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THE C. NEUMANN PROBLEM AS A COMPLETELY INTEGRABLE SYSTEM ON AN ADJOINT ORBIT

BY

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ABSTRACT. It is shown by purely Lie algebraic methods that the C. Neumann problem–the motion of a material point on a sphere under the influence of a quadratic potential–is a completely integrable system of Euler-Poisson equations on a minimal-dimensional orbit of a semidirect product of Lie algebras.

1. The C. Neumann problem. The motion of a point on the sphere S^{n-1} under the influence of a quadratic potential $U(\mathbf{x}) = \frac{1}{2}A\mathbf{x} \cdot \mathbf{x}$, $\mathbf{x} \in \mathbf{R}^n$, $A = \operatorname{diag}(a_1, \ldots, a_n)$ is a completely integrable Hamiltonian system. For $n = 3$ this has been shown by C. Neumann in 1859 **[12]** and for arbitrary n by K. Uhlenbeck **[16]**, R. Devaney **[3]**, J. Moser **[10]**, **[11]**, M. Adler, and P. van Moerbeke **[2]**. In this paper we show how this problem fits naturally in the framework of Euler-Poisson equations **[4]**, **[5]**, **[14]**, **[17]** proving that the C. Neumann problem is a Hamiltonian system on a minimal-dimensional adjoint orbit in a semidirect product of Lie algebras. Thus its complete integrability will follow entirely from Lie algebraic considerations.

The equations of motion are

$$\ddot{x}_i = -a_i x_i + \lambda x_i, \qquad i = 1, \ldots, n, \tag{1.1}$$

where the Lagrange multiplier $\lambda = A\mathbf{x} \cdot \mathbf{x} - \|\mathbf{x}\|^2$ is chosen such that $\mathbf{x} \in S^{n-1}$ during the motion. Set $\dot{\mathbf{x}} = \mathbf{y}$ and get the equivalent system to (1.1)

$$\dot{x}_i = y_i, \qquad \dot{y}_i = -a_i x_i + \left(A\mathbf{x} \cdot \mathbf{x} - \|\mathbf{y}\|^2\right) x_i, \qquad \|\mathbf{x}\| = 1, \qquad \mathbf{x} \cdot \mathbf{y} = 0. \tag{1.2}$$

The following crucial remark that motivated the present investigation is due to K. Uhlenbeck and can be verified without any difficulties.

LEMMA 1.1. *Put* $X = (x_i x_j)$, $P = (y_i x_j - x_i y_j)$. *System* (1.2) *is equivalent to*

$$\dot{X} = [P, X], \qquad \dot{P} = [X, A], \qquad \|\mathbf{x}\| = 1, \qquad \mathbf{x} \cdot \mathbf{y} = 0. \tag{1.3}$$

Remark that if one replaces X and A by $X - \mathrm{Id}/n$ and $A - (\mathrm{Tr}(A))\mathrm{Id}/n$ respectively, where Id is the $n \times n$ identity matrix, equations (1.3) remain unchanged. From now on we shall assume that in (1.3) this change has been made so that $X, P, A \in sl(n)$. The next section gives a Lie algebraic interpretation to these equations.

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2. The equations of motion as a Hamiltonian system on an adjoint orbit. We start by recalling a few facts about the ad-semidirect product $\mathfrak{G}_{\text{ad}} \times \overline{\mathfrak{G}}$ of a semisimple Lie algebra $\mathfrak{G}$ with the abelian Lie algebra $\overline{\mathfrak{G}}$ having underlying vector space $\mathfrak{G}$. If $(\xi_1, \eta_1), (\xi_2, \eta_2) \in \mathfrak{G}_{\text{ad}} \times \overline{\mathfrak{G}}$, their bracket is defined by

$$[(\xi_1, \eta_1), (\xi_2, \eta_2)] = ([\xi_1, \xi_2], [\xi_1, \eta_2] - [\xi_2, \eta_1]). \tag{2.1}$$

If κ denotes a bilinear, symmetric, nondegenerate, bi-invariant, two-form on $\mathfrak{G}$, the form κ_s, called the semidirect product of κ with itself and defined by

$$\kappa_s((\xi_1, \eta_1), (\xi_2, \eta_2)) = \kappa(\xi_1, \eta_2) + \kappa(\xi_2, \eta_1) \tag{2.2}$$

is a bilinear, symmetric, nondegenerate, bi-invariant, two-form on $\mathfrak{G}_{\text{ad}} \times \overline{\mathfrak{G}}$.

