On the Shifted Convolution Problem

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May 5, 2003

1. The Shifted convolution Problem (SCP)

Given $g(z)$ a primitive modular form of some level D and nebentypus χ_D, and Hecke-eigenvalues $\lambda_g(n)$ the Shifted convolution problem consists in estimating non trivially the following kind of sums: for $h \neq 0$,

$$\Sigma(g, l_1, l_2, h) := \sum_{l_1 m - l_2 n = h} \overline{\lambda_g}(m)\lambda_g(n)V(m, n)$$

where $l_1, l_2 \geqslant 1$ and (for simplicity) V is smooth compactly supported in $[M, 2M] \times [N, 2N]$ (for application one needs l_1, l_2 to be as large as (very) small power of $M + N$). The trivial bound for this sum is

$$\Sigma(g, l_1, l_2, h) \ll_\varepsilon (MN)^\varepsilon \max(M, N).$$

In many applications M and N are about the same size:
The Shifted Convolution Problem (SCP): Find $\delta > 0$ s.t.

$$\Sigma(g, l_1, l_2, h) = \text{Main Term}(h) + O(M^{1-\delta}),$$

here Main Term(h) is a main term (non-zero only if g is an eisenstein series) and the remaining terms is an error term.

Remark. If $h = 0$ this sum is essentially a Rankin/Selberg type partial sum and can be evaluate by standard (contour shift methods) from the analytic properties of $L(g \times \tilde{g}, s)$.

The interesting case is when $h \neq 0$. The problem goes back to Ingham with

$$g(z) = \frac{\partial}{\partial s} E(z,s)_{|s=1/2} = 2y^{1/2}\log(e^{\gamma}y/4\pi) + 4y^{1/2}\sum_{n\geqslant 1}\tau(n)\cos(2\pi nx)K_0(2\pi ny)$$

with $\tau(n) = \sum_{d|n} 1$ is the divisor function. In that case the SCP is know as the **Shifted Divisor Problem**.

Ingham (1927):

$$\sum_{n\leqslant N}\tau(n)\tau(n+h) \sim Cst(h)N\log^2 N,\ \ N \to +\infty$$

Remark. If $h = 0$,

$$\sum_{n\leqslant N}\tau(n)^2 \sim CstNP_3(\log N) + O(N^{1/2}),\ \ N \to +\infty$$

P_3 a polynomial of degree 3.

Estermann (1931): Using Kloostermann's non trivial bound of Kloostermann's sums

$$\sum_{n\leqslant N}\tau(n)\tau(n+h) \sim P_{2,h}(\log N)N + O(N^{1-1/12+\varepsilon}),\ \ N \to +\infty$$

where $P_{2,h}$ is a polynomial of degree 2. Weil's bound allows to improve the error term substantially and stronger results have been obtained by Motohashi, Jutila, Tatakjan/Vinogradov using automorphic forms.

2. The Subconvexity Problem (ScP)

The **SCP** is a rather technical looking problem but it has quite substantial applications.
Given $L(\pi, s)$ an automorphic representation, the L-function of π is given by the product of the (finite) local L-factors

$$L(\pi, s) = \prod_p L(\pi_p, s) = \sum_n \lambda_\pi(n) n^{-s},$$

and is completed by a factor at ∞

$$L_\infty(\pi, s) = \prod_{i=1}^{d} \Gamma(s - \mu_{i,\pi}),\ \Gamma_{\mathbf{R}}(s) = \pi^{-s/2}\Gamma(\frac{s}{2}).$$

