Physica 9D (1983) 300–323
North-Holland Publishing Company

KAC–MOODY LIE ALGEBRAS AND SOLITON EQUATIONS

II. LAX EQUATIONS ASSOCIATED WITH $A_1^{(1)}$

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Received 29 November 1982
Revised 20 June 1983

The soliton equations associated with sl(2) eigenvalue problems polynomial in the eigenvalue parameter are given a unified treatment; they are shown to be generated by a single family of commuting Hamiltonians on a subalgebra of the loop algebra of sl(2). The conserved densities and fluxes of the usual ANKS hierarchy are identified with conserved densities and fluxes for the polynomial eigenvalue problems. The Hamiltonian structures of the soliton equations associated with the polynomial eigenvalue problems are given a unified treatment.

1. Introduction

1.1. *Background*

This paper is concerned with the role of Lie algebras in soliton theory. We concentrate on the soliton equations associated with the AKNS eigenvalue problem

$$V_x(x,\zeta)=\left[\begin{pmatrix}-\mathrm{i} & 0\\ 0 & \mathrm{i}\end{pmatrix}\zeta+\begin{pmatrix}0 & q(x)\\ r(x) & 0\end{pmatrix}\right]V(x,\zeta) \tag{1}$$

and on their relationship with the "loop algebra" of formal series $\Sigma X_k\zeta^{-k}$, $X_k\in \mathrm{sl}(2,\mathbb{C})$. Our results, which are new even for this familiar example, will be described shortly. First, we will explain the motivation for our study.

There had been a long-standing belief that Lie algebras are important in soliton theory. Such results as could be obtained in the early days generally reflected little more than the fact that, for instance, the matrices in (1) belong to sl(2) with x-dependent coefficients; the deeper theory of Lie algebras was not brought in. The situation has changed dramatically in the last two or three years, and the fundamental role of Lie algebras and their representations in soliton theory is now appreciated.

We find it useful to classify the current Lie-algebraic explanations of soliton phenomena into two (admittedly oversimplified) types.

In "type I", the point of view is that soliton equations are evolution equations for functions of x. Lie algebras arise because one looks at matrix-functions of x, and they play the role of phase space for the evolution equations. These explanations have proved to be very successful in classifying Lax equations $\dot{L}=[B,L]$ for scalar operators, and in unravelling the Hamiltonian structure (in a functional-derivative sense) of those equations. See [1, 2, 3].

The "type II" explanations – at least when they introduce Lie algebras – do not single out the variable x, and indeed do not deal with evolution equations. The basic object is the τ-function, alias Hirota's new dependent variable [4]. It is a function of infinitely many variables: the original x, and infinitely many time variables, one for each equation in the soliton hierarchy. The phase space, such as it is, consists of τ-functions, and as far as is

* Supported in part by DOA Contract DAAG29-82-K-0068 and NSF grant MCS-81-02748A01.
** Supported in part by DOA Contract DAAG29-82-K-0068, NSF grant MCS75-07548A01, and ONR Contract N0014-76-C-0867.
*** Supported in part by DOA Contract DAAG29-82-K-0068 and NSF grant MCS81-06142.

0167-2789/83/0000–0000/$03.00

known, is not equipped with a Poisson bracket. All these variables play similar roles. The Lie algebra, rather than providing the phase space, now acts on it as an algebra of symmetries. In going from type I to type II, one moves from a Lie algebra to a representation space. This kind of explanation makes sense out of Hirota's method and Bäcklund transformations. See the original work in [5].

It seemed puzzling to us that there were two different manifestations of the same Lie algebra. The ideal theory of soliton equations, therefore, should have the capacity to meld these two points of view. Furthermore, it should also permit one to deduce all the miracles of soliton theory completely systematically from a single starting point. Amongst the more spectacular of these miracles we count:

1) nontriviality of the Wahlquist–Estabrook prolongation;
2) existence of infinitely many local conservation laws;
3) the auxiliary spectral problem;
4) existence of a natural Hamiltonian structure;
5) Bäcklund and Miura transformations;
6) Hirota bilinearization;
7) the Painlevé property.

Our overall project is to incorporate all these features into a systematic and coherent picture. We still have a way to go before realizing our goals, but we do have results we find interesting, and we believe that our point of view will turn out to be correct for a comprehensive soliton theory. In this paper, we will explain our point of view and describe new results that give validity to our approach independently of the wider framework into which we hope to place it eventually.

The precise results obtained in this paper are summarized next. After that, at the end of this introduction, we indulge in a little speculation and explain what we think might happen in the future.

It should be made clear once and for all that we are interested in the *formal properties* of soliton equations. That means: we deal with symbols subject to certain algebraic rules. For example, we will use the language of *differential algebras*. Those consist of symbols such as q, r, q_x, r_x, q_{xx}, ... that may be added and multiplied, and multiplied by constants, together with an operator $\mathrm{d}/\mathrm{d}x$ that takes q to q_x, q_x to q_{xx}, etc. This concept legitimizes the formal manipulations that are so useful in soliton theory. Another structure we need is the Lie algebra of formal series $\Sigma X_k\zeta^{-k}$, $X_k \in \mathrm{sl}(2, \mathbb{C})$. Convergence is not required. Throughout, it may be convenient to use language that sounds analytical, but no analytical properties are implied, and none are true without severe supplemental hypotheses [6, 7].

1.2. *Results*

Hoping to establish a direct connection between the type I and type II explanations mentioned above, we decided to reformulate the theory of AKNS soliton equations, as follows. The unknowns, $q(x)$ and $r(x)$, are usually taken to be functions of x, *prescribed arbitrarily* at time zero. We think of them instead as *determined* by a system of ordinary differential equations, in x, in infinitely many unknowns (rather like an analytic function of x is determined by all its derivatives at $x = 0$). The time-dependence is also given by a system of ordinary differential equations, in t. The use of this technique in formal soliton theory is new, as are the results we obtain from it, but a similar idea forms the basis for the Deift–Trubowitz study of KdV [8]. For them, the x-dependence of the initial $q(x)$ is (rigorously) determined by a Neumann oscillator system. We take it to be given formally by a kind of stationary equation. This will be explained presently.

We begin with a short review of the standard approach to the AKNS system (1).

The theory starts with

$$V_x = \begin{pmatrix} -\mathrm{i} & \\ & \mathrm{i} \end{pmatrix}\zeta + \begin{pmatrix} & q \\ r & \end{pmatrix} V \underset{\mathrm{def}}{=} PV. \qquad (1)$$

ζ is the eigenvalue*; q, r are functions of x, the

* The unfortunate use of $\begin{pmatrix} -\mathrm{i} & \\ & \mathrm{i}\end{pmatrix}$ in place of the more sensible $\begin{pmatrix} 1 & \\ & -1\end{pmatrix}$ is a decade-long habit with some of us. The continuous spectrum of (1), in inverse scattering analyses, will lie on the real axis with our convention.

"potentials." Adjoin to (1) a time-evolution for V,

$$V_t = Q^{(N)} V, \tag{2}$$

where $Q^{(N)} = Q_0\zeta^N + Q_1\zeta^{N-1} + \cdots + Q_N$ is polynomial in ζ, with coefficients being trace zero (i.e., sl(2)) matrices of polynomials in q, r and their x-derivatives.

$V_{xt} = V_{tx}$ implies

$$P_t - Q_x^{(N)} + [P, Q^{(N)}] = 0. \tag{3}$$

By setting the coefficient of each power of ζ equal to zero, one gets from (3) ($P_0 = \begin{pmatrix} -\mathrm{i} & \\ & \mathrm{i}\end{pmatrix}$, $P_1 = \begin{pmatrix} & q \\ r & \end{pmatrix}$):

$$\zeta^{N+1}:\quad [P_0, Q_0] = 0, \tag{4$_{N+1}$}$$

$$\zeta^{N}\ :\quad [P_0, Q_1] + [P_1, Q_0] = Q_{0_x}, \tag{4$_N$}$$

$$\zeta^{N-k}:\quad [P_0, Q_{k+1}] + [P_1, Q_k] = Q_{k_x} \quad (k < N), \tag{4$_{N-k}$}$$

$$\zeta^{0}\ :\quad P_{1_t} - Q_{N_x} + [P_1, Q_N] = 0. \tag{4$_0$}$$

From (4), one finds $Q_0, \ldots, Q_N$ recursively. The ζ^0-equation is then a p.d.e., q_t = stuff, r_t = stuff, where "stuff" means a polynomial in q, r and their x derivatives. Those are the soliton p.d.e.'s solved by the inverse scattering theory for (1).

There are many solutions Q_j of (4), differing from each other by integration constants. The standard normalization is defined as follows: assign weight 1 to q, r and each differentiation with respect to x. Then require that Q_j be homogeneous of weight j. The first few Q_j are

$$Q_0 = \begin{pmatrix} -\mathrm{i} & \\ & \mathrm{i}\end{pmatrix} (= P_0), \quad Q_1 = \begin{pmatrix} & q \\ r & \end{pmatrix} (= P_1),$$

$$Q_2 = \begin{pmatrix} -(\mathrm{i}/2)qr & (\mathrm{i}/2)q_x \\ -(\mathrm{i}/2)r_x & (\mathrm{i}/2)qr \end{pmatrix},$$

$$Q_3 = \begin{pmatrix} -\frac{1}{4}(qr_x - rq_x) & -\frac{1}{4}q_{xx} + \frac{1}{2}q^2 r \\ -\frac{1}{4}r_{xx} + \frac{1}{2}qr^2 & \frac{1}{4}(qr_x - rq_x) \end{pmatrix}.$$

For later reference, we list the p.d.e.'s $(4)_0$ corresponding to $N = 0, 1, 2, 3$ in (2) (it is best to use a different symbol for the time variable in each, since eventually we will want to treat these equations simultaneously):

$(N=0)$ $q_{t_0} = -2\mathrm{i}q, \quad r_{t_0} = 2\mathrm{i}r$
(scaling of q, r);

$(N=1)$ $q_{t_1} = q_x, \quad r_{t_1} = r_x$
(translation $x \mapsto x + t_1$);

$(N=2)$ $q_{t_2} = \frac{\mathrm{i}}{2} q_{xx} - \mathrm{i}q^2 r,$

$r_{t_2} = -\frac{\mathrm{i}}{2} r_{xx} + \mathrm{i}qr^2$

(nonlinear Schrödinger);

$(N=3)$ $q_{t_3} = -\frac{1}{4} q_{xxx} + \frac{3}{2} qq_x r,$

$r_{t_3} = \frac{1}{4} r_{xxx} + \frac{3}{2} qrr_x$ (modified KdV),

In the usual formal theory, one now proves the following theorems:

(1) All equations $P_{t_N} = Q_x^{(N)} + [Q^{(N)}, P]$ commute, i.e., $P_{t_N t_M} = P_{t_M t_N}$.

(2) All these equations are Hamiltonian, i.e.,

$$q_{t_N} = \frac{\delta \mathscr{F}_N}{\delta r}, \quad r_{t_N} = -\frac{\delta \mathscr{F}_N}{\delta q},$$

where $\mathscr{F}_N$ is a certain functional of q, r, and δ/δ denotes the functional (or "Fréchet", or "variational") derivative. E.g., for $N = 2$, $\mathscr{F}_2 = \mathrm{i}/6 \int (r_{xx}q + q_{xx}r - q_x r_x - 3r^2 q^2)\,\mathrm{d}x$.

(3) The $\mathscr{F}_N$ are integrals $\int F_N$ of certain polynomials in q, r and x-derivatives, and there is a relation of the form

$$\frac{\partial F_N}{\partial t_j} = \frac{\partial G_{Nj}}{\partial x},$$

called a *conservation law*. The G_{Nj} are again differential polynomials. These results are usually proved via formal asymptotic expansions for the eigenfunction V of (1) and/or a symbolic calculus for the inverse of d/dx.

We will get those results, *and the analogous statements for any eigenvalue problem polynomial in* ζ,

$$V_x = \left(\sum_{j=0}^{k} \zeta^{k-j} P_j\right) V, \tag{5}$$

from a single starting point: the Adler–Kostant–Symes theorem in the loop algebra $\mathrm{sl}(2) \otimes \mathbb{C}[\zeta, \zeta^{-1}]$.

To this end, we first eliminate the special role assigned in the usual theory to "x", which means that we no longer take the eigenvalue problem (1) as basic. The idea is this.

All the Q_j that could ever be generated from (4) can be determined from the single equation

$$Q_x = [P, Q] \tag{6}$$

in which Q is now an infinite series $\Sigma_{j=0}^{\infty} Q_j \zeta^{-j}$. The coefficient of ζ^{-k} in (6) is ($P_0 = Q_0$, $P_1 = Q_1$)

$$Q_{k_x} = [Q_0, Q_{k+1}] + [Q_1, Q_k], \tag{7}$$

which agrees with (4); there is, however, no ζ^0 or P_t term to end the recursion.

Suppose one now calculates all the Q_j as matrices of (weighted homogeneous) differential polynomials in x. Let q, r satisfy the t_2 or t_3 equation above. What equation does the series Q obey?

