AN ALGEBRAIC MODEL FOR MOD 2 TOPOLOGICAL CYCLIC HOMOLOGY

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Abstract. For any space X with the homotopy type of simply-connected, finite-type CW-complex, we construct an associative cochain algebra $fls^*(X)$ such that $\mathrm{H}^*(fls^*(X)) \cong \mathrm{H}^*(\mathcal{L}X)$ as algebras, where $\mathcal{L}X$ denotes the free loop space on X. Under additional conditions on X, e.g., when X is a wedge of spheres, we define a cochain complex $hos^*(X)$ by twisting together $fls^*(X)$ and $\mathrm{H}^*(BS^1)$ and prove that $\mathrm{H}^*(hos^*(X)) \cong \mathrm{H}^*(\mathcal{L}X_{hS^1})$ as graded modules. We also show that there is a cochain map from $fls^*(X)$ to itself that is a good model of the p^{th}-power operation on $\mathcal{L}X$. Finally, we define $tc^*(X)$ to be the mapping cone of the composite of the projection map from $hos^*(X)$ to $fls^*(X)$ with the model of the p^{th}-power map (for $p=2$), so that the mod 2 spectrum cohomology of $TC(X;2)$ is isomorphic to $\mathrm{H}^*(tc^*(X)\otimes\mathbb{F}_2)$. We conclude by calculating $\mathrm{H}^*(TC(S^{2n+1};2);\mathbb{F}_2)$.

Preface

Bökstedt, Hsiang and Madsen introduced *topological cyclic homology (TC)* as a topological version of Connes' cyclic homology in [BHM]. The topological cyclic homology of a space X at a prime p, denoted $TC(X;p)$, is a spectrum that is the target of the *cyclotomic trace map*

$$Trc : A(X) \longrightarrow TC(X;p),$$

the source of which is Waldhausen's algebraic K-theory spectrum of X, to which $TC(X;p)$ provides a useful approximation. The cyclotomic trace map is analogous to the Dennis trace map $K_*(A) \to HH_*(A)$. Very little is known about the $TC(X;p)$ when X is not a singleton.

Key words and phrases. Free loop space, homotopy orbits, algebraic model, topological cyclic homology.

Waldhausen's algebraic K-theory itself approximates the smooth and topological Whitehead spectra $Wh^d(X)$ and $Wh^t(X)$. There are natural cofiber sequences of spectra

$$\Sigma^\infty X_+ \xrightarrow{\eta_X} A(X) \longrightarrow Wh^d(X)$$

and

$$A(*) \wedge X_+ \xrightarrow{a_X^A} A(X) \longrightarrow Wh^t(X)$$

respectively, where η_X is the unit map and a_X^A is the A-theory assembly map [W1, 3.3.1]. Here, Y_+ denotes the space Y with an extra basepoint, and $\Sigma^\infty Y$ denotes the suspension spectrum of Y.

By Waldhausen's stable parametrized h-cobordism theorem [W2], there are homotopy equivalences

$$\mathcal{H}^d(M) \simeq \Omega\Omega^\infty Wh^d(M)$$

and

$$\mathcal{H}^t(M) \simeq \Omega\Omega^\infty Wh^t(M)$$

when M is a smooth, respectively topological, compact manifold. Here $\mathcal{H}^d(M)$ is the stable smooth h-cobordism space of M, which in a stable range carries information about the homotopy type of the topological group $\mathrm{Diff}(M)$ of self-diffeomorphisms of M. Likewise $\mathcal{H}^t(M)$ is the stable topological h-cobordism space of M, which in a stable range carries information about the topological group $\mathrm{Homeo}(M)$ of self-homeomorphisms of M.

Any information we may obtain about the topological cyclic homology of a smooth or topological manifold will therefore give us some indication of the nature of its stable h-cobordism space.

For further explanation of the role of trace maps in K-theory and (topological) cyclic and Hochschild homology, as well as about Whitehead spectra, we refer the reader to [Be] and [R].

The goal of this chapter, as well as of the minicourse upon which the chapter is based, is to construct a cochain complex $tc^*(X)$ such that $\mathrm{H}^*(tc^*(X) \otimes \mathbb{F}_p)$ is isomorphic to the mod p spectrum cohomology of $TC(X;p)$. For reasons of ease of notation and computation, we will limit ourselves to $p = 2$ in this article.

There are several equivalent definitions of $TC(X;p)$. The definition that is best suited to algebraic modeling can be stated as follows [BHM]. Let $\mathcal{L}X$ be the free loop space on X, i.e., the space of unbased maps from the circle S^1 into X, which admits a natural S^1-action, by rotation of loops. Let $\mathcal{L}X_{hS^1} = ES^1 \underset{S^1}{\times} \mathcal{L}X$ denote the homotopy orbit space of this action. Let $\lambda^p : \mathcal{L}X \longrightarrow \mathcal{L}X$ denote the

p^{th}-power map, defined by $\lambda^p(\ell)(z) = \ell(z^p)$ for all $\ell \in \mathcal{L}X$ and all $z \in S^1$. There is a homotopy pullback of spectra

$$\begin{array}{ccc} TC(X;p) & \longrightarrow & \Sigma^\infty \mathcal{L}X_+ \\ \downarrow & & \downarrow {\scriptstyle \Sigma^\infty(Id-\lambda^p)} \\ \Sigma^\infty\big(\Sigma(\mathcal{L}X_{hS^1})_+\big) & \xrightarrow{trf_{S^1}} & \Sigma^\infty \mathcal{L}X_+ \end{array}$$

where trf_{S^1} is the S^1-transfer map associated to the principal S^1-bundle

$$ES^1 \times \mathcal{L}X \longrightarrow \mathcal{L}X_{hS^1}.$$

It is therefore clear that $TC(X;p)$ is the homotopy fiber of the composition

$$\Sigma^\infty \mathcal{L}X_+ \xrightarrow{\Sigma^\infty(Id-\lambda^p)} \Sigma^\infty \mathcal{L}X_+ \xrightarrow{\iota} \text{hocofib}(trf_{S^1})\ .$$

Motivated by this characterization of $TC(X;p)$, we apply the following method to constructing $tc^*(X)$. We first define an associative cochain algebra $fls^*(X)$ together with a cochain map

$$\Upsilon : fls^*(X) \longrightarrow CU^*\mathcal{L}X$$

inducing an isomorphism of algebras in cohomology, where CU^* denotes the reduced cubical cochains. We then twist together $fls^*(X)$ and $\mathrm{H}^*(BS^1)$, obtaining a new cochain complex $hos^*(X)$ that fits into a commuting diagram

$$\begin{array}{ccc} hos^*(X) & \xrightarrow{\pi} & fls^*(X) \\ \downarrow {\scriptstyle \overline{\Upsilon}} & & \downarrow {\scriptstyle \Upsilon} \\ CU^*(\mathcal{L}X_{hS^1}) & \xrightarrow{CU^*c} & CU^*(\mathcal{L}X) \end{array}$$

where π is the projection map, $c : \mathcal{L}X \longrightarrow \mathcal{L}X_{hS^1}$ is the map induced by the inclusion $\mathcal{L}X \longrightarrow ES^1 \times \mathcal{L}X$ and $\overline{\Upsilon}$ induces an isomorphism in cohomology. The projection map $\pi : hos^*(X) \longrightarrow fls^*(X)$ is then a model for the inclusion

$$\Sigma^\infty \mathcal{L}X_+ \xrightarrow{\iota} \text{hocofib}(trf_{S^1}).$$

Finally, we define a cochain map $\mathfrak{l}^p : fls^*(X) \longrightarrow fls^*(X)$ such that $\Upsilon \circ \mathfrak{l}^p$ and $CU^*\lambda^p \circ \Upsilon$ are chain homotopic. Thus $Id - \mathfrak{l}^p$ is a model for $\Sigma^\infty(Id - \lambda^p)$. As explained carefully and in detail in [HR], if we set $tc^*(X)$ equal to the mapping cone of the composition $(Id-\mathfrak{l}^p)\pi$, then $\mathrm{H}^*(tc^*(X)\otimes\mathbb{F}_p)\cong\mathrm{H}^*(TC(X;p);\mathbb{F}_p)$, as desired. Here we remark only that there is a Thom isomorphism involved in the identification of the cohomology of the homotopy cofiber of the S^1-transfer map and the cohomology of the S^1-homotopy orbits of the free loop space. Furthermore, the fact that the projection map π is a model for the inclusion ι requires an analysis of these transfers and isomorphisms

The article is organized as follows. We begin in section 0 by reminding the reader of certain algebraic and topological notions and constructions. In section 1 we study free loop spaces and their algebraic models, beginning by defining a simplicial set that models $\mathcal{L}X$, which we then apply to constructing $fls^*(X)$ via a refined version of methods from [DH1-4]. Section 2 is devoted to the study of homotopy orbit spaces of circle actions. We first treat the general case, twisting together $\mathrm{H}^*(BS^1)$ and CU^*Y to obtain a large but attractive cochain complex $HOS^*(Y)$ for calculating $\mathrm{H}^*(Y_{hS^1})$, when Y is any S^1-space. Specializing to the case $Y=\mathcal{L}X$, we show how to twist together $\mathrm{H}^*(BS^1)$ and $fls^*(X)$ to build $hos^*(X)$, so that we obtain a complex equivalent to $HOS^*(\mathcal{L}X)$. Finally, in section 3 we define and study our model for the p^{th}-power map (for $p=2$) , then apply it, together with the results of the preceding chapters, to the construction of $tc^*(X)$. We conclude by applying our model to the calculation of the mod 2 spectrum cohomology of $TC(S^{2n+1};2)$.

