Halving lines and perfect cross-matchings

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Abstract

It is shown that a set $P$ of $2n$ points in general position in the plane admits a perfect matching with pairwise crossing segments if and only if it has precisely $n$ halving lines. As a consequence, one can give a $O(n \log n)$-time algorithm which decides whether there exists such a matching in $P$ and, if so, finds it.

1 Preliminaries

Let $P = \{ p_1, p_2, \ldots, p_{2n} \}$ be a set of $2n$ points in the plane in general position, i.e., no three points are collinear. A line $p_ip_j$ is said to be a halving line of $P$ if both open half-planes bounded by $p_ip_j$ contain precisely $n - 1$ points. The number of halving lines of $P$ is denoted by $h(P)$.

Taking an arbitrary line through any point of $P$ and turning it around by at most 180 degrees, it always arrives at a position where it becomes a halving line. Thus, we have $h(P) \geq n$, and equality holds, e.g., when $P$ is the vertex set of a convex $2n$-gon.

It is an intriguing open problem to determine the asymptotic behavior of $h(n) = \max_P h(P)$, where the maximum is taken over all $2n$-element sets in general position in the plane. It is known that

$$c_1 n \log n \leq h(n) \leq c_2 n^{4/3}$$

for suitable constants $c_1, c_2 > 0$ (see [L], [EL], [D]). This function plays an important role in the analysis of many algorithms in computational geometry (cf. [E]).

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We say that two segments cross if they have an interior point in common. Let \( c(P) \) denote the maximum number of pairwise crossing segments \( p_i p_j \) whose endpoints belong to \( P \). Obviously, \( c(P) \leq n \) holds for every \( 2n \)-element set \( P \). If \( c(P) = n \), we say that \( P \) has a perfect cross-matching. This is the case, for example, when \( P \) is the vertex set of a convex \( 2n \)-gon.

Let \( c(n) = \min_P c(P) \), where the minimum is taken over all \( 2n \)-element sets in general position in the plane. We have

\[
c(n) \geq c_3 \sqrt{n},
\]

for some positive constant \( c_3 \), but there is no sublinear upper bound known for \( c(n) \) (see [A],[P]). In fact, in [A] it was shown that every \( 2n \)-element set in general position has an at least \( c_3 \sqrt{n} \)-element subset which not only admits a perfect cross-matching, but also satisfies a much stronger condition. In this strong sense the result is best possible [V]. It looked difficult to improve the lower bound on \( c(n) \), because we had no good characterization of perfectly cross-matchable sets.

The aim of this note is to give such a good characterization and to design an efficient algorithm which decides whether a set admits a perfect cross-matching.

## 2 Characterization of perfectly cross-matchable sets

In this section, we would like to point out a simple relation between \( c(P) \) and \( h(P) \): the first quantity attains its maximum if and only if the second attains its minimum. More precisely, we have the following.

**Theorem 1.** A set of \( 2n \) points in general position in the plane admits a perfect cross-matching if and only if it has precisely \( n \) halving lines.

**Proof:** Suppose first that \( P = \{p_1, p_2, \ldots, p_{2n}\} \) has a perfect cross-matching (i.e., \( n \) pairwise crossing segments) \( \overline{p_{2i-1}p_{2i}} \), \( 1 \leq i \leq n \). The extension of each of these segments is a halving line, because each of them separates the two endpoints of all other segments \( \overline{p_{2i-1}p_{2i}} \). We will show that \( P \) has no other halving lines.

Assume, in order to obtain a contradiction, that (say) \( p_1 p_3 \) is also a halving line. We may suppose without loss of generality that \( p_1 p_2 \) is horizontal, \( p_2 \) is to the right of \( p_1 \), and that \( p_{2i} \) is below and \( p_{2i-1} \) is above \( p_1 p_2 \), for every \( 2 \leq i \leq n \). Since each segment \( \overline{p_{2i-1}p_{2i}} \) (\( 3 \leq i \leq n \)) has to cross \( \overline{p_1p_2} \), if \( \overline{p_{2i-1}p_{2i}} \) has an endpoint to the left of \( p_1 p_3 \), then its other endpoint must lie to the right of \( p_1 p_3 \). However, both \( p_2 \) and \( p_4 \) are on the right-hand
side of \(p_1p_3\). This implies that the number of points to the right of \(p_1p_3\) exceeds by at least 2 the number of points to the left of it, contradicting our assumption that \(p_1p_3\) is a halving line.