Let G be a Lie group with Lie algebra $\mathfrak{G}$. The Ad-semidirect product $G_{\text{Ad}} \times \overline{\mathfrak{G}}$ is a Lie group with underlying manifold $G \times \mathfrak{G}$ and composition law

$$(g_1, \xi_1)(g_2, \xi_2) = (g_1 g_2, \xi_1 + \text{Ad}_{g_1}\xi_2). \tag{2.3}$$

Note that the identity element is $(e, 0)$ and the inverse $(g, \xi)^{-1} = (g^{-1}, -\text{Ad}_{g^{-1}}\xi)$. The Lie algebra of $G_{\text{Ad}} \times \overline{\mathfrak{G}}$ is $\mathfrak{G}_{\text{ad}} \times \overline{\mathfrak{G}}$. The adjoint action of the Lie group $G_{\text{Ad}} \times \overline{\mathfrak{G}}$ on $\mathfrak{G}_{\text{ad}} \times \overline{\mathfrak{G}}$ is given by

$$\text{Ad}_{(g,\theta)}(\xi, \eta) = (\text{Ad}_g\xi, \text{Ad}_g\eta + [\theta, \text{Ad}_g\xi]). \tag{2.4}$$

In the considerations that follow, the orbit symplectic structure plays a central role (see Abraham and Marsden **[1]** and Ratiu **[14]** for proofs). If a Lie algebra $\mathfrak{G}$ has a bilinear, symmetric, nondegenerate, bi-invariant two-form κ,

$$\omega_\xi(\text{Ad}_g\xi)([\eta, \text{Ad}_g\xi], [\zeta, \text{Ad}_g\xi]) = -\kappa([\eta, \zeta], \text{Ad}_g\xi) \tag{2.5}$$

for $\xi, \eta, \zeta \in \mathfrak{G}$, $g \in G$, defines the canonical symplectic structure on the adjoint orbit $G \cdot \xi$ through ξ. If E, E': $\mathfrak{G} \to \mathbf{R}$, the Hamiltonian vector field of $E|G \cdot \xi$ is given by

$$X_{E|G\cdot\xi}(\text{Ad}_g\xi) = -[(\text{grad}\, E)(\text{Ad}_g\xi), \text{Ad}_g\xi] \tag{2.6}$$

and the Poisson bracket of $E|G \cdot \xi$, $E'|G \cdot \xi$ is

$$\{E|G \cdot \xi, E'|G \cdot \xi\}(\text{Ad}_g\xi) = -\kappa([(\text{grad}\, E)(\text{Ad}_g\xi), (\text{grad}\, E')(\text{Ad}_g\xi)], \text{Ad}_g\xi) \tag{2.7}$$

where grad denotes the gradient with respect to κ.

For the semidirect product these formulas become

$$\begin{aligned} &\omega(\xi, \eta)([(\xi, \eta), (\zeta_1, \zeta_1')], [(\xi, \eta), (\zeta_2, \zeta_2')]) \\ &\qquad = -\kappa_s((\xi, \eta), [(\zeta_1, \zeta_1'), (\zeta_2, \zeta_2')]), \end{aligned} \tag{2.8}$$

$$X_E(\xi, \eta) = -([(\text{grad}_2\, E)(\xi, \eta), \xi], [(\text{grad}_2\, E)(\xi, \eta), \eta] + [(\text{grad}_1\, E)(\xi, \eta), \xi]), \tag{2.9}$$

$$\begin{aligned} \{E, E'\}(\xi, \eta) = &-\kappa(\xi, [(\text{grad}_2\, E)(\xi, \eta), (\text{grad}_1\, E')(\xi, \eta)]) \\ &-\kappa(\xi, [(\text{grad}_1\, E)(\xi, \eta), (\text{grad}_2\, E')(\xi, \eta)]) \\ &-\kappa(\eta, [(\text{grad}_2\, E)(\xi, \eta), (\text{grad}_2\, E')(\xi, \eta)]), \end{aligned} \tag{2.10}$$

where $(\text{grad}_1, \text{grad}_2)$ denotes the usual gradient with respect to $\kappa \times \kappa$; note that the gradient with respect to κ_s is $(\text{grad}_2, \text{grad}_1)$.

Assume that $\mathfrak{G} = \mathfrak{K} \oplus \mathfrak{N}$, where $\mathfrak{K}$ is a vector subspace and $\mathfrak{N}$ a Lie subalgebra of $\mathfrak{G}$, $\mathfrak{N}$ having N as underlying closed Lie subgroup of G. Denote by $\Pi_{\mathfrak{K}}$, $\Pi_{\mathfrak{N}}$ the projections of $\mathfrak{G}$ on $\mathfrak{K}$ and $\mathfrak{N}$ respectively. Then $\mathfrak{G}^* = \mathfrak{K}^* \oplus \mathfrak{N}^*$. The nondegeneracy of κ on $\mathfrak{G}$ defines the isomorphisms $\mathfrak{N}^\perp \cong \mathfrak{K}^*$, $\mathfrak{K}^\perp \cong \mathfrak{N}^*$ and thus the coadjoint action of N on $\mathfrak{N}^*$ induces an action of N on $\mathfrak{K}^\perp$ given by