One can verify that

$$Q_{t_2} = [Q_0\zeta^2 + Q_1\zeta + Q_2, Q],$$

$$Q_{t_3} = [Q_0\zeta^3 + \cdots + Q_3, Q],$$

and generally,

$$Q_{t_N} = [Q^{(N)}, Q].$$

Now we can get rid of x. Q is a (formal) series

$$Q = \sum_{j=0}^{\infty} Q_j \zeta^{-j} = \sum_{j=0}^{\infty} \begin{pmatrix} h_j & e_j \\ f_j & -h_j \end{pmatrix} \zeta^{-j}, \tag{8}$$

and it satisfies the equations

$$Q_{t_k} = [Q^{(k)}, Q], \quad Q^{(k)} = \sum_{j=0}^{k} Q_j \zeta^{k-j}. \tag{9}$$

The e_j, f_j, h_j are considered to be independent variables, and *not* differential polynomials. x is now the time variable t_1 in (9), no different from t_2 or t_{10} in its role in the theory.

An analogy with another feature of soliton theory may make this step clearer for some readers. Set $P_t = 0$ in eq. (3). The result, $Q_x^{(N)} = [P, Q^{(N)}]$, is an o.d.e. in x for time-independent, or stationary, solutions of (3). $q(x)$ and $r(x)$ determined from it are known to be multi-soliton or "N-gap" quasi-periodic initial conditions for all the equations $(4)_0$. Our eq. (9), for $k = 1$, is the stationary equation corresponding to $N = \infty$ in (3). Formally (but not analytically) it defines $q(x)$, $r(x)$ as functions of $x = t_1$; they are "∞-gap potentials."

Our first result will be seen to prove commutativity for all isospectral deformations of (1) or (5) in a single stroke.

Main result 1. Eq. (9) are commuting Hamiltonian flows on the Lie algebra $\{\Sigma_0^{\infty} X_j \zeta^{-j},\ X_j \in \mathrm{sl}(2, \mathbb{C})\}$, with respect to a natural Lie–Poisson bracket.

Our next theorem shows that soliton equations in this generalized sense still have conservation laws. Indeed, the result provides, again in a single stroke, the local conservation laws for *all* isospectral deformations of *all* the eigenvalue problems (5).

Main result 2. There exist polynomials F_{kj} in the e_i, f_i, h_i, such that

$$\frac{\partial F_{kj}}{\partial t_l} = \frac{\partial F_{lk}}{\partial t_j}. \tag{10}$$

First consequence. The usual conservation laws are

$$\frac{\partial}{\partial t_k}(\text{density}) = \frac{\partial}{\partial x}(\text{flux}):$$

$\partial/\partial x$ is $\partial/\partial t_1$, i.e., $j = 1$ in (10). But (10) suggests that if we care to view all variables as *functions of* t_j *for* $j \neq 1$, we can do so and still have conservation laws. We return to this shortly.

Second consequence. More than (10) is in fact true:

$$\frac{\partial F_{kj}}{\partial t_l} \text{ is symmetric under permutations of } kjl.$$

This suggests that $F_{kj} = \partial^2/\partial t_k \partial t_j$ of something. "Something" turns out to be $\log \tau$ of [5]. This will be shown in paper IV.

Suppose we now want to put x back into the setup. We would do this by focusing on the case $k = 1$ in (9); this just gives the recursion (7) from which the familiar Q_j are determined in terms of q, r, q_x, $r_x, \ldots$ We would use the symbols q and r for e_1 and f_1, respectively. The first equations contained in (9) are $e_{1,t_1} = -2ie_2$, $f_{1,t_1} = 2if_2$. With $t_1 = x$, this gives $e_2 = (i/2)q_x$, $f_2 = -(i/2)r_x$. Continuing in this way, we would view each individual equation in (9) as defining the next e_j, f_j, h_j in terms of the preceding ones and their t_1-or x-derivatives, and arrive eventually at the usual expression for the Q_j. (This is done in more detail later on). We could, however, equally well focus on $k = 2$:

$$Q_{t_2} = [Q_0\zeta^2 + Q_1\zeta + Q_2, Q]\,.$$

This, too, leads to a recursion (similar to, but more complicated than (7)) and defines – it turns out – all Q_j in terms of e_1, e_2, f_1, f_2 and their t_2-derivatives. Here e_1, f_1, resp. e_2, f_2, are entries in Q_1, Q_2 – see above, eq. (8).

In effect, we have taken t_2 to be our x; this amounts to positing the eigenvalue problem

$$V_x = \left[\begin{pmatrix} -i & \\ & i \end{pmatrix}\zeta^2 + \begin{pmatrix} & e_1 \\ f_1 & \end{pmatrix}\zeta + \begin{pmatrix} -(i/2)e_1f_1 & e_2 \\ f_2 & (i/2)e_1f_1 \end{pmatrix}\right]V,$$

and looking for its isospectral flows. Of course, there is nothing sacred about t_2.

Main result 3. Any t_N may be taken as the special x. This amounts to positing the eigenvalue problem (5),

$$V_x = \left(\sum_{j=0}^{N} \zeta^{N-j} Q_j\right)V.$$

$e_1, \ldots, e_N$, $f_1, \ldots, f_N$ in the Q_j are $2N$ potentials depending on x (alias t_N); the h_j turn out to be determined. All the flows (9) are Hamiltonian; the Hamiltonian for $(9)_k$ is $(2i)/(k+1)\int F_{N,k+1}\,dt_N$, where $F_{N,k+1}$ is the generalized density/flux from (10), viewed as differential polynomial in $e_1, \ldots, f_N$, with respect to $x = t_N$.

Actually, the correct conjugate variables turn out to be not the e_i, $f_i(i = 1 \ldots N)$], but certain differential polynomials of them.

This theorem is the most important and surprising one; it should perhaps be explained in words. Main result 1 gave the soliton equations as Hamiltonian o.d.e.'s in x and the various t's, but it must be remembered that the dependent variables e_j, f_j, h_j are complex numbers and not functions of x. (One could think of them as the values of q, r and all derivatives at a particular $x = x_0$, if that helps to clarify the idea). This Hamiltonian structure has *nothing* to do with the usual one ($q_t = \delta \int F/\delta r$), and the Hamiltonians themselves have *nothing* to do with the familiar ones from inverse scattering theory. To recover the standard theory, one must view e_j, f_j, h_j as polynomials in q, r, q_x, $r_x, \ldots$ as explained above. Result 3 says that the Hamiltonians then are given by integrals over certain fluxes from result 2, and that indeed the Hamiltonians for *any* isospectral deformation of *any* eigenvalue problem (5) are given by fluxes from that one basic family.

These, then, are the main steps. We view the AKNS equations as commuting systems of o.d.e.'s for scalar unknowns rather than as evolution equations for functions of x. In that setting, we get a very simple proof of the commutativity of those flows. We then recover the conservation laws, and in fact discover much more structure to them than was previously known. When we try to return from our setup to the usual one, we find that our abstraction contains not only the AKNS soliton equations, but those of all polynomial eigenvalue problems (5) as well.

1.3. *Further discussion.*

In the summary above we have confined ourselves to the actual results presented. It is important to us, however, to view those results as part of a yet unfinished picture, and so we want to state briefly what must still be done and what we hope to do.

First of all, this paper has been kept on a concrete and computational level, in part because all our formulas derive from, and can be compared with, the well-studied AKNS theory. As a result, the role of the Lie algebra $sl(2,\mathbb{C})\otimes\mathbb{C}[\zeta,\zeta^{-1}]$ appears at times to be less central than it really is. At present, we can get main results 1 and 2, but not the crowning 3, for any semisimple algebra in place of $sl(2,\mathbb{C})$. This will be commented upon in the appropriate place. We still need to find a complete and general treatment for all twisted loop algebras. The computational evidence indicates that such a treatment will involve the full affine algebra, rather than just the loop algebra as in this paper, and this poses some additional obstacles.

We should explain that $sl(2,\mathbb{C})\otimes\mathbb{C}(\zeta,\zeta^{-1})$ is the most obvious concrete example to choose because it turns up in connection with so many of the canonical solvable equations of mathematical physics. When we say 'turn up' we mean a lot more than observing that the Lax pairs of these equations can be expressed as systems of equations with 2×2 trace free matrices. In paper I, we show how the proper applications of the ideas of Wahlquist–Estabrook leads naturally to the choice of the phase space $sl(2,\mathbb{C})\otimes\mathbb{C}[\zeta^{-1}]$.

Our real goal as we indicated in subsection 1.1, is to see the τ-function, Hirota's method, the vertex-operator representation of Bäcklund transformation, and the Painlevé property as consequences of the framework introduced here. Throughout our calculations, we find formulas that beg for Hirota bilinearization, or for the introduction of a (formal) function that is known (from other work) to be τ. We will point out some of those suggestive facts in due time. As one referee of an earlier version of this paper put it, however, the τ-function is for us still a computational fact of life rather than a Lie-algebraic necessity.

1.4. *Related papers*

We want to draw attention to several related works about which we have learned in the course of our study.

The paper of Dhooghe [9] develops Hamiltonian systems on loop algebras essentially as we do here. Connections with evolution equations for functions of x are established in a different way. The relations between conservation laws and variational Hamiltonians are given more emphasis in our work.

Segal and Wilson [6] put the τ-function into a rigorous analytical framework. That paper, and a forthcoming work by Wilson (private communication) should go a long way towards uniting the new ideas of [5] with earlier soliton theory. We hope that our approach may eventually complement theirs by showing that all the main soliton ideas are related on a purely algebraic level as well as analytically.

A somewhat different analytical theory of τ-functions is being worked out by J. Palmer and D. Pickerell [7].

2. Notation and basic algebraic ideas

We will be dealing with two setups.

i) Lie algebras of formal power series, and vector fields on them. The concern is with – for instance – commutators of such vector fields, which make sense on a formal level, and not with their integral curves, which only exist under many additional hypotheses.

ii) Differential algebras. Those are polynomial algebras generated by symbols q, r, q_x, r_x, q_{xx}, ... They are equipped with a distinguished derivation ∂, thought of as $\mathrm{d}/\mathrm{d}x$:

$$\partial q = q_x, \quad \text{etc.}$$

We now set down our notation, and certain basic facts, for the two algebraic systems just mentioned.

2.1. *The loop algebra of $A_1^{(1)}$*

In this paper, the Lie algebra $\mathscr{G}$ of formal series

$$X = \sum_{j=-\infty}^{N} X_j\zeta^j, \quad X_j \in \mathrm{sl}(2, \mathbb{C}), \quad N < +\infty ,$$

will play the basic role. For a more general treatment, it becomes necessary to add to $\mathscr{G}$ two special elements (center and derivation), but in this paper they will not be required.

The commutator in $\mathscr{G}$ is defined by

$$[X, Y] = \sum_k \sum_{i+j=k} [X_i, Y_j]\zeta^k ;$$

it makes sense because the series X, Y contain only finitely many positive powers of ζ.

On $\mathscr{G}$, define the symmetric bilinear forms

$$\langle X, Y\rangle_k = \sum_{i+j=k} \mathrm{Tr}\, X_i Y_j .$$

The case $k = 0$ will be most important in this paper; $k = -1$ will play a role in paper III.

Certain subalgebras of $\mathscr{G}$ are needed:

$$\mathscr{K} = \left\{\sum_{-\infty}^{-1} X_j\zeta^j\right\},$$

$$\mathscr{N} = \left\{\sum_{0}^{\infty} X_j\zeta^j\right\} \quad \text{(remember: only finitely many positive powers of } \zeta\text{)} .$$

So $\mathscr{G} = \mathscr{K} \oplus \mathscr{N}$.

Let $\Pi_{\mathscr{K}}$, $\Pi_{\mathscr{N}}$ be the projections from $\mathscr{G}$ to $\mathscr{K}$ and $\mathscr{N}$. The annihilators of $\mathscr{K}$ and $\mathscr{N}$ with respect to $\langle\ ,\ \rangle_0$ are

$$\mathscr{K}^{\perp_0} = \left\{\sum_{-\infty}^{0} X_j\zeta^j\right\}, \quad \mathscr{N}^{\perp_0} = \left\{\sum_{1}^{\infty} X_j\zeta^j\right\},$$

and with respect to $\langle\ ,\ \rangle_{-1}$

$$\mathscr{K}^{\perp_{-1}} = \mathscr{K}, \quad \mathscr{N}^{\perp_{-1}} = \mathscr{N} .$$

We may and do identify the dual of $\mathscr{N}$ with $\mathscr{K}^{\perp}$. Depending on the choice of $\langle\ ,\ \rangle_k$, $k = 0$ or -1, $\mathscr{N}^*$ can be realized concretely as $\mathscr{K}^{\perp_0}$ or $\mathscr{K}$. Likewise, $\mathscr{K}^*$ is identified either with $\mathscr{N}^{\perp_0}$ or $\mathscr{N}$. In this paper,

$$\mathscr{N}^* \cong \mathscr{K}^{\perp_0}$$

is used almost exclusively, and we henceforth omit the subscript 0 or -1 on $\langle\ ,\ \rangle$ unless some confusion is possible.

