EULER-POISSON EQUATIONS ON LIE ALGEBRAS AND THE N-DIMENSIONAL HEAVY RIGID BODY

By TUDOR RATIU

Abstract. The classical Euler-Poisson equations describing the motion of a heavy rigid body about a fixed point are generalized to arbitrary Lie algebras as Hamiltonian systems on coadjoint orbits of a tangent bundle Lie group. The N-dimensional Lagrange and heavy symmetric top are thereby shown to be completely integrable and a new family of completely integrable systems on semisimple Lie algebras is found.

Introduction. In recent years a lot of attention has been drawn to the problem of complete integrability of Hamiltonian systems. The most widely used method for finding conserved quantities is to represent the system in the form of a Lax equation $\dot{L} = [L, A]$, for L, A matrices or differential operators. Each time this can be done, the problem is a Hamiltonian system on adjoint orbits of a Lie group on its Lie algebra and the integrals are intimately connected to the ring of invariants; see [2], [3], [4], [19], [27], [37], [38], [44]. A famous problem which was solved by this method is the complete integrability of the free rigid N-dimensional body and its Lie algebraic analogues ([3], [27], [37]). It is based on the crucial observation of Arnold [6] that the second component of the geodesic spray of a left-invariant metric on a Lie group in body coordinates (i.e. trivialising the tangent bundle via left translations), the Euler vector field, is a Lax equation.

It is important however to consider more general Hamiltonians of the form kinetic (of a left invariant metric) plus potential energy. The typical problem we have in mind is the rigid body motion about a fixed point under the influence of gravity. The potential is no longer left-invariant and thus the Hamiltonian vector field in body coordinates mixes the group and algebra variables ([1, Section 4.4]). Nevertheless, for the heavy rigid body, Euler and Poisson deduced equations of motion only in vector form,

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i.e. the Euler-Poisson equations live on the product Lie algebra. This indicates that these equations are obtained by another technique, rather than just computing body coordinates, as was the case for the free rigid body. Assume (as in the case of the heavy rigid body) that the potential is invariant under the isotropy subgroup of the adjoint action at a certain element (the Oz-axis for the rigid body).

If one carries out the Marsden-Weinstein reduction of the Hamiltonian system with respect to this subgroup and its momentum map [26], the Euler-Poisson equations drop out. Moreover, the reduced manifold—the new phase space for the problem—turns out to be a symplectic covering of the adjoint orbit of the Lie algebra of the tangent bundle Lie group; the latter is the semidirect product of the group with its Lie algebra regarded as abelian group, under the Ad-action. Generically this covering is a diffeomorphism. This generalizes a result of Marsden-Weinstein [26] and is proved in the natural setting of duals for any Lie algebra in Section 2. In this way it is shown that Euler-Poisson equations are Hamiltonian on adjoint orbits of tangent bundle Lie groups. The Hamiltonian character of Euler-Poisson equations has been observed independently by Adler-van Moerbeke [3] and Iacob-Sternberg [15]. Section 1 presents background material relating especially to the Kirillov-Kostant-Souriau theorem and displays the relevant formulas to be used throughout the paper.

For the three-dimensional heavy rigid body, it has been noted [40] that the Euler-Poisson equations can be written in Lax form with variables formal matrix polynomials if and only if the equations describe the Lagrange (two of the principal moments of inertia are equal and the center of mass is on the axis of symmetry of the body) or the heavy symmetric top (all three moments of inertia are equal). After defining a general N-dimensional heavy rigid body in $so(N)$, the same result is proven in Section 4 and the necessary number of integrals is found in order to make these problems completely integrable. Section 3 presents the relevant material on Hamiltonian structures, Kac-Moody Lie algebras (Lie algebras of formal polynomials with coefficients in a given Lie algebra) and the Kostant-Symes involution theorem. Together with an involution theorem on semidirect products, these are used in Section 4 to prove the involution of the integrals and to find Lenard recursion relations between their Hamiltonian vector fields.

In Section 5 it is shown that the symmetric heavy top is a restriction to $so(N)$ of a more general Euler-Poisson equation on $sl(N; \mathbf{C})$ which is also shown to be completely integrable. The methods used here generalize directly to any semisimple Lie algebra. All these problems have linear flows on complex cylinders. We do not explicitly linearize them since this follows either directly from theorems in Adler-van Moerbeke [4], or from an obvious extension of the method in Ratiu-van Moerbeke [40].

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1. Preliminaries. In this section we concentrate the basic theorems and formulas to be used throughout the paper. All facts stated here are found either in the literature (specific references are included) or can be easily worked out from the given information. We suppose the reader is familiar with momentum maps and the Marsden-Weinstein reduction procedure; accounts thereof can be found in e.g. [1, Sections 4.2, 4.3], [25], [26], [17], [33], [46], [38], [37], [41]. In the sequel G denotes a Lie group and $\mathfrak{g}$ its Lie algebra with dual $\mathfrak{g}^*$.

1.1. Let $G_\nu = \{g \in G \mid \mathrm{Ad}^*_{g^{-1}}\nu = \nu\}$ be the isotropy subgroup at $\nu \in \mathfrak{g}^*$ of the co-adjoint action $g \mapsto \mathrm{Ad}^*_{g^{-1}}$ of G on $\mathfrak{g}^*$ and let $\mathfrak{g}_\nu = \{\xi \in \mathfrak{g} \mid (\mathrm{ad}\,\xi)^*\nu = 0\}$ be its Lie algebra; $(\mathrm{ad}\,\xi)^*\nu\cdot\eta = \nu([\xi, \eta])$. It is well-known that the set $\mathcal{O} = \{\nu \in \mathfrak{g}^* \mid \dim(\mathfrak{g}_\nu)$ is minimal$\}$ is Zariski open, hence open and dense in $\mathfrak{g}^*$ and that for any $\nu \in \mathcal{O}$, $\mathfrak{g}_\nu$, G_ν are abelian ([10], [35]). For any $\nu \in \mathfrak{g}^*$ we have

$$\{(\mathrm{ad}\,\xi)^*\nu \mid \xi \in \mathfrak{g}\} = \{\mu \in \mathfrak{g}^* \mid \mu \mid \mathfrak{g}_\nu \equiv 0\}. \tag{1.1}$$

For any $\nu \in \mathfrak{g}^*$, the map $g \mapsto \mathrm{Ad}^*_{g^{-1}}\nu$ gives an injective immersion $G/G_\nu \to \mathfrak{g}^*$ whose image is the co-adjoint orbit $G\cdot\nu \subset \mathfrak{g}^*$. Endow $G\cdot\nu$ with the manifold structure given by this immersion. This topology is in general finer than the one induced by $\mathfrak{g}^*$ and these coincide if and only if the induced topology is locally compact. This happens for instance when the co-adjoint action is proper, e.g. when G is compact.

1.2. Let $\bar{\mathfrak{g}}$ denote the underlying vector space of $\mathfrak{g}$ and denote by $S = G\ {}_{\mathrm{Ad}}\!\times \bar{\mathfrak{g}}$ the semidirect product of G with the abelian Lie group $\bar{\mathfrak{g}}$ by the adjoint representation $g \mapsto \mathrm{Ad}_g$ of G on $\bar{\mathfrak{g}}$. S is a Lie group with composition law $(g_1, \xi_1)(g_2, \xi_2) = (g_1g_2, \xi_1 + \mathrm{Ad}_{g_1}\xi_2)$, identity element $(e, 0)$, and inverse $(g, \xi)^{-1} = (g^{-1}, -\mathrm{Ad}_{g^{-1}}\xi)$. Its Lie algebra $\mathfrak{s}$ is the semidirect product $\mathfrak{g}_{\mathrm{ad}} \times \bar{\mathfrak{g}}$ of $\mathfrak{g}$ with $\bar{\mathfrak{g}}$ by the ad-representation of $\mathfrak{g}$ on $\bar{\mathfrak{g}}$, $(\mathrm{ad}\,\xi)(\eta) = [\xi, \eta]$. The bracket in $\mathfrak{s}$ is

$$[(\xi_1, \eta_1), (\xi_2, \eta_2)] = ([\xi_1, \xi_2], [\xi_1, \eta_2] + [\eta_1, \xi_2]), \tag{1.2}$$

and the adjoint action of S on $\mathfrak{s}$ is given by

$$\mathrm{Ad}_{(g,\zeta)}(\xi, \eta) = (\mathrm{Ad}_g \xi, \mathrm{Ad}_g \eta + [\zeta, \mathrm{Ad}_g \xi]). \tag{1.3}$$

Thus the Ad* and ad* actions of S and $\mathfrak{s}$ and $\mathfrak{s}^*$ are

$$\mathrm{Ad}^*_{(g,\zeta)^{-1}}(\mu, \nu) = (\mathrm{Ad}^*_{g^{-1}}\mu - (\mathrm{ad}\,\zeta)^*(\mathrm{Ad}^*_{g^{-1}}\nu), \mathrm{Ad}^*_{g^{-1}}\nu), \tag{1.4}$$

$$(\mathrm{ad}(\xi, \eta))^*(\mu, \nu) = ((\mathrm{ad}\,\xi)^*\mu + (\mathrm{ad}\,\eta)^*\nu, (\mathrm{ad}\,\xi)^*\nu), \tag{1.5}$$

for $g \in G$, $\xi, \eta, \zeta \in \mathfrak{g}$, $\mu, \nu \in \mathfrak{g}^*$.

If κ denotes a bilinear, symmetric, non-degenerate, bi-invariant (i.e. $\kappa([\xi, \eta], \zeta) = \kappa(\xi, [\eta, \zeta])$ for all $\xi, \eta, \zeta \in \mathfrak{g}$) two-form on $\mathfrak{g}$, the two-form

$$\kappa_s((\xi_1, \eta_1), (\xi_2, \eta_2)) = \kappa(\xi_1, \eta_2) + \kappa(\xi_2, \eta_1) \tag{1.6}$$

satisfies the same properties on $\mathfrak{s}$. Note that $\kappa \times \kappa$ is not bi-invariant.

The tangent bundle TG is a Lie group with composition law $(v_h, w_g) \mapsto T_h R_g(v_h) + T_g L_h(w_g)$, inverse $v_h \mapsto -T_h(L_{h^{-1}} \circ R_{h^{-1}})(v_h)$, and identity $0_e \in T_e G$. The space coordinate map ρ (see 1.4.) is a Lie group isomorphism of TG with S.

1.3. The Kirillow-Kostant-Souriau theorem states that any co-adjoint orbit $G \cdot \mu \subset \mathfrak{g}^*$ is symplectic with form

$$\omega_\mu(\bar{\mu})((\mathrm{ad}\,\xi)^*\bar{\mu}, (\mathrm{ad}\,\eta)^*\bar{\mu}) = -\bar{\mu}([\xi, \eta]) \tag{1.7}$$

for $\xi, \eta \in \mathfrak{g}$, $\bar{\mu} \in G \cdot \mu$. Marsden and Weinstein [26] showed that $G \cdot \mu$ is symplectically diffeomorphic to the reduced manifold $J^{-1}(\mu)/G_\mu$, where $G_\mu = \{g \in G \mid \mathrm{Ad}_{g^{-1}}\mu = \mu\}$ is the isotropy subgroup at μ of the co-adjoint action, and $J: T^*G \to \mathfrak{g}^*$, $J(\alpha_g) = T_e^* R_g(\alpha_g)$, $R_g(h) = hg$, is the momentum map of the lift to T^*G of the left-translation L_g on G. Moreover, using theorem 4.3.3. of [1], it can be easily shown that the co-adjoint orbit $G \cdot \mu$ is symplectically embedded onto a subbundle over G/G_μ of

$T^*(G/G_\mu)$ endowed with the non-canonical symplectic form $\omega_0 + d\hat{\alpha}_\mu$, where ω_0 is the canonical symplectic form of the co-tangent bundle and $\hat{\alpha}_\mu$ is the pull-back to $T^*(G/G_\mu)$ of a certain one-form on G/G_μ which we now describe: $J^{-1}(\mu) = \text{graph}(\alpha_\mu)$, where $\alpha_\mu = \mu \circ T_g R_{g^{-1}}$ is a right-invariant one-form on G, invariant under G_μ and thus induces a one-form on G/G_μ. This embedding is a diffeomorphism if and only if $\mathfrak{g}_\mu = \mathfrak{g}$.

If $f, f' : \mathfrak{g} \to \mathbf{R}$, the Hamiltonian vector field of $f|G\cdot\mu$ and the Poisson bracket of $f|G\cdot\mu$, $f'|G\cdot\mu$ are

(1.8) $$X_{f|G\cdot\mu}(\overline{\mu}) = \text{ad}(df(\overline{\mu}))^*\overline{\mu}$$

(1.9) $$\{f|G\cdot\mu, f'|G\cdot\mu\}(\overline{\mu}) = -\overline{\mu}([df(\overline{\mu}), df'(\overline{\mu})])$$

where $df(\overline{\mu}), df'(\overline{\mu}) \in \mathfrak{g}^{**} = \mathfrak{g}$. In the case of $\mathfrak{s}^*$, the dual of the semidirect product $\mathfrak{s}$, these formulas become

(1.10) $$\omega_{(\mu,\nu)}(\overline{\mu}, \overline{\nu})((\text{ad}(\xi, \eta))^*(\overline{\mu}, \overline{\nu}), (\text{ad}(\xi', \eta'))^*(\overline{\mu}, \overline{\nu})) = -\overline{\mu}([\xi, \xi']) - \overline{\nu}([\xi, \eta']) - \overline{\nu}([\eta, \xi'])$$

(1.11) $$X_{f|S\cdot(\mu,\nu)}(\overline{\mu}, \overline{\nu}) = ((\text{ad}(d_1 f(\overline{\mu}, \overline{\nu})))^*\overline{\mu} + (\text{ad}(d_2 f(\overline{\mu}, \overline{\nu})))^*\overline{\nu} + (\text{ad}(d_1 f(\overline{\mu}, \overline{\nu})))^*\overline{\nu})$$

(1.12) $$\{f|S\cdot(\mu, \nu), f'|S\cdot(\mu, \nu)\} = -\overline{\mu}([d_1 f(\overline{\mu}, \overline{\nu}), d_1 f'(\overline{\mu}, \overline{\nu})]) - \overline{\nu}([d_1 f(\overline{\mu}, \overline{\nu}), d_2 f'(\overline{\mu}, \overline{\nu})]) - \overline{\nu}([d_2 f(\overline{\mu}, \overline{\nu}), d_1 f'(\overline{\mu}, \overline{\nu})])$$

where $d_i f(\overline{\mu}, \overline{\nu})$, $i = 1, 2$ denote the partial derivatives of $df(\overline{\mu}, \overline{\nu})$.

If $\mathfrak{g}$ carries a bilinear, symmetric, non-degenerate, bi-invariant two-form κ, then co-adjoint and adjoint orbits are diffeomorphic, and thus any adjoint orbit $G\cdot\xi$ is symplectic with form

(1.13) $$\omega_\xi(\overline{\xi})([\eta, \overline{\xi}], [\zeta, \overline{\xi}]) = -\kappa([\eta, \zeta], \overline{\xi})$$

for $\overline{\xi} \in G\cdot\xi$. The formulas for the Hamiltonian vector field and Poisson bracket are

$$(1.14)\qquad X_{f|G\cdot\xi}(\bar{\xi}) = -[(\operatorname{grad} f)(\bar{\xi}), \bar{\xi}]$$

$$(1.15)\qquad \{f|G\cdot\xi, f'|G\cdot\xi\}(\bar{\xi}) = -\kappa([(\operatorname{grad} f)(\bar{\xi}), (\operatorname{grad} f')(\bar{\xi})], \bar{\xi})$$

where grad denotes the κ-gradient, i.e. $df(\xi)(\eta) = \kappa((\operatorname{grad} f)(\xi), \eta)$ for any $\xi, \eta \in \mathfrak{g}$.