Remark. In these notes, complete proofs are provided only of those results that have not yet appeared elsewhere and that are due to the author. Furthermore, some results that have yet to be published are not proved completely here, if the complete proof is excessively technical. We hope in such cases to have provided enough detail to convince the reader of the truth of the statement. The reader who is curious about the details is refered to articles that should appear soon.

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0. Preliminaries

We begin here by recalling certain elementary definitions and constructions and fixing our basic notation and terminology. We then remind the reader of the construction of the canonical, enriched Adams-Hilton model of a simplicial set, which is the input data for our free loop space model. We conclude this section with a description of our general method for constructing algebraic models of fiber squares, which we then apply in section 1 to building our free loop space model.

0.1 Elementary definitions, terminology and notation.
Throughout this paper we work over $\mathbb{Z}$, the ring of integers, unless stated otherwise.

Given chain complexes (V, d) and (W, d), the notation $f : (V, d) \xrightarrow{\simeq} (W, d)$ indicates that f induces an isomorphism in homology. In this case we refer to f as a *quasi-isomorphism*.

If $V = \bigoplus_{i\in\mathbb{Z}} V_i$ is a graded module, then $s^{-1}V$ and sV denote the graded modules with, respectively, $(s^{-1}V)_i \cong V_{i+1}$ and $(sV)_i \cong V_{i-1}$. Given a homogeneous element v in V, we write $s^{-1}v$ and sv for the corresponding elements of $s^{-1}V$ and sV. If the gradings are written as upper indices, i.e., $V = \bigoplus_{i\in\mathbb{Z}} V^i$, then $(s^{-1}V)^i \cong V^{i-1}$ and $(sV)^i \cong V^{i+1}$.

Dualization is indicated throughout the paper by a $\sharp$ as superscript. The degree of an element x in a graded module is denoted $|x|$, unless it is used as an exponent, in which case the bars may be dropped.

A graded $\mathbb{Z}$-module $V = \bigoplus_{i\in\mathbb{Z}} V_i$ is *connected* if $V_{<0} = 0$ and $V_0 \cong \mathbb{Z}$. It is *simply connected* if, in addition, $V_1 = 0$. We write V_+ for $V_{>0}$. Let V be a positively-graded, free $\mathbb{Z}$-module. The free associative algebra on V is denoted TV, i.e.,

$$TV \cong \mathbb{Z} \oplus V \oplus (V \otimes V) \oplus (V \otimes V \otimes V) \oplus \cdots .$$

A typical basis element of TV is denoted $v_1 \cdots v_n$, i.e., we drop the tensors from the notation. The product on TV is then defined by

$$\mu(u_1 \cdots u_m \otimes v_1 \cdots v_n) = u_1 \cdots u_m v_1 \cdots v_n.$$

The cofree, coassociative coalgebra on V, denoted $\perp V$ in this article, is isomorphic as a graded $\mathbb{Z}$-module to TV. We write $\perp^n V = \bigotimes^n V$, of which a typical basis element is denoted $v_1|\cdots|v_n$. The coproduct on $\perp V$ is then defined in the obvious manner by

$$\begin{aligned}\Delta(v_1|\cdots|v_n) = v_1|\cdots|v_n \otimes 1 + 1 \otimes v_1|\cdots|v_n \\ + \sum_{i=1}^{n-1} v_1|\cdots|v_i \otimes v_{i+1}|\cdots|v_n.\end{aligned}$$

Let (C, d) be a simply-connected (co)chain coalgebra with reduced coproduct $\overline{\Delta}$. The *cobar construction* on (C, d), denoted $\Omega(C, d)$, is the (co)chain algebra $(Ts^{-1}(C_+), d_\Omega)$, where $d_\Omega = -s^{-1}ds + (s^{-1} \otimes s^{-1})\overline{\Delta}s$ on generators.

Let (A, d) be a connected chain algebra or a simply-connected cochain algebra over R, and let $\bar{A}$ be the component of A of positive degree. The *bar construction* on (A, d), denoted $\mathcal{B}(A, d)$, is a differential graded coalgebra $(\perp(s\bar{A}), D_\mathcal{B})$. Let $(D_\mathcal{B})_1$ denote the linear part of the differential, i.e., $(D_\mathcal{B})_1 = \pi D_\mathcal{B}$, where $\pi : \perp V \to V$ is the natural projection. The linear part of $D_\mathcal{B}$ specifies the entire differential and is given by

$$(D_\mathcal{B})_1(sa_1|\cdots|sa_n) = \begin{cases} -s(da_1) & \text{if} \quad n = 1 \\ (-1)^{a_1+1}s(a_1 \cdot a_2) & \text{if} \quad n = 2 \\ 0 & \text{if} \quad n > 2. \end{cases}$$

Definition. Let $f, g : (A, d) \longrightarrow (B, d)$ be two maps of chain (respectively, cochain) algebras. An (f, g)-*derivation homotopy* is a linear map $\varphi : A \longrightarrow B$ of degree $+1$ (respectively, -1) such that $d\varphi + \varphi d = f - g$ and $\varphi\mu = \mu(\varphi \otimes g + f \otimes \varphi)$, where μ denotes the multiplication on A and B.

If f and g are maps of (co)chain coalgebras, there is an obvious dual definition of an (f, g)-*coderivation homotopy*.

We often apply Einstein's summation convention in this chapter. When an index appears as both a subscript and a superscript in an expression, it is understood that we sum over that index. For example, given an element c of a coalgebra (C, Δ), the notation $\Delta(c) = c_i \otimes c^i$ means $\Delta(c) = \sum_{i \in I} c_i \otimes c^i$.

Another convention used consistently throughout this chapter is the Koszul sign convention for commuting elements of a graded module or for commuting a morphism of graded modules past an element of the source module. For example, if V and W are graded algebras and $v \otimes w, v' \otimes w' \in V \otimes W$, then

$$(v \otimes w) \cdot (v' \otimes w') = (-1)^{|w| \cdot |v'|} vv' \otimes ww'.$$

Futhermore, if $f : V \longrightarrow V'$ and $g : W \longrightarrow W'$ are morphisms of graded modules, then for all $v \otimes w \in V \otimes W$,

$$(f \otimes g)(v \otimes w) = (-1)^{|g| \cdot |v|} f(v) \otimes g(w).$$

The source of the Koszul sign convention is the definition of the twisting isomorphism

$$\tau : V \otimes W \longrightarrow W \otimes V : v \otimes w \mapsto (-1)^{|v| \cdot |w|} w \otimes v.$$

We assume throughout this chapter that the reader is familiar with the elements of the theory of simplicial sets and of model categories. We recall here only a few very basic definitions, essentially to fix notation and terminology, and refer the reader to, e.g., [May] and [GJ] for simplicial theory and to [Ho], [DS] and [H] for model category theory.

Definition. Let K be a simplicial set, and let $\mathcal{F}_{ab}$ denote the free abelian group functor. For all $n > 0$, let $DK_n = \cup_{i=0}^{n-1} s_i(K_{n-1})$, the set of degenerate n-simplices of K. The *normalized chain complex* on K, denoted $C_*(K)$, is given by

$$C_n(K) = \mathcal{F}_{ab}(K_n)/\mathcal{F}_{ab}(DK_n).$$

Given a map of simplicial sets $f : K \longrightarrow L$, the induced map of normalized chain complexes is denoted $C_* f$.

Recall that $H_*(C_*(K)) \cong H_*(|K|)$ as graded coalgebras, where $|K|$ denotes the geometric realization of K.

Definition. Let K be a reduced simplicial set, and let $\mathcal{F}$ denote the free group functor. The *loop group* GK on K is the simplicial group such that $(GK)_n = \mathcal{F}(K_{n+1} \smallsetminus Ims_0)$, with faces and degeneracies specified by

$$\begin{aligned} \partial_0 \bar{x} &= (\,\overline{\partial_0 x}\,)^{-1}\overline{\partial_1 x} \\ \partial_i \bar{x} &= \overline{\partial_{i+1} x} \quad \text{for all } i > 0 \\ s_i \bar{x} &= \overline{s_{i+1} x} \quad \text{for all } i \geq 0 \end{aligned}$$

where $\bar{x}$ denotes the class in $(GK)_n$ of $x \in K_{n+1}$.

Recall that $H_*(GK) \cong H_*(\Omega|K|)$ as graded Hopf algebras.

In any model category we use the notation $\rightarrowtail$ for cofibrations, $\twoheadrightarrow$ for fibrations and $\xrightarrow{\sim}$ for weak equivalences.

0.2 The canonical, enriched Adams-Hilton model.
We recall in this section the construction given in [HPST] of the canonical, enriched Adams-Hilton model of a 1-reduced simplicial set K, upon which our free loop space model construction is based. We begin by reminding the reader of the theories that are essential to this construction. We first sketch briefly the classical and crucial theory of twisting cochains, which goes back to work of E. Brown [Br]. We then outline the theory of strongly homotopy coalgebra maps. We conclude this section by presenting the canonical Adams-Hilton model.

Twisting cocochains.