$\Lambda(\pi, s) := L_\infty(\pi, s)L(\pi, s)$ has meromorphic continuation to $\mathbf{C}$ and satisfies the functional equation

$$\Lambda(\pi, s) = w(\pi) q_\pi^{(1-2s)/2} \Lambda(\tilde{\pi}, 1 - s)$$

here $|w_\pi| = 1$ and $q_\pi \geqslant 1$ is the arithmetic conductor. One defines the *analytic conductor* of π (Iwaniec/Sarnak) as the following function which plays the role of a normalizing factor and a way to measure the ”size” of π:

$$Q_\pi(s) = q_\pi \prod_{i=1}^{d} (1 + |s - \mu_{\pi,i}|).$$

The subconvexity problem address the question of the size of $L(\pi, s)$ when s is on the critical line. Unconditionally one has the following bound which is a consequence of the functional equation (and of the general theory of L-functions of pairs, plus an additional tricks of Iwaniec)
Convexity Bound: $\forall\varepsilon > 0$, and for $\Re es = 1/2$,

$$L(\pi, s) \ll_{\varepsilon} Q_{\pi}(s)^{1/4+\varepsilon}.$$

One expects that $1/4$ above can be replaced by 0 (GLH) and the **SCP** ask simply for any non-trivial improvement over the convexity bound.
Subconvexity Problem (ScP): Find $\delta > 0$ (absolute) such that

$$L(\pi, s) \ll Q_{\pi}(s)^{1/4-\delta}.$$

Remarkably, this problem is meaningful (cf. Sarnak lectures in april).

Many instances of the **ScP** are now solved for the case of GL_1, GL_2 and $GL_2 \times GL_2$ L-functions and ALL can be reduced to an instance of the **SCP** . Even, if some cases ad-hoc methods are more efficient, (ie. Burgess method for $L(\chi, s)$ for example) these are not available in general and reduction to the **SCP** via the method of families and moments is so far the most robust method.

3. Solving the SCP I: the δ-symbol method

We recall that we wish to evaluate

$$\Sigma(g, l_1, l_2, h) := \sum_{l_1 m - l_2 n = h} \overline{\lambda_g}(m)\lambda_g(n)V(m, n).$$

The main issues is to have a manageable expression for the condition

$$l_1 m - l_2 n - h = 0.$$

A possibility is to use additive characters:

$$\delta_{c|n} = \frac{1}{c}\sum_{a(c)} e(an/c)$$

and for $|n| < c$,

$$\delta_{c|n} = \delta_{n=0},$$

so has a priori a good expression, but the problem is that is requires characters of large conductor compared with the size of the variable, and this is not admissible.
Building on the fact that the set of divisors of an integer is symmetric

$$c|n \Leftrightarrow (n/c)|n \text{ so that either } c \text{ or } n/c \leqslant \sqrt{n},$$

Iwaniec has given an expression of $\delta_{n=0}$ in terms of additive characters of modulus $\ll \sqrt{n}$.

We consider for $R \geqslant 1/2$ and a smooth function ω such that

$$\omega \in C_c^\infty([-R,-R/2] \cup [R/2,R]),\ \text{(so that } \omega(0)=0),\ \omega \text{ even},\ \sum_{d\geqslant 1}\omega(d)=1$$

then

$$\delta_{n=0} = \sum_{d|n} \omega(d) - \omega(n/d)$$

and one detects the condition $d|n$ via additive characters, this leads to

$$\delta_{n=0} = \sum_{c\geqslant 1} \Delta_c(n) r(n;c)$$

where

$$\Delta_c(x) = \sum_{k\geqslant 1} \frac{1}{ck}\big(\omega(ck) - \omega(x/ck)\big) \text{ and } r(n;c) = \sum_{\substack{a(c)\\(a,c)=1}} e(an/c)$$

is the Ramanujan sum. Set $N = R^2/2$, one see that for $|n| \leqslant N$, $\Delta_c(n) = 0$ unless $c \leqslant R \ll \sqrt{N}$, thus one has an expression of $\delta_{n=0}$ in terms of additive character of much smaller modulus.

