega)]\delta\rho \\ &+ \rho\Big(\dot{\Phi} + T - \frac{1}{\rho}\boldsymbol{\omega}\cdot\nabla\Phi'\Big)\delta\eta + [\rho\boldsymbol{v} - \Phi''(\eta, \Omega)\nabla\eta\times\nabla\Omega]\cdot\delta\boldsymbol{v}\Big\}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \\ &+ \oint_{\partial D} \Phi'(\eta, \Omega)(\delta\boldsymbol{v}\times\nabla\eta + \boldsymbol{\omega}\delta\eta)\cdot\boldsymbol{n}\,\mathrm{d}S \\ &+ \mu\oint_{\partial D}(\delta\boldsymbol{v}\times\nabla\eta + \boldsymbol{\omega}\delta\eta)\cdot\boldsymbol{n}\,\mathrm{d}S\,,\end{aligned} \tag{8FV1}$$

where $\dot{\Phi}(\eta, \Omega)$ denotes the partial derivative of $\Phi(\eta, \Omega)$ with respect to its first argument and $\Phi'(\eta, \Omega)$ denotes the partial derivative with respect to its second argument; $\boldsymbol{n}$ is the outward unit vector normal to the surface element $\mathrm{d}S$, on ∂D. The following proposition is essentially due to Dikii [1965b] (who also admitted the possibility of a uniformly rotating coordinate system).

Proposition. Within the class of smooth solutions with velocities parallel to ∂D, a stationary solution $\boldsymbol{v}_e$, ρ_e, η_e of the adiabatic fluid equations in three dimensions with Ω_e and η_e constant on ∂D and satisfying $\Omega_e \neq 0$, $\nabla\Omega_e \times \nabla\eta_e \neq \mathbf{0}$, is a critical point of $H_C = H + C_\Phi + C_\mu$, where Φ is determined from the steady flow quantities by

$$K(\alpha, \beta) + \Phi(\alpha, \beta) - \beta\Phi'(\alpha, \beta) = 0\,, \quad \text{i.e.,} \quad \Phi(\alpha, \beta) = \beta\Big(\int^{\beta} \frac{K(\alpha, t)}{t^2}\,\mathrm{d}t + \Psi(\alpha)\Big)\,, \tag{8FV2}$$

for arbitrary arguments α, β and an arbitrary function $\Psi(\alpha)$, and the constant μ is determined by

$$\mu = -\Phi'(\eta_e, \Omega_e)|\partial D\,, \tag{8FV3}$$

with Φ given by (8FV2). The function K is constant on streamlines and is determined by Bernoulli's Law (8S1). Conversely, any critical point of H_C is a stationary solution.

Proof. Since η_e and Ω_e are constant on ∂D, for stationary quantities in (8FV1), one can factor $\Phi'(\eta_e, \Omega_e)$ out of the first boundary integral and hence the two boundary terms cancel if condition (8FV3) holds. The volume integrals will vanish for stationary flows, provided that

$$\tfrac{1}{2}|\boldsymbol{v}_e|^2 + h(\rho_e, \eta_e) + \Phi(\eta_e, \Omega_e) - \Omega_e\Phi'(\eta_e, \Omega_e) = 0\,, \tag{8FV4}$$

$$\rho_e\boldsymbol{v}_e = \Phi''(\eta_e, \Omega_e)\nabla\eta_e \times \nabla\Omega_e\,, \tag{8FV5}$$

and

$$\dot{\Phi}(\eta_e, \Omega_e) + T_e - \frac{1}{\rho_e}\boldsymbol{\omega}_e \cdot \nabla\Phi'(\eta_e, \Omega_e) = 0\,. \tag{8FV6}$$

Relation (8FV4) is equivalent via Bernoulli's Law (8S1) to condition (8FV2). We now show that (8FV5) and (8FV6) follow from (8FV4). To prove (8FV5), substitute (8FV2) into the gradient of Bernoulli's Law, take into account (8T2), the relation $(\boldsymbol{v}\cdot\nabla)\boldsymbol{v} = \boldsymbol{\omega} \times \nabla\frac{1}{2}|\boldsymbol{v}|^2$, and the first equation (8EM) for stationary solutions, and find

$$\boldsymbol{v}_e \times \boldsymbol{\omega}_e = -(\dot{\Phi}(\eta_e, \Omega_e) + T_e - \Omega_e\dot{\Phi}'(\eta_e, \Omega_e))\nabla\eta_e + \Omega_e\Phi''(\eta_e, \Omega_e)\nabla\Omega_e\,. \tag{8V1}$$

Vector multiplication of this relation by $\nabla\eta_e$ then gives

$$\nabla\eta_e \times (\boldsymbol{v}_e \times \boldsymbol{\omega}_e) = \boldsymbol{v}_e(\boldsymbol{\omega}_e \cdot \nabla\eta_e) - \boldsymbol{\omega}_e(\boldsymbol{v}_e \cdot \nabla\eta_e) = \Omega_e\Phi''(\eta_e, \Omega_e)\nabla\eta_e \times \nabla\Omega_e\,. \tag{8V2}$$

The term $\boldsymbol{v}_e \cdot \nabla\eta_e$ vanishes for stationary flows and $\boldsymbol{\omega}_e \cdot \nabla\eta_e = \rho_e\Omega_e$ by the definition of Ω. Thus, if $\Omega_e \neq 0$ we get (8FV5). Note that (8FV5) determines the function λ in (8S2) from (8V2). To get (8FV6), vector multiply formula (8V1) by $\nabla\Omega_e$ and find that

$$\begin{aligned}\nabla\Omega_e \times (\boldsymbol{v}_e \times \boldsymbol{\omega}_e) &= \boldsymbol{v}_e(\boldsymbol{\omega}_e \cdot \nabla\Omega_e) - \boldsymbol{\omega}_e(\boldsymbol{v}_e \cdot \nabla\Omega_e) \\ &= [\dot{\Phi}(\eta_e, \Omega_e) + T_e - \Omega_e\dot{\Phi}'(\eta_e, \Omega_e)](\nabla\eta_e \times \nabla\Omega_e)\,.\end{aligned}$$

The term $\boldsymbol{v}_e \cdot \nabla\Omega_e$ vanishes for stationary flows. Using (8FV5) and $\nabla\eta_e \times \nabla\Omega_e \neq \mathbf{0}$, we get

$$0 = \dot{\Phi}(\eta_e, \Omega_e) + T_e - \Omega_e \dot{\Phi}'(\eta_e, \Omega_e) - \frac{1}{\rho_e} \Phi''(\eta_e, \Omega_e)\boldsymbol{\omega}_e \cdot \nabla\Omega_e$$

$$= \dot{\Phi}(\eta_e, \Omega_e) + T_e - \frac{1}{\rho_e}\boldsymbol{\omega}_e \cdot \nabla\Phi'(\eta_e, \Omega_e)$$

which is (8FV6) and so the proposition is proved. ■

Remark. For flow on a planar surface of constant η, the gradient $\nabla\eta$ is a vector normal to the plane, and $\Omega = \omega/\rho$, up to a constant factor. The proposition then reduces to the planar barotropic case, treated in Holm et al. [1983] and section 3.4.

Remark D. Formal stability. For three-dimensional adiabatic flows,

$$H_C(\boldsymbol{v}, \rho, \eta) = \int_D \{\tfrac{1}{2}\rho|\boldsymbol{v}|^2 + \varepsilon(\rho, \eta) + \rho[\Phi(\eta, \Omega) + \mu]\}\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z\,. \tag{8HC}$$

It will be convenient to write (8FV1) in the form

$$\delta H_C := \mathrm{D}H_C(\boldsymbol{v}_e, \rho_e, \eta_e)\cdot(\delta\boldsymbol{v}, \delta\rho, \delta\eta)$$

$$= \int_D \{\tfrac{1}{2}|\boldsymbol{v}_e|^2\delta\rho + \rho_e\boldsymbol{v}_e\cdot\delta\boldsymbol{v} + \delta\varepsilon(\rho_e, \eta_e) + [\Phi(\eta_e, \Omega_e) + \mu\Omega_e]\delta\rho + \rho_e\dot{\Phi}(\eta_e, \Omega_e)\delta\eta$$

$$+ [\Phi'(\eta_e, \Omega_e) + \mu]\,\delta\Omega\}\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z\,,$$

Where $\delta\varepsilon(\rho_e, \eta_e) := \varepsilon_\rho(\rho_e, \eta_e)\delta\rho + \varepsilon_\eta(\rho_e, \eta_e)\delta\eta$. The second variation is given by

$$\delta^2 H_C := \mathrm{D}^2 H_C(\boldsymbol{v}_e, \rho_e, \eta_e)\cdot(\delta\boldsymbol{v}, \delta\rho, \delta\eta)^2$$

$$= \int_D \{2\boldsymbol{v}_e\cdot\delta\boldsymbol{v}\delta\rho + \rho_e|\delta\boldsymbol{v}|^2 + \delta^2\varepsilon(\rho_e, \eta_e) + 2\dot{\Phi}(\eta_e, \Omega_e)(\delta\eta)(\delta\Omega)$$

$$+ \rho_e[\ddot{\Phi}(\eta_e, \Omega_e)(\delta\eta)^2 + \Phi''(\eta_e, \Omega_e)(\delta\Omega)^2 + 2\dot{\Phi}'(\eta_e, \Omega_e)(\delta\eta)(\delta\Omega)]$$

$$+ 2(\Phi'(\eta_e, \Omega_e) + \mu)(\delta\rho)(\delta\Omega) + \rho_e(\Phi'(\eta_e, \Omega_e) + \mu)\delta^2\Omega\}\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z\,, \tag{8SV1}$$

where

$$\delta^2\varepsilon(\rho_e, \eta_e) := \varepsilon_{\rho\rho}(\rho_e, \eta_e)(\delta\rho)^2 + 2\varepsilon_{\rho\eta}(\rho_e, \eta_e)(\delta\rho)(\delta\eta) + \varepsilon_{\eta\eta}(\rho_e, \eta_e)(\delta\eta)^2\,,$$

and where $\delta^2\Omega$ is computed as follows:

$$\Omega := \rho^{-1}(\operatorname{curl} \boldsymbol{v}) \cdot \nabla\eta ,$$

$$\delta\Omega = -\rho_e^{-1}\Omega_e\delta\rho + \rho_e^{-1}(\operatorname{curl}\delta\boldsymbol{v}) \cdot \nabla\eta_e + \rho_e^{-1}(\operatorname{curl}\boldsymbol{v}_e) \cdot \nabla\delta\eta ,$$

$$\delta^2\Omega = \rho_e^{-2}\Omega_e(\delta\rho)^2 - \rho_e^{-1}(\delta\Omega)(\delta\rho) - \rho_e^{-2}\delta\rho[(\operatorname{curl}\delta\boldsymbol{v}) \cdot \nabla\eta_e + \operatorname{curl}\boldsymbol{v}_e \cdot \nabla\delta\eta] + 2\rho_e^{-1}(\operatorname{curl}\delta\boldsymbol{v}) \cdot \nabla\delta\eta$$
$$= -2\rho_e^{-1}(\delta\rho)(\delta\Omega) + 2\rho_e^{-1}(\operatorname{curl}\delta\boldsymbol{v}) \cdot \nabla\delta\eta .$$

Consequently,

$$\rho_e\delta^2\Omega + 2(\delta\rho)(\delta\Omega) = 2(\operatorname{curl}\delta\boldsymbol{v}) \cdot \nabla\delta\eta ,$$

and so that the last two terms in (8SV1) can be integrated by parts using $\Phi'(\eta_e, \Omega_e)|_{\partial D} + \mu = 0$ to give

$$\delta^2 H_C = \int_D \{\rho_e|\delta\boldsymbol{v}|^2 + 2\delta\boldsymbol{v} \cdot (\boldsymbol{v}_e\delta\rho + \nabla\Phi'(\eta_e, \Omega_e) \times \nabla\delta\eta) + \delta^2\varepsilon(\rho_e, \eta_e) + 2\dot{\Phi}(\eta_e, \Omega_e)(\delta\eta)(\delta\Omega)$$
$$+ \rho_e[\ddot{\Phi}(\eta_e, \Omega_e)(\delta\eta)^2 + \Phi''(\eta_e, \Omega_e)(\delta\Omega)^2 + 2\dot{\Phi}'(\eta_e, \Omega_e)(\delta\eta)(\delta\Omega)]\}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z . \tag{8SV2}$$

Completing the square in the first two terms of this expression yields

$$\delta^2 H_C = \int_D \{\rho_e|\delta\boldsymbol{v} + \rho_e^{-1}\boldsymbol{v}_e\delta\rho + \rho_e^{-1}\nabla\Phi'(\eta_e, \Omega_e) \times \nabla\delta\eta|^2 - \rho_e^{-1}|\boldsymbol{v}_e\delta\rho + \nabla\Phi'(\eta_e, \Omega_e) \times \nabla\delta\eta|^2$$
$$+ \delta^2\varepsilon(\rho_e, \eta_e) + 2\dot{\Phi}(\eta_e, \Omega_e)\delta\eta\delta\Omega + \rho_e[\ddot{\Phi}(\eta_e, \Omega_e)(\delta\eta)^2$$
$$+ \Phi''(\eta_e, \Omega_e)(\delta\Omega)^2 + 2\dot{\Phi}'(\eta_e, \Omega_e)\delta\eta\delta\Omega]\}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z . \tag{8SV3}$$

This quadratic form has indefinite sign; for example, variations with $\delta\rho = 0$, $\delta\eta = 0$ and $\operatorname{curl}\delta\boldsymbol{v} = \boldsymbol{0}$ (so $\delta\Omega = 0$) give a positive expression. To get a negative expression, choose a variation with $\delta\eta$ small compared to $\nabla\delta\eta$, $\nabla\delta\eta$ in the plane orthogonal to $\operatorname{curl}\boldsymbol{v}_e$, and with $\delta\boldsymbol{v}$ the curl free part of $-\nabla\Phi'(\eta_e, \Omega_e) \times \nabla\delta\eta$ in the L^2 orthogonal Hodge decomposition with a ρ_e weight (Marsden [1976]). (Note that $\operatorname{curl}\boldsymbol{v}_e \cdot \nabla\delta\eta = 0$, $\delta\rho = 0$, and $\operatorname{curl}\delta\boldsymbol{v} = 0$ gives $\delta\Omega = 0$.)

We next show that $\delta^2 H_C$ becomes positive definite for states satisfying

$$|\nabla\delta\eta|^2 \leq k_+^2(\delta\eta)^2 , \tag{8BD}$$

where k_+ will be determined below in terms of equilibrium flow quantities. We will get conditional stability results: that is, our bounds will hold as long as (8BD) is not violated. This is consistent with the fact that generally, shocks can develop and violate (8BD). This conditional stability is similar to that for compressible barotropic flow (see section 3.4). It should be kept in mind that when (8BD) is violated, the model itself may become invalid because of heat diffusion that should be accounted for if $\nabla\eta$ is large.