Let ϕ be a $\mathbb{C}$-valued function on $\mathscr{G}$. Typically, $\phi(X)$ will be a polynomial in finitely many of the entries of the $X_j(X = \Sigma\, X_j\zeta^j)$. The tangent space to $\mathscr{G}$ at X is identified, as usual, with $\mathscr{G}$ itself. Let $\dot{X} \in \mathscr{G} \cong T_X\mathscr{G}$, and take the derivative of ϕ in the direction $\dot{X}$:

$$\frac{\mathrm{d}}{\mathrm{d}\epsilon}\phi(X + \epsilon\dot{X})\big|_{\epsilon=0}$$

is a linear function of $\dot{X}$. The *gradient of ϕ at X* is the element grad $\phi(X) \in \mathscr{G}$ such that

$$\frac{\mathrm{d}}{\mathrm{d}\epsilon}\phi(X + \epsilon\dot{X})\big|_{\epsilon=0} = \langle \mathrm{grad}\, \phi(X), \dot{X}\rangle .$$

This definition has an analytical flavor, but it is possible to rephrase everything quite algebraically in terms of the polynomial algebra generated by the entries of the X_j.

2.2. *Differential algebra*

Let $\mathscr{B}$ be the polynomial algebra (over $\mathbb{C}$) generated by symbols $q_j^{(\alpha)}$, $r_j^{(\alpha)}(j = 1, \ldots, N, \alpha \geqslant 0)$. Write $q_j^{(0)} = q_j$, $r_j^{(0)} = r_j$. Suppose there exists a derivation ∂ such that

$$\partial : q_j^{(\alpha)} \to q_j^{(\alpha+1)}, \quad r_j^{(\alpha)} \to r_j^{(\alpha+1)}$$

and such that $\partial : \mathbb{C} \to \{0\}$.

One thinks of q_j, r_j as functions of a variable x, of $q_j^{(\alpha)}$ as $\mathrm{d}^\alpha q_j/\mathrm{d}x^\alpha$, and of ∂ as $\mathrm{d}/\mathrm{d}x$.

For $F \in \mathscr{B}$, define

$$\frac{\delta F}{\delta s} = \sum_{l=0}^{\infty} \left(- \frac{\mathrm{d}}{\mathrm{d}x} \right)^l \frac{\partial F}{\partial s^{(l)}} ;$$

s is one of the q_j, r_j. $\delta F / \delta s$ is the *functional* (or *variational*) derivative of F with respect to s. It can be proved [18] that $\delta F / \delta q_j = \delta F / \delta r_j = 0$ for all j if, and only if, $F \in \partial \mathscr{B} + \mathbb{C}$.

We now describe what we will call the *variational Hamiltonian formalism*.

Let $\bar{\mathscr{B}}$ be the quotient $\mathscr{B}/\mathrm{Image}\, \partial$. An equivalence class $F + \mathrm{Image}\, \partial$ will be denoted by $\bar{F}$. $\delta/\delta s$ $(s = q_j, r_j)$ acts from $\bar{\mathscr{B}}$ to $\mathscr{B}$,

$$\frac{\delta}{\delta s} : \bar{F} \to \frac{\delta F}{\delta s} .$$

This is well defined, since if F_1, $F_2 \in \bar{F}$, $\delta(F_1 - F_2)/\delta s = 0$ because $F_1 - F_2 \in \mathrm{Image}\, \partial$.

Let $\bar{F} \in \bar{\mathscr{B}}$. A derivation $\partial_{\bar{F}}$ is defined on $\mathscr{B}$ as follows. Set

$$\partial_{\bar{F}} q_j = \frac{\delta \bar{F}}{\delta r_j}, \quad \partial_{\bar{F}} r_j = - \frac{\delta \bar{F}}{\delta q_j}, \quad \partial_{\bar{F}} : \mathbb{C} \to \{0\} . \tag{11}$$

Extend $\partial_{\bar{F}}$ to all of $\mathscr{B}$ by requiring it to commute with $\partial : \partial_{\bar{F}} q_j^{(\alpha+1)} = \partial \partial_{\bar{F}} q_j^{(\alpha)}$, etc. Then $\partial_{\bar{F}}$ restricts to the quotient $\bar{\mathscr{B}}$. The result, still denoted by $\partial_{\bar{F}}$ since only pedants will be confused, is the *Hamiltonian derivation with Hamiltonian* $\bar{F}$.

It is usually simpler to think of (11) as a system of partial differential equations, with $\partial_{\bar{F}} q_j$ being $\partial q_j / \partial t$ (but in some of the work described later on we found it helpful to be clear about what lives in which space). All the formulas, of course, are set up so as to mimick the usual derivatives of functionals of q_j, r_j. F above, is concretely, the integrand of such a functional, and (11) is often written

$$\frac{\partial q_j}{\partial t} = \frac{\delta}{\delta r_j} \int F \,\mathrm{d}x, \quad \frac{\partial r_j}{\partial t} = - \frac{\delta}{\delta q_j} \int F \,\mathrm{d}x .$$

Note that if F is a perfect derivative, $F = G_x$, then $\int F \,\mathrm{d}x = 0$. Passage to $\bar{\mathscr{B}}$ takes care of this.

It turns out, as expected, that the "Hamiltonian" of (11) is "constant": $\partial_{\bar{F}} \bar{F} = 0$. A *conserved density* for (11) is a $\bar{G} \in \bar{\mathscr{B}}$ for which $\partial_{\bar{F}} \bar{G} = 0$. There are alternative ways of saying this:

i) if $G \in \bar{G}$, then in $\mathscr{B}$, $\partial G / \partial t = \partial H / \partial x$ for some H (the flux); to different G's correspond different H's.

ii) $\bar{F}$ and $\bar{G}$ are in involution with respect to the Poisson bracket in $\bar{\mathscr{B}}$,

$$\{\bar{F}, \bar{G}\} = \sum_{j=1}^{N} \frac{\delta \bar{F}}{\delta r_j} \frac{\delta \bar{G}}{\delta q_j} - \frac{\delta \bar{F}}{\delta q_j} \frac{\delta \bar{G}}{\delta r_j} , \tag{12}$$

i.e., $\{\bar{F}, \bar{G}\} = 0$.

3. Lax equations on $\mathscr{K}^\perp$

In this section, we give *two* Lie-algebraic interpretations of the AKNS soliton equations in the version (9). We will not repeat the explanation of the relation between our abstract setup and the familiar language; that should be clear after reference to the introduction.

3.1. 1*st Lie-algebraic interpretation*

The Lie-theoretic setup in this subsection is directly inspired by M. Adler and P. van Moerbeke's paper [10] (which in turn goes back to Adler [11]). The decompositions we use, and the involution theorem, come from the section on "spinning tops" in [10]. We observed some time ago that "spinning tops" were relevant to stationary soliton equations, and what we develop below is a formal infinite-dimensional generalization of the stationary equations. See the comments in subsection 1.2, and also paper III of this series.

$\mathscr{K}^\perp$, as the dual of $\mathscr{N}$, has a natural Poisson bracket, the so-called Lie–Poisson bracket. It is defined by†

$$\{\phi, \psi\}(X) = -\langle X, [\Pi_{\mathcal{N}} \operatorname{grad} \phi(X), \Pi_{\mathcal{N}} \operatorname{grad} \psi(X)]\rangle, \quad X \in \mathcal{K}^{\perp}. \tag{12}$$

grad $\phi(X)$ and grad $\psi(X)$ are computed with reference to the full Lie algebra $\mathcal{G}$, even though ϕ and ψ may only be defined on $\mathcal{K}^{\perp}$.

To a function ϕ there is associated a Hamiltonian vectorfield

$$\mathcal{X}_{\phi}(X) = -\Pi_{\mathcal{K}^{\perp}}[\Pi_{\mathcal{N}} \operatorname{grad} \phi(X), X]. \tag{13}$$

Although the term "field" implies some smoothness, we only mean that $\mathcal{X}_{\phi}(X) \in T_X\mathcal{K}^{\perp} \cong \mathcal{K}^{\perp}$.

If ϕ is ad-invariant, meaning that

$$[\operatorname{grad} \phi(X), X] = 0, \quad X \in \mathcal{G}, \tag{14}$$

then (13) becomes simpler:

$$\mathcal{X}_{\phi}(X) = [\Pi_{\mathcal{K}} \operatorname{grad} \phi(X), X], \quad X \in \mathcal{K}^{\perp}. \tag{15}$$

The Adler–Kostant–Symes theorem [11, 12, 13] says that if ϕ, ψ are both ad-invariant, (14), then

i) $\{\phi, \psi\} \equiv 0$ on $\mathcal{K}^{\perp}$;

ii) the vectorfields (15) commute.

(For proofs and further references, see the survey [14]).

The AKNS equations, in the form (9), arise via this theory from the simplest ad-invariant functions.

Let $\phi_k(X) = -\frac{1}{2}\langle S^k X, X\rangle$, where S^k is the shift $\Sigma X_j \zeta^j \mapsto \Sigma X_j \zeta^{j+k}$, i.e., "multiplication by ζ^k".

An easy calculation shows that

$$\operatorname{grad} \phi_k(X) = -S^k X, \quad X \in \mathcal{G}.$$

Since $[S^k X, X] = 0$, $\phi_k(X)$ is ad-invariant. The formulas listed at the beginning immediately imply the following theorem:

† Throughout subsection 3.1, $\langle\ ,\ \rangle$ means $\langle\ ,\ \rangle_0$ and $\perp$ means $\perp_0$.

Theorem 1. The functions $\phi_k(X) = -\frac{1}{2}\langle S^k X, X\rangle$, $k \geqslant 0$, restricted to $X \in \mathcal{K}^{\perp}$, are in involution. Their Hamiltonian vector fields have the form

$$\mathcal{X}_{\phi_k}(X) = -[\Pi_{\mathcal{K}} S^k X, X], \tag{16}$$

or, since $[S^k X, X] = 0$,

$$\mathcal{X}_{\phi_k}(X) = [\Pi_{\mathcal{N}} S^k X, X]. \tag{17}$$

Note that (17) is essentially the version (9) of the AKNS equations, with X standing for Q. We take $k \geqslant 0$ only, because otherwise $[\Pi_{\mathcal{K}} S^k X, X] = 0$ for $X \in \mathcal{K}^{\perp}$, which is not interesting.

Warning: This theorem does *not* give the commutativity of the usual AKNS flows without a lot of further work. The Hamiltonians ϕ_k are completely different from the ones encountered in inverse scattering. See below, remark 2 following proposition 5.

It will be useful to have a variety of formulas that allow one to find Poisson brackets between general, but explicitly given, functions. The explicit nature of the formulas that follow is a fortunate accident due to our choice of sl(2) as the basic Lie algebra. Even for sl(n) things get much more involved, and indeed the detailed information one can extract in the present case is a useful guide to the general situation. Remarks on a "coordinate-free" description are given at the end of this section.

We write $X \in \mathcal{K}^{\perp}$ as

$$\sum_{j=0}^{\infty} \begin{pmatrix} h_j & e_j \\ f_j & -h_j \end{pmatrix} \zeta^{-j},$$

and will work with the "coordinates on $\mathcal{K}^{\perp}$", e_j, f_j, h_j, as well as with the series

$$e = e(\zeta) = \sum_{j=0}^{\infty} e_j \zeta^{-j},$$

f or $f(\zeta)$, and h or $h(\zeta)$ (defined similarly). These formal series make certain calculations easier.

Proposition 1. $\phi_k(X)$ is the coefficient of ζ^{-k} in the series $-(h^2+ef)$,

$$\phi_k(X) = -\sum_{i=0}^{k} h_i h_{k-i} + e_i f_{k-i}. \tag{18}$$

Proof. an easy calculation.

Proposition 2. The Poisson brackets between the e_j, f_j, and h_j (viewed as functions on $\mathscr{K}^{\perp}$) are

$$\{h_i, e_j\} = e_{i+j}, \quad \{h_i, f_j\} = -f_{i+j}, \quad \{e_i, f_j\} = 2h_{i+j}. \tag{19}$$

Proof. For the moment, we write $e_j(X)$ to indicate that e_j is a function of X. Now

$$\frac{\mathrm{d}}{\mathrm{d}\epsilon} e_j(X+\epsilon\dot{X})\big|_{\epsilon=0} = e_j(\dot{X}),$$

(this is the (1, 2) entry in the coefficient of ζ^{-j} in $\dot{X}\in\mathscr{G}$). Since

$$e_j(\dot{X}) = \left\langle \zeta^j\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}, \dot{X}\right\rangle,$$

grad $e_j(X) = \left(\begin{smallmatrix}0 & 0\\ 1 & 0\end{smallmatrix}\right)\zeta^j$. Likewise,

$$\operatorname{grad} f_j(X) = \zeta^j\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix},$$

$$\operatorname{grad} h_j(X) = \frac{1}{2}\zeta^j\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}.$$

So

$$\begin{aligned}\{h_i, e_j\}(X) &= -\left\langle X, \left[\frac{1}{2}\zeta^i\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, \zeta^j\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}\right]\right\rangle\\ &= \left\langle X, \zeta^{i+j}\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}\right\rangle\\ &= e_{i+j}(X),\end{aligned}$$

and similarly for the other brackets. ■

The formulas (19) are easiest to use when expressed through "generating functions."