Definition. Let (C, d) be a chain coalgebra with coproduct Δ, and let (A, d) be a chain algebra with product μ. A *twisting cochain* from (C, d) to (A, d) is a degree -1 map $t : C \to A$ of graded modules such that

$$dt + td = \mu(t \otimes t)\Delta.$$

The definition of a twisting cochain $t : C \to A$ is formulated precisely so that the following two constructions work smoothly. First, let $(A, d) \otimes_t (C, d) = (A \otimes C, D_t)$, where $D_t = d \otimes 1_C + 1_A \otimes d - (\mu \otimes 1_C)(1_A \otimes t \otimes 1_C)(1_A \otimes \Delta)$. It is easy to see that $D_t^2 = 0$, so that $(A, d) \otimes_t (C, d)$ is a chain complex, which extends (A, d), i.e., of which (A, d) is subcomplex. Second, if C is connected, let $\tilde{t} : Ts^{-1}C_+ \to A$ be the algebra map given by $\tilde{t}(s^{-1}c) = t(c)$. Then $\tilde{t}$ is in fact a chain algebra map $\tilde{t} : \Omega(C, d) \to (A, d)$. It is equally clear that any algebra map $\theta : \Omega(C, d) \to (A, d)$ gives rise to a twisting cochain via the composition

$$C_+ \xrightarrow{s^{-1}} s^{-1}C_+ \hookrightarrow Ts^{-1}C_+ \xrightarrow{\theta} A.$$

Furthermore, the complex $(A, d) \otimes_t (C, d)$ is acyclic if and only if $\tilde{t}$ is a quasi-isomorphism.

The twisting cochain associated to the cobar construction is a fundamental example of this notion. Let (C, d, Δ) be a simply-connected chain coalgebra. Consider the linear map

$$t_\Omega : C \to \Omega C : c \to s^{-1}c.$$

It is an easy exercise to show that t_Ω is a twisting cochain and that $\tilde{t}_\Omega = 1_{\Omega C}$. Thus, in particular, $(\Omega C, d) \otimes_{t_\Omega} (C, d)$ is acyclic; this is the well-known acyclic cobar construction.

Strongly homotopy coalgebra and comodule maps. In [GM] Gugenheim and Munkholm showed that Cotor was natural with respect to a wider class of morphisms than the usual morphisms of chain coalgebras. Given two chain coalgebras (C, d, Δ) and (C', d', Δ'), a *strongly homotopy coalgebra (SHC) map* $f : (C, d, \Delta) \Rightarrow (C', d', \Delta')$ is a chain map $f : (C, d) \to (C', d')$ together with a family of $\mathbb{Z}$-linear maps

$$\mathfrak{F}(f) = \{F_k : C \to (C')^{\otimes k} \mid \deg F_k = k - 1,\ k \geq 1\}$$

satisfying

(1) $F_1 = f$ and
(2) for all $k \geq 2$

$$\begin{aligned} F_k d - \sum_{i+j=k-1} &(-1)^j (1_{C'}^{\otimes i} \otimes d' \otimes 1_{C'}^{\otimes j}) F_k \\ &= \sum_{i+j=k} (-1)^j (F_i \otimes F_j)\Delta + \sum_{i+j=k-2} (-1)^j (1_{C'}^{\otimes i} \otimes \Delta' \otimes 1_{C'}^{\otimes j}) F_{k-1}. \end{aligned}$$

We call $\mathfrak{F}(f)$ an *SHC family* for f.

An SHC map is thus a coalgebra map, up to an infinite family of homotopies. In particular if f is a map of chain coalgebras, then it can be seen as an SHC map, with $F_k = 0$ for all $k > 1$. Furthermore if $f : (C, d, \Delta) \Rightarrow (C', d', \Delta')$ is an SHC map and $g : (C', d', \Delta') \to (C'', d'', \Delta'')$ is a strict coalgebra map, then gf is an SHC map, where $\mathfrak{F}(gf) = \{g^{\otimes k} F_k \mid k \geq 1\}$.

Observe that the existence of $\mathfrak{F}(f)$ is equivalent to the existence of a chain algebra map $\widetilde{\Omega} f : \Omega(C, d) \to \Omega(C', d')$ such that $\widetilde{\Omega} f(s^{-1}c) - s^{-1}f(c) \in T^{\geq 2} s^{-1} C'_+$. Given $\mathfrak{F}(f)$, we can define $\widetilde{\Omega} f$ by setting

$$\widetilde{\Omega} f(s^{-1}c) = \sum_{k \geq 1} (s^{-1})^{\otimes k} F_k(c)$$

and extending to a map of algebras. Condition (2) above then implies that $\widetilde{\Omega} f$ is a differential map as well.

Note that if f is a strict coalgebra map, seen as an SHC map with trivial SHC family, then $\widetilde{\Omega} f = \Omega f$. More generally, if $f : (C, d, \Delta) \Rightarrow (C', d', \Delta')$ is an SHC map and $g : (C', d', \Delta') \to (C'', d'', \Delta'')$ is a strict coalgebra map, seen as an SHC map with trivial SHC family, then there is an SHC family for gf such that $\widetilde{\Omega}(gf) = \Omega g \circ \widetilde{\Omega} f$.

Similarly, given $\widetilde{\Omega} f$, we can define F_k via the composition

$$C_+ \xrightarrow{s^{-1}} s^{-1}C_+ \hookrightarrow Ts^{-1}C_+ \xrightarrow{\widetilde{\Omega} f} Ts^{-1}C'_+ \xrightarrow{proj} \big(s^{-1}C'_+\big)^{\otimes k} \xrightarrow{s^{\otimes k}} (C')^{\otimes k}.$$

Gugenheim and Munkholm proved in [GM] that the usual simplicial Alexander-Whitney map

$$f_{K,L} : C_*(K \times L) \longrightarrow C_*(K) \otimes C_*(L)$$

defined by $f_{K,L}(x, y) = \sum_{i=0}^{n} \partial_{i+1} \cdots \partial_n x \otimes \partial_0^i y$ is naturally an SHC map.

The canonical Adams-Hilton model. For every pair of simply-connected chain coalgebras (C, d) and (C', d'), Milgram proved that there is a quasi-isomorphism of chain algebras

$$\rho : \Omega\big((C, d) \otimes (C', d')\big) \to \Omega(C, d) \otimes \Omega(C', d') \tag{0.2.1}$$

specified by $\rho\big(s^{-1}(x \otimes 1)\big) = s^{-1}x$, $\rho\big(s^{-1}(1 \otimes y)\big) = s^{-1}y$ and $\rho\big(s^{-1}(x \otimes y)\big) = 0$ for all $x \in C_+$ and $y \in C'_+$ [Mi].

In [S] Szczarba gave an explicit formula for a natural transformation of functors from simplicial sets to chain algebras

$$\theta : \Omega C_*(-) \to C_*(G(-))$$

such that $\theta_K : \Omega C_*(K) \to C_*(GK)$ is a quasi-isomorphism of chain algebras for every 1-reduced simplicial set K. Since $C_*(GK)$ is in fact a chain Hopf algebra, it is reasonable to ask whether $\Omega_* C(K)$ can be endowed with a coproduct with respect to which θ_K is a quasi-isomorphism of chain Hopf algebras.

Let $\psi : \Omega C_*(-) \to \Omega C_*(-) \otimes \Omega C_*(-)$ denote the natural transformation given for each 1-reduced simplicial set K by the composition

$$\Omega C_*(K) \xrightarrow{\Omega(\Delta_K)_\sharp} \Omega C_*(K \times K) \xrightarrow{\widetilde{\Omega} f_{K,K}} \Omega\big(C_*(K) \otimes C_*(K)\big) \xrightarrow{\rho} \Omega C_*(K) \otimes \Omega C_*(K).$$

The coproduct $\psi_K : \Omega C_*(K) \to \Omega C_*(K) \otimes \Omega C_*(K)$ is called the *Alexander-Whitney (A-W) cobar diagonal*. In [HPST] Hess, Parent, Scott and Tonks proved that for all 1-reduced K, the Alexander-Whitney cobar diagonal is strictly coassociative and cocommutative up to derivation homotopy, which we call Θ. They established furthermore that Szczarba's equivalence θ_K is an SHC map with respect to ψ_K and the usual coproduct on $C_*(GK)$.

In [B] Baues provided a purely combinatorial definition of strictly coassociative coproduct and of a derivation homotopy for cocommutativity on $\Omega C_*(K)$ for any 1-reduced simplicial set K, but without giving a map from $\Omega C_*(K)$ to $C_*(GK)$. In [HPST] it is shown that the Alexander-Whitney cobar diagonal is the same as Baues's coproduct, which implies that

$$\mathrm{Im}\,\overline{\psi}_K \subseteq T^{\geq 1}s^{-1}C_+K \otimes s^{-1}C_+K,$$

where $\overline{\psi}_K$ is the reduced coproduct.

Henceforth we refer to $\theta_K : \Omega C_*(K) \to C_*(GK)$ as the *canonical Adams-Hilton model* and to $\psi_K : \Omega C_*(K) \to \Omega C_*(K) \otimes \Omega C_*(K)$ as its *canonical enrichment*.

0.3 Noncommutative algebraic models of fiber squares.
We review in this section the bare essentials of noncommutative modeling of fiber squares, as developed in [DH1]. Note that this theoretical framework is highly analogous to the theory of KS-extensions in rational homotopy theory. See [DH1]and [FHT] for more details.

We first define the classes of morphisms with which we work throughout the remainder of this article.

Definition. Let (B,d) and (C,d) be bimodules over an associative cochain algebra (A,d). A cochain map $f:(B,d)\longrightarrow(C,d)$ is a *quasi-bimodule map* if H^*f is a map of $\mathrm{H}^*(A,d)$-bimodules. If (A,d) and (A',d') are associative cochain algebras, then a cochain map $f:(A,d)\longrightarrow(A',d')$ is a *quasi-algebra map* if H^*f is a map of algebras.

Noncommutative cochain algebra models of topological spaces are defined in terms of quasi-algebra maps.

Definition. Let X be a topological space. An (integral) *noncommutative model* of X consists of an associative cochain algebra over $\mathbb{Z}$, (A,d), together with a quasi-algebra quasi-isomorphism

$$\alpha:(A,d)\xrightarrow{\simeq}C^*(X),$$

where α is called a *model morphism*.