We introduce the notation

$$a^2 = \rho_e^{-1}|\nabla\Phi'(\eta_e, \Omega_e)|^2, \qquad A = \varepsilon_{\rho\rho}(\rho_e, \eta_e) - \rho_e^{-1}|v_e|^2, \qquad B = \varepsilon_{\eta\eta}(\rho_e, \eta_e) + \rho_e\ddot{\Phi}(\eta_e, \Omega_e),$$

$$C = \varepsilon_{\rho\eta}(\rho_e, \eta_e) + \dot{\Phi}(\eta_e, \Omega_e) = \rho_e^{-1}\omega_e \cdot \nabla\Phi'(\eta_e, \Omega_e) = T_e + \dot{\Phi}(\eta_e, \Omega_e), \qquad \text{[by (8FV5)]}$$

$$D = \rho_e\Phi''(\eta_e, \Omega_e), \qquad F = \rho_e\dot{\Phi}'(\eta_e, \Omega_e), \qquad y = \rho_e^{-1}\nabla\Phi'(\eta_e, \Omega_e) \times v_e.$$

Thus, $\delta^2 H_C$ in (8SV3) can be written (using the vector identity $|\alpha \times \beta|^2 = |\alpha|^2|\beta|^2 - (\alpha \cdot \beta)^2$ as

$$\delta^2 H_C = \int_D \{\rho_e|\delta v + \rho_e^{-1}v_e\delta\rho + \rho_e^{-1}\nabla\Phi'(\eta_e, \Omega_e) \times \nabla\delta\eta|^2 + \rho_e^{-1}(\nabla\delta\eta \cdot \nabla\Phi'(\eta_e, \Omega_e))^2 + \mathcal{Q}\}\,dx\,dy\,dz, \tag{8SV4}$$

where the quadratic form $\mathcal{Q}$ is given by

$$\mathcal{Q} = \begin{bmatrix}\delta\rho \\ \delta\Omega \\ \delta\eta\end{bmatrix}^t \begin{bmatrix} A + |y|^2/a^2 & 0 & C \\ 0 & D & F \\ C & F & B \end{bmatrix} \begin{bmatrix}\delta\rho \\ \delta\Omega \\ \delta\eta\end{bmatrix} - a^2|\nabla\delta\eta - a^{-2}y\delta\rho|^2. \tag{8$\mathcal{Q}$1}$$

Sufficient conditions will be found for $\mathcal{Q}$ in (8$\mathcal{Q}$1) to be positive. To do this, we first look at two special cases: (i) choose $\delta\rho = 0$, and take δv and $\delta\eta$ such that $\delta\Omega = 0$. Then

$$\mathcal{Q} = B(\delta\eta)^2 - a^2|\nabla\delta\eta|^2,$$

so $\mathcal{Q} > 0$ in this case if $|\nabla\delta\eta|^2 < a^{-2}B(\delta\eta)^2$; and (ii) choose $\nabla\delta\eta = 0$, and take $\delta\rho$ and δv such that $\delta\Omega = 0$. Then

$$\mathcal{Q} = \begin{pmatrix}\delta\rho \\ \delta\eta\end{pmatrix}^t \begin{pmatrix} A & C \\ C & B \end{pmatrix} \begin{pmatrix}\delta\rho \\ \delta\eta\end{pmatrix},$$

and $\mathcal{Q} > 0$ in this case if $A > 0$ and $AB - C^2 > 0$. In (8$\mathcal{Q}$1) we write $\nabla\delta\eta = k\delta\eta$, which defines the vector k and recombine to find

$$\mathcal{Q} = \begin{bmatrix}\delta\rho \\ \delta\Omega \\ \delta\eta\end{bmatrix}^t \begin{bmatrix} A & 0 & C + k\cdot y \\ 0 & D & F \\ C + k \cdot y & F & B - |k|^2a^2 \end{bmatrix} \begin{bmatrix}\delta\rho \\ \delta\Omega \\ \delta\eta\end{bmatrix}. \tag{8$\mathcal{Q}$2}$$

For positivity of the quadratic form (8$\mathcal{Q}$2) we need each subdeterminant to be positive, i.e.,

$$A > 0, \qquad D > 0, \qquad D[A(B - |k|^2a^2) - (C + k \cdot y)^2] - AF^2 > 0.$$

The last condition for positivity can be rewritten as

$$-AB + Aa^2|k|^2 + (C + k\cdot y)^2 + AF^2/D < 0,$$

or, on expanding the square,

$$Aa^2|\boldsymbol{k}|^2+(\boldsymbol{k}\cdot\boldsymbol{y})^2+2C\boldsymbol{k}\cdot\boldsymbol{y}-(AB-C^2-AF^2/D)<0\,.$$

By the Schwartz inequality, $\boldsymbol{k}\cdot\boldsymbol{y}\le|\boldsymbol{k}||\boldsymbol{y}|$; if $C\ge 0$, It suffices for positivity of $\mathcal{Q}$ in (8$\mathcal{Q}$2) that

$$(Aa^2+|\boldsymbol{y}|^2)|\boldsymbol{k}|^2+2C|\boldsymbol{y}||\boldsymbol{k}|-(AB-C^2-AF^2/D)<0\,. \tag{8KI}$$

The product of the roots of this quadratic expression in $|\boldsymbol{k}|$ is $-(AB-C^2-AF^2/D)/(Aa^2+|\boldsymbol{y}|^2)$, so there is one positive real root, k_+, if

$$AB-C^2-AF^2/D>0\,.$$

(If $C\ge 0$, the discriminant is trivially positive, so the roots are real.) The value of k_+ is given by the quadratic formula:

$$k_+=[Aa^2+|\boldsymbol{y}|^2]^{-1}\{-C|\boldsymbol{y}|+\sqrt{C^2|\boldsymbol{y}|^2+(Aa^2+|\boldsymbol{y}|^2)(AB-C^2-AF^2/D)}\}\,, \tag{8k_+}$$

and the inequality (8KI) is satisfied for

$$0\le|\boldsymbol{k}|<k_+\,.$$

This calculation using the Schwartz inequality shows that the quadratic form $\mathcal{Q}$ in (8$\mathcal{Q}$1) is positive definite provided

$$A>0\,,\qquad D>0\,,\qquad C\ge 0\,,$$

and

$$AB-C^2-AF^2/D\ge 0$$

for specific entropy variations satisfying

$$|\nabla\delta\eta|<k_+|\delta\eta|\,, \tag{8SEV}$$

with k_+ given by (8k_+) in terms of equilibrium flow quantities. These positivity conditions for $\mathcal{Q}$ imply positivity of the second variation $\delta^2 H_C$ in (8SV4). Hence, as long as (8SEV) is satisfied, sufficient conditions for formal stability of three-dimensional equilibrium flows of an adiabatic fluid are expressible (by using (8T1–3), (8FV5), and the definitions of A, B, C, D, and F in (8D)) as

$$A:=\varepsilon_{\rho\rho}(\rho_e,\eta_e)-\rho_e^{-1}|\boldsymbol{v}_e|^2=\rho_e^{-1}(c_e^2-|\boldsymbol{v}_e|^2)>0\,, \tag{8FS1}$$

$$\rho_e\Phi''(\eta_e,\Omega_e)=\frac{\rho_e^2\boldsymbol{v}_e\cdot\nabla\eta_e\times\nabla\Omega_e}{-\;|\nabla\eta_e\times\nabla\Omega_e|^2}>0\,, \tag{8FS2}$$

$$C := \rho_e^{-1}\boldsymbol{\omega}_e \cdot \nabla\Phi'(\eta_e, \Omega_e) \geq 0\,, \tag{8FS3}$$

$$B := \varepsilon_{\eta\eta}(\rho_e, \eta_e) + \rho_e\ddot{\Phi}(\eta_e, \Omega_e) > \rho_e^{-2}(\boldsymbol{\omega}\cdot\nabla\Phi')^2 + (c_e^2 - |\boldsymbol{v}_e|^2(\dot{\Phi}')^2/\Phi''$$
$$=: (C^2 + AF^2/D)/A\,, \tag{8FS4}$$

where c_e is the equilibrium sound speed, $\varepsilon_{\eta\eta}(\rho_e, \eta_e) = \rho_e T_e/c_v^e$, and $c_v = T/(\partial T/\partial\eta)\rho$ is the specific heat at constant specific volume. The quantities ρ_e, T_e, c_e^2, and c_v^e are all assumed to be positive, on physical grounds. The formal stability conditions (8FS1–3) can also be expressed simply as

$$c_e^2 - |\boldsymbol{v}_e|^2 > 0\,, \tag{8FS1$'$}$$

$$\boldsymbol{v}_e \cdot \nabla\eta_e \times \nabla\Omega_e > 0\,, \tag{8FS2$'$}$$

$$\boldsymbol{\omega}_e \cdot \nabla\Phi'(\eta_e, \nabla_e) \geq 0\,. \tag{8FS3$'$}$$

In the isentropic planar limit, the fluid flow takes place on planes in which η_e is constant, so that if these planes have coordinates (x, y), $\nabla\eta_e \parallel \hat{z}$. Among the formal stability conditions in this limit, the subsonic condition (8FS1′) remains, but (8FS4) and (8SEV) are absent, since $\delta\eta = 0 = \nabla\delta\eta$. The Casimir variable, $\Omega = \rho^{-1}\operatorname{curl}\boldsymbol{v}\cdot\nabla\eta$, in this limit becomes the scalar specific vorticity, $\omega/\rho = \rho^{-1}\hat{z}\cdot\operatorname{curl}\boldsymbol{v}$; so the first geometrical condition for formal stability (8FS2′) becomes $\boldsymbol{v}_e\cdot\hat{z}\times\nabla(\omega_e/\rho_e) > 0$ and the second geometrical condition (8FS3′) is satisfied identically $[\omega_e\hat{z}\cdot\nabla\Phi'(\omega_e/\rho_e) = 0$, since ω_e and ρ_e are functions of $(x, y)]$. Thus, the formal stability conditions for three-dimensional adiabatic fluids reduce to those for two-dimensional barotropic fluids (see section 3.4) in the isentropic planar limit.

D. *Convexity estimates*

In (8HC) we have $H_C = H + C$, with

$$H(\boldsymbol{v}, \rho, \eta) = \int_D \{\tfrac{1}{2}\rho|\boldsymbol{v}|^2 + \varepsilon(\rho, \eta)\}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,, \qquad C(\boldsymbol{v}, \rho, \eta) = \int_D \rho\{\Phi(\eta, \Omega) + \mu\Omega\}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,.$$

Consequently,

$$\hat{H}(\Delta\boldsymbol{v}, \Delta\rho, \Delta\eta) := H(\boldsymbol{v}_e + \Delta\boldsymbol{v}, \rho_e + \Delta\rho, \eta_e + \Delta\eta) - H(\boldsymbol{v}_e, \rho_e, \eta_e) - \mathrm{D}H(\boldsymbol{v}_e, \rho_e, \eta_e)\cdot(\Delta\boldsymbol{v}, \Delta\rho, \Delta\eta)$$
$$= \int_D \{\tfrac{1}{2}(\rho_e + \Delta\rho)|\Delta\boldsymbol{v}|^2 + \Delta\rho(\boldsymbol{v}_e\cdot\Delta\boldsymbol{v}) + \hat{\varepsilon}(\Delta\rho, \Delta\eta)\}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,,$$

where

$$\hat{\varepsilon}(\Delta\rho, \Delta\eta) := \varepsilon(\rho_e + \Delta\rho, \eta_e + \Delta\eta) - \varepsilon(\rho_e, \eta_e) - \varepsilon_\rho(\rho_e, \eta_e)\Delta\rho - \varepsilon_\eta(\rho_e, \eta_e)\Delta\eta\,. \tag{8D1}$$

Likewise,

$$\hat{C}(\Delta v, \Delta\rho, \Delta\eta) := C(v_e + \Delta v, \rho_e + \Delta\rho, \eta_e + \Delta\eta) - C(v_e, \rho_e, \eta_e) - DC(v_e, \rho_e, \eta_e) \cdot (\Delta v, \Delta\rho, \Delta\eta)$$

$$= \int_D \{(\rho_e + \Delta\rho)[\Phi(\eta_e + \Delta\eta, \Omega_e + \Delta\Omega) + \mu(\Omega_e + \Delta\Omega)] - \rho_e[\Phi(\eta_e, \Omega_e) + \mu\Omega_e]$$

$$- [\Phi(\eta_e, \Omega_e) + \mu\Omega_e)\Delta\rho - \rho_e\Phi_\eta(\eta_e, \Omega_e)\Delta\eta - \rho_e[\Phi_\Omega(\eta_e, \Omega_e)$$

$$+ \mu]D\Omega \cdot (\Delta v, \Delta\rho, \Delta\eta)\}\, dx\, dy\, dz\,,$$

where, since $\Omega = \rho^{-1}(\text{curl}\, v) \cdot \nabla\eta$,

$$D\Omega \cdot (\Delta v, \Delta\rho, \Delta\eta) = -\rho_e^{-1}\Omega_e\Delta\rho + \rho_e^{-1}(\text{curl}\, \Delta v) \cdot \nabla\eta_e + \rho_e^{-1}(\text{curl}\, v_e) \cdot \nabla\Delta\eta\,,$$

and

$$\Delta\Omega := (\rho_e + \Delta\rho)^{-1}\, \text{curl}(v_e + \Delta v) \cdot (\eta_e + \Delta\eta) - \rho_e^{-1}\, \text{curl}\, v_e \cdot \nabla\eta_e$$

$$= \frac{\rho_e}{\rho_e + \Delta\rho} D\Omega \cdot (\Delta v, \Delta\rho, \Delta\eta) + \frac{1}{\rho_e + \Delta\rho} (\text{curl}\, \Delta v) \cdot \nabla\Delta\eta\,.$$

Comparison of these expressions for $\Delta\Omega$ and $D\Omega \cdot (\Delta v, \Delta\rho, \Delta\eta)$ gives

$$\rho_e D\Omega \cdot (\Delta v, \Delta\rho, \Delta\eta) = (\rho_e + \Delta\rho)\Delta\Omega - (\text{curl}\, \Delta v) \cdot \nabla\Delta\eta$$

which, when substituted into the previous equation for $\hat{C}$, leads to

$$\hat{C}(\Delta v, \Delta\rho, \Delta\eta) = \int_D \{(\rho_e + \Delta\rho)\hat{\Phi}(\Delta\eta, \Delta\Omega) + \dot{\Phi}(\eta_e, \Omega_e)\Delta\rho\, \Delta\eta$$

$$+ \Delta v \cdot \nabla\Phi'(\eta_e, \Omega_e) \times \nabla\Delta\eta\}\, dx\, dy\, dz\,,$$

where

$$\hat{\Phi}(\Delta\eta, \Delta\eta) := \Phi(\eta_e + \Delta\eta, \Omega_e + \Delta\Omega) - \Phi(\eta_e, \Omega_e) - \dot{\Phi}(\eta_e, \Omega_e)\Delta\eta - \Phi'(\eta_e, \Omega_e)\Delta\Omega\,. \tag{8D2}$$

Adding together $\hat{H} + \hat{C}$ yields

$$\hat{H} + \hat{C} = \int_D \{\tfrac{1}{2}(\rho_e + \Delta\rho)|\Delta v|^2 + \Delta v \cdot (v_e\Delta\rho + \nabla\Phi'(\eta_e, \Omega_e) \times \nabla\Delta\eta)$$

$$+ \hat{\varepsilon}(\Delta\rho, \Delta\eta) + (\rho_e + \Delta\rho)\hat{\Phi}(\Delta\eta, \Delta\Omega) + \dot{\Phi}(\eta_e, \Omega_e)\Delta\rho\,\Delta\eta\}\, dx\, dy\, dz\,,$$

whereupon completing the square and using the relations

$$\Delta v + v_e \Delta\rho/(\rho_e + \Delta\rho) = \Delta M/(\rho_e + \Delta\rho)\,, \qquad \Delta M := (\rho_e + \Delta\rho)(v_e + \Delta v) - \rho_e v_e\,,$$

leads to the expression in terms of ΔM,

$$\hat{H} + \hat{C} = \int_D \Big\{ \frac{1}{2(\rho_e \mp \Delta\rho)} |\Delta M + \nabla\Phi'(\eta_e, \Omega_e) \times \nabla\Delta\eta|^2 - \frac{1}{2(\rho_e + \Delta\rho)} |v_e \Delta\rho + \nabla\Phi'(\eta_e, \Omega_e) \times \nabla\Delta\eta|^2$$

$$+ \hat{\varepsilon}(\Delta\rho, \Delta\eta) + (\rho_e + \Delta\rho)\hat{\Phi}(\Delta\eta, \Delta\Omega) + \dot{\Phi}(\eta_e, \Omega_e)\Delta\rho\Delta\eta \Big\}\, dx\, dy\, dz\,.$$

This expression can be compared with the second variation (8SV3), to which it reduces (up to an inessential factor of 2) for infinitesimal values of $\{\Delta v, \Delta\rho, \Delta\eta\}$.

Assume now that for certain positive constants ρ_{min}, ρ_{max}, e_i, E_i, a_i, A_i, $i = 1, 2, 3$, we have the following bounds and convexity properties:

$$0 < \rho_{min} \le \rho \le \rho_{max} < \infty\,,$$

$$0 < e_1 \le \varepsilon_{\rho\rho}(\rho_e, \eta_e) < \infty\,, \qquad 0 < a_1 \le (\rho_e + \Delta\rho)\Phi''(\eta_e, \Omega_e) < \infty\,,$$

$$\tfrac{1}{2}\begin{pmatrix}\Delta\rho\\ \Delta\eta\end{pmatrix}^t \begin{pmatrix} e_1 & e_3\\ e_3 & e_2\end{pmatrix}\begin{pmatrix}\Delta\rho\\ \Delta\eta\end{pmatrix} \le \hat{\varepsilon}(\Delta\eta, \Delta\Omega) \le \tfrac{1}{2}\begin{pmatrix}\Delta\rho\\ \Delta\eta\end{pmatrix}^t \begin{pmatrix} E_1 & E_3\\ E_3 & E_2\end{pmatrix}\begin{pmatrix}\Delta\rho\\ \Delta\eta\end{pmatrix},$$

where $\hat{\varepsilon}$ and $\hat{\Phi}$ are given in (8D1) and (8D2), respectively.