For the semidirect product $\mathfrak{s}$, the two-form κ_s gives the formulas

$$(1.16)\qquad \omega(\xi, \eta)([(\xi, \eta), (\zeta_1, \zeta_2)], [(\xi, \eta), (\zeta_1', \zeta_2')])$$
$$= -\kappa([\zeta_1, \zeta_1'], \eta) - \kappa([\zeta_1, \zeta_2'], \xi) - \kappa([\zeta_2, \zeta_1'], \xi)$$

$$(1.17)\qquad X_f(\xi, \eta)$$
$$= -([(\operatorname{grad}_2 f)(\xi, \eta), \xi], [(\operatorname{grad}_2 f)(\xi, \eta), \eta] + [(\operatorname{grad}_1 f)(\xi, \eta), \xi])$$

$$(1.18)\qquad \{f, f'\}(\xi, \eta) = -\kappa(\xi, [(\operatorname{grad}_2 f)(\xi, \eta), (\operatorname{grad}_1 f')(\xi, \eta)])$$
$$- \kappa(\xi, [(\operatorname{grad}_1 f)(\xi, \eta), (\operatorname{grad}_2 f')(\xi, \eta)])$$
$$- \kappa(\eta, [(\operatorname{grad}_2 f)(\xi, \eta), (\operatorname{grad}_2 f')(\xi, \eta)]),$$

where $(\operatorname{grad}_1, \operatorname{grad}_2)$ denotes the usual gradient with respect to $\kappa \times \kappa$; note that the gradient with respect to κ_s is $(\operatorname{grad}_2, \operatorname{grad}_1)$.

1.4. Throughout this paper it will be computationally convenient to realize TG and T^*G as $G \times \mathfrak{g}$ and $G \times \mathfrak{g}^*$ respectively. We shall say that

$$\lambda: TG \to G \times \mathfrak{g} \qquad \lambda(v_g) = (g, T_eL_g^{-1}(v_g)) \qquad v_g \in T_gG$$

$$\rho: TG \to G \times \mathfrak{g} \qquad \rho(v_g) = (g, T_eR_g^{-1}(v_g)) \qquad v_g \in T_gG$$

$$\bar{\lambda}: T^*G \to G \times \mathfrak{g}^* \qquad \bar{\lambda}(\alpha_g) = (g, T_eL_g^*(\alpha_g)) \qquad \alpha_g \in T_gG$$

$$\bar{\rho}: T^*G \to G \times \mathfrak{g}^* \qquad \bar{\rho}(\alpha_g) = (g, T_eR_g^*(\alpha_g)) \qquad \alpha_g \in T_gG$$

define the body $(\lambda, \bar{\lambda})$ and space $(\rho, \bar{\rho})$ coordinates on TG and T^*G respectively. The canonical one (θ) and two forms $(\omega = -d\theta)$ of T^*G have the following expressions in body coordinates:

$$(1.19)\qquad \theta(g, \mu)(v_g, \rho) = \mu(T_gL_{g^{-1}}(v_g)),$$

$$(1.20)\qquad \omega(g, \mu)((v_g, \rho), (w_g, \sigma)) = -\rho(T_gL_{g^{-1}}(w_g) + \sigma(T_gL_{g^{-1}}(v_g)) + \mu([T_gL_{g^{-1}}(v_g), T_gL_{g^{-1}}(w_g)])$$

for $g \in G$, $\mu, \rho, \sigma \in \mathfrak{g}^*$, $v_g, w_g \in T_gG$. ([1], Propositon 4.4.1)

If $H: T^*G \to \mathbf{R}$ is a left-invariant Hamiltonian on T^*G, i.e. $H \circ T^*L_g = H$ for all $g \in G$, the expression of the Hamiltonian vector field X_H in body coordinates is $(\bar{\lambda}_* H_H)(g, \mu) = (\bar{X}(g, \mu), \mu, \mathrm{ad}^*(dH(\mu))\cdot\mu)$, where $g \mapsto \bar{X}(g, \mu)$ is a family of left-invariant Hamiltonian vector fields on G depending smoothly on $\mu \in \mathfrak{g}^*$ and $\bar{Y}(\mu) = \mathrm{ad}^*(dH(\mu))\cdot\mu$ is known as the *cotangent Euler vector field* ([1] Section 4.4, [37] Section 2). It is shown in Ratiu [37] that $\bar{Y}|G\cdot\mu$ is Hamiltonian and that it corresponds to the reduction of X_H to $J^{-1}(\mu)/G_\mu \approx G\cdot\mu$ (see (1.8)). Since the motion of the reduced Hamiltonian system determines uniquely the motion of the original Hamiltonian system on level sets of the momentum map ([1], page 305), it follows that all the mechanical information is carried by the cotangent Euler vector field $\bar{Y}: \mathfrak{g}^* \to \mathfrak{g}^*$.

Assume G has a left-invariant metric $\langle\ ,\ \rangle$ and that $E: TG \to \mathbf{R}$ is an arbitrary left-invariant energy function. Taking on TG the symplectic structure induced from T^*G by $\langle\ ,\ \rangle$, the expression of the canonical one (Θ) and two-forms $(\Omega = -d\Theta)$ in body coordinates are ([1], Proposition 4.4.2)

$$(1.21)\qquad \Theta(g, \xi)(v_g, \zeta) = \langle T_gL_{g^{-1}}(v_g), \xi\rangle_e$$

(1.22)

$$\Omega(g, \xi)((v_g, \zeta), (w_g, \eta)) = -\langle \zeta, T_gL_{g^{-1}}(w_g)\rangle_e + \langle \eta, T_gL_{g^{-1}}(v_g)\rangle_e + \langle \xi, [T_gL_{g^{-1}}(v_g), T_gL_{g^{-1}}(w_g)]\rangle_e.$$

Let $E: TG \to \mathbf{R}$ be an arbitrary left-invariant energy function and assume that X_E is a second order equation on TG. Its expression in body coordinates is given by $(\lambda_* X_E)(g, \xi) = (T_eL_g(\xi), \xi, Y(\xi))$, where $Y: \mathfrak{g} \to \mathfrak{g}$, called the *Euler vector field* is characterized by

$$(1.23)\qquad \langle Y(\xi), \eta\rangle = \langle [\xi, \eta], \xi\rangle$$

for any $\xi, \eta \in \mathfrak{g}$. Since the last formula determines Y uniquely in terms of $\langle\ ,\ \rangle$, and is independent of E, it follows that the *geodesic sprays of left-invariant metrics are the unique left-invariant vector fields on TG which are also second order* ([1], Section 4.4).

If $\mathfrak{g}$ carries a bilinear, symmetric, non-degenerate, bi-invariant two form κ, there exists a unique linear κ-symmetric isomorphism $I:\mathfrak{g} \to \mathfrak{g}$ such that $\kappa(I\cdot, \cdot) = \langle \cdot, \cdot \rangle$. (1.23) becomes $IY(\xi) = [I\xi, \xi]$ and thus putting $L = I^{-1}$, the change of variables $\eta = I\xi$ defines a vector field

$$Z(\eta) = [\eta, L\eta] \tag{1.24}$$

which is Hamiltonian on $G\cdot\eta$ with Hamiltonian $\eta \mapsto (1/2)\kappa(\eta, L\eta)$.

2. Co-adjoint orbits of semidirect products as reduced manifolds and the Euler-Poisson equations. This section shows how generic co-adjoint orbits $S\cdot(\mu, \nu) \subset \mathfrak{s}^*$ naturally arise as reduced manifolds in $T^*G \approx G \times \mathfrak{g}^*$ and represents a generalization of a result of Marsden and Weinstein [26]. Hamiltonian vector fields on these orbits generalize the classical Euler-Poisson equations.

2.1. Two linear functionals (μ_0, ν_0) on the Lie algebra $\mathfrak{g}$ determine three symplectic manifolds.

(1) The co-adjoint orbit $(S\cdot(\mu_0, \nu_0), \omega_{(\mu_0,\nu_0)})$ of S in $\mathfrak{s}^*$.

(2) Let $\nu_0 \in \mathfrak{g}^*$ be fixed and $G_{\nu_0} = \{g \in G \mid \mathrm{Ad}^*_{g^{-1}}\nu_0 = \nu_0\}$ its isotropy subgroup under the co-adjoint action. G_{ν_0} acts freely and properly on $G \times \mathfrak{g}^*$ by $(h, (g, \alpha)) \mapsto (hg, \alpha)$. The infinitesimal generator of this action defined by $\xi \in \mathfrak{g}_{\nu_0} = \{\zeta \in \mathfrak{g} \mid (\mathrm{ad}\, \zeta)^*\nu_0 = 0\}$ is $(g, \alpha) \mapsto (T_eR_g(\xi), 0)$ and thus this action has a momentum map $J:G \times \mathfrak{g}^* \to \mathfrak{g}^*_{\nu_0}$ given by ([1, Theorem 4.2.10]) $J(g, \alpha)\cdot\xi = \theta(g, \alpha)(T_eR_g(\xi), 0) = (\mathrm{Ad}^*_{g^{-1}}\alpha)(\xi)$, i.e.

$$J(g, \alpha) = (\mathrm{Ad}^*_{g^{-1}}\alpha) \mid \mathfrak{g}_{\nu_0}.$$

Since the isotropy group at any (g, α) of the G_{ν_0}-action is trivial, J has no critical points ([5, Theorem 1]).

Let $\mu_0 \in \mathfrak{g}^*$, denote $\overline{\mu}_0 = \mu_0 \mid \mathfrak{g}_{\nu_0} \in \mathfrak{g}^*_{\nu_0}$, and form the reduced symplectic manifold $(J^{-1}(\overline{\mu}_0)/(G_{\nu_0})_{\overline{\mu}_0}, \sigma)$. The symplectic form σ is uniquely determined by $i^*\omega = \pi^*\sigma$, where $i:J^{-1}(\overline{\mu}_0) \hookrightarrow G \times \mathfrak{g}^*$ is the inclusion and $\pi:J^{-1}(\overline{\mu}_0) \to J^{-1}(\overline{\mu}_0)/(G_{\nu_0})_{\overline{\mu}_0}$ the canonical projection.

(3) For $\overline{\mu}_0 \in \mathfrak{g}^*_{\nu_0}$ the one-form $\alpha_{\overline{\mu}_0}(g) = \overline{\mu}_0 \circ T_gR_{g^{-1}}$ (R_g = right

translation) on G is right-invariant and $(G_{\nu_0})_{\overline{\mu}_0}$-left-invariant thus inducing a one-form on the quotient $G/(G_{\nu_0})_{\overline{\mu}_0}$. Denote by $\hat{\alpha}_{\overline{\mu}_0}$ its pull-back to $T^*(G/(G_{\nu_0})_{\overline{\mu}_0})$ and form the symplectic manifold $(T^*(G/(G_{\nu_0})_{\overline{\mu}_0}), \omega_0 + d\hat{\alpha}_{\overline{\mu}_0})$, where ω_0 is the canonical symplectic form on $T^*(G/(G_{\nu_0})_{\overline{\mu}_0})$ (see 1.3.).

The first goal of this section is to relate these three symplectic manifolds.

2.2. The map $F: G \times \mathfrak{g}^* \to \mathfrak{g}^* \times G\cdot\nu_0$, $F(g, \alpha) = (\alpha, \mathrm{Ad}_g^*\nu_0)$ induces a smooth map $\hat{F}([g], \alpha) = (\alpha, \mathrm{Ad}_g^*\nu_0)$, $\hat{F}: G/(G_{\nu_0})_{\overline{\mu}_0} \times \mathfrak{g}^* \to \mathfrak{g}^* \times G\cdot\nu_0$. Remark that if $(G_{\nu_0})_{\overline{\mu}_0} = G_{\nu_0}$, $\hat{F}$ is a diffeomorphism. Let $M(\overline{\mu}_0, \nu_0) = \{(\mu, \nu) \in \mathfrak{s}^* \mid$ there exists $g \in G$ such that $\nu = \mathrm{Ad}_{g^{-1}}^*\nu_0$, $(\mathrm{Ad}_g^*\mu) \mid \mathfrak{g}_{\nu_0} = \overline{\mu}_0\} \subset \mathfrak{s}^*$. A straightforward verification shows that

$$\hat{F}(J^{-1}(\overline{\mu}_0)/(G_{\nu_0})_{\mu_0}) = F(J^{-1}(\overline{\mu}_0)) = M(\overline{\mu}_0, \nu_0).$$

LEMMA 2.1. *$M(\overline{\mu}_0, \nu_0) = S\cdot(\mu_0, \nu_0)$ for any $\mu_0 \in \mathfrak{g}^*$ satisfying $\mu \mid \mathfrak{g}_{\nu_0} = \overline{\mu}_0$.*

Proof. Let $\mu \in \mathfrak{g}^*$ satisfy $\mu \mid \mathfrak{g}_{\nu_0} = \overline{\mu}_0$. Then by (1.1) there exists $\zeta \in \mathfrak{g}$ such that $\mu = \mu_0 - (\mathrm{ad}\,\zeta)^*\nu_0$, and (1.4) shows that $S\cdot(\mu, \nu_0) = S\cdot(\mu_0, \nu_0)$.

It is straightforward to check that $S\cdot(\mu, \nu_0) \subseteq M(\overline{\mu}_0, \nu_0)$ for any $\mu \in \mathfrak{g}^*$ satisfying $\mu \mid \mathfrak{g}_{\nu_0} = \overline{\mu}_0$ and hence $\cup_{\mu \mid \mathfrak{g}_{\nu_0} = \overline{\mu}_0} S\cdot(\mu, \nu_0) \subseteq M(\overline{\mu}_0, \nu_0)$. Conversely, if $(\mu, \nu) \in M(\overline{\mu}_0, \nu_0)$, then there exists $g \in G$ such that $\nu = \mathrm{Ad}_{g^{-1}}^*\nu_0$ and $(\mathrm{Ad}_g^*\mu) \mid \mathfrak{g}_{\nu_0} = \overline{\mu}_0$. Put $\mu' = \mathrm{Ad}_g^*\mu$, notice that $\mu' \mid \mathfrak{g}_{\nu_0} = \overline{\mu}_0$, and that $(\mathrm{Ad}_{g^{-1}}^*\mu', \mathrm{Ad}_{g^{-1}}^*\nu_0) = (\mu, \nu)$, i.e. $(\mu, \nu) \in S\cdot(\mu', \nu_0)$ by (1.4) and thus $M(\overline{\mu}_0, \nu_0) = \cup_{\mu \mid \mathfrak{g}_{\nu_0} = \overline{\mu}_0} S\cdot(\mu, \nu_0)$. $M(\overline{\mu}_0, \nu_0) = S\cdot(\mu, \nu_0)$ for $\mu \mid \mathfrak{g}_{\nu_0} = \overline{\mu}_0$ follows now from the first step in the proof. ▽

LEMMA 2.2. *$\hat{F}: (J^{-1}(\overline{\mu}_0)/(G_{\nu_0})_{\overline{\mu}_0}, \sigma) \to (S\cdot(\mu_0, \nu_0), \omega_{(\mu_0, \nu_0)})$ is a symplectic covering map.*

Proof. Since σ is defined by $i^*\omega = \pi^*\sigma$, the relation $\hat{F}^*\omega_{(\mu_0, \nu_0)} = \sigma$ is equivalent to $(F \mid J^{-1}(\overline{\mu}_0))^*\omega_{(\mu_0, \nu_0)} = i^*\omega$.