Proposition 3. Write $e = e(\zeta)$, $e' = e(\eta)$, and so forth for f, h, where ζ, η are formal parameters. Then

$$\begin{aligned}\{h, e'\} &= \frac{\zeta^{-1}e - \eta^{-1}e'}{\zeta^{-1}-\eta^{-1}},\\ \{h, f'\} &= -\frac{\zeta^{-1}f - \eta^{-1}f'}{\zeta^{-1}-\eta^{-1}},\\ \{e, f'\} &= 2\frac{\zeta^{-1}h - \eta^{-1}h'}{\zeta^{-1}-\eta^{-1}}.\end{aligned} \tag{20}$$

Proof.

$$\begin{aligned}\{h, e'\} &= \sum_{i,j=0}^{\infty} \{h_i, e_j\}\zeta^{-i}\eta^{-j}\\ &= \sum_{i,j=0}^{\infty} e_{i+j}\zeta^{-i}\eta^{-j} = \sum_{k=0}^{\infty} e_k \sum_{i+j=k} \zeta^{-i}\eta^{-j}\\ &= \sum_{k=0}^{\infty} e_k \frac{\zeta^{-(k+1)} - \eta^{-(k+1)}}{\zeta^{-1}-\eta^{-1}}\\ &= \frac{\zeta^{-1}\sum_{k=0}^{\infty} e_k\zeta^{-k} - \eta^{-1}\sum_{k=0}^{\infty} e_k\eta^{-k}}{\zeta^{-1}-\eta^{-1}}\\ &= \frac{\zeta^{-1}e - \eta^{-1}e'}{\zeta^{-1}-\eta^{-1}},\end{aligned}$$

etc. ■

Now, using the fact that the derivative of a function ψ along a Hamiltonian vectorfield $\mathscr{X}_\phi$ is given by $\{\psi, \phi\}$, we obtain equations for the derivatives $\partial e_j/\partial t_k, \ldots$, of $e_j, \ldots$, along the vectorfields $\mathscr{X}_{\phi_k}$.

Proposition 4.

$$\begin{aligned}\frac{\partial e_j}{\partial t_k} &= 2\sum_{i=0}^{\min(j-1,k)} h_i e_{k+j-i} - e_i h_{k+j-i},\\ \frac{\partial f_j}{\partial t_k} &= 2\sum_{i=0}^{\min(j-1,k)} h_i f_{k+j-i} - e_i h_{k+j-i},\\ \frac{\partial h_j}{\partial t_k} &= \sum_{i=0}^{\min(j-1,k)} e_i f_{k+j-i} - f_i e_{k+j-i}.\end{aligned} \tag{21}$$

Proof. Straightforward but tedious, using (19), or by directly calculating the ζ^{-j} component of $[\Pi_{\mathcal{N}}S^kX, X]$. ■

Corollary. h_0, e_0, f_0, h_1 are independent of all t_k.

Proposition 5. Formulas (21) are expressed by generating functions, as follows:

$$\{h^2+ef, e'\} = \frac{2\eta^{-1}}{\zeta^{-1}-\eta^{-1}}(eh'-e'h),$$
$$\{h^2+ef, f'\} = -\frac{2\eta^{-1}}{\zeta^{-1}-\eta^{-1}}(fh'-f'h), \qquad (22)$$
$$\{h^2+ef, h'\} = \frac{\eta^{-1}}{\zeta^{-1}-\eta^{-1}}(e'f-ef').$$

Proof. Use Proposition 3. Note that the generating function for $\partial e_j/\partial t_k$ is $\{e', -(h^2+ef)\}$, since the Hamiltonians ϕ_k are coefficients in $-(h^2+ef)$. Switching the order in $\{\ \}$ removes the $(-)$ sign. ■

From (21) we easily get the following.

Corollary.

$$\frac{\partial e_{j+1}}{\partial t_k} = \frac{\partial e_{k+1}}{\partial t_j},$$
$$\frac{\partial f_{j+1}}{\partial t_k} = \frac{\partial f_{k+1}}{\partial t_j}, \qquad (23)$$
$$\frac{\partial h_{j+1}}{\partial t_k} = \frac{\partial h_{k+1}}{\partial t_j}.$$

Remark 1. This corollary is extremely important. It shows, for example, that $e_{j+1}, f_{j+1}, h_{j+1}$ are the $\partial/\partial t_j$ derivatives of functions of all the time variables. We will write (see eq. (29)) $h_{j+1} = \partial/\partial t_j \cdot ((i/2)\partial \ln \tau/\partial t_1)$; τ turns out to be the τ-function of central importance to the theory. Similarly, the potentials σ/τ, ρ/τ of which the sequences e_j, f_j are gradients, are "auxiliary" τ-functions and can be generated by applying Schlesinger transformations to the original τ. When one writes the equations of motion (21) in the potential variables τ, σ, ρ one obtains the Hirota equations. These results are elaborated on in paper IV.

Remark 2. The Hamiltonians ϕ_k *are not the familiar Hamiltonians of inverse scattering theory, and carry little information.* In inverse scattering, the Hamiltonians (integrals of conserved densities) have numerical values expressible in terms of the scattering data. One can, for example, relate the values of those Hamiltonians to soliton speeds. Our ϕ_k, on the other hand, seem to carry no information in any numerical value that might be assigned to them. We now describe the implications of "$\phi_k = \text{const.}$"

Suppose we want to recover the usual AKNS q, r, q_x, r_x, ... formulation of the soliton equations. We should then interpret the t_1-equation

$$\frac{\partial X}{\partial t_1} = [\Pi_{\mathcal{N}}SX, X]$$

as the recursion relation it once was (see (4)). X will be thought of as the $Q = \begin{pmatrix} -i & \\ & i \end{pmatrix} + \begin{pmatrix} & q \\ r & \end{pmatrix}\zeta^{-1} + \cdots$ of (6). So we should set $h_0 = -i$, $h_1 = e_0 = f_0 = 0$ in X, and write $e_1 = q$, $f_1 = r$. Eq. (21) gives

$$\frac{\partial e_1}{\partial t_1} = -2ie_2;$$

with $t_1 = x$, this says $e_2 = (i/2)q_x$. Likewise, $f_2 = -(i/2)r_x$.

Next, we have $\partial e_2/\partial t_1 = -2ie_3 - 2e_1h_2$, or $(i/2)q_{xx} = -2ie_3 - 2qh_2 = 0$. To find e_3 in terms of q, r, we need h_2.

h_2 comes from $\phi_2 = \text{const} = -c_2$, say:

$$-2ih_2 + h_1^2 + e_0f_2 + e_1f_1 + e_2f_0 = c_2,$$

or

$$h_2 = c_2 - \frac{i}{2}qr.$$

Then $e_3 = -\frac{1}{4}q_{xx} + \frac{1}{2}q^2 r + \mathrm{i}c_2 q$. We also get f_3 in an analogous way, find h_3 from $\phi_3 = -c_3$, and so on.

Hence, a particular choice $\phi_k = -c_k (k \geqslant 2)$ only defines the h_k as certain linear combinations of homogeneous differential polynomials in q, r. It has no influence whatsoever on the dynamics or on the character of the solutions $q(x, t)$, $r(x, t)$; by a linear change of time variables, one can recover the homogeneous series Q familiar in the AKNS theory.

In order to set up a simple and direct correspondence between our phase-space and the differential algebra language, we will often find it convenient to restrict attention to a certain set $\mathscr{M} \subset \mathscr{K}^\perp$:

Definition.

$$\mathscr{M} = \{X \in \mathscr{K}^\perp \mid h_0 = -\mathrm{i}, h_1 = e_0 = f_0 = 0, h^2 + ef = -1\}.$$

Remark 3. Although they are never used in this paper, it is worth noting that we may recover the AKNS "eigenfunctions" V, and the equations they satisfy, as follows. The commuting family of flows (17),

$$Q_{t_k} = [Q^{(k)}, Q]$$

(we use the notation of (9)) admit a formal solution. Let

$$Q = VQ_0V^{-1};$$

it then follows that Q_0 is constant in all times t_k if

$$V_{t_k} = Q^{(k)}V - VN^{(k)},$$

and if Q_0 and $N^{(k)}$ (the latter is used to normalize the eigenfunctions in a suitable way) commute. In the present context, both $N^{(k)}$ and Q_0 are proportional to $H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Note from $Q = VQ_0V^{-1}$ that $h^2 + ef = -\det Q = -\det Q_0 = \text{constant}$.

3.2. *2nd Lie algebraic interpretation*

There is another Hamiltonian structure that also gives the AKNS equations in the form (9). The Hamiltonians, however, are not those of subsection 3.1: the $\partial/\partial t_k$ vectorfield $[\Pi_{\mathscr{N}} S^k X, X]$ will come from the Hamiltonian ϕ_{k+1}, not from ϕ_k.

This second Hamiltonian structure is interesting for various reasons.

i) It arises from a quite different construction, namely, from a version of Kostant's translated-invariant theorem [12, 14].

ii) Many soliton systems have two distinct Hamiltonian structures; there result the so-called *Lenard relations* that play an important role in the theory. Apparently, the AKNS system in its usual guise does not admit two (local) Hamiltonian structures. It is curious that our point of view provides two of them. We have looked for, but could not find, any more.

All arguments in this paper could be done with either structure; we will use interpretation 3.1, because the formulas are simpler.

The general Lie-algebra result is the following [14].

Let $\mathscr{G} = \mathscr{K} \oplus \mathscr{N}$, and let $\langle\ ,\ \rangle$ be a non-degenerate symmetric, ad-invariant bilinear form on $\mathscr{G}$. There is a Hamiltonian structure on $\mathscr{K}^\perp$ (as in subsection 2.1); hence, by translation there will be one on the set $\mathscr{K}^\perp + \epsilon$, where $\epsilon \in \mathscr{G}$ is a *fixed* element. For certain ϵ, special formulas hold.

Suppose

$$\epsilon \perp [\mathscr{K}, \mathscr{K}] \quad \text{and} \quad \epsilon \perp [\mathscr{N}, \mathscr{N}]. \tag{24}$$

Suppose furthermore that ϕ is ad-invariant on $\mathscr{G}$. The Hamiltonian vectorfield is then given by

$$\mathscr{X}_\phi(X + \epsilon) = [\Pi_{\mathscr{K}} \operatorname{grad} \phi(X + \epsilon), X + \epsilon]. \tag{25}$$

Moreover, if ϕ, ψ are ad-invariant then they are in involution with respect to this Poisson bracket on $\mathscr{K}^\perp + \epsilon$.

We apply this result with $\mathscr{G}$, $\mathscr{K}$, $\mathscr{N}$ as before, but with $\langle\ ,\ \rangle = \langle\ ,\ \rangle_{-1}$. For ϵ, we take $\begin{pmatrix} -\mathrm{i} & 0 \\ 0 & \mathrm{i} \end{pmatrix}\zeta^0$.

It is easy to see that (24) holds for this ϵ. As Hamiltonians, we take, as before, $\phi_k = -\frac{1}{2}\langle S^k X, X\rangle_0 = -\frac{1}{2}\langle S^{k-1}X, X\rangle_{-1}$, $k \geqslant 1$. The gradient of ϕ_k is now taken with respect to $\langle\ ,\ \rangle_{-1}$, so $\operatorname{grad}\phi_k(X) = -S^{k-1}X$.

The Hamiltonian vectorfields (25) will then be, for $X \in \mathscr{K}$:

$$\begin{aligned}\mathscr{X}_{\phi_k}(\epsilon + X) &= -[\Pi_{\mathscr{K}} S^{k-1}(\epsilon + X), \epsilon + X] \\ &= [\Pi_{\mathscr{N}} S^{k-1}(\epsilon + X), \epsilon + X]\,. \end{aligned} \tag{26}$$

There are two differences between this result and formula (17) in subsection 3.1:

1) Most importantly, the Hamiltonians are shifted: the vectorfield

$$[\Pi_{\mathscr{N}} S^k X, X], \quad X \in \mathscr{K}^{\perp_0} \quad \text{or} \quad X \in \mathscr{K}^{\perp_{-1}} + \epsilon$$

has Hamiltonian ϕ_k in (17) and ϕ_{k+1} in (26).

2) In (17), the term X_0 in X is constant by the corollary to prop. 4. If we choose $X_0 = \begin{pmatrix} -\mathrm{i} & \\ & \mathrm{i}\end{pmatrix}$, the vectorfields (17) and (26) agree. The phase space $\mathscr{K}^{\perp_{-1}} + \epsilon$ is more restrictive in that the ζ^0 coefficient is fixed once and for all, but that might actually be convenient for application to the AKNS system.

One can find analogs of the propositions in subsection 1. The basic formulas are

$$\{h_i, e_j\} = e_{i+j-1}\,,$$

$$\{h_i, f_j\} = -f_{i+j-1}\,,$$

$$\{e_i, f_j\} = 2h_{i+j-1}\,,$$

with the understanding that $h_0 = -\mathrm{i}$, $e_0 = 0$, $f_0 = 0$. From these, the rest of the paper could be built up, but as stated above, we will use formulas (19) instead.

Remark. A coordinate-free description. The formulas of 3.1 can be adopted to loop algebras $\mathscr{g} \otimes \mathbb{C}[\zeta, \zeta^{-1}]$, $\mathscr{g}$ a semisimple Lie algebra, in a fairly direct way. The basic Poisson brackets (19) could be thought of as dual to the commutation relations for a Weyl basis of $\mathscr{g}$, but that seems to become very messy. Without choosing any particular basis of $\mathscr{g}$, one can argue as follows.