Then $\hat{H} + \hat{C}$ is bounded below by

$$\hat{H} + \hat{C} \ge \tfrac{1}{2} \int_D \Big\{ \frac{1}{\rho_{max}} |\Delta M + \nabla\Phi'(\eta_e, \Omega_e) \times \nabla\Delta\eta|^2 - \frac{1}{\rho_{min}} |v_e \Delta\rho + \nabla\Phi'(\eta_e, \Omega_e) \times \nabla\Delta\eta|^2$$

$$+ \begin{bmatrix}\Delta\rho\\ \Delta\Omega\\ \Delta\eta\end{bmatrix}^t \begin{bmatrix} e_1 & 0 & e_3 + \dot{\Phi}\\ 0 & a_1 & a_3\\ e_3 + \dot{\Phi} & a_3 & a_2 + e_2\end{bmatrix} \begin{bmatrix}\Delta\rho\\ \Delta\Omega\\ \Delta\eta\end{bmatrix} \Big\}\, dx\, dy\, dz\,. \tag{8D4}$$

Expanding the square, we get

$$-\frac{1}{\rho_{min}} |v_e \Delta\rho + \nabla\Phi'(\eta_e, \Omega_e) \times \nabla\Delta\eta|^2 = -\frac{1}{\rho_{min}} |v_e|^2 (\Delta\rho)^2 + \frac{1}{\rho_{min}} (\nabla\Phi'(\eta_e, \Omega_e) \cdot \nabla\Delta\eta)^2$$

$$+ \frac{2\Delta\rho}{\rho_{min}} \nabla\Delta\eta \cdot \nabla\Phi'(\eta_e, \Omega_e) \times v_e - \frac{1}{\rho_{min}} |\nabla\Phi'(\eta_e, \Omega_e)|^2 |\nabla\Delta\eta|^2\,,$$

which we substitute into (8D4) giving

$$\hat{H} + \hat{C} \ge \tfrac{1}{2} \int_D \Big\{ \frac{1}{\rho_{max}} |\Delta M + \nabla\Phi'(\eta_e, \Omega_e) \times \nabla\Delta\eta|^2 + \frac{1}{\rho_{min}} (\nabla\Delta\eta \cdot \nabla\Phi'(\eta_e, \Omega_e))^2 + \hat{\mathscr{Q}} \Big\}\, dx\, dy\, dz\,. \tag{8D5}$$

The quadratic form $\hat{\mathcal{Q}}$ in (8D5) is given by an expression similar to (8$\mathcal{Q}$1):

$$\hat{\mathcal{Q}} = \begin{bmatrix} \Delta\rho \\ \Delta\Omega \\ \Delta\eta \end{bmatrix}^t \begin{bmatrix} e_1 - \rho_{\min}^{-1}|v_e|^2 + |\tilde{y}|^2/\tilde{a}^2 & 0 & e_3 + \dot{\Phi} \\ 0 & a_1 & a_3 \\ e_3 + \dot{\Phi} & a_3 & a_2 + e_2 \end{bmatrix} \begin{bmatrix} \Delta\rho \\ \Delta\Omega \\ \Delta\eta \end{bmatrix} - |\tilde{a}^2| \nabla\Delta\eta - \tilde{a}^{-2}\tilde{y}\Delta\rho|^2 , \tag{8$\hat{\mathcal{Q}}$1}$$

where $\tilde{a}^2$ and $\tilde{y}$ are defined by

$$\tilde{a}^2 = \rho_{\min}^{-1}|\nabla\Phi'(\eta_e, \Omega_e)|^2$$

$$\tilde{y} = \rho_{\min}^{-1}\nabla\Phi'(\eta_e, \Omega_e) \times v_e .$$

Writing the quadratic form $\hat{\mathcal{Q}}$ in terms of a vector k by setting $\nabla\delta\eta = k\delta\eta$ yields

$$\hat{\mathcal{Q}} = \begin{bmatrix} \Delta\rho \\ \Delta\Omega \\ \Delta\eta \end{bmatrix}^t \begin{bmatrix} A & 0 & C + k\cdot\tilde{y} \\ 0 & D & F \\ C + k\cdot\tilde{y} & F & B - |k|\tilde{a}^2 \end{bmatrix} \begin{bmatrix} \Delta\rho \\ \Delta\Omega \\ \Delta\eta \end{bmatrix} , \tag{8$\hat{\mathcal{Q}}$2}$$

which has the same form as (8$\mathcal{Q}$2), but now with the following interpretations:

$$A = e_1 - \rho_{\min}^{-1}|v_e|^2 , \qquad B = a_2 + e_2 , \qquad C = e_3 + \dot{\Phi}(\eta_e, \Omega_e) , \qquad D = a_1 , \qquad F = a_3 .$$

Consequently, upon retracing the argument using the Schwartz inequality for positivity of the quadratic form $\mathcal{Q}$ in (8$\mathcal{Q}$2), we find that the quadratic form $\hat{\mathcal{Q}}$ in (8$\hat{\mathcal{Q}}$2) is positive if

$$\begin{aligned} &A = e_1 - \rho_{\min}^{-1}|v_e|^2 > 0 , \qquad D = a_1 > 0 , \qquad C = e_3 + \dot{\Phi}(\eta_e, \Omega_e) > 0 , \\ &A(B - F^2/D) - C^2 = (e_1 - |v_e|^2/\rho_{\min})(a_2 + e_2 - a_3^2/a_1) - (e_3 + \dot{\Phi}(\eta_e, \Omega_e))^2 > 0 , \end{aligned} \tag{8D6}$$

provided the specific entropy variation $\Delta\eta$ satisfies

$$|\nabla\Delta\eta| < k_+|\Delta\eta| , \tag{8D7}$$

with k_+ given now by ($8k_+$) in terms of $\rho_{\min}$, e_i, a_i, $i = 1, 2, 3$ and equilibrium flow quantities.

E. *A priori estimates*

Under the assumptions of (8D3) and the conditions (8D6–7) for positive definiteness of the quadratic form in (8$\hat{\mathcal{Q}}$1), the following estimate holds:

$$\tfrac{1}{2}\int_D \left\{ \frac{1}{\rho_{\max}} |\Delta M + \nabla\Phi'(\eta_e, \Omega_e) \times \nabla\Delta\eta|^2 + \frac{1}{\rho_{\min}} (\nabla\Delta\eta \cdot \nabla\Phi'(\eta_e, \Omega_e))^2 \right.$$

$$+\begin{bmatrix}\Delta\rho\\ \Delta\Omega\\ \Delta\eta\end{bmatrix}^t\begin{bmatrix} e_1-|v_e|^2/\rho_{\min} & 0 & e_3+\dot{\Phi}(\eta_e,\Omega_e)+k\cdot\tilde{y}\\ 0 & a_1 & a_3\\ e_3+\dot{\Phi}(\eta_e,\Omega_e)+k\cdot\tilde{y} & a_3 & a_2+e_2-|k|^2\tilde{a}^2\end{bmatrix}\begin{bmatrix}\Delta\rho\\ \Delta\Omega\\ \Delta\eta\end{bmatrix}\Bigg\}\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

$$\le \hat{H}+\hat{C}=:\hat{H}_C$$

$$= H_C(v_e+\Delta v(0),\rho_e+\Delta\rho(0),\eta_e+\Delta\eta(0))-H_C(v_e,\rho_e,\eta_e) \tag{8E}$$

(the last equality uses the proposition concerning vanishing of the first variation).

F. *Nonlinear stability*

The left-hand side of the inequality (8E) defines a norm $\|(\Delta M,\Delta\rho,\Delta\Omega,\Delta\eta)\|^2$ in which the estimate (8E) may be expressed as

$$\hat{H}_C(\Delta v(0),\Delta\rho(0),\Delta\eta(0))\ge\|(\Delta M(t),\Delta\rho(t),\Delta\Omega(t),\Delta\eta(t))\|^2,$$

and (8F1)

$$\hat{H}_C(\Delta v(t),\Delta\rho(t),\Delta\eta(t))\le(\text{const})\|(\Delta M(0),\Delta\rho(0),\Delta\Omega(0),\Delta\eta(0))\|^2,$$

since $\hat{H}_C$ is a constant of the motion and the norm of the initial perturbation is bounded for functions v_e, ρ_e, η_e defined in a finite domain (or having appropriate decay properties at infinity, if D is not bounded). The estimate (8F1) expresses Liapunov stability as summarized by the following:

Adiabatic stability theorem. Let the function Φ be related to the Bernoulli function K by

$$\Phi(\alpha,\beta)=\beta\Big(\int^{\beta}\frac{K(\alpha,t)}{t^2}\,\mathrm{d}t+\Psi(\alpha)\Big)$$

for an arbitrary function Ψ and let $\mu=-\Phi'(\eta_e,\Omega_e)|\partial D$, where v_e, ρ_e, η_e, is a stationary solution of the adiabatic fluid equations (8EM) satisfying $\Omega_e\neq 0$ and $\nabla\eta_e\times\nabla\Omega_e\neq 0$. Assume that Φ and the internal energy density ε for the flow satisfy the convexity conditions (8D3). In addition, assume that the quadratic form $(8\hat{\mathscr{Q}}1)$ is positive definite. Then the stationary solution v_e, ρ_e, η_e is conditionally stable, i.e., stable for solutions that (i) are sufficiently smooth and that satisfy $|\nabla\Delta\eta|<k_+|\Delta\eta|$ with k_+ given by $(8k_+)$ in terms of constants $\rho_{\min}$, e_i, a_1, $i=1,2,3$ and equilibrium flow quantities as in (8D6); and (ii) satisfy $0\le\rho_{\min}\le\rho\le\rho_{\max}<\infty$.

Remarks (1) For specific examples, the arbitrariness of the function Ψ might be helpful since it gives additional freedom in the (1, 2) and (2, 1) entries of the matrix as well as in the inequalities involving η-derivatives of Φ.

(2) Estimates on (div v) are absent, another indication that shock formation is not prevented. In the isentropic planar case, the present result reduces to the estimate of Holm et al. [1983] in section 3.4.

(3) A subcase of the flows treated in this section occurs when $\Omega \equiv 0$. In this case, there is the additional conserved quantity $\int_D \boldsymbol{v} \cdot \operatorname{curl} \boldsymbol{v} \, d^3x$. However, the corresponding class of flows with $\rho \boldsymbol{v} + \lambda \operatorname{curl} \boldsymbol{v} = 0$, $\lambda = \text{const}$ (compressible Beltrami flows), are not even formally stable, as can be readily shown.

9. Adiabatic MHD

This section studies the stability of static three-dimensional MHD equilibria. As we shall see, the methods used in this paper are consistent with Bernstein's δW method. They also reduce to the corresponding two-dimensional results considered in sections 5 and 6.

A. *Equations of motion and Hamiltonian*

In adiabatic magnetohydrodynamics (MHD), quasineutral plasma motion is described in terms of the following physical variables: ρ, the mass density; $\boldsymbol{M}$, the fluid momentum density; η, the specific entropy; and $\boldsymbol{B}$, the magnetic field. The three-dimensional MHD equations are

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} = -\frac{1}{\rho} \nabla p + \frac{1}{\rho} \boldsymbol{J} \times \boldsymbol{B},$$

$$\frac{\partial \rho}{\partial t} = -\operatorname{div}(\rho \boldsymbol{v}), \qquad \frac{\partial \eta}{\partial t} = -\boldsymbol{v} \cdot \nabla \eta, \qquad \frac{\partial \boldsymbol{B}}{\partial t} = \operatorname{curl}(\boldsymbol{v} \times \boldsymbol{B}), \tag{9EM}$$

where $\boldsymbol{v} = \boldsymbol{M}/\rho$ is the fluid velocity, $\boldsymbol{J} = \operatorname{curl} \boldsymbol{B}$ is the current density and p is a given function ρ and η. The electric field $\boldsymbol{E} = -\boldsymbol{v} \times \boldsymbol{B}$ has been eliminated. The pressure p may be determined from an internal energy function $e = e(\rho, \eta)$ by the first law

$$de = T \, d\eta + (p/\rho^2) \, d\rho,$$

which also defines the temperature, T. The boundary conditions are taken to be $\boldsymbol{v} \cdot \boldsymbol{n}|_{\partial D} = 0$ and $\boldsymbol{B} \cdot \boldsymbol{n}|_{\partial D} = 0$, for an impermeable fixed boundary ∂D of our domain D. The divergenceless condition $\operatorname{div} \boldsymbol{B} = 0$ is preserved by the dynamics.

The (3dMHD) equations conserve the total energy

$$H(\boldsymbol{v}, \rho, \eta, \boldsymbol{B}) = \int_D [\tfrac{1}{2}\rho |\boldsymbol{v}|^2 + \rho e(\rho, \eta) + \tfrac{1}{2}|\boldsymbol{B}|^2] \, d^3x, \tag{9H}$$

where $\varepsilon = \rho e$. The space P consists of octuples $(\boldsymbol{v}, \rho, \eta, \boldsymbol{B})$ with $\operatorname{div} \boldsymbol{B} = 0$ and are assumed to have certain differentiability properties (and to decay at infinity if D is unbounded).

Remark A. Poisson structure. The Hamiltonian structure for the (MHD) equations was introduced in Morrison and Greene [1980] and connected to semidirect Lie–Poisson brackets in Holm and Kupershmidt [1983]. It was derived by reduction from canonical brackets in Lagrangian coordinates by Marsden et al. [1983], Holm, Kupershmidt and Levermore [1983], and Marsden, Ratiu and Weinstein [1984]. These equations are Lie–Poisson equations on the dual of the Lie algebra of the semidirect product

$\mathscr{X}$ Ⓢ $(\Lambda^0 \oplus \Lambda^3 \oplus \Lambda^1)$ of vector fields and the direct sum of functions, densities, and one-forms. Dual coordinates are: $\boldsymbol{M}$ is dual to vector fields; ρ is dual to functions; η is dual to densities; and $\boldsymbol{B}$ is dual to two-forms. The Poisson bracket is given by

$$\{F, G\} = \int_D \Big\{ M_i \Big[\frac{\delta G}{\delta \boldsymbol{M}} \cdot \nabla \frac{\delta F}{\delta M_i} - \frac{\delta F}{\delta \boldsymbol{M}} \cdot \nabla \frac{\delta G}{\delta M_i}\Big] + \rho \Big[\frac{\delta G}{\delta \boldsymbol{M}} \cdot \nabla \frac{\delta F}{\delta \rho} - \frac{\delta F}{\delta \boldsymbol{M}} \cdot \nabla \frac{\delta G}{\delta \rho}\Big]$$

$$+ \eta \operatorname{div} \Big[\frac{\delta G}{\delta \boldsymbol{M}} \frac{\delta F}{\delta \eta} - \frac{\delta F}{\delta \boldsymbol{M}} \frac{\delta G}{\delta \eta}\Big] + B_i \Big[\frac{\delta G}{\delta \boldsymbol{M}} \cdot \nabla \frac{\delta F}{\delta B_i} - \frac{\delta F}{\delta \boldsymbol{M}} \cdot \nabla \frac{\delta G}{\delta B_i}\Big]$$

$$+ B_i \Big[\frac{\delta F}{\delta B_j} \partial_i \frac{\delta G}{\delta M_j} - \frac{\delta G}{\delta B_j} \partial_i \frac{\delta F}{\delta M_j}\Big]\Big\} \, \mathrm{d}^3 x \,. \tag{9PB}$$

The equations of motion (9EM) are equivalent to $\dot{F} = \{F, H\}$ with H given in (9H).

B. *Constants of motion*

The equations (9EM) have a geometric reformulation which facilitates the search for their constants of motion. For ρ, η, and $\boldsymbol{B}/\rho$ we may write

$$(\partial_t + \mathscr{L}_v)(\rho \, \mathrm{d}^3 x) = 0 \,, \qquad (\partial_t + \mathscr{L}_v)\eta = 0 \,, \qquad (\partial_t + \mathscr{L}_v)(\rho^{-1}\boldsymbol{B} \cdot \mathrm{d}\boldsymbol{x}) = 0 \,,$$

where $\mathscr{L}_v$ denotes the Lie differentiation along the velocity field $\boldsymbol{v}$, and $\boldsymbol{B} \cdot \mathrm{d}\boldsymbol{x} = \Sigma B_i \, \mathrm{d}x^i$ is $\boldsymbol{B}$ thought of as a one form. The last equation is also expressible as a commutator relation

$$[\partial_t + \mathscr{L}_v, \mathscr{L}_{\boldsymbol{B}/\rho}] = 0 \,, \tag{9CR}$$

where $\mathscr{L}_{\boldsymbol{B}/\rho}$ is the Lie derivative by $\boldsymbol{B}/\rho$. This commutator relation and $(\partial_t + \mathscr{L}_v)\eta = 0$ imply there are constants along flow lines,

$$\Omega^{(n)} = (\mathscr{L}_{\boldsymbol{B}/\rho})^n \eta \,; \qquad n = 0, 1, 2, \ldots .$$

See also Henyey [1982], where these constants along flow lines are derived from symmetry under particle relabeling.