The formulae

$$T_g(\mathrm{Ad}_{\cdot}^*\alpha)(v_g) = \mathrm{ad}(T_g L_{g^{-1}}(v_g))^*\mathrm{Ad}_g^*\alpha$$

$$T_g(\mathrm{Ad}_{\cdot^{-1}}^*\alpha)(v_g) = -(\mathrm{ad}\, T_g R_{g^{-1}}(v_g))^*\mathrm{Ad}_{g^{-1}}^*\alpha$$

for $g \in G$, $\alpha \in \mathfrak{g}^*$, $v_g \in T_gG$ imply

$$(T_{(g,\alpha)}F)(v_g, \beta) = (\beta, (\operatorname{ad} T_gL_{g^{-1}}(v_g))^*\operatorname{Ad}_g^*\nu_0)$$

$$(T_{(g,\alpha)}J)(v_g, \beta) = -((\operatorname{ad} T_gR_{g^{-1}}(v_g))^*\operatorname{Ad}_{g^{-1}}^*\alpha)\,|\,\mathfrak{g}_{\nu_0} + (\operatorname{Ad}_{g^{-1}}^*\beta)\,|\,\mathfrak{g}_{\nu_0}$$

(2.1)

$$T_{(g,\alpha)}(J^{-1}(\bar{\mu}_0)) = \operatorname{Ker}(T_{(g,\alpha)}J) = \{(v_g, \beta) \in T_gG \times \mathfrak{g}^* \,|\, (\operatorname{Ad}_{g^{-1}}^*\beta)\,|\,\mathfrak{g}_{\nu_0} = ((\operatorname{ad} T_gR_{g^{-1}}(v_g)^*\operatorname{Ad}_{g^{-1}}^*\alpha)\,|\,\mathfrak{g}_{\nu_0}\}.$$

Thus if $(g, \alpha) \in J^{-1}(\bar{\mu}_0)$, (v_g, β), $(w_g, \gamma) \in T_{(g,\alpha)}(J^{-1}(\bar{\mu}_0))$ we have

$$(F^*\omega_{(\mu_0,\nu_0)})(g, \alpha)((v_g, \beta), (w_g, \gamma))$$
$$= \omega_{(\mu_0,\nu_0)}(\alpha, \operatorname{Ad}_g^*\nu_0)((\beta, (\operatorname{ad} T_gL_{g^{-1}}(v_g))^*\operatorname{Ad}_g^*\nu_0),$$
$$(\gamma, (\operatorname{ad} T_gL_{g^{-1}}(w_g))^*\operatorname{Ad}_g^*\nu_0)).$$

Put $\xi = T_gL_{g^{-1}}(v_g)$, $\xi' = T_gL_{g^{-1}}(w_g)$. Determine $\eta, \eta' \in \mathfrak{g}$ from the equations

$$(\operatorname{ad} \operatorname{Ad}_g\eta)^*\nu_0 = \operatorname{Ad}_{g^{-1}}^*\beta - \operatorname{Ad}_{g^{-1}}^*(\operatorname{ad}\xi)^*\alpha$$
$$= \operatorname{Ad}_{g^{-1}}^*\beta - (\operatorname{ad} \operatorname{Ad}_g\xi)^*\operatorname{Ad}_{g^{-1}}^*\alpha$$
$$(\operatorname{ad} \operatorname{Ad}_g\eta')^*\nu_0 = \operatorname{Ad}_{g^{-1}}^*\gamma - \operatorname{Ad}_{g^{-1}}^*(\operatorname{ad}\xi')^*\alpha$$
$$= \operatorname{Ad}_{g^{-1}}^*\gamma - (\operatorname{ad} \operatorname{Ad}_g\xi')^*\operatorname{Ad}_{g^{-1}}^*\alpha$$

which is possible by (2.1) and (1.1). Then (1.5) shows that

$$(\beta, (\operatorname{ad} T_gL_{g^{-1}}(v_g))^*\operatorname{Ad}_g^*\nu_0) = (\operatorname{ad}(\xi, \eta))^*(\alpha, \operatorname{Ad}_g^*\nu_0)$$

$$(\gamma, (\operatorname{ad} T_gL_{g^{-1}}(w_g))^*\operatorname{Ad}_g^*\nu_0) = (\operatorname{ad}(\xi', \eta'))^*(\alpha, \operatorname{Ad}_g^*\nu_0).$$

Thus by (1.10) we have

$$(F^*\omega_{(\mu_0,\nu_0)})(g, \alpha)((v_g, \beta), (w_g, \gamma)) = -\alpha([\xi, \xi']) - \operatorname{Ad}_g^*\nu_0([\xi, \eta'])$$
$$- \operatorname{Ad}_g^*\nu_0([\eta, \xi']).$$

But

$$\begin{aligned}\mathrm{Ad}_g^*\nu_0([\eta, \xi']) &= (\mathrm{ad}\,\mathrm{Ad}_g\eta)^*\nu_0\cdot \mathrm{Ad}_g\xi' \\ &= \mathrm{Ad}^*_{g^{-1}}\beta\cdot \mathrm{Ad}_g\xi' - \mathrm{Ad}^*_{g^{-1}}(\mathrm{ad}\,\xi)^*\alpha\cdot \mathrm{Ad}_g\xi' \\ &= \beta(\xi') - \alpha([\xi, \xi']) \\ &= \beta(T_gL_{g^{-1}}(w_g)) - \alpha([\xi, \xi']).\end{aligned}$$

Similarly

$$\mathrm{Ad}_g^*\nu_0([\xi, \eta']) = -\gamma(T_gL_{g^{-1}}(v_g)) + \alpha([\xi', \xi])$$

and hence

$$\begin{aligned}&(F^*\omega_{(\mu_0,\nu_0)})(g, \alpha)((v_g, \beta), (w_g, \gamma)) \\ &\quad = -\beta(T_gL_{g^{-1}}(w_g)) + \gamma(T_gL_{g^{-1}}(v_g)) + \alpha([T_gL_{g^{-1}}(v_g),\ T_gL_{g^{-1}}(w_g)])\end{aligned}$$

i.e. $F^*\omega_{(\mu_0,\nu_0)} = \sigma$ by (1.20).

The formula for $T_{(g,\alpha)}F$, (2.1), and (1.5) prove that F and hence $\hat{F}: J^{-1}(\bar{\mu}_0)/(G_{\nu_0})_{\bar{\mu}_0} \to M(\bar{\mu}_0, \nu_0) = S\cdot(\mu_0, \nu_0)$ is a surjective submersion, and hence an open mapping ([9]). Since $\hat{F}$ is symplectic, it is an immersion and thus locally injective, i.e. $\hat{F}$ is a symplectic covering map. ∇

Recall from 1.1. that there exists an open dense set $\mathcal{O} \subseteq \mathfrak{g}^*$ such that if $\nu_0 \in \mathcal{O}$, G_{ν_0} is abelian. Thus

$$\{(\mu_0, \nu_0) \mid (G_{\nu_0})_{\bar{\mu}_0} = G_{\nu_0}\} \supseteq (\mathfrak{g}^* \times \mathcal{O}) \cup (\{0\} \times \mathfrak{g}^*)$$

and hence the condition $(G_{\nu_0})_{\bar{\mu}_0} = G_{\nu_0}$ under which $\hat{F}$ is a symplectic diffeomorphism is generic.

THEOREM 2.3. *Let* $\mu_0, \nu_0 \in \mathfrak{g}^*$, $\bar{\mu}_0 = \mu_0 \mid \mathfrak{g}_{\nu_0}$.

(A) *The reduced symplectic manifold* $(J^{-1}(\bar{\mu}_0)/(G_{\nu_0})_{\bar{\mu}_0}, \sigma)$ *is a symplectic covering of the coadjoint orbit* $(S\cdot(\mu, \nu_0), \omega_{(\mu,\nu_0)})$ *and symplectically embeds onto a subbundle over* $G/(G_{\nu_0})_{\bar{\mu}}$ *of* $(T^*(G/(G_{\nu_0})_{\bar{\mu}}, \omega_0 + d\hat{\alpha}_{\bar{\mu}})$, *for any* $\mu \in \mathfrak{g}^*$ *satisfying* $\mu \mid \mathfrak{g}_{\nu_0} = \bar{\mu} = \bar{\mu}_0$.

(B) *Under the generic assumption* $(G_{\nu_0})_{\bar{\mu}_0} = G_{\nu_0}$ *the three manifolds*

$(S\cdot(\mu, \nu_0), \omega_{(\mu,\nu_0)})$, $(J^{-1}(\overline{\mu}_0)/G_{\nu_0}, \sigma)$, *and* $(T^*(G/G_{\nu_0}), \omega_0 + d\hat{\alpha}_{\overline{\mu}})$ *are symplectically diffeomorphic for any* $\mu \in \mathfrak{g}^*$ *satisfying* $\mu | \mathfrak{g}_{\nu_0} = \overline{\mu} = \overline{\mu}_0$.

Proof. (A) The first part is lemma 2.2. The second part follows from theorem 4.3.3. of [1] provided $\alpha_{\overline{\mu}}$ has values in $J^{-1}(\overline{\mu}_0)$. But this is immediate since the expression of $\alpha_{\overline{\mu}}(g)$ in body coordinates is $\overline{\lambda}(\alpha_{\overline{\mu}}(g)) = (g, \mathrm{Ad}_g^*\overline{\mu})$ (see 1.4) and thus $J(g, \mathrm{Ad}_g^*\overline{\mu}) = \overline{\mu} | \mathfrak{g}_{\nu_0} = \overline{\mu}_0$.

(B) This follows directly from (A), the fact that now $\hat{F}$ is a diffeomorphism, and theorem 4.3.3 of [1]. □

Remarks. (1) If $\mathfrak{g}$ carries a bilinear, symmetric, bi-invariant, nondegenerate two-form, all results above can be easily reformulated in terms of adjoint orbits of S on $\mathfrak{s}$. If $\mathfrak{g}$ is semisimple, the generic set includes the open dense set {regular semisimple elements of $\mathfrak{g}$} $\times\ \mathfrak{g}$.

(2) Assume now that $G_{\nu_0} = G$. Then $S\cdot(\mu_0, \nu_0) = G\cdot\mu_0 \times \{\nu_0\}$ and it is easy to see that for any $(\mu, \nu) \in S\cdot(\mu_0, \nu_0)$, $\hat{F}^{-1}(\mu, \nu) = ([e], \mu)$, i.e. $\hat{F}$ is a diffeomorphism and we recovered the Marsden-Weinstein theorem ([26], [1, page 303]), which states that $J^{-1}(\mu_0)/G_{\mu_0}$ is symplectically diffeomorphic to the co-adjoint orbit $G\cdot\mu_0$.

(3) The map $F: G \times \mathfrak{g}^* \to \mathfrak{g}^* \times \mathfrak{g}^*$ has the following interpretation ([12], [13]). S acts on the *right* on $G \times \mathfrak{g}^*$ by $((g, \zeta), (h, \alpha)) \mapsto (hg, \mathrm{Ad}_g^*\alpha + \mathrm{Ad}_g^*(\mathrm{ad}\,\zeta)^*\mathrm{Ad}_h^*\nu_0)$, $(g, \zeta) \in S$, $(h, \alpha) \in G \times \mathfrak{g}^*$, with infinitesimal generator $(\xi, \eta)_{G\times\mathfrak{g}^*}(h, \alpha) = (T_eL_h(\xi), (\mathrm{ad}\,\xi)^*\alpha + (\mathrm{ad}\,\eta)^*\mathrm{Ad}_h^*\nu_0)$ for $(\xi, \eta) \in \mathfrak{s}$. It is readily verified that this action is symplectic admitting an Ad*-equivariant (on the *right*) momentum map which is F. Starting from this observation, Marsden [13] gives an alternate proof of Lemma 2.2 using general facts about collective Hamiltonians [12].

(4) The above theorem and proof have a word by word generalization to the case when S is the semidirect product of G with a vector space V on which G is linearly represented. In this form (with $G = SO(n)$, $V =$ symmetric traceless matrices) the theorem appears already in Ratiu [39]. The present paper will not use this straightforward generalization.

(5) In general $d\hat{\alpha}_{\overline{\mu}} \neq 0$. If $\mu = 0$ then $\alpha_{\overline{\mu}}$ itself vanishes; this is important for a large class of completely integrable systems ([3], [4], [34], [39]). For $G = SO(3)$ a quite unpleasant computation using Euler angles shows that $d\hat{\alpha}_{\overline{\mu}} = 0$ ([13], [40]).

2.3. The geometric result in theorem 2.3 has a mechanical counterpart. Assume $(G_{\nu_0})_{\overline{\mu}_0} = G_{\nu_0}$ and that the Hamiltonian $H: G \times \mathfrak{g}^* \to \mathbf{R}$ is G_{ν_0}-invariant, i.e. $H(g, \alpha) = H(hg, \alpha)$ for all $h \in G_{\nu_0}$. Then H induces a

Hamiltonian system on the reduced manifold ([1, Section 4.3], [25, Section 5], [26]) and thus by $\hat{F}$ one on $S \cdot (\mu_0, \nu_0)$ whose expression is $H_s(\alpha, \mathrm{Ad}_g^* \nu_0) = H(g, \alpha)$, for $(\mathrm{Ad}_{g^{-1}}^* \alpha) | \mathfrak{g}_{\nu_0} = \bar{\mu}_0$. Regarding ν_0 as variable parameter, H_s defines a function on $\mathfrak{s}^*$ whose Hamiltonian equations on each co-adjoint orbit of S in $\mathfrak{s}^*$ are the generalized *cotangent Euler-Poisson equations* (1.11). The original mechanical problem is thus embedded in a larger one, in which "the axis of symmetry" ν_0 becomes a variable parameter.

Assume now that H defines a *simple mechanical system* ([1, Section 4.5], [41]), i.e. $H(g, \mu) = K(\mu) + U(g)$, where $K(\mu) = (1/2)\langle \mu, \mu \rangle$ is the expression in body coordinates of the kinetic energy $K(\alpha_g) = (1/2)\langle \alpha_g, \alpha_g \rangle_g$ of the left-invariant metric $\langle \ , \ \rangle$ on G in cotangent formulation and the potential U is a real-valued function on G. If U is G_{ν_0}-invariant, it induces a map $V: G \cdot \nu_0 \to \mathbf{R}$, $V(\nu) = U(g)$ for $\nu = \mathrm{Ad}_{g^{-1}}^* \nu_0$. The equations of motion (1.11) become in this case the *cotangent Euler-Poisson equations*

$$
(2.2) \qquad \begin{aligned} \dot{\mu} &= (\mathrm{ad}(\langle \mu, \cdot \rangle))^* \mu + (\mathrm{ad}(dV(\nu)))^* \nu \\ \dot{\nu} &= (\mathrm{ad}(\langle \mu, \cdot \rangle))^* \nu \end{aligned}
$$

where $\langle \mu, \cdot \rangle \in \mathfrak{g}^{**} = \mathfrak{g}$. Note that on the manifold $T^*(G/(G_{\nu_0})_{\bar{\mu}_0})$ (see theorem 2.3.B), the kinetic energy is induced from the kinetic energy of the metric $\langle \ , \ \rangle$ on G and the potential energy is induced from the G_{ν_0}-invariant *effective potential* ([41]) $V_{\bar{\mu}_0}(g) = U(g) + K(\alpha_{\bar{\mu}}(g)) = U(g) + K(\bar{\mu} \circ T_g R_{g^{-1}})$ ([1, theorem 4.5.6]). Thus on the manifold $T^*(G/(G_{\nu_0})_{\bar{\mu}_0})$ the motion can be regarded as taking place under the combined influence of a potential $V_{\bar{\mu}_0}$ and a *field potential* $\alpha_{\bar{\mu}}$. In case of the heavy rigid body, $\alpha_{\bar{\mu}}$ represents the gravitational field potential (Section 4).