$$N \times \mathfrak{K}^\perp \ni (n, \xi) \mapsto \Pi_{\mathfrak{K}^\perp} \operatorname{Ad}_n \xi \in \mathfrak{K}^\perp \tag{2.11}$$

where $\Pi_{\mathfrak{K}^\perp}: \mathfrak{G} \to \mathfrak{K}^\perp$ denotes the canonical projection defined by the direct sum decomposition $\mathfrak{G} = \mathfrak{K}^\perp \oplus \mathfrak{N}^\perp$. Thus the orbit $N \cdot \xi$ equals

$$N \cdot \xi = \{\Pi_{\mathfrak{K}^\perp}(\operatorname{Ad}_n \xi) | n \in N\} \subseteq \mathfrak{K}^\perp, \qquad \xi \in \mathfrak{K}^\perp, \tag{2.12}$$

whose tangent space at $\bar{\xi} \in N \cdot \xi$ is

$$T_{\bar{\xi}}(N \cdot \xi) = \{\Pi_{\mathfrak{K}^\perp}[\bar{\xi}, \eta] | \eta \in \mathfrak{N}\} \subseteq \mathfrak{K}^\perp. \tag{2.13}$$

The symplectic structure on $N \cdot \xi$ equals, by (2.5),

$$\omega_\xi(\bar{\xi})(\Pi_{\mathfrak{K}^\perp}[\eta, \bar{\xi}], \Pi_{\mathfrak{K}^\perp}[\zeta, \bar{\xi}]) = -\kappa([\eta, \zeta], \bar{\xi}), \qquad \bar{\xi} \in N \cdot \xi \subseteq \mathfrak{K}^\perp, \tag{2.14}$$

and the Hamiltonian vector field defined by $E | N \cdot \xi$, $E: \mathfrak{G} \to \mathbf{R}$, is, by (2.6),

$$X_{E|N\cdot\xi}(\bar{\xi}) = -\Pi_{\mathfrak{K}^\perp}[\Pi_{\mathfrak{N}}(\operatorname{grad} E)(\bar{\xi}), \bar{\xi}], \qquad \bar{\xi} \in N \cdot \xi \subseteq \mathfrak{K}^\perp. \tag{2.15}$$

Finally the Poisson bracket of $E | G \cdot \xi$, $E' | G \cdot \xi$ is given by (2.7),

$$\{E | G \cdot \xi, E' | G \cdot \xi\}(\bar{\xi}) = -\kappa([\Pi_{\mathfrak{N}}(\operatorname{grad} E)(\bar{\xi}), \Pi_{\mathfrak{N}}(\operatorname{grad} E')(\bar{\xi})], \bar{\xi}), \tag{2.16}$$

for $\bar{\xi} \in N \cdot \xi \subseteq \mathfrak{K}^\perp$. All previous considerations naturally live on the duals but this is the form we shall use for the C. Neumann problem; see Ratiu **[14]** for a parallel description on duals.

We shall apply all previous results to a specific Lie algebra. Let $\mathfrak{G} = sl(n)_{\text{ad}} \times \overline{sl(n)}$, $G = Sl(n)_{\text{Ad}} \times \overline{sl(n)}$, $\mathfrak{N} = so(n) \times sym$, $N = SO(n) \times sym$, $\mathfrak{K} = sym \times so(n)$, where $sym \subset sl(n)$ denotes the vector space of all symmetric matrices. Clearly $\mathfrak{N}$ is a Lie subalgebra and $\mathfrak{K}$ a vector subspace of $\mathfrak{G}$, N a Lie subgroup of G with Lie algebra $\mathfrak{N}$. Thus by our general considerations N acts on $\mathfrak{K}^\perp$. It is easy to check that with respect to κ_s, where $\kappa(A, B) = -\frac{1}{2}\operatorname{Tr}(AB)$, $\mathfrak{K}^\perp = \mathfrak{K}$, $\mathfrak{N}^\perp = \mathfrak{N}$. In what follows we shall determine explicitly a particular N-orbit; note first that in the case above $\Pi_{\mathfrak{K}^\perp}$ in formula (2.11) is not necessary, i.e. the action of N on $\mathfrak{K}^\perp$ is given by (2.4).

