The Motion of the Free n-dimensional Rigid Body

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§1. Introduction. This paper deals with the complete integrability of the equations of motion of a free n-dimensional rigid body about a fixed point. The Euler vector fields on a Lie algebra and its dual are reviewed in §2. Euler's equations of motion for the n-dimensional free rigid body about a fixed point as a Hamiltonian system on adjoint orbits of $so(n)$ are derived in §3 and Manakov's constants of the motion are given. Inspired by the proof of the Kostant-Symes involution theorem, §4 gives a Lie algebraic proof of the involution of Manakov's integrals. Based on work of Kupershmidt, Manin, Adler and van Moerbeke, a Hamiltonian structure different from the one defined by the decomposition of $so(n)$ in its adjoint orbits is given in §5 and Lenard type recursion relations between the two families of Hamiltonian vector fields are found. This enables us to give a second proof—entirely similar to the one for the Korteweg-de Vries equation—of the involution of Manakov's integrals. §6 deals with another set of integrals of the Euler equations found earlier by Mishchenko. Finally, §7 extends a matrix equation of Dubrovin to a Hamiltonian system on adjoint orbits of $s\ell(n)$ which, when restricted to $so(n)$, gives the rigid body equations; Manakov and Mishchenko type integrals are shown to be in involution.

When this present work was completed, B. Kupershmidt and V. Kac mentioned the announcement [17] and the paper [18] respectively of which we were not aware. There, Mishchenko and Fomenko generalize the rigid body equations to any semi-simple Lie algebra in the spirit of our §7 and prove complete integrability. Our proofs of involution are however different but the independence in which we used crucially the Lenard relations, turns out to be very similar to their proof; we refer the reader to [18, Section 4] for this proof. For an algebraic geometrical proof of independence and the linearization of the flow, see Adler-van Moerbeke [3].

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§2. The Euler vector fields. In this section we give a quick review of the tangent and cotangent Euler vector fields on arbitrary Lie algebras (see Arnold [2]). Almost all statements in this section are proved either in Abraham, Marsden [1, Chapter 4], or in Ratiu [20]; we shall give the specific reference each time.

If G is a finite dimensional Lie group and $\mathfrak{g}$ its Lie algebra, the tangent bundle TG can be identified with $G \times \mathfrak{g}$ in two ways:

$$\lambda\colon TG \to G \times \mathfrak{g}, \quad \lambda(v_g) = (g, T_e L_g^{-1}(v_g)), \qquad v_g \in T_g G$$

$$\rho\colon TG \to G \times \mathfrak{g}, \quad \rho(v_g) = (g, T_e R_g^{-1}(v_g)), \qquad v_g \in T_g G$$

where $L_g, R_g\colon G \to G$, $L_g(h) = gh$, $R_g(h) = hg$ are left and right translations by $g \in G$. $\lambda(v_g)$ and $\rho(v_g)$ are said to represent the vector $v_g \in T_g G$ in *body* and *space coordinates* respectively. The transition from body to space coordinates is given by the map $\rho \circ \lambda^{-1}$:

$$(\rho \circ \lambda^{-1})(g, \xi) = (g, \mathrm{Ad}_g \xi), \qquad g \in G,\ \xi \in \mathfrak{g}.$$

$\mathrm{Ad}_g = T_e(L_g \circ R_{g^{-1}})\colon \mathfrak{g} \to \mathfrak{g}$ is the adjoint action of G on $\mathfrak{g}$. Similarly the cotangent bundle T^*G is isomorphic in two ways, by $\bar{\lambda}$ and $\bar{\rho}$, to $G \times \mathfrak{g}^*$:

$$\bar{\lambda}\colon T^*G \to G \times \mathfrak{g}^*, \quad \bar{\lambda}(\alpha_g) = (g, T_e^* L_g(\alpha_g)), \qquad \alpha_g \in T_g^* G$$

$$\bar{\rho}\colon T^*G \to G \times \mathfrak{g}^*, \quad \bar{\rho}(\alpha_g) = (g, T_e^* R_g(\alpha_g)), \qquad \alpha_g \in T_g^* G.$$

As before, we shall say that $\bar{\lambda}(\alpha_g)$, $\bar{\rho}(\alpha_g)$ represent α_g in *body* and *space coordinates* respectively. We have

$$(\bar{\rho} \circ \bar{\lambda}^{-1})(g, \mu) = (g, \mathrm{Ad}^*_{g^{-1}}(\mu)), \qquad g \in G,\ \mu \in \mathfrak{g}^*,$$

where $\mathrm{Ad}^*_{g^{-1}}$ is the co-adjoint action of G on $\mathfrak{g}^*$.

On T^*G with canonical symplectic form ω_0 consider the left invariant Hamiltonian $H\colon T^*G \to \mathbf{R}$. The expression of the Hamiltonian vector field X_H with flow F_t is given by $(\bar{\lambda}_* X_H)(g,\mu) = (\bar{X}(g,\mu), \mu, \bar{Y}(\mu))$, where $g \mapsto \bar{X}(g,\mu)$ is a family of left-invariant vector fields on G depending smoothly on $\mu \in \mathfrak{g}^*$ and $\bar{Y}\colon \mathfrak{g}^* \to \mathfrak{g}^*$, called the *cotangent Euler vector field,* equals

$$\bar{Y}(\mu) \cdot \eta = \left[\left. \frac{dx}{dt} \right|_{t=0}, \eta \right] \tag{1.1}$$

for all $\mu \in \mathfrak{g}^*$, $\eta \in \mathfrak{g}$; here $x(t) = \tau^*(F_t(\mu))$ and $\tau^*\colon T^*G \to G$ is the canonical projection. $\bar{\lambda}_* H$ is a Hamiltonian vector field on $(G \times \mathfrak{g}^*, \bar{\lambda}_* \omega_0)$ with Hamiltonian $\bar{\lambda}_* H$ and has flow $\bar{H}_t(\nu) = F_t(\nu) \circ T_e L_{x(t)}$. This is the content of Theorems 4.4.5 and 4.4.6 in [1] in cotangent formulation.

We shall simplify (1.1). Clearly $dx/dt\,|_{t=0} = (T\tau^* \circ X_H)(\mu)$, so that picking a canonical chart in T^*G around $\mu \in T_e^* G = \mathfrak{g}^*$, $\mu = (\mathbf{0},\boldsymbol{\mu})$ and denoting by $(\mathbf{x},\boldsymbol{\nu},A(\mathbf{x},\boldsymbol{\nu}),B(\mathbf{x},\boldsymbol{\nu}))$ the local representative of $X_H(\nu_g)$, $\nu_g = (\mathbf{x},\boldsymbol{\nu})$, we have $A(\mathbf{0},\boldsymbol{\mu}) \in \mathfrak{g}$, $B(\mathbf{0},\boldsymbol{\mu}) \in \mathfrak{g}^*$ and for any $\alpha \in T_e^* G$, $\alpha = (\mathbf{0},\boldsymbol{\alpha})$, $\alpha(T\tau^* \circ X_H)(\mu) = \boldsymbol{\alpha} \cdot A(\mathbf{0},\boldsymbol{\mu})$. Let i: $\mathfrak{g}^* \to T^*G$ be the inclusion so that $T_\mu i$ is in the local chart $(\mathbf{0},\boldsymbol{\alpha}) \mapsto (\mathbf{0},\boldsymbol{\mu},\mathbf{0},\boldsymbol{\alpha})$. Since ω_0 is in the local chart equal to $\omega_0(\mathbf{x},\boldsymbol{\nu})((\mathbf{x},\boldsymbol{\nu},\mathbf{e}_1,\boldsymbol{\beta}_1),(\mathbf{x},\boldsymbol{\nu},\mathbf{e}_2,\boldsymbol{\beta}_2)) = \boldsymbol{\beta}_2(\mathbf{e}_1) - \boldsymbol{\beta}_1(\mathbf{e}_2)$, we conclude that $d(H|\mathfrak{g}^*)(\mu) \cdot \alpha = (dH(\mu) \circ T_\mu i)(\alpha) = \omega_0(\mu)(X_H(\mu), T_\mu i(\alpha)) = \omega_0(\mathbf{0},\boldsymbol{\mu})((\mathbf{0},\boldsymbol{\mu},A(\mathbf{0},\boldsymbol{\mu}),B(\mathbf{0},\boldsymbol{\mu})),(\mathbf{0},\boldsymbol{\mu},\mathbf{0},\boldsymbol{\alpha})) = \boldsymbol{\alpha} \cdot A(\mathbf{0},\boldsymbol{\mu})$. We showed hence that $dx/dt|_{t=0} = d(H|\mathfrak{g}^*)(\mu)$ with the right hand side interpreted as an element of $\mathfrak{g} \cong \mathfrak{g}^{**}$. Hence

$$\bar{Y}(\mu) = \mathrm{ad}^*(d(H|\mathfrak{g}^*)(\mu)) \cdot \mu \tag{1.2}$$

for all $\mu \in \mathfrak{g}^*$.