Let J_μ, $\mu = 1, \ldots, R = \operatorname{rank}$ of $\mathscr{g}$, be a basis of homogeneous polynomials for the invariant polynomials on $\mathscr{g}$. For $X = \Sigma X_j \zeta^{-j}$, $X_j \in \mathscr{g}$, let $J_\mu^k(X) = (-1) \times \zeta^{-k}$ – coefficient of the formal series $J_\mu(X)$. The vectorfields

$$\frac{\partial X}{\partial t_{\mu,k}} = [\Pi_{\mathscr{K}} \operatorname{grad} J_\mu^k(X), X]$$

then commute (analog of theorem 1).

Let $V \in \mathscr{g}$. Then

$$\frac{\partial}{\partial t_{\mu,k}}\langle \operatorname{grad} J_\lambda^{l+1}(X), V\rangle = \frac{\partial}{\partial t_{\lambda,l}}\langle \operatorname{grad} J_\mu^{k+1}(X), V\rangle\,.$$

This generalizes the relations (23). Since $R = 1$ for $\mathscr{g} = \mathrm{sl}(2)$, the indices λ, μ may be dropped in this case, and J^k is our earlier $-\phi_k$. With $V = \begin{pmatrix} 1 & \\ & -1\end{pmatrix}$, $\begin{pmatrix} 0 & 1 \\ 0 & 0\end{pmatrix}$, $\begin{pmatrix} 0 & 0 \\ 1 & 0\end{pmatrix}$, we recover (23).

Again, we will not spend much time on the general case, because we do not know a conceptual proof of theorem 2, section 7.

4. Conservation laws

In this section, we derive conservation laws:

$$\frac{\partial}{\partial t_j}(\text{density}) = \frac{\partial}{\partial x}(\text{flux})\,,$$

with $x = t_1$ as explained earlier. "Density" and "flux" are polynomials in the e_i, f_i, h_i. More generally, we obtain relations

$$\frac{\partial}{\partial t_j}(\text{density}) = \frac{\partial}{\partial t_l}(\text{flux})\,.$$

In our setup, none of the usual significance attaches to conservation laws. The variables are not functions of x, so $\int \ldots \mathrm{d}x$ has no meaning. It is all the more remarkable that conservation laws do exist, and in such generality. Moreover, the generalized conservation laws will be seen to be crucial to the variational Hamiltonian descriptions later on.

The formulas we want involve 3 indices. Correspondingly, there are generating series in 3 parameters: ζ, η, ξ. We write $e = e(\zeta)$, $e' = e(\eta)$, $e'' = e(\xi)$, and so on. Define

$$F(\eta, \xi) = \frac{1}{2h_0}\frac{\eta^{-1}\xi^{-1}}{(\eta^{-1} - \xi^{-1})^2} \times [e'f'' + e''f' + 2h'h'' - (h'^2 + e'f') - (h''^2 + e''f'')] \tag{27}$$

Lemma. $F(\eta, \xi)$ is a formal power series in η^{-1}, ξ^{-1}.

Proof. The only thing to check is that the expression inside [. . .] is divisible by $(\eta^{-1} - \xi^{-1})^2$. This is so because [. . .], $\partial[\ldots]/\partial\eta$, and $\partial[\ldots]/\partial\xi$ all vanish when $\xi = \eta$. ■

Write $F(\eta, \xi) = \Sigma_{k,j=1}^{\infty} F_{kj}\eta^{-k}\xi^{-j}$. Since $F(\eta, \xi) = F(\xi, \eta)$, we have this

Lemma.

$$F_{kj} = F_{jk}\,. \tag{28}$$

Lemma.

$$\frac{\partial h_{j+1}}{\partial t_k} = \frac{\partial F_{kj}}{\partial t_1} \tag{29}$$

Remark. Eqs. (29) are the usual AKNS conservation laws. The h_{j+1} are the conserved densities, and the F_{kj} are the corresponding fluxes.

Proof. From (22),

$$\{h^2 + ef, h'\} = \frac{\eta^{-1}}{\zeta^{-1} - \eta^{-1}}(e'f - ef')\,.$$

On the other hand, $\partial\,\eta^{-1}F(\xi, \eta)/\partial t_1 = \{2h_0h_1 + e_0f_1 + e_1f_0,\ \eta^{-1}F(\zeta, \eta)\}$ can be evaluated from formulas (19). One finds that

$$\{h^2 + ef, h'\} = \frac{\partial}{\partial t_1}\eta^{-1}F(\zeta, \eta)\,,$$

or

$$\sum \frac{\partial h_j}{\partial t_k}\zeta^{-k}\eta^{-j} = \frac{\partial}{\partial t_1}\sum F_{kj}\zeta^{-k}\eta^{-(j+1)}\,,$$

from which the desired result is read off. ■

Proposition 6.

$$F_{kl} = \frac{1}{2h_0}\sum_{i=0}^{k-1}(k - i)(2h_ih_{k+l-i} + e_if_{k+l-i} + f_ie_{k+l-i}) + k\phi_{k+l}\,. \tag{30}$$

Proof. The coefficient of x^k in a formal series $(x - y)^{-2}\Sigma\, a_{ij}x^iy^j$ is

$$\frac{1}{k!}\frac{\partial^k}{\partial x^k}(x - y)^{-2}\sum a_{ij}x^iy^j\,|_{x=0}\,,$$

which is calculated to be

$$\sum_{\alpha=0}^{k}(k - \alpha + 1)\sum_{j=0}^{\infty}a_{\alpha j}y^{j-(k-\alpha+2)} = \sum_{l=0}^{\infty}\left(\sum_{\alpha=0}^{k}(k - \alpha + 1)a_{\alpha,l+k-\alpha+2}\right)y^l$$

Now set $a_{ij} = (1/2h_0)(2h_{i-1}h_{j-1} + e_{i-1}f_{j-1} + e_{j-1}f_{i-1} + \phi_{i-1}\delta_{0,j-1} + \phi_{j-1}\delta_{0,i-1})$ to get the result ($a_{0j} = a_{i0} = 0$). ■

Remark. Since $\partial\phi_k/\partial t_j = 0$ for any j, the ϕ_{k+l} term in (30) may be left off without affecting any formula with the exception of (28).

Remark. $F_{1,l} = h_{l+1} + \phi_{l+1}$. When combined with (29), this just says that $\partial h_{l+1}/\partial t_1 = \partial h_{l+1}/\partial t_1$.

From (22), we get

Proposition 7.

$$\{h^2 + ef, F(\eta, \xi)\}$$
$$= \frac{\zeta^{-1}\eta^{-1}\xi^{-1}}{h_0(\xi^{-1}-\eta^{-1})(\zeta^{-1}-\xi^{-1})(\zeta^{-1}-\eta^{-1})}$$
$$\times [h(e'f'' - e''f') + h'(e''f - ef'') + h''(ef' - e'f)] \,. \quad (31)$$

The only point of this result is: the right side is symmetric in ζ, η, ξ. Hence, so is the left side. Therefore, we have this

Corollary. $\partial F_{kj}/\partial t_i$ is symmetric under permutations of ijk.

Now there are the two consequences already mentioned in the introduction:

(1) By symmetry, $\partial F_{kj}/\partial t_i = \partial F_{ik}/\partial t_j$ etc. This generalizes the conservation law (29) to arbitrary t_j instead of t_1.

(2) The symmetry of ∂F_{kj} suggests the introduction of a function of the t's – we will call it $\log \tau(t_1, t_2, \ldots)$ – such that

$$F_{kj} = \frac{\partial^2 \log \tau}{\partial t_k \partial t_j}\,.$$

This direction will be explored in paper IV.

Remark. A coordinate-free description.

We continue the notation of the remark at the end of section 3. In the more general case $\mathscr{g} \otimes \mathbb{C}[\zeta, \zeta^{-1}]$, set

$$F_{\mu,k;\lambda,l}(X) = \langle [D, \Pi_{\mathscr{N}} \,\mathrm{grad}\, J_\mu^k(X)], \mathrm{grad}\, J_\lambda^l(X) \rangle \,.$$

Here D is the derivation $\zeta\, \mathrm{d}/\mathrm{d}\zeta$. One can then verify:

(i) "Symmetry": $F_{\mu,k;\lambda,l} = F_{\lambda,l;\mu,k}$ + a term annihilated by all $\partial/\partial t_{\nu,m}$.

(ii) Analog of (29):

$$\frac{\partial}{\partial t_{\nu,1}} F_{\mu,k;\lambda;l}(X) = \frac{\partial}{\partial t_{\mu,k}} \langle \mathrm{grad}\, J_\lambda^{l+1}(X), \mathrm{grad}\, J_\mu^{k+1}(X_0) \rangle$$

(X_0 is the coefficient of ζ^0 in X; it is taken to lie in a fixed Cartan subalgebra of $\mathscr{g}$)

(iii) Analog of the last corollary:

$$\frac{\partial}{\partial t_{\nu,m}} F_{\mu,k;\lambda,l} = \frac{\partial}{\partial t_{\mu,k}} F_{\nu,m;\lambda,l}\,.$$

Here, incidentally, is another reason for our avoiding general semisimple Lie algebras. The definition of $F_{\mu,k;\lambda,l}$ makes it embarrassingly clear that everything should take place on the *affine algebra* $\mathscr{g} \otimes \mathbb{C}[\zeta, \zeta^{-1}]$ + center + derivation. Our attempts to put the Hamiltonian system directly on this extension of the loop algebra have failed so far. It should be remarked that *all* formulas involving fluxes of conservation laws or τ-functions implicitly bring in the derivation $\zeta\, \mathrm{d}/\mathrm{d}\zeta$. For example, our F_{kj}'s above can be written (eq. (30))

$$F_{kj} = \mathrm{Tr}(kQ_0 Q_{k+j} + (k-1)Q_1 Q_{k+j-1} + \cdots + Q_{k-1}Q_{j+1}) + k\phi_{k+l}\,.$$

The k, $(k-1), \ldots$ came from $\zeta\, \mathrm{d}\zeta^k/\mathrm{d}\zeta = k\zeta^k$, etc. Or, the "vertex operator" of [5] has the general form

$$\exp\left(\sum \zeta^k t_k + \sum \frac{1}{k} \zeta^{-k} \frac{\partial}{\partial t_k}\right).$$

Again, the $1/k$ comes from $\zeta\, \mathrm{d}/\mathrm{d}\xi$, but in *all* studies to date, the derivation (and hence the complete affine Lie algebra) has only played an implicit role.

We consider this to be the main obstacle to a comprehensive theory.

5. Vectorfields that commute with $\mathscr{X}_{\phi_N}$

We now turn to the translation of the Lie-algebraic framework above to the more familiar language of differential algebra. At the end of section 4.1, we saw that it was natural to introduce new variables (on the subset $\mathscr{M} \subset \mathscr{K}^\perp$, at least):

$$e_1 = q,\quad f_1 = r,\quad e_2 = \frac{\mathrm{i}}{2} q_x,\quad f_2 = \frac{\mathrm{i}}{2} r_x, \ldots\,.$$

There is no problem as long as q, q_x, q_{xx} etc., are thought of as mere symbols. If q_x is to mean $\partial q/\partial x$, however, a difficulty does arise.

Let ψ be any polynomial in the e's, f's and h's. If it be taken as Hamiltonian, there will be the equations

$$\dot{e}_j = \{e_j, \psi\}$$

and so on for f, h.

These will make sense in the variational framework, where $e_1 = q$ and $f_1 = r$ are functions of x, when relations like

$$\left(\frac{\partial e_1}{\partial x}\right)^{\cdot} = \frac{\partial \dot{e}_1}{\partial x},$$

$$\left(\frac{\partial \dot{e}_1}{\partial x}\right)^{\cdot} = \frac{\partial \ddot{e}_1}{\partial x},$$

hold. That is necessary because all Hamiltonian derivations

$$\dot{q} = \frac{\delta F}{\delta r}, \quad \dot{r} = -\frac{\delta F}{\delta q}$$

commute with $\partial/\partial x$. Hence, a vectorfield on $\mathscr{K}^\perp$ can be meaningful in the differential algebra interpretation of AKNS only when it commutes with $\partial/\partial x$, i.e., with the vectorfield $\mathscr{X}_{\phi_1}$.

We will want to have the freedom of choosing any t_N as the distinguished x, so we must understand vectorfields on $\mathscr{K}^\perp$ commuting with a fixed t_N. That is the question taken up in this section. From here on, our results are known only for sl(2), and we can make no more comments about other Lie algebras.

First, we prove that the Hamiltonians ϕ_k are a maximal commuting family in a very strong sense.

Proposition 8. Let ψ be a polynomial in e's, f's, and h's that commutes with a fixed $\mathscr{X}_{\phi_N}$. Then ψ is a polynomial of the ϕ_k, $k \geqslant 1$, and of h_0, h_0^{-1}, e_0, f_0.

Proof. Instead of e_j, f_j, h_j, $j \geqslant 0$, it will be convenient to use, as coordinates on $\mathscr{K}^\perp$,

$$e_j, f_j, j \geqslant 0;\ h_0;\ \phi_j, j \geqslant 1\,.$$

To see that this is sensible, remember the definition

$$-\phi_k = \sum_{j=0}^{k} h_i h_{k-1} + e_i f_{k-1}\,. \tag{32}$$

$-\phi_0 = h_0^2 + e_0 f_0$, so it is not possible to replace h_0 by ϕ_0, e_0, f_0. But from h_1 on, one can solve (32) for h_k in terms of ϕ_k and the preceding h_j, which – with exception of h_0 – are expressed in terms of $\phi_j, j < k$.

