JAMES COOK MATHEMATICAL NOTES

Volume 5 Issue number 47

October 1988

Editor and Publisher: B. C. Rennie

Address: 66, Hallett Road,
Burnside, SA 5066,
Australia

The "James Cook Mathematical Notes" is published in three issues per year, in February, June and October.

The normal subscription rate for one year (3 issues) is 30 Singapore dollars, and for retired or unemployed subscribers it is 15 Singapore dollars. Cheques (or other forms of money) in any currency are acceptable, the exchange calculated at the rate current when the money is sent. The subscription rate includes air mail postage to anywhere outside the country where the issue is posted (now Australia). Some exchange rates at the time of writing (September 1988) are

One Singapore dollar = 61 cents Australian
= 29 pence United Kingdom
= 49 cents U.S.A.

We do not send out invoices or subscription reminders. Please make cheques payable to B.C.Rennie.

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PROBLEM ON CONVEX POLYGONS (JCMN 46, p.5093)

Janos Pach

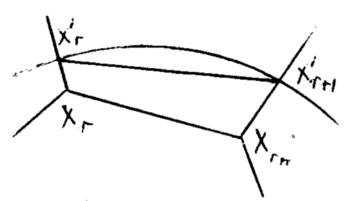
(Translated from the Hungarian by Esther Szekeres)

The problem was about a plane convex polygon C_n with vertices $X_1, X_2, \dots X_n$. To exclude trivial cases we assume that $n \geq 3$ and that the vertices are not collinear. Let S be the circumference (denoted by $S(C_n)$ in the original problem) and let Q be the centre of mass (of the polygon regarded as consisting of equal point masses at each vertex). For any two points A and B let d(A, B) be their distance apart and let $p_i(A, B)$ be the perpendicular projection of the directed line segment AB on to the line X_iQ, and put $d_i(A, B) = |p_i(A, B)|$.

The inequality $\qquad d_i(A, B) \leq d(A, B) \qquad \dots (1)$

can be strengthened to a strict inequality unless AB is parallel to X_iQ.

Lemma 1 If the polygon has all vertices in or on a circle of radius R then $S \leq 2Rn\sin\pi/n$.



Proof Firstly if one or more of the vertices are strictly inside the circle we can construct another polygon $X'_1, \dots X'_n$ with all its vertices on the circle, see the sketch, where $X'_rX'_{r+1} > X_rX_{r+1}$, the line $X_rX'_r$ bisects the angle of the polygon. Secondly, of all the n-gons inscribed in the circle, the regular n-gon has the greatest circumference because of the concavity of the sine function.

Lemma 2 $\quad n\,d(X_i, Q) = \sum_{j=1}^n p_i(X_i, X_j) \leq \sum_{j=1}^n d_i(X_i, X_j)$.

Proof from the definition of centre of mass.

Lemma 3 For some i, $\sum_{j=1}^n d_i(X_i, X_j) \geq \frac{1}{2}S/\sin\pi/n$.

Proof Use reductio ad absurdum. Suppose the result untrue.

$$\sum_{j=1}^n d_i(X_i, X_j) < \tfrac{1}{2}S/\sin\pi/n \quad \text{for all } i.$$

By Lemma 2 it follows that $d(X_i, Q) < S/(2n\sin\pi/n) = R$ for all i. This means that the polygon is strictly inside a circle with centre Q and radius R, and is therefore inside a circle of some radius $R' < R$. By Lemma 1 the polygon must have circumference $\leq 2R'n\sin\pi/n = SR'/R < S$, this is the required contradiction.

Theorem For some i, $\sum_{j=1}^n d(X_i, X_j) > \frac{1}{2}S/\sin\pi/n$.

Proof This comes from Lemma 3 and the inequality (1). The inequality becomes strict because there is at least one vertex X_j with the side X_iX_j not parallel to X_iQ.

In the notation of the original problem the result proved is that $f(C_n) > S(C_n)/(2\sin\pi/n)$. In consequence $f(C_n) > S(C_n)$ for all $n \geq 6$, which answers one of the questions.

CONGRATULATIONS

Terry Tao won a gold medal at the July 1988 International Mathematical Olympiad in Canberra.

THREE SQUARES IN ARITHMETIC PROGRESSION (JCMN 46, p.5098)

J. B. Parker

If $a < b < c$ are coprime positive integers and $a^2 + c^2 = 2b^2$, then what are the possible values of a? Firstly a and c are both odd. Therefore $a^2 + c^2 \equiv 2 \bmod 4$, and so b must also be odd. Put $u = \frac{1}{2}c - \frac{1}{2}a$ and $v = b - \frac{1}{2}a - \frac{1}{2}c$.

$$2a^2 + 2c^2 = 4b^2 = (a+c)^2 + 4v(a+c) + 4v^2$$

$$u^2 = v(a+c+v)$$

$$c+a = u^2/v - v \quad \text{and} \quad c-a = 2u$$

$$a = \tfrac{1}{2}u^2/v - \tfrac{1}{2}v - u$$

$$b = \tfrac{1}{2}u^2/v + \tfrac{1}{2}v$$

$$c = \tfrac{1}{2}u^2/v - \tfrac{1}{2}v + u$$

Since $u+v = b-a$ is even, u and v are of the same parity. Also $u > v(1+\sqrt{2})$ to make a positive.