Of course, we must also define what it means to model a continuous map, if we wish to model pull-backs of fibrations.

Definition. Let $f:Y\to X$ be a continuous map. A *noncommutative model* of f consists of a commuting diagram

$$\begin{array}{ccc}(A,d) & \xrightarrow{\varphi} & (B,d)\\ \simeq\big\downarrow\alpha & & \simeq\big\downarrow\beta\\ C^*X & \xrightarrow{C^*f} & C^*Y\end{array}$$

in which α and β are model morphisms, and φ is a quasi-algebra map.

Remarks.

(1) In most applications of noncommutative models, α and φ are strict morphisms of algebras, while it is usually impossible for β to be a strict morphism of algebras.
(2) It is often difficult to use a noncommutative model for constructions or calculations, unless A is a free algebra. It is easy to see, however, that every space X possesses such a noncommutative model.

There is a special class of strict algebra maps, known as *twisted algebra extensions*, that are used for modeling topological fibrations. Roughly speaking, a twisted algebra extension of one algebra by another is a tensor product of the two algebras in which both the differential and multiplication are perturbed from the usual tensor-product differential and multiplication.

Definition. Let (A, d) and (B, d) be a cochain algebra and a cochain complex over $\mathbb{Z}$, respectively. A *twisted bimodule extension* of (A, d) by (B, d) is an (A, d)-bimodule (C, D) such that

(1) $C \cong A \otimes B$ as graded modules;
(2) the right action of A on C is free, i.e., $(a \otimes b) \cdot a' = (-1)^{a'b} aa' \otimes b$;
(3) the left action of A on C commutes with the right action, i.e.,

$$(a \cdot c) \cdot a' = a \cdot (c \cdot a')$$

for all $a, a' \in A$ and $c \in C$, and satisfies

$$a \cdot (1 \otimes b) - a \otimes b \in A^+ \otimes B^{<b}$$

for all a in A and b in B; and
(4) the inclusion map $(A, d) \longrightarrow (C, D)$ and the projection map $(C, D) \longrightarrow (B, d)$ are both (A, d)-bimodule maps, where (B, d) is considered with the trivial (A, d)-bimodule structure, so that, in particular,

$$D(1 \otimes b) - 1 \otimes db \in A^+ \otimes B$$

for all b, b' in B.

If (B, d) is a cochain algebra, then a twisted bimodule extension (C, D) of (A, d) by (B, d) is a *twisted algebra extension* if the bimodule structure of (C, D) extends to a full algebra structure such that the inclusion and projection maps above are maps of cochain algebras. In particular,

$$(1 \otimes b)(1 \otimes b') - 1 \otimes bb' \in A^+ \otimes B$$

for all b, b' in B.

Notation. We write $(A,d)\widetilde{\otimes}(B,d)$ to denote a twisted bimodule extension of (A,d) by (B,d) and $(A,d)\odot(B,d)$ to denote a twisted algebra extension.

The proposition below, which is the noncommutative analogue of a well-known result concerning KS-extensions, states that twisted algebra extensions have the left lifting property with respect to surjective quasi-algebra quasi-isomorphisms. Since it is natural to think of surjective quasi-algebra morphisms as fibrations of cochain algebras, Proposition 0.3.1 implies that we can think of twisted extensions as cofibrations. In other words, twisted algebra extensions are plausible models of topological fibrations, since the cochain functor is contravariant.

Proposition 0.3.1. *Let $\iota : (A,d) \to (A,d)\odot(B,d)$ be a twisted algebra extension. Given a commuting diagram*

$$\begin{array}{ccc} (A,d) & \xrightarrow{f} & (C,d) \\ \downarrow{\scriptstyle \iota} & & \simeq\downarrow{\scriptstyle p} \\ (A,d)\odot(B,d) & \xrightarrow{g} & (E,d) \end{array}$$

in which f is a right (A,d)-module map, p is a surjective quasi-algebra quasi-isomorphism, and g is a quasi-algebra map, there exists a quasi-algebra map,

$$h : (A,d)\odot(B,d) \longrightarrow (C,d)$$

which is a right (A,d)-module map, as well as a lift of g through p and an extension of f, i.e., $ph = g$ and $h\iota = f$.

This proposition is a simplified version of a result that first appeared in [DH1], but that we do not need in its full generality here.

Proof. Since $(A,d)\odot(B,d)$ is semifree as a right (A,d)-module, the lift h exists as a map of right (A,d)-modules. In cohomology $\mathrm{H}^* p\,\mathrm{H}^* h = \mathrm{H}^* g$, which implies that $\mathrm{H}^* h = (\mathrm{H}^* p)^{-1}\,\mathrm{H}^* g$, since $\mathrm{H}^* p$ is an isomorphism. Hence $\mathrm{H}^* h$ is an algebra map, as it is a composition of algebra maps. □

Let us see how to model pull-backs of fibrations in this context. Consider a pull-back square of simply-connected spaces

$$\begin{array}{ccc} E\underset{B}{\times}X & \xrightarrow{\bar f} & E \\ \downarrow{\scriptstyle \bar q} & & \downarrow{\scriptstyle q} \\ X & \xrightarrow{f} & B \end{array}$$

in which q is a fibration and f an arbitrary continuous map. Suppose that

(0.3.1)
$$\begin{array}{ccc} (A,d) & \xrightarrow{\varphi} & (\bar{A},\bar{d}) \\ \simeq\downarrow\alpha & & \simeq\downarrow\gamma \\ C^*B & \xrightarrow{C^*f} & C^*X \end{array}$$

and

(0.3.2)
$$\begin{array}{ccc} (A,d) & \xrightarrow{\iota} & (A,d)\odot(C,e) \\ \simeq\downarrow\alpha & & \simeq\downarrow\beta \\ C^*B & \xrightarrow{C^*q} & C^*E \end{array}$$

are noncommutative models of f and q, where ι is a twisted algebra extension of (A,d). We assume that α and γ are algebra maps, while β may be only a quasi-algebra map.

The following theorem provides the theoretical underpinnings for noncommutative modeling of fiber squares. It states that under certain reasonable conditions, there exists a sort of push-out of ι and φ that is a model of the pull-back $E \underset{B}{\times} X$.

Theorem 0.3.2 [DH1]**.** *Given a commuting diagram over a field $\Bbbk$, with squares as in diagrams (0.3.1) and (0.3.2),*

$$\begin{array}{ccccc} (A,d)\odot(C,e) & \xleftarrow{\iota} & (A,d) & \xrightarrow{\varphi} & (\bar{A},\bar{d}) \\ \simeq\downarrow\beta & & \simeq\downarrow\alpha & & \simeq\downarrow\gamma \\ C^*E & \xleftarrow{C^*q} & C^*B & \xrightarrow{C^*f} & C^*X \end{array}$$

in which $\bar{A}$ is a free algebra and φ admits a cochain algebra section σ, there exist a twisted algebra extension

$$\bar{\iota} : (\bar{A},\bar{d}) \to (\bar{A},\bar{d}) \odot (C,e)$$

and a noncommutative model over $\Bbbk$

$$\delta : (\bar{A},\bar{d}) \odot (C,e) \xrightarrow{\simeq} C^*(E \underset{B}{\times} X)$$

such that

(1) $(\bar{a}\otimes 1)(1\otimes c) = (\varphi\otimes 1)\big((\sigma(\bar{a})\otimes 1)(1\otimes c)\big)$ *for all $\bar{a}$ in $\bar{A}$ and c in C;*
(2) *if D is the differential on $A\otimes C$, then the differential $\bar{D}$ on $\bar{A}\otimes C$ commutes with the right action of $\bar{A}$ and is specified by $\bar{D}(1\otimes c) = (\varphi\otimes 1)\big(D(1\otimes c)\big)$ for all c in C;*
(3) $\delta(\bar{a}\otimes c) = C^*\bar{f}\circ\beta(\sigma(\bar{a})\otimes c)$ *for all $\bar{a}$ in $\bar{A}$ and c in C.*

When we are not working over a field, as in this chapter, we cannot apply this theorem directly but have to employ more ad hoc methods, in order to obtain a result of this type. In particular, defining the full algebra structure on $(\bar{A},\bar{d})\otimes(C,e)$ and then showing that δ is a quasi-algebra map can be delicate.

Related work. As mentioned at the beginning of this subsection, our approach to algebraic modeling of fiber squares is analogous to the KS-extensions of rational homotopy theory, as developed by Sullivan [FHT]. The *Adams-Hilton model*, which to any 1-reduced CW-complex X associates a chain algebra $(AH(X),d)$ quasi-isomorphic to the cubical chains on ΩX [AH], is another particularly useful tool for algebraic modeling. The algebra $AH(X)$ is free on generators in one-to-one correspondence with the cells of X, and the differential d encodes the attaching maps.

In [An], Anick showed that the Adams-Hilton model could be endowed with a coproduct ψ, so that it became a *Hopf algebra up to homotopy*. He showed furthermore that if X is a finite r-connected CW complex of dimension at most rp, then there is a commutative cochain algebra $A(X)$ that is quasi-isomorphic to $C^*(X;\mathbb{F}_p)$, the algebra of mod p cochains on X. Using Anick's result, Menichi proved in [Me] that if $i:X\hookrightarrow Y$ is an inclusion of finite r-connected CW complexes of dimension at most rp and F is the homotopy fiber of i, then the mod p cohomology of F is isomorphic as an algebra to $\mathrm{Tor}^{A(Y)}(A(X),\mathbb{F}_p)$.