Duke/Friedlander/Iwaniec used the δ-symbol method to solve the **SCP** : from the expression for the δ-symbol the shifted convolution sum equals

$$\Sigma(g, l_1, l_2, h) = \sum_{c \ll (LM)^{1/2}} \sum_{\substack{a(c) \\ (a,c)=1}} e(\frac{-ah}{c})$$

$$\times \sum_m \overline{\lambda_g}(m) e(\frac{l_1 am}{c}) \sum_n e(\frac{-l_2 an}{c}) \lambda_g(n) V(m,n) \Delta_c(l_1 m - l_2 n - h)$$

Next the m and n sums are transformed by means of a Voronoi type summation formula; for example if g holomorphic of weight k, and for $(a,c) = 1$ and $D|c$

$$c \sum_{n \geqslant 1} \lambda_g(n) e(n\frac{a}{c}) W(n) = 2\pi i^k \overline{\chi}_g(a) \sum_{n \geqslant 1} \lambda_g(n) e(-n\frac{\overline{a}}{c}) \int_0^\infty W(x) J_{k-1}(\frac{4\pi\sqrt{nx}}{c}) dx;$$

If g has level one (so that the condition $D|c$ is always valid) an application of this formula to each variable transform $\Sigma(g, l_1, l_2, h)$ into a sum of the form

$$\sum_{c \ll (LM)^{1/2}} \sum_{m,n} \overline{\lambda_g}(m)\lambda_g(n)S(-l_1m + l_2n, -h; c)\widetilde{V}(m, n, c)$$

where $S(-l_1m + l_2n, -h; c)$ is the standard Kloostermann sum and $\widetilde{V}(m, n, c)$ is a double Bessel transform. In general (when g is an Eisenstein series) other terms occurs coming from the constant term. The resolution of the **SCP** comes from the analysis of $\widetilde{V}$ and from Weil's bound

$$S(-l_1m + l_2n, -h; c) \ll_\varepsilon (-l_1m + l_2n, -h, c)^{1/2}c^{1/2+\varepsilon}$$

Remark. Any bound for the Kloostermann sum with $1/2 + \varepsilon$ replaced by $\theta < 1$ would be sufficient to solve the **SCP** .

4. Application to the ScP

By means of the method of moments and the amplification method, Duke/Friedlander/Iwaniec reduced several instances of the **ScP** to the **SCP** above: this enabled them to prove

Theorem. *(Duke/Friedlander/Iwaniec) Fix g of level 1, then for any Dirichlet character χ of level q or for any holomorphic form f of level q with trivial nebentypus, one has for $\Re e s = 1/2$,*

$$L(\chi \times g, s),\ L(f \times g, s) \ll_{g,k_f} q^{1/2-\delta},$$

for some absolute $\delta > 0$. In particular, taking $g = E'(z, 1/2)$ one has

$$L(f, s)^2 = L(f \times g, s) \ll_{g,k_f} q^{1/2-\delta}.$$

The limitation to level 1 is here because of the condition $D|c$ in the Voronoi summation formula. One can establish more general (but much more messy) summations formulas for g of any level and these yield to

Theorem. *(Kowalski/Michel/Vanderkam, Michel) Fix g of any level D, then for any Dirichlet character χ of modulus q or for any holomorphic form f of level q such that the conductor of the nebentypus of χ_f is less than $q^{1/2-\beta}$ for some $\beta > 0$, then, for $\Re e s = 1/2$ one has*

$$L(\chi \times g, s),\ L(f \times g, s) \ll_{g,k_f} q^{1/2-\delta},$$

for some positive $\delta(\beta) > 0$.

5. Jutila's variant of the δ-symbol method

A technical problem with the δ-symbol method is that one cannot impose much constraints on the moduli c of the exponential sums. Elaborating a variant of Kloostermann's method, Jutila could settle this question: the starting point of the method is the identity

$$\delta_{n=0} = \int_{[0,1]} e(nx)dx.$$

Jutila's method consists in approximating the characteristic function of the unit interval $[0,1]$ by an average of characteristic function of (overleaping) intervals of length some small parameter $2\delta > 0$ centered at rational points of the form a/c with $1 \leqslant a \leqslant c$, $(a,c) = 1$ where c belong to some subset $\mathcal{C}$ of the integers $\leqslant C$. He showed that for any sufficiently dense subset $\mathcal{C} \subset [1,C]$, the function $\delta_{n=0}$ well approximated by the average

$$\delta_{\mathcal{C}}(n) := \frac{1}{|\{(a,c)=1,\ 1 \leqslant a \leqslant c,\ c \in \mathcal{C}\}|} \sum_{c \in \mathcal{C}} \sum_{\substack{a(c) \\ (a,c)=1}} e(\frac{an}{c}) I_\delta(n)$$

with

$$I_\delta(n) = \frac{1}{2\delta} \int_{-\delta}^{\delta} e(nx)dx,$$

when C is large and $C^{-2} \leqslant \delta \leqslant C^{-1}$.