There are also constants of motion which depend on the magnetic vector potential $\boldsymbol{A}$; for example, the magnetic helicity $\int_D \boldsymbol{A} \cdot \boldsymbol{B} \, \mathrm{d}^3 x$, where $\boldsymbol{B} = \operatorname{curl} \boldsymbol{A}$. In terms of the one-form $A = \boldsymbol{A} \cdot \mathrm{d}\boldsymbol{x}$, the evolution equation for the magnetic field equation may be written as $(\partial_t + \mathscr{L}_v) \, \mathrm{d}A = 0$. Choosing a time-dependent gauge $\phi = \boldsymbol{v} \cdot \boldsymbol{A}$ for the electrostatic potential ϕ satisfies the MHD requirement $\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B} = 0$, provided $(\partial_t + \mathscr{L}_v)A = 0$, so in that gauge

$$(\partial_t + \mathscr{L}_v)A \wedge \mathrm{d}A = 0 \,,$$

or, equivalently by the continuity equation,

$$(\partial_t + \mathscr{L}_v)(\rho^{-1}\boldsymbol{B} \cdot \boldsymbol{A}) = 0 \,.$$

By the commutator relation (9CR) the following quantities are also constants along flow lines

$$\Lambda^{(n)} = (\mathscr{L}_{B/\rho})^n(\rho^{-1}\boldsymbol{B}\cdot\boldsymbol{A})\,.$$

Consequently, any function $\Phi(\{\Omega^{(n)}\}, \{\Lambda^{(n)}\})$ is constant along flow lines and the following quantities are conserved by the equations (9EM):

$$C_\Phi = \int_D \rho\Phi(\{\Omega^{(n)}\}, \{\Lambda^{(n)}\})\,\mathrm{d}^3x\,, \qquad n = 0, 1, 2, \ldots\,. \tag{9C}$$

Among those conserved quantities C_Φ depending on $\Lambda^{(n)}$, only the magnetic helicity is gauge invariant (this uses $\boldsymbol{B}\cdot\boldsymbol{n}|_{\partial D} = 0$).

The constants $\Omega^{(n)}$, $\Lambda^{(n)}$, are independent of velocity $\boldsymbol{v}$. Seeking additional constants depending on $\boldsymbol{v}$, we rewrite the velocity equation of motion as

$$\partial_t\boldsymbol{v} + \boldsymbol{\omega}\times\boldsymbol{v} + \nabla(v^2/2 + h) - T\nabla\eta = \frac{1}{\rho}\operatorname{curl}\boldsymbol{B}\times\boldsymbol{B}\,.$$

This is the vector form of the Lie derivative relation

$$(\partial_t + \mathscr{L}_v)(\boldsymbol{v}\cdot\mathrm{d}\boldsymbol{x}) + \mathrm{d}(-|\boldsymbol{v}|^2/2 + h + |\boldsymbol{B}|^2/\rho - T\,\mathrm{d}\eta = \mathscr{L}_{B/\rho}(\boldsymbol{B}\cdot\mathrm{d}\boldsymbol{x})\,.$$

Two Lie derivatives appear in the (3dMHD) motion equation: one by $\boldsymbol{v}$, and one by $\boldsymbol{B}/\rho$. Still, the so-called "cross helicity" $\int_D \boldsymbol{v}\cdot\boldsymbol{B}\,\mathrm{d}^3x$ is conserved for flows with $\Omega^{(1)} = 0$, i.e., with $\Omega^{(1)}$ vanishing throughout the domain of flow, and $\boldsymbol{v}\cdot\boldsymbol{n}|_{\partial D} = 0$ and $\boldsymbol{B}\cdot\boldsymbol{n}|_{\partial D} = 0$, by the equation

$$\partial_t(\boldsymbol{v}\cdot\boldsymbol{B}) = -\operatorname{div}[\boldsymbol{v}(\boldsymbol{v}\cdot\boldsymbol{B}) + \boldsymbol{B}(-|\boldsymbol{v}|^2/2 + h)] + \rho T\Omega^{(1)}\,,$$

together with constancy of $\Omega^{(1)}$, on flow lines. This equation follows readily either from (9EM) directly, or by taking the inner product of the vector field $\boldsymbol{B}/\rho$ with the motion equation written in differential geometric form above.

Remark B. Casimirs. The functionals (9C) are Casimirs for the Poisson bracket in Holm and Kupershmidt [1983] in terms of $(\boldsymbol{M}, \rho, \eta, \boldsymbol{A})$. The functionals (9C) without the $\Lambda^{(n)}$ dependence are Casimirs for (9PB). The cross helicity $\int \boldsymbol{v}\cdot\boldsymbol{B}\,\mathrm{d}^3x$ is a sub-Casimir of (9PB) (Weinstein [1984] and appendix B), subject to vanishing of $\Omega^{(1)}$ and the boundary conditions $\boldsymbol{v}\cdot\boldsymbol{n}|_{\partial D} = 0$, $\boldsymbol{B}\cdot\boldsymbol{n}|_{\partial D} = 0$.

C. *First variation*

The equilibrium states ρ_e, η_e, $\boldsymbol{v}_e$, $\boldsymbol{B}_e$ of the system (9EM) satisfy the relations

$$\operatorname{div}\rho_e\boldsymbol{v}_e = 0\,, \qquad \operatorname{curl}\boldsymbol{v}_e\times\boldsymbol{B}_e = 0\,, \qquad \boldsymbol{v}_e\cdot\nabla\eta_e = 0\,, \qquad \boldsymbol{v}_e\cdot\nabla\Omega_e = 0\,,$$
$$\boldsymbol{\omega}_e\times\boldsymbol{v}_e + \nabla(\tfrac{1}{2}|\boldsymbol{v}_e|^2 + h(\rho_e, \eta_e)) - T_e\nabla\eta_e = \rho_e^{-1}(\operatorname{curl}\boldsymbol{B}_e)\times\boldsymbol{B}_e\,,$$

where $\Omega_e = \Omega_e^{(1)} = \boldsymbol{B}_e\cdot\nabla\eta_e/\rho_e$, without superscript in the remainder of this section. For *static* equilibria the above relations reduce to the single condition

$$(\operatorname{curl} \boldsymbol{B}_e) \times \boldsymbol{B}_e = \nabla p(\rho_e, \eta_e) . \tag{9SC}$$

To get a variational principle for static equilibria, we need only use the Casimirs depending on $\Omega^{(0)}$ and $\Omega^{(1)}$, i.e., on η and $\Omega = (1/\rho)\boldsymbol{B} \cdot \nabla\eta$.

Proposition. For smooth solutions satisfying $\operatorname{div} \boldsymbol{B} = 0$ and $\boldsymbol{B} \cdot \boldsymbol{n} = 0$ on the boundary, a static equilibrium ($\boldsymbol{v}_e \equiv \boldsymbol{0}, \rho_e, \eta_e, \boldsymbol{B}_e$) of the ideal three-dimensional MHD equations is a critical point of

$$H_C = H + C_\Phi , \tag{9H$_C$}$$

where $C_\Phi = \int_D \rho\Phi(\eta, \Omega)\, d^3x$, provided the function Φ satisfies the following relations in terms of the equilibrium solutions:

$$h(\rho_e, \eta_e) + \Phi(\eta_e, \Omega_e) - \Omega_e \Phi_\Omega(\eta_e, \Omega_e) = 0 , \tag{9FV2}$$

$$\boldsymbol{B}_e + \Phi_\Omega(\eta_e, \Omega_e)\nabla\eta_e = 0 , \tag{9FV4}$$

where $h(\rho_e, \eta_e)$ is the specific enthalpy and T_e the temperature at equilibrium.

Proof. The conserved functional in the proposition is

$$H_C(\boldsymbol{v}, \rho, \eta, \boldsymbol{B}) = \int_D [\tfrac{1}{2}\rho|\boldsymbol{v}|^2 + \rho e(\rho, \eta) + \tfrac{1}{2}|\boldsymbol{B}|^2 + \rho\Phi(\eta, \Omega)]\, d^3x .$$

This functional has the same form as the corresponding conserved quantities in section 8 for adiabatic flow. We obtain the following expression for DH_C:

$$\begin{aligned} DH_C(\boldsymbol{v}_e, \rho_e, \eta_e)(\delta\boldsymbol{v}, \delta\rho, \delta\eta, \delta\boldsymbol{B}) = \int_D d^3x \Big\{ & \rho_e\boldsymbol{v}_e \cdot \delta\boldsymbol{v} + [\tfrac{1}{2}|\boldsymbol{v}_e|^2 + h_e + \Phi - \Omega\Phi_\Omega]\delta\rho \\ & + \rho_e[T_e + \Phi_\eta - \rho_e^{-1}\boldsymbol{B}_e \cdot \nabla\Phi_\Omega]\delta\eta + [\boldsymbol{B}_e + \Phi_\Omega\nabla\eta_e]\cdot\delta\boldsymbol{B} \Big\} \\ & + \oint_{\partial D} \delta\eta\Phi_\Omega\boldsymbol{B}_e \cdot \boldsymbol{n}\, dS , \end{aligned} \tag{9DH}$$

where dS is the surface element on the boundary and $\boldsymbol{n}$ is the unit vector normal to the boundary. The boundary integrals in (9DH) vanish by $\delta\boldsymbol{B} \cdot \boldsymbol{n}|_{\partial D} = 0$ and $\boldsymbol{B}_e \cdot \boldsymbol{n}|_{\partial D} = 0$. Throughout (9DH), suppressed arguments of functions are to be evaluated at equilibrium. For DH_C to vanish at equilibrium requires each coefficient to vanish, i.e.,

$$\delta\ :\ \ \boldsymbol{v}_e = 0 , \tag{9FV1}$$

$$\delta\rho: \quad h(\rho_e, \eta_e) + \Phi(\eta_e, \Omega_e) - \Omega_e \Phi_\Omega(\eta_e, \Omega_e) = 0\,, \tag{9FV2}$$

$$\delta\eta: \quad T_e + \Phi_\eta - \rho_e^{-1} \boldsymbol{B}_e \cdot \nabla \Phi_\Omega(\eta_e, \Omega_e) = 0\,, \tag{9FV$_{\delta\eta}$3}$$

$$\delta \boldsymbol{B}: \quad \boldsymbol{B}_e + \Phi_\Omega(\eta_e, \Omega_e) \nabla \eta_e = 0\,. \tag{9FV$_{\delta B}$) = (9FV4}$$

Substituting (9SC) into the gradient of (9FV2) and using (9FV4) shows that (9FV$_{\delta\eta}$3) holds.

To determine Φ from the static equilibrium, one possibility is to *assume* a functional relationship $\rho_e = F(\eta_e, \Omega_e)$. (If $\boldsymbol{v}_e$ were not zero, this would follow if div $\boldsymbol{v}_e = 0$). Then (9FV2) determines Φ in the usual way. Let $\boldsymbol{v}_e = \boldsymbol{0}$ and let the functions η_e and Ω_e be arbitrarily specified; let $\rho_e = F(\eta_e, \Omega_e)$ and let B_e be defined by (9FV3). This gives an equilibrium solution and Φ.

Note that (9FV3) implies

$$\boldsymbol{J}_e = \operatorname{curl} \boldsymbol{B}_e = \Phi_{\Omega\Omega}(\eta_e, \Omega_e) \nabla \eta_e \times \nabla \Omega_e\,,$$

so the current follows lines of intersection of surfaces of constant η_e and Ω_e. Also

$$\Phi_{\Omega\Omega}(\eta_e, \Omega_e) = (\boldsymbol{J}_e \cdot \nabla \eta_e \times \nabla \Omega_e)/|\nabla \eta_e \times \nabla \Omega_e|^2\,, \tag{9SVC}$$

so $\Phi_{\Omega\Omega}(\eta_e, \Omega_e) > 0$ as long as $\boldsymbol{J}_e$, $\nabla \eta_e$ and $\nabla \Omega_e$ form a right-hand triad.

Formal stability and stability of MHD

In the proposition, H_C is identical in form to the corresponding quantity for three-dimensional adiabatic fluids in section 8, except for quadratic pieces which do not harm stability, and the reinterpretation of Ω. *Therefore the corresponding results for stability in the adiabatic stability theorem in section* 8 *apply for MHD.* For a static MHD equilibrium the subsonic condition is automatically satisfied, of course.

There also exist stationary, non-static MHD equilibria in three dimensions; for example, the aligned flows, with $\boldsymbol{v}_e \times \boldsymbol{B}_e = 0$, $\boldsymbol{v}_e \neq 0$. These flows are extremal points of

$$H_C = H + C_\Phi + C_B + C_\lambda\,,$$

where $\Omega_e = 0$ identically and C_λ is the sub-Casimir for this case:

$$C_\lambda = \lambda \int \boldsymbol{v} \cdot \boldsymbol{B}\, d^3x\,.$$

Unfortunately, the resulting aligned flows are not formally stable, since the quadratic form $D^2 H_C$ in this case is indefinite.

Further remarks

(1) Notice that in the two-dimensional limit in which we pass to a surface of constant η_e, so $\nabla \eta_e \to \hat{\boldsymbol{z}}$, the three-dimensional second variation condition (9SVC) corresponds to the two-dimensional one ((c′) of section 6.2).

(2)* *The* δW method of Bernstein [1958, 1983] is related to $\delta^2 H_C$ by reduction from Lagrangian to Eulerian coordinates. In fact δW is the second variation of the potential energy for MHD written in Lagrangian coordinates. For MHD the Casimirs are given via Lie derivatives as we have seen, and Eulerian and Lagrangian variations of a quantity differ by a Lie derivative (the coadjoint action generator on a general Lie group). In this case one arrives at,

$$\delta^2 H_C \cdot (\delta \boldsymbol{\xi})^2_{\text{Euler}} = |\delta \dot{\boldsymbol{\xi}}|^2_{\text{Lagr}} \rho + \delta W_{\text{Lagr}} \,, \tag{9δW}$$

where $\delta\boldsymbol{\xi}$ is the Lagrangian displacement from the equilibrium trajectory, so that formal stability in terms of δW implies that for H_C. We shall explore this relationship at greater length in another publication.

(3) In appendix A, we recall Bernstein's logarithmic convexity argument which shows that definiteness of the second variation is necessary as well as sufficient for linearized stability, in Lagrangian coordinates.

10. Adiabatic multifluid plasmas

A. *Equations of motion and Hamiltonian*

The physical variables are (as in section 7, but suppressing species indices): ρ, the mass density; $\boldsymbol{M}$, the fluid momentum density; η, the specific entropy; $\boldsymbol{E}$, the electric field; and $\boldsymbol{B}$, the magnetic field. The velocity $\boldsymbol{v}$ is related to momentum density by $\boldsymbol{v} = \boldsymbol{M}/\rho$. The three-dimensional MFP equations are

$$\partial_t \boldsymbol{v} = -(\boldsymbol{v}\cdot\nabla)\boldsymbol{v} - \frac{1}{\rho}\nabla p + a(\boldsymbol{E} + \boldsymbol{v}\times\boldsymbol{B}) \,, \qquad \partial_t \rho = -\operatorname{div} \rho\boldsymbol{v} \,,$$

$$\partial_t \eta = -\boldsymbol{v}\cdot\nabla\eta \,, \qquad \partial_t \boldsymbol{B} = -\operatorname{curl}\boldsymbol{E} \,, \qquad \partial_t \boldsymbol{E} = \operatorname{curl}\boldsymbol{B} - \sum a\rho\boldsymbol{v} \,, \tag{10EM}$$

where p is the pressure, the parameter a is the species charge-to-mass ratio, and Σ indicates a sum over species. The remaining Maxwell equations

$$\operatorname{div}\boldsymbol{B} = 0 \,, \qquad \operatorname{div}\boldsymbol{E} - \sum a\rho = 0 \tag{10SME}$$

are preserved by the dynamics and, thus, can be treated as initial conditions. The equation of state for specific internal energy $e = e(\rho, \eta)$ determines the pressure through the first law

$$\mathrm{d}e = T\,\mathrm{d}\eta + (p/\rho^2)\,\mathrm{d}\rho \,,$$

where T is temperature. Boundary conditions are $\boldsymbol{v}\cdot\boldsymbol{n}|_{\partial D} = 0$, $\boldsymbol{B}\cdot\boldsymbol{n}|_{\partial D} = 0$, and $\boldsymbol{E}\times\boldsymbol{n}|_{\partial D} = \boldsymbol{0}$.

For a single fluid species and when $\boldsymbol{E}$ and $\boldsymbol{B}$ are absent, these equations reduce to the equations for a three-dimensional adiabatic fluid, whose stability conditions for stationary flows are discussed in section 8.

* We thank Philippe Similon and Phil Morrison for discussions on this point.