If $\mathbf{y}, \mathbf{z} \in \mathbf{R}^n$, denote by $\mathbf{y} \otimes \mathbf{z}$ the matrix having entries $y_i z_j$ and remark that if $g \in SO(n)$, $g(\mathbf{y} \otimes \mathbf{z})g^{-1} = (g\mathbf{y}) \otimes (g\mathbf{z})$. Let $\mathbf{z} = (1, \ldots, 1)/\sqrt{n}$ and take $(\mathbf{z} \otimes \mathbf{z} - \operatorname{Id}/n, 0) \in \mathfrak{K}^\perp$. Let $g \in SO(n)$ be arbitrary and denote $\mathbf{x} = g\mathbf{z}$. Then $\|\mathbf{x}\| = \|\mathbf{z}\| = 1$ and $g(\mathbf{z} \otimes \mathbf{z} - \operatorname{Id}/n)g^{-1} = \mathbf{x} \otimes \mathbf{x} - \operatorname{Id}/n$ which is a matrix X having all off-diagonal entries equal to $x_i x_j$ and diagonal entries $x_i^2 - 1/n$. Thus the first component of the N-orbit through $(\mathbf{z} \otimes \mathbf{z} - \operatorname{Id}/n, 0)$ is the matrix X occurring in Lemma 1.1. We compute the second component. If $\theta \in sym$, $X_{ij} = x_i x_j$, $X_{ii} = x_i^2 - 1/n$, then

$$[\theta, X]_{ij} = \left(\sum_{k=1}^n \theta_{ik} x_k - x_i C(\mathbf{x}, \theta)\right) x_j - x_i \left(\sum_{k=1}^n \theta_{jk} x_k - x_j C(\mathbf{x}, \theta)\right),$$

where $C(\mathbf{x}, \theta) = \Sigma_{k,i} x_i x_k \theta_{ik}$. Put $y_i = \Sigma_{k=1}^n (\theta_{ik} x_k - x_i C(\mathbf{x}, \theta))$ and remark that if $\mathbf{y} = (y_1, \ldots, y_n)$, $\mathbf{x} \cdot \mathbf{y} = 0$ since $\|\mathbf{x}\| = 1$. Thus the second component of the N-orbit consists of matrices $P \in so(n)$, $P_{ij} = y_i x_j - x_j y_i$, $\mathbf{x} \cdot \mathbf{y} = 0$. We showed hence that this N-orbit consists of pairs $(X, P) \in sym \times so(n)$ with X, P defined as in Lemma 1.1.

Remark that the correspondence $(X, P) = \lambda(\mathbf{x}, \mathbf{y})$ defines a diffeomorphism of this orbit onto the tangent bundle TS^{n-1} of the unit sphere S^{n-1} in $\mathbf{R}^n$. A tangent vector at (X, P) to this orbit is $[(X, P), (\xi, \eta)]$ for $(\xi, \eta) \in so(n) \times sym$ and is of the form $(V, W) \in sym \times so(n)$, where $V_{ij} = v_j x_i + x_j v_i$, $v_i = \Sigma_{k=1}^n x_k \xi_{ki}$, $W_{ij} = w_i x_j - x_i w_j + y_i v_j - y_j v_i$, $w_i = \Sigma_{k=1}^n (y_k \xi_{ki} - x_k \eta_{ki}) + x_i \Sigma_{k,i} x_i x_k \eta_{ik}$ as a short calculation shows. Thus the tangent map of λ is given by $(V, W) \mapsto (\mathbf{v}, \mathbf{w})$, where $\mathbf{v} = (v_1, \ldots, v_n)$, $\mathbf{w} = (w_1, \ldots, w_n)$.

TS^{n-1} has a natural symplectic structure induced by the canonical symplectic form $-\Sigma_{i=1}^n dx_i \wedge dy_i$ of $\mathbf{R}^{2n}$. By (2.8) the canonical symplectic structure ω on the orbit is given by

$$\omega(X, P)\big([(X, P), (\xi^1, \eta^1)], [(X, P), (\xi^2, \eta^2)]\big) = \kappa_s\big([(\xi^2, \eta^2), (\xi^1, \eta^1)], (X, P)\big).$$

Let V^i, W^i, $\mathbf{v}^i$, $\mathbf{w}^i$ be defined by ξ^i, η^i, $i = 1, 2$. We have by bi-invariance of κ_s, antisymmetry of ξ^2, symmetry of η^1, and by (2.8), (2.14)

$$\begin{aligned}
(\lambda_* \omega)(\mathbf{x}, \mathbf{y})((\mathbf{v}^1, \mathbf{w}^1), (\mathbf{v}^2, \mathbf{w}^2)) &= \omega(X, P)((V^1, W^1), (V^2, W^2)) \\
&= -\kappa_s((\xi^2, \eta^2), (V^1, W^1)) \\
&= \frac{1}{2} \operatorname{Tr}(\xi^2 W^1) + \frac{1}{2} \operatorname{Tr}(\eta^2 V^1) \\
&= \sum_{k=1}^n \left(w_k^1 \sum_{i=1}^n x_i \xi_{ik}^2 \right) - \sum_{k=1}^n \left(v_k^1 \sum_{i=1}^n y_i \xi_{ik}^2 \right) + \sum_{k=1}^n \left(v_k^1 \sum_{i=1}^n x_i \eta_{ik}^2 \right) \\
&= -\sum_{k=1}^n (v_k^1 w_k^2 - w_k^1 v_k^2) \\
&= \left(-\sum_{k=1}^n dx_k \wedge dy_k \right)((\mathbf{v}^1, \mathbf{w}^1), (\mathbf{v}^2, \mathbf{w}^2)).
\end{aligned}$$