Case 1, u and v both odd. Case 1a with $v = 1$ is simple, put $u = 2k+1$, then $a = 2k^2-1$, giving the values 1, 7, 17, 31, ...

Case 1b with odd u and odd $v \geq 3$. Clearly v must divide u^2. Suppose that v has the prime factor p to the power $s \geq 1$. Then u must have the factor p to a power $r \geq \frac{1}{2}s$. If $2r > s$ then a, b and c would all have the factor p, impossible. Therefore $2r = s$; this is for all p, so that v must be a square. Put $v = n^2$ and $u = mn$.

$$a = \tfrac{1}{2}m^2 - \tfrac{1}{2}n^2 - mn$$

$$b = \tfrac{1}{2}m^2 + \tfrac{1}{2}n^2$$

$$c = \tfrac{1}{2}m^2 - \tfrac{1}{2}n^2 + mn$$

where m and n are odd coprime positive integers with $m/n > 1+\sqrt{2}$. Put $m = 2k + n$. The solutions are

$$a = 2k^2 - n^2$$

$$b = 2k^2 + 2kn + n^2$$

$$c = 2k^2 + 4kn - n^2$$

for coprime n and k with n odd. Values of $a = 2k^2 - n^2$ are given below, the values for n = 1 cover Case 1a.

n \ k	1	2	3	4	5	6	7	8
1	1	7	17	31	49	71	97	127
3	-7	1		23	41		89	119
5	-23	-17	-7	7		47	73	103
7	-97	-41	-31	-17	-1	23		79

Case 2 Where u and v are both even put $u = 2x$ and $v = 2y$.

$$a = x^2/y - y - 2x$$

$$b = x^2/y + y$$

$$c = x^2/y - y + 2x$$

Reasoning about prime factors as above shows that we may put $x = mn$ and $y = n^2$. Then $a = 2m^2 - (m+n)^2$, so that this case gives us no new values of a.

Finally, are there, for each a, infinitely many ways of choosing b and c? Note that $2b^2 - c^2 = 2(3b+2c)^2 - (3c+4b)^2$. With $b_0 = b$ and $c_0 = c$, put $b_{n+1} = 3b_n + 2c_n$ and $c_{n+1} = 3c_n + 4b_n$. Then a, b_n, and c_n have their squares in arithmetic progression for all n = 1, 2, 3, ... Also the b_n are all unequal because they form an increasing sequence.

BINOMIAL IDENTITY 24

Marta Sved

It was a day in ancient times
A peaceful day of summer, hot,
When trumpets, bugles, bells and chimes
Summoned the knights of Camelot.

News came to the lofty wall,
News of wild marauding strangers.
Here was the need, here was the call,
Here the challenge, here the dangers.

— Forward — shouted every knight,
Overwhelming was their zeal
To meet the enemy, to fight.
But from the King came the appeal

— A fighting corps will be selected,
Our knightage cannot spare your lot,
A leader then shall be selected.
The rest shall stay in Camelot. —

Who should go and who should stay?
Who can count up all the choices?
Everybody had a say,
A medley of conflicting voices.

How to choose the one to lead?
All knights had equality!
But who were those who had indeed
That more equal quality?

The younger set was clamouring:
— We are stronger, braver, bolder! —
— Wisdom counts more, — ruled the King
— Your leader must come from the older. —

Merlin had his word at last
To still the altercation,
And for the future from the past
Presented this equation:

$$\sum_{j=1}^{m} \binom{n+j-1}{r} = \sum_{j=1}^{m} \binom{n}{r+1-j}\binom{m}{j}$$

So here the formulae appear,
Just as the old sage divined.
Can we of this age make it clear
What Merlin had in mind?

THE KNIGHTS OF THE ROUND TABLE

The n knights were all gathered at Camelot for the feast of Pentecost, which was celebrated with great solemnity every year. King Arthur was discussing with Merlin plans for the procession from the Castle to the Church. This year there was a difficulty, there was some repair work being done and consequently the stalls in which the knights usually sat would not all be available. "Suppose that only m stalls can be used," said Merlin "then the first m in the procession must go there, and the rest must sit in the nave with the people of the village." "Then, " answered the King "perhaps we should have the procession in order of seniority, with the oldest knights in front." "But we have the Round Table to remind us that all are of equal status." objected Merlin. The King thought for a bit and then came to his decision. "We must arrange the procession so that, whatever m may be, the first m knights will have their average age no less than that of the other $n-m$. There must surely be several ways of doing that." "Yes, sire," answered Merlin "there are certainly (n-1)! ways, and possibly even more." "That is hard to believe," said King Arthur "but I have learnt to trust you in such matters. What should be done?"