This variant was used by Harcos to provide a simplified method for the **ScP** for twisted L-function $L(\chi \otimes g, s)$: by selecting the frequencies satisfying $c \equiv 0(Dl_1l_2)$ (the set of such c is dense enough since D is fixed and l_1l_2 are small) appearing in the approximation to $\delta_{l_1m-l_2n-h=0}$, he could avoid the advanced forms of Voronoi summation formulas and stick with the simplest one.
We will see later some other advantages of this method.

6. Solving the SCP II: Spectral methods

In the sixties, Selberg suggested a spectral method to handle shifted convolution sums. One suppose that g is n holomorphic cusp form of weight k and one consider the $\Gamma_0(Dl_1l_2)$-invariant function on the upper-half plane

$$V(z) = (\Im m l_1 z)^{k/2}\overline{g}(z)(\Im m l_2 z)^{k/2} g(z).$$

Let $U_h(z,s)$ be the non-holomorphic Poincare series

$$U_h(z,s) = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} (\Im m\gamma z)^s e(h\Re e s\gamma z),$$

which is absolutely convergent for $\Re e s > 1$ and has meromorphic continuation to $\mathbf{C}$. By the unfolding method one finds that

$$I(s) := \int_{\Gamma\backslash\mathbf{H}} \overline{V}(z)U_h(z,s)d\mu(z) = \langle V, U_h\rangle = \frac{l_1l_2\Gamma(s+k-1)}{(2\pi)^{s+k-1}} D(g,l_1,l_2,h;s)$$

where

$$D(g,l_1,l_2,h;s) = \sum_{l_1m-l_2n=h} \overline{\lambda_g}(m)\lambda_g(n)\Big(\frac{\sqrt{l_1l_2mn}}{l_1m+l_2n}\Big)^{k-1}\frac{1}{(l_1m+l_2n)^s}.$$

$$D(g,l_1,l_2,h;s)=\sum_{l_1m-l_2n=h}\overline{\lambda_g}(m)\lambda_g(n)\Big(\frac{\sqrt{l_1l_2mn}}{l_1m+l_2n}\Big)^{k-1}\frac{1}{(l_1m+l_2n)^s}.$$

This series is absolutely convergent for $\Re es>1$ and since the shifted convolution sums $\Sigma(g,l_1,l_2,h)$ can be expressed as inverse Mellin transforms involving $D(s)$. An analytic continuation of $D(s)$ beyond the trivial range and a non trivial bound are sufficient to solve the **SCP** .

Theorem. *(Sarnak) $D(s)$ extends analytically to $\Re es>1/2+\theta$ and for any $\varepsilon>0$, and $\Re es\geqslant 1/2+\theta+\varepsilon$*

$$D(s)\ll_\varepsilon\sqrt{l_1l_2}|s|^3|h|^{1/2+\theta+\varepsilon-\Re es},$$

where θ is an approximation to the Ramanujan/Petersson conjecture for Maass forms of weight 0.

Remark. By Kim/Shahidi and Kim/Sarnak, $\theta=7/64$ is available, however any fixed $\theta<1/2$ would be sufficient for solving the **SCP** .

Some advantages

- This result retrieves the subconvexity bounds for $L(f \times g, s)$ and $L(\chi \times g, s)$, when g is holomorphic, with much stronger subconvexity exponents. Using it, Sarnak and then liu/Ye solved the **ScP** for Rankin/ Selberg L-functions $L(f \times g, s)$ when g is fixed and when the spectral parameter of f goes to $+\infty$ (probably this is accessible to the δ-symbol method too).

- The quality of the subconvexity exponents is directly linked with the progress on the Ramanujan/Petersson conjecture.