This shows that λ is a symplectic diffeomorphism:

$$\lambda_* \omega = \left(-\sum_{i=1}^n dx_i \wedge dy_i \right) \Big| TS^{n-1}.$$

Let L: $sl(n) \to sl(n)$ be given by $L(\xi) = -\xi$. L is clearly a κ-symmetric isomorphism. The following Euler-Poisson Hamiltonian (see **[4]**, **[5]**, **[15]**, **[17]** for motivations) $E(\xi, \eta) = \frac{1}{2}\kappa(\eta, L(\eta)) + \kappa(A, \xi)$ for $A \in sl(n)$ a fixed diagonal matrix, induces a Hamiltonian vector field on the N-orbit through $(\mathbf{z} \otimes \mathbf{z} - \mathrm{Id}/n, 0)$ given by (2.15), $(X, P) \mapsto ([P, X], [X, A])$, i.e. we get equations (1.3). Hence we proved the following.

THEOREM 2.1. *The* $N = so(n) \times sym$*-orbit through* $(\mathbf{z} \otimes \mathbf{z} - \mathrm{Id}/n, 0)$ *in* $\mathfrak{K}^{\perp} = \mathfrak{K} = sym \times so(n)$, $\mathbf{z} = (1, \ldots, 1)/\sqrt{n}$, *consists of all pairs* (X, P), $X_{ij} = x_i x_j$ *for* $i \neq j$, $X_{ii} = x_i^2 - 1/n$, $P_{ij} = y_i x_j - x_i y_j$, $\|\mathbf{x}\| = 1$, $\mathbf{x} \cdot \mathbf{y} = 0$. *With the Kirillov-Kostant-Souriau symplectic structure this* $(2n - 2)$*-dimensional orbit is symplectically diffeomorphic via* $(X, P) \mapsto (\mathbf{x}, \mathbf{y})$ *to* TS^{n-1} *with the symplectic structure induced from* $\mathbf{R}^{2n}$ *by* $-\sum_{i=1}^{n} dx_i \wedge dy_i$. *The Hamiltonian* $E(X, P) = -\frac{1}{2}\kappa(P, P) + \kappa(A, X)$ *defines on this orbit the equations of motion of the C. Neumann problem*

$$\dot{X} = [P, X], \qquad \dot{P} = [X, A], \qquad \|\mathbf{x}\| = 1, \qquad \mathbf{x} \cdot \mathbf{y} = 0. \tag{2.17}$$

REMARK. M. Adler and P. van Moerbeke [2] have independently observed that (2.17) is a Hamiltonian system in a semidirect product.

3. The complete set of integrals and their involution.

LEMMA 3.1. *The equations* $\dot{X} = [P, X]$, $\dot{P} = [X, A]$, $\|\mathbf{x}\| = 1$, $\mathbf{x} \cdot \mathbf{y} = 0$ *are equivalent to*

$$(-X + P\lambda + A\lambda^2)^{\cdot} = [-X + P\lambda + A\lambda^2, -P - A\lambda] \tag{3.1}$$

for any parameter λ.

The proof is a straightforward verification. It follows that the functions $(1/2(k+1))\mathrm{Tr}(-X + P\lambda + A\lambda^2)^{k+1}$ are conserved on the flow of (3.1). If $t \mapsto (X(t), P(t))$ denotes the flow of (2.17), then $t \mapsto -X(t) + P(t)\lambda + A\lambda^2$ is the flow of (3.1) and we conclude that the coefficients of λ in the expansion of $(1/2(k+1))\mathrm{Tr}(-X + P\lambda + A\lambda^2)^{k+1}$ are conserved along the flow of (2.17). Let $f_k(X, P)$ be the coefficient of λ^{2k} in this expansion for $k = 1, \ldots, n-1$. We shall prove in this section that all f_k are in involution. The method of the proof follows Ratiu [13], [15] closely.

THEOREM 3.2. *Let* $\mathfrak{G}$ *be a Lie algebra with a bilinear, symmetric, nondegenerate, bi-invariant, two-form* κ. *Let* $f, g\colon \mathfrak{G} \to \mathbf{R}$ *satisfy* $[(\mathrm{grad}\, f)(\xi), \xi] = 0$, $[(\mathrm{grad}\, g)(\xi), \xi] = 0$ *for all* $\xi \in \mathfrak{G}$. *Denote* $f_a(\xi, \eta) = f(\xi + a\eta + a^2\varepsilon)$, $g_b(\xi, \eta) = g(\xi + b\eta + b^2\varepsilon)$ *for* $\varepsilon \in \mathfrak{G}$ *fixed and* a, b *arbitrary parameters. Then* f_a, g_b *Poisson commute in the bracket of* $\mathfrak{G}\ {}_{\mathrm{ad}}\!\times \overline{\mathfrak{G}}$ *defined by its symplectic decomposition in adjoint orbits.*