Now take the polynomial ψ to be a polynomial in $e_j, f_j, j \geqslant 0$; $\phi_j, j \geqslant 1$; h_0, h_0^{-1} (h_0^{-1} enters when one solves (32) for h_k).

ψ contains finitely many e_j only. Let e_R be the e_j with highest subscript in ψ. So ψ is of the form

$$\psi = \sum_\alpha e_R^\alpha \rho_\alpha\,,$$

where ρ_α is a polynomial involving at most e_j with $j \leqslant R-1$. From $\{\psi, \phi_N\} = 0$ follows

$$0 = \sum_\alpha \alpha \{e_R, \phi_N\} e_R^{\alpha-1} \rho_\alpha + e_R^\alpha \{\rho_\alpha, \phi_N\}\,.$$

There is a term $2h_0 \Sigma_\alpha \alpha e_{R+N} e_R^{\alpha-1} \rho_\alpha$ on the right, coming from $\partial e_R/\partial t_N$ (cf. (21)). The rest of the right side involves at worst e_j with $j \leqslant R+N-1$. So

$$\sum \alpha e_R^{\alpha-1} \rho_\alpha = 0\,,$$

because there is no other way to cancel the e_{R+N}. Thus

$$\frac{\partial}{\partial e_R} \sum e_R^\alpha \rho_\alpha = 0\,,$$

which means that e_R in fact does not occur in ψ for $R \geqslant 1$. Likewise, f_R, $R \geqslant 1$, is absent. ∎

Remark. This argument does not exclude h_0, e_0, f_0, since these are in involution with ϕ_N.

It follows from this proposition that of the many variational equations for the $q(x)$, $r(x)$ in the AKNS setup ($x = t_1$), *only the soliton hierarchy* (9) *is Hamiltonian in the* $\mathscr{K}^{\perp}$ *language*. If we are interested in vectorfields (on $\mathscr{K}^{\perp}$) that commute with $\mathscr{X}_{\phi_N}$, therefore, we unavoidably leave our Hamiltonian framework.

Let $\mathscr{X}$ be a vectorfield commuting with $\mathscr{X}_{\phi_N}$ (on $\mathscr{K}^{\perp}$). Suppose that the components of $\partial/\partial e_j$, $\partial/\partial f_j$, $\partial/\partial h_j$, $j = 1, \ldots, N$, are polynomial in e, f, h. For $j = 0$, they are necessarily zero.

A new set of coordinates will be required for the description of such a vectorfield. This idea is to treat $e_1, \ldots, e_N, f_1, \ldots, f_N$ as though they are functions of t_N, and to think of all other variables as differential polynomials (w.r. to t_N) in these basic ones. It will be sufficient for our purposes to take the vectorfields only on the subset $\mathscr{M} \subset \mathscr{K}^{\perp}$ ($h_0 = -\mathrm{i}$, $e_0 = f_0 = h_1 = 0$, $h^2 + ef = -1$, see 3.1). On $\mathscr{M}$, one just needs the coordinates e_j, f_j, or the differential polynomials we will substitute for them. The relations $\phi_k = 0$ imply inductively that the h_k are differential polynomials of the desired form, once this has been proved for the e's and f's.

Lemma. Eq. (21) may be solved to express the derivatives

$$\frac{\partial^\alpha e_j \text{ or } f_j}{\partial t_N^\alpha}, \quad j = 1, \ldots, N, \quad \alpha = 0, 1, 2, \ldots$$

as polynomials in e_i, f_i, $i \geqslant 1$; conversely the e_i, f_i, $i \geqslant 1$, are polynomials in these derivatives.

The proof is a straightforward induction (remember that we are working on $\mathscr{M}$). The vectorfield $\mathscr{X}$ on $\mathscr{M}$ is now written in the coordinates

$$e_j^{(\alpha)} = \frac{\partial^\alpha e_j}{\partial t_N^\alpha}, \quad f_j^{(\alpha)}, j = 1, \ldots, N, \alpha \geqslant 0:$$

$$\mathscr{X} = \sum_{j,\alpha} X_{j\alpha}^e \frac{\partial}{\partial e_j^{(\alpha)}} + X_{j\alpha}^f \frac{\partial}{\partial f_j^{(\alpha)}}.$$

This much could be done for any $\mathscr{X}$. Since $[\mathscr{X}, \partial/\partial t_N] = 0$, however, the coefficients $X_{j\alpha}$ are quite special. For instance, the "integral curves" of $\mathscr{X}$ are defined by

$$(e_j^{(\alpha)})^{\cdot} = X_{j\alpha}^e.$$

But

$$\frac{\partial}{\partial t_N}(e_j^{(\alpha)\cdot}) = \left(\frac{\partial}{\partial t_N} e_j^{(\alpha)}\right)^{\cdot} = (e_j^{(\alpha+1)})^{\cdot},$$

so

$$\frac{\partial}{\partial t_N} X_{j\alpha}^e = X_{j,\alpha+1}^e.$$

$\mathscr{X}$ is therefore determined by $2N$ differential polynomials, which we will call A_j, B_j ($j = 1, \ldots, N$), and has the form

$$\sum_{\alpha=0}^{\infty} \sum_{j=1}^{N} A_j^{(\alpha)} \frac{\partial}{\partial e_j^{(\alpha)}} + B_j^{(\alpha)} \frac{\partial}{\partial f_j^{(\alpha)}}, \tag{33}$$

superscript α denoting the αth t_N derivative.

In the next section, we study a formal variational calculus associated with vectorfields of the form (33).

6. Lie algebras of vectorfields and a variational calculus

This section is independent of the preceding ones, so we will use a neutral notation: $q_j^{(\alpha)}$, $r_j^{(\alpha)}$ instead of the $e_j^{(\alpha)}, f_j^{(\alpha)}$ in (33).

Let V be the vectorspace generated by $q_j^{(\alpha)}$, $r_j^{(\alpha)}$, $j = 1, \ldots, N$, $\alpha \geqslant 0$, let $\mathscr{B}$ be the polynomial algebra generated by these symbols, and let $\partial (= \partial/\partial t_N)$ be the derivation of $\mathscr{B}$ defined by

$$\partial q_j^{(\alpha)} = q_j^{(\alpha+1)}, \quad \partial r_j^{(\alpha)} = r_j^{(\alpha+1)}.$$

As in section 2, let $\overline{\mathscr{B}} = \mathscr{B}/\text{Image}\,\partial$.

We want to associate Hamiltonian derivations on $\overline{\mathscr{B}}$ to certain vectorfields on V in such a way that commuting vectorfields go to commuting derivations. This was done by Gel'fand–Dikii [15] in connection with KdV equation; they dealt with a differential algebra with one generator. Our generalization is conceptually straightforward, but a more awkward notation is necessary. This whole section is needed only to tie down some loose logical ends at the conclusion of the paper. The reader who skips directly to section 7 will miss little of the idea and will avoid some messy formulas.

For two vector fields* on V,

$$\xi = \sum_{\alpha=0}^{\infty}\sum_{j=1}^{N}\left(a_j^{(\alpha)}\frac{\partial}{\partial q_j^{(\alpha)}} + b_j^{(\alpha)}\frac{\partial}{\partial r_j^{(\alpha)}}\right)$$

and

$$\eta = \sum_{\alpha=0}^{\infty}\sum_{j=1}^{N}\left(c_j^{(\alpha)}\frac{\partial}{\partial q_j^{(\alpha)}} + d_j^{(\alpha)}\frac{\partial}{\partial r_j^{(\alpha)}}\right),$$

the commutator $[\xi,\eta]$ has $\partial/\partial q_j^{(\alpha)}$ component

$$\sum_{\beta=0}^{\infty}\sum_{i=0}^{N}\left(a_i^{(\beta)}\frac{\partial c_j^{(\alpha)}}{\partial q_i^{(\beta)}} - c_i^{(\beta)}\frac{\partial a_j^{(\alpha)}}{\partial q_i^{(\beta)}} + b_i^{(\beta)}\frac{\partial c_j^{(\alpha)}}{\partial r_i^{(\beta)}} - d_i^{(\beta)}\frac{\partial a_j^{(\alpha)}}{\partial r_i^{(\beta)}}\right) \tag{34}$$

and $\partial/\partial r_j^{(\alpha)}$ component

$$\sum_{\beta=0}^{\infty}\sum_{i=0}^{N}\left(b_i^{(\beta)}\frac{\partial d_j^{(\alpha)}}{\partial r_i^{(\beta)}} - d_i^{(\beta)}\frac{\partial b_j^{(\alpha)}}{\partial r_i^{(\beta)}} + a_i^{(\beta)}\frac{\partial d_j^{(\alpha)}}{\partial q_i^{(\beta)}} - c_i^{(\beta)}\frac{\partial b_j^{(\alpha)}}{\partial q_i^{(\beta)}}\right). \tag{35}$$

Now we transfer vectorfields and their commutators from V to $\bar{\mathscr{B}}$. Let $\alpha:\mathscr{B}\to\bar{\mathscr{B}}$ be the canonical projection, and write $\alpha(F)=\bar{F}$.

A vectorfield ξ on V gives rise to a derivation $\bar{\xi}$ on $\bar{\mathscr{B}}$ via

$$\bar{\xi}\bar{F} = \alpha(\xi F). \tag{36}$$

This definition is correct, because ξ takes $\partial\mathscr{B}$ into itself. Note that (36) implies

$$\overline{[\xi,\eta]} = [\bar{\xi},\bar{\eta}]. \tag{37}$$

Lemma 1.

$$\bar{\xi}\bar{F} = \alpha\left(\sum_{j=1}^{N} a_j^{(0)}\frac{\delta F}{\delta q_j} + b_j^{(0)}\frac{\delta F}{\delta r_j}\right) \tag{38}$$

Proof.

$$\begin{aligned}\xi F &= \sum_{\alpha=0}^{\infty}\sum_{j=1}^{N}\left(a_j^{(\alpha)}\frac{\partial F}{\partial q_j^{(\alpha)}} + b_j^{(\alpha)}\frac{\partial F}{\partial r_j^{(\alpha)}}\right)\\ &= \sum_{j=1}^{N}\sum_{\alpha=0}^{\infty} a_j^{(0)}(-\partial)^{\alpha}\frac{\partial F}{\partial q_j^{(\alpha)}} + b_j^{(0)}(-\partial)^{\alpha}\frac{\partial F}{\partial r_j^{(\alpha)}} + \partial P\end{aligned}$$

(for a certain $P\in\partial B$)

$$= \sum_{j=1}^{N}\left(a_j^{(0)}\frac{\delta F}{\partial q_j} + b_j^{(0)}\frac{\delta F}{\delta r_j}\right) + \partial P. \quad \blacksquare$$

Definition.

$$\Omega^{(1)}(\bar{\xi}) = \alpha\left(\sum_{j=1}^{N} r_j a_j^{(0)}\right),$$

$$\Omega^{(2)}(\bar{\xi},\bar{\eta}) = -\alpha\left(\sum_{j=1}^{N} b_j^{(0)}c_j^{(0)} - a_j^{(0)}d_j^{(0)}\right).$$

(Note that $\Omega^{(2)}$ is weakly non-degenerate on $\bar{\xi},\bar{\eta}$).

Lemma 2. $-\mathrm{d}\Omega^{(1)} = \Omega^{(2)}$.

Proof.

$$\begin{aligned}\mathrm{d}\Omega^{(1)}(\bar{\xi},\bar{\eta}) &= \bar{\xi}\Omega^{(1)}(\bar{\eta}) - \bar{\eta}\Omega^{(1)}(\bar{\xi}) - \Omega^{(1)}([\bar{\xi},\bar{\eta}])\\ &= \bar{\xi}\alpha\left(\sum_{j=1}^{N} r_j c_j^{(0)}\right) - \bar{\eta}\alpha\left(\sum_{j=1}^{N} r_j a_j^{(0)}\right)\\ &\quad -\alpha\left(\sum_{j=1}^{N} r_j\sum_{k=1}^{N}\sum_{\alpha=0}^{\infty}\left(a_k^{(\alpha)}\frac{\partial c_j^{(0)}}{\partial q_k^{(\alpha)}} - c_k^{(\alpha)}\frac{\partial a_j^{(0)}}{\partial q_k^{(\alpha)}} + b_k^{(\alpha)}\frac{\partial c_j^{(0)}}{\partial r_k^{(\alpha)}} - d_k^{(\alpha)}\frac{\partial a_j^{(0)}}{\partial r_k^{(\alpha)}}\right)\right.\end{aligned}$$

* From now on, all "vectorfields" are of this form: see [15] for the official language.

$$= \alpha\left(\sum_{j=1}^{N} r_j \left(\sum_{k=1}^{N} \sum_{\alpha=0}^{\infty} a_k^{(\alpha)} \frac{\partial c_j^{(0)}}{\partial q_k^{(\alpha)}} + b_k^{(\alpha)} \frac{\partial c_j^{(0)}}{\partial r_k^{(\alpha)}} \right) + \sum_{j=1}^{N} b_j^{(0)} c_j^{(0)} \right) - \alpha\left(\sum_{j=1}^{N} r_j \left(\sum_{k=1}^{N} \sum_{\alpha=0}^{\infty} c_k^{(\alpha)} \frac{\partial a_j^{(0)}}{\partial q_k^{(\alpha)}} + d_k^{(\alpha)} \frac{\partial a_j^{(0)}}{\partial r_k^{(\alpha)}} \right) + \sum_{j=1}^{k} a_j^{(0)} d_j^{(0)} \right)$$

$- \alpha$(last term in preceding equality)

$$= \alpha\left(\sum_{j=1}^{N} b_j^{(0)} c_j^{(0)} - a_j^{(0)} d_j^{(0)} \right). \quad ■$$

Definition. For $F \in \mathscr{B}$, let

$$\xi_F = \sum_{\alpha=0}^{\infty} \sum_{j=1}^{N} \left(\frac{\delta F}{\delta r_j}\right)^{(\alpha)} \frac{\partial}{\partial q_j^{(\alpha)}} - \left(\frac{\delta F}{\delta q_j}\right)^{(\alpha)} \frac{\partial}{\partial r_j^{(\alpha)}}.$$

Lemma 3. $\bar{\eta}\bar{F} = \Omega^{(2)}(\xi_F, \bar{\eta})$, i.e., ξ_F is a Hamiltonian vectorfield for the formal symplectic form $\Omega^{(2)}$.