In applications one encounters matrix equations, i.e. equations in $\mathfrak{g}$, not in $\mathfrak{g}^*$. It is therefore important to reformulate the above results on adjoint orbits of $\mathfrak{s}$. The basic assumption is that $\mathfrak{g}$ carries a bilinear, symmetric, non-degenerate, bi-invariant two-form κ. For example, if $\mathfrak{g}$ is semisimple, κ is some multiple of the Killing form. Let $I: \mathfrak{g} \to \mathfrak{g}$ be the κ-symmetric, positive isomorphism given by $\kappa(I \cdot, \cdot) = \langle \cdot, \cdot \rangle$ and $L = I^{-1}$ i.e. $\kappa(\cdot, \cdot) = \langle L \cdot, \cdot \rangle$. The expression of the above Hamiltonian on $\mathfrak{s}^*$ via the isomorphism $\mathfrak{s}^* \cong \mathfrak{s}$ defined by κ_s (see (1.6)) is $H(\xi, \eta) = (1/2)\kappa(L\eta, \eta) + V(\xi)$. Hamilton's equations of motion on adjoint orbits of

S in $\mathfrak{s}$ (the counterpart of (2.2)) are the *Euler-Poisson equations,* equal by (1.17) to

$$\dot{\xi} = [\xi, L\eta], \qquad \dot{\eta} = [\eta, L\eta] + [\xi, (\text{grad } V)(\xi)] \tag{2.3}$$

Such equations appear for example in the study of the magnetohydrodynamics of an incompressible, inviscid, ideal conducting fluid in a compact region of $\mathbf{R}^3$ with smooth boundary and its analogue in $SO(n)$ (where $V(\xi) = (1/2)\kappa(L\xi, \xi)$) ([45]).

If V is linear there exists a unique χ such that $V = \kappa(\chi, \cdot)$ and thus equations (2.3) become the *classical Euler-Poisson equations*

$$\dot{\xi} = [\xi, L\eta], \qquad \dot{\eta} = [\eta, L\eta] + [\xi, \chi]. \tag{2.4}$$

Summarizing, Hamiltonian systems on a Lie group G with energy of the form kinetic plus potential, the kinetic energy defined by a left-invariant metric and the potential invariant under some isotropy subgroup of the (co-)adjoint action, naturally generalize to Hamiltonian systems on orbits of the semidirect product S. It is thus not surprising that a variety of completely integrable systems have been found to have for phase space adjoint orbits of S in $\mathfrak{s}$; see [3], [4], [15], [40], [33], [34], [39]. In applications it is usually easier to work with the generalized Hamiltonian systems on orbits of S rather than with the original one; see the prior references and Section 4, Section 5.

2.4. In the sequel the complete integrability of a certain Euler-Poisson equation on $so(N)$ will be discussed. We shall prove below a general involution theorem on adjoint orbits of S on $\mathfrak{s}$. Of course the theorem naturally lives on duals, but this is the form to be used later on (see Section 4).

THEOREM 2.4. *Let $\mathfrak{g}$ be a Lie algebra carrying a bilinear, symmetric, bi-invariant, non-degenerate two form. Let $f, g : \mathfrak{g} \to \mathbf{R}$ satisfy $[(\text{grad } f)(\xi), \xi] = 0$, $[(\text{grad } g)(\xi), \xi] = 0$ for all $\xi \in \mathfrak{g}$ and denote $f_\alpha(\xi, \eta) = f(\xi + \alpha\eta + \alpha^2\epsilon)$, $g_\beta(\xi, \eta) = g(\xi + \beta\eta + \beta^2\epsilon)$, for $\epsilon \in \mathfrak{g}$ fixed, $\alpha, \beta \in \mathbf{R}$. Then the Poisson bracket $\{f_\alpha, g_\beta\}$ vanishes on all adjoint orbits of S in $\mathfrak{s}$.*

Proof. Since

$$(\text{grad}_2 f_\alpha)(\xi, \eta) = \alpha(\text{grad } f)(\xi + \alpha\eta + \alpha^2\epsilon)$$

$$(\text{grad}_1 f_\alpha)(\xi, \eta) = (\text{grad } f)(\xi + \alpha\eta + \alpha^2\epsilon),$$

for $a \neq b$, writing

$$(a+b)\xi + ab\eta = (a^2/(a-b))(\xi + b\eta + b^2\epsilon) - (b^2/(a-b))(\xi + a\eta + a^2\epsilon)$$

we get from (1.18)

$$\begin{aligned}
&\{f_a, g_b\}(\xi, \eta)\\
&\quad = -\kappa((a+b)\xi + ab\eta, [(\operatorname{grad} f)(\xi + a\eta + a^2\epsilon), (\operatorname{grad} g)(\xi + b\eta + b^2\epsilon)])\\
&\quad = (a^2/(a-b))\kappa([\xi + b\eta + b^2\epsilon, (\operatorname{grad} g)(\xi + b\eta + b^2\epsilon)],\\
&\qquad\qquad (\operatorname{grad} f)(\xi + a\eta + a^2\epsilon))\\
&\qquad + (b^2/(a-b))\kappa([\xi + a\eta + a^2\epsilon, (\operatorname{grad} f)(\xi + a\eta + a^2\epsilon)],\\
&\qquad\qquad (\operatorname{grad} g)(\xi + b\eta + b^2\epsilon)) = 0.
\end{aligned}$$

By continuity $\{f_a, g_b\} = 0$ holds for $a = b$ too. □

3. Hamiltonian structures and Kac-Moody Lie algebras. This section is a brief review of Hamiltonian structures, Kac-Moody Lie algebras and the Kostant-Symes involution theorem with special emphasis on the semidirect product $\mathfrak{s}$. The concepts and formulas of this section will be applied to prove the complete integrability of certain Euler-Poisson equations in the rest of the paper.

3.1. Let M be a smooth manifold and $\mathfrak{F}(M)$, $\mathfrak{X}(M)$ the algebra of smooth functions, respectively the Lie algebra of smooth vector fields on M. A *Hamiltonian* (*Poisson*, or *cosymplectic*) *structure* ([21], [22], [24]) on M is

(i) a bracket $\{\ ,\ \}$ on $\mathfrak{F}(M)$ making $\mathfrak{F}(M)$ into a Lie algebra,

(ii) a Lie algebra anti-homomorphism $X:(\mathfrak{F}(M), \{\ ,\ \}) \to (\mathfrak{X}(M), [\ ,\ ])$ given by $X_f(g) = -\{f, g\}$, i.e. $X_{\{f,g\}} = -[X_f, X_g]$.

For $f \in \mathfrak{F}(M)$, X_f is called the *Hamiltonian vector field* of the *Hamiltonian* f, and $\{\ ,\ \}$ the *Poisson bracket*.

Since $X_f(p)$, $\{f, g\}(p)$, $p \in M$ depend only on df_p, dg_p, the Hamiltonian structure defines a contravariant antisymmetric two-tensor field Λ on M by $\Lambda(p)(df_p, dg_p) = \{f, g\}(p)$. The following has a straightforward proof ([38]).

PROPOSITION 3.1. *The Hamiltonian structure* $(M, \{\ ,\ \}, X)$ *defines a contravariant antisymmetric two-tensor field* Λ *on* M. *If* k *denotes the minimal dimension of its kernel and* $n = \dim(M)$, *the set* $\mathcal{A} = \{p \in M \mid \dim \mathcal{H}(p) = n - k\}$ *is open in* M, *where* $\mathcal{H}(p) = \operatorname{span}\{X_f(p) \mid f \in \mathcal{F}(M)\}$. *If there exists a dense set in* M *on which* $\dim \mathcal{H}(p) =$ constant, this set is necessarily included in $\mathcal{A}$. *Finally,* $\mathcal{A}$ *is foliated by symplectic manifolds, called the generic symplectic leaves of the Hamiltonian structure.*

The last statement follows by the Frobenius theorem applied to the $(n - k)$-dimensional involutive distribution $\mathcal{H}$ on $\mathcal{A}$.

A submanifold $N \subset M$ is *invariant* if $q \in N$ implies $X_f(q) \in T_qN$ for all $f \in \mathcal{F}(M)$. Clearly N is in this case itself Hamiltonian.

Let $(M, \{\ ,\ \}, X)$, $(M, \{\ ,\ \}', X')$ be two Hamiltonian structures on the same manifold M and $\mathcal{M} \subset \mathcal{F}(M)$. A bijective map $\Phi : \mathcal{M} \to \mathcal{M}$ satisfying $X_{\Phi f} = X_f'$ is called *Lenard relations in* $\mathcal{M}$. Remark that $\{f, g\}' = -X_f'(g) = -X_{\Phi f}(g) = \{\Phi f, g\} = \{f, \Phi g\}$ so that $\mathcal{M}$ commutes in $\{\ ,\ \}$ if and only if it commutes in $\{\ ,\ \}'$. By abuse of language, in the literature $X_{\Phi f} = X_f'$ are often called Lenard relations themselves, due to the fact that Φ has been discovered in this form as a recursion relation for the integrals of the *KdV* equation by Lenard.

3.2. Hamiltonian structures on M with non-degenerate Poisson bracket (i.e. $\{f, g\} = 0$ for all $g \in \mathcal{F}(M)$ implies $f =$ constant on connected components of M) coincide with symplectic structures of M. This theorem is due to Jost [16]; see also [38] for a proof.

Let $\mathfrak{g}$ be a Lie algebra. A whole family of Hamiltonian structures is defined on $\mathfrak{g}^*$ by

$$\{f, g\}_\Psi(\mu) = -\Psi(\mu)([df(\mu), dg(\mu)]) \tag{3.1}$$

$$X_f^\Psi(\mu) = (\operatorname{ad}(df(\mu)))^*\Psi(\mu) \tag{3.2}$$

where $\Psi : \mathfrak{g}^* \to \mathfrak{g}^*$ is a smooth map satisfying the following identity

$$(d\Psi(\mu)\cdot(\operatorname{ad}\zeta)^*\Psi(\mu))\cdot[\eta, \xi] + (d\Psi(\mu)\cdot(\operatorname{ad}\eta)^*\Psi(\mu))\cdot[\xi, \zeta] + (d\Psi(\mu)\cdot(\operatorname{ad}\xi)^*\Psi(\mu))\cdot[\zeta, \eta] = 0$$

for all $\xi, \eta, \zeta \in \mathfrak{g}$, $\mu \in \mathfrak{g}^*$. If $\Psi =$ identity, one gets the *Kirillov-Kostant-*

Souriau structure on $\mathfrak{g}^*$ and (3.1), (3.2) become (1.9), (1.8). This structure is determined actually by the decomposition of $\mathfrak{g}^*$ in co-adjoint orbits. The set $\mathcal{A}$ of proposition 3.1 is in this case open and dense and coincides with $\mathcal{O}$ of 1.1.

If $\Psi(\mu) = \nu$ for all $\mu \in \mathfrak{g}^*$, one obtains the *modified Kirillov-Kostant-Souriau structure on* $\mathfrak{g}^*$.

Assume $\mathfrak{g}$ carries a bilinear, symmetric, non-degenerate, bi-invariant two-form κ. Then all Hamiltonian structures above induce isomorphic structures on $\mathfrak{g}$. For example, the *Kirillov-Kostant-Souriau structure on* $\mathfrak{g}$ is given by (1.14), (1.15), and if $\mathfrak{g}$ is semisimple, $\mathcal{A}$ = regular semisimple elements in $\mathfrak{g}$, whereas for $\epsilon \in \mathfrak{g}$ fixed, the *modified Kirillov-Kostant-Souriau structure on* $\mathfrak{g}$ is defined by

$$\{f, g\}_\epsilon(\xi) = -\kappa([(\operatorname{grad} f)(\xi), (\operatorname{grad} g)(\xi)], \epsilon) \tag{3.3}$$

$$X_f^\epsilon(\xi) = -[(\operatorname{grad} f)(\xi), \epsilon]. \tag{3.4}$$

Define the *Euler-Poisson structure on* $\mathfrak{g}^* \times \mathfrak{g}^*$ ($\mathfrak{g} \times \mathfrak{g}$) as the Kirillov-Kostant-Souriau structure on $\mathfrak{s}^*$ ($\mathfrak{s}$); it is hence given by (1.11), (1.12) (respectively by (1.17), (1.18)) and determined thus by the co-adjoint (adjoint) orbits in $\mathfrak{s}^*$ ($\mathfrak{s}$). For $\alpha \in \mathfrak{g}^*$, $\epsilon \in \mathfrak{g}$ fixed, the *modified Euler-Poisson structures* on $\mathfrak{g}^* \times \mathfrak{g}^*$ and $\mathfrak{g} \times \mathfrak{g}$ are given respectively by

$$\begin{aligned}\{f, g\}_\alpha(\mu, \nu) = &-\alpha([d_1 f(\mu, \nu), d_2 g(\mu, \nu)]) \\ &- \alpha([d_2 f(\mu, \nu), d_1 g(\mu, \nu)]) \\ &- \nu([d_2 f(\mu, \nu), d_2 g(\mu, \nu)])\end{aligned} \tag{3.5}$$

$$X_f^\alpha(\mu, \nu) = (\operatorname{ad}(d_2 f(\mu, \nu)))^*\alpha, (\operatorname{ad}(d_1 f(\mu, \nu)))^*\alpha + (\operatorname{ad}(d_2 f(\mu, \nu)))^*\nu) \tag{3.6}$$

for $f, g : \mathfrak{g}^* \times \mathfrak{g}^* \to \mathbf{R}$, $\mu, \nu \in \mathfrak{g}^*$, $d_i f(\mu, \nu)$, $d_i g(\mu, \nu) \in \mathfrak{g}$, $i = 1, 2$ and

$$\begin{aligned}\{f, g\}_\epsilon(\xi, \eta) = &-\kappa(\epsilon, [\operatorname{grad}_2 f(\xi, \eta), \operatorname{grad}_1 g(\xi, \eta)]) \\ &- \kappa(\epsilon, [\operatorname{grad}_1 f(\xi, \eta), \operatorname{grad}_2 g(\xi, \eta)]) \\ &- \kappa(\eta, [\operatorname{grad}_1 f(\xi, \eta), \operatorname{grad}_1 g(\xi, \eta)])\end{aligned} \tag{3.7}$$

$$X_f^\epsilon(\xi, \eta) = ([\eta, \operatorname{grad}_1 f(\xi, \eta)] + [\epsilon, \operatorname{grad}_2 f(\xi, \eta)], [\epsilon, \operatorname{grad}_1 f(\xi, \eta)]) \tag{3.8}$$

for $f, g: \mathfrak{g} \times \mathfrak{g} \to \mathbf{R}$, $\xi, \eta \in \mathfrak{g}$. The set $\mathfrak{A}$ depends on α (on ϵ respectively) and has to be determined case by case.

We shall prove below that the two above modified Hamiltonian structures on $\mathfrak{g}$ and $\mathfrak{g} \times \mathfrak{g}$ are induced by an orbit decomposition in an infinite-dimensional Lie algebra.

3.3. If $\mathfrak{g}$ is a Lie algebra, let $\tilde{\mathfrak{g}}$ denote the vector space of all formal finite sums $\tilde{\xi} = \Sigma_{n\in\mathbf{Z}} \xi_n h^n$ with the usual addition and scalar multiplication on components. $\tilde{\mathfrak{g}}$ becomes a Lie algebra with respect to the bracket

$$\left[\sum_{n\in\mathbf{Z}} \xi_n h^n, \sum_{p\in\mathbf{Z}} \eta_p h^p\right] = \sum_{r\in\mathbf{Z}} \left(\sum_{n+p=r} [\xi_n, \eta_p]\right) h^r. \tag{3.9}$$

The Lie algebra $\tilde{\mathfrak{g}}$ is called the *Kac-Moody extension of* $\mathfrak{g}$ ([3], [29], [31], [32]).