The equations (10EM) conserve the total energy

$$H(\boldsymbol{v}, \rho, \eta, \boldsymbol{E}, \boldsymbol{B}) = \sum \int_D [\tfrac{1}{2}\rho|\boldsymbol{v}|^2 + \rho e(\rho, \eta)]\, \mathrm{d}^3x + \int_D (\tfrac{1}{2}|\boldsymbol{E}|^2 + \tfrac{1}{2}|\boldsymbol{B}|^2)\, \mathrm{d}^3x\,, \tag{10H}$$

where $\varepsilon(\rho, \eta) = \rho e(\rho, \eta)$ is the internal energy density. The space P consists of $(\boldsymbol{v}, \rho, \eta)$ for each species, and $\boldsymbol{E}$, $\boldsymbol{B}$, all of which are assumed to be suitably smooth and to satisfy (10SME).

Remark A. The Hamiltonian structure for the MFP equations is due to Iwinski and Turski [1976] and is discussed further in Spencer [1982], Holm and Kupershmidt [1983], Marsden et al. [1983], Holm, Kupershmidt and Levermore [1983], and Marsden, Ratiu and Weinstein [1984]. For functionals F, G of $(\boldsymbol{M}, \rho, \eta, \boldsymbol{E}, \boldsymbol{B})$ with boundary conditions as above, the Poisson bracket is given by

$$\begin{aligned}\{F, G\} = -\sum \int_D \Big\{ & M_i\Big[\Big(\frac{\delta G}{\delta \boldsymbol{M}}\cdot \nabla\Big)\frac{\delta F}{\delta M_i} - \Big(\frac{\delta F}{\delta \boldsymbol{M}}\cdot \nabla\Big)\frac{\delta G}{\delta M_i}\Big] + \rho\Big[\frac{\delta G}{\delta \boldsymbol{M}}\cdot \nabla\frac{\delta F}{\delta \rho} - \frac{\delta F}{\delta \boldsymbol{M}}\cdot \nabla\frac{\delta G}{\delta \rho}\Big] \\ & + \eta \operatorname{div}\Big(\frac{\delta G}{\delta \boldsymbol{M}}\frac{\delta F}{\delta \eta} - \frac{\delta F}{\delta \boldsymbol{M}}\frac{\delta G}{\delta \eta}\Big) + a\rho\Big(\frac{\delta F}{\delta \boldsymbol{M}}\cdot\frac{\delta G}{\delta \boldsymbol{E}} - \frac{\delta G}{\delta \boldsymbol{M}}\cdot\frac{\delta F}{\delta \boldsymbol{E}} + \boldsymbol{B}\cdot\frac{\delta F}{\delta \boldsymbol{M}}\times\frac{\delta G}{\delta \boldsymbol{M}}\Big)\Big\}\, \mathrm{d}^3x \\ & + \int_D \Big(\frac{\delta F}{\delta \boldsymbol{E}}\cdot \operatorname{curl}\frac{\delta G}{\delta \boldsymbol{B}} - \frac{\delta G}{\delta \boldsymbol{E}}\cdot \operatorname{curl}\frac{\delta F}{\delta \boldsymbol{B}}\Big)\, \mathrm{d}^3x\,. \end{aligned} \tag{10PB}$$

The equations of motion (10EM) are equivalent to $\dot F = \{F, H\}$, with H given in (10H).

B. *Constants of motion*

A direct way to find the constants of motion associated with the "freezing-in" of fluid quantities is to notice that the velocity equation in (10EM) can be written in the following suggestive form by using $\boldsymbol{E} = \dot{\boldsymbol{A}} - \nabla\phi$, $\boldsymbol{B} = \operatorname{curl} \boldsymbol{A}$, and the first law expressed in terms of specific enthalpy, $h = e + p/\rho$,

$$\partial_t(\boldsymbol{v} + a\boldsymbol{A}) - \boldsymbol{v}\times \operatorname{curl}(\boldsymbol{v} + a\boldsymbol{A}) + \nabla(v^2/2 + h + a\phi) - T\nabla\eta = 0\,. \tag{10EMa'}$$

This is the vector form of the differential relation

$$(\partial_t + \mathscr{L}_{\boldsymbol{v}})q - \mathrm{d}(v^2/2 - h - a\phi + a\boldsymbol{v}\cdot\boldsymbol{A}) - T\,\mathrm{d}\eta = 0\,, \tag{10EMa''}$$

where $\mathscr{L}_{\boldsymbol{v}}$ is the Lie derivative with respect to the vector field with components v^i, $q = (v_i + aA_i)\,\mathrm{d}x^i$ is the "circulation" one-form, and d is exterior derivative. Recall that for any differential form, Q, the Lie derivative is given by $\mathscr{L}_{\boldsymbol{v}}Q = \boldsymbol{v}\cdot \mathrm{d}Q + \mathrm{d}(\boldsymbol{v}\cdot Q)$, where the dot "·" means substitution, i.e., inner product of a vector field and a differential form. Taking the exterior derivative of (10EMa''), using $\mathrm{d}^2 = 0$, and $[\mathrm{d}, \mathscr{L}_{\boldsymbol{v}}] = 0$ gives

$$(\partial_t + \mathscr{L}_{\boldsymbol{v}})\,\mathrm{d}q - \mathrm{d}T\wedge \mathrm{d}\eta = 0\,.$$

Since $(\partial_t + \mathscr{L}_{\boldsymbol{v}})\eta = 0$ for the scalar η, one finds

$$(\partial_t + \mathcal{L}_v)(\mathrm{d}q \wedge \mathrm{d}\eta) = 0\,,$$

and combining with the continuity equation

$$(\partial_t + \mathcal{L}_v)(\rho\, \mathrm{d}^3x) = 0\,,$$

yields

$$(\partial_t + \boldsymbol{v}\cdot\boldsymbol{\nabla})\Omega = 0\,, \tag{10Ω}$$

where

$$\Omega = \rho^{-1}\,\mathrm{curl}(\boldsymbol{v} + a\boldsymbol{A})\cdot\boldsymbol{\nabla}\eta = \rho^{-1}(\boldsymbol{\omega} + a\boldsymbol{B})\cdot\boldsymbol{\nabla}\eta\,.$$

Likewise,

$$(\partial_t + \mathcal{L}_v)(q \wedge \mathrm{d}q) = \mathrm{d}(\beta\,\mathrm{d}q + T\,\mathrm{d}\eta \wedge q) + 2T\,\mathrm{d}\eta \wedge \mathrm{d}q\,,$$

where

$$\beta = v^2/2 - h - a\phi + a\boldsymbol{v}\cdot\boldsymbol{A}\,,$$

yields an equation for helicity, $\int_D (\boldsymbol{v} + a\boldsymbol{A})\cdot(\boldsymbol{\omega} + a\boldsymbol{B})\,\mathrm{d}^3x$, namely,

$$\partial_t \int_D (\boldsymbol{v} + a\boldsymbol{A})\cdot(\boldsymbol{\omega} + a\boldsymbol{B})\,\mathrm{d}^3x = \int_D 2\rho T\Omega\,\mathrm{d}^3x = 0 \quad \text{for} \quad \Omega = 0\,,$$

provided $\boldsymbol{v}\cdot\boldsymbol{n}|_{\partial D} = 0$, $(\boldsymbol{\omega} + a\boldsymbol{B})\cdot\boldsymbol{n}|_{\partial D} = 0$ and $\boldsymbol{\nabla}\eta \times \boldsymbol{n}|_{\partial D} = 0$. Note that by (10$\Omega$) if Ω is initially zero throughout the domain of flow, it will remain zero.

The eq. (10Ω), the entropy equation, and the continuity equations imply that the quantity

$$C_\Phi(\eta, \Omega) := \int \rho\Phi(\eta, \Omega)\,\mathrm{d}^3x \tag{10C}$$

is conserved by the MFP equations (10EM) for every function $\Phi(\eta, \Omega)$.

Remark B. Casimirs. The functionals (10C) are Casimirs for the Poisson bracket (10PB). The helicity integral $\int (\boldsymbol{v} + a\boldsymbol{A})\cdot(\boldsymbol{\omega} + a\boldsymbol{B})\,\mathrm{d}^3x$ is a sub-Casimir, subject to vanishing of Ω and the boundary conditions mentioned previously.

C. *First variation*

The equilibrium states ρ_e, η_e, $\boldsymbol{v}_e$, $\boldsymbol{E}_e$, $\boldsymbol{B}_e$, of the system (10EM) are the stationary, three-dimensional, adiabatic MFP flows. Such stationary flows satisfy the relations

$$\boldsymbol{E}_e = -\boldsymbol{\nabla}\phi_e\,, \tag{10SRa}$$

$$\operatorname{div} \boldsymbol{E}_e = \sum a\rho_e\,, \tag{10SRb}$$

$$\operatorname{div} \boldsymbol{B}_e = 0\,, \tag{10SRc}$$

$$\operatorname{curl} \boldsymbol{B}_e = \sum a\rho_e \boldsymbol{v}_e\,, \tag{10SRd}$$

$$\operatorname{div} \rho_e \boldsymbol{v}_e = 0\,, \tag{10SRe}$$

$$\boldsymbol{v}_e \cdot \nabla\eta_e = 0\,, \tag{10SRf}$$

$$\boldsymbol{v}_e \cdot \nabla(\tfrac{1}{2}|\boldsymbol{v}_e|^2 + h(\rho_e, \eta_e) + a\phi_e) = 0\,, \tag{10SRg}$$

$$\boldsymbol{v}_e \cdot \nabla\Omega_e = 0\,. \tag{10SRh}$$

A sufficient condition for the last three relations is the *Bernoulli law*

$$\tfrac{1}{2}|\boldsymbol{v}_e|^2 + h(\rho_e, \eta_e) + a\phi_e = K(\eta_e, \Omega_e) \tag{10C1}$$

for a Bernoulli function K. Thus, by the stationary equation (10EMa′), we have

$$\boldsymbol{v}_e \times (\boldsymbol{\omega}_e + a\boldsymbol{B}_e) = \nabla K(\eta_e, \Omega_e) - T_e \nabla\eta_e\,.$$

Vector multiplying this by $\nabla\eta_e$ gives

$$\rho_e \boldsymbol{v}_e = \frac{K_\Omega}{\Omega_e} \nabla\eta_e \times \nabla\Omega_e\,, \tag{10C2}$$

where $K_\Omega := \partial K(\eta_e, \Omega_e)/\partial\Omega_e$. Note the divergence of (10C2) vanishes, as required by (10SRe). Similarly, vector multiplying by $\nabla\Omega_e$ gives

$$\frac{\nabla\Omega_e \cdot (\boldsymbol{\omega}_e + a\boldsymbol{B}_e)}{\nabla\eta_e \cdot (\boldsymbol{\omega}_e + a\boldsymbol{B}_e)} = \frac{T_e - K_\eta}{K_\Omega}\,, \tag{10C3}$$

where $K_\eta := \partial K(\eta_e, \Omega_e)/\partial\eta_e$. Relations (10C1)–(10C3) will be useful in demonstrating the following proposition.

Proposition. For smooth solutions satisfying $\boldsymbol{v} \cdot \boldsymbol{n} = 0$, $\boldsymbol{B} \cdot \boldsymbol{n} = 0$ and $\boldsymbol{E} \times \boldsymbol{n} = \boldsymbol{0}$ on the boundary, a stationary solution $(\boldsymbol{v}_e, \rho_e, \eta_e, \boldsymbol{E}_e, \boldsymbol{B}_e)$ of the ideal three-dimensional MFP equations is a critical point of

$$H_C = H + \sum C_\Phi + \sum \mu \int \rho\Omega \, d^3x\,,$$

where μ is a constant (separated out for convenience), provided for each species,

$$\Phi(\eta, \zeta) = \zeta\left(\int^{\zeta} \frac{K(\eta, t)}{t^2}\,\mathrm{d}t + \text{const}\right),$$

and $\mu = -\Phi_\Omega|_{\partial D}$, K being the Bernoulli function of a given species.

Proof. The conserved functional in the proposition is

$$H_C(\boldsymbol{v}, \rho, \eta, \boldsymbol{E}, \boldsymbol{B}) = \sum \int_D [\tfrac{1}{2}\rho|\boldsymbol{v}|^2 + \rho e(\rho, \eta)]\,\mathrm{d}^3x + \tfrac{1}{2}\int_D (|\boldsymbol{E}|^2 + |\boldsymbol{B}|^2)\,\mathrm{d}^3x$$
$$+ \sum \int_D [\rho\Phi(\eta, \Omega) + \mu\rho\Omega]\,\mathrm{d}^3x\,.$$

Except for the electromagnetic pieces, the expression for H_C is identical to the corresponding conserved quantity in section 8 for adiabatic flow. Integrating by parts leads to the following expression for the derivative $\mathrm{D}H_C$:

$$\mathrm{D}H_C(\boldsymbol{v}_e, \rho_e, \eta_e, \boldsymbol{E}_e, \boldsymbol{B}_e)\cdot(\delta\boldsymbol{v}, \delta\rho, \delta\eta, \delta\boldsymbol{E}, \delta\boldsymbol{B}) = \sum\int_D \{[\tfrac{1}{2}|\boldsymbol{v}_e|^2 + h(\rho_e, \eta_e) + \Phi - \Omega_e\Phi_\Omega]\delta\rho$$
$$+ (\rho_e\boldsymbol{v}_e - \nabla\eta_e \times \nabla\Phi_\Omega)\cdot\delta\boldsymbol{v} + \rho_e[T_e + \Phi_\eta - \rho_e^{-1}(\boldsymbol{\omega}_e + a\boldsymbol{B}_e)\cdot\nabla\Phi_\Omega]\delta\eta\}\,\mathrm{d}^3x$$
$$+ \int_D \left[\left(\boldsymbol{B}_e + \sum a\Phi_\Omega\nabla\eta_e\right)\cdot\delta\boldsymbol{B} + (\boldsymbol{E}_e\cdot\delta\boldsymbol{E})\right]\mathrm{d}^3x + \sum\oint_{\partial D}(\mu + \Phi_\Omega)[\delta\boldsymbol{v}\times\nabla\eta_e + \delta\eta(\boldsymbol{\omega}_e + a\boldsymbol{B}_e)]\cdot\boldsymbol{n}\,\mathrm{d}S, \tag{10DH}$$

where $\mathrm{d}S$ is the surface element on the boundary and $\boldsymbol{n}$ is its unit normal vector. Throughout (10DH), Φ and its partial derivatives Φ_η and Φ_Ω are to be evaluated at equilibrium (η_e, Ω_e). The $\delta\boldsymbol{E}$ coefficient in (DH) is transformed to a $\delta\rho$ piece by using $\operatorname{div}\delta E = \sum a\delta\rho$, as in section 7. For a stationary solution, the connected components of the boundary are both streamlines and (by $\boldsymbol{E}_e\times\boldsymbol{n}|_{\partial D} = \boldsymbol{0}$) equipotential surfaces. Thus, η_e and Ω_e are all constants on the boundary. The boundary integrals in (10DH) vanish upon choosing $\mu + \Phi_\Omega = 0$ on ∂D and noting that $\delta\boldsymbol{B}\cdot\boldsymbol{n}|_{\partial D} = 0$. The remaining coefficients in (10DH) vanish for stationary flows by virtue of the single relation

$$K(\eta_e, \Omega_e) + \Phi(\eta_e, \Omega_e) - \Omega_e\Phi_\Omega(\eta_e, \Omega_e) = 0\,, \tag{10K}$$

which is the same as the relation between K and Φ in the proposition. Given (10K), the $\delta\rho$ coefficient in (10DH) vanishes for stationary flows according to (10C1). Since (10K) implies that

$$K_\Omega(\eta_e, \Omega_e)/\Omega_e = \Phi_{\Omega\Omega}(\eta_e, \Omega_e)$$

the $\delta \boldsymbol{v}$ coefficient vanishes by (10C2′). Upon substituting (10K) into it, the $\delta\eta$ coefficient becomes

$$T_e - K_\eta - \Phi_{\Omega\Omega}\rho_e^{-1}(\boldsymbol{\omega}_e + a\boldsymbol{B}_e)\cdot \nabla\Omega_e ,$$

which vanishes by (10C3). Finally, the $\delta \boldsymbol{B}$ coefficient vanishes by using (10K) and (10C2), to imply

$$\rho \boldsymbol{v}_e = -\mathrm{curl}(\Phi_\Omega \nabla\eta_e) .$$

Hence, (10CRd) gives

$$\boldsymbol{B}_e + \sum a\Phi_\Omega \nabla\eta_e = \boldsymbol{0} ,$$

for the $\delta \boldsymbol{B}$ coefficient. ■

Formal stability and stability for MFP

Observe that H_C in the proposition is identical in form to its counterpart for three-dimensional adiabatic flow in section 8, except for quadratic electromagnetic terms which cause no difficulties, and reinterpretation of the quantity Ω to incorporate the magnetic field. Consequently, *both the formal stability conditions* (8*a*, *b*, *c*) *and the convexity estimates* (8*E*) *may be taken over directly for MFP*; *so the stability theorem of section* 8 *applies for MFP, as well.*

Part III. Plasma Systems

In this part we apply the energy-Casimir method to the Poisson–Vlasov and Maxwell–Vlasov systems in one, two, and three dimensions. Our main result is an a priori estimate that is in agreement with the formal stability results of Newcomb [1958], Gardner [1963] and Rosenbluth [1964]. In the homogeneous case, this theorem applies if the equilibrium plasma density function f_e is a function of the particle velocity that is isotropic and monotone decreasing. Because of technical difficulties for large values of the velocity, a full nonlinear stability result does not appear possible using these methods.