Recall that by the Kirillov-Kostant-Souriau theorem any co-adjoint orbit $G \cdot \mu$ has a canonical symplectic structure ω_μ given by

$$\omega_\mu(\bar{\mu})((\mathrm{ad}\,\eta)^*\bar{\mu}, (\mathrm{ad}\,\zeta)^*\bar{\mu}) = -\bar{\mu}([\eta,\zeta]) \tag{1.3}$$

for $\eta,\zeta \in \mathfrak{g}$, $\bar{\mu} = \mathrm{Ad}^*_{g^{-1}}(\mu) \in G \cdot \mu$; we shall soon explain this in the more general framework of reduction. We used here the fact that the tangent space at $\bar{\mu}$ to $G \cdot \mu$ equals $T_{\bar{\mu}}(G \cdot \mu) = \{(\mathrm{ad}\,\xi)^*\bar{\mu}:\xi \in \mathfrak{g}\}$. Moreover, if f,f': $\mathfrak{g}^* \to \mathbf{R}$ are smooth, the Hamiltonian vector field of $f|G \cdot \mu$ is given by

$$X_{f|G\cdot\mu}(\bar{\mu}) = \mathrm{ad}^*(df(\bar{\mu}))(\bar{\mu}) \tag{1.4}$$

and the Poisson bracket of $f|G \cdot \mu$ and $f'|G \cdot \mu$ by

$$\{f|G \cdot \mu, f'|G \cdot \mu\}(\bar{\mu}) = -\bar{\mu}([df(\bar{\mu}), df'(\bar{\mu})]) \tag{1.5}$$

where $df(\bar{\mu})$, $df'(\bar{\mu})$ are thought of as elements of $\mathfrak{g} = \mathfrak{g}^{**}$; formulas (1.4) (1.5) are proved in Ratiu [20]. From (1.2) and (1.4) we draw the following conclusion. *If H: $\mathfrak{g}^* \to \mathbf{R}$, its extension by left-translations defines a left-invariant Hamiltonian vector field on T^*G whose Euler vector field on $\mathfrak{g}^*$ when restricted to an arbitrary co-adjoint orbit is itself Hamiltonian with Hamiltonian* $H|G \cdot \mu$. The question naturally arises to what extent does the Euler vector field determine the motion. In order to understand this, a short review on momentum maps and the Marsden-Weinstein reduction procedure [15] are necessary; for proofs see [1], §4.2, 4.3.

Let G be a Lie Group with Lie algebra $\mathfrak{g}$, exponential map exp: $\mathfrak{g} \to G$, P a smooth manifold and $\Phi: G \times P \to P$ a smooth action of G on P. $\xi_P(p) = d/dt|_{t=0}\Phi(\exp(t\xi),p), p \in P, \xi \in \mathfrak{g}, t \in \mathbf{R}$ will denote the infinitesimal generators of this action. If $G \cdot p = \{\Phi(g,p):g \in G\}$ denotes the G-orbit through $p \in P$, its tangent space at p is $T_p(G \cdot p) = \{\xi_P(p):\xi \in \mathfrak{g}\}$. Later on three actions will be important.

—The action of G on itself by left-multiplication L: $G \times G \to G$, $L(g,h) = gh$; its infinitesimal generator is $\xi_G(g) = T_e R_g(\xi)$, where $R_g(k) = kg$ denotes right multiplication in G by g;

—The adjoint action Ad: $G \times \mathfrak{g} \to \mathfrak{g}$ given by $\mathrm{Ad}_g = T_e(R_{g^{-1}} \circ L_g)$; its infinitesimal generator is $\xi_{\mathfrak{g}} = \mathrm{ad}(\xi)$, where $(\mathrm{ad}\ \xi)\eta = [\xi,\eta]$, $[\ ,\ ]$ denoting the Lie bracket in $\mathfrak{g}$;

—The co-adjoint action of G on $\mathfrak{g}^*$ is the dual of the adjoint action and is given by $g \mapsto \mathrm{Ad}^*_{g^{-1}}$; its infinitesimal generator is $\xi_{\mathfrak{g}^*} = -(\mathrm{ad}\ \xi)^*$.

Let (P,ω) be a symplectic manifold and Φ: $G \times P \to P$ a symplectic action, i.e. $\Phi_g^* \omega = \omega$ for all $g \in G$. The map J: $P \to \mathfrak{g}^*$ is *a momentum mapping* for this action if

$$T_p J(v_p) \cdot \xi = \omega_p(\xi_P(p), v_p)$$

for every $\xi \in \mathfrak{g}$, $p \in P$, $v_p \in T_pP$. Denoting by $\hat{J}(\xi)$: $P \to \mathbf{R}$ the map defined by $\hat{J}(\xi)(p) = J(p) \cdot \xi$, the definition above says that ξ_P is a Hamiltonian vector field on P with Hamiltonian $\hat{J}(\xi)$, i.e. $X_{\hat{J}(\xi)} = \xi_P$ for all $\xi \in \mathfrak{g}$. We shall call (P,ω,Φ,J) a *Hamiltonian G-space.* Since not every locally Hamiltonian vector field is globally Hamiltonian, not every action admits a momentum map. However, if a momentum map exists, it is uniquely determined up to constants in $\mathfrak{g}^*$.

Momentum maps are important since they give conserved quantities. More precisely, if H: $P \to \mathbf{R}$ is a G-invariant Hamiltonian of (P,ω,Φ,J), i.e. $H \circ \Phi_g = H$ for all $g \in G$, then J is constant on the flow of the Hamiltonian vector field X_H.

The momentum map J: $P \to \mathfrak{g}^*$ is said to be Ad^**-equivariant* if $J(\Phi_g(p)) = \mathrm{Ad}^*_{g^{-1}} J(p)$ for all $p \in P$, $g \in G$. In this case it is shown that

$$\{\hat{J}(\xi), \hat{J}(\eta)\} = \hat{J}\,[\xi,\eta]$$

for all $\xi,\eta \in \mathfrak{g}$, where $\{\ ,\ \}$ denotes the Poisson bracket in P.

The following criterion gives a formula for the momentum mapping for exact symplectic manifolds. Assume $\omega = -d\theta$ and that $\Phi_g^*\theta = \theta$ for all $g \in G$. Then J: $P \to \mathfrak{g}^*$, $J(p) \cdot \xi = (i_{\xi_P}\theta)(p)$ is an Ad^*-equivariant momentum map for the action Φ. Here i_{ξ_P} denotes the interior product of a form with the vector field ξ_P. Two special cases of this theorem will be important.

—$P = T^*Q$ with the canonical symplectic structure, when G acts on Q. The lift Φ^{T^*}: $G \times T^*Q \to T^*Q$, $\Phi_g^{T^*} = T^*\Phi_{g^{-1}}$ has a momentum mapping J: $T^*Q \to \mathfrak{g}^*$

$$J(\alpha_q) \cdot \xi = \alpha_q \cdot \xi_Q(q)$$

for all $q \in Q$, $\alpha_q \in T_q^*Q$, $\xi \in \mathfrak{g}$.

—$P = TQ$ endowed with the symplectic structure induced from T^*Q via a pseudo-Riemannian metric $\langle\ ,\ \rangle$ on Q. Let G act on Q by isometries and lift this action to TQ by Φ^T: $G \times TQ \to TQ$, $\Phi_g^T = T\Phi_g$. Φ^T has a momentum mapping J: $TQ \to \mathfrak{g}^*$ given by

$$J(v_q) \cdot \xi = \langle v_q, \xi_Q(q)\rangle_q$$

for all $q \in Q$, $v_q \in T_qQ$, $\xi \in \mathfrak{g}$. Particular examples are:

a) $Q = \mathbf{R}^n$, $G = (\mathbf{R}^n,+)$, $(s,q) \in G \times Q \overset{\Phi}{\mapsto} s + q \in Q$, $\xi_Q(q) = \xi$ for $\xi \in \mathfrak{g} = \mathbf{R}^n$; the momentum map of Φ^{T^*} is $J(q,p) = p$;

b) $Q = G$, $\Phi = L$, $\xi_G(g) = T_e R_g(\xi)$; the action Φ^{T^*} has a momentum map $J(\alpha_g) = (T_e^* R_g)(\alpha_g)$ for $g \in G$, $\alpha_g \in T_g^* G$.

Let (P,ω,Φ,J) be a Hamiltonian G-space with Ad^*-equivariant momentum map. Denote by $G_\mu = \{g \in G : \mathrm{Ad}^*_{g^{-1}}\mu = \mu\}$ the isotropy subgroup of the co-adjoint action at $\mu \in \mathfrak{g}^*$. Assume that μ is a regular value for J so that $J^{-1}(\mu)$ is a $(\dim P - \dim G)$-dimensional submanifold of P. By Ad^*-equivariance, G_μ acts on $J^{-1}(\mu)$. Assume that this action is proper and free so that $P_\mu = J^{-1}(\mu)/G_\mu$, the G_μ-orbit space of $J^{-1}(\mu)$, is a smooth $(\dim P - \dim G - \dim G_\mu)$-dimensional manifold with the canonical projection $\pi_\mu \colon J^{-1}(\mu) \to P_\mu$ a surjective submersion. The theorem of Marsden and Weinstein [15] states then that P_μ has a unique symplectic structure ω_μ satisfying $\pi_\mu^* \omega_\mu = i_\mu^* \omega$, where $i_\mu \colon J^{-1}(\mu) \to P_\mu$ is the canonical inclusion. P_μ is called the *reduced phase space.*

Under all the hypotheses above, let $H : P \to \mathbf{R}$ be a G-invariant Hamiltonian. Then the flow F_t of X_H leaves $J^{-1}(\mu)$ invariant (since J is a conserved quantity) and commutes with the G_μ-action on $J^{-1}(\mu)$ (since $\Phi_g^* X_H = X_H$) so it induces canonically a flow $H_t \colon P_\mu \to P_\mu$ defined by $\pi_\mu \circ F_t = H_t \circ \pi_\mu$. Then the theorem of Marsden and Weinstein [15] asserts that H_t is a Hamiltonian flow on P_μ with the Hamiltonian H_μ induced by H, i.e. $H_\mu \circ \pi_\mu = H \circ i_\mu$ and Hamiltonian vector field X_{H_μ} on P_μ which is π_μ-related to $X_H | J^{-1}(\mu)$, i.e. $T\pi_\mu \circ X_H = X_{H_\mu} \circ \pi_\mu$. H_μ is called the *reduced Hamiltonian* and X_{H_μ} the reduced *Hamiltonian vector field.* Let us denote by $\{\ ,\ \}^\mu$ the Poisson bracket of P_μ. Then if $f,g \colon P \to \mathbf{R}$ are G-invariant, $\{f,g\}$ is G-invariant and $\{f,g\}_\mu = \{f_\mu,g_\mu\}^\mu$. In particular if f,g Poisson commute in P, then they Poisson commute in P_μ. Thus the flow of the reduced system completely determines the motion of the original Hamiltonian system on $J^{-1}(\mu)$; for an algorithm see [1, page 305]. Moreover, if one wishes to prove complete integrability, it suffices to show it on P_μ for all $\mu \in \mathfrak{g}^* \setminus S$, where $\bigcup_{\mu \in S} J^{-1}(\mu)$ has dense complement in P.