PROOF. Clearly $(\mathrm{grad}_1 f_a)(\xi, \eta) = (\mathrm{grad}\, f)(\xi + a\eta + a^2\varepsilon)$, $(\mathrm{grad}_2 f_a)(\xi, \eta) = a(\mathrm{grad}\, f)(\xi + a\eta + a^2\varepsilon)$ and similarly for g_b. By (2.10)

$$\begin{aligned}
\{f_a, g_b\}(\xi, \eta) &= -\kappa([(\mathrm{grad}_1 f_a)(\xi, \eta), (\mathrm{grad}_2 g_b)(\xi, \eta)], \xi) \\
&\quad - \kappa([(\mathrm{grad}_2 f_a)(\xi, \eta), (\mathrm{grad}_1 g_b)(\xi, \eta)], \xi) \\
&\quad - \kappa([(\mathrm{grad}_2 f_a)(\xi, \eta), (\mathrm{grad}_2 g_b)(\xi, \eta)], \eta) \\
&= -\kappa\big((a + b)\xi + ab\eta, [(\mathrm{grad}\, f)(\xi + a\eta + a^2\varepsilon), \\
&\qquad\qquad (\mathrm{grad}\, g)(\xi + b\eta + b^2\varepsilon)]\big) \\
&= (b^2/(a - b))\kappa\big([\xi + a\eta + a^2\varepsilon, (\mathrm{grad}\, f)(\xi + a\eta + a^2\varepsilon)], \\
&\qquad\qquad (\mathrm{grad}\, g)(\xi + b\eta + b^2\varepsilon)\big) \\
&\quad + (a^2/(a - b))\kappa\big([\xi + b\eta + b^2\varepsilon, (\mathrm{grad}\, g)(\xi + b\eta + b^2\varepsilon)], \\
&\qquad\qquad (\mathrm{grad}\, f)(\xi + a\eta + a^2\varepsilon)\big) = 0
\end{aligned}$$

by hypothesis. By continuity $\{f_a, g_b\} = 0$ holds also for $a = b$. $\square$

REMARK. The condition $[(\text{grad} f)(\xi), \xi] = 0$ for all $\xi \in \mathfrak{G}$ is the infinitesimal version of Ad-invariance of f as an easy computation shows.

THEOREM 3.3. *Let G be a Lie group, N a closed subgroup, with Lie algebras $\mathfrak{G}$ and $\mathfrak{N}$ respectively. Assume $\mathfrak{G} = \mathfrak{K} \oplus \mathfrak{N}$, $\mathfrak{K}$ a vector subspace, $[\mathfrak{K}, \mathfrak{N}] \subseteq \mathfrak{K}$, and that $\mathfrak{G}$ has a bilinear, symmetric, nondegenerate, bi-invariant, two-form κ. Assume that f, g: $\mathfrak{G} \to \mathbf{R}$ Poisson commute on $\mathfrak{G}$, i.e.*

$$\kappa([(\text{grad} f)(\xi), (\text{grad}\, g)(\xi)], \xi) = 0,$$

for all $\xi \in \mathfrak{G}$. If either

(1) $\mathfrak{K}$ is a Lie subalgebra, or

(2) $\Pi_{\mathfrak{N}}[\Pi_{\mathfrak{K}}(\text{grad} f)(\eta), \Pi_{\mathfrak{K}}(\text{grad}\, g)(\eta)] = 0$ for all $\eta \in \mathfrak{K}^{\perp}$,

then on any N-orbit in $\mathfrak{K}^{\perp}$, the functions f, g Poisson commute.

PROOF. Let $\eta \in \mathfrak{K}^{\perp}$. By hypothesis and (2.16) we get

$$\begin{aligned} 0 &= -\kappa([(\text{grad} f)(\eta), (\text{grad}\, g)(\eta)], \eta) \\ &= \{f|N\cdot\eta, g|N\cdot\eta\}(\eta) - \kappa(\Pi_{\mathfrak{N}}[\Pi_{\mathfrak{K}}(\text{grad} f)(\eta), \Pi_{\mathfrak{K}}(\text{grad}\, g)(\eta)], \eta). \end{aligned}$$

The second term vanishes in either hypothesis 1 or 2. $\square$

REMARK. Both theorems have identical versions on duals and κ is not needed there.