11. The Poisson–Vlasov system

Here we consider the Poisson–Vlasov (or collisionless Boltzmann, or Jeans) equation and the nonlinear stability of equilibrium densities that are functions of the velocity. Stability results in the context of stellar dynamics are due to Jeans [1902, 1919]. Formal stability in a spirit similar to Arnold [1965a] was considered by Newcomb [1958] for a Maxwellian equilibrium and by Gardner [1963] and Rosenbluth [1964] for a general monotonic decreasing equilibrium. The Casimir one uses to obtain formal stability is essentially the entropy of the equilibrium solution. Here we provide the convexity estimates needed to bound the growth of perturbations.

For one dimensional plasmas, Penrose [1960] and Case [1958] studied neutral stability and linearized stability. (See, for instance, Clemmow and Dougherty [1969], Krall and Trivelpiece [1973], and Ichimaru [1973].) These results provide neutral stability for equilibrium densities that are not monotone, but may have two humps. When dissipation is present, these results are very important for bifurcation analysis, as in Crawford [1983]. In the conservative case, however, we conjecture that these equilibria are nonlinearly unstable through the mechanism of Arnold diffusion. It is conceivable that the numerical work of Berman et al. [1982] is evidence for this. (The stability results of Rowlands [1966] do not appear to be correct.) It would be of interest to understand the Penrose criterion from the point of view of Krein's [1950] Hamiltonian spectral theory.

A. *Equations of motion and Hamiltonian*

Consider a collisionless plasma consisting of several species with charges q^s and masses m^s, where s is the species label, $1 \leq s \leq S$. The particles move in n-space $\mathbb{R}^n$, $n \geq 1$. Let $\boldsymbol{x}, \boldsymbol{v} \in \mathbb{R}^n$ denote the position and velocity of particles in the plasma and $f^s(\boldsymbol{x}, \boldsymbol{v}, t)$ be the phase space density of species s. We assume that f^s is either periodic in $\boldsymbol{x}$ or has appropriate asymptotic behavior as $\boldsymbol{x} \to \infty$ and decays as $\boldsymbol{v} \to \infty$ (for example, f^s may belong to function spaces governed by the existence theory described in Batt [1977, 1980], Ukai-Okabe [1978], Horst [1980, 1982], Wollman [1980–1984], Cooper and Klimas [1980], Bardos and Degond [1983], and Glassey and Strauss [1984]).

In the case of one species, we assume in addition that $\int\int f^s(\boldsymbol{x}, \boldsymbol{v})\, \mathrm{d}^n x \mathrm{d}^n v = 1$ and that the plasma is moving in a background static ion field.

The Poisson–Vlasov equations are

$$\frac{\partial f^s}{\partial t} + \boldsymbol{v} \cdot \frac{\partial f^s}{\partial \boldsymbol{x}} - \frac{q^s}{m^s} \frac{\partial \phi_f}{\partial \boldsymbol{x}} \cdot \frac{\partial f^s}{\partial \boldsymbol{v}} = 0\,, \qquad \nabla^2 \phi_f = -\rho_f := -\sum_s q^s \int f^s(\boldsymbol{x}, \boldsymbol{v})\, \mathrm{d}^n v\,, \tag{11EM}$$

where $\partial/\partial \boldsymbol{x}$, $\partial/\partial \boldsymbol{v}$ denote the gradients with respect to $\boldsymbol{x}$ and $\boldsymbol{v}$ respectively, ∇^2 is the Laplacian in the $\boldsymbol{x}$-variable, $\boldsymbol{f} = (f^1, \ldots, f^s)$, and ρ_f is the total charge density of the plasma. If we are dealing with a one-species plasma, the right-hand side of the Poisson equation in (11EM) is replaced by $q(\int f(x, v)\, \mathrm{d}v - 1)$. Let

$$\{g, h\}(\boldsymbol{x}, \boldsymbol{v}) = \frac{1}{m}\left(\frac{\partial g}{\partial \boldsymbol{x}} \cdot \frac{\partial h}{\partial \boldsymbol{v}} - \frac{\partial h}{\partial \boldsymbol{x}} \cdot \frac{\partial q}{\partial \boldsymbol{v}}\right)$$

be the canonical Poisson bracket in $(\boldsymbol{x}, \boldsymbol{v})$ space. A direct verification shows that the dynamic equation for the species s in (11EM) has the expression

$$\partial f^s/\partial t = \{\mathcal{H}_{f^s} f^s\}\,, \tag{11EM)$'$}$$

where

$$\mathcal{H}_{f^s}(\boldsymbol{x}, \boldsymbol{v}) = \tfrac{1}{2} m^2 |\boldsymbol{v}|^2 + q^s \phi_f(\boldsymbol{x})\,.$$

The total energy of the system has the expression

$$H(f^s) = \sum_s \left[\tfrac{1}{2} \int\int m^s |\boldsymbol{v}|^2 f^s(\boldsymbol{x}, \boldsymbol{v})\, \mathrm{d}^n x\, \mathrm{d}^n v\right] + \tfrac{1}{2} \int \phi_f \rho_f\, \mathrm{d}^n x\,. \tag{11H}$$

It is easily verified that H is conserved along trajectories of (11EM) by using the following relation (proved by integration by parts):

$$\int\int f\{g, h\}\, \mathrm{d}^n x\, \mathrm{d}^n v = \int\int \{f, g\} h\, \mathrm{d}^n x\, \mathrm{d} n^v .$$

Remark A. The eqs. (11EM) are Hamiltonian with respect to the Lie–Poisson bracket

$$\{F, G\}(f) = \sum_s \int\int f^s(x, v) \left\{\frac{\delta F}{\delta f^s}, \frac{\delta G}{\delta f^s}\right\} \mathrm{d}^n x\, \mathrm{d}^n v \qquad \text{(11PB)}$$

on the dual of the direct sum of functions of (x, v). This can be easily verified by showing first that $\partial H/\partial f^s = \mathcal{H}_{f^s}$ and using (11EM)′. The bracket (11PB) was introduced by Morrison [1980] and Gibbons [1981], and its interpretation as a Lie–Poisson bracket and derivation from the canonical bracket in Lagrangian representation is due to Marsden and Weinstein [1982].

B. *Constants of the motion*

From equation (11EM)′ it follows that for every function $\Phi^s : \mathbb{R} \to \mathbb{R}$ the functional

$$C^s(f^s) = \int\int \Phi^s(f^s)\, \mathrm{d}^n x\, \mathrm{d}^n v \qquad \text{(11C)}$$

is a conserved quantity for (11EM).

Remark B. The coadjoint action of the group of symplectic diffeomorphisms of (x, v) space on the dual of its Lie algebra, which consists of functions of (x, v), is given by push-forward (see Marsden and Weinstein [1982] or Marsden et al. [1983]). It is easily verified that C^s is invariant under this action and thus it is a Casimir function. Alternatively, one can check directly that the Poisson bracket (11PB) of C^s with any other function vanishes identically.

C. *First variation*

Let $C = \Sigma_s C^s$. A short computation shows that the derivative of $H_C := H + C$ at a stationary solution $f_e = (f_e^1, \ldots, f_e^s)$ equals

$$\frac{\partial H_C}{\partial f^s}(f_e) = \mathcal{H}_{f_e^s} + \Phi^{s\prime}(f_e^s) .$$

This vanishes if and only if

$$\mathcal{H}_{f_e^s} = -\Phi^{s\prime}(f_e^s)$$

for every s. But by (11EM)′, for stationary solutions we have

$$\frac{\partial \mathcal{H}_{f_e^s}}{\partial x} \cdot \frac{\partial f_e^s}{\partial v} = \frac{\partial f_e^s}{\partial x} \cdot \frac{\partial \mathcal{H}_{f_e^s}}{\partial v} .$$

A sufficient condition for this to hold is the functional relationship

$$\mathcal{H}_{f_e^s} = \Psi^s(f_e^s)$$

for every s. Thus we get the following.

Proposition. Stationary solutions f_e^s of the Poisson–Vlasov equation satisfying $\mathcal{H}_{f_e^s} = \Psi^s(f_e^s)$ for every s, are critical points $H_C := H + C$ where $\Phi^s(\xi) = -\int^{\xi} \Psi^s(u)\, du$. Conversely, critical points of H_C are stationary solutions.

Remark D. Formal stability. To compute $D^2H_C(f_e)\cdot(\delta f)^2$, we need the expression of the derivative of ϕ_f with respect to f. Since $\nabla^2\phi_f = \rho_f$ and

$$D\rho_f \cdot \delta f = \sum_s q^s \int \delta f^s(x, v)\, d^n v\,,$$

we have

$$D\phi_f \cdot \delta f = -\sum_s q^s \int \delta f^s(x, v)\, d^n v\,.$$

This formula and

$$DH(f)\cdot\delta f = \sum_s \left[\tfrac{1}{2}\int\int m^s|v|^2 \delta f^s(x, v)\, d^n x\, d^n v + q^s \int \phi_f(x)\left(\int \delta f^s(x, v)\, d^n v\right) d^n x\right]$$

yield

$$D^2H(f_e)\cdot(\delta f)^2 = \int D\phi_f\cdot\delta f\left(\sum_s q^s \int \delta f^s\, d^n v\right) d^n x$$

$$= -\int (\nabla^2)^{-1}\left(\sum_s q^s \int \delta f^s\, d^n v\right)\cdot\left(\sum_s q^s \int \delta f^s\, d^n v\right) d^n x\,.$$

Since ∇^2 is negative definite it follows that for perturbations $\delta f^s(x, v)$ such that $\sum_s q^s \int \delta f^s(x, v)\, d^n v \neq 0$, we have $D^2H(f_e)\cdot(\delta f)^2 > 0$.

Since

$$D^2C(f_e)\cdot(\delta f)^2 = \sum_s \int\int \Phi^{s\prime\prime}(f_e^s)(\delta f^s(x, v))^2\, d^n x\, d^n v\,,$$

the second variation D^2H_C will be positive definite if $\Phi^{s\prime\prime}(f_e^s) > 0$ for all s; that is, $\Psi^{s\prime}(f_e^s) < 0$ for all s. Taking the gradient of $\mathcal{H}_{f_e^s} = \Psi^s(f_e^s)$, it follows that $\Psi^{s\prime}(f_e^s) = \nabla\mathcal{H}_{f^s}/\nabla f_e^s$, provided $\nabla f_e^s \neq 0$ for all s. We have therefore proved the following stability result of Newcomb [1958], Gardner [1963] and Rosenbluth [1964]:

A stationary solution f_e *with* $\nabla f_e^s \neq 0$ *for all* s *and for which* $\mathcal{H}_{f_e^s}$ *is a function of* f_e *is formally (and hence linearly) stable, if* $\nabla\mathcal{H}_{f_e^s}/\nabla f_e^s < 0$ *for all* s.

D. *Convexity estimates*

Let $f := f_e + \Delta f$; then we have

$$\hat{H}(\Delta f) := H(f_e + \Delta f) - H(f_e) - \mathrm{D}H(f_e)\cdot\Delta f = \tfrac{1}{2}\int |\nabla\phi_{\Delta f}|^2\,\mathrm{d}^n x\,,$$

since $f \mapsto \phi_f$ is linear and the kinetic energy term of H is linear in f. Consequently, we can take

$$Q_1(\Delta f) = \tfrac{1}{2}\int |\nabla\phi_{\Delta f}|^2\,\mathrm{d}^n x\,,$$

and condition (CH) of section 2 holds. For (CC) we require

$$Q_2(\Delta f) \leq \int\int \sum_s \left(\Phi^s(f_e^s + \Delta f^s) - \Phi^s(f_e^s) - \Phi^{s\prime}(f_e^s)\Delta f^s\right)\mathrm{d}^n x\,\mathrm{d}^n v$$

which holds with

$$Q_2(\Delta f) = \tfrac{1}{2}a\int\int \sum_s (\Delta f^s)^2\,\mathrm{d}^n x\,\mathrm{d}^n v\,,$$

provided

$$\Phi^{s\prime\prime}(\zeta) \geq a \text{ for all s and all } \zeta.$$

Condition (D) requires

$$a > 0\,.$$

E. *A priori estimates*

We have the following a priori estimate on $\Delta f = f - f_e$:

$$Q_1(\Delta f(t)) + Q_2(\Delta f(t)) = \tfrac{1}{2}\int |\nabla\phi_{\Delta f}|^2\,\mathrm{d}^n x + \tfrac{1}{2}a\int\int \sum_s (\Delta f^s)^2\,\mathrm{d}^n x\,\mathrm{d}^n v$$
$$\leq H_C(f(0)) - H_C(f_e)\,.$$

F. *Nonlinear stability*

For $a > 0$ we set

$$\|\Delta f\|^2 = \tfrac{1}{2}\int |\nabla\phi_{\Delta f}|^2\,\mathrm{d}^n x + \tfrac{1}{2}a\int\int \sum_s (\Delta f^s)^2\,\mathrm{d}^n x\,\mathrm{d}^n v \qquad \text{(11N)}$$

(note that this norm is bounded below by the L^2-norm of f). Condition (CH)′ is satisfied. A sufficient condition for (CC)′ to hold is

$$\Phi^{s\prime\prime}(\zeta) \le A < +\infty,$$

for all ζ and all s, i.e.,

$$-\infty < -A \le \Psi^{s\prime}(f_e^s) = \nabla\mathcal{H}_{f_e^s} / \nabla f_e^s \le -a < 0$$

provided $\nabla f_e^s \neq 0$ for all s. We have proved the following.

Poisson–Vlasov stability theorem. Let f_e be a stationary solution of the Poisson–Vlasov equations. Assume that $\mathcal{H}_{f_e^s} = \Psi^s(f_e^s)$ for all s, and for a constant $a > 0$,

$$0 < a \le -\Psi^{s\prime}(\zeta) \tag{11SC$_1$}$$

for all ζ and s; i.e., that

$$\nabla\mathcal{H}_{f_e^s} / \nabla f_e^s \le -a < 0\,.$$

Then the a priori estimate (11E) holds, as long as C^1 solutions exist (which is automatic in one dimension).

If

$$-\Psi^{s\prime}(\zeta) \le A < \infty\,, \tag{11SC$_2$}$$

then the f_e is stable in the norm (11N).

Corollary. Let f_e be a spatially homogeneous spherically symmetric stationary solution of the one-species Poisson–Vlasov equation, i.e. $f_e(\boldsymbol{x}, \boldsymbol{v}) = g(|\boldsymbol{v}|)$ for all s, with g a real-valued C^2 function. Assume that $g'(\lambda) < 0$ for $\lambda > 0$ and $g''(0) < 0$. Then f_e satisfies the a priori estimates (E) of section 2 (as long as smooth solutions exist).

Proof. If f_e is a spatially homogeneous solution, it follows from Poisson's equation in (11EM) (with the charge modified to $q(\int f(x, v)\,\mathrm{d}v - 1)$) that ϕ_{f_e} is a constant which we can take to be zero. Since g is monotonically decreasing on the positive semiaxis, it is invertible. If $\zeta = g(|\boldsymbol{v}|)$, we have

$$\mathcal{H}_{f_e} = \tfrac{1}{2}m|\boldsymbol{v}|^2 + q\phi_{f_e} = \tfrac{1}{2}m(g^{-1}(\zeta))^2\,.$$

Therefore $\Psi(\zeta) = \frac{1}{2}((g)^{-1}(\zeta))^2$ satisfies $\mathcal{H}_{f_e} = \Psi(f_e)$ and

$$\Psi'(\zeta) = m\,\frac{g^{-1}(\zeta)}{g'(g^{-1}(\zeta))} = m\,\frac{|\boldsymbol{v}|}{g'(|\boldsymbol{v}|)} < 0\,.$$

Since $g''(0) < 0$, it follows that $\Psi'(\zeta) \le -a < 0$ for some $a > 0$, so we get the result. ∎

In the stability conditions $(11SC)_2$, the upper estimate on $-\Psi^{s\prime}$, and in this corollary, the behavior for large $\boldsymbol{v}$ is a serious problem. Indeed, while one has the a priori estimate (E) of section 2, the upper estimate (CC)′ is not possible. In fact, it is easy to see that in one dimension for the Maxwell distribution, where $\Phi(\zeta) = \zeta \log \zeta$, the functional C is not continuous in the L^2-norm at the Maxwellian $f_e = \text{const} \cdot \exp(-\sigma|v|^2)$, nor in the norm (11N) (consider perturbations of f_e that are like $1/v$ for large v). Thus, the full stability result one might like does not seem possible by these methods. More delicate estimates at large v are needed. The present results require control on a norm of the perturbations that is stronger than the norm guaranteed by the a priori estimates. Of course one can still interpret this as a nonlinear stability result, but it is not as clean as one would have hoped.