Let us carry out the reduction process for the lift to T^*G of the left multiplication on G. Recall that this action has the momentum map $J \colon T^*G \to \mathfrak{g}^*$, $J(\alpha_g) = (T_e^* R_g)(\alpha_g)$. Each $\mu \in \mathfrak{g}^*$ is clearly a regular value of J and $J^{-1}(\mu) = \{\alpha_g \in T^*G : \alpha_g \circ T_e R_g = \mu\}$ = graph of the right-invariant one-form whose value at e is μ. Thus the isotropy subgroup G_μ of the co-adjoint action acts on $J^{-1}(\mu)$ by left-translations on the base point and we get that $J^{-1}(\mu)/G_\mu \approx G/G_\mu \approx G \cdot \mu \subseteq \mathfrak{g}^*$, the diffeomorphism being given by $\pi_\mu(\mu \circ TR_{g^{-1}}) \mapsto \mathrm{Ad}_g^*(\mu)$. Thus the co-adjoint orbits are symplectic manifolds of $\mathfrak{g}^*$. Computing the symplectic form on $G \cdot \mu$ induced by the above diffeomorphism one gets formula (1.3); see [1, page 303]. If $H \colon T^*G \to \mathbf{R}$ is left-invariant, i.e. $H \circ T^*L_g = H$ for all $g \in G$, its reduction H_μ to $G \cdot \mu$ is given by $H_\mu = H | G \cdot \mu$. Thus the restriction of the Euler vector field to $G \cdot \mu$ coincides with the reduction of the original Hamiltonian system and we conclude that $\bar{Y}$ characterizes the motion. Moreover, because $J^{-1}(\mu) =$ graph of $\mu \circ TR_{g^{-1}}$, it follows that in order to prove complete integrability of X_H, it suffices to show it for $\bar{Y}|G \cdot \mu$ for almost all $\mu \in \mathfrak{g}^*$.

We pass now to the tangent formulation of the same results. The reason for doing this, is that in concrete examples one encounters matrix equations, i.e. equations in a Lie algebra, and in order to apply the previous results one has to identify $\mathfrak{g}^*$ with $\mathfrak{g}$ by means of a non-degenerate two-form on $\mathfrak{g}$. As it will turn out, one has to work simultaneously with two such forms, one of them bi-invariant, if $\mathfrak{g}$ is reductive. This will force us to slightly modify the Euler vector field and a change of sign will be introduced.

Let G be a Lie group, $\mathfrak{g}$ its Lie algebra, $\langle\ ,\ \rangle$ a left-invariant metric on G, and $E\colon TG \to \mathbf{R}$ an arbitrary left-invariant smooth energy function. Taking on TG the symplectic structure induced from T^*G by $\langle\ ,\ \rangle$, the Hamiltonian vector field X_E on TG is a left-invariant second-order equation on G. Its expression in body coordinates is given by $(\lambda_* X_E)(g,\xi) = (T_e L_g(\xi), \xi, Y(\xi))$, $g \in G$, $\xi \in \mathfrak{g}$, where $Y\colon \mathfrak{g} \to \mathfrak{g}$, called the *Euler vector field*, is characterized by

$$\langle Y(\xi),\eta\rangle = \langle [\xi,\eta],\xi\rangle \tag{1.6}$$

for any ξ, $\eta \in \mathfrak{g}$. Since the last formula determines Y uniquely in terms of $\langle\ ,\ \rangle$ and is independent of E, it follows that the geodesic sprays of left-invariant metrics are the unique left-invariant Hamiltonian vector fields on TG which are also second order equations; the energy function is obtained by left-translating $K(\xi) = (1/2)\langle\xi,\xi\rangle$. This is the content of [1, Theorems 4.4.5, 4.4.6] in tangent formulation.

If $\mathfrak{g}$ has a bilinear, symmetric, non-degenerate form $(\ ,\ )$ invariant under the Ad-action (i.e. $(\mathrm{Ad}_g\xi, \mathrm{Ad}_g\eta) = (\xi,\eta)$), then there exists a unique linear, $(\ ,\ )$-symmetric, positive isomorphism $\tilde{J}\colon \mathfrak{g} \to \mathfrak{g}$ such that $(\tilde{J}\ \cdot,\cdot) = \langle\cdot,\cdot\rangle$. It is also easy to see—using Ad-invariance of $(\ ,\ )$—that for any $\xi \in \mathfrak{g}$, $\mathrm{ad}\,\xi\colon \mathfrak{g} \to \mathfrak{g}$, $(\mathrm{ad}\,\xi)(\eta) = [\xi,\eta]$ is $(\ ,\ )$-skew-symmetric and thus the defining equation for Euler's vector field becomes

$$\tilde{J}Y(\xi) = [\tilde{J}\xi,\xi] \tag{1.7}$$

or in terms of integral curves,

$$(\tilde{J}\xi)^{\cdot} = [\tilde{J}\xi,\xi]. \tag{1.8}$$

Denoting $L = \tilde{J}^{-1}$, equation (1.8) becomes

$$\dot{\eta} = -[L\eta,\eta] \tag{1.9}$$

and we defined a vector field $Z(\eta) = -[L\eta,\eta]$ on $\mathfrak{g}$ equivalent to Y by the change of variables $\eta = \tilde{J}\xi$. We shall prove below that Z is Hamiltonian on each adjoint orbit.

The bi-invariant, symmetric, non-degenerate, two-form $(\cdot,\cdot)$ defines an equivariant diffeomorphism between co-adjoint and adjoint orbits, inducing thus a symplectic structure on the adjoint orbits $G \cdot \xi$. Denote by $\bar{\xi} = \mathrm{Ad}_g \xi$ an arbitrary element of the adjoint orbit $G \cdot \xi$ through ξ. The tangent space at $\bar{\xi} \in G \cdot \xi$ is $T_{\bar{\xi}}(G \cdot \xi) = \{[\eta,\bar{\xi}] \mid \eta \in \mathfrak{g}\}$ and the symplectic form ω_ξ is

$$\omega_\xi(\bar{\xi})([\eta,\bar{\xi}],[\zeta,\bar{\xi}]) = -([\eta,\zeta],\bar{\xi}). \tag{1.10}$$

where $(\cdot,\cdot)$ is a bi-invariant pseudo-metric on G. If $f\colon \mathfrak{g} \to \mathbf{R}$ is smooth, denote by $\operatorname{grad} f$ the gradient of f with respect to $(\cdot,\cdot)$, i.e. $df(\xi) \cdot \eta = (\operatorname{grad} f(\xi),\eta)$. The Hamiltonian vector field X_f on $G \cdot \xi$ is given by

$$X_{f|G\cdot\xi}(\bar{\xi}) = -[(\operatorname{grad} f)(\bar{\xi}),\bar{\xi}]. \tag{1.11}$$

Finally, if $f'\colon \mathfrak{g} \to \mathbf{R}$ is another smooth function on $\mathfrak{g}$, the Poisson bracket of $f|G \cdot \xi, f'|G \cdot \xi$ is

$$\{f|G \cdot \xi, f'|G \cdot \xi\}(\bar{\xi}) = -([(\operatorname{grad} f)(\bar{\xi}),(\operatorname{grad} f')(\bar{\xi})],\bar{\xi}). \tag{1.12}$$

(1.11) replaces (1.2); note the change in sign introduced by $(\cdot,\cdot)$. The function $H(\eta) = (1/2)(L\eta,\eta)$ has gradient $L\eta$ and thus $X_{H|G\cdot\eta}(\bar{\eta}) = -[L\bar{\eta},\bar{\eta}]$, which equals $Z|G \cdot \eta$. Thus to prove the complete integrability of the geodesic spray of $\langle\ ,\ \rangle$, it suffices to show it for $Z|G \cdot \eta$ for all regular semi-simple elements $\eta \in \mathfrak{g}$ (if $\mathfrak{g}$ is semi-simple). That is exactly what we shall do in this paper for the groups $so(n)$ and $s\ell(n)$.

§3. Derivation of the rigid body equation as a Hamiltonian system on adjoint orbits of $so(n)$. The problem under consideration is the free rotation of a rigid body about a fixed point, which we assume to be the origin in $\mathbf{R}^n$. "Rigid" means that the distances between the points of the body are unchanged during the motion. Let $f(t,\mathrm{x})$ denote the position of the particle of the body at time t which was at x at time zero; rigidity means that $f(t,\mathrm{x}) = A(t)\mathrm{x}$, where $A(t)$ is an orthogonal matrix. We assume the motion to be smooth. Since $f(0,\mathrm{x}) = A(0)\mathrm{x} = \mathrm{x}$ for every $\mathrm{x} \in \mathbf{R}^n$, $A(0) =$ identity matrix, so that $A(t) \in SO(n)$.