"Choose one of the squires good at arithmetic." answered Merlin "Tell him first to calculate the average age A of all the knights. When the knights are all sitting at the Round Table, order a waxed tablet to be put in front of each one. The squire must start at one place, add that knight's age to 100 years, subtract A, and then write the result on the tablet. Then he must walk clockwise round the table and at each place

he must add the number on the previous tablet to the knight's age, subtract A and write the result on the tablet. When all this is done you must call on the knight with the highest number in front of him to lead the procession, he is to be followed by the neighbour on his right, and so on in the order in which they are sitting round the table, so that the knight on the left of the chosen one will bring up the rear of the procession." "Excellent," said the King "and now I see why there are so many possible outcomes, for we all know that there are (n-1)! ways in which the knights can sit at a round table. If two knights both have the maximum total then either may lead the procession, and there would be more than (n-1)! ways. But, tell me Merlin, why did you want the squire to start with 100 years before doing all his additions and subtractions? Surely it can make no difference to the result." "That is a matter of history, your Majesty." answered Merlin "We cannot use negative numbers, for it will be hundreds of years before they are invented. And I remember another bit of history, this combinatorial problem that we have solved being raised in JCMN 46, page 5103, by Ross Talent in AD 1988."

BINOMIAL IDENTITY 25

A. Brown

$$\sum_{r=0}^{2m+j} (-1)^r \binom{4m+p}{2r} / (2r+1) = (-4)^m k / (4m+p+1)$$

where for any p = 0, 1, 2 or 3 the numbers j and k are as follows

p	0	1	2	3
j	0	0	1	1
k	1	2	2	0

BINOMIAL IDENTITY 23 (JCMN 46, p.5102)

Cecil Rousseau

$$\sum_{q=0}^{[p/2]} (-1)^q \binom{n}{q}\binom{2n-2q}{p-2q} = 2^p\binom{n}{p}$$

Proof. The 2n knights of King Arthur are in n pairs and the king wants to choose p knights so that no two knights of a pair are both chosen. To compute the number of ways this can be done, the king uses one of his favorite methods, inclusion-exclusion. He thus finds that there are

$$\sum_q (-1)^q\binom{n}{q}\binom{2n-2q}{p-2q}$$

choices. But then he realizes that he might as well choose p of the pairs of knights and then choose one from each of these pairs. He concludes that the result above holds.

Another one of the king's favorite results is the binomial theorem. To compute the coefficient of x^{2n-p} in the polynomial $((x+1)^2 - 1)^n$, he uses the binomial theorem to write

$$((x+1)^2 - 1)^n = \sum_q (-1)^q\binom{n}{q}(x+1)^{2n-2q},$$

from which he concludes that the desired coefficient is

$$\sum_q (-1)^q\binom{n}{q}\binom{2n-2q}{2n-p} = \sum_q (-1)^q\binom{n}{q}\binom{2n-2q}{p-2q}.$$

On the other hand, the polynomial is just $x^n(x+2)^n$, and it is clear that the coefficient of x^{2n-p} is $2^p\binom{n}{p}$. Having found that both inclusion-exclusion and the binomial theorem lead to the conclusion that

$$\sum_{q=0}^{[p/2]} (-1)^q\binom{n}{q}\binom{2n-2q}{p-2q} = 2^p\binom{n}{p},$$

the king is prepared to give this result his royal seal of approval.

SEQUENCES WITHOUT ARITHMETIC PROGRESSIONS

George Szekeres

In JCMN 46 "Extrapolation" p.5107, Basil Rennie constructs a sequence $(a_1, a_2, \ldots) = (1, 2, 4, 5, \ldots)$ which contains no 3-term arithmetic progression, and asks whether his sequence has the largest possible sum $\sum 1/a_n$. It is easy to check that the sequence $a_1-1, a_2-1, \ldots$ consists of precisely those integers (including 0) whose ternary representation does not contain the digit 2. Quite generally one can see without much trouble that the sequence of integers whose (prime) base p representation has no digit (p-1), contains no p-term arithmetic progression.

How are these sequences obtained? Start with 0 and keep all integers that do not form an arithmetic progression of length p with the integers already chosen. This is what is known as the greedy algorithm: you grab an integer at the first available opportunity if it does not conflict with the condition stated. A trivial example is the following: suppose for given $N > 0$ we want to find the largest possible number of positive integers $n_i \leqslant N$ so that n_i and n_j are relatively prime for all $i \neq j$. Starting with 1, 2, 3, we have to discard 4 since it is divisible by 2, so the next term is 5, etc. Clearly what we get are just the primes $\leqslant N$, and it is easy to see that this is the best possible such sequence, as follows. For any other sequence a_i satisfying the condition just replace a_i by its smallest prime factor p_i (the p_i must be distinct if $(a_i, a_j) = 1$ for $i \neq j$) and the set $\{p_i\}$ is a subset of the set of all primes $\leqslant N$.