Proof.

$$\Omega^{(2)}(\xi_F, \bar{\eta}) = \alpha\left(\sum_{j=1}^{N} \frac{\delta F}{\delta q_j} c_j^{(0)} + \frac{\delta F}{\delta r_j} d_j^{(0)} \right) = \bar{\eta}\bar{F},$$

by lemma 1. ■

Corollary. On $\bar{\mathscr{B}}$, we have a Poisson bracket

$$\{\bar{F}, \bar{G}\} = \Omega^{(2)}(\xi_F, \xi_G) = \alpha\left(\sum_{j=1}^{N} \frac{\delta F}{\delta q_j}\frac{\delta G}{\delta r_j} - \frac{\delta F}{\delta r_j}\frac{\delta G}{\delta q_j} \right).$$

The map $\bar{F} \mapsto \xi_F$ gives a Hamiltonian structure,

$$\xi_F(\bar{G}) = -\{\bar{F}, \bar{G}\}.$$

Proof. We compute $x_j^{(\alpha)}, y_j^{(\alpha)}$ in

$$[\xi_F, \xi_G] = \sum_{\alpha=0}^{\infty} \sum_{j=1}^{N} \left(x_j^{(\alpha)} \frac{\partial}{\partial q_j^{(\alpha)}} + y_j^{(\alpha)} \frac{\partial}{\partial r_j^{(\alpha)}} \right).$$

Since $\Omega^{(2)} = -\mathrm{d}\Omega^{(1)}$ is closed, $\mathrm{d}\Omega^{(2)} = 0$. So

$$\begin{aligned}
0 &= \mathrm{d}\Omega^{(2)}(\xi_F, \xi_G, \bar{\eta}) = \xi_F \Omega^{(2)}(\xi, \bar{\eta}) - \xi_G \Omega^{(2)}(\xi_F, \bar{\eta}) \\
&\quad + \bar{\eta}\Omega^{(2)}(\xi_F, \xi_G) - \Omega^{(2)}([\xi_F, \xi_G], \bar{\eta}) \\
&\quad + \Omega^{(2)}([\xi_F, \bar{\eta}], \xi_G) - \Omega^{(2)}([\xi_G, \bar{\eta}], \xi_F) \\
&= -\xi_F \bar{\eta}\bar{G} + \xi_G \bar{\eta}\bar{F} + \bar{\eta}\{\bar{F}, \bar{G}\} \\
&\quad + \alpha\left(\sum_{j=1}^{N} y_j^{(0)} c_j^{(0)} - x_j^{(0)} d_j^{(0)} \right) + [\xi_F, \bar{\eta}]\bar{G} - [\xi_G, \bar{\eta}]\bar{F} \\
&= -\bar{\eta}\xi_F \bar{G} + \bar{\eta}\xi_G \bar{F} + \bar{\eta}\{\bar{F}, \bar{G}\} \\
&\quad + \alpha\left(\sum_{j=1}^{N} y_j^{(0)} c_j^{(0)} - x_j^{(0)} d_j^{(0)} \right) \\
&= -\bar{\eta}\alpha\left(\sum_{j=1}^{N} \left(-\frac{\delta F}{\delta r_j}\frac{\delta G}{\delta q_j} + \frac{\delta F}{\delta q_j}\frac{\delta G}{\delta r_j} + \frac{\delta G}{\delta r_j}\frac{\delta F}{\delta q_j} - \frac{\delta G}{\delta q_j}\frac{\delta F}{\delta r_j} - \frac{\delta F}{\delta q_j}\frac{\delta G}{\delta r_j} + \frac{\delta F}{\delta r_j}\frac{\delta G}{\delta q_j} \right) \right) \\
&\quad + \alpha\left(\sum_{j=1}^{N} y_j^{(0)} c_j^{(0)} - x_j^{(0)} d_j^{(0)} \right) \\
&= -\bar{\eta}\{\bar{F}, \bar{G}\} + \alpha\left(\sum_{j=1}^{N} y_j^{(0)} c_j^{(0)} - x_j^{(0)} d_j^{(0)} \right) \\
&= (\text{by lemma 2}) - \alpha\left(\sum_{j=1}^{N} c_j^{(0)} \frac{\delta\{F, G\}}{\delta q_j} + d_j^{(0)} \frac{\delta\{F, G\}}{\delta r_j} - y_j^{(0)} c_j^{(0)} + x_j^{(0)} d_j^{(0)} \right).
\end{aligned}$$

All this started out as zero in $\bar{\mathscr{B}}$, with $c_j^{(0)}$ and $d_j^{(0)}$ arbitrary. So,

$$c_j^{(0)}\left(\frac{\delta\{F, G\}}{\delta q_j} - y_j^{(0)} \right)$$

is a perfect ∂-derivative for any $c_j^{(0)} \in \mathscr{B}$, as is $d_j^{(0)}(\delta\{F, G\}/\delta r_j + x_j^{(0)})$. It follows that

$$x_j^{(0)} = -\frac{\delta\{F, G\}}{\delta r_j}, \quad y_j^{(0)} = \frac{\delta\{F, G\}}{\delta g_j}. \quad ■ \tag{39}$$

Proposition 9. $[\xi_F, \xi_G] = 0$ iff $\{\bar{F}, \bar{G}\} = 0$.

Proof. By lemma 3 and eq. (39)

$$[\xi_F, \xi_G] = 0 \quad \text{iff} \frac{\delta\{F, G\}}{\delta q_j} = \frac{\delta\{F, G\}}{\delta r_j} = 0,$$

$$j = 1, \ldots, N,$$

so that

$$[\xi_F, \xi_G] = 0 \quad \text{iff}\, \{F, G\} \in \text{Image}\, \partial \text{ iff } \{\bar{F}, \bar{G}\} = 0\,. \qquad \blacksquare$$

7. Lax equations as Hamiltonian derivations

The theory of the last section is now applied to certain vectorfields on $\mathscr{K}^\perp$ that commute with $\mathscr{X}_{\phi_N}$: namely, vectorfields of the form ξ_F, with F a differential polynomial in $e_1, \ldots, e_N$, $f_1, \ldots, f_N$ (playing the role of q_j, r_j). In particular, all X_{ϕ_j} will be shown to be of this form, so that (by prop. 9) they map to commuting Hamiltonian derivations on $\bar{\mathscr{B}}$ (the derivation ∂ is $\partial/\partial t_N$, as explained already).

It turns out that the most convenient variables q_j, r_j are in fact *not* the e_j, f_j themselves, but rather certain polynomials in the e_j, f_j. Define formal series

$$\tilde{e} = \frac{e}{\sqrt{\mathrm{i} - h}} = \sum_{j=1}^{\infty} \tilde{e}_j \zeta^{-j}, \quad \tilde{f} = \frac{f}{\sqrt{\mathrm{i} - h}} = \sum_{j=1}^{\infty} \tilde{f}_j \zeta^{-j}, \tag{40}$$

subject to $h^2 + ef = -1$ and $h_0 = -\mathrm{i}$, $e_0 = f_0 = h_1 = 0$. It is clear that $\tilde{e}_k$ and $\tilde{f}_k$ are polynomials in $e_j, f_j, j \leqslant k$, and conversely e_k and f_k are polymomials in $\tilde{e}_j, \tilde{f}_j, j \leqslant k$. Specifically,

$$\tilde{e}_k = \frac{1}{\sqrt{2\mathrm{i}}} e_k + \text{polynomial in } e_j, f_j, \quad j \leqslant k - 1\,,$$

and likewise for $\tilde{f}_k$. We may therefore use the $\tilde{e}_k, \tilde{f}_k$ as coordinates on $\mathscr{M}$, and along the lines of section 6, we may write vectorfields commuting with $\mathscr{X}_{\phi_N}$ in the form (33) with twiddles over $e_j^{(\alpha)}$ and $f_j^{(\alpha)}$.

$\tilde{e}$ and $\tilde{f}$ are not particularly convenient for the study of the Hamiltonian structure on $\mathscr{K}^\perp$, but they are fundamental in the variational Hamiltonian framework, as we shall see in this section†.

Our object is to calculate the variational derivatives of the F_{kj} with respect to $\tilde{e}_j, \tilde{f}_j$, $j = 1, \ldots, N$, where the F_{kj} are considered to be differential polynomials in t_N-derivatives of these $\tilde{e}_j$, $\tilde{f}_j$. This will be done in one fell swoop, via generating series. The calculation is based on a characterization of functional derivatives used to great advantage by Gel'fand–Dikii in the study of stationary KdV equations [16].

Let $F \in \mathscr{B}$, the (polynomial) differential algebra with generators $q_j, r_j, j = 1, \ldots, N$ and derivation $\mathrm{d}/\mathrm{d}x$. There exists a unique 1-form $\omega^{(1)}$, and unique $A_j, B_j \in B$, such that

$$\mathrm{d}F = \sum_{j=1}^{N} A_j \,\mathrm{d}q_j + B_j \,\mathrm{d}r_j + \frac{\mathrm{d}}{\mathrm{d}x} \omega^{(1)}\,. \tag{41}$$

(The exterior derivative d has the natural properties, and it commutes with $\mathrm{d}/\mathrm{d}x$; see [17] for definitions and a proof of (41)). Furthermore,

$$A_j = \frac{\delta F}{\delta q_j}, \quad B_j = \frac{\delta F}{\delta r_j},$$

according to our earlier definition in section 2.

The idea behind (41) is of course the familiar calculation in the calculus of variations,

$$\delta \int F \,\mathrm{d}x = \int \left(\sum \frac{\delta F}{\delta q_j} \delta q_j + \frac{\delta F}{\delta r_j} \delta r_j + \text{perfect } x\text{-derivative} \right) \mathrm{d}x\,.$$

We will prove the following formula:

Theorem 2.

$$\mathrm{d}F_{Nk} = -\frac{\mathrm{i}}{2} k \sum_{j+l=N+1} \frac{\partial \tilde{f}_j}{\partial t_{k-1}} \mathrm{d}\tilde{e}_l - \frac{\partial \tilde{e}_j}{\partial t_k} \mathrm{d}\tilde{f}_l + \frac{\partial}{\partial t_N} \left(-\frac{\mathrm{i}}{2} (\tilde{f}_\zeta \,\mathrm{d}\tilde{e} - \tilde{e}_\zeta \,\mathrm{d}\tilde{f}) \right)_{k+1}. \tag{42}$$

† We discovered $\tilde{e}$ and $\tilde{f}$ in the course of work on the stationary AKNS equations in 1980, more or less by accident, after long calculations with formal series. There is still no satisfactory explanation of their origin, nor a good theoretical reason for their significance. They appear again in paper III.

Explanation. $F_{Nk}, k \geqslant 1$, is thought of as differential t_N-polynomial in $\tilde{e}_j, \tilde{f}_j, j=1,\dots,N$. Subscript $(\ )_{k+1}$ denotes the coefficient of ζ^{-k-1} in a formal series, and subscript ζ denotes $d/d\zeta$. $\partial\tilde{e}_j/\partial t_{k-1}$ denotes the differential t_N-polynomial obtained when this derivative is calculated according to (21) and (40). *Always,* $h_0 = -i$, $h_1 = e_0 = f_0 = 0$, $h^2 + ef = -1$.

From (42) and (41), we get

$$\frac{\partial \tilde{f}_j}{\partial t_{k-1}} = \frac{2i}{k}\frac{\delta F_{Nk}}{\delta \tilde{e}_{N+1-j}}, \qquad \frac{\partial \tilde{e}_j}{\partial t_{k-1}} = -\frac{2i}{k}\frac{\delta F_{Nk}}{\delta \tilde{f}_{N+1-j}}. \tag{43}$$

So we have – informally – the *third main result* advertised in the introduction: the t_{k-1}-flow $(k \geqslant 1)$ is Hamiltonian in the differential algebra of "functions of t_N". The Hamiltonian is $(2i/k)F_{Nk}$, and the conjugate variables are $\tilde{f}_j$ and $\tilde{e}_{N+1-j}$, $j=1,\dots,N$. Some details are needed to connect the various algebraic structures in a rigorous way.