Other interesting algebraic models include the *SHC-algebras* studied by Ndombol and Thomas [NT] and *E_∞-algebras*, shown by Mandell to serve as models for p-complete homotopy theory [Man]. In particular, Mandell proved that the cochain functor $C^*(-;\overline{\mathbb{F}_p})$ embeds the category of nilpotent p-complete spaces onto a full subcategory of E_∞-algebras. He also characterized those E_∞-algebras that are weakly equivalent to the cochains on a p-complete space.

1. Free loop spaces

Consider the free loop fiber square for a simply-connected CW-complex of finite type, X.

$$\begin{array}{ccc} \mathcal{L}X & \xrightarrow{j} & X^I \\ \downarrow{\scriptstyle e} & & \downarrow{\scriptstyle (\mathrm{ev}_0,\mathrm{ev}_1)} \\ X & \xrightarrow{\Delta} & X\times X \end{array}$$

Here ev_t is defined by $\mathrm{ev}_t(\ell)=\ell(t)$, and Δ is the diagonal.

Our goal in this chapter is to construct canonically an associative cochain algebra $fls^*(X)$ together with a quasi-isomorphism $fls^*(X)\xrightarrow{\simeq}C^*\mathcal{L}X$ that induces an isomorphism of algebras in cohomology.

We construct the noncommutative model of $\mathcal{L}X$ over $\mathbb{Z}$ in three steps. First we find a model of Δ, a relatively easy exercise. The second step, in which we define a twisted extension of cochain algebras that is a model of the topological fibration $(\mathrm{ev}_0, \mathrm{ev}_1)$, requires considerably more work. Once we have obtained the models of Δ and $(\mathrm{ev}_0, \mathrm{ev}_1)$, we show that they can be "twisted together," leading to a model for $\mathcal{L}X$.

Since it is much easier to obtain precise, natural algebraic models for simplicial sets than for topological spaces, we begin this section by constructing a useful, canonical simplicial model of the free loop space. We then apply the general theory of section 0.3 to building the desired algebraic free loop space model.

1.1 A simplicial model for the free loop space.

The general model. Let $\mathcal{S}_\bullet : \mathcal{T}op \longrightarrow s\mathcal{S}et$ denote the singular simplicial set functor, which is a right adjoint to the geometric realization functor, with which it forms a Quillen equivalence. Let $\eta : Id \longrightarrow \mathcal{S}_\bullet|\cdot|$ denote the unit of the adjunction. Recall that η_L is always a weak equivalence [May].

Let X be a 1-connected space, and let K be a 1-reduced Kan complex such that $|K|$, the geometric realization of K, has the homotopy type of X. For example, we could take $K = \mathcal{S}_\bullet(X)$. Let

$$\begin{array}{ccc} K & \xrightarrow{\quad\Delta\quad} & K \times K \\ & {\scriptstyle i}\searrow_{\simeq} \qquad \nearrow{\scriptstyle p} & \\ & P & \end{array}$$

be a path object on K. Set $\mathcal{L}(K, P) := P \underset{K\times K}{\times} K$. Let $\bar{p} : \mathcal{L}(K, P) \longrightarrow K$ and $\overline{\Delta} : \mathcal{L}(K, P) \longrightarrow P$ denote the canonical maps, i.e.,

$$\begin{array}{ccc} \mathcal{L}(K, P) & \xrightarrow{\overline{\Delta}} & P \\ \downarrow{\scriptstyle \bar{p}} & & \downarrow{\scriptstyle p} \\ K & \xrightarrow{\Delta} & K \times K \end{array}$$

is the pullback diagram.

Proposition 1.1.1. *There is a weak equivalence* $\mathcal{L}(K, P) \longrightarrow \mathcal{S}_\bullet(\mathcal{L}X)$.

Proof. Since $\mathcal{S}_\bullet$ is a right adjoint, it preserves limits. It also preserves fibrations, as the right half of a Quillen equivalence. There is therefore a pullback diagram

$$\begin{array}{ccc} \mathcal{S}_\bullet(\mathcal{L}|K|) & \xrightarrow{\mathcal{S}_\bullet(j)} & \mathcal{S}_\bullet(|K|^I) \\ \downarrow{\scriptstyle \mathcal{S}_\bullet(e)} & & \downarrow{\scriptstyle (\mathcal{S}_\bullet(\mathrm{ev}_0), \mathcal{S}_\bullet(\mathrm{ev}_1))} \\ \mathcal{S}_\bullet(|K|) & \xrightarrow{\mathcal{S}_\bullet(\Delta)} & \mathcal{S}_\bullet(|K|) \times \mathcal{S}_\bullet(|K|) \end{array}$$

since $\mathcal{S}_\bullet(|K|\times|K|)\cong\mathcal{S}_\bullet(|K|)\times\mathcal{S}_\bullet(|K|)$. Consider the following diagram, which commutes by naturality of η,

$$\xymatrix{K \ar[r]^{\eta_K}_{\simeq} \ar@{>->}[d]_{\simeq}^{i} \ar[dr]^{\Delta} & \mathcal{S}_\bullet(|K|) \ar[rr]^{\mathcal{S}_\bullet(s)}_{\simeq} \ar[drr]^{\mathcal{S}_\bullet(\Delta)} && \mathcal{S}_\bullet(|K|^I) \ar@{->>}[d]^{(\mathcal{S}_\bullet(\mathrm{ev}_0),\mathcal{S}_\bullet(\mathrm{ev}_1))}\\ P \ar@{->>}[r]^{p} & K\times K \ar[rr]^{\eta_{K^2}}_{\simeq} && \mathcal{S}_\bullet(|K|)\times\mathcal{S}_\bullet(|K|)}$$

where $s:|K|\longrightarrow|K|^I$ is the usual section, sending an element x of $|K|$ to the constant path at x. Since i is an acyclic cofibration and $(\mathcal{S}_\bullet(\mathrm{ev}_0),\mathcal{S}_\bullet(\mathrm{ev}_1))$ is a fibration, there is a simplicial map $\bar\eta:P\longrightarrow\mathcal{S}_\bullet(|K|^I)$ such that

$$\xymatrix{K \ar[rr]^{\mathcal{S}_\bullet(s)\eta_K}_{\simeq} \ar@{>->}[d]_{\simeq}^{i} && \mathcal{S}_\bullet(|K|^I) \ar@{->>}[d]^{(\mathcal{S}_\bullet(\mathrm{ev}_0),\mathcal{S}_\bullet(\mathrm{ev}_1))}\\ P \ar[urr]^{\bar\eta} \ar[rr]^{\eta_{K^2}\circ p} && \mathcal{S}_\bullet(|K|)\times\mathcal{S}_\bullet(|K|)}$$

commutes. Note that by "2-out-of-3" $\bar\eta$ is a weak equivalence.

We have therefore a commutative diagram

$$\xymatrix{\mathcal{L}(K,P) \ar[drrr]^{\bar\eta\overline{\Delta}} \ar[ddr]_{\eta_K\bar p} &&&\\ & \mathcal{S}_\bullet(\mathcal{L}|K|) \ar[rr]^{\mathcal{S}_\bullet(j)} \ar@{->>}[d]^{\mathcal{S}_\bullet(e)} && \mathcal{S}_\bullet(|K|^I) \ar@{->>}[d]^{(\mathcal{S}_\bullet(\mathrm{ev}_0),\mathcal{S}_\bullet(\mathrm{ev}_1))}\\ & \mathcal{S}_\bullet(|K|) \ar[rr]^{\mathcal{S}_\bullet(\Delta)} && \mathcal{S}_\bullet(|K|)\times\mathcal{S}_\bullet(|K|)}$$

which implies that there is a simplicial map

$$\mathcal{L}(K,P)\xrightarrow{\hat\eta}\mathcal{S}_\bullet(\mathcal{L}|K|)$$

such that $\mathcal{S}_\bullet(j)\hat\eta=\bar\eta\overline{\Delta}$ and $\mathcal{S}_\bullet(e)\hat\eta=\eta_K\bar p$. By the "Cogluing Lemma" (cf. [GJ: §II.8]), it is clear that $\hat\eta$ is a weak equivalence. $\square$

For explicit computations to be possible, it is important to be able to build a simplicial model for $\mathcal{L}X$ from a simplicial set that isn't necessarily Kan. Suppose therefore that K is any 1-reduced simplicial set such that $|K|\simeq X$. Let

$$K\overset{j}{\underset{\simeq}{\rightarrowtail}}K'\twoheadrightarrow\{*\}$$

be a fibrant replacement of K.

Given path objects on K and on K',

$$K \underset{\simeq}{\overset{i}{\rightarrowtail}} P \overset{p}{\twoheadrightarrow} K \times K$$

and

$$K' \underset{\simeq}{\overset{i'}{\rightarrowtail}} P' \overset{p'}{\twoheadrightarrow} K' \times K',$$

consider the commuting square

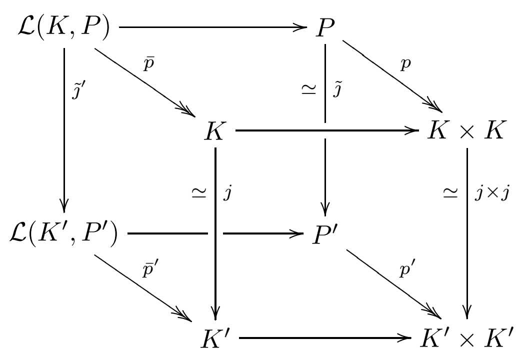
$$\begin{array}{ccccc} K & \underset{\simeq}{\xrightarrow{j}} & K' & \underset{\simeq}{\xrightarrow{i'}} & P' \\ \simeq \big\downarrow i & & & & \big\downarrow p' \\ P & \xrightarrow{p} & K \times K & \underset{\simeq}{\xrightarrow{j \times j}} & K' \times K' \end{array}$$

in which $j \times j$ is a weak equivalence. Since i is an acyclic cofibration and p' is a fibration, there is a simplicial map $\tilde{\jmath} : P \longrightarrow P'$ such that

$$\begin{array}{ccc} K & \underset{\simeq}{\xrightarrow{i'j}} & P' \\ \simeq \big\downarrow i & \overset{\tilde{\jmath}}{\nearrow} & \big\downarrow p' \\ P & \xrightarrow{(j \times j)p} & K' \times K' \end{array}$$

commutes. By "2-out-of-3", $\tilde{\jmath}$ is also weak equivalence.