The greedy algorithm often succeeds but not always, and the problem of 3-term arithmetic progressions is an intriguing example to illustrate just that. What is the size of the longest sequence of distinct integers up to N that contains no 3-term arithmetic progression? The problem has a long history, going back to a 1936 paper of Erdös and Turán in the Journal of the London Mathematical Society; it even found its way into the International Mathematical Olympiad as Problem 5 in 1983 in Paris. The obvious conjecture that the greedy algorithm gives the best answer was disproved by Salem and Spencer in 1942 (Proc. Nat. Acad. Sci. (USA) 28, pp.561-563). Surprisingly, not even the order of magnitude given by the greedy algorithm (roughly $N^{\log 2/\log 3}$ for numbers up to N) is correct. In fact Salem and Spencer's construction gives more than $N^{1-a/\log\log N}$ elements for a suitable positive a, which (for large N) is more than any N^b for a fixed $b < 1$. The construction is a clever modification of the greedy algorithm. Take an integer $d > 2$ and a multiple $n = kd$. The following n-digit integers in base 2d-1,

$$a_1 + a_2(2d-1) + a_3(2d-1)^2 + \dots + a_n(2d-1)^{n-1},$$

with exactly k of the digits $a_i = 0$, k of the digits $a_i = 1$, ... and k of the digits $a_i = d-1$, can be shown to contain no 3-term arithmetic progression. Now these numbers are all $< N = (2d-1)^n$, and there are $n!/(k!)^d$ of them. The estimate of Salem and Spencer is obtained by judiciously choosing k for a given large d ($k \simeq \log^2 d$ will do). They also state (though without proof) that by modifying the construction one can obtain an infinite (N-independent) sequence with no 3-term arithmetic progression and with density $N^{-a/\log\log N}$.

Although the Salem-Spencer sequence contains (for large N) more terms than the greedy sequence, it does not at all follow that the corresponding $\sum 1/a_i$ is also greater. On the contrary,

it seems likely, and it would be interesting to prove, that the greedy solution is indeed the best for Basil's problem.

FROM CAPTAIN COOK'S JOURNAL

Sunday, 15th. October 1769

At 8 a.m. being abreast of the S.W. point of the Bay, some fishing Boats came off to us and sold us some stinking fish; however it was such as they had, and we were glad to enter into Traffick with them upon any Terms. These People behaved at first very well, until a large Arm'd boat, wherein were 22 Men, came alongside. We soon saw that this Boat had nothing for Traffick, yet as they came boldly alongside we gave them 2 or 3 pieces of Cloth, Articles they seem'd the most fond off. One Man in this Boat had on him a black skin, something like a Bear Skin, which I was desirous of having that I might be a better judge what sort of Animal the first Owner was. I offered him for it a piece of Red Cloth, which he seem'd to jump at by immediately putting off the Skin and holding it up to us, but would not part with it until he had the Cloth in his possession and after that not at all, but put off the Boat and went away, and with them all the rest. But in a very short time they return'd again, and one of the fishing Boats came alongside and offer'd us some more fish. The Indian Boy Tiata, Tupia's Servant, being over the side, they seiz'd hold of him, pull'd him into the Boat and endeavoured to carry him off; this obliged us to fire upon them, which gave the Boy an opportunity to jump overboard. We brought the Ship too, lower'd a Boat into the Water, and took him up unhurt. Two or 3 paid for this daring attempt with the loss of their lives, and many more would have suffer'd had it not been for fear of killing the Boy. This affair occasioned my giving this point of land the name of Cape Kidnapper. It is remarkable on account of 2 White rocks in form of Haystacks standing very near it. On each side of the Cape are Tolerable high white steep Cliffs, Latitude 39^{o} 43' S.; Longitude 182^{o} 24' W.

BINOMIAL IDENTITY 22 (JCMN 45 p.5081)

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \ldots \\ 1 & 1 & 0 & 0 & 0 & \ldots \\ 1 & 2 & 1 & 0 & 0 & \ldots \\ 1 & 3 & 3 & 1 & 0 & \ldots \\ 1 & 4 & 6 & 4 & 1 & \ldots \\ \ldots & & & & & \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \ldots \\ -1 & 1 & 0 & 0 & 0 & \ldots \\ 1 & -2 & 1 & 0 & 0 & \ldots \\ -1 & 3 & -3 & 1 & 0 & \ldots \\ 1 & -4 & 6 & -4 & 1 & \ldots \\ \ldots & & & & & \end{pmatrix}$$

Are these matrices inverses of one another? The m, n element of the product is $\sum(-1)^{n-j}\binom{m-1}{j-1}\binom{j-1}{n-1}$ where the summation is over all j for which the binomial coefficients are non-zero. Clearly it is 0 if $m < n$ and is 1 if $m = n$. If $m > n$ then use the identity $\binom{a}{b}\binom{b}{c} = \binom{a}{c}\binom{a-c}{b-c}$. This gives $\binom{m-1}{n-1}\sum(-1)^{n-j}\binom{m-n}{j-n} = 0$.