Assume that κ is a bilinear, symmetric, bi-invariant, non-degenerate two-form on $\mathfrak{g}$ and let $k \in \mathbf{Z}$. Define $\tilde{\kappa}_k$ on $\tilde{\mathfrak{g}} \times \tilde{\mathfrak{g}}$ by

$$\tilde{\kappa}_k\left(\sum_{n\in\mathbf{Z}} \xi_n h^n, \sum_{p\in\mathbf{Z}} \eta_p h^p\right) = \sum_{n+p=k} \kappa(\xi_n, \eta_p). \tag{3.10}$$

$\tilde{\kappa}_k$ is clearly bilinear, symmetric and bi-invariant. It is also easy to see that $\tilde{\kappa}_k(\tilde{\xi}, \tilde{\eta}) = 0$ for all $\tilde{\eta} \in \tilde{\mathfrak{g}}$ implies $\tilde{\xi} = 0$, but that $\tilde{\kappa}_k$ does not induce an isomorphism between $\tilde{\mathfrak{g}}$ and $\tilde{\mathfrak{g}}^*$, i.e. $\tilde{\kappa}_k$ is only *weakly non-degenerate.* Thus not every real-valued function on $\tilde{\mathfrak{g}}$ has a gradient and the Kirillov-Kostant-Souriau structure is defined only on $\mathcal{G}(\tilde{\mathfrak{g}}) = \{f \in \mathfrak{F}(\tilde{\mathfrak{g}}) \mid f \text{ has a gradient}\}$ by formulas (1.14), (1.15).

3.4. We recall the Kostant-Symes involution theorem; for proofs see Kostant [19], Symes [44], Ratiu [38], Adler-van Moerbeke [3].

Let $\mathcal{g}$ be a Lie algebra and assume $\mathcal{g} = \mathcal{k} \oplus \mathcal{n}$, where $\mathcal{k}$ is a vector subspace and $\mathcal{n}$ a Lie subalgebra of $\mathcal{g}$. Let κ be a bilinear, symmetric, bi-invariant, weakly non-degenerate two-form on $\mathcal{g}$. Assume that $\mathcal{g} = \mathcal{k}^\perp \oplus \mathcal{n}^\perp$; this is automatic if κ is non-degenerate on $\mathcal{g}$. The Kirillov-Kostant-Souriau structure on $\mathcal{k}^\perp$ induced from $\mathcal{n}^*$ can be expressed only in terms of elements of $\mathcal{g}$ in the following way:

$$\{f \mid \mathcal{k}^\perp, g \mid \mathcal{k}^\perp\}(\xi) = -\kappa([\Pi_n(\operatorname{grad} f)(\xi), \Pi_n(\operatorname{grad} g)(\xi)], \xi) \tag{3.11}$$

$$X_{f|k^\perp}(\xi) = -\Pi_{k^\perp}([\Pi_n(\text{grad } f)(\xi), \xi]), \tag{3.12}$$

for $f, g \in \mathcal{G}(g)$, $\xi \in k^\perp$; $\Pi_n : g \to n$, $\Pi_k : g \to k$, $\Pi_{k^\perp} : g \to k^\perp$, $\Pi_{n^\perp} : g \to n^\perp$ denote the canonical projections defined by the splittings $g = k \oplus n$, and $g = k^\perp \oplus n^\perp$. Assume that either 1) k is a subalgebra of g, or 2) $[k, n] \subseteq k$. Then if f is ad-invariant, i.e. $[(\text{grad } f)(\xi), \xi] = 0$ for all $\xi \in g$, (3.12) becomes

$$X_{f|k^\perp}(\xi) = \begin{cases} [\Pi_k(\text{grad } f)(\xi), \xi], \text{ in case } 1 \\ -[\Pi_n(\text{grad } f)(\xi), \xi], \text{ in case } 2. \end{cases} \tag{3.13}$$

Moreover, the Kostant-Symes involution theorem states that in hypotheses 1 or 2, if f, g are both ad-invariant on g, then they commute in the bracket of $k^\perp$, i.e. $\{f|k^\perp, g|k^\perp\} = 0$. All the above statements remain unchanged if $k^\perp$ is replaced with an invariant submanifold of $k^\perp$.

3.5. We return to the Kac-Moody extension $\tilde{\mathfrak{g}}$ of $\mathfrak{g}$ and remark that $\tilde{\mathfrak{g}} = \mathcal{K} \oplus \mathcal{N}$, where $\mathcal{K} = \{\Sigma_{n=0}^{\infty} \xi_n h^n \,|\, \xi_n \in \mathfrak{g}, \text{ finitely many } \xi_n \neq 0\}$, $\mathcal{N} = \{\Sigma_{n=-\infty}^{-1} \xi_n h^n \,|\, \xi_n \in \mathfrak{g}, \text{ finitely many } \xi_n \neq 0\}$ are Lie subalgebras of $\tilde{\mathfrak{g}}$. For $k = -1$, we have $\mathcal{K}^\perp = \mathcal{K}$, $\mathcal{N}^\perp = \mathcal{N}$ and thus $\tilde{\mathfrak{g}} = \mathcal{K}^\perp \oplus \mathcal{N}^\perp$. Consider for $\epsilon \in \mathfrak{g}$ fixed the submanifolds $R_\epsilon = \{\xi + \epsilon h \,|\, \xi \in \mathfrak{g}\} \subset \mathcal{K}^\perp$, $Q_\epsilon = \{\xi + \eta h + \epsilon h^2 \,|\, \xi, \eta \in \mathfrak{g}\} \subset \mathcal{K}^\perp$. If $f \in \mathcal{G}(\tilde{\mathfrak{g}})$, denote $(\text{grad } f)(\tilde{\xi}) = \Sigma_{n \in \mathbf{Z}} f_n h^n$, for $\tilde{\xi} \in \tilde{\mathfrak{g}}$. Using (3.11), (3.13), it is easily verified that R_ϵ, Q_ϵ are invariant submanifolds of $\mathcal{K}^\perp$. We shall deal now only with Q_ϵ, the proofs for R_ϵ being identical. The Kirillov-Kostant-Souriau structure on $\mathcal{K}^\perp$ induces a Hamiltonian structure on Q_ϵ given by

$$\{f|Q_\epsilon, g|Q_\epsilon\}(\tilde{\xi}) = -\kappa(\epsilon, [f_{-2}, g_{-1}]) - \kappa(\epsilon, [f_{-1}, g_{-2}]) - \kappa(\eta, [f_{-1}, g_{-1}]) \tag{3.14}$$

$$X_{f|Q_\epsilon}(\tilde{\xi}) = [\eta, f_{-1}] + [\epsilon, f_{-2}] + [\epsilon, f_{-1}]h \tag{3.15}$$

for $\tilde{\xi} \in Q_\epsilon$.

Q_ϵ is canonically diffeomorphic to $\mathfrak{g} \times \mathfrak{g}$ by $\psi(\xi + \eta h + \epsilon h^2) = (\xi, \eta)$. If $f : \mathfrak{g} \times \mathfrak{g} \to \mathbf{R}$, note that $f \circ \psi \in \mathcal{G}(\tilde{\mathfrak{g}})$ and $\text{grad}(f \circ \psi)(\tilde{\xi}) = \text{grad}_1 f(\xi, \eta) h^{-1} + \text{grad}_2 f(\xi, \eta) h^{-2}$. Then (3.14), (3.15) imply that $\psi^*\{f, g\}_\epsilon = \{\psi^* f|Q_\epsilon, \psi^* g|Q_\epsilon\}$, $\psi^* X_f^\epsilon = X_{\psi^* f|Q_\epsilon}$, i.e. the modified Euler-Poisson structure of $\mathfrak{g} \times \mathfrak{g}$ and the Hamiltonian structure of Q_ϵ are isomorphic.

PROPOSITION 3.2. *The invariant submanifold* $Q_\epsilon(R_\epsilon)$ of $\mathcal{K}^\perp$ *with respect to the Kirillov-Kostant-Souriau structure is isomorphic to* $\mathfrak{g} \times \mathfrak{g}$ *with the modified Euler-Poisson structure* (*to* $\mathfrak{g}$ *with the modified Kirillov-Kostant-Souriau structure*).

Remark that this proposition proves in a different way that the modified structures are indeed Hamiltonian.

4. The N-dimensional Lagrange and heavy symmetric top. This section is devoted to Euler-Poisson equations in $so(N) \times so(N)$. It is shown that they can be written in Lax form in the Kac-Moody extension $\widetilde{so(N)}$ of $so(N)$ if and only if they describe the motion of the free, Lagrange, or heavy symmetric top. The right number of conserved quantities is found and three different Lie algebraic proofs of their involution in both Euler-Poisson structures of $so(N) \times so(N)$ is given. Lenard relations in the familiar form of recursion relations are established.

4.1. We start by recalling briefly the free N-dimensional rigid body problem [37]. On the Lie algebra $so(N)$ define the inner products κ and $\langle\ ,\ \rangle$ by $\kappa(A, B) = -(1/2)\mathrm{Tr}(AB)$, $\langle A, B\rangle = \kappa(L(A), B)$, where $L: so(N) \to so(N)$ is the positive κ-symmetric isomorphism with inverse $L^{-1}(C) = CJ + CJ$, $J = \mathrm{diag}(J_1, \ldots, J_N)$ a fixed diagonal matrix satisfying $J_i + J_j > 0$ for $i \neq j$. The Hamiltonian $M \to (1/2)\kappa(M, L(M))$ defines on adjoint orbits in $so(N)$ the equations of motion $\dot{M} = [M, L(M)]$ which is the Euler vector field of the geodesic spray of the metric $\langle\ ,\ \rangle$ on $SO(N)$. These equations are derived from physical principles in Ratiu [37] and they represent the motion of the free N-dimensional rigid body about a fixed point, the origin of $\mathbf{R}^N$. The $N(N-1)/2$ positive numbers $J_i + J_j$, $i \neq j$ are the principal moments of inertia of the body.

If $N = 3$ and the center of mass χ is not the origin, the fixed point about which the body of unit weight moves, a potential $V(A) = \kappa(\chi, \mathrm{Ad}_{A^{-1}}\epsilon)$ must be added to the kinetic energy $(1/2)\kappa(M, L(M))$ in the expression of the total energy and it represents the height of χ over the horizontal plane; ϵ is the unit vector of the Oz-axis (see [40]). Since gravity in $\mathbf{R}^N$ is meaningless, we *define* the rigid body Hamiltonian in $SO(N) \times so(N)$ by

$$E(A, M) = \frac{1}{2}\kappa(M, \Omega) + \kappa(\chi, \mathrm{Ad}_{A^{-1}}\epsilon), \qquad L^{-1}(\Omega) = M,$$

for χ, $\epsilon \in so(N)$ fixed. The potential V is invariant under the isotropy subgroup $SO(N)_\epsilon$ of the adjoint action. On the reduced manifold (Section

2) it has the expression $V(\Gamma) = \kappa(\chi, \Gamma)$ for $\Gamma = \mathrm{Ad}_{A^{-1}}\epsilon$ (see 2.3.) and is thus linear in Γ. The Euler-Poisson equations in $so(N) \times so(N)$ induced by this Hamiltonian are by (2.4)

$$\dot{\Gamma} = [\Gamma, \Omega], \qquad \dot{M} = [M, \Omega] + [\Gamma, \chi], \tag{4.1}$$

for $M = \Omega J + J\Omega$, χ a fixed matrix in $so(N)$. The Hamiltonian of (4.1) is $H(\Gamma, M) = (1/2)\kappa(M, \Omega) + \kappa(\Gamma, \chi)$.

The *Lagrange top* is defined by $a = J_1 = J_2$, $b = J_3 = \cdots = J_N$, $\chi_{12} \neq 0$, $\chi_{ij} = 0$ for all $i, j \neq 1, 2$, $i < j$. The *heavy symmetric top* is defined by $a = b$, $\chi \in so(N)$ arbitrary. In this case $J = aId$, $M = 2a\Omega$ and (4.1) become

$$\dot{\Gamma} = [\Gamma, \Omega], \qquad 2a\dot{\Omega} = [\Gamma, \chi].$$

For $N = 3$, due to the fact that the adjoint action of $SO(3)$ on the two-sphere in $\mathbf{R}^3 \cong so(3)$ is transitive, the symmetric case reduces to a special Lagrange top by an orthonormal change of basis [40]. For general N this is no longer true.

4.2. In order to find integrals of motion, it is convenient to write (5.1) in Lax form if possible. This trick due to Manakov [23] led to the complete integrability of the free N-dimensional rigid body problem [27], [3], [4], [37]. The following crucial observation is due for $N = 3$ to Ratiu and van Moerbeke [40].

THEOREM 4.1. *Assume* $\chi_{12} \neq 0$. *The Euler-Poisson equations* (4.1) *can be written in the form*

$$(\Gamma + Mh + Ch^2)^{\cdot} = [\Gamma + Mh + Ch^2, \Omega + \chi h] \tag{4.2}$$

if and only if (4.1) *describe the N-dimensional Lagrange or heavy symmetric top, in which case* $C = (a + b)\chi$.

Proof. (4.2) is equivalent to (4.1) plus $[C, \chi] = 0$, $[C, \Omega] + [M, \chi] = 0$. The second relation is equivalent to

$$\begin{cases} C_{j\ell} = (J_i + J_j)\chi_{j\ell}, & \ell \neq i,j \\ C_{i\ell} = (J_i + J_j)\chi_{i\ell}, & \ell \neq i,j \\ C_{ki} = (J_i + J_j)\chi_{ki}, & k \neq i,j \\ C_{kj} = (J_i + J_j)\chi_{kj}, & k \neq i,j \end{cases}, \qquad i < j \tag{4.3}$$

A somewhat lengthy but straightforward computation shows that (4.3) are satisfied if and only if the matrices J, χ have the Lagrange or heavy symmetric top restrictions in which case $C = (\alpha + \beta)\chi$. □

Remarks. (1) We could have assumed $\chi_{ij} \neq 0$ for another pair $(i, j) \neq (1, 2)$ $i < j$. This would have lead to "another" Lagrange top, which, after a change of basis is reduced to $\chi_{12} \neq 0$.

(2) If $\chi = 0$, then $C = 0$ and we get the free rigid body which is known to be completely integrable ([3], [4], [27], [37]).

(3) From now on (4.2) becomes the basic equation and in all that follows we shall not use the form of χ or C. This enables us to treat the Lagrange and symmetric top simultaneously.

4.3. Note that from (4.2) it follows that $\mathrm{Tr}(\Gamma + Mh + Ch^2)^k$ (and $\mathrm{Pf}(\Gamma + Mh + Ch^2)$ if $N =$ even, where $\mathrm{Pf}(X) = (\det X)^{1/2}$ is the Pfaffian) are conserved along the flow of (4.2). If $t \mapsto (\Gamma(t), M(t))$ denotes the flow of (4.1), then clearly $t \mapsto \Gamma(t) + M(t)h + Ch^2$ is the flow of (4.2). Thus the coefficients of h in the expansion of $\mathrm{Tr}(\Gamma + Mh + Ch^2)^k$ (and $\mathrm{Pf}(\Gamma + Mh + Ch^2)$ if $N =$ even) are conserved along the flow of (4.1). Note that for $k =$ odd all these coefficients vanish. It can be easily checked directly (or use the Lenard relations in theorem 4.5) that the coefficients of h^0, h, h^{2k} (and h^0, h, h^N in case of the Pfaffian) give identically zero Hamiltonian vector fields in the Euler-Poisson structure of $so(N) \times so(N)$. Thus the total number of non-trivial integrals is:

—if $N = 2r + 1$

$$\sum_{\substack{k=2 \\ k=\text{even}}}^{2r} (2k - 2) = 2r^2 = N(N-1)/2 - [N/2]$$

—If $N = 2r$

$$\sum_{\substack{k=2 \\ k=\text{even}}}^{2r-2} (2k - 2) + 2r - 2 = 2r^2 - 2r = N(N-1)/2 - [N/2],$$

i.e. half the dimension of the generic symplectic leaf in the Euler-Poisson structure of $so(N) \times so(N)$ (the generic adjoint orbit in the semidirect product $so(N) \times \overline{so(N)}$; see 1.2.).

For the classical case $N = 3$ there are two integrals, the coefficients of

h^2 and h^3 for $k = 2$, and they are polynomials in the total energy and the momentum of the body along its symmetry axis ([40], [3], [13]).