We assume that the mass distribution of the body is described by a positive measure μ on $\mathbf{R}^n$ whose support is not in a one-dimensional subspace. Thus, the kinetic energy of the body is given by

$$K(t) = \frac{1}{2}\int_{\mathbf{R}^n} \|\dot{f}(t,\mathrm{x})\|^2\, d\mu(\mathrm{x})$$

where $\|\cdot\|$ denotes Euclidean norm on $\mathbf{R}^n$. A short computation gives $\dot{f}(t,\mathrm{x}) = \Omega_s(t)f(t,\mathrm{x})$, where $\Omega_s(t) = \rho(\dot{A}(t)) \in so(n)$ is the vector $\dot{A}(t) \in T_{A(t)}SO(n)$ expressed in space coordinates. Thus the integrand of $K(t)$ is $\|A(t)^{-1}\Omega_s(t)A(t)\mathrm{x}\|^2$. But $A(t)^{-1}\Omega_s(t)A(t) = \mathrm{Ad}_{A(t)^{-1}}\Omega_s(t) = \lambda(\dot{A}(t)) = \Omega(t)$ is the expression of the vector $\dot{A}(t)$ in body coordinates. Thus the kinetic energy has the form:

$$K(t) = \frac{1}{2} \int_{\mathbf{R}^n} \|\Omega(t)\mathbf{x}\|^2 \, d\mu(\mathbf{x}).$$

For $A,\ B \in so(n)$,

$$\langle A,B \rangle = \int_{\mathbf{R}^n} A\mathbf{x} \cdot B\mathbf{x} \, d\mu(\mathbf{x})$$

defines an inner product on $so(n)$, where $\mathbf{x} \cdot \mathbf{y}$ denotes the usual dot-product on $\mathbf{R}^n$ (i.e. $\mathbf{x} \cdot \mathbf{x} = \|\mathbf{x}\|^2$). Thus

$$K(t) = \frac{1}{2} \langle \Omega(t), \Omega(t) \rangle.$$

$\langle \cdot,\cdot \rangle$ defines by left translations a left invariant metric on $SO(n)$ whose geodesic spray is the Hamiltonian vector field of the free rigid body motion about a fixed point.

Define on $so(n)$ the following bilinear Ad-invariant form:

$$(A,B) = -\frac{1}{2} \operatorname{Tr}(AB).$$

The motivation for this choice comes from the well-known case $n = 3$ where one uses in the derivation of the Euler equations the dot product on $\mathbf{R}^3$ which is invariant under the usual action of $SO(3)$ on $\mathbf{R}^3$. This usual action is equivariant to the adjoint action of $SO(3)$ on $so(3)$ under the standard isomorphism of $\mathbf{R}^3$ with $so(3)$:

$$\mathbf{x} = (x_1,x_2,x_3) \leftrightarrow \hat{\mathbf{x}} = \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix}.$$

A straightforward computation shows that $\mathbf{x} \cdot \mathbf{y} = -(1/2)\operatorname{Tr}(\hat{\mathbf{x}}\hat{\mathbf{y}})$ whence the above choice of $(\ ,\ )$ on $so(n)$.

Thus to write Euler's equations we have to determine the operator $\tilde{J}$: $so(n) \to so(n)$ given by the condition

$$(\tilde{J}A,B) = -\frac{1}{2} \operatorname{Tr}((\tilde{J}A)B) = \langle A,B \rangle = \int_{\mathbf{R}^n} A\mathbf{x} \cdot B\mathbf{x} \, d\mu(\mathbf{x}).$$

Let e_{ij} denote the matrix all of whose entries are zero except the (i,j) entry which is 1. The above condition can be rewritten as

$$\frac{1}{2} \operatorname{Tr}((\tilde{J}A)B) = \int_{\mathbf{R}^n} AB\mathbf{x} \cdot \mathbf{x} \, d\mu(\mathbf{x}).$$

Put $B = e_{ij} - e_{ji}$ and get for the left-hand side, using antisymmetry of $\tilde{J}A$,

$(1/2)\mathrm{Tr}((\tilde{J}A)(e_{ij} - e_{ji})) = (1/2)((\tilde{J}A)_{ji} - (\tilde{J}A)_{ij}) = (\tilde{J}A)_{ji}$, where $(\tilde{J}A)_{ji}$ denotes the (j,i)-entry of $\tilde{J}A$. As for the right hand side, remark first that $Ae_{ij}\mathbf{x} = (A_{1i}x_j,\ldots,A_{ni}x_j)$ so that denoting

$$J_{jk} = \int_{\mathbf{R}^n} x_j x_k \, d\mu(\mathbf{x}), \qquad \mathbf{x} = (x_1,\ldots,x_n)$$

we have $J_{jk} = J_{kj}$ and

$$\int_{\mathbf{R}^n} Ae_{ij}\,\mathbf{x}\, d\mu(\mathbf{x}) = \int_{\mathbf{R}^n} \left(\sum_{k=1}^{n} A_{ki} x_j x_k\right) d\mu(\mathbf{x}) = \sum_{k=1}^{n} J_{jk} A_{ki}.$$

Similarly

$$\int_{\mathbf{R}^n} Ae_{ji}\,\mathbf{x}\, d\mu(\mathbf{x}) = \sum_{k=1}^{n} J_{ik} A_{kj}$$

so that finally

$$\begin{aligned}(\tilde{J}A)_{ji} &= \sum_{k=1}^{n} J_{jk}A_{ki} - \sum_{k=1}^{n} J_{ik}A_{kj} = \sum_{k=1}^{n} (J_{jk}A_{ki} + A_{jk}J_{ki}) \\ &= (JA + AJ)_{ji}\end{aligned}$$

where $J = (J_{ji})$. Thus

$$\tilde{J}A = AJ + JA.$$

Since J is symmetric, there is a $g \in SO(n)$ such that $D = gJg^{-1}$ is diagonal. Define a measure ν on $\mathbf{R}^n$ by $\nu(\mathbf{x}) = \mu(g^{-1}\mathbf{x})$ and an operator $\tilde{\tilde{J}}$: $so(n) \to so(n)$ by $\tilde{\tilde{J}}(A) = g\tilde{J}(g^{-1}Ag)g^{-1}$; $\tilde{\tilde{J}}$ is a linear, (,)-symmetric, positive isomorphism. We have after a short computation

$$\tilde{\tilde{J}}(A) = AD + DA,$$

and

$$(\tilde{\tilde{J}}(A),B) = \int_{\mathbf{R}^n} A\mathbf{x} \cdot B\mathbf{x}\, d\nu(\mathbf{x})$$

Thus there is a new orthonormal basis of $\mathbf{R}^n$ having the same orientation as the initial one in which the operator $\tilde{J}$ is diagonal. Choose hence this coordinate system which is completely determined by the mass distribution of the body as the initial one in $\mathbf{R}^n$ and obtain

$$\tilde{J}A = AJ + JA$$

for $J = \mathrm{diag}(\lambda_1,\ldots,\lambda_n)$. Since $e_{ij} - e_{ji}$ for $i < j$ is a basis of $so(n)$, the relations $\tilde{J}(e_{ij} - e_{ji}) = (\lambda_i + \lambda_j)(e_{ij} - e_{ji})$ imply that the canonical basis of $so(n)$ is a basis of eigenvectors of $\tilde{J}$. In particular $\lambda_i + \lambda_j > 0$ since $\tilde{J}$ is positive definite. By analogy with the case $n = 3$, the $n(n-1)/2$ numbers $\lambda_i + \lambda_j$, $i \neq j$ are called the *principal moments of inertia.*

We have proved the following.

Theorem 3.1. *Given a rigid body in* $\mathbf{R}^n$ *there exists an orthonormal basis in* $\mathbf{R}^n$ *completely determined by the mass-distribution of the body in which the equations of motion of the free rigid body about a fixed point (the origin) have the form*

$$\dot{M} = [M,\Omega] \tag{3.1}$$

where M, $\Omega \in so(n)$, $M = \Omega J + J\Omega$, *for* J *a diagonal matrix* $J = \operatorname{diag}(\lambda_1,\ldots,\lambda_n)$ *satisfying* $\lambda_i + \lambda_j > 0$ *for* $i \neq j$. *These equations are Hamiltonian on each adjoint orbit of* $so(n)$ *defined by initial conditions with Hamiltonian* $H(M) = (1/2)(M,\Omega) = -(1/4)\operatorname{Tr}(M\Omega)$.

The second part of this theorem follows from the last remarks of §1.

For $n = 3$ we regain the usual Euler equations. If $\mathbf{x}$, $\mathbf{y} \in \mathbf{R}^3$ and $\mathbf{x} \times \mathbf{y}$ denotes their cross-product, then $(\mathbf{x} \times \mathbf{y})\hat{} = [\hat{\mathbf{x}},\hat{\mathbf{y}}]$. Let $\mathbf{I}$: $\mathbf{R}^3 \to \mathbf{R}^3$ be the linear map defined by

$$(\mathbf{Ix})\hat{} = \tilde{J}\hat{\mathbf{x}} = \hat{\mathbf{x}}J + J\hat{\mathbf{x}}.$$

Note that if $e_1 = (1,0,0)$, $e_2 = (0,1,0)$, $e_3 = (0,0,1)$, then $\hat{e}_1 = -(e_{23} - e_{32})$, $\hat{e}_2 = (e_{13} - e_{31})$, $\hat{e}_3 = -(e_{12} - e_{21})$, so that $(\mathbf{I}e_1)\hat{} = (\lambda_2 + \lambda_3)\hat{e}_1$, $(\mathbf{I}e_2)\hat{} = (\lambda_1 + \lambda_3)\hat{e}_2$, $(\mathbf{I}e_3)\hat{} = (\lambda_1 + \lambda_2)\hat{e}_3$ and hence $\mathbf{I}$: $\mathbf{R}^3 \to \mathbf{R}^3$ is a 3×3 diagonal matrix

$$\mathbf{I} = \operatorname{diag}(\lambda_2 + \lambda_3, \lambda_1 + \lambda_3, \lambda_1 + \lambda_2) = \operatorname{diag}(\mathbf{I}_1,\mathbf{I}_2,\mathbf{I}_3).$$

Thus Euler's equations become

$$(\mathbf{Ix})^{\cdot} = (\mathbf{Ix}) \times \mathbf{x}$$

or

$$\mathbf{I}_1\dot{x}_1 - (\mathbf{I}_2 - \mathbf{I}_3)x_2x_3 = 0$$
$$\mathbf{I}_2\dot{x}_2 - (\mathbf{I}_3 - \mathbf{I}_1)x_3x_1 = 0$$
$$\mathbf{I}_3\dot{x}_3 - (\mathbf{I}_1 - \mathbf{I}_2)x_1x_2 = 0$$

with $\mathbf{I}_1$, $\mathbf{I}_2$, $\mathbf{I}_3$ principal moments of inertia. That's why we defined in $so(n)$, $\lambda_i + \lambda_j$, $i \neq j$, as principal moments of inertia, in analogy with this classical 3-dimensional case.