MONTE CARLO INTEGRATION

(JCMN 46, pp. 5104-5106)

In our last issue were some numerical results about

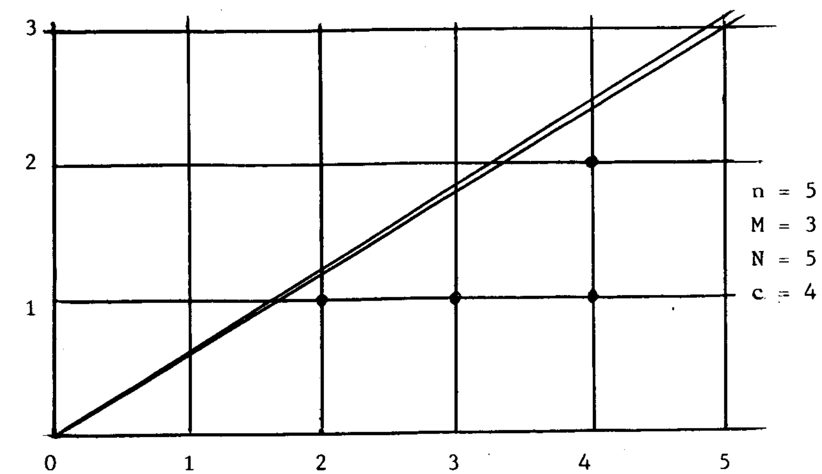
$$S(N) = \sum_{r=1}^{N}\left(f(\{rg\}) - \int_0^1 f(x)dx\right)$$

where $g = \frac{1}{2}\sqrt{5} - \frac{1}{2}$. This S(N) is N times the error in a Monte Carlo estimate for the integral. Since then some more information has emerged. Gerry Myerson has pointed out that the conjecture of S(N) being bounded is wrong. Define the sequence:

n	4	5	6	7	8	9	10
H_n	3	5	9	15	24	39	64

with $H_n = H_{n-4} + F(n)$. Then for $f(x) = x$, $S(H_n)$ appears (by convincing numerical evidence) to be $(-1)^n n/20 + O(1)$. For the case where N is a Fibonacci number, however, several of the results suggested by the numerical work can now be proved.

1 Exact result for f(x) = x



Let M = F(n-1) and N = F(n) be two consecutive Fibonacci numbers and put $x_r = \{rg\}$. In the x-y plane consider two right-angled triangles, the first where $0 < x < N$ and $0 < y < gx$, and the second where $0 < x < N$ and $0 < y < xM/N$. The first is bigger or smaller than the second according to whether n is odd or even. Let c be the number of lattice points inside the second triangle. Since there are $(N-1)(M-1)$ in the rectangle where $0 < x < N$ and $0 < y < M$, and there are none on the diagonal, $c = \frac{1}{2}(N-1)(M-1)$, as may also be calculated by Pick's Theorem. The first triangle also has c lattice points inside because M/N is a best rational approximation to g.

For any $r = 1, 2, \ldots N-1$ the number of lattice points on the line $x = r$ inside the first triangle is $rg - x_r$, and so

$$c = \sum_{r=1}^{N-1}(rg - x_r) = \tfrac{1}{2}N(N-1)g - \sum_{r=1}^{N-1} x_r$$

$$\sum_{r=1}^{N-1} x_r = \tfrac{1}{2}N(N-1)g - c = \tfrac{1}{2}(N-1)(Ng-M+1) = \tfrac{1}{2}(N-1)(1-(-g)^n)$$

$$x_N = \{Ng\} = \{M-(-g)^n\} = \tfrac{1}{2} + (-1)^n(\tfrac{1}{2}-g^n)$$

$$S(N) = (-1)^n(\tfrac{1}{2} - g^n - \tfrac{1}{2}(N-1)g^n) = \tfrac{1}{2}(-1)^n(1-(N+1)g^n)$$

and because $\sqrt{5}\,N g^n = 1 - (-1)^n g^{2n}$,

$$S(N) = (-1)^n(2-g)/5 - \tfrac{1}{2}(-g)^n + \tfrac{1}{2}g^{2n}/\sqrt{5}.$$

2 Exact result for $f(x) = 6x(1-x)$

Notation: $g^* = M/N$, $x_r^* = \{rg^*\}$, $N = F(n)$, $M = F(n-1)$

Lemma 1 $\sum_{r=1}^{N-1} f(x_r^*) = \sum_{r=1}^{N-1} f(r/N) = N - 1/N.$

Proof: The numbers M and N are coprime, and so the x_r^* are a permutation of the r/N.

Lemma 2 $\sum_{r=1}^{N-1} r\,x_r^* = N(N-1)/4 - (1+(-1)^n)(N-M)/12.$

This can be proved by induction, using the results for the two preceding Fibonacci numbers, we spare our readers the details (and spare ourselves the typing of them).

Lemma 3 $\sum_{r=1}^{N-1} f(x_r^*) - f(x_r) = g^{2n}(2-1/N)(N-1) + (1+(-1)^n)g^n(1-M/N)$

The proof is simple after noting that $x_r^* - x_r = r(g^*-g) = r(-g)^n/N$.

Gathering together the results we find

$$S(N) = 6g^n - 1/N - g^{2n}(2+1/N)(N+1) - (1+(-1)^n)g^n(1-M/N).$$

All four terms are of order 1/N.