If, as above, $\mathcal{L}(K, P) := P \underset{K \times K}{\times} K$ and $\mathcal{L}(K', P') := P' \underset{K' \times K'}{\times} K'$, then by the universal property of pull-backs, there is a simplicial map $\tilde{\jmath}' : \mathcal{L}(K, P) \longrightarrow \mathcal{L}(K', P')$ such that the following cube commutes.



As in the proof of Proposition 1.1.1, the "Cogluing Lemma" implies that $\tilde{\jmath}'$ is a weak equivalence. In particular, $\mathrm{H}^*(\mathcal{L}(K, P)) \cong \mathrm{H}^*(\mathcal{L}(K', P'))$, as algebras. We have thus established the following result.

Theorem 1.1.2. *Let X be a 1-connected space, and let $\mathcal{L}X$ be the free loop space on X. Let K be any 1-reduced simplicial set such that $|K|$ has the homotopy type of X. Let $K \underset{\simeq}{\overset{i}{\rightarrowtail}} P \overset{p}{\twoheadrightarrow} K \times K$ be a path object on K, and set $\mathcal{L}(K,P) := P \underset{K\times K}{\times} K$. Then there is simplicial weak equivalence*

$$\mathcal{L}(K,P) \xrightarrow{\simeq} \mathcal{S}_\bullet(\mathcal{L}X).$$

In particular, $\mathrm{H}^*(\mathcal{L}(K,P)) \cong \mathrm{H}^*(\mathcal{L}X)$ as algebras.

Choosing the free loop model functorially. Let K be a 1-reduced simplicial set. Let GK denote the Kan loop group on K, as defined in the preface. Let PK denote the twisted cartesian product with structure group $GK \times GK$

$$PK := GK \underset{\tau}{\times} (K \times K),$$

where $\tau : K \times K \longrightarrow GK \times GK : (x,y) \longmapsto (\bar{x}, \bar{y})$ and $GK \times GK$ acts on GK by $(v,w)\cdot u := vuw^{-1}$. It is easy to verify that τ satisfies the conditions of a twisted cartesian product. Observe that, in particular,

$$\partial_0(u,(x,y)) = (\bar{x}\cdot\partial_0 u\cdot\bar{y}^{-1}, (\partial_0 x, \partial_0 y)).$$

Proposition 1.1.3. *Let $p : PK \longrightarrow K \times K$ denote the projection map, and let*

$$i : K \longrightarrow PK : x \longmapsto (e,x,x).$$

Then $K \xrightarrow{i} PK \xrightarrow{p} K \times K$ is a path object on K.

Proof. Since PK is a twisted cartesian product and GK is a Kan complex, p is a Kan fibration. Furthermore, i is a simplicial cofibration, since it is obviously injective. We need therefore only to show that i is a weak equivalence.

If K is a Kan complex, then there is a long exact sequence of homotopy groups

$$\cdots \longrightarrow \pi_{n+1}(K\times K) \xrightarrow{\delta_\tau} \pi_n(GK) \longrightarrow \pi_n(PK) \longrightarrow \pi_n(K\times K) \longrightarrow \cdots,$$

where δ_τ is the connecting homomorphism, defined by $\delta_\tau\big([x,y]\big) = [\bar{x}\cdot\bar{y}^{-1}]$ (cf., [GJ:§I.7]). Comparing this long exact sequence with that obtained from the universal acyclic twisted cartesian product $GK \underset{\eta}{\times} K = EK$

$$\cdots \longrightarrow \pi_{n+1}(EK) \longrightarrow \pi_{n+1}(K) \xrightarrow{\delta_\eta} \pi_n(GK) \longrightarrow \pi_n(EK) \longrightarrow \cdots,$$

we obtain that δ_η is an isomorphism given by $\delta_\eta\big([x]\big) = [\bar{x}]$, and therefore that δ_τ is surjective. Furthermore, $\ker \delta_\tau = \operatorname{Im} \pi_{n+1}\Delta$, since

$$[x, y] \in \ker \delta_\tau \Longleftrightarrow [\bar{x} \cdot \bar{y}^{-1}] = [e] \Longleftrightarrow [\bar{x}] = [\bar{y}] \Longleftrightarrow [x] = [y] \Longleftrightarrow [x, y] = [x, x]$$

where the second equivalence comes from multiplying both sides of the equation by $[\bar{y}]$, using the multiplication induced by that on GK, while the third equivalence is due to the fact that δ_η is an isomorphism.

The first long exact sequence therefore breaks up into short exact sequences

$$0 \longrightarrow \pi_{n+1}(PK) \xrightarrow{\pi_{n+1}p} \pi_{n+1}(K \times K) \xrightarrow{\delta_\tau} \pi_n(GK) \longrightarrow 0,$$

which are split, since δ_τ has an obvious section σ defined by $\sigma\big([\bar{x}]\big) = [x, *]$, where $*$ is (an iterated degeneracy of) the basepoint of K. Consequently $\pi_{n+1}p$ has a left inverse ρ, and so

$$\pi_{n+1}i = \rho\pi_{n+1}p\pi_{n+1}i = \rho\pi_{n+1}\Delta.$$

Since $\operatorname{Im} \pi_{n+1}\Delta = \ker \delta_\tau = \operatorname{Im} \pi_{n+1}p$ and the restriction of ρ to $\operatorname{Im} \pi_{n+1}p$ is surjective, $\pi_{n+1}i$ is surjective as well. Furthermore $\pi_{n+1}i$ is necessarily injective, as $\pi_{n+1}pr_1\pi_{n+1}p$ is a left inverse to $\pi_{n+1}i$, where $pr_1 : K \times K \longrightarrow K$ is the projection map onto the first factor. Thus $\pi_{n+1}i$ is an isomorphism for all n, i.e., i is a weak equivalence.

If K is not a Kan complex, consider a fibrant replacement $K \underset{\simeq}{\overset{j}{\rightarrowtail}} K' \twoheadrightarrow \{*\}$, which, by the naturality of the P-construction, induces a map of twisted cartesian products

$$\begin{array}{ccccc} GK & \longrightarrow & PK & \overset{p}{\twoheadrightarrow} & K \\ \simeq \downarrow Gj & & \downarrow Pj & & \simeq \downarrow j \\ GK' & \longrightarrow & PK' & \twoheadrightarrow & K' \end{array}$$

in which Gj is an acyclic cofibration, since G is the left member of a Quillen equivalence. A 5-Lemma argument applied to the long exact sequences in homotopy of the realizations of these fibrations then shows that Pj is a weak equivalence. Again by naturality, the square

$$\begin{array}{ccc} K & \xrightarrow{i} & PK \\ \simeq \downarrow j & & \simeq \downarrow Pj \\ K' & \underset{\simeq}{\xrightarrow{i'}} & PK' \end{array}$$

commutes and so, by "2-out-ot-3", i is also a weak equivalence. □

We now use this functorial path object construction to define a functorial, simplicial free loop space model.

Definition. Given a 1-reduced simplicial set K, the *canonical free loop construction* on K is the the simplicial set

$$\mathcal{L}K := \mathcal{L}(K, PK) = PK \underset{K\times K}{\times} K.$$

Observe that $\mathcal{L}K$ is the twisted cartesian product $GK \underset{\bar\tau}{\times} K$, with structure group $GK \times GK$, where $GK \times GK$ acts on GK as before and

$$\bar\tau : K \longrightarrow GK \times GK : x \longmapsto (\bar x, \bar x)$$

so that

$$\partial_0(w, x) = (\bar x \cdot \partial_0 w \cdot \bar x^{-1}, \partial_0 x).$$

According to Theorem 1.1.2, there is a weak equivalence $\mathcal{L}K \xrightarrow{\simeq} \mathcal{S}_\bullet(\mathcal{L}|K|)$ and therefore an algebra isomorphism $\mathrm{H}^*(\mathcal{L}K) \cong \mathrm{H}^*(\mathcal{L}|K|)$.

1.2 The multiplicative free loop space model. In this section we apply the methods described in section 0.3 to constructing naturally a noncommutative model for the canonical free loop construction $\mathcal{L}K$ on a 1-reduced simplicial set of finite-type K. By Theorem 1.1.2 we obtain therefore a noncommutative model for the free loop space $\mathcal{L}X$ on a 1-connected space X with the homotopy type of a finite-type CW-complex.

We begin by constructing specific explicit models of the diagonal map

$$\Delta : K \longrightarrow K \times K$$

and of the path fibration

$$p : PK \longrightarrow K \times K,$$

which we then twist together appropriately, in order to obtain a model of the free loop construction.