We will first prove theorem 2; this requires a lengthy and – for a change – somewhat subtle calculation. Then we bring in the variational Hamiltonian framework of the preceding section to explain more fully the transition from $\mathscr{K}^\perp$ to a differential algebra.

Proof of theorem 2.† We introduce the differentials de_j, df_j and their generating series de, df. It will be necessary to calcuate expressions of the form

$$\{h^2 + ef, de'\}$$

$(e' = e(\eta))$. By general principles, d commutes with any Lie derivative $\mathscr{L}_{\mathscr{X}}$, so

$$\{h^2 + ef, de'\} = \sum \mathscr{L}_{\mathscr{X}_{\phi_k}} de_j \zeta^{-k}\eta^{-j} = \sum d\frac{\partial e_j}{\partial t_k}\zeta^{-k}\eta^{-j} = d\{h^2 + ef, e'\}. \tag{44}$$

The first relation to be derived is

$$-\frac{2\eta^{-2}\zeta^{-1}}{(\zeta^{-1}-\eta^{-1})^2} d(ef' + e'f + 2hh' + 2) = \{h^2 + ef, f'\}_\eta \frac{de}{h} - \{h^2 + ef, e'\}_\eta \frac{df}{h} + \left\{h^2 + ef, \frac{e'_\eta\, df' - f'_\eta\, de'}{h'}\right\}. \tag{45}$$

To check this, calculate $\{h^2 + ef, (e'_\eta\, df' - f'_\eta\, de')/h'\}$ using (22) and (44). (Subscript η denotes $d/d\eta$). Calculate the other terms from (22), and simplify, using the relations

$$2h\, dh + e\, df + f\, de = 0, \quad 2h'h'_\eta + e'f'_\eta + f'e'_\eta = 0$$

that follow from $h^2 + ef = -1$.

Remark. The tricky part of (45) is guessing the 1-form $(h')^{-1}(e'_\eta\, df' - f'_\eta\, de')$. It is the counterpart of the 1-form used by Gel'fand–Dikii in [16]; compare their equation (4.4).

The left side of (45) is $4i\eta^{-1}$ times d of the generating function (27) of the F_{kj}, which is what is wanted on the left side of (42).

We must now replace e', f' on the right by $\tilde{e}'$, $\tilde{f}'$.

To this end, we observe that $\eta\{h^2 + ef, e'\} = 2(eh' - e'h)/(\zeta^{-1} - \eta^{-1})$ is symmetric in ζ, η, so that

$$\eta\{h^2 + ef, e'\} = \zeta\{h'^2 + e'f', e\},$$

with a similar expression for $\{\ , f'\}$.

The right side of (45) becomes, after $\{h^2 + ef, e'\} \mapsto \eta^{-1}\zeta\{h'^2 + e'f', e\}$,

$$\left[\eta^{-1}\zeta\left(\{h'^2 + e'f', f\}\frac{de}{h} - \{h'^2 + e'f', e\}\frac{df}{h}\right)\right]_\eta + \left\{h^2 + ef, \frac{e'_\eta\, df' - f'_\eta\, de'}{h'}\right\}. \tag{46}$$

† The proof has nothing to do with Lie algebras; we realize that an intrinsic argument is desirable, and hope to be able to provide one eventually. This is the only important result of the paper we do not yet have for arbitrary simple $\mathscr{g}$ in place of sl(2).

Now replace e by $\tilde{e}(\mathrm{i}-h)^{1/2}$:

$$\{h'^2+e'f',e\}=(\mathrm{i}-h)^{1/2}\{h'^2+e'f',\tilde{e}\} -\tfrac{1}{2}(\mathrm{i}-h)^{-1/2}\tilde{e}\{h'^2+e'f',h\}\,, \quad (47)$$

$$\mathrm{d}e=(\mathrm{i}-h)^{1/2}\,\mathrm{d}\tilde{e}-\tfrac{1}{2}(\mathrm{i}-h)^{-1/2}\tilde{e}\,\mathrm{d}h\,. \quad (48)$$

From $h^2+ef=-1$ one finds

$$h+\mathrm{i}=\frac{ef}{\mathrm{i}-h}=\tilde{e}\tilde{f},$$

whence

$$\mathrm{d}h=\tilde{e}\,\mathrm{d}\tilde{f}+\tilde{f}\,\mathrm{d}\tilde{e}\,,$$

which is plugged into (48).

Another consequence of $h^2+ef=-1$ is

$$\{h'^2+e'f',h\}=\tilde{e}\{h'^2+e'f',\tilde{f}\}+\tilde{f}\{h'^2+e'f',\tilde{e}\}\,,$$

which is plugged into (47). Similar things are done to $f=\tilde{f}(\mathrm{i}-h)^{1/2}$.

A straightforward calculation now turns (46) into

$$[-2\eta^{-1}\zeta(\{h'^2+e'f',\tilde{f}\}\,\mathrm{d}\tilde{e}-\{h'^2+e'f',\tilde{e}\}\,\mathrm{d}\tilde{f})]_\eta + 2\{h^2+ef,\tilde{f}_\eta'\,\mathrm{d}\tilde{e}'-\tilde{e}_\eta'\,\mathrm{d}\tilde{f}'\}\,. \quad (49)$$

This is substituted for the right side of (45). Next, multiply (45) by $-\mathrm{i}/4$; the left side is then

$$\sum_{k=1,j=1}^{\infty}\mathrm{d}F_{kj}\eta^{-k-1}\zeta^{-j}\,.$$

On the new right side, substitute the series

$$\{h'^2+e'f',\tilde{e}\}=\sum_{k=0,j=1}^{\infty}\frac{\partial\tilde{e}_j}{\partial t_k}\eta^{-k}\zeta^{-j}, \quad \text{etc.},$$

and match powers. The desired formula (42) drops out. ■

There is a point of logic to be disposed of before theorem 2 can be applied. In (42), d is the exterior derivative with respect to the variables $e_j, f_j, j\geqslant 1$, on $\mathscr{M}$. In (41), with which (42) was compared to give the Hamiltonian equations (43), d is an exterior derivative defined on the generators q_j, r_j and extended to all of $\mathscr{B}$ so as to commute with the distinguished derivation $\mathrm{d}/\mathrm{d}x$. It had better be the case, therefore, that the differentials $\mathrm{d}F_{Nk}$ and $(\partial/\partial t_N)(\tilde{f}_\zeta\,\mathrm{d}\tilde{e}-\tilde{e}_\zeta\,\mathrm{d}\tilde{f})_{k+1}$ in (42) can be calculated in two ways:

Express them in terms of $\mathrm{d}\tilde{e}_j, \mathrm{d}\tilde{f}_j$, $j\geqslant 1$, then replace the $\tilde{e}_j$, $\tilde{f}_j$ by appropriate differential polynomials and pass to the differential-algebra basis $\mathrm{d}\tilde{e}_j^{(\alpha)}, \mathrm{d}\tilde{f}_j^{(\alpha)}$, $j=1,\dots,N$, or

Express everything as polynomials in $\tilde{e}_j^{(\alpha)}$, $\tilde{f}_j^{(\alpha)}$, $j=1,\dots,N$, and then apply d, requiring it to commute with $\partial/\partial t_N$.

The reader may convince himself that these procedures are indeed equivalent; the key is the commutativity of d and $\mathscr{L}_{\mathscr{X}_{\phi_N}}$.

Now we can describe quite precisely what happens to our t_k-flows when t_N is singled out.

$\mathscr{X}_{\phi_k}$ is a vectorfield that commutes with $\mathscr{X}_{\phi_N}$, and so it has the form (33),

$$X_{\phi_k}=\sum_{\alpha=0}^{\infty}\sum_{j=1}^{N}A_j^{(\alpha)}\frac{\partial}{\partial\tilde{e}_j^{(\alpha)}}+B_j^{(\alpha)}\frac{\partial}{\partial\tilde{f}_j^{(\alpha)}}\,.$$

The coefficients $A_j^{(0)}$, $B_j^{(0)}$ are of course exactly the differential polynomials we have denoted by

$$\frac{\partial\tilde{e}_j}{\partial t_k},\ \frac{\partial\tilde{f}_j}{\partial t_k}.$$

By (43), these are functional derivatives

$$\frac{\partial\tilde{e}_j}{\partial t_k}=-\frac{2\mathrm{i}}{k+1}\frac{\delta F_{N,k+1}}{\delta\tilde{f}_{N+1-j}},$$

$$\frac{\partial\tilde{f}_j}{\partial t_k}=\frac{2\mathrm{i}}{k+1}\frac{\delta F_{N,k+1}}{\delta\tilde{e}_{N+1-j}}.$$

Hence the vectorfield $\mathscr{X}_{\phi_k}$ on $\mathscr{K}^\perp$ gives rise to a vectorfield ξ_{F_k} on the differential algebra $\mathscr{B}$ (cf section 6), with $F_k = 2i/(k+1)F_{N,k+1}$. This induces a Hamiltonian derivation ξ_{F_k} on $\overline{\mathscr{B}}$. Finally, since the $\mathscr{X}_{\phi_k}$ commute with each other on $\mathscr{K}^{\perp}$, the corresponding ξ_{F_k} commute on $\mathscr{B}$, and by proposition 9, the variational Hamiltonians $\bar{F}_k$ are in involution with respect to the natural Poisson bracket on $\overline{\mathscr{B}}$.

8. Summary

We conclude this paper with a restatement of the overall logical thread of our approach.

1) The starting point is an infinite-dimensional Lie algebra. We think of this as having been obtained from a given equation by use of the ideas of Wahlquist–Estabrook. It is generally thought that the existence of a nontrivial infinite prolongation algebra indicates that the equation belongs to an infinite family of commuting flows.

2) The Lie algebra is written as a direct sum, an inner product is introduced and certain ad-invariant functions are identified; this is the Adler–Kostant–Symes setup. These functions are the Hamiltonians for our infinite set of commuting flows, $Q_{t_k} = [Q^{(k)}, Q]$, $k = 0, 1, \ldots$.

3) The Lax form of these equations immediately suggests that Q may be written $Q = VQ_0V^{-1}$, where $V_{t_k} = Q^{(k)}V$, a form which reveals the group theoretic nature of V. Thus, the auxiliary functions V (the old eigenfunctions) are introduced *after* the commuting flows. If the Lie algebra is graded, as it is in this paper, ζ, which is introduced as the grading parameter, becomes the "eigenvalue". However, up to this point no particular t_k is distinguished. If we distinguish $t_1 = x$ and take $V_{t_1} = Q^{(1)}V$ to be the eigenvalue problem, then we recover the AKNS flows. If we were to distinguish t_2 and pick $V_{t_2} = Q^{(2)}V$ to be the eigenvalue problem, there are *four* dependent variables (e_1, e_2, f_1, f_2) and a different set of flows is found, in a new function space. As one example, we take the special case $e_2 = f_2 = 0, f_1 = -\bar{e}_1$, with everything independent of t_{2N+1}, $N = 0, 1, \ldots$. Then the t_4 flow is the derivative nonlinear Schrödinger equation [19]

$$q_{t_4} = \mathrm{i}q_{t_2t_2} + (q^2\bar{q})_{t_2},$$

when

$$e_1 = q\,\mathrm{e}^{-\mathrm{i}\int_{-\infty}^{\infty} q\bar{q}\,\mathrm{d}t_2}.$$

4) The properties 1), 3), 5) and (we conjecture) 7) follow without resort to the auxiliary V equations. Properties 1), 3) have already been discussed in this paper, and the Hirota τ-function was introduced (sketchily). In paper IV, we discuss how the equations (21) for the h_j, e_j, f_j, with each variable thought of as $\partial/\partial t_j$-derivative of a potential (see (23)), can be written equivalently as equations for the potentials τ, $\sigma = e_1\tau$, $\rho = f_1\tau$. These are the Hirota equations.

5) On the other hand, results about Bäcklund transformations follow more naturally from the auxiliary equations for the V's. One knows, for example, what property V must have if it is to be derived from a Q which contains a soliton with given parameters. Multiplying V on the left by a matrix R to add a soliton is equivalent to multiplying Q on the left by R and on the right by R^{-1}. The new $\tilde{Q} = RQR^{-1}$ obeys the same equations in $t_1, t_2, \ldots$ as the old Q. Thus the equation $RQ = \tilde{Q}R$ provides the Bäcklund transformation between solutions of all equations in the family. We develop this point in more detail, and work out examples, in paper IV.

It is not yet clear to us whether this material generalizes beyond replacement of sl(2) by $\mathscr{g}$. Since our considerations are entirely local in the independent variables, there is in principle no exclusion of "higher space dimension" problems. The work of the Kyoto group on the Kadomtsev–Petviashvilii hierarchy indicates very clearly that the appropriate phase space is a subalgebra of gl(∞). We hope that eventually we will be able to identify from an algorithmic standpoint the appropriate algebra, its splitting, and the Hamiltonian framework, for all the soliton equations.

Acknowledgements

This work is an outgrowth of our efforts to relate the new theory of τ-functions and vertex operators to earlier soliton mathematics; the problems might never have been posed had not Flaschka been at the Kyoto University RIMS when the results reported in [5] were being discovered. Flaschka and (through him) the other authors are indebted to E. Date, M. Jimbo, M. Kashiwara, T. Miwa, and M. and Y. Sato for many explanations. We also wish to acknowledge the support of NSF and DOA, which made our collaboration possible.

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