4.4. We shall interpret (4.2) as a Hamiltonian system on the invariant submanifold $Q_C \subset \mathcal{K}$, $\mathcal{K} = \{\Sigma_{n=0}^{\infty} \xi_n h^n \mid \text{finitely many } \xi_n \in so(N) \text{ non-zero}\}$ (see 3.5.). By (3.13), the Hamiltonian of (4.2) must be an ad-invariant function H such that $\Pi_{\mathcal{K}}(\text{grad } H)(\Gamma + Mh + Ch^2) = -(\Omega + \chi h)$.

In the case of the *heavy symmetric top* $M = 2_\alpha \Omega$, $C = 2_\alpha \chi$ and thus

$$\Omega + \chi h = \frac{1}{2_\alpha}(M + Ch) = \frac{1}{2_\alpha}\Pi_{\mathcal{K}}((\Gamma + Mh + Ch^2)h^{-1})$$

which implies

$$\tilde{H}(\tilde{\xi}) = -\frac{1}{2}\tilde{\kappa}_{-1}\left(\tilde{\xi}^2, \frac{1}{2_\alpha}\text{Id}\right)$$

for $\tilde{\xi} \in \widetilde{so(N)}$.

Let C be regular semisimple in $so(N)$. A straightforward computation shows that the set $\mathfrak{A}$ (proposition 3.1) for the modified Euler-Poisson structure on $so(N) \times so(N)$ defined by C equals $so(N) \times so(N)$ and that the dimension of the symplectic leaf is $N(N - 1) - 2[N/2]$.

For the *Lagrange top* more care has to be taken with Q_C itself due to the fact that in this case C is highly degenerate. By (3.15), for $f \in \mathcal{G}(\widetilde{so(N)})$, $(\text{grad } f)(\Gamma + Mh + Ch^2) = \Sigma_{n \in \mathbf{Z}} f_n h^n$, $X_{f|Q_C}(\Gamma + Mh + Ch^2) = [M, f_{-1}] + [C, f_{-2}] + [C, f_{-1}]h$. Note now that $[C, f_{-1}]$ has zero components in the upper left 2×2 and lower right $(N - 2) \times (N - 2)$ blocks and that $\kappa(C, [M, f_{-1}] + [C, f_{-2}]) + \kappa([C, f_{-1}], M) = 0$. Moreover, the lower right $(N - 2) \times (N - 2)$ block of $[M, f_{-1}] + [C, f_{-2}]$ is the bracket of M_2, the lower right $(N - 2) \times (N - 2)$ block of M, with an arbitrary matrix in $so(N - 2)$. This proves that Q_C is decomposed in lower dimensional invariant submanifolds of the form $\{\Gamma + Mh + Ch^2 \mid (1/2)\kappa(M, M) + \kappa(C, \Gamma) = \text{constant}, M_2 = \text{constant}, M_{12} = \text{constant}$, lower right-hand $(N - 2) \times (N - 2)$ block of Γ equals $[M_2, A], A \in so(N - 2)\}$. Assuming M_2 regular semisimple in $so(N - 2)$, we see that the dimension of this invariant submanifold equals $N(N - 1) - 2 - (N - 2)(N - 3)/2 - [(N - 2)/2] = (N - 2)(N - 3)/2 - [(N - 2)/2] + 4(N - 2)$. It is easy to see that the set $\mathfrak{A} = \{(\Gamma, M) \mid M_2 \in so(N - 2) \text{ regular semisimple}\}$ of propositions 3.1, is open, dense, and foliated by symplectic manifolds which are the images under ψ of the above invariant submanifolds (see 3.5.). Thus the dimension of the generic symplectic leaf for the modified Euler-Poisson structure defined by C is $(N-2)(N-3)/2 - [(N-2)/2] + 4(N-2)$.

Remark that (4.1) trivially implies $M_{12} = \text{constant}$, $M_2 = \text{constant}$. Thus denoting by $\overline{M}$ the constant matrix having all entries zero except the lower right $(N-2) \times (N-2)$ block which equals M_2,

$$\Omega + \chi h = \frac{1}{\alpha + \beta}(M + Ch) + \left(1 - \frac{2\alpha}{\alpha+\beta}\right) \cdot \frac{M_{12}}{2\alpha(\alpha+\beta)\chi_{12}} C + \left(1 + \frac{2\beta}{\alpha+\beta}\right)\overline{M}$$

$$= \frac{1}{\alpha+\beta}\Pi_{\mathcal{K}}((\Gamma + Mh + Ch^2)h^{-1}) + \left(1 - \frac{2\alpha}{\alpha+\beta}\right) \cdot \frac{M_{12}}{2\alpha(\alpha+\beta)\chi_{12}}\Pi_{\mathcal{K}}((\Gamma + Mh + Ch^2)h^{-2}) + \left(1 - \frac{2\beta}{\alpha+\beta}\right)\overline{M}$$

which implies the following formula for the Hamiltonian $\tilde{H}$ of (4.2)

$$\tilde{H}(\xi) = -\frac{1}{2}\tilde{\kappa}_{-1}\left(\xi^2, \frac{1}{\alpha+\beta}\operatorname{Id} h^{-1} + \left(1 - \frac{2\alpha}{\alpha+\beta}\right)\frac{M_{12}}{2\alpha(\alpha+\beta)\chi_{12}}\operatorname{Id} h^{-2}\right) - \tilde{\kappa}_{-1}\left(\xi, \left(1 - \frac{2\beta}{\alpha+\beta}\right)\overline{M}\right).$$

where $\xi \in so(N)$.

4.5. Denote by $u_{k+1,j}(\Gamma, M)$ (resp. $p_j(\Gamma, M)$) the coefficient of h^j_{k+1} (resp. h^j) in $f_{k+1}(\Gamma, M) = (1/(k+1)) \operatorname{Tr}(\Gamma + Mh_{k+1} + Ch^2_{k+1})^{k+1}$ (resp. in $P(\Gamma, M) = Pf(\Gamma + Mh + Ch^2)$) for an arbitrary parameter h_{k+1} (resp. h). We shall prove now that $u_{k+1,i}$, p_j are in involution in the Euler-Poisson structure of $so(N) \times so(N)$.

By theorem 2.4, $\{f_{k+1}, f_{\ell+1}\} = 0$ ($\{f_{\ell+1}, P\} = 0$, if N = even) on $so(N) \times so(N)$, for any parameters h_{k+1}, $h_{\ell+1}$, i.e. $f_{\ell+1}$ is constant on the flow defined by f_{k+1} (and P) no matter what $h_{\ell+1}$, h_{k+1} (resp h) are. This says that all $u_{\ell+1,i}$ are constant on the flow defined by f_{k+1} (and P) for all $i = 1, \ldots, 2k$ and h_{k+1} (resp. h). Hence $\{u_{\ell+1}, f_{k+1}\} = 0$ ($\{u_{\ell+1}, P\} = 0$) for any h_{k+1} and repeating the prior argument, $\{u_{k+1,j}, u_{\ell+1,i}\} = 0$ ($\{u_{k+1,j}, p_i\} = 0$).

We prove now that $u_{k+1,i}$, p_j are in involution also in the modified Euler-Poisson structure of $so(N) \times so(N)$ determined by C (see 3.2).

On $\widetilde{gl(N)}$ define a trace functional by $\mathrm{Tr}(\tilde{\xi}) = \mathrm{Tr}(\xi_{-1})$, where the right-hand side denotes the usual matrix trace of $\xi_{-1} \in gl(N)$. If one defines for $\tilde{\xi}$, $\tilde{\eta} \in \widetilde{gl(N)}$, $\tilde{\xi}\tilde{\eta} = \Sigma_{n \in \mathbf{Z}} (\Sigma_{p+r=n} \xi_p \eta_r) h^n$, it is clear that $[\tilde{\xi}, \tilde{\eta}] = \tilde{\xi}\tilde{\eta} - \tilde{\eta}\tilde{\xi}$, and that $\tilde{\xi}h = h\tilde{\xi}$ for any $\tilde{\xi} \in \widetilde{gl(N)}$. Take $\mathfrak{g} = so(N)$ and remark that $\tilde{\kappa}_{-1}(\tilde{\xi}, \tilde{\eta}) = -(1/2)\mathrm{Tr}(\tilde{\xi}\tilde{\eta})$. The real-valued functions on $\widetilde{so(N)}$, $f_{k+1,j}(\tilde{\xi}) = (1/(k+1))\mathrm{Tr}(\tilde{\xi}^{k+1} h^{-j})$ are clearly ad-invariant on $\widetilde{so(N)}$ (since their gradients with respect to $\tilde{\kappa}_{-1}$ equal $-2\tilde{\xi}^k h^{-j}$) and thus, by the Kostant-Symes theorem (see 3.4.), they are in involution on the invariant submanifold $Q_C = \{\Gamma + Mh + Ch^2 \mid \Gamma, M \in so(N)\}$ of $\mathcal{K}^{\perp} = \mathcal{K}$. Since $f \circ \psi = u_{k+1,2k+1-j}$, it follows that all $u_{k+1,j}$ are in involution in the modified Euler-Poisson structure of $so(N) \times so(N)$ defined by $C \in so(N)$. If $N =$ even, we can recover the involution of the integrals defined by the Pfaffian in the following way. Since all $f_{k+1,j}$ commute and $\det(\tilde{\xi})$ is a polynomial in the powers of traces, we conclude that all $f_{k+1,j}$ will commute with $\det(\tilde{\xi})$ and hence with $\mathrm{Pf}(\tilde{\xi}) = \det(\tilde{\xi})^{1/2}$ also, i.e. $f_{k+1,j}$ and $q_j =$ coefficient of h^j in $\mathrm{Pf}(\tilde{\xi})$ will commute on Q_C. In particular q_j commutes with $\det(\tilde{\xi})$, hence with $\mathrm{Pf}(\tilde{\xi})$ i.e. it commutes with all q_i. Since $p_i = q_i \circ \psi$ we conclude that for $N =$ even, $u_{k+1,j}$, p_i all commute in the modified Euler-Poisson structure of $so(N) \times so(N)$ defined by C.

THEOREM 4.2. *The $N(N-1)/2 - [N/2]$ integrals $\{u_{k+1,j} \mid j = 2, \ldots, 2k+1, k = 1, \ldots, N-2, k = \text{odd}\}$, if $N =$ odd; $\{u_{k+1,j}, p_i \mid j = 2, \ldots, 2k+1, k = 1, \ldots, N-3, k = \text{odd}, i = 2, \ldots, N-1\}$, if $N =$ even, are in involution in both Euler-Poisson structures of $so(N) \times so(N)$.*

4.6. We proceed to search for Lenard relations.

LEMMA 4.3. (a) *Let U_{kj} be the coefficient of h^j in the development of $(\Gamma + Mh + Ch^2)^k$ in powers of h, $k =$ odd. Then*

$$\mathrm{grad}_1 u_{k+1,j} = \begin{cases} -2U_{kj} & , \quad \text{for } j = 0, \ldots, 2k \\ 0 & , \quad \text{for } j = 2k+1, 2k+2 \end{cases}$$

$$\operatorname{grad}_2 u_{k+1,j} = \begin{cases} 0 & , \quad \text{for } j = 0,\ 2k+2 \\ -2U_{k,j-1}, & \quad \text{for } j = 1, \ldots, 2k+1 \end{cases}$$

(b) *If* $N =$ *even, let* $P_j \in so(N)$ *be defined by* (grad Pf)$(\Gamma + Mh + Ch^2) = \Sigma_{j=0}^{N} P_j(\Gamma, M)h^j$. *Then* $P_N = P_{N-1} = 0$ *and*

$$\operatorname{grad}_1 p_j = \begin{cases} P_j & , \quad \textit{for } f = 0, \ldots, N-2 \\ 0 & , \quad \textit{for } j = N-1, N \end{cases}$$

$$\operatorname{grad}_2 p_j = \begin{cases} 0 & , \quad \textit{for } j = 0, N \\ P_{j-1} & , \quad \textit{for } j = 1, \ldots, N-1 \end{cases}$$

Proof. (a) The equalities involving zeros are obvious. For the rest, write $(1/(k+1))\operatorname{Tr}(\Gamma + Mh + Ch^2)^{k+1} = \Sigma_{j=0}^{2k+2} u_{k+1,j}(\Gamma, M)h^j$ and take differentials of both sides; remark first that for $k =$ odd, $U_{kj} \in so(N)$. The factors -2 appear due to the definition $\kappa(A, B) = -(1/2)\operatorname{Tr}(AB)$ for $A, B \in so(N)$.

(b) Proceed as above taking the differential of the identity $\operatorname{Pf}(\Gamma + Mh + Ch^2) = \Sigma_{j=0}^{N} p_j(\Gamma, M)h^j$, to obtain the identity

$$\sum_{j=0}^{N} \kappa(P_j, A)h^j + \sum_{j=1}^{N+1} \kappa(P_{j-1}, B)h^j$$
$$= \sum_{j=0}^{N} \kappa(\operatorname{grad}_1 p_j, A)h^j + \sum_{j=0}^{N} \kappa(\operatorname{grad}_2 p_j, B)h^j$$

for any $A, B \in so(N)$. We conclude $\operatorname{grad}_1 p_j = P_j, j = 0, \ldots, N$, $\operatorname{grad}_2 p_0 = 0$ (known already since $p_0(\Gamma, M) = (\det \Gamma)^{1/2}$), $\operatorname{grad}_2 p_j = P_{j-1}, j = 1, \ldots, N$, and $P_N = 0$ (which is also implied by $\operatorname{grad}_1 p_N = 0$ since $p_N = (\det C)^{1/2}$), $P_{N-1} = \operatorname{grad}_2 p_N = 0$. ▽

LEMMA 4.4. *Let* $k =$ odd. $U_{k,0} = \Gamma^k$, $U_{k,2k} = C^k$,

$$[\Gamma, U_{k,1}] + [M, U_{k,0}] = 0$$

$$[M, U_{k,2k}] + [C, U_{k,2k-1}] = 0$$

$$[\Gamma, U_{k,j}] + [M, U_{k,j-1}] + [C, U_{k,j-2}] = 0, \qquad \textit{for } j = 2, \ldots, 2k.$$

If $N =$ even, $[\Gamma, P_0] = 0$, $P_N = 0$, $P_{N-1} = 0$,

$$[\Gamma, P_j] + [M, P_{j-1}] + [C, P_{j-2}] = 0, \qquad \textit{for } j = 2, \dots, N$$

$$[\Gamma, P_1] + [M, P_0] = 0.$$

Proof. The first two identities are obvious. The other equalities are the coefficients of h in the trivial identities

$$[\Gamma + Mh + Ch^2, (\Gamma + Mh + Ch^2)^k] = 0,$$

$$\left[\Gamma + Mh + Ch^2, \sum_{j=0}^{N} P_j(\Gamma, M)h^j\right] = 0$$

(the last follows since Pf is Ad-invariant, i.e. $[(\text{grad Pf})(X), X] = 0$). □

In order to simplify notations make the convention that any U_{kj} involving an index j which is <0 or $>2k$ vanishes identically. For the relations involving the Pfaffian, the convention is that $P_j \equiv 0$ for $j < 0, j > N - 2$. Lemma 4.4 can be written as

$$\begin{cases} [\Gamma, U_{kj}] + [M, U_{k,j-1}] + [C, U_{k,j-2}] = 0, \text{for } j = 0, \dots, 2k+2, k = \text{odd} \\ [\Gamma, P_j] + [M, P_{j-1}] + [C, P_{j-2}] = 0, \text{for } j = 0, \dots, N \\ U_{k,0} = \Gamma^k, \; U_{k,2k} = C^k. \end{cases}$$

(4.4)

Consider now the modified Euler-Poisson structure on $so(N) \times so(N)$ defined by C. With the above conventions, (4.4) and Lemma 4.3 imply

$$\begin{aligned} X_{u_{k+1,j}}(\Gamma, M) &= -([-2U_{k,j-1}, \Gamma], [-2U_{kj}, \Gamma] + [-2U_{k,j-1}, M]) \\ &= -([M, -2U_{k,j-2}] + [C, -2U_{k,j-3}], [C, -2U_{k,j-2}]) \\ &= -X^C_{u_{k+1,j-2}}(\Gamma, M), \qquad \text{for } j = 0, \dots, 2k+2 \\ X_{p_j}(\Gamma, M) &= -([P_{j-1}, \Gamma], [P_j, \Gamma] + [P_{j-1}, M]) \\ &= -([M, P_{j-2}] + [C, P_{j-3}], [C, P_{j-2}]) \\ &= -X^C_{p_{j-2}}(\Gamma, M), \qquad \text{for } j = 0, \dots, N. \end{aligned}$$

(4.4), Lemma 4.3, and (3.8) clearly imply $X^C_{u_{k+1,2k+1}} = X^C_{u_{k+1,2k}} = X^C_{p_{N-1}} = X^C_{p_N} = 0$. We get thus the following *Lenard relations.*

THEOREM 4.5. *With the convention that all vector fields involving an index j which is <0 or $>2k$ (and in the case $N =$ even, $X_{p_j} \equiv 0$ for $j < 0, j > N-2$) the following Lenard relations hold.*

(a) *If* $N =$ odd

$$X_{u_{k+1,j}} = -X^C_{u_{k+1,j-2}} \tag{4.5}$$

for $k =$ odd, $k = 1, \ldots, N-1, j = 0, \ldots, 2k+4$.