3 Extension to more general functions

Under certain conditions it is possible to prove:-

$$S(F(n)) \sim (-1)^n(f(1)-f(0))(2-g)/5.$$

Below is outlined a method using Fourier series. Another method has been suggested by Gerry Myerson.

Lemma 3 If $r \geq 2$ is an integer then $|\sin \pi rg| > 1/r$.

Proof We use the algebraic property of the Golden Ratio. Let $rg = k+x$ where k is an integer and $|x| < \frac{1}{2}$.

$r^2 = rg(r + rg) = (k + x)(r + k + x) = k(r + k) + x(2k + r + x)$ and so $1/|x| \leq 2k + r + x < 3r$, and the result follows.

Lemma 4 If $|u(m, n)| < a(n)$ where $\sum a(n)$ converges, and if $u(m, n) \to b(n)$ as $m \to \infty$, then $\sum_{n=1}^{\infty} u(m, n) \to \sum_{n=1}^{\infty} b(n)$.

This is a useful classical result which can be regarded as a corollary to Weierstrass's M-test or as a corollary to Lebesgue's theorem on dominated convergence. Does it have a name?

Theorem Let f(x) have third derivative continuous on the closed unit interval. Then $(-1)^n S(F(n)) \to (f(1)-f(0))(2-g)/5$

Proof On the closed unit interval define

$$g(x) = f(x) - A - Bx - Cx(1-x)$$

where A, B and C are such that $g(1)-g(0) = g'(1)-g'(0) = 0$ and $\int_0^1 g(x)\,dx = 0$. Now extend $g(x)$ for all real x to have unit period. It will have a Fourier expansion

$$g(x) = \sum_{r=1}^{\infty} a_r \cos 2\pi rx + b_r \sin 2\pi rx$$

where the coefficients a_r and b_r are both $O(r^{-3})$.
Because $\sum_{m=1}^{N} \cos 2\pi rmg = \cos(N+1)\pi rg \sin N\pi rg \operatorname{cosec} \pi rg$
and $\sum_{m=1}^{N} \sin 2\pi rmg = \sin(N+1)\pi rg \sin N\pi rg \operatorname{cosec} \pi rg$
we may put (for the function g) $S(N) = \sum_{m=1}^{N} g(x_m)$
$= \sum_{r=1}^{\infty} (a_r \cos(N+1)\pi rg + b_r \sin(N+1)\pi rg) \sin N\pi rg \operatorname{cosec} \pi rg$.
By Lemma 3 $(a_r \cos(N+1)\pi rg + b_r \sin(N+1)\pi rg) \operatorname{cosec} \pi rg = O(r^{-2})$ and so we may use Lemma 4 to give $S(N) \to 0$.

The S(N) for the function $f(x) = g(x) + A + Bx + Cx(1-x)$ is the sum of the contributions from the four functions on the right-hand side, and the only non-zero contribution to $\lim S(F(n))$ is from the term Bx. As $B = f(1)-f(0)$ the required result follows.

4 Use of extrapolation

For a calculation of $\int_0^1 f(x)dx$ we can use the fact that $S(F(n)) = (-1)^n(f(1)-f(0))(2-g)/5 + o(1)$ where the o(1) can with some extra work be improved to $O(1/F(n))$. We can therefore estimate the integral as (putting N for F(n)):

$$\frac{1}{N}\sum_{m-1}^{N} f(x_m) - (-1)^n \frac{2-g}{5N} (f(1)-f(0))$$

with error $O(1/N^2)$, the same order of magnitude as with the trapezoidal rule for numerical integration.

With a little imagination it is possible to do better. Recall that $N = F(n) = (g^{-n} - (-g)^n)/\sqrt{5}$, then looking at S(N) in the two cases for which we have an exact expression, it is

plausible that for any well-behaved function on [0, 1], S(N) has an asymptotic expansion

$$(-1)^n \frac{2-g}{5} (f(1)-f(0)) + Ag^n + (-1)^n Bg^n + Cg^{2n} + (-1)^n Dg^{2n} + \ldots$$

To eliminate the $(-1)^n$ factors we shall from now on consider the odd-numbered and even-numbered Fibonacci numbers separately.

The raw result of our Monte Carlo calculation is

$$J(N) = \frac{1}{N}\sum_{m=1}^{N} f(x_m) = \int_0^1 f(x)\,dx + S(N)/N$$

If our guess about the asymptotic form is correct then $J(F(2n))$ and $J(F(2n-1))$ each has an asymptotic expansion of the form

$$A + Bg^{2n} + Cg^{4n} + Dg^{6n} + \ldots$$

where A is the integral that we want.

Lemma If $K(n) \sim A + Br^n + Cr^{2n} + Dr^{3n} + \ldots$ then the asymptotic estimates for A from successive values of K(n) are

First order: $A = (K(n) - rK(n-1))/(1-r) + O(r^{2n})$

Second order: $A = \dfrac{K(n)-(r+r^2)K(n-1)+r^3K(n-2)}{(1-r)(1-r^2)} + O(r^{3n})$

Third: $A = \dfrac{K(n)-(r+r^2+r^3)K(n-1)+(r^3+r^4+r^5)K(n-2)-r^6K(n-3)}{(1-r)(1-r^2)(1-r^3)} + O(r^{4n})$

Proof Let E be the operator defined by $EK(n)=K(n+1)$, For the first order estimate operate with E-r, which annihilates the Br^n term, giving $K(n+1)-rK(n) = (1-r)A + O(r^{2n})$. For the second order estimate use the operator $(E-r)(E-r^2)$ which annihilates both Br^n and Cr^{2n}. The general case should now be clear.