Remarks. 1) The n-dimensional Euler equations, though not in the above form, appear for the first time in Weyl [23, Chapter 1, Section 6]; I owe this information to the referee.

2) The equations of the rigid body motion under the influence of gravity (the Euler-Poisson equations) have been also generalized to arbitrary Lie algebras independently in [6], [7], [21]. They turn out to be Hamiltonian

vector fields on (co-)adjoint orbits of a semi-direct product.

The dimension of the generic adjoint orbit in any semi-simple Lie algebra $\mathfrak{g}$ is equal to $\dim \mathfrak{g} - \operatorname{rank} \mathfrak{g}$. If $\mathfrak{g} = so(n)$, $\operatorname{rank} \mathfrak{g} = [n/2]$ where $[n/2]$ is the biggest natural number less than or equal to $n/2$. Thus the dimension of the generic adjoint orbit of $so(n)$ is equal to $n(n-1)/2 - [n/2]$.

The first step towards the complete integrability of (3.1) was done by Manakov [13] who observed that (3.1) can be written as

$$(M + J^2\lambda)^{\cdot} = [M + J^2\lambda, \Omega + J\lambda] \tag{3.2}$$

for any parameter λ, concluding thus that the functions $(1/2k)\mathrm{Tr}(M + J^2\lambda)^k$, $k = 2, \ldots, n$ are constant on the flow of (3.2). (This is clear, since if $\dot{X} = [X, Y]$, $\mathrm{Tr}\, X^k$ are constant on its flow.) Thus, denoting by $t \mapsto M(t)$ the flow of (3.1), $t \mapsto M(t) + J^2\lambda$ is the flow of (3.2) and hence the coefficients of λ in the expansion of $(1/2k)\mathrm{Tr}(M + J^2\lambda)^k$ will be constant on the flow of (3.1). We count now the coefficients leading to non-zero Hamiltonian vector fields. First, note that the coefficient of λ^j is identically zero whenever $k - j =$ odd. Second, the coefficient of λ^k is $(1/2k)\mathrm{Tr}\, J^{2k} =$ constant, so it yields a zero Hamiltonian vector field. Third, the coefficient of λ^0 is $(1/2k)\mathrm{Tr}\, M^k =$ constant on the orbit; this is an *orbit invariant* and hence leads again to a zero Hamiltonian vector field. Thus $(1/2k)\mathrm{Tr}(M + J^2\lambda)^k$ has $[(k-1)/2]$ coefficients leading to Hamiltonian vector fields not vanishing identically on the orbit and hence the total number of conserved quantities is equal to $\Sigma_{k=2}^n [(k-1)/2] = (1/2)(n(n-1)/2 - [n/2])$ as an easy computation shows. Let c_{kj} be the coefficient of λ^j in $(1/2k)\mathrm{Tr}(M + J^2\lambda)^k$, $k = 2, \ldots, n$; the constants of the motion $\{c_{kj}\}$ are called the *Manakov integrals* and their number equals half the dimension of the generic orbit making them candidates for the generically independent integrals in involution of (3.1).

§4. Involution of Manakov's integrals. In this section a Lie algebraic interpretation will be given to a modification of Manakov's equation (3.2) which together with a Lax equation trick will enable us to prove the involution of Manakov's integrals $c_{i,j}$. Observe that a typical c_{kj} is $(1/2k)$ times the trace of a sum of products of M and J^2 in which all combinations with $(k-j)$ M's and j J^2's occur. Let now $\Gamma = J - ((1/n)\mathrm{Tr}\, J)I$, $\Lambda = J^2 - ((1/n)\mathrm{Tr}\, J^2)I$, $I =$ identity matrix. Both $\Gamma, \Lambda \in s\ell(n)$. Denote by $d_{k,j}$ the coefficient of λ^j in the expansion of $(1/2k)\mathrm{Tr}(M + \Lambda\lambda)^k$. Again, since $M \in so(n)$, Λ is diagonal, $d_{k,j} \equiv 0$ for k, j having different parity, $d_{k,0} = (1/2k)\mathrm{Tr}\, M^k = c_{k,0}$, $d_{k,k} = (1/2k)\mathrm{Tr}\, \Lambda^k$. In the explicit expression of c_{kj} replace every J^2 by $\Lambda + ((1/n)\mathrm{Tr}\, J^2)I$ and develop the products; obtain thus a linear combination of terms, each term being a product containing $(k-j)$ M's and j Λ's. Thus

$$c_{kj} = \sum_{p=0}^{j} a_{kp}\, d_{k-p, j-p}, \qquad a_{k,p} \in \mathbf{R}.$$

(Note that "half" of the terms are zero.) Similarly

$$d_{kj} = \sum_{r=0}^{j} b_{k,r} c_{k-r,j-r}, \qquad b_{k,r} \in \mathbf{R}.$$

This proves the first part of the following proposition.

Proposition 4.1. *If $c_{k,j}$ (respectively $d_{k,j}$) denotes the coefficient of λ^j in $(1/2k)\mathrm{Tr}(M + J^2\lambda)^k$ (respectively $(1/2k)\mathrm{Tr}(M + \Lambda\lambda)^k$), then*

$$c_{kj} = \sum_{p=0}^{j} a_{kp} d_{k-p,j-p}, \qquad a_{kp} \in \mathbf{R}$$

$$d_{kj} = \sum_{r=0}^{j} b_{kr} c_{k-r,j-r}, \qquad b_{kr} \in \mathbf{R}.$$

Thus the family $\{c_{kj}\}$ is in involution iff $\{d_{kj}\}$ is. Also $\{c_{kj}\}$ are independent iff $\{d_{kj}\}$ are.

Proof. The involution part is clear by the above expressions. As for independence, since $X_{c_{k,0}} = X_{c_{kk}} = X_{d_{k,0}} = X_{d_{kk}} = 0$,

$$X_{c_{kj}} = \sum_{p=1}^{j-1} a_{kp} X_{d_{k-p,j-p}}, \quad X_{d_{k,j}} = \sum_{r=1}^{j-1} b_{k,r} X_{c_{k-r,j-r}}$$

proving that if $X_{c_{k,j}}(M)$ is a basis of the tangent space to the $SO(n)$ adjoint orbit through M so is $X_{d_{k,j}}(M)$ and conversely. ■

In view of this proposition it is enough to work only with $d_{k,j}$. The following is easy to prove.

Lemma 4.2. *The equation $\dot{M} = [M,\Omega]$ is equivalent to*

$$(M + \Lambda\lambda)^{\cdot} = [M + \Lambda\lambda, \Omega + \Gamma\lambda]. \tag{4.1}$$

Thus, by the same reasoning as at the end of §3 we conclude that $d_{k,j}$ are conserved along the flow of $\dot{M} = [M,\Omega]$. Inspired by the proof of the Kostant-Symes theorem we state the following (see Kostant [12], Symes [22] for the original and Ratiu [20] for comments on and proofs of Kostant-Symes type theorems).

Theorem 4.3. *Let G be a Lie group with Lie algebra $\mathfrak{g}$, $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{n}$, $\mathfrak{n}$ the Lie algebra of the closed subgroup N, $\mathfrak{k}$ a vector subspace of $\mathfrak{g}$, the splitting satisfying $[\mathfrak{k},\mathfrak{n}] \subseteq \mathfrak{k}$. Let $(\ ,\ )$ be a symmetric, bi-invariant, non-degenerate 2-form on $\mathfrak{g}$. Assume that f, g: $\mathfrak{g} \to \mathbf{R}$ Poisson commute on $\mathfrak{g}$, i.e.,*

$$-([\operatorname{grad} f(\xi), \operatorname{grad} g(\xi)], \xi) = 0, \qquad \text{for all} \quad \xi \in \mathfrak{g}.$$

Assume that either

1) $\mathfrak{k}$ is a Lie subalgebra, or

2) $\Pi_{\mathfrak{n}}[\Pi_{\mathfrak{k}}(\operatorname{grad} f(\eta),\Pi_{\mathfrak{k}}(\operatorname{grad} g)(\eta)] = 0$, *for all* $\eta \in \mathfrak{k}^{\perp}$, *where* $\Pi_{\mathfrak{n}}$, $\Pi_{\mathfrak{k}}$ *are the canonical projections* $\mathfrak{g} \to \mathfrak{n}$, $\mathfrak{g} \to \mathfrak{k}$ *given by the splitting* $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{n}$. *Then*

$$\{f|\mathfrak{k}^{\perp}, g|\mathfrak{k}^{\perp}\}(\eta) = -([\Pi_{\mathfrak{n}}(\operatorname{grad} f)(\eta),\Pi_{\mathfrak{n}}(\operatorname{grad} g)(\eta)],\eta) = 0$$

for all $\eta \in \mathfrak{k}^{\perp}$, *i.e.* $f|\mathfrak{k}^{\perp}$, $g|\mathfrak{k}^{\perp}$ *Poisson commute in the bracket of* $\mathfrak{k}^{\perp}$.