(b) *If* $N =$ even

$$X_{u_{k+1,j}} = -X^C_{u_{k+1,j-2}} \tag{4.6}$$

for $k =$ odd, $k = 1, \ldots, N-2, j = 0, \ldots, 2k+4$

$$X_{p_j} = -X^C_{p_{j-2}} \tag{4.7}$$

for $j = 0, \ldots, N+2$.

Thus, in particular $X_{u_{k+1,0}} = X_{u_{k+1,1}} = X^C_{u_{k+1,2k+1}} = X^C_{u_{k+1,2k+2}} = 0$. Since $u_{k+1,2k+2} = \operatorname{Tr} C^k =$ constant, we also have $X_{u_{k+1,2k+2}} = X^C_{u_{k+1,2k}} = 0$. Similarly, $X_{p_0} = X_{p_1} = X^C_{p_N} = X^C_{p_{n-1}} = 0$ and since $p_N = (\det C)^{1/2} =$ constant, $X_{p_N} = X^C_{p_{N-2}} = 0$. In the terminology of 3.6. $\Phi(u_{k+1,j-2}) = -u_{k+1,j}$, $\Phi(p_{j-2}) = -p_j$.

Let $k, \ell =$ odd. We have by (4.6)

$$\begin{aligned}
\{u_{k+1,j}, u_{\ell+1,i}\} &= -X_{u_{k+1,j}}(u_{\ell+1,i}) \\
&= X^C_{u_{k+1,j-2}}(u_{\ell+1,i}) \\
&= -\{u_{k+1,j-2}, u_{\ell+1,i}\}_C \\
&= -X^C_{u_{\ell+1,i}}(u_{k+1,j-2}) \\
&= X_{u_{\ell+1,i+2}}(u_{k+1,j-2}) \\
&= \{u_{k+1,j-2}, u_{\ell+1,i+2}\}, \qquad \text{i.e.}
\end{aligned}$$

(4.8) $$\{u_{k+1,j}, u_{\ell+1,i}\} = -\{u_{k+1,j-2}, u_{\ell+1,i}\}_C$$

(4.9) $$\{u_{k+1,j}, u_{\ell+1,i}\} = \{u_{k+1,j-2}, u_{\ell+1,i+2}\}.$$

Similarly if $N =$ even, we have

(4.10) $$\{p_i, p_j\} = -\{p_{i-2}, p_j\}_C$$

(4.11) $$\{p_i, p_j\} = \{p_{i-2}, p_{j+2}\}$$

Relations (4.8), (4.10) say that involution in $\{\ ,\ \}$ and $\{\ ,\ \}_C$ are equivalent. Relations (4.9), (4.11), and the prior remarks on vanishing Hamiltonian vector fields give a second proof of the involution of the integrals $u_{k+1,j}$ (and p_j if $N =$ even): consecutive application of (4.9), (4.11) comes to a stop as soon as either $j = 0, 1$, or $i = 0, 1$, in which cases the respective Poisson brackets vanish. We recovered theorem 4.2. The phenomenon observed above is general: Lenard relations imply involution in two Hamiltonian structures concomitently.

5. A completely integrable Euler-Poisson equation in $sl(N; \mathbf{C})$ and the N-dimensional heavy symmetric top. In this section an Euler-Poisson equation in $sl(N; \mathbf{C})$ is considered which induces on $so(N) \times so(N)$ the equations of motion of a heavy symmetric top. Its complete integrability is shown.

5.1. On $sl(N; \mathbf{C})$ with the bilinear, symmetric, non-degenerate, bi-invariant two-form $\kappa(A, B) = -(1/2)\mathrm{Tr}(AB)$, define the κ-symmetric isomorphism $L: sl(N; \mathbf{C}) \to sl(N; \mathbf{C})$ by $L(A) = (a_{ij}/(J_i + J_j)) - (\Sigma_{i=1}^N a_{ii}/2NJ_i)\mathrm{Id}$, where $A = (a_{ij})$ and $J = \mathrm{diag}(J_1, \ldots, J_N)$, $J_i > 0$ for all $i = 1, \ldots, N$, is a fixed diagonal matrix. The Euler-Poisson equations for the Hamiltonian $H(A, B) = (1/2)\kappa(B, L(B)) + \kappa(A, \chi)$, $\chi \in sl(N; \mathbf{C})$ a fixed non-zero matrix, are by (2.4)

(5.1) $$\dot{A} = [A, L(B)], \qquad \dot{B} = [B, L(B)] + [A, \chi].$$

Remark that the restriction of this problem to $so(N) \times so(N)$ gives the N-dimensional rigid body under gravity for which all $J_i > 0$.

Following the ideas of Section 4, we expect (5.1) to be completely integrable whenever there exists a constant matrix $C \in sl(N; \mathbf{C})$ such that the equation

$$(A + Bh + Ch^2)^{\cdot} = [A + Bh + Ch^2, L(B) + \chi h] \tag{5.2}$$

holds in the Kac-Moody extension of $sl(N; \mathbf{C})$. This is equivalent to (5.1) plus $[C, \chi] = 0$, $[B, \chi] + [C, L(B)] = 0$ for all $B \in sl(N; \mathbf{C})$. A straightforward analysis of these conditions yields the following.

THEOREM 5.1. *The Euler-Poisson equations* (5.1) *for* $\chi \neq 0$ *are equivalent to* (5.2) *if and only if either*

(a) $J_1 = \cdots = J_N = \alpha > 0$, $C = 2\alpha\chi$, $B = 2\alpha L(B)$, *in case that some off-diagonal entry of* χ *is non-zero, or*

(b) *the relations*

$$(J_j - J_k)\chi_{ii} + (J_k - J_i)\chi_{jj} + (J_i - J_j)\chi_{kk} = 0$$

holds for all triplets (i, j, k) *of distinct numbers* $1 \leq i, j, k \leq N$, *and* C *is given by*

$$\begin{aligned} C_{ii} = &-\frac{1}{N}(J_N + J_{N-1})(\chi_{NN} - \chi_{N-1,N-1}) - \cdots \\ &- \frac{N-i}{N}(J_{i+1} + J_i)(\chi_{i+1,i+1} - \chi_{ii}) \\ &+ \frac{i-1}{N}(J_i + J_{i-1})(\chi_{ii} - \chi_{i-1,i-1}) + \cdots \\ &+ \frac{1}{N}(J_2 + J_1)(\chi_{22} - \chi_{11}). \end{aligned}$$

if χ *is diagonal.*

As in Section 4 equation (5.2) is Hamiltonian on the invariant submanifold Q_C in the Kac-Moody extension $\widetilde{sl(N; \mathbf{C})}$ of $sl(N; \mathbf{C})$ (see 3.4.) and their Hamiltonian functions can be easily computed; they will not be used in the sequel.

5.2. Both Hamiltonian systems have a family of integrals $u_{k+1,j}$, the coefficients of h^j in $(1/(k + 1))\mathrm{Tr}(A + Bh + Ch^2)^{k+1}$ which follow from

(5.2), $k = 1, \ldots, N-1$. Their involution in both Hamiltonian structures is shown exactly as in 4.5 and the Lenard relations

$$(5.3) \qquad X_{u_{k+1,j}} = -X^{C}_{u_{k+1,j-2}}, \qquad j = 0, \ldots, 2k+4$$

where any vector field involving an index j which is <0 or $>2k+2$ is identically zero, are found as in 4.6. Since $X_{u_{k+1,0}}, X_{u_{k+1,1}}, X_{u_{k+1,2k+2}}$ vanish, the total number of integrals in involution is $\Sigma_{k=1}^{N-1} 2k = N(N-1)$, equalling half the dimension of the generic symplectic leaf in the Euler-Poisson structure of $sl(N; \mathbf{C}) \times sl(N; \mathbf{C})$. Here and in all that follows dimension is taken over the complex numbers.

THEOREM 5.2. *The Euler-Poisson equations* (5.1) *with the conditions of theorem* 7.1 *in* $sl(N; \mathbf{C}) \times sl(N; \mathbf{C})$ *for* χ, $C \in sl(N; \mathbf{C})$ *fixed matrices having distinct eigenvalues, are completely integrable Hamiltonian systems in both Euler-Poisson structures of* $sl(N; \mathbf{C}) \times sl(N; \mathbf{C})$.

Proof. We still have to prove the independence of the vector fields $X_{u_{k+1,j}}$ on a dense set. Let sgrad $= (\text{grad}_2, \text{grad}_1)$ denote the gradient with respect to κ_s (see (1.6)) and denote by $\mathcal{I} = \{u_{k+1,j} \mid k = 1, \ldots, N-1, j = 0, \ldots, 2k+2\}$ the set of integrals of (5.2), $\mathcal{A} = \text{span}\{(\text{sgrad} f)(A, B) \mid f \in \mathcal{I}\}$, $\mathcal{V} = \text{span}\{X_f(A, B) \mid f \in \mathcal{I}\}$, for a fixed pair (A, B). Throughout this proof, brackets of pairs are taken in the semidirect product $sl(N; \mathbf{C})\ {}_{\text{ad}}\!\times \overline{sl(N; \mathbf{C})}$; see (1.2). Clearly $\dim(\mathcal{V}) \le N(N-1)$, so we have to show that on a dense set, $\dim(\mathcal{V}) \ge N(N-1)$. Assume that we know $\dim(\mathcal{A}) \ge (N+2)(N-1)$. Then the κ_s-orthogonal $\mathcal{A}^\perp$ has dimension $\dim(\mathcal{A}^\perp) = 2(N^2-1) - \dim(\mathcal{A}) \le 2(N^2-1) - (N+2)(N-1) = N(N-1)$. But

$$\begin{aligned}
\mathcal{V}^\perp &= \{(A', B') \in sl(N; \mathbf{C}) \times sl(N; \mathbf{C}) \mid \kappa_s((A', B'), [(A, B), (\text{sgrad} f)(A, B)] \\
&= 0, \qquad \text{for all} f \in \mathcal{I}\} \\
&= \{(A', B') \in sl(N; \mathbf{C}) \times sl(N; \mathbf{C}) \mid [(A', B'), (A, B)] \in \mathcal{A}^\perp\} \\
&= \text{ad}^{-1}_{(A,B)}(\mathcal{A}^\perp),
\end{aligned}$$

so that $\text{ad}_{(A,B)}: \mathcal{V}^\perp \to \mathcal{A}^\perp$ is surjective and we conclude that for A, B regular semisimple elements in $sl(N; \mathbf{C})$

$$\dim(\mathcal{V}^\perp) = \dim(\mathcal{A}^\perp) + \dim(\text{Ker}(\text{ad}_{(A,B)} \mid \mathcal{V}^\perp))$$

$$\leq \dim(\mathcal{A}^{\perp}) + \dim(\mathrm{Ker}(\mathrm{ad}_{(A,B)}))$$

$$\leq N(N-1) + 2(N-1) = (N+2)(N-1)$$

so that finally

$$\dim(\mathcal{V}) = 2(N^2-1) - \dim(\mathcal{V}^{\perp}) \geq N(N-1).$$

Hence we reduced the problem to the proof of

$$\dim(\mathcal{A}) \geq (N+2)(N-1) \tag{5.4}$$

on a dense subset of $\{(A, B) | A, B \text{ regular semisimple}\}$. Conjugating all equations and gradients involved with a matrix in $SL(N; \mathbf{C})$ which diagonalizes C, and taking as new variables in (5.2) the conjugates of the old ones, we can assume that in (5.2) C is diagonal; the entries of C are hence distinct by hypothesis. The proof of (5.4) will consist of a series of technical lemmas.

Let D_i denote the vector space of matrices in $sl(N; \mathbf{C})$ all of whose entries are zero except on the i^{th} upper diagonal; thus D_0 = diagonal matrices, ..., D_{N-1} = matrices with all entries zero except eventually the $(1, N)$-entry, and $\dim(D_0) = N-1$, $\dim(D_i) = N-i$ for $i \geq 1$.

LEMMA 5.3. *For $i \geq 0$, let $T: D_i \to D_{i+1}$ be a surjective linear map. There exists an open dense set $Z_T \subset D_2 \times D_1$ such that for $(A, B) \in Z_T$, $S_{A,B}: D_i \to D_{i+2}$, $S_{A,B} = \mathrm{ad}_A + \mathrm{ad}_B \circ T$ is surjective.*

Proof. If $A \in D_2$, $B \in D_1$ have upper diagonals consisting of non-zero entries only, then it is easy to see that $\mathrm{ad}_A: D_i \to D_{i+2}$, $\mathrm{ad}_B: D_i \to D_{i+1}$ are surjective. The set $V = \{B \in D_1 |$ upper diagonal entries of B are all $\neq 0$, the first $(N-i-2) \times (N-i-2)$ block in the matrix representation of $\mathrm{ad}_B \circ T$ has all eigenvalues different from 0 and $-1\}$ is clearly open and dense in D_1. Let $U = \{A \in D_2 |$ all entries of A on the second diagonal are $\neq 0\}$, U is open and dense in D_2 and thus $U \times V$ is open and dense in $D_2 \times D_1$. We claim that if $(A, B) \in U \times V$, $S_{A,B}$ is surjective. To see this, remark that the matrix of $\mathrm{ad}_A: D_i \to D_{i+2}$ has $N-i$ columns and $N-i-2$ rows, the only non-zero entries being $-A_{1+i,3+i}, \ldots, -A_{N-2,N}$ on the diagonal and $A_{13}, \ldots, A_{n-i-2,N-i}$ on the second upper diagonal. Elementary transformations bring this matrix to one having only $1, \ldots, 1$ on the diagonal and all other entries equal to zero. Performing the same elementary transformations on the matrix of $\mathrm{ad}_B \circ T$, we still have the determinant of the first $(N-i-2) \times (N-i-2)$-block $\neq 0$. Moreover, since -1 is not an eigenvalue of this square matrix, it follows that the determinant of the first block in the matrix of $\mathrm{ad}_A + \mathrm{ad}_B \circ T$ transformed is $\neq 0$, i.e. $\mathrm{ad}_A + \mathrm{ad}_B \circ T$ is surjective. ∇

LEMMA 5.4. *Let $(A, B) \in D_2 \times D_1$ and $U_{kj} \in sl(N; \mathbf{C})$ be the coefficient of h^j in $(A + Bh + Ch^2)^k$ minus* $((1/N)\mathrm{Tr}(U_{kj})\mathrm{Id}$. $\{U_{k,2k} \mid k = 1, \ldots, N-1\}$ *generates D_0 and if all upper diagonal entries of B are $\neq 0$,* $\{U_{k,2k-1} \mid k = 1, \ldots, N-1\}$ *generates D_1.*

This is a classical result in invariant theory and can be proved in this simple case by a direct verification.