These two general theorems prove the involution of the functions f_k in the following way. In Theorem 3.2 take $\mathfrak{G} = sl(n)$ and let

$$\phi_k(\xi, \eta) = (1/2(k+1))\, Tr(-\xi + \eta\lambda_{k+1} + A\lambda_{k+1}^2)^{k+1}.$$

Then $\{\phi_k, \phi_l\} = 0$ on $sl(n)_{\text{ad}} \times \overline{sl(n)}$ for any parameters $\lambda_{k+1}, \lambda_{l+1}$, i.e. ϕ_{k+1} is constant on the flow defined by ϕ_{l+1} no matter what $\lambda_{k+1}, \lambda_{l+1}$ are, i.e. the coefficients of λ_{k+1} in ϕ_{k+1} are constant on the flow defined by ϕ_{l+1} for all λ_{l+1}. Hence $\{f_k, \phi_l\} = 0$ for all λ_l and thus $\{f_k, f_l\} = 0$ for any k, l. In Theorem 3.3 take $\mathfrak{G} = sl(n)_{\text{ad}} \times \overline{sl(n)}$, $\mathfrak{K} = sym \times so(n)$, $\mathfrak{N} = so(n) \times sym$ and remark that $[\mathfrak{K}, \mathfrak{N}] \subseteq \mathfrak{K}$. f_k, f_l Poisson commute on $\mathfrak{G}$ by what we just proved, so in order to conclude that they Poisson commute on the N-orbit through $(\mathbf{z} \otimes \mathbf{z} - \text{Id}/n, 0)$ we have to check condition (2) of Theorem 3.3 for $\eta = (X, P) \in \mathfrak{K}^{\perp} = \mathfrak{K}$. An easy computation shows that

$$f_k(X, P) = \frac{1}{2(k+1)} \text{Tr}\left[-\sum_{i=0}^{k} A^i X A^{k-i} + \sum_{\substack{i+j+l=k-1 \\ i,j,l>0}} A^i P A^j P A^l\right] \tag{3.2}$$

so that the gradient of f_k with respect to κ_s is

$$(\text{grad} f_k)(X, P) = \left(-\sum_{i=0}^{k-1} A^i P A^{k-1-j}, A^k\right) \in so(n) \times \text{sym} = \mathfrak{N} \tag{3.3}$$

and hence $\Pi_{\mathfrak{K}}(\text{grad} f_k)(X, P) = 0$.

THEOREM 3.4. *The functions f_k, $k = 1, \ldots, n-1$, are constants of the motion in involution for the C. Neumann problem. $f_1 = -E$, $E =$ energy function.*

REMARK. Equation (3.1) is Hamiltonian in the Kac-Moody extension of $sl(n)$; see Adler and van Moerbeke [2].

4. Independence. Throughout this section we assume that $A = \operatorname{diag}(a_1, \ldots, a_n)$ has all entries distinct.

Let $\mathcal{V} = \operatorname{span}\{X_{f_k}(X, P) | k = 1, \ldots, n-1\}$. We have to show that generically $\dim(\mathcal{V}) = n - 1$.

Denote by U_{ki} the coefficient of λ^i in the expansion of $(-X + P\lambda + A\lambda^2)^k$. From (3.3) it follows that $(\operatorname{grad} f_k)(X, P) = (-U_{k,2k-1}, U_{k,2k})$, so that $\mathcal{V} = \operatorname{ad}_{(X,P)}\mathcal{Q}_0$, where $\mathcal{Q}_0 = \operatorname{span}\{(-U_{k,2k-1}, U_{k,2k}) | k = 1, \ldots, n-1\}$. Since $U_{k,2k} = A^k$ and A has all entries distinct we conclude that $\{A^k | k = 1, \ldots, n-1\}$ are linearly independent in $sl(n)$ and thus $\dim(\mathcal{Q}_0) = n - 1$; in particular $\dim(\mathcal{V}) \leqslant \dim(\mathcal{Q}_0) = n - 1$ which was already obvious from the definition of $\mathcal{V}$.

Let $\mathcal{Q}_j = \operatorname{span}\{(-U_{k,2k-1-2j}, U_{k,2k-2j}) | k = j, \ldots, n-1\}$ where we make the convention that any U_{ki} with $i < 0$ is identical zero; thus $\dim(\mathcal{Q}_j) \leqslant n - j$, $j = 1, \ldots, n-1$. Denote $\mathcal{V}_j = \operatorname{ad}_{(X,P)}\mathcal{Q}_j$, $j = 0, 1, \ldots, n-1$, so that $\mathcal{V} = \mathcal{V}_0$.

LEMMA 4.1. *The linear map $f_{A,P}\colon sl(n) \times sl(n) \to sl(n) \times sl(n)$ defined by*

$$f_{A,P}(\xi, \eta) = ([\eta, P] - [\xi, A], [A, \eta])$$

is injective on all $\mathcal{Q}_j$, $j = 1, \ldots, n-1$, for generic (X, P).

This is a direct, but somewhat lengthy verification (see [15] for a more complicated similar proof).