  Remark. The exponents given by the δ- symbol method can be obtained by taking $\theta = 1/4$ (Selberg-Gelbart/Jacquet bound).This is not a coincidence : the proof by Selberg of $\theta = 1/4$ used Weil's bound for Kloostermann sums.

- This method is purely automorphic, smooth and generalizable to number fields (so far the δ-symbol method does not) cf. the works of Petridis/Sarnak and Cogdell/Piatetski-Shapiro/Sarnak.

- Even over $\mathbf{Q}$, with a supplementary input, this method enable to solve subconvexity problems not accessible to the δ-symbol type method (see below).

But one drawback...

- There are still some (hopefully purely technical) problems when g is a Maass form. In that case, $D(s)$ is given by changing the term

$$\Big(\frac{\sqrt{l_1 l_2 mn}}{l_1 m + l_2 n}\Big)^{k-1} \frac{1}{(l_1 m + l_2 n)^s}$$

into

$$(\frac{\sqrt{|\ell_1 \ell_2 mn|}}{|\ell_1 m| + |\ell_2 n|})^{2it_g} \times F_{2,1}(\frac{s+2it_g}{2}, \frac{1+2it_g}{2}, \frac{s+1}{2}; (\frac{|\ell_1 m| - |\ell_2 n|}{|\ell_1 m| + |\ell_2 n|})^2) \frac{1}{(|\ell_1 m| + |\ell_2 n|)^s}.$$

In some important cases it is unclear how to relate $\Sigma(g, l_1, l_2, h)$ to $D(g, l_1, l_2, h; s)$. In particular, for the moment one has not solved the **ScP** for $L(f, s)$ by this method when the level of f gets large. We will see how to get around this problem.

7. Sketch of the proof of Sarnak's theorem

The method consists in expanding spectrally $U_h(z,s)$ and $V(z)$: If $\{\varphi_j\}$ denotes an orthonormal Hecke-eigen basis of Maass forms for $\Gamma_0(Dl_1l_2)$, one has by Plancherel

$$\langle \overline{V}, U_h\rangle = \sum_j \langle \overline{V}, \varphi_j\rangle\langle \varphi_j, U_h\rangle + \text{ Eisenstein Spectrum.}$$

The Poincaré series captures the fourier coefficients of φ_j:

$$\langle \varphi_j, U_h\rangle = L_\infty(\varphi_j, s)|h|^{1-s}\overline{\rho}_j(h),$$

from the factor $L_\infty(\varphi_j, s)$ it follows that $D(s)$ is holomorphic for $\Re es > 1/2+\theta$. If φ_j is a primitive form

$$\rho_j(h) = \pm\rho_j(1)|h|^{-1/2}\lambda_j(|h|) \ll \frac{|h|^{\theta-1/2}e^{\pi t_j/2}}{\sqrt{Dl_1l_2}}.$$

to conclude one a bound for $\langle \overline{V}, \varphi_j\rangle$ with an exponential decay in t_j is needed (to compensate the exponential factor $e^{\pi t_j/2}$ above). The exponential decay was established in full generality by Sarnak; later Bernstein/ Reznikov and Krötz/Stanton gave other proofs by different representation theoretic methods:

$$\langle \overline{V}, \varphi_j\rangle = \int (\Im m l_1 z)^{k/2}\overline{g}(z)(\Im m l_2 z)^{k/2}g(z)\varphi_j(z)d\mu(z) \ll \sqrt{Dl_1l_2}(1+|t_j|)^A e^{-\pi t_j/2}.$$

8. The ScP for primitive nebentypus

An instance of the **ScP** that we have not addressed so far is the case where the varying form f, has nebentypus with large conductor. This problem cannot be addressed by the methods given so far. In the case of principal L-functions Duke/ Friedlander/Iwaniec proved the very difficult

Theorem. *(Duke/Friedlander/Iwaniec) There exists $\delta > 0$ such that for f a primitive form of level q with primitive nebentypus, one has for $\Re e s = 1/2$*

$$L(f, s) \ll_{s,f_\infty} q^{1/4-\delta}.$$

The proof uses a very difficult variant of the **SCP** proven earlier by D/F/I. Using the spectral method, the **ScP** could be solved for the following type of Rankin/Selberg L-functions:

Theorem. *(Michel) There exists $\delta > 0$ such that for f a primitive form of level q with non-trivial nebentypus, and g a primitive holomorphic cusp form one has for $\Re e s = 1/2$*

$$L(f \times g, s) \ll_{s,f_\infty,g} q^{1/2-\delta}.$$

Proof. (Sketch) We note χ the nebentypus of f and q_χ its conductor. This instance of the **ScP** is essentially the consequence of the following variant of the **SCP**

$$\sum_{h\neq 0} G(h,\chi)\Sigma(g,l_1,l_2,h) \ll M^{2-\delta}$$

where $M \sim N \sim q$. Note first that when $q_\chi \sim q$, the individual resolution of the **SCP** for $\Sigma(g,l_1,l_2,h)$ is not sufficient to solve this variant, even under the maximal admissible cancellation $\Sigma(g,l_1,l_2,h) \ll M^{1/2}$ because the Gauss sum equals $\overline{\chi}(h)G(1;\chi)$ has size $\sqrt{q_\chi} \sim M^{1/2}$. One has to exploit the oscillation of the Gauss sums. For this the spectral method is better adapted: expressing $\Sigma(g,l_1,l_2,h)$ in terms of $D(g,l_1,l_2,h,s)$ and expanding spectrally the series, one see that the resolution amounts to show that

$$\sum_{h\sim M} \overline{\chi}(h)\lambda_j(h) \ll M^{1/2-\delta},$$

which is interpreted as the **ScP** for $L(\overline{\chi}\varphi_j,s)$ which was proved before ! Similarly the contribution of the continuous spectrum amounts essentially to the **ScP** for $L(\chi,s)$! $\square$

9. A unification of the δ and the spectral methods

It seems difficult to use the δ-symbol method alone to solve the above variant of the **SCP** (when f has a nebentypus with large conductor). On the other hand it is (so far) difficult to solve the **SCP** by the spectral method when the fixed form g is a Maass form. Fortunately, there is a way to combine both approaches: one express each $\Sigma(g, l_1, l_2, h)$ as a sum of Kloostermann sum using Jutila's variant of the δ-symbol method by selecting the frequencies $c \equiv 0(Dl_1l_2)$: one gets this way

$$\Sigma(g, l_1, l_2, h) = \sum_{c\equiv 0(Dl_1l_2)} \sum_{m,n} \overline{\lambda_g}(m)\lambda_g(n)S(-l_1m + l_2n, -h; c)\widetilde{V}(m, n, c)$$

then one can use Kuznetzov type formulas *backwards* to express the above sums in term of Fourier coefficients of modular forms on $\Gamma_0(Dl_1l_2)$ getting

$$\sum_j \sum_{m,n} \overline{\lambda_g}(m)\lambda_g(n)\overline{\rho}_j(-l_1m + l_2n)\rho_j(-h)\widetilde{\widetilde{V}}_{it_j}(h, m, n)$$
$$+ \text{ Eisenstein spectrum + Discrete Series Spectrum.}$$

Then one incorporate, the sum over h and reduce the problem to the **ScP** for the $L(\overline{\chi} \times \varphi_j, s)$ and for the twisted L-functions of Eisenstein and Discrete series. We expect to prove the

Theorem. *(Michel/Harcos-in progress) There exist $\delta > 0$ such that for any primitive form g (cuspidal or Eisenstein) and for any primitive cuspidal f of level q with non-trivial nebentypus, one has*

$$L(f \times g, s) \ll_{g, f_\infty} q^{1/2-\delta}.$$

In particular this would generalize the D/F/I theorem for $L(f, s)$ in the q-aspect assuming only that χ_f is non- trivial.