Proof. Let $\eta \in \mathfrak{k}^{\perp}$. By hypothesis

$$\begin{aligned} 0 &= -((\operatorname{grad} f)(\eta),(\operatorname{grad} g)(\eta)) \\ &= \{f|\mathfrak{k}^{\perp}, g|\mathfrak{k}^{\perp}\}(\eta) - (\Pi_{\mathfrak{n}}[\Pi_{\mathfrak{k}}(\operatorname{grad} f)(\eta),\Pi_{\mathfrak{k}}(\operatorname{grad} g)(\eta)],\eta). \end{aligned}$$

The second term is zero in hypothesis (1) or (2). ■

Remark. It can be shown that the Hamiltonian vector field $X_{f|\mathfrak{k}^{\perp}}$ for $f\colon \mathfrak{g} \to \mathbf{R}$ is given by

$$X_{f|\mathfrak{k}^{\perp}}(\eta) = -\Pi_{\mathfrak{k}^{\perp}}[\Pi_{\mathfrak{n}}(\operatorname{grad} f)(\eta),\eta], \qquad \eta \in \mathfrak{k}^{\perp}.$$

See Ratiu [20] for a proof. This result will not be used here but it was the key to the guess of the "right" involution theorem above for (3.1). Also, this theorem naturally lives on duals ([7], [20]) but this is the form we shall use here.

We now start the actual proof of the involution of the $d_{k+1,j}$. First, define $d_{k+1,j}$ on $s\ell(n)$ by the same formula, namely the coefficient of λ^j in the expression of $(1/2(k+1))\operatorname{Tr}(A+\Lambda\lambda)^{k+1}$, $A \in s\ell(n)$. Choose in the above theorem $\mathfrak{g} = s\ell(n)$, $\mathfrak{n} = so(n)$, $\mathfrak{k} =$ symmetric $n \times n$ matrices and $(A,B) = -(1/2)\operatorname{Tr}(AB)$. Then $\mathfrak{n}^{\perp} = \mathfrak{k}$, $\mathfrak{k}^{\perp} = \mathfrak{n}$, $[\mathfrak{k},\mathfrak{n}] \subseteq \mathfrak{k}$, $[\mathfrak{k},\mathfrak{k}] \subseteq \mathfrak{n}$, $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{n}$ and all conditions regarding $\mathfrak{g}$ are satisfied. If A^* denotes the transpose of A, $\Pi_{\mathfrak{n}}(A) = (1/2)(A - A^*)$, $\Pi_{\mathfrak{k}}(A) = (1/2)(A + A^*)$.

Lemma 4.4. *Denote by* $A_{k,j}$ *the coefficient of* λ^j *in the expansion of* $(A+\Lambda\lambda)^k$. $A_{k,j}$ *is a symmetric polynomial in* A *and its powers and contains* $(k-j)$ *factors of* $A \in s\ell(n)$. *We have*

$$(\operatorname{grad} d_{k+1,j})(A) = A_{k,j} - \left(\frac{1}{n}\operatorname{Tr} A_{k,j}\right) I.$$

Proof. Differentiate both sides of the equality

$$\frac{1}{2(k+1)}\operatorname{Tr}(A+\Lambda\lambda)^{k+1} = \sum_{j=0}^{k+1} d_{k+1,j}(A)\lambda^j$$

with respect to A and obtain for any $B \in s\ell(n)$

$$\sum_{j=0}^{k} (\operatorname{grad} d_{k+1,j}(A),B)\lambda^j = \frac{1}{2} d\left(\frac{1}{k+1}\operatorname{Tr}(A+\Lambda\lambda)^{k+1}\right)\cdot B$$

$$= \sum_{j=0}^{k} (A_{k,j}, B)\lambda^j.$$

But $A_{k,j}$ is not of trace zero so we must project it off the center of $g\ell(n)$ to get the desired result. ■

Recall that for $k + 1 - j = $ odd, $d_{k+1,j}|so(n)$ is identically zero. Thus, if $k + 1 - j = $ even, $M \in so(n)$, (grad $d_{k+1,j}$)$(M) = A_{k,j} = $ symmetric polynomial in M with $k - j = $ odd numbers of factors A and hence $A_{k,j} \in so(n)$. In particular $\Pi_{\mathfrak{k}}\, A_{k,j} = 0$ and condition (2) of Theorem 4.3 is satisfied. Thus if we know that on $s\ell(n)$, $d_{k,j}$ Poisson commute, by Theorem 4.3 it follows that the family $\{d_{k,j}\}$ is in involution on $so(n)$. (Remark that the Poisson bracket of the $d_{k,j}$'s for $\mathfrak{k}^{\perp}$ coincides with the Poisson bracket on adjoint orbits of $so(n)$ given in §2 since $A_{k,j} \in so(n)$ for $k - j = $ odd.)

Assume for the moment that $f_{k+1}(A) = (1/2(k+1))\mathrm{Tr}(A + \Lambda\lambda_{k+1})^{k+1}$ Poisson commute on $s\ell(n)$ for $\lambda_{k+1} \in \mathbf{R}$ arbitrary. Then the Hamiltonian vector field $X_{f_{k+1}}$ would have as constants of the motion any $f_{\ell+1}$ for any $\lambda_{\ell+1} \in \mathbf{R}$, i.e. $d_{\ell+1,j}$, the coefficients of $\lambda^j_{\ell+1}$ in $f_{\ell+1}$, would be constant on the flow of $X_{f_{k+1}}$ and hence $\{d_{k+1,j}, f_{k+1}\} = 0$ on $s\ell(n)$. But since $f_{k+1} = \Sigma_{j=0}^{k+1} d_{k+1,i}\lambda^i_{k+1}$ this implies that $\{d_{\ell+1,j}, d_{k+1,i}\} = 0$ on $s\ell(n)$.

Thus it remains to be shown that $\{f_{\ell+1}, f_{k+1}\} = 0$ on $s\ell(n)$ for any $\lambda_i \in \mathbf{R}$, $f_{k+1}(A) = (1/2(k+1))\mathrm{Tr}(A + \Lambda\lambda_{k+1})^{k+1}$. To prove this, remark first that for $\lambda_{k+1} = \lambda_{\ell+1}$, (grad f_{k+1})(A), (grad $f_{\ell+1}$)(A) commute and hence $\{f_{k+1}, f_{\ell+1}\} = 0$ on $s\ell(n)$. If now $\lambda_{k+1} \neq \lambda_{\ell+1}$ let μ, $\nu \in \mathbf{R}$ be arbitrary but such that $\lambda_{k+1} + \mu = \lambda_{\ell+1} + \nu$. Then $\{\phi,\psi\} = 0$ on $s\ell(n)$ where $\phi(A) = (1/2(k+1))\mathrm{Tr}(A + (\lambda_{k+1} + \mu)\Lambda)^{k+1} = f_{k+1}(A) + (\mathrm{Tr}\, B)\mu$, $\psi(A) = (1/2(\ell + 1))\mathrm{Tr}(A + (\lambda_{\ell+1} + \nu)\Lambda)^{\ell+1} = f_{\ell+1}(A) + (\mathrm{Tr}\, C)\nu$. Thus

$$0 = \{f_{k+1}, f_{\ell+1}\} - \{f_{\ell+1}, \mathrm{Tr}\, B\}\mu + \{f_{k+1}, \mathrm{Tr}\, C\}\nu + \{\mathrm{Tr}\, B, \mathrm{Tr}\, C\}\mu\nu.$$

Take for (μ,ν) the following four pairs $(0, \lambda_{k+1} - \lambda_{\ell+1})$, $(\lambda_{\ell+1} - \lambda_{k+1}, 0)$, $(s, \lambda_{k+1} - \lambda_{\ell+1} + s)$, $(\lambda_{\ell+1} - \lambda_{k+1} + s, s)$ where $s \in \mathbf{R}$, $s \neq 0$, is arbitrary and get a 4×4 homogeneous system whose determinant is $2s(\lambda_{k+1} - \lambda_{\ell+1})(\lambda_{k+1} - \lambda_{\ell+1} - s)(\lambda_{k+1} - \lambda_{\ell+1} + s)$. If $s = 1$ and $\lambda_{k+1} - \lambda_{\ell+1} \neq \pm 1$, the determinant is nonzero and we have $\{f_{k+1}, f_{\ell+1}\} = 0$ on $s\ell(n)$. Now for $s \neq 1$ and $\lambda_{k+1} - \lambda_{\ell+1} \neq \pm s$ we have again $\{f_{k+1}, f_{\ell+1}\} = 0$, i.e. $\{f_{k+1}, f_{\ell+1}\} = 0$ on $s\ell(n)$ for any λ_{k+1}, $\lambda_{\ell+1} \in \mathbf{R}$.

We have proved

Theorem 4.5. *The families of functions $\{c_{k,j}\}$ and $\{d_{k,j}\}$ are in involution on $so(n)$.*

A direct proof of their independence and the linearization of the flow of (3.1) is given in Adler-van Moerbeke [3].

§5. Lenard relations. Let M be a manifold and $\mathscr{F}(M)$, $\mathscr{X}(M)$ the algebra of smooth functions, respectively the Lie algebra of smooth vector fields on M. We shall say that M is endowed with a *Hamiltonian structure* (Kupershmidt-Manin [11], Manin [14]), if the following data are given:

(1) a bracket $\{\ ,\ \}$ on $\mathcal{F}(M)$ making $\mathcal{F}(M)$ into a Lie algebra;

(2) A Lie algebra anti-homomorphism X: $(\mathcal{F}(M),\{\ ,\ \}) \to (\mathcal{X}(M),[\ ,\ ])$ given by $X_f(g) = -\{f,g\}$.

For $f \in \mathcal{F}(M)$, X_f will be called the *Hamiltonian vector field* of the *Hamiltonian* f. Thus condition (2) is

$$X_{\{f,g\}} = -[X_f, X_g].$$

Hamiltonian structures are widely called Poisson or cosymplectic structures too.