12. The Maxwell–Vlasov system

Here we extend the analysis given in section 11 to include electrodynamics rather than just electrostatics A number of previous investigations of formal stability cited in section 11 also apply to this case; see, for example, Gardner [1963].

A. *Equations of motion and Hamiltonian*

Consider a multispecies collisionless plasma consisting of particles with charges q^s and masses m^s, moving in 3-space $\mathbb{R}^3$, with positions $\boldsymbol{x}$ and velocities $\boldsymbol{v}$. We shall assume that the plasma densities $f^s(\boldsymbol{x}, \boldsymbol{v}, t)$ as well as the electric and magnetic fields $\boldsymbol{E}$ and $\boldsymbol{B}$ are either periodic in $\boldsymbol{x}$, or have asymptotic decay sufficient to justify integration by parts. We shall also assume that f^s decay to zero in the $\boldsymbol{v}$ variable at ∞ at a sufficient rate that makes all subsequent integrals convergent. (The weighted spaces of Cantor [1979] and Christodoulou [1981] may be appropriate here.) The Maxwell–Vlasov equations are:

$$\frac{\partial f^s}{\partial t} + \boldsymbol{v} \cdot \frac{\partial f^s}{\partial \boldsymbol{x}} + \frac{q^s}{m^s}\left(\boldsymbol{E} + \frac{\boldsymbol{v} \times \boldsymbol{B}}{c}\right) \cdot \frac{\partial f^s}{\partial \boldsymbol{v}} = 0\,, \qquad \frac{1}{c}\frac{\partial \boldsymbol{B}}{\partial t} = -\operatorname{curl} \boldsymbol{E}\,,$$

$$\frac{1}{c}\frac{\partial \boldsymbol{E}}{\partial t} = \operatorname{curl} \boldsymbol{B} = \sum_s \frac{q^s}{c} \int \boldsymbol{v} f^s(\boldsymbol{x}, \boldsymbol{v})\, \mathrm{d}^3 v\,, \qquad \operatorname{div} \boldsymbol{E} = \rho_f = \sum_s q^s \int f^s(\boldsymbol{x}, \boldsymbol{v})\, \mathrm{d}^3 v\,, \qquad \operatorname{div} \boldsymbol{B} = 0\,, \tag{12EM}$$

where we denote by $\boldsymbol{f}$ the vector (f^s) of plasma densities for every species. If we deal with only one species, we assume in addition that there is a constant background electrostatic field and replace the right-hand side of the equation $\operatorname{div} \boldsymbol{E} = \rho_f$ by $q(1 - \int f(\boldsymbol{x}, \boldsymbol{v})\, \mathrm{d}^3 v)$ where f satisfies $\int\int f(\boldsymbol{x}, \boldsymbol{v})\, \mathrm{d}^3 v\, \mathrm{d}^3 x = 1$.

The total energy of the system is the conserved Hamiltonian

$$H(f, \boldsymbol{E}, \boldsymbol{B}) = \tfrac{1}{2}\sum_s \int\int |\boldsymbol{v}|^2 f^s(\boldsymbol{x}, \boldsymbol{v})\, \mathrm{d}^3 x\, \mathrm{d}^3 v + \tfrac{1}{2}\int [|\boldsymbol{E}|^2 + |\boldsymbol{B}|^2]\, \mathrm{d}^3 x \tag{12H}$$

Remark A. The equations of motion (12EM) are Hamiltonian with respect to the following Poisson bracket found by reduction from Lagrangian to Eulerian coordinates by Marsden and Weinstein [1982], inspired by Morrison [1980]. (The bracket seems to have been first given by Iwinski and Turski [1976]):

$$\{F, G\}(f, \boldsymbol{E}, \boldsymbol{B}) = \sum_s \int\int \left[f^s \left\{ \frac{\delta F}{\delta f^s}, \frac{\delta G}{\delta f^s} \right\} + \left(\frac{\delta F}{\delta \boldsymbol{E}} \cdot \frac{\partial f}{\partial \boldsymbol{v}} \frac{\delta G}{\delta f^s} - \frac{\delta G}{\delta \boldsymbol{E}} \cdot \frac{\partial f}{\partial \boldsymbol{v}} \frac{\partial F}{\partial f^s} \right) \right.$$
$$\left. + f^s \boldsymbol{B} \cdot \left(\frac{\partial}{\partial \boldsymbol{v}} \frac{\delta F}{\delta f^s} \times \frac{\partial}{\partial \boldsymbol{v}} \frac{\delta G}{\delta f^s} \right) \right] \mathrm{d}^3x\, \mathrm{d}^3v + \int \left(\frac{\delta F}{\delta \boldsymbol{E}} \cdot \operatorname{curl} \frac{\delta G}{\delta \boldsymbol{B}} - \frac{\delta G}{\delta \boldsymbol{E}} \cdot \operatorname{curl} \frac{\delta F}{\delta \boldsymbol{B}} \right) \mathrm{d}^3x . \tag{12PB}$$

The first double integral in (12PB) represents a Lie–Poisson bracket on a semidirect product. This bracket is a canonical one in the quotient space of the cotangent bundle of a principal bundle by its structure group; for the general theory see Marsden, Ratiu and Weinstein [1984] and Montgomery, Marsden and Ratiu [1984].

B. *Constants of the motion*

A direct computation shows that the quantities

$$C^s(f) = \int\int \Phi^s(f^s(\boldsymbol{x}, \boldsymbol{v}))\, \mathrm{d}^3x\, \mathrm{d}^3v \tag{12C}$$

are conserved along trajectories of (12EM), for any smooth functions $\Phi^s : \mathbb{R} \to \mathbb{R}$.

Remark B. The functionals C^s are Casimirs for the Lie–Poisson bracket given by the *first* double integral in (12PB). When (12PB) is written in momentum representation they become Casimirs for the entire bracket; see Marsden and Weinstein [1982].

C. *First variation*

Let $H_C := H + C$, where $C := \sum_s C^s$. The derivative of H_C at a stationary solution $\boldsymbol{f}_e$, $\boldsymbol{E}_e$, $\boldsymbol{B}_e$ equals

$$\mathrm{D}H_C(\boldsymbol{f}_e, \boldsymbol{E}_e, \boldsymbol{B}_e) \cdot (\delta f, \delta \boldsymbol{E}, \delta \boldsymbol{B}) = \int\int \sum_s (\tfrac{1}{2}|\boldsymbol{v}|^2 + \Phi^{s\prime}(f^s_e(\boldsymbol{x}, \boldsymbol{v}))\delta f^s(\boldsymbol{x}, \boldsymbol{v})\, \mathrm{d}^3x\, \mathrm{d}^3v$$
$$+ \tfrac{1}{2} \int (\boldsymbol{E}_e \cdot \delta \boldsymbol{E} + \boldsymbol{B}_e \cdot \delta \boldsymbol{B})\, \mathrm{d}^3x . \tag{12FV1}$$

This vanishes if

$$\tfrac{1}{2}|\boldsymbol{v}|^2 + \Phi^{s\prime}(f^s_e(\boldsymbol{x}, \boldsymbol{v})) = 0 , \qquad \boldsymbol{E}_e = \boldsymbol{0} , \qquad \boldsymbol{B}_e = \boldsymbol{0} . \tag{12FV2}$$

Consequently we shall consider stationary solutions of the following kind:

$\boldsymbol{f}_e$ is spatially homogeneous, i.e., $\boldsymbol{f}_e$ is independent of $\boldsymbol{x}$;

$\boldsymbol{f}_e$ is spherically symmetric, i.e. $f^s_e(v) = g^s(|\boldsymbol{v}|)$ for all s, where g^s are some real-valued functions on $[0, +\infty)$.

Note that these conditions imply that the total current is $\Sigma_s q^s \int \boldsymbol{v} f^s_e(\boldsymbol{x}, \boldsymbol{v})\, \mathrm{d}^3v = \Sigma_s q^s \int \boldsymbol{v} g^s(|\boldsymbol{v}|)\, \mathrm{d}^3v = 0$. The second condition is satisfied, e.g., for Maxwellian density distributions.

For such stationary solutions the first variation of H_C vanishes if

$$\tfrac{1}{2}|\boldsymbol{v}|^2 + \Phi^{s\prime}(g^s(|\boldsymbol{v}|)) = 0 ,$$

so that, if each g^{s} is invertible, we get

$$\Phi^{s\prime}(\zeta)=\tfrac{1}{2}((g^{s})^{-1}(\zeta))^{2}.$$

Proposition. Spatially homogeneous spherically symmetric solutions of the form $f_{e}^{s}(\boldsymbol{v})=g^{s}(|\boldsymbol{v}|)$ for some invertible functions $g^{s}:[0,\infty)\to\mathbb{R}$ of the Maxwell–Vlasov equations with zero electromagnetic field are critical points of $H_C:=H+C$ provided

$$\Phi^{s}(\zeta)=\tfrac{1}{2}\int^{\zeta}(g^{s})^{-1}(u)^{2}\,\mathrm{d}u$$

for every s.

Remark D. Formal stability. The second variation of H_C at a stationary solution $(\boldsymbol{f}_e,\boldsymbol{0},\boldsymbol{0})$ considered above equals

$$\mathrm{D}^{2}H_C(\boldsymbol{f}_e,\boldsymbol{0},\boldsymbol{0})\cdot(\delta f,\delta\boldsymbol{E},\delta\boldsymbol{B})^{2}=\sum_{s}\int\int\Phi^{s\prime\prime}(f_{e}^{s})(\delta f^{s})\,\mathrm{d}^{3}x\,\mathrm{d}^{3}v+\tfrac{1}{2}\int(|\delta\boldsymbol{E}|^{2}+|\delta\boldsymbol{B}|^{2})\,\mathrm{d}^{3}x.$$

This is positive definite if $\Phi^{s\prime\prime}(g_{e}^{s})>0$ for all s. Taking the gradient with respect to $\boldsymbol{v}$ of the defining relation for Φ^{s},

$$\tfrac{1}{2}|\boldsymbol{v}|^{2}+\Phi^{s\prime}(f_{e}^{s}(\boldsymbol{v}))=0,$$

it follows that

$$\boldsymbol{v}+\Phi^{s\prime\prime}(f_{e}^{s})\partial f_{e}^{s}/\partial\boldsymbol{v}=0,$$

i.e., for $\partial f_{e}^{s}/\partial\boldsymbol{v}\neq0$ we get $\Phi^{s\prime\prime}(f_e)=-\boldsymbol{v}/(\partial f_{e}^{s}/\partial\boldsymbol{v})$. We have proved the following result (cf. Gardner [1963]);

A stationary solution of the Maxwell–Vlasov equations as in the above proposition with $\partial f_{e}^{s}/\partial\boldsymbol{v}\neq0$ is formally and hence linearly stable if $\boldsymbol{v}/(\partial f_{e}^{s}/\partial\boldsymbol{v})<0$.

D. *Convexity estimates*

Let $(\boldsymbol{f}_e,\boldsymbol{0},\boldsymbol{0})$ be a stationary solution, as in the Proposition. We choose

$$Q_1(\Delta\boldsymbol{E},\Delta\boldsymbol{B})=\tfrac{1}{2}\int(|\Delta\boldsymbol{E}|^{2}+|\Delta\boldsymbol{B}|^{2})\,\mathrm{d}^{3}x,$$

so that we have

$$\hat{H}(\Delta f,\Delta\boldsymbol{E},\Delta\boldsymbol{B}):=H(\boldsymbol{f}_e+\Delta f,\Delta\boldsymbol{E},\Delta\boldsymbol{B})-H(\boldsymbol{f}_e,\boldsymbol{0},\boldsymbol{0})-\mathrm{D}H(\boldsymbol{f}_e,\boldsymbol{0},\boldsymbol{0})\cdot(\Delta f,\Delta\boldsymbol{E},\Delta\boldsymbol{B})=Q_1(\Delta\boldsymbol{E},\Delta\boldsymbol{B}).\tag{12CH}$$

Similarly,

$$\hat{C}(\Delta f, \Delta \boldsymbol{E}, \Delta \boldsymbol{B}) := C(f_e + \Delta f, \Delta \boldsymbol{E}, \Delta \boldsymbol{B}) - C(f_e, \boldsymbol{0}, \boldsymbol{0}) - \mathrm{D}C(f_e, \boldsymbol{0}, \boldsymbol{0}) \cdot (\delta f, \Delta \boldsymbol{E}, \Delta \boldsymbol{B})$$
$$= \sum_s \int\int [\Phi^s(f_e^s + \Delta f^s) - \Phi^s(f_e^s) - \Phi^{s\prime}(f_e^s)\Delta f^s]\, \mathrm{d}^3x\, \mathrm{d}^3v \geq Q_2(\Delta f), \tag{12CC}$$

where

$$Q_2(\Delta f) = \tfrac{1}{2} a \sum_s \int\int |\Delta f^s(\boldsymbol{x}, \boldsymbol{v})|^2\, \mathrm{d}^3x\, \mathrm{d}^3v,$$

provided

$$\Phi^{s\prime\prime}(\zeta) \geq a, \quad \text{for all} \quad \zeta \quad \text{and all} \quad s.$$

Then

$$Q_1(\Delta \boldsymbol{E}, \Delta \boldsymbol{B}) + Q_2(\Delta f) > 0 \tag{12D}$$

holds if $a > 0$.

E. A priori estimates

In the hypotheses of the previous step we have the following estimate:

$$Q_1(\Delta \boldsymbol{E}, (t), \Delta \boldsymbol{B}(t) + Q_2(\Delta f(t)) \leq H_C(f(t)) - H_C(f_e), \tag{12E}$$

where $f = f_e + \Delta f$.

F. *Nonlinear stability*

Let

$$\|(\Delta f, \Delta \boldsymbol{E}, \Delta \boldsymbol{B})\|^2 = Q_1(\Delta \boldsymbol{E}, \Delta \boldsymbol{B}) + Q_2(\Delta f) = \tfrac{1}{2}\|(\Delta f, \Delta \boldsymbol{E}, \Delta \boldsymbol{B})\|_{L^2}^2 \tag{12N}$$

so that (12E) gives an a priori estimate in this norm. However, as in section 11, H_C is not usually continuous in this norm at the solution $(f_e, \boldsymbol{0}, \boldsymbol{0})$, again for technical reasons involving the behavior at large velocities.

Maxwell–Vlasov stability theorem. Let $(f_e, \boldsymbol{0}, \boldsymbol{0})$ be a spatially homogeneous, spherically symmetric solution of the Maxwell–Vlasov equations with $f_e(\boldsymbol{v}) = g^s(|\boldsymbol{v}|)$ where g^s is invertible. If $\partial f_e^s/\partial \boldsymbol{v} \neq 0$ and if

$$\boldsymbol{v}/(\partial f_e^s/\partial \boldsymbol{v}) \leq -a < 0,$$

then the solution $(f_e, \boldsymbol{0}, \boldsymbol{0})$ satisfies the a priori estimate (12E) which limits the growth of perturbations from equilibrium.

Appendix A. The linearized equations

In this appendix, we show that the equations of motion linearized about an equilibrium solution of a Lie–Poisson system are Hamiltonian with respect to a constant coefficient Lie–Poisson bracket. The Hamiltonian for these linearized equations is $\delta^2 H_C(\mu_e)$, which is the quadratic functional obtained by taking the second derivative of the Hamiltonian plus conserved quantities for the nonlinear equations, when evaluated at the equilibrium solution μ_e (where the first variation of H_C vanishes). An immediate consequence is that the linearized dynamics preserves $\delta^2 H_C(\mu_e)$. We will also show that formal stability of the equilibrium solution implies its linearized stability. Finally, the Rayleigh equation will be derived, using this Hamiltonian formalism in the example of ideal planar incompressible flow. The Rayleigh equation is a linearized fluid perturbation equation, whose spectrum will be compared to the condition for positivity of the second variation.