In our application of this extrapolation formula we have $r = g^2$, and the coefficients simplify as follows:

First order: $A = K(n)(1+g)-K(n-1)g+O(g^{4n})$

Second order: $A = K(n)(7+4g)/5-K(n-1)+K(n-2)(3-4g)/5 + O(g^{6n})$

Third: $A = \frac{29+18g}{20}K(n)-\frac{4+3g}{5}K(n-1)+\frac{3g-1}{5}K(n-2)-\frac{18g-11}{20}K(n-3)+O(g^{8n})$

From the computational point of view the assumption about the asymptotic formula for J(N) is a safe one, because if it is wrong the fact will show in the extrapolation calculation.

As an example, let us calculate π by numerical integration of $4/(1+x^2)$ in the unit interval. In the table below is shown (after the Fibonacci number N which is the number of points at which the function is calculated) firstly the sum J(N), then the same corrected by $(-1)^n(f(1)-f(0))\frac{2-g}{5N}$, and finally the third order estimate derived from four successive values of J(N), as explained above. The first part of the table uses the even-numbered Fibonacci numbers, and the second the odd-numbered.

n	N			
2	1	2·894427190999916	3·447213595499958	
4	3	2·998700629934308	3·182962764767655	
6	8	3·078551854173998	3·147650154736503	
8	21	3·116158109036674	3·142481271155724	3·141505660786111
10	55	3·131671985824063	3·141722647724063	3·141590437350810
12	144	3·137772845962427	3·141611640438122	3·141592601278958
14	377	3·140129148001813	3·141595424936827	3·141592652425868
16	987	3·141032990712550	3·141593057991678	3·141592653564580
18	2584	3·141378785967902	3·141592712595031	3·141592653589258
20	6765	3·141510949500386	3·141592662198760	3·141592653589794

n	N			
1	1	2·894427190999916	2·341640786499874	
3	2	3·341640786499874	3·065247584249853	
5	5	3·246944410674675	3·136387129774666	
7	13	3·183736088550812	3·141214057435424	3·138506403672553
9	34	3·157817548166064	3·141559124504298	3·141550951015649
11	89	3·147800059416273	3·141588976219643	3·141592258136363
13	233	3·143964658642356	3·141592184803300	3·141592649767558
15	610	3·142498796190776	3·141592588970284	3·141592653546714
17	1597	3·141938784888633	3·141592644372352	3·141592653589192
19	4181	3·141724866177909	3·141592652256718	3·141592653589792

A more refined method is to take advantage of the fact that we know f(1)-f(0), and therefore know the first term in the asymptotic expansion of J(N) - integral. For a sequence $K(m) = A + Cg^{4m} + Dg^{6m} + Eg^{8m} + \ldots$ the third order asymptotic estimate for A is

$$A = (g^{-9}K(m) - 4g^{-3}K(m-1) + 4g^{3}K(m-2) - g^{9}K(m-3))/60$$

Below are the estimates obtained in this way.

n	N		n	N	
8	21	3·141591831597904	7	13	3·141556968238019
10	55	3·141592653015055	9	34	3·141591908943207
12	144	3·141592653602078	11	89	3·141592646616433
14	377	3·141592653589946	13	233	3·141592653531655
16	987	3·141592653589796	15	610	3·141592653589318
18	2584	3·141592653589797	17	1597	3·141592653589792

ANALYTIC GEOMETRY FOR SPHERICAL TRIANGLES

Starting with the unit sphere, we modify it by identifying pairs of opposite points. This gives what is called the "elliptic plane", which geometrically is something like a hemisphere with half its boundary, or something like the real projective plane, or something like the Euclidean plane with a line at infinity added. For a fuller discussion see H.S.M.Coxeter's "Introduction to Geometry" §6.9, page 92. The points of the space can be regarded as the straight lines through a fixed point, the centre of the sphere. A point of the sphere may be represented by the unit vector from the centre, so that a point in our geometry may be represented by a unit vector, or any positive or negative multiple of it. A great circle on the sphere, which may be called a line in our space, can also be represented by a vector or any positive or negative multiple of it, namely a vector representing the two poles of the great circle, a vector normal to the plane of the great circle.

Take a fixed non-degenerate spherical triangle on the unit sphere, let $\underline{a}$, $\underline{b}$ and $\underline{c}$ be the vectors to the vertices A, B and C from the centre of the sphere. Any point $\underline{u}$ of the space may be denoted by the family of three coordinates $x = \underline{u}.\underline{a}$, $y = \underline{u}.\underline{b}$ and $z = \underline{u}.\underline{c}$. We treat (x, y, z) as homogeneous coordinates, so that (tx, ty, tz) for any $t \neq 0$ denotes the same point. For example, the circumcentre of ABC has the coordinates (1, 1, 1).