Suppose next that \(P\) has precisely \(n\) halving lines. Since there is at least one halving line through every point \(p_k\), we obtain that there must be exactly one. Thus, we can assume without loss of generality that the complete list of halving lines is \(p_{2i-1}p_{2i}\) (\(1 \leq i \leq n\)). We will show that the segments \(p_{2i-1}p_{2i}\) (\(1 \leq i \leq n\)) are pairwise crossing.

Assume, for contradiction, that \(p_1p_2\) and \(p_3p_4\) have no interior point in common. By renumbering the points if necessary, we may also suppose that \(p_1p_2\) is horizontal, \(p_2\) is to the right of \(p_1\), \(p_3p_4\) is entirely above the line \(p_1p_2\), and that \(p_3\) is closer to \(p_1p_2\) than \(p_4\) is. Notice that a minor counterclockwise turn around \(p_3\) will bring the line \(\ell = p_3p_4\) into a position, where there are precisely \(n\) points on its right-hand side. (Indeed, \(p_4\) will be added to the set of points to the right of \(p_3p_4\).) If we continue to turn \(\ell\) around \(p_3\) in the counterclockwise direction, we arrive at a position where \(\ell\) becomes parallel to \(p_1p_2\), i.e., it becomes horizontal. At that moment, there are at most \(n - 2\) points above \(\ell\) (these points form a subset of the set of all points different from \(p_3\) which lie above the halving line \(p_1p_2\)). Hence, there is an intermediate position \(\ell = p_3p_k\) for some \(k \neq 4\), in which the number of points on the right-hand side of \(\ell\) is precisely \(n - 1\). This means that \(p_3p_k\) is a halving line which does not appear in the complete list of halving lines, \(p_{2i-1}p_{2i}\) (\(1 \leq i \leq n\)). Contradiction. \(\Box\)

Actually, this argument also yields the uniqueness of the perfect cross-matching.

**Theorem 2.** Any set of points in general position in the plane admits at most one perfect cross-matching.

**Proof:** As we have shown, every perfect cross-matching of \(P\) consists of exactly those segments between two points of \(P\), whose extensions are halving lines of \(P\). \(\Box\)

### 3 Algorithm

The above characterization of perfectly cross-matchable sets allows us to design an \(O(n \log n)\)-time algorithm which decides whether a set of \(2n\) points satisfies this property and, if so, finds a perfect cross-matching for it.

Let \(P\) be a \(2n\)-element point set in general position in the plane, which is the union of two \(n\)-element sets, \(P_1\) and \(P_2\), separated by a straight line (say, by the \(y\)-axis). For any non-vertical line \(\ell\), let \(P_1(\ell^+)\) (resp. \(P_1(\ell^-)\))
denote the set of points in $P_i$ lying above (resp. below) $\ell$. A line $\ell$ not passing through any point of $P$ is called a \emph{ham-sandwich cut} for $P$, if
\[ |P_1(\ell^+)| = |P_2(\ell^-)| = [n/2]. \]

It was shown by Megiddo [M] that one can always find such a line $\ell$ in $O(n)$ steps (see also [LM]).

Any matching of $P$ that has a segment to the left of the $y$-axis, has another one to the right of it, and these two segments cannot cross. Thus, if there exists a perfect cross-matching for $P$, then all of its segments must cross the $y$-axis and, similarly, they must also cross the ham-sandwich cut $\ell$. Consequently, a perfect cross-matching $M$ for $P$ is the union of a perfect cross-matching $M_1$ for $P_1(\ell^+) \cup P_2(\ell^-)$ and a perfect cross-matching $M_2$ for $P_1(\ell^-) \cup P_2(\ell^+)$. Let $M_i^+$ and $M_i^-$ denote the \emph{upper envelope} and the \emph{lower envelope} (i.e., the pointwise maximum and pointwise minimum) of the lines supporting the segments of $M_i$, respectively ($i = 1, 2$). Clearly, $M_i^+$ and $M_i^-$ are unbounded convex polygonal paths, with at most $[n/2]$ vertices each. (See Figure 1.)