10. The SCP and uniformity w.r. the parameters of g

So far g was considered as fixed. One may wonder how much one can solve of the **SCP** when one allows the parameters of g to get closer than the main variables M, N. Here we consider more specifically the uniformity w.r. to the level D. Some examples of this type are know already:

Example 1. (D/F/I) Let χ be a primitive character of modulus D. Let $M_1 = M_2 = N_1 = N_2 = \sqrt{D}$ and $V(x, y, z, t)$ be smooth compactly supported on $[1/2, 1]^4$, then

$$\sum_{m_1m_2 - n_1n_2} \overline{\chi}(m_1)\chi(n_1)V(\frac{m_1}{\sqrt{D}}, \frac{m_1}{\sqrt{D}}, \frac{m_1}{\sqrt{D}}, \frac{m_1}{\sqrt{D}})$$

$$= MT(h) + O(D^{1-1/1200})$$

where $MT(h)$ is an explicit main term. This can be related to a **SCP** once one has remarked that the arithmetic function $\sum_{m_1m_2=m} \chi(m_1)$ is the m-fourier coefficient of an Eisenstein series of level D and nebentypus χ. This example is in some sense the model for the (much more complicated) **SCP** solved by D/F/I to handle the **ScP** for $L(f, s)$ and f with primitive nebentypus.

The second example comes from the Gross/Zagier type formulas

Example 2. (Michel/Ramakrishnan) Let q, D be primes, such that q is inert in $K := \mathbf{Q}(\sqrt{-D})$. Let χ_K be the Legendre character associated to K, For η a character of $Pic(O_K)$ set $r_\chi(n) = \sum_{N_{K/\mathbf{Q}}(\mathcal{A})=n} \chi(\mathcal{A})$, the $r_\chi(n)$ are Fourier coefficients of a modular form g_χ of weight 1, level D and nebentypus χ_K. One has

$$\sum_{n=1}^{D/q} r_\chi(D-qn) r_{\chi_0}(n) = \delta_{\chi=\chi_0} ?MT + \sum_{f \in S_2^p(q)} \frac{D^{1/2} L(g_\chi \times f, 1/2)}{\langle f, f \rangle}$$

and since the **ScP** is know for $L(g_\chi \times f, 1/2)$, the above sum equals

$$\sum_{n=1}^{D/q} r_\chi(n) r_{\chi_0}(D-qn) = \delta_{\chi=\chi_0} ?MT + O_q(D^{1-\delta}).$$

In particular when $\chi = \chi_0$ is the trivial character it shows that the class number is well approximated by a shifted convolution sum.

11. Uniform SCP and uniform bounds for triple products.

To deal with the problem of uniformity in g for the **SCP** (hence for the **ScP**), we use the spectral method. So in the sequel we assume that g is an holomorphic cusp form. Our main objective is to solve new cases of the **ScP** for Rankin/Selberg L-function $L(f \times g, s)$.
We denote by q the level of f, D the level of g and we assume that both nebentypus are trivial. To simplify the discussion, we assume that q, D are primes (which reduce to the two most extreme cases $q = D$ and $(q, D) = 1$). The conductor of $L(f \times g, s)$ is then $Q = [q, D]^2$.
By the amplification method the **ScP** for $L(f \times g, s)$ reduces essentially to the following **SCP** : $M = N = \sqrt{Q}$

$$\sum_{h \neq 0} r(h; [q, D]) \Sigma(g, l_1, l_2, h) \ll [q, D]^{2-\delta}$$

for some $\delta > 0$, uniformly for $l_1 l_2 \ll Q^{1/100000}$ say.

12. Bounds for triple products

Applying the spectral method to the given **SCP** lead inexorably to the question of the dependence in D of the triple product $\langle V, \varphi_j \rangle$. Sarnak's method gives the following estimate for $T \geqslant 1$

$$\sum_{|t_j| \leqslant T} |\langle V, \varphi_j \rangle| e^{\pi |t_j|/2} \ll_\varepsilon (DT)^\varepsilon (l_1 l_2) T^{k+3} D^{5/2}.$$

It is not difficult to work out the dependence in D for the method of Bernstein/Reznikov-Krötz/Stanton; this has been done by Kowalski (the argument are general and not arithmetic): one gets the stronger estimate

$$\sum_{|t_j| \leqslant T} |\langle V, \varphi_j \rangle| e^{\pi |t_j|/2} \ll_\varepsilon (DT)^\varepsilon (l_1 l_2) T^{k+1} D^{3/2}.$$

Neither of these bounds use strongly the arithmetic nature of the situation: it is likely however that the dependency can be much improved.