LEMMA 5.5. *There exists an open dense set $\mathcal{S} \subset D_2 \times D_1$ such that if $(A, B) \in \mathcal{S}$, $\{U_{k,2k-j}(A, B) \mid k = 1, \ldots, N-1\}$ generates D_j.*

Proof. If $j = 0, 1$ the statement is true by lemma 5.4. Assume inductively that the lemma holds for all $i \leq j, j \geq 1$ on a set $\mathcal{S}_j$; we want to prove that there exists an open dense set $\mathcal{S}_{j+1} \subset D_2 \times D_1$ such that for $(A, B) \in \mathcal{S}_{j+1}$, $\{U_{k,2k-j-1}(A, B) \mid k = 1, \ldots, N-1\}$ generates D_{j+1}. Define $T_j: D_{j-1} \to D_j$ by $T_j(U_{k,2k-j+1}(A, B)) = U_{k,2k-j}(A, B)$, $k = [(j-1)/2], \ldots, N-1$, with the convention that any U_{kj} involving a negative index is identically zero. By the inductive hypothesis, if $(A, B) \in \mathcal{S}_j$, $D_{j-1} = \mathrm{span}\{U_{k,2k-j+1}(A, B) \mid k = 1, \ldots, N-1\}$, $D_j = \mathrm{span}\{U_{k,2k-j}(A, B) \mid k = 1, \ldots, N-1\}$, so that T_j is a surjective linear map. Thus by Lemma 5.3 for $(A, B) \in \mathcal{S}_{j+1} = \mathcal{S}_j \cap Z_{T_j}$, the map $S^{j+1}_{A,B}: D_{j-1} \to D_{j+1}$, $S^{j+1}_{A,B}(X) = [A, X] + [B, T_j(X)]$ is surjective; $\mathcal{S}_{j+1}$ is clearly open and dense in $D_2 \times D_1$. On the system of generators of D_{j-1}, the Lenard relations (5.3) imply that $S^{j+1}_{A,B}(U_{k,2k-j+1}(A, B)) = [A, U_{k,2k-j+1}] + [B, U_{k,2k-j}] = -[C, U_{k,2k-j-1}]$, so that $\{U_{k,2k-j-1} \mid k = 1, \ldots, N-1\}$ generates D_{j+1}, since for C having all eigenvalues distinct, $\mathrm{ad}_C: D_{j+1} \to D_{j+1}$ is an isomorphism. Thus, by induction it follows that for $(A, B) \in \mathcal{S} = \mathcal{S}_{N-1}$ the statement of the lemma holds. ∇

Since A, B strictly upper triangular imply $U_{kj}(A, B)$ upper triangular, by the lemma above there exists an open dense set $\mathfrak{I}$ in the set of pairs of strictly upper triangular matrices such that if $(A, B) \in \mathfrak{I}$, $\{U_{kj}(A, B) \mid k = 1, \ldots, N-1, j = 0, \ldots, 2k\}$ generates the set of upper triangular matrices; $\mathfrak{I}$ is formed by all strictly upper triangular (A, B) such that $(A_2, B_1) \in \mathcal{S}$, where $A_2 \in D_2$, $B_1 \in D_1$ and the non-zero entries of

A_2, B_1 coincide with the corresponding entries of A, B. As in lemma 4.3 it can be shown that

$$\mathrm{grad}_1 u_{k+1,j} = \begin{cases} -2U_{kj} & , \quad \text{for } j = 0, \ldots, 2k \\ 0 & , \quad \text{for } j = 2k+1, 2k+2 \end{cases}$$

$$\mathrm{grad}_2 u_{k+1,j} = \begin{cases} 0 & , \quad \text{for } j = 0, 2k+2 \\ -2U_{k,j-1}, & \quad \text{for } j = 1, \ldots, 2k+1. \end{cases}$$

Since sgrad $u_{k+1,j} = (\mathrm{grad}_2 u_{k+1,j}, \mathrm{grad}_1 u_{k+1,j})$, we conclude that for (A, B) in an open dense set $\mathfrak{J}$ in the set of pairs of strictly upper triangular matrices $\mathfrak{B} = \mathrm{span}\{(\mathrm{sgrad}\, u_{k+1,j})(A, B) \mid k = 1, \ldots, N-1, j = 0, \ldots, 2k\}$ has projection on the second factor equal to the set of upper triangular matrices in $sl(N; \mathbf{C})$, i.e.

$$\dim(\Pi_2(\mathfrak{B})) = (N-1) + N(N-1)/2.$$

We shall denote in all that follows by Π_1, Π_2, the projections on the first and second factor respectively.

Carry out everything done so far for strictly lower triangular matrices to conclude that there exists an open dense set $\mathfrak{J}'$ in the set of pairs of strictly lower triangular matrices such that if $(A, B) \in \mathfrak{J}'$, $\dim(\Pi_2(\mathfrak{B}')) = (N-1) + N(N-1)/2$, where $\mathfrak{B}' = \mathrm{span}\{(\mathrm{sgrad}\, u_{k+1,j})(A, B) \mid k = 1, \ldots, N-1, j = 0, \ldots, 2k\}$. The set $\mathfrak{U} = \{(A, B) \mid A, B$ regular semisimple, strictly upper triangular part of $(A, B) \in \mathfrak{J}$, strictly lower triangular part of $(A, B) \in \mathfrak{J}'\}$ is open and dense in $sl(N; \mathbf{C}) \times sl(N; \mathbf{C})$. Clearly $\Pi_2(\mathfrak{B}) \cap \Pi_2(\mathfrak{B}') =$ diagonal matrices, so that for $(A, B) \in \mathfrak{U}$

$$\begin{aligned} &\dim(\mathrm{span}\{(\mathrm{sgrad}\, u_{k+1,j})(A, B) \mid k = 1, \ldots, N-1, j = 0, \ldots, 2k\}) \\ &\qquad \geq \dim(\mathfrak{B} + \mathfrak{B}') \geq \dim(\Pi_2(\mathfrak{B}) + \Pi_2(\mathfrak{B}')) \\ &\qquad = \dim(\Pi_2(\mathfrak{B})) + \dim(\Pi_2(\mathfrak{B}')) - \dim(\Pi_2(\mathfrak{B}) \cap \Pi_2(\mathfrak{B}')) \\ &\qquad = (N-1) + N(N-1). \end{aligned}$$

Since $(\mathrm{sgrad}\, u_{k+1,2k+1})(A, B) = (C^k - ((1/N)\mathrm{Tr} C^k)\mathrm{Id}, 0)$,

$$\mathrm{span}\{(\mathrm{sgrad}\, u_{k+1,2k+1})(A, B) \mid k = 1, \ldots, N-1\} = D_0 \times 0.$$

Remark that $\Pi_1(\mathcal{B}) \subseteq$ strictly upper triangular matrices, $\Pi_1(\mathcal{B}') \subseteq$ strictly lower triangular matrices, so that $(\mathcal{B} + \mathcal{B}') \cap (D_0 \times 0) = \{0\}$ and thus since $\mathcal{A} = \text{span}\{(\text{sgrad}\, u_{k+1,j})(A, B) \mid k = 1, \ldots, N-1, j = 0, \ldots, 2k+2\}$, for any $(A, B) \in \mathcal{U}$

$$\dim(\mathcal{A}) \geq \dim(\mathcal{B} + \mathcal{B}' + (D_0 \times 0)) \geq N(N-1) + (N-1) + \dim(D_0)$$

$$= (N+2)(N-1).$$

Theorem 5.2. is proven. □

The flow of this Hamiltonian system can be linearized with the aid of Theorem 1 of Adler-van Moerbeke [4] which is directly applicable in this case. The algebraic curve $Q(z, h) = \det(A + Bh + Ch^2 - z\text{Id}) = 0$ for C = constant, having all eigenvalues distinct, has genus $(N-1)^2$. The coefficients of $Q(z, h)$ are polynomials in the integrals u_{ij} and thus they commute. $(N-1)^2$ among them lead to independent linear flows on the Jacobian of $Q(z, h) = 0$ (in the form of sums of Abelian integrals). $N - 1$ additional independent flows are generated by conjugation with diagonal matrices and can be taken in the form

$$(A + Bh + Ch^2) = [A + Bh + Ch^2, C^k], \qquad k = 1, \ldots, N-1$$

Note that from this argument it follows once again that (5.2) is completely integrable; see Adler-van Moerbeke [4, Section 3] for details.

5.3. Since the $N(N-1)$ Hamiltonian vector fields $\{X_{u_{k+1,j}}(A, B) \mid k = 1, \ldots, N-1, j = 2, \ldots, 2k+1\}$ for (A, B) in a dense open set in the generic $N(N-1)$-dimensional symplectic leaf are tangent to this leaf and generate the tangent space, they are necessarily independent. Thus, since the above Hamiltonian system induces on $so(N) \times so(N)$ the N-dimensional symmetric rigid body under gravity, if N = odd, we conclude that the $N(N-1)/2 - [N/2]$ Hamiltonian vector fields $\{X_{u_{k+1,j}}(\Gamma, M) \mid k = \text{odd}, k = 1, \ldots, N-2, j = 2, \ldots, 2k+1\}$ are generically independent. If N = even, since Pf is functionally generated by traces of powers, we conclude that the Hamiltonian vector fields $\{X_{u_{k+1,j}}(\Gamma, M), X_{p_i}(\Gamma, M) \mid k = \text{odd}, k = 1, \ldots, N-3, j = 2, \ldots, N-1, i = 2, \ldots, N-1\}$ generically span the tangent space to the generic symplectic leaf and thus they must be independent. This and theorem 4.2 prove the following.

THEOREM 5.6. *The symmetric N-dimensional rigid body under gravity for $\chi \in so(N)$ having all eigenvalues distinct is a completely integrable Hamiltonian system in both Euler-Poisson structures of $so(N) \times so(N)$.*

The linearization of these flows follows again from the general method in Adler-van Moerbeke [4]. If $N =$ even, their method in example 2, Section 3 (the linearization of the free rigid body motion) can be applied directly and the flows linearize on the Prym-variety of $Q(z, h) = \det(\Gamma + Mh + Ch^2 - z\mathrm{Id}) = 0$. If $N =$ odd, C has always a zero eigenvalue and the curve becomes singular. The way out is to introduce an arbitrary parameter ϵ in the diagonal form of C in place of the zero eigenvalue and to linearize this new problem with the aid of the general Adler-van Moerbeke method. One obtains then time in form of sums of Abelian integrals. Pick now from the basis of holomorphic differentials those which in the limit $\epsilon \to 0$ remain holomorphic on $Q(z, h) = 0$. In the limit $\epsilon \to 0$ one gets the curve $Q(z, h) = 0$, time as sums of the remaining Abelian integrals, and the problem is thus linearized. This represents the direct extension of the method with which the flow of the 3-dimensional Lagrange top has been linearized in [4] and [40].

Remark. It is clear that the techniques used throughout the paper work for any semisimple complex Lie algebra and its real forms. The role of the traces in finding the integrals of Euler-Poisson equations is taken by a basis of the ring of invariants and our matrix computations are replaced by formulations in terms of a root-space decomposition. Kostant [19], Mishchenko, and Fomenko [27] have carried this out for other classical systems: the non-periodic Toda lattice and the n-dimensional free rigid body problem; see also Adler-van Moerbeke [3] for the periodic classical and non-symmetric Toda systems.

5.4. We discuss now briefly the Lagrange top. C being in this case highly degenerate, the above proofs do not work, except for $N = 3, 4$ where the generic independence of the integrals can be easily shown directly [40, Section 2]. The solution for general N proceeds via a modified version of the van Moerbeke-Mumford method [30]. Since this is done in great detail in [40, Section 3] for $N = 3$ and the extension for general N follows step by step this proof, we only point out a difference necessary in the proof and presenting interest in itself.

The unitary matrix

$$U = \begin{bmatrix} 0\cdots 0 & i/\sqrt{2} & 1/\sqrt{2} \\ 0\cdots 0 & 1/\sqrt{2} & i/\sqrt{2} \\ 0\cdots 1 & 0 & 0 \\ \cdots & \cdots & \cdots \\ 1\cdots 0 & 0 & 0 \end{bmatrix}$$

brings C in diagonal form. A straightforward computation shows that

$$U^{-1}(\Gamma + Mh + Ch^2)U = \begin{bmatrix} \tilde{\Gamma}_2 + \tilde{M}_2 h & \beta & i\beta^* \\ -\beta^{*t} & -\omega & 0 \\ i\beta^t & 0 & \omega \end{bmatrix} \tag{5.5}$$

where β is an $(N - 2)$-column vector whose entries are $y_k + hx_k$, $k = 3, \ldots, N$, $x_k = -(iM_{1k} + M_{2k})/\sqrt{2}$, $y_k = -(i\Gamma_{1k} + \Gamma_{2k})/\sqrt{2}$, $\beta^* = \bar{y} + h\bar{x}$ ($^-$ is complex conjugation of each component), and ^{t} denotes taking the transpose; $\omega = i(\Gamma_{12} + M_{12}h + C_{12}h^2)$; $\tilde{\Gamma}_2$ and $\tilde{M}_2$ are the $(N - 2) \times (N - 2)$ matrices

$$\tilde{\Gamma}_2 = \begin{bmatrix} \Gamma_{NN} & \cdots & \Gamma_{N3} \\ \cdots & \cdots & \cdots \\ \Gamma_{3N} & \cdots & \Gamma_{33} \end{bmatrix} \qquad \tilde{M}_2 = \begin{bmatrix} M_{NN} & \cdots & M_{N3} \\ \cdots & \cdots & \cdots \\ M_{3N} & \cdots & M_{33} \end{bmatrix}$$

Let P diagonalize $\tilde{M}_2$ and conjugate the above resulting matrix by

$$V = \begin{bmatrix} P & 0 \\ 0 & \mathrm{Id} \end{bmatrix}$$

This will affect only $\tilde{\Gamma}_2$, $\tilde{M}_2$ and nothing else. Thus in (5.5) $\tilde{M}_2$ will be now diagonal.

Performing the same operations on $\Omega + \chi h$ we get a matrix of the same type as (5.1). We alter now the diagonal entries of this new matrix so as to get $2\mathfrak{a}/(\mathfrak{a} + \mathfrak{b})$ times the lower 2×2-block and $2\mathfrak{b}/(\mathfrak{a} + \mathfrak{b})$ times the upper $(N - 2) \times (N - 2)$-block in the coefficient of h^0. This is always possible since conjugation by a time-dependent diagonal matrix does not modify the flow on the Jacobian. Another reason is that in any Lax equation $\dot{L} = [L, A]$ conjugation with a time-dependent diagonal matrix alters *only* elements of the diagonal of A, since $(DLD^{-1})^{\cdot} = [DLD^{-1}, DAD^{-1} - \dot{D}D^{-1}]$ as a simple computation shows.

Conjugate now this resulting equation by V^{-1} and U^{-1} call the new variables again M, Γ, Ω and observe that in this new equation $(\Gamma + Mh + Ch^2)^{\cdot} = [\Gamma + Mh + Ch^2, \Omega + \chi h]$, C, χ are unchanged and $M = (\alpha + \beta)\Omega$. In other words, after a time-dependent change of variables, *the Lagrange top reduces to a degenerate symmetric top.* It is this flow which is linearized with the method of Section 3 of [40], by putting ϵ in the upper $(N - 2) \times (N - 2)$-diagonal, using Theorem 1 of Section 3 in [4] and then letting $\epsilon \to 0$; for details see [40, Section 3].

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