Examples. (1) If M is symplectic it carries a Hamiltonian structure defined by the symplectic form. Moreover, by a theorem of Jost [8], there is a bijective correspondence between Hamiltonian structures on M for which the bracket is non-degenerate in the sense that if $\{f,g\} = 0$ for all $g \in \mathcal{F}(M)$ implies f is constant on connected components of M, and symplectic structures on M.

(2) A whole family of Hamiltonian structures are defined on $\mathfrak{g}^*$ by the bracket

$$\{f,g\}_\Psi(\mu) = -\Psi(\mu) \cdot [df(\mu), dg(\mu)]$$

for f,g: $\mathfrak{g}^* \to \mathbf{R}$, where Ψ: $\mathfrak{g}^* \to \mathfrak{g}^*$ is a smooth map satisfying certain conditions, and $df(\mu)$, $dg(\mu)$ are thought of as elements of $\mathfrak{g} = \mathfrak{g}^{**}$. If $\Psi = \mathrm{id}$ we obtain the Hamiltonian structure given by the decomposition of $\mathfrak{g}^*$ in its co-adjoint orbits. Fixing $\nu \in \mathfrak{g}^*$ and taking $\Psi(\mu) = \nu$ for all $\mu \in \mathfrak{g}^*$ yields a Hamiltonian structure responsible for Lenard relations. I owe this example for general Ψ to the referee.

(3) The two particular examples of above translate on $\mathfrak{g}$ in the following two Hamiltonian structures we shall use in this paper. The Kirillov-Kostant-Souriau structure on $\mathfrak{g}$ is given by

$$\{f,g\}(\xi) = -([(\operatorname{grad} f)(\xi), (\operatorname{grad} g)(\xi)], \xi)$$
$$X_f(\xi) = -[(\operatorname{grad} f)(\xi), \xi].$$

Fix $\varepsilon \in \mathfrak{g}$ and put

$$\{f,g\}_\varepsilon(\xi) = -([(\operatorname{grad} f)(\xi), (\operatorname{grad} g)(\xi)], \varepsilon)$$
$$X_f^\varepsilon(\xi) = -[(\operatorname{grad} f)(\xi), \varepsilon].$$

Here $(\ ,\ )$ is a symmetric, bilinear bi-invariant two-form on $\mathfrak{g}$.

Lenard relations on a set of functions on $\mathfrak{g}$ are identities which relate the Hamiltonian vector fields in the two Hamiltonian structures of $\mathfrak{g}$. Such relations were first discovered by Lenard for the Korteweg-de Vries integrals of motion.

Return now to the notations of §4. For $k - j = $ odd we remarked after the proof of Lemma 4.4 that if $M \in so(n)$, $(\text{grad } d_{k+1,j})(M) = A_{k,j} \in so(n)$. Thus $([A_{k,j}, A_{\ell,i}], \Lambda) = 0$. We have hence:

Proposition 5.1. *For $k - j = odd$, $d_{k+1,j}|so(n)$ commute in the bracket $\{\ ,\ \}_\Lambda$.*

Λ hence defines the Hamiltonian structure of Example 3 above on $s\ell(n)$. Since $\{d_{k+1,j}|so(n)\}$ commute in both brackets $\{\ ,\ \}$ and $\{\ ,\ \}_\Lambda$ one expects some relation between the corresponding Hamiltonian vector fields.

Lemma 5.2. *Let $M \in s\ell(n)$. Then*

$$[M,(M + \Lambda\lambda)^k] = \lambda(A_0 + A_1\lambda + \dots + A_{k-1}\lambda^{k-1}) \equiv \lambda A[\lambda]$$
$$[(M + \Lambda\lambda)^k,\Lambda] = B_0 + B_1\lambda + \dots + B_{k-1}\lambda^{k-1} \equiv B[\lambda]$$

and $A[\lambda] \equiv B[\lambda]$.

Proof. The first term in the development of $[M,(M + \Lambda\lambda)^k]$ is just $[M,M^k] = 0$, which proves that $[M,(M + \Lambda\lambda)^k]$ has no free term. Similarly it is seen that $[(M + \Lambda\lambda)^k,\Lambda]$ is a polynomial of degree $k - 1$. To prove now that $A[\lambda]$ is identical to $B[\lambda]$ it suffices to show that $f(\lambda) - \lambda g(\lambda)$ is identically zero, where $f(\lambda) = [M,(M + \Lambda\lambda)^k]$, $g(\lambda) = [(M + \Lambda\lambda)^k,\Lambda]$. We have

$$f(\lambda) - \lambda g(\lambda) = [M + \Lambda\lambda,(M + \Lambda\lambda)^k] \equiv 0. \qquad \blacksquare$$

Restating the lemma we get

$$[M,A_{k,j}] = [A_{k,j-1},\Lambda], \qquad k = 1, \dots, n-1, \quad j = 1, \dots, k. \tag{5.1}$$

Now if $M \in so(n)$ and $k - j = $ odd we get the Lenard relations for the integrals $\{d_{k+1,j}\}$:

$$\begin{aligned} X_{d_{k+1,j}}(M) &= [A_{k,j-1},\Lambda] \\ X^\Lambda_{d_{k+1,j}}(M) &= -[M,A_{k,j+1}]. \end{aligned} \tag{5.2}$$

We can now prove again, purely mechanically as in the case of the Korteweg-de Vries equation, the involution of $\{d_{k+1,j} \colon k - j = \text{odd}\}$ on $so(n)$. Even though this proof is easier than the one given in §4, it has the disadvantage of ignoring the whole underlying structure which actually makes it work.

Theorem 5.3. *$\{d_{k+1,j}|k - j = \text{odd}\}$ commute on $so(n)$ in the Poisson bracket $\{\ ,\ \}$.*

Proof. Since $(\ ,\ )$ is bi-invariant, for f, g: $so(n) \to \mathbf{R}$, $M \in so(n)$, we have

$$\{f,g\}(M) = ((\text{grad} f)(M),X_g(M)) = -((\text{grad } g)(M),X_f(M)).$$

Hence, for $k - j$, $\ell - i$ odd, we obtain successively

$$\{d_{k+1,j},d_{\ell+1,i}\}(M) = ((\text{grad } d_{k+1,j})(M),X_{d_{\ell+1,i}}(M))$$

$$
\begin{aligned}
&= -((\mathrm{grad}\ d_{k+1,j})(M), [\Lambda, A_{\ell,i-1}]) \\
&= -([A_{k,j}, \Lambda], A_{\ell,i-1}) \\
&= -([M, A_{k,j+1}], A_{\ell,i-1}) \\
&= (A_{k,j+1}, [A_{\ell,i-2}, \Lambda]) \\
&= -([M, A_{k,j+2}], A_{\ell,i-2}) \\
&= -(X_{d_{k+1,j+2}}(M), (\mathrm{grad}\ d_{\ell+1,i-2})(M)) \\
&= \{d_{k+1,j+2}, d_{\ell+1,i-2}\}(M).
\end{aligned}
$$

Now repeat this procedure until either j increases to reach $k + 1$ or i decreases to reach 1 or 0. If first j reaches $k + 1$, then $\{d_{k+1,j}, d_{\ell+1,i}\} = \{d_{k+1,k+1}, d_{\ell+1,i-(k+1-j)}\} = 0$ since $d_{k+1,k+1} = (1/2)(k + 1)\mathrm{Tr}\ \Lambda^{k+1} = \text{constant}$. If first i reaches zero, then necessarily ℓ is odd and as before $\{d_{k+1,j}, d_{\ell+1,i}\} = \{d_{k+1,j+i}, d_{\ell+1,0}\} = 0$, since $d_{\ell+1,0} = (1/2(\ell + 1))\mathrm{Tr}\ M^{\ell+1} = \text{constant}$ on the orbit, i.e. $X_{d_{\ell+1,0}} \equiv 0$. If first i reaches 1, then necessarily ℓ is even and we have

$$
\begin{aligned}
\{d_{k+1,j}, d_{\ell+1,i}\}(M) &= \{d_{k+1,j+i-1}, d_{\ell+1,1}\}(M) \\
&= (A_{k,j+i-1}, [A_{\ell,0}, \Lambda]) \\
&= -([M, A_{k,j+i}], M^{\ell}) \\
&= (A_{k,j+i}, [M, M^{\ell}]) = 0. \qquad \blacksquare
\end{aligned}
$$

§6. Mishchenko's integrals. Manakov's integrals do not contain the Hamiltonian $H(M) = (1/2)(M,\Omega) = -(1/4)\mathrm{Tr}(M\Omega)$ of (3.1). In [16] Mishchenko found a whole series of integrals of which H is the first one and proved their generic independence; their number turns out to be half of the dimension of the adjoint orbit in the case $n = 4$ and in this case only. Mishchenko's integrals are:

$$
m_k(M) = -\frac{1}{4}\mathrm{Tr}\sum_{p=1}^{k} J^{p-1}MJ^{k-p}L(M); \qquad m_1 = H,\ L(M) = \Omega,
$$

with gradients defined by the form (,)

$$
(\mathrm{grad}\ m_k)(M) = \sum_{p=1}^{k} J^{k-p}L(M)J^{p-1} \in so(n)
$$

(since $L(J^{k-p}MJ^{p-1}) = J^{k-p}L(M)J^{p-1}$) and Hamiltonian vector fields

$$
\dot{M} = X_{m_k}(M) = \left[M, \sum_{p=1}^{k} J^{k-p}L(M)J^{p-1}\right]. \tag{6.1}
$$

Using the relation $\mathrm{Tr}(J^{p-1}MJ^{\ell-p}L(N)) = \mathrm{Tr}(J^{p-1}L(M)J^{\ell-p}N)$ and the fact that (3.1) is equivalent to $\dot{M} = [J,\Omega^2]$, an easy computation shows that

$\{m_k,H\} = 0$. To prove however by this direct method that $\{m_k,m_j\} = 0$ is extremely laborious; it has been sketched by Dikii [4]. We shall prove this here in a different way which is considerably easier.