LEMMA 4.2. *The following relations hold for any $k = 1, \ldots, n-1$:*

$$-[U_{k,2k-j}, X] + [U_{k,2k-j-1}, P] + [U_{k,2k-j-2}, A] = 0.$$

This is obvious if one notes that the expression above is the coefficient of λ^j in the expansion of $[(-X + P\lambda + A\lambda^2)^k, -X + P\lambda + A\lambda^2] \equiv 0$.

We have thus by the two prior lemmas

$$\begin{aligned} &\operatorname{ad}_{(X,P)}(-U_{k,2k-1-2j}, U_{k,2k-2j}) \\ &\quad = ([U_{k,2k-1-2j}, X], [X, U_{k,2k-2j}] + [U_{k,2k-1-2j}, P]) \\ &\quad = ([U_{k,2k-2-2j}, P] + [U_{k,2k-3-2j}, A], [A, U_{k,2k-2j-2}]) \\ &\quad = f_{A,P}(-U_{k,2k-2j-3}, U_{k,2k-2j-2}), \end{aligned}$$

i.e. $f_{A,P}(\mathcal{Q}_{j+1}) \subseteq \operatorname{ad}_{(X,P)}(\mathcal{Q}_j) = \mathcal{V}_j$. $f_{A,P}$ injective implies $\dim(\mathcal{V}_j) \geqslant \dim(\mathcal{Q}_{j+1})$, $j = 0, 1, \ldots, n-1$. Assume from now on that for any $j = 1, \ldots, n-1$, $X^j \neq 0$; this condition is generically satisfied. Since $U_{j,0} = (-1)^j X^j$ we conclude $\operatorname{ad}_{(X,P)}(0, U_{j,0}) = (0, 0)$ and hence $\dim(\mathcal{Q}_j) \geqslant 1 + \dim(\mathcal{V}_j)$ for $j \geqslant 1$. Hence we obtain

$$\dim(\mathcal{Q}_j) \geqslant 1 + \dim(\mathcal{Q}_{j+1}), \qquad j = 1, \ldots, n-1, \ \mathcal{Q}_n = 0. \tag{4.1}$$

Clearly $\mathcal{Q}_{n-1} = \operatorname{span}(0, U_{n-1,0})$ so that $\dim(\mathcal{Q}_{n-1}) = 1$. Repeated application of (4.1) yields then $\dim(\mathcal{Q}_1) \geqslant n - 1$, which combined with $n - 1 \geqslant \dim(\mathcal{V}_0) \geqslant \dim(\mathcal{Q}_1)$ gives $\dim(\mathcal{V}) = n - 1$.

THEOREM 4.3. *Let* $A = \operatorname{diag}(a_1, \ldots, a_n)$ *have all entries distinct. The C. Neumann problem is a completely integrable Hamiltonian system,* $n - 1$ *generically independent integrals in involution being given by*

$$f_{k+1}(X, P) = \frac{1}{2(k+1)} \operatorname{Tr}\left[-\sum_{i=0}^{k} A^i X A^{k-i} + \sum_{\substack{i+j+l=k-1 \\ i,j,l \geq 0}} A^i P A^j P A^l \right].$$

REMARKS. (1) The geodesic spray on an ellipsoid in $\mathbf{R}^n$ with all axes distinct is also completely integrable and given by the Euler-Poisson equations on the *same* orbit

$$\dot{X} = [Q, X], \qquad \dot{P} = [Q, P] + [X, A^{-1}],$$

for $Q_{ij} = -P_{ij}/a_i a_j$, with Hamiltonian $E(X, P) = -\frac{1}{2}\kappa(P, Q) + \kappa(X, A^{-1})$. It has the same integrals f_k since the previous equations can be written as

$$(-X + P\lambda + A\lambda^2)^{\cdot} = [-X + P\lambda + A\lambda^2, -Q - A^{-1}\lambda].$$

This follows easily from the work of Moser **[10]**, **[11]** and has been independently observed by Adler and van Moerbeke **[2]** who also linearize the flow.

(2) The geodesic spray on S^{n-1} corresponds to $A = 0$ in the C. Neumann problem, or to $A = \operatorname{Id}$ in the ellipsoidal problem. The Euler-Poisson equations on the *same* orbit are $\dot{X} = [P, X]$, $\dot{P} = 0$ and the integrals in involution are

$$f_k(X, P) = \begin{cases} \kappa\left(P^k - \left(\frac{1}{n} \operatorname{Tr} P^k\right)\operatorname{Id}, X\right), & k = \text{even}, \\ \frac{1}{2(k+1)} \operatorname{Tr}(P^{k+1}), & k = \text{odd}. \end{cases}$$

The Hamiltonian is $-f_1(X, P) = -\frac{1}{2}\kappa(P, P)$. The prior proof of independence can be easily modified step-by-step to show that X_{f_k}, $k = 1, \ldots, n - 1$, are generically independent.

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