<u>Theorem 1</u> Any linear homogeneous equation in x, y and z (with coefficients not all zero) represents a great circle, and vice versa.

Proof Since $\underline{a}$, $\underline{b}$ and $\underline{c}$ are linearly independent, any point P

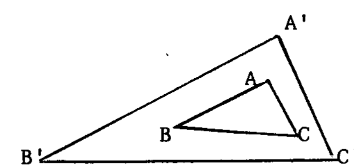
of the 3-dimensional space has position vector $l\underline{a} + m\underline{b} + n\underline{c}$. Any point $\underline{u}$ is on the great circle that has P as pole if and only if $\underline{u}.(l\underline{a} + m\underline{b} + n\underline{c}) = 0$, that is $lx + my + nz = 0$. Q.E.D.

Now we introduce the parameters $p = \underline{b}.\underline{c}$, $q = \underline{c}.\underline{a}$ and $r = \underline{a}.\underline{b}$. They are the cosines of the sides of ABC.

The mid-point of the side BC is represented by the vector $\underline{b} + \underline{c}$, etc. so that the centroid (the intersection of the medians) is represented by the vector $\underline{a} + \underline{b} + \underline{c}$, and has the coordinates (1+q+r, p+1+r, p+q+1).

Convexity theory tells us that a non-degenerate spherical triangle ABC may be enclosed in an open hemisphere in such a way that the mid-point of the hemisphere is inside the triangle (any closed convex set without a pair of diametrically opposite points has this property). The poles of the side BC are the points with position vectors $\pm \underline{b}\times\underline{c} / ||\underline{b}\times\underline{c}||$, and one of these two will be in our open hemisphere, the other not.

Starting with ABC on the sphere we construct the polar triangle A'B'C' by taking A' to be the pole of BC that is on the same side as A, and similarly B' and C'. Both ABC and A'B'C' are in our chosen open hemisphere. When we draw a picture



we may think of it as representing either the elliptic plane or the open hemisphere. A note on drawing — we often draw a spherical triangle with curved sides, to remind the reader that it is not a plane triangle, but it is legitimate to draw it with straight sides because the open hemisphere can be mapped on the Euclidean plane (projecting from the centre) so that great circles become straight lines.

The relation between a spherical triangle and its polar triangle is mutual, A is the pole of B'C' because AB' and AC' are both right angles.

Now to find the orthocentre of ABC. Because every great circle through A' is perpendicular to BC, the altitude from A to BC is the great circle AA'. Since the vectors for A and A' are $\underline{a}$ and $\underline{b} \times \underline{c}$ respectively, the vector for AA' is $\underline{a} \times (\underline{b} \times \underline{c}) = \underline{b}(\underline{a}.\underline{c}) - \underline{c}(\underline{a}.\underline{b}) = q\underline{b} - r\underline{c}$, so that the equation for the altitude is $qy - rz = 0$. The other two altitudes are $rz - px = 0$ and $px - qy = 0$. They all meet at the point with coordinates (qr, rp, pq). It may be remarked that a triangle with two sides of 90° (and therefore also two angles of 90°), although quite a respectable member of the community of spherical triangles in most ways, does not have an orthocentre; if the angles at B and C are right angles then every point on BC has the properties of the orthocentre.

Since AA' is perpendicular to B'C', etc. the two triangles ABC and A'B'C' have the same orthocentre, which is also their centre of perspective.

In a plane triangle the circumcentre, orthocentre and centroid are collinear, but in a spherical triangle the three points are not always on a great circle. Instead we have

<u>Theorem 2</u> If the circumcentre, centroid and orthocentre of a spherical triangle are on a great circle then the triangle is isosceles.

Proof The circumcentre is (1, 1, 1).

The centroid is $(q+r, r+p, p+q)$.

The orthocentre is (qr, rp, pq).

If they are on a great circle then

$$\begin{vmatrix} 1 & 1 & 1 \\ q+r & r+p & p+q \\ qr & rp & pq \end{vmatrix} = 0$$

This determinant is one of those that one gives to first year students to simplify. With a little care or a lot of work they find it to equal $(r-q)(p-r)(q-p)$; if it is zero then the triangle is isosceles. QED

The exceptional triangles in which two sides are right angles can be regarded as coming within the scope of the theorem for the circumcentre and centroid are on a great circle with one of the possible orthocentres.

This theorem solves the problem in "Isosceles spherical triangles" JCMN 46, p.5098.

OLD GEOMETRY QUESTION

From the First Year Problems paper of Clare College, Gonville and Caius College, Trinity Hall and King's College. Wednesday June 4, 1902. 9-12

Q1. A straight line ABCD cuts two fixed circles X and Y so that the chord AB of X is equal to the chord CD of Y. The tangents to X at A and B meet the tangents to Y at C and D in four points P, Q, R, S. Show that P, Q, R, S. lie on a fixed circle.