Indeed assuming $l_1 l_2 = 1$, that φ_j is a primitive form, formulas of Kudla/Harris, Gross/Kudla, Böcherer/Schultze-Pillot and Watson relate the triple products to the central value of the triple product L-function $L(g \times g \times \varphi_j, s)$ more precisely (the precise relation in that case is due to T. Watson)

$$|\frac{\langle V, \varphi_j \rangle}{\langle g, g \rangle}|^2 \ll D^{-2} \frac{\Lambda(g \times g \times \varphi_j, 1/2)}{\Lambda(\mathrm{sym}^2 \pi_g, 1)^2 \Lambda(\mathrm{sym}^2 \varphi_j, 1)}.$$

The triple product L-function in that case has conductor D^5 and splits as

$$L(g \times g \times f, s) = L(\mathrm{sym}^2 g \times f, s) L(f, s).$$

One can then represent these central values by sums of length $\simeq D^2(1+|t_j|)^3$ and $\simeq D^{1/2}(1+|t_j|)$ respectively use Cauchy/Schwarz and the large sieve inequality of modular forms to get a good bound. By generalization of this to the case $l_1 l_2$ arbitrary, one is led to the following conjecture

Conjecture 1.

$$\sum_{|t_j| \leqslant T} |\langle V, \varphi_j \rangle| e^{\pi |t_j|/2} \ll_\varepsilon (DT)^\varepsilon (l_1 l_2)^{100000} T^{k+1} D^{5/4}.$$

(100000 should possibily be replaced by 1 but thisq is not important); in fact a new technique of Bernstein/Reznikov suggest that the exponent $9/8 < 5/4$ could be accessible.

Using these bounds, as well as known cases of subconvexity for L-function of lower degree, one obtain the following:

Theorem. *(Michel/Iwaniec) Assume that* (q, D) *are prime and distinct. Set* $D = q^\eta$*, if* $\eta < (1/2-\theta)/(1/2+\theta)$*, there exists* $\delta(\eta) > 0$ *such that*

$$L(f \times g, s) \ll (qD)^{1/2-\delta}.$$

Assuming Conjecture 1, if $\eta < (1/2-\theta)/(1/4+\theta)$*, there exists* $\delta(\eta) > 0$ *such that*

$$L(f \times g, s) \ll (qD)^{1/2-\delta}.$$

Note that for $\theta = 7/63$,

$$(1/2-\theta)/(1/2+\theta) = 25/39 < 1 \text{ while} (1/2-\theta)/(1/4+\theta) = 25/23 > 1,$$

hence Conjecture 1 would be sufficient to solve the **ScP** when f and g are both holomorphic have coprime and prime levels of arbitrary size .

When $q = D$ however a problem arise: for $h = q$ the Ramanujan sum degenerate to $\varphi(q)$ and the Fourier coefficient of the Eisenstein series $E_0(z, 1/2 + it)$ at the cusp 0 is not small; moreover the triple product $\langle V, \varphi_j \rangle$ is essentially equal to the symmetric square $L(\mathrm{sym}^2 g, 1/2 + it)$. It follows that, under Conjecture 1 the **ScP** for $L(f \times g, s)$ follows from the **ScP** for $L(\mathrm{sym}^2 g, s)$. More precisely

Theorem. *Assume conjecture (1). Given g holomorphic of prime level q with trivial nebentypus, suppose there exist $\delta > 0$ such that $L(\mathrm{sym}^2 g, s) \ll |s|^{1000} q^{1/2-\delta}$ on the critical line, then for any primitive f of level q with trivial nebentypus, there exists $\delta' > 0$ such that $L(f \times g, s) \ll q^{1/2-\delta'}$.*

Ironically, our investigation started with the objective of solving the **ScP** for $L(\mathrm{sym}^2 g, s)$ by solving it for $L(g \times g, s) = \zeta(s) L(\mathrm{sym}^2 g, s)$ as a special case of $L(f \times g, s)$.