A1. *The Hamiltonian structure of linearized Lie–Poisson equations*

From Marsden et al. [1983] for example, we recall that for a Lie algebra $\mathfrak{G}$, the Lie–Poisson bracket is defined on $\mathfrak{G}^*$, a space paired with $\mathfrak{G}$ by a weakly nondegenerate pairing $\langle\ ,\ \rangle$ between $\mathfrak{G}^*$ and $\mathfrak{G}$, by

$$\{F, G\}(\mu) = \left\langle \mu, \left[\frac{\delta F}{\delta \mu}, \frac{\delta F}{\delta \mu}\right]\right\rangle , \tag{A1}$$

where $\mu \in \mathfrak{G}^*$ and $\delta F/\delta\mu \in \mathfrak{G}$ is determined by

$$\mathrm{D}F(\mu)\cdot\delta\mu = \left\langle \frac{\delta F}{\delta\mu}, \delta\mu\right\rangle , \tag{A2}$$

for any $\delta\mu \in \mathfrak{G}^*$, when such an element $\delta F/\delta\mu$ exists. The equations of motion $\dot{F} = \{F, H\}$ are equivalent to

$$\frac{\mathrm{d}\mu}{\mathrm{d}t} = -\mathrm{ad}\left(\frac{\delta H}{\delta\mu}\right)^{*}\mu , \tag{A3}$$

where $H:\mathfrak{G}^*\to\mathbb{R}$ is the Hamiltonian and $\mathrm{ad}(\xi):\mathfrak{G}\to\mathfrak{G}$ is the adjoint action $\mathrm{ad}(\xi)\cdot\eta = [\xi, \eta]$ for any $\eta\in\mathfrak{G}$; $\mathrm{ad}(\xi)^*:\mathfrak{G}^*\to\mathfrak{G}^*$ is the dual of $\mathrm{ad}(\xi)$. Let μ_e be an equilibrium solution of (A3). The linearized equation of (A3) at μ_e is obtained by expanding all quantities in a Taylor expansion with small parameter ε and taking $\mathrm{d}/\mathrm{d}\varepsilon|_{\varepsilon=0}$ of the resulting equations. For $\mu = \mu_e + \varepsilon\delta\mu$, using Taylor's theorem gives

$$\frac{\delta H}{\delta\mu} = \frac{\delta H}{\delta\mu_e} \times \varepsilon\mathrm{D}\left(\frac{\delta H}{\delta\mu}\right)(\mu_e)\cdot\delta\mu + \mathrm{O}(\varepsilon^2) . \tag{A4}$$

The derivative $\mathrm{D}(\delta H/\delta\mu)(\mu_e)\cdot\delta\mu$ is the linear functional

$$\nu\in\mathfrak{G}^*\mapsto \mathrm{D}^2 H(\mu_e)\cdot(\delta\mu, \nu)\in\mathbb{R} , \tag{A5}$$

using the definition and (A2). Since $\delta^2 H = D^2H(\mu_e)\cdot(\delta\mu, \delta\mu)$, it follows that the functional (A5) equals $\frac{1}{2}\delta(\delta^2 H)/\delta(\delta\mu)$. Consequently (A4) becomes

$$\frac{\delta H}{\delta\mu} = \frac{\delta H}{\delta\mu_e} + \frac{\varepsilon}{2}\frac{\delta(\delta^2 H)}{\delta(\delta\mu)} + O(\varepsilon^2), \tag{A6}$$

and the Lie–Poisson equations (A3) yield

$$\frac{d\mu_e}{dt} + \varepsilon\frac{d(\delta\mu)}{dt} = -\mathrm{ad}\left(\frac{\delta H}{\delta\mu_e}\right)^*\mu_e - \varepsilon\left[\tfrac{1}{2}\mathrm{ad}\left(\frac{\delta(\delta^2 H)}{\delta(\delta\mu)}\right)^*\mu_e + \mathrm{ad}\left(\frac{\delta H}{\delta\mu_e}\right)^*\delta\mu\right] + O(\varepsilon^2).$$

Thus, the linearized equations are

$$\frac{d(\delta\mu)}{dt} = -\tfrac{1}{2}\mathrm{ad}\left(\frac{\delta(\delta^2 H)}{\delta(\delta\mu)}\right)^*\mu_e - \mathrm{ad}\left(\frac{\delta H}{\delta\mu_e}\right)^*\delta\mu. \tag{A7}$$

Letting C be a conserved functional* for (A3) satisfying $\delta(H + C)/\delta\mu_e = 0$ and replacing H in (A7) by $H_C := H + C$ we get

$$\frac{d(\delta\mu)}{dt} = -\tfrac{1}{2}\mathrm{ad}\left(\frac{\delta(\delta^2 H_C)}{\delta(\delta\mu)}\right)^*\mu_e. \tag{A8}$$

This equation is Hamiltonian with respect to the Poisson bracket

$$\{F, G\}(\mu) = \left\langle\mu_e, \left[\frac{\delta F}{\delta\mu}, \frac{\delta G}{\delta\mu}\right]\right\rangle. \tag{A9}$$

That this bracket satisfies the Jacobi identity is readily checked. (See, for example, Guillemin and Sternberg [1983].) The Poisson bracket (A9) differs from the Lie–Poisson bracket in that it is *constant* in the argument. In fact, as shown in Ratiu [1982], the Poisson bracket (A9) is also Lie–Poisson, but on a Poisson submanifold of the dual of a Lie subalgebra of the loop algebra defined by $\mathfrak{G}$. With respect to (A9), Hamilton's equations given by $\delta^2 H_C$ are (A8), as a verification shows.

Finally, note that if $\delta^2 H_C$ is definite, i.e., either $\delta^2 H_C$ or $-\delta^2 H_C$ is positive definite, it defines a norm on the space of perturbations $\delta\mu$. Being the Hamiltonian function for (A8), $\delta^2 H_C$ is conserved. Thus, any solution of (A8) starting on an energy surface of $\delta^2 H_C$, i.e. on a sphere in this norm, stays on it and, hence, the zero solution of (A8) is Liapunov stable. Thus, *formal stability* (i.e. $\delta^2 H_C$ *definite*) *implies linearized* (*and hence spectral*) *stability.*

It should be noted, however, that the conditions for definiteness of $\delta^2 H_C$ are different from the conditions for spectral stability i.e. that the operator acting on $\delta\mu$ given by the right-hand side of (A8) have purely imaginary spectrum ("normal mode stability"). In particular, as noted already in section 1, having purely imaginary spectrum for the linearized equations does *not* produce Liapunov stability of the linearized equations in general.

We shall now make explicit the difference between $\delta^2 H_C$ and the operator in (A8). Assume that the pairing $\langle\,,\rangle$ identifies $\mathfrak{G}^*$ with $\mathfrak{G}$ itself, i.e. $\langle\,,\rangle$ is a symmetric pairing on $\mathfrak{G}$. Then

* Such a C may not exist if μ_e lies on a singular symplectic leaf (see appendix B).

$$\tfrac{1}{2}D^2 H_G \cdot (\delta\mu, \delta\nu) = \langle \delta\mu, L\delta\nu \rangle \,, \tag{A.10}$$

defines a linear operator $L: \mathfrak{G} \to \mathfrak{G}$ symmetric with respect to $\langle\,,\rangle$, i.e., $\langle \alpha, L\beta \rangle = \langle L\alpha, \beta \rangle$ for all α, $\beta \in \mathfrak{G}$. Then the linear operator in (A8) becomes

$$\delta\mu \mapsto [L\delta\mu, \mu_e] \,, \tag{A.11}$$

which, of course, differs from L in general. However, note that the kernel of L is included in the kernel of the linear operator (A.11), i.e. the zero eigenvalue of L gives rise to "neutral modes" in the spectral analysis of (A.11).

A2. *The Rayleigh equation*

We shall now derive the Rayleigh equation for linearized incompressible planar parallel shear flow from the Hamiltonian formalism described in A1. We shall assume, for simplicity, that $D \subset \mathbb{R}^2$ is simply connected, so that the equations of motion

$$\frac{\partial \omega}{\partial t} = \{\omega, \psi\} \,, \qquad -\nabla^2 \psi = \omega \,, \tag{A.12}$$

are Hamiltonian with total energy

$$H(\omega) = \tfrac{1}{2} \int |\nabla\psi|^2 \, dx \, dy \,, = -\tfrac{1}{2} \int \omega (\nabla^2)^{-1} \omega \, d\omega \, dy \,, \tag{A.13}$$

with respect to the Lie–Poisson bracket (see the cautionary remarks in section 3.3)

$$\{F, G\}(\omega) = \int_D \omega \left\{ \frac{\delta F}{\delta \omega}, \frac{\delta F}{\delta \omega} \right\} dx \, dy \,. \tag{A.14}$$

The Casimir functions are

$$C_\Phi(\omega) = \int_D \Phi(\omega) \, dx \, dy \,. \tag{A15}$$

As we saw in section 3.3, the equilibrium conditions

$$\psi_e = \Psi(\omega_e) \tag{A16}$$

imply the vanishing of the first variation of $H_C(\omega) := H(\omega) + C_\Phi(\omega) + \lambda \int_D \omega \, dx \, dy$, provided that

$$\Phi' = -\Psi \,, \quad \text{and} \quad \lambda = \Phi(\omega_e | \partial D) \,. \tag{A17}$$

Now let D be a strip of finite height in the xy plane, and impose either x-periodic boundary conditions on the velocity, or sufficiently rapid decay of the velocity at infinity in the x-direction. Let $\psi_e = \bar{\psi}(y)$ be the parallel shear flow solution of the equation (A.12) and let $v_e(U(y), 0)$, with $U(y) = \bar{\psi}'(y)$. The second variation of H_C equals

$$\delta^2 H_C = \int_D \left[\delta\omega(-\nabla^2)^{-1}\delta\omega + \frac{U(y)}{U''(y)}(\delta\omega)^2\right] dx\, dy$$

so that the operator L is given by

$$L = -(\nabla^2)^{-1} + U(y)/U''(y)\,. \tag{A18}$$

Let $\phi = -(\nabla^2)^{-1}\delta\omega$ denote the perturbed stream function. Then the eigenvalue probem for L can be written as

$$(-\nabla^2\phi)\left(\frac{U(y)}{U''(y)} - \lambda\right) + \phi = 0\,, \quad \text{or} \quad -\nabla^2\phi + \frac{U''(y)}{U(y) - \lambda U''(y)}\phi = 0\,. \tag{A19}$$

Now set $\phi(x, y) = \exp[\mathrm{i}k(x - ct)]\chi(y)$ to get

$$(k^2 - \partial_y^2)\chi(y) + \frac{U''(y)}{U(y) - \lambda U''(y)}\chi(y) = 0\,. \tag{A20}$$

We shall compare this equation with the Rayleigh equation, obtained from the operator L in (A18) via the formula (A11). We have

$$\begin{aligned}\{L\delta\omega, \omega_e\} &= \left\{-(\nabla^2)^{-1}\delta\omega + \frac{U(y)}{U''(y)}\delta\omega, U'(y)\right\} \\ &= \left\{\phi - \frac{U(y)}{U''(y)}\nabla^2\phi, U'(y)\right\} \\ &= \left(\partial_x\phi - \frac{U(y)}{U''(y)}\nabla^2\phi_x\right)U''(y)\,,\end{aligned}$$

so that the linearized equations are

$$\nabla^2\frac{\partial\phi}{\partial t} = (\partial_x\phi)U''(y) - U(y)\nabla^2\partial_x\phi\,. \tag{A21}$$

Now set as before $\phi(x, y) = \exp[\mathrm{i}k(x - ct)]\chi(y)$ to get, after a few manipulations from (A21),

$$\left[(k^2 - \partial_y^2) + \frac{U''(y)}{U(y) - c}\right]\chi(y) = 0\,. \tag{A22}$$

This is the Rayleigh equation. In this case, the eigenvalue problem for the second variation and the linearized equations are remarkably similar. In fact the zero eigenvalue ($\lambda = 0$) case of (A20) coincides with the normal mode ($c = 0$) case of (A22). This is not true in general; see Abarbanel et al. [1984] for a discussion of the Taylor–Goldstein equation for linearized incompressible ideal stratified planar shear flow in this context.

A3. *Bernstein's logarithmic convexity result* (Bernstein [1983])

We now want to show that in some circumstances, indefiniteness of the second variation of the potential energy implies instability of the linearized equations. The argument as formulated below is given for linear equations second order in time, so it applies to the linearization of systems in Lagrangian coordinates at a static equilibrium.

In Hilbert space $\mathscr{H}$ we consider an equation

$$\ddot{u} = Au$$

where A is a self-adjoint operator. As in Marsden and Hughes [1983], this is Hamiltonian with energy

$$H(u, \dot{u}) = \tfrac{1}{2}\|\dot{u}\|^2 - \tfrac{1}{2}\langle u, Au\rangle, \quad \text{where } \|\dot{u}\|^2 = \langle \dot{u}, \dot{u}\rangle .$$

Let

$$I = \tfrac{1}{2}\langle u, u\rangle .$$

One computes that

$$\frac{d^2}{dt^2}(\log I) = \frac{1}{I}\left(\ddot{I} - \frac{\dot{I}^2}{I}\right) \geq \frac{1}{I}(\ddot{I} - 2\|\dot{u}\|^2)$$

by the Schwarz inequality. Thus

$$(d^2/dt^2)(\log I) \geq -2H/I .$$

Suppose we look at the static equilibrium satisfying $\dot{u}_e = 0$ and $Au_e = 0$. Suppose there is an initial condition $(u(0), \dot{u}(0))$ such that

$$H(u(0), \dot{u}(0)) = -\lambda < 0 .$$

Then

$$d^2 \log I/dt^2 \geq 2\lambda/I > 0 .$$

Using elementary differential inequalities one finds

$$I \geq I_0 + e^t(\langle u(0), \dot{u}(0)\rangle + 2\lambda) - 2\lambda t,$$

so I grows exponentially with t, and one concludes instability of $(u_e, \dot{u}_e)$.

Appendix B. Symplectic leaves and Casimirs

There is a limitation of practical importance to the energy–Casimir method that is geometric in nature. To understand it, we recall first some facts from the theory of finite dimensional semisimple Lie algebras. If $\mathfrak{G}$ is semisimple, the dual $\mathfrak{G}^*$ is identified with $\mathfrak{G}$ via the Killing form and thus the ad* and ad actions are also identified. The polynomial Casimirs of $\mathfrak{G}$ are generated by rank ($\mathfrak{G}$) (= dimension of a Cartan subalgebra of $\mathfrak{G}$) functionally independent homogeneous polynomials on $\mathfrak{G}$ which generate a ring called the *ring of invariants of* $\mathfrak{G}$. Every generic adjoint orbit is characterized by rank ($\mathfrak{G}$) values of the basis of the ring of invariants; a generic adjoint orbit is an adjoint orbit through a regular semisimple element of $\mathfrak{G}$. Thus, the tangent space to a generic adjoint orbit at $x \in \mathfrak{G}$ coincides with $\ker_x\{C\} := \{v \in \mathfrak{G} | DC(x) \cdot v = 0$ for all Casimirs C of $\mathfrak{G}\}$. But the generic adjoint orbits, which are maximal dimensional, form only an open dense subset of $\mathfrak{G}$, so that lower dimensional orbits are distinguished by additional functions on $\mathfrak{G}$ which commute only on the manifold of lower dimensional orbits. Motivated by these facts, we define a *regular symplectic leaf* S of a Poisson manifold P to be a submanifold S of P satisfying

$$\ker_x\{C\} := \{v \in T_x\mathrm{P} | T_xC(v) = 0 \quad \text{for all Casimirs} \quad C\} = T_x\mathrm{S}.$$

The union of all regular symplectic leaves forms the open set R of regular points of P. The set P\R is called the set of *singular points* of P. Note that for *any* point $x \in \mathrm{P}$ we have $T_x\mathrm{S} \subset \ker_x\{C\}$ where S is the symplectic leaf through x, equality holding if and only if x is regular. If Q is a subset of P, a function $K: \mathrm{P} \to \mathbb{R}$ is called a *sub-Casimir* for Q if $\{\bar{K}, G\}$ is zero on Q for every (smooth) extension $\bar{K}$ of K to P and for every function G on P.

For example, the orbits in the dual of divergence free vector fields on the domain $D \subset \mathbb{R}^2$ formed by point vortices, vortex filaments and vortex patches are irregular orbits (Marsden and Weinstein [1983]). The strengths of the individual vortices are sub-Casimirs on the manifold of point vortices. This brings us to a practical limitation of the energy-Casimir stability method. If the equilibrium solution happens to lie on an irregular leaf, to characterize it as a critical point, one needs to know the sub-Casimirs of that leaf. If this characterization is not feasible, to prove stability other direct estimates are needed (as in Wan [1984], Wan and Pulvirente [1984], Tang (1984) and Weinstein [1984]).

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