The diagonal map. Let $\varepsilon : \Omega\mathcal{B} \longrightarrow Id$ denote the counit of the cobar-bar adjunction, which is a quasi-isomorphism for each 1-connected cochain algebra (A, d). Let

$$\gamma = \varepsilon_K : \Omega\mathcal{B}C^*K \xrightarrow{\simeq} C^*K.$$

Let α denote the following composition.

$$\begin{array}{ccccc} \Omega(\mathcal{B}C^*K \otimes \mathcal{B}C^*K) & \xrightarrow{\Omega(\rho^\sharp)} & \Omega\mathcal{B}(C^*K \otimes C^*K) & \xrightarrow{\Omega(\tilde\Omega f_{K,K})^\sharp} & \Omega\mathcal{B}C^*(K \times K) \\ & & & & \simeq \big\downarrow \varepsilon_{K\times K} \\ & & \xrightarrow{\qquad\alpha\qquad} & & C^*(K \times K) \end{array}$$

Here $\rho^\sharp$ is the dual of the map defined in equation (0.2.1), while $f_{K,K}$ is again the Alexander-Whitney map, as in section 0.2. Note that we are relying on the fact that K is of finite-type, in writing, e.g., $\mathcal{B}C^*K$ for the dual of ΩC_*K.

Given these definitions, it is an easy exercise, using the naturality of ε, to show that

$$\begin{array}{ccc} \Omega(\mathcal{B}C^*K\otimes\mathcal{B}C^*K) & \xrightarrow{\Omega\psi_K^\sharp} & \Omega\mathcal{B}C^*K \\ \simeq\downarrow\alpha & & \simeq\downarrow\gamma \\ C^*(K\times K) & \xrightarrow{C^*\Delta} & C^*K \end{array} \tag{1.2.1}$$

commutes and is therefore a model of Δ. Observe that if

$$\iota_k : \Omega\mathcal{B}C^*K \longrightarrow \Omega(\mathcal{B}C^*K\otimes\mathcal{B}C^*K)$$

is the inclusion of the kth tensor factor ($k=1,2$), then $C^*pr_k\circ\gamma=\alpha\iota_k$.

The path fibration. Henceforth, we employ the following useful notation.

Notation. For any $a,b\in\mathcal{B}C^*K$, let $a\star b:=\psi_K^\sharp(a\otimes b)$. Furthermore, let $\varphi=\Omega\psi_K^\sharp$.

We begin by defining a certain twisted extension of $\Omega(\mathcal{B}C^*K\otimes\mathcal{B}C^*K)$ by $\mathcal{B}C^*K$. Since the former is a model of $K\times K$, while the latter is a model of GK, it is reasonable to expect to be able to construct a model of PK of this form. Once we have the explicit definition of the twisted extension, we show that it is indeed a model of PK.

Definition of $\Omega(\mathcal{B}C^*K\otimes\mathcal{B}C^*K)\odot\mathcal{B}C^*K$.
Let $\Omega(\mathcal{B}C^*K\otimes\mathcal{B}C^*K)\odot\mathcal{B}C^*K$ be the twisted algebra extension determined by the following conditions.

(1) The left action of $\Omega(\mathcal{B}C^*K\otimes\mathcal{B}C^*K)$ is specified recursively by

$$\begin{aligned}\big(s^{-1}(a\otimes b)\otimes 1\big)&\cdot(1\otimes sx_1|\cdots|sx_n)\\ =&s^{-1}(a\otimes b)\otimes sx_1|\cdots|sx_n\\ &+(-1)^{b\cdot\theta_n}\Big(s^{-1}\big(a\otimes(sx_1|\cdots|sx_n\star b)\big)-s^{-1}\big((a\star sx_1|\cdots|sx_n)\otimes b\big)\Big)\\ &+\sum_{j=1}^{n-1}\Big[(-1)^{b\cdot\theta_j}s^{-1}\big((a\star sx_1|\cdots|sx_j)\otimes b\big)\cdot(1\otimes sx_{j+1}|\cdots|sx_n)\\ &\quad+(-1)^{(b+\theta_j)(\theta_n-\theta_j)}s^{-1}\big(a\otimes(sx_{j+1}|\cdots|sx_n)\star b\big)\otimes sx_1|\cdots|sx_j\Big]\end{aligned}$$

where $\theta_j=j+\sum_{i\le j}|x_i|$.

(2) The restriction of the differential D of the twisted extension to $1\otimes \mathcal{B}C^*K$ is specified recursively by

$$
\begin{aligned}
D(1\otimes sx_1|\cdots|sx_n) =&1\otimes d_{\mathcal{B}}(sx_1|\cdots|sx_n)\\
&+s^{-1}(sx_1|\cdots|sx_n\otimes 1)-s^{-1}(1\otimes sx_1|\cdots|sx_n)\\
&+\sum_{j=1}^{n-1}\big[s^{-1}(sx_1|\cdots|sx_j\otimes 1)\cdot(1\otimes sx_{j+1}|\cdots|sx_n)\\
&\quad -(-1)^{\theta_j(\theta_n-\theta j)}s^{-1}(1\otimes sx_{j+1}|\cdots|sx_n)\otimes sx_1|\cdots|sx_j\big]
\end{aligned}
$$

with θ_j as above.

Observe that according to this definition, if $c=sx_1|\cdots|sx_n$, then $D(1\otimes c)$ is of the following form.

$$D(1\otimes c)=1\otimes d_{\mathcal{B}}c+\sum_{i<j}s^{-1}\big(\lambda_{ij,k}(c)\otimes\lambda_{ij}^k(c)\big)\otimes sx_i|\cdots|sx_j$$

where we are applying the Einstein summation convention (cf., section 0.1).

(3) Let $a=sx_1|\cdots|sx_m, b=sy_1|\cdots|sy_n\in\mathcal{B}C^*K$. Then we define

$$
\begin{aligned}
&(1\otimes a)(1\otimes b)\\
&\quad=1\otimes a\star b\\
&\quad\quad+\sum_{\substack{i<j\\p<q}}\Big[\pm s^{-1}\big(\Theta^\sharp(\lambda_{ij,k}(a)\otimes\lambda_{pq,r}(b))\otimes\lambda_{ij}^k(a)\star\lambda_{pq}^r(b)\big)\\
&\quad\quad\quad\pm s^{-1}(\lambda_{ij,k}(a)\star\lambda_{pq,r}(b)\otimes\Theta^\sharp(\lambda_{ij}^k(a)\otimes\lambda_{pq}^r(b)))\\
&\quad\quad\quad\pm s^{-1}\big(\Theta^\sharp(\lambda_{ij,k}(a)_s\otimes\lambda_{pq,r}(b)_t)\otimes\lambda_{ij}^k(a)_u\star\lambda_{pq}^r(b)_v\big)\cdot\\
&\quad\quad\quad\quad\cdot s^{-1}(\lambda_{ij,k}(a)^s\star\lambda_{pq,r}(b)^t\otimes\Theta^\sharp(\lambda_{ij}^k(a)^u\otimes\lambda_{pq}^r(b)^v))\Big]\\
&\quad\quad\quad\quad\quad\otimes(sx_i|\cdots|sx_j)\star(sy_p|\cdots|sy_q)
\end{aligned}
$$

where we have suppressed the relatively obvious, though horrible to specify, signs given by the Koszul convention (cf., section 0.1). Here $\Theta^\sharp$ is the coderivation homotopy for the commutativity of the multiplication in $\mathcal{B}C^*K$, dual of the derivation homotopy Θ for the cocommutativity of ψ_K, and the extra lower and upper indices in the last summand indicate factors of the coproduct evaluated on the terms in question, e.g., $\Delta(\lambda_{ij,k}(a))=\lambda_{ij,k}(a)_s\otimes\lambda_{ij,k}(a)^s$.

Observe that, in particular,

$$(1\otimes a)(1\otimes b)\in T^{\leq 2}s^{-1}\mathcal{B}C^*K\otimes\mathcal{B}C^*K.$$

Dupont and Hess showed in [DH4] that the conditions above specify an associative cochain algebra, when the product $\psi_K^\sharp$ is strictly commutative, so that condition (3) reduces to $(1\otimes a)\cdot(1\otimes b) = 1\otimes(a\star b)$. The general, homotopy-commutative case was established by Blanc in his thesis [Bl].

Extend $\varphi : \Omega(\mathcal{B}C^*K\otimes\mathcal{B}C^*K)\longrightarrow\Omega\mathcal{B}C^*K$ to

$$\overline{\varphi} : \Omega(\mathcal{B}C^*K\otimes\mathcal{B}C^*K)\odot\mathcal{B}C^*K\longrightarrow\Omega\mathcal{B}C^*K$$

by $\overline{\varphi}(1\otimes w) = 0$ for all $w\in(\mathcal{B}C^*K)^+$. It is easy to see that $\overline{\varphi}$ is a differential map and that, as explained in [DH2], a straightforward spectral sequence argument shows as well that $\overline{\varphi}$ is a quasi-isomorphism. If $\psi_K^\sharp$ is strictly commutative, then $\overline{\varphi}$ is a strict algebra map, as proved in [DH4]. More generally, Blanc showed in [Bl] that $\overline{\varphi}$ is a quasi-algebra map when $\psi_K^\sharp$ is commutative up to a derivation homotopy. We have thus established the following result.

Lemma 1.2.1. *There is a commuting diagram*

$$\begin{array}{ccc}
\Omega(\mathcal{B}C^*K\otimes\mathcal{B}C^*K) & \xrightarrow{\qquad\qquad\varphi\qquad\qquad} & \Omega\mathcal{B}C^*K \\
& {\searrow}^{\iota} \qquad\qquad {\nearrow}^{\overline{\varphi}}_{\simeq} & \\
& \Omega(\mathcal{B}C^*K\otimes\mathcal{B}C^*K)\odot\mathcal{B}C^*K &
\end{array}$$

where ι is the inclusion, $\overline{\varphi}(1\odot w) = 0$ for all $w\in(\mathcal{B}C^*K)^+$, and $\overline{\varphi}$ is a quasi-algebra quasi-isomorphism.