Lemma 6.1. *Equation (6.1) is equivalent to*

$$(6.2)\qquad (M + J^2\lambda)^{\cdot} = \left[M + J^2\lambda, \sum_{p=1}^{k} J^{k-p}\Omega J^{p-1} + J^k\lambda \right].$$

This can be easily seen by showing that the coefficient of λ in the expansion of the bracket in the right hand side vanishes.

Thus $(1/2k)\mathrm{Tr}(M + J^2\lambda)^k$ are constant on the flow defined by (6.2), i.e. $c_{k,j}$ are constant on the flow of X_{m_j}. This proves the first part of the following

Theorem 6.2. *(i) Manakov's and Mishchenko's integrals Poisson commute.*
(ii) Mishchenko's integrals Poisson commute.

Proof of (ii). Let c be any Manakov integral. By the Jacobi identity and (i) we conclude $\{\{m_k,m_j\},c\} = 0$. But the Hamiltonian vector fields corresponding to the Manakov integrals span the tangent space of the generic adjoint orbit and hence $\{m_k,m_j\}$ = constant on the generic orbit. A simple computation shows that $\{m_j,m_k\}$ is a homogeneous function of degree 3 in M and thus $\{m_k,m_j\} = 0$ on the generic orbit so that $\{m_j m_k\} \equiv 0$ on $so(n)$. ■

§7. **The extended Dubrovin equation.** In the review article [5] of Dubrovin, Matveev and Novikov the following matrix equation of Dubrovin is considered:

$$(7.1)\qquad [\mathscr{a},V]^{\cdot} = [[\mathscr{a},V],[\mathscr{b},V]]$$

where $\mathscr{a} = \mathrm{diag}(a_1,\dots,a_n)$, $\mathscr{b} = \mathrm{diag}(b_1,\dots,b_n)$ are constant diagonal matrices and V is a matrix with zeros on the diagonal. Using algebraic geometry and inverse scattering it is claimed that this equation is Hamiltonian and completely integrable. We show here that it is a projection of a Hamiltonian system on adjoint orbits of $s\ell(n)$, find its integrals and show they are in involution.

It should be noted first that (7.1) leaves $so(n)$ invariant. Also, following Manakov [13], put $\mathscr{a} = J^2$, $\mathscr{b} = J$, $[\mathscr{a},V] = M$, $[\mathscr{b},M] = \Omega$ to regain (3.1). Thus the rigid body equation is the restriction to $so(n)$ of Dubrovin's equation.

In all that follows $\mathscr{a}$, $\mathscr{b}$ will be assumed constant diagonal matrices in $s\ell(n)$, $\mathscr{a}$ having distinct entries. Let $\mathfrak{d}$ denote the diagonal matrices in $s\ell(n)$ and $\mathfrak{g}$ the matrices with zeros on the diagonal in $s\ell(n)$. Then

$$s\ell(n) = \mathfrak{d} \oplus \mathfrak{g}.$$

Given $\mathscr{a}$, any matrix in $\mathfrak{g}$ can be written in the form $[\mathscr{a},V]$ and thus (7.1) is a differential equation in $\mathfrak{g}$. We shall add to (7.1) another trivial differential equation in $\mathfrak{d}$ and the resulting equation will be a Hamiltonian system in $s\ell(n)$.

For this purpose, let $(\ ,\ )$ be defined as before by $(A,B) = -(1/2)\mathrm{Tr}(AB)$. Let the symmetric linear operator L: $s\ell(n) \to s\ell(n)$ be given by:

$$L\,|\,\mathfrak{d}: \mathfrak{d} \to \mathfrak{d} \text{ arbitrary symmetric}$$
$$(L\,|\,\mathfrak{g})(A) = \left(\frac{b_i - b_j}{a_i - a_j}\, a_{ij}\right),$$

where $A = (a_{ij}) \in \mathfrak{g}$. Finally, define the Hamiltonian $H(M) = (1/2)(M,L(M))$, $M \in s\ell(n)$ whose gradient with respect to $(\ ,\ )$ is $(\mathrm{grad}\ H)(M) = L(M)$ and Hamiltonian vector field $X_H(M) = [M,L(M)]$. Note that if $M = D + A$, $D \in \mathfrak{d}$, $A \in \mathfrak{g}$, then Hamilton's equations are

$$(D + A)^{\cdot} = [D + A, L(D) + L(A)] = [A,L(D)] + [D,L(A)] + [A,L(A)].$$

Since $[\mathfrak{d},\mathfrak{d}] = 0$, $[\mathfrak{d},s\ell(n)] \subseteq \mathfrak{g}$, $[A,L(A)] \in \mathfrak{g}$ for all $A \in \mathfrak{g}$,

$$\dot{D} = 0, \quad \dot{A} = [A,L(A)] + [D,L(A)] + [A,L(D)].$$

Putting $A = [a, V]$, $D = 0$, the second equation becomes Dubrovin's equation (7.1).

From now on we shall refer to

$$\dot{M} = [M,L(M)], \quad M \in s\ell(n) \tag{7.2}$$

as the *extended Dubrovin equation* whose complete integrability we now prove. First, note that from the definition of L one has $[a,L(A)] = [\ell,A]$ for all $A \in \mathfrak{g}$ and hence $[a,L(M)] = [\ell,M]$, $M \in s\ell(n)$, whence equation (7.2) has the equivalent form

$$(M + a\lambda)^{\cdot} = [M + a\lambda, L(M) + \ell\lambda]. \tag{7.3}$$

Thus the coefficients $d_{k+1,j}$ of λ in $(1/2(k+1))\mathrm{Tr}(M + a\lambda)^{k+1}$, $k = 1, \ldots, n-1$ are conserved along the flow of (7.2) for each $\lambda \in \mathbf{R}$ since $(1/2(k+1))\mathrm{Tr}(M + a\lambda)^{k+1}$ is conserved along the flow of (7.3). There are $k + 2$ coefficients, the first $(1/2(k+1))\mathrm{Tr}\,M^{k+1}$ and the last $(1/2(k+1))\mathrm{Tr}\,a^{k+1}$ leading to identically zero Hamiltonian vector fields on the adjoint orbit through M. Thus, the number of coefficients leading to non-zero Hamiltonian vector fields is $\Sigma_{k=1}^{n-1} k = n(n-1)/2$.

On the other hand, the dimension of the generic adjoint orbit of $s\ell(n)$ is equal to $(n^2 - 1) - (n - 1) = n(n - 1)$ and thus the integrals $d_{k+1,j}$ of above are candidates for a complete set of integrals in involution. Their involution was proved already in §4.

Also, everything in §5 goes through. The Lenard relations are

$$X_{d_{k+1,j}} = -X^{a}_{d_{k+1,j-1}}$$

and it follows from here again that the integrals $d_{k+1,j}$ are in involution in the usual Poisson bracket $\{\ ,\ \}$ of $s\ell(n)$ given by its adjoint orbits as well as in the Poisson bracket $\{\ ,\ \}_a$. It should be mentioned here that $\{\ ,\ \}_a$ has also a different Lie algebraic interpretation: it is the Poisson bracket of a certain adjoint orbit of the one-tiered Kac-Moody Lie algebra of $s\ell(n)$ with a certain trace functional as it is shown in Adler-van Moerbeke [3]. The independence of $\{d_{k+1,j}\}$ follows essentially from the Lenard relations above. Since our proof is very similar to the one in §4 of [18] we only sketch it here briefly and refer the reader to the above paper for details. One has to prove that for a regular semisimple element M the space generated by $X_{d_{k+1,j}}(M)$ has dimension $\geq n(n-1)/2$. For this it is sufficient to show that the space generated by grad $d_{k+1,j}(M)$ has dimension $\geq (1/2)(n-1)(n+2)$. Now conjugate M to an upper triangular matrix first, an operation which does not change the dimension, and then show—using the Lenard relations and the fact that $A_{k,k}$ is diagonal—that if M is upper diagonal the estimate above holds and that if M is upper triangular grad $d_{k+1,j}(M)$ is too. Finally, note that the independence just shown gives another proof (without the use of algebraic geometry) of the independence of Manakov's integrals for the n-dimensional rigid body problem.

The extended Dubrovin equation has also Mishchenko type integrals. They are

$$m_k(M) = -\frac{1}{4}\operatorname{Tr}\sum_{p=1}^{k} \ell^{p-1} M \ell^{k-p} L(M), \qquad m_1 = H$$

with gradients

$$(\operatorname{grad} m_k)(M) = \sum_{p=1}^{k} \ell^{k-p} L(M) \ell^{p-1} - \left(\frac{1}{n}\operatorname{Tr}\sum_{p=1}^{k} \ell^{k-p} L(M)\ell^{p-1}\right) I$$

and Hamiltonian vector fields

$$\dot{M} = X_{m_k}(M) = \left[M, \sum_{p=1}^{k} \ell^{k-p} L(M) \ell^{k-p} \right]. \tag{7.4}$$

Equation (7.4) is easily seen to be equivalent to

$$(M + \lambda a)^{\cdot} = \left[M + \lambda a, \sum_{p=1}^{k} \ell^{k-p} L(M)\ell^{p-1} + \ell^{k}\lambda \right]$$

since the coefficient of λ in the right hand side is zero. Proceeding as in §6 one shows $\{d_{k+1,j}, m_k\} = 0$, $\{m_k, m_j\} = 0$ on $s\ell(n)$. We have proved hence the following.

Theorem 7.1. *The extended Dubrovin equation is a completely integrable Hamiltonian system on adjoint orbits of $s\ell(n)$. For $a = J^2$, $\ell = J$, this system leaves $so(n)$ invariant and its restriction to $so(n)$ is the n-dimensional free rigid body equation. Manakov's and Mishchenko's integrals for the rigid body problem are restrictions to $so(n)$ of similar integrals for the extended Dubrovin equation.*

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