![Figure 1.](image)

We need the following corollary of Theorem 2.
Claim. The set $P$ admits a perfect cross-matching $M$ if and only if the following conditions are satisfied.

1. $P_1(\ell^+ \cup P_2(\ell^-)$ admits a perfect cross-matching $M_1$ and $P_1(\ell^- \cup P_2(\ell^+)$ admits a perfect cross-matching $M_2$.

2. The convex hull $\text{conv} P_2(\ell^+)$ is above the polygonal path $M_1^+$, and $\text{conv} P_1(\ell^-)$ is below $M_1^-$. Similarly, $\text{conv} P_1(\ell^+)$ is above $M_2^+$, and $\text{conv} P_2(\ell^-)$ is below $M_2^-$. Then, we have $M = M_1 \cup M_2$.

Proof. We have seen before that if $P$ admits a perfect cross-matching $M$, then it satisfies condition (1) and $M = M_1 \cup M_2$ holds. By Theorem 2, $M_1$ and $M_2$ are uniquely determined. To see that (2) is necessary, too, assume that (say) $P_2(\ell^+)$ has a point $p$ below $M_1^+$. Then $p$ lies below the supporting line of at least one segment $\overline{qq'} \in M_1$. Let $p'$ denote the element of $P_1(\ell^-)$ connected to $p$ in $M_2$. Then $\overline{pp'} \cap \overline{qq'} = \emptyset$, contradicting our assumption that any two segments of $M$ cross.

Suppose next that conditions (1) and (2) are satisfied. Then $M = M_1 \cup M_2$ is a perfect cross-matching for $P$. Indeed, if there were two disjoint segments, $\overline{pp'} \in M_1$ and $\overline{qq'} \in M_2$, such that (say) $\overline{qq'}$ is below (resp. above) the line $pp'$, then $\text{conv} P_2(\ell^+)$ would not lie above the polygonal path $M_1^+$ (resp. $\text{conv} P_1(\ell^-)$ would not lie below $M_1^-$), contradicting condition (2). □

Let $M^+$ and $M^-$ denote the upper and the lower envelope of all lines supporting the segments of $M = M_1 \cup M_2$, respectively. Clearly, $M^+$ can be obtained as the upper envelope of $M_1^+$ and $M_2^+$, and $M^-$ can be obtained as the lower envelope of $M_1^-$ and $M_2^-$. It is well known that one can compute the union and the intersection of two convex polygons of at most $n$ sides in time $O(n)$ ([PH], [S]). Thus, if we know $\text{conv} P_1(\ell^+), \text{conv} P_1(\ell^-), M_1^+$, and $M_1^-$ for $i = 1, 2$, then in linear time we can determine $\text{conv} P_i(i = 1, 2), M^+$ and $M^-$. If any of the conditions of the Claim is not satisfied, we conclude that $P$ does not admit a perfect cross-matching.

So one can use a divide-and-conquer algorithm to decide whether $P = P_1 \cup P_2$ admits a perfect cross-matching and, if yes, to compute it simultaneously with $\text{conv} P_i$ $(i = 1, 2), M^+$ and $M^-$. At each stage it takes linear time to find a ham-sandwich cut $\ell$ and to do the merge step.

We obtained the following.

Theorem 3. There is an $O(n \log n)$ time, $O(n)$ space algorithm which decides whether a set of $2n$ points in general position in the plane admits a perfect cross-matching and, if so, computes it.
Clearly, any decision tree that determines the perfect cross-matching of a planar point set of \(2n\) points (if it exists) has height \(\Omega(n \log n)\). In this sense Theorem 3 is asymptotically tight.

References