Now consider the following commutative diagram

$$\begin{array}{ccccc}
\Omega(\mathcal{B}C^*K\otimes\mathcal{B}C^*K) & \xrightarrow[\simeq]{\alpha} & C^*(K\times K) & \xrightarrow{C^*p} & C^*PK \\
\downarrow{\scriptstyle\iota} & & & & \downarrow{\scriptstyle C^*i} \\
\Omega(\mathcal{B}C^*K\otimes\mathcal{B}C^*K)\odot\mathcal{B}C^*K & \xrightarrow[\simeq]{\overline{\varphi}} & \Omega\mathcal{B}C^*K & \xrightarrow[\simeq]{\gamma} & C^*K
\end{array}$$

which satisfies the conditions of Proposition 0.3.1. Hence, we can lift $\overline{\varphi}\gamma$ through C^*i, obtaining a quasi-algebra map

$$\beta : \Omega(\mathcal{B}C^*K\otimes C^*K)\odot\mathcal{B}C^*K\longrightarrow C^*PK$$

such that $\beta\iota = C^*p\circ\alpha$ and $C^*i\beta = \gamma\circ\overline{\varphi}$. We can therefore take

$$\begin{array}{ccc}
\Omega(\mathcal{B}C^*K\otimes\mathcal{B}C^*K) & \xrightarrow{\iota} & \Omega(\mathcal{B}C^*K\otimes\mathcal{B}C^*K)\odot\mathcal{B}C^*K \\
\simeq\downarrow{\scriptstyle\alpha} & & \simeq\downarrow{\scriptstyle\beta} \\
C^*(K\times K) & \xrightarrow{\qquad C^*p\qquad} & C^*PK
\end{array}\tag{1.2.2}$$

as an algebraic model of the path fibration.

The free loop space model. We now twist together the models (1.2.1) and (1.2.2), in the spirit of Theorem 0.3.2.

Theorem 1.2.2 [DH4],[Bl]**.** *There is a twisted algebra extension*

$$\bar{\iota}:\Omega\mathcal{B}C^*K \longrightarrow \Omega\mathcal{B}C^*K\odot \mathcal{B}C^*K$$

and a quasi-algebra quasi-isomorphism

$$\delta:\Omega\mathcal{B}C^*K\odot \mathcal{B}C^*K \xrightarrow{\simeq} C^*(\mathcal{L}K)$$

defined as follows.

(1) *The left action of $\Omega\mathcal{B}C^*K$ is specified recursively by*

$$\begin{aligned}\big(s^{-1}a\otimes 1\big)&\cdot(1\otimes sx_1|\cdots|sx_n)\\ &=s^{-1}a\otimes sx_1|\cdots|sx_n\\ &\quad-\sum_{j=1}^{n-1}\Big[s^{-1}\big(a\star(sx_1|\cdots|sx_j)\big)\cdot(1\otimes sx_{j+1}|\cdots|sx_n)\\ &\qquad+(-1)^{(\theta_j)(\theta_n-\theta_j)}s^{-1}\big(a\star(sx_{j+1}|\cdots|sx_n)\big)\otimes sx_1|\cdots|sx_j\Big]\end{aligned}$$

where $\theta_j=j+\sum_{i\leq j}|x_i|$.

(2) *The restriction of the differential $\overline{D}$ of the twisted extension to $1\otimes\mathcal{B}C^*K$ is specified recursively by*

$$\begin{aligned}\overline{D}(1\otimes sx_1|\cdots|sx_n)=&1\otimes d_{\mathcal{B}}(sx_1|\cdots|sx_n)\\ &+\sum_{j=1}^{n-1}\big[(s^{-1}(sx_1|\cdots|sx_j)\otimes 1)\cdot(1\otimes sx_{j+1}|\cdots|sx_n)\\ &\quad-(-1)^{\theta_j(\theta_n-\theta j)}s^{-1}(sx_{j+1}|\cdots|sx_n)\otimes sx_1|\cdots|sx_j\big]\end{aligned}$$

with θ_j as above.

(3) *Let $a=sx_1|\cdots|sx_m, b=sy_1|\cdots|sy_n\in\mathcal{B}C^*K$. Then we define*

$$\begin{aligned}(1\otimes a)&(1\otimes b)\\ &=1\otimes a\star b\\ &\quad+\sum_{\substack{i<j\\ p<q}}(-1)^{\zeta_{ij,pq}}s^{-1}\Big(\Theta^\sharp\big(\lambda_{ij,k}(a)\star\lambda^k_{ij}(a)\otimes\lambda_{pq,r}(b)\big)\star\lambda^r_{pq}(b)\Big)\\ &\qquad\otimes(sx_i|\cdots|sx_j)\star(sy_p|\cdots|sy_q).\end{aligned}$$

Here we are using the same notation as in the definition of the path space model, and

$$\zeta_{ij,pq}=\big(|\lambda_{pq,r}(b)|+|\lambda^r_{pq}(b)|\big)\cdot(\theta_j-\theta_{i-1}).$$

(4) *For all* $a, b \in \mathcal{B}C^*K$,

$$\begin{aligned}\delta(s^{-1}a \otimes b) :&= C^*\overline{\Delta} \circ \beta(s^{-1}(a \otimes 1) \otimes b)\\ &= C^*\overline{\Delta} \circ \beta(1 \otimes b) \cdot C^*\bar{p} \circ \gamma(s^{-1}a).\end{aligned}$$

This theorem was proved in the strictly commutative case in [DH4] and in the homotopy-commutative case in [Bl].

In the article [BH] based on part of Blanc's thesis, we use this model to compute the free loop space cohomology algebra of a space X not in the "Anick" range, i.e., where the space does not have a strictly commutative model. This is the first time an explicit calculation of this type has been carried out.

Related work. Our free loop space model construction is patterned on the construction for rational spaces, due to Sullivan and Vigué [SV]. They used their construction to prove the rational *Closed Geodesic Conjecture*.

Kuribayashi, alone [K] and with Yamaguchi [KY], has applied the Eilenberg-Moore spectral sequence to calculating the cohomology algebra of certain free loop spaces.

Let $\mathfrak{C}_*(A)$ denote the Hochschild complex of a (co)chain algebra A. Let N^*X denote the normalized singular cochains on a topological space X with coefficients in a field $\Bbbk$. Ndombol and Thomas have applied SHC-algebra methods to showing that there is a natural map of cochain complexes $\mathfrak{C}_*(N^*X) \longrightarrow C^*(\mathcal{L}X)$ inducing an isomorphism of graded algebras in cohomology. In a similar vein, Menichi proved in [Me2] that $\mathrm{H}^*(\mathcal{L}X)$ is isomorphic to the Hochschild cohomology of the singular chains on ΩX.

Very recently Bökstedt and Ottosen have developed yet another promising method for calculating free loop space cohomology, via a Bousfield spectral sequence [BO1]. Their method, which they have applied explicitly to spaces X with $\mathrm{H}^*(X; \mathbb{F}_2)$ a truncated polynomial algebra on one generator, allows them to obtain the module structure of $\mathrm{H}^*(X; \mathbb{F}_2)$ over the Steenrod algebra.

1.3 The free loop model for topological spaces.

In this section we adapt the free loop model above to the case of topological spaces, in order to facilitate construction of a model of the S^1-homotopy orbits on the free loop space. We sacrifice perhaps a bit of the multiplicative structure of the model in section 1.2, but since we are interested in this chapter only in the linear structure of the homotopy orbit cohomology, this is not a great loss.

As usual let X be a 1-connected space with the homotopy type of a finite-type CW-complex, and let K be a finite-type, 1-reduced simplicial set such that $|K| \simeq X$. We again adopt the notation φ for $\Omega\psi_K^\sharp$, where $(\Omega C_*K, \psi_K)$ is the canonical Adams-Hilton model of section 0.2. Recall furthermore that ε_K denotes the unit of the cobar-bar adjunction.

The unit η of the $(|\cdot|, \mathcal{S}_\bullet)$-adjunction induces an injective quasi-isomorphism of chain coalgebras

$$C_*\eta_K : C_*K \xrightarrow{\simeq} S_*|K|$$

that dualizes to a surjective quasi-isomorphism of cochain algebras

$$C^*\eta_K : S^*|K| \xrightarrow{\simeq} C^*K\ .$$

Furthermore, for all spaces Y, subdivision of cubes defines a quasi-isomorphism of chain coalgebras

$$CU_*Y \xrightarrow{\simeq} S_*Y,$$

which, upon dualization, gives rise to a quasi-isomorphism of cochain algebras

$$S^*Y \xrightarrow{\simeq} CU^*Y.$$

We begin the topological adaptation of the model for the simplicial free loop space by carefully specifying a model of the diagonal map, directly in the cubical cochains, which is where we know how to work explicitly with circle actions, as we show in section 2.

Theorem 1.3.1. *There is a noncommutative model of the diagonal map*

$$\begin{array}{ccc} \Omega(\mathcal{B}C^*K\otimes \mathcal{B}C^*K) & \xrightarrow{\varphi} & \Omega\mathcal{B}C^*K \\ \simeq\downarrow\tilde\alpha & & \simeq\downarrow\tilde\gamma \\ CU^*(X\times X) & \xrightarrow{CU^*\Delta} & CU^*X \end{array}$$

such that

(1) $\tilde\alpha\circ\iota_k = C^*pr_k\circ\tilde\gamma$ *for* $k=1,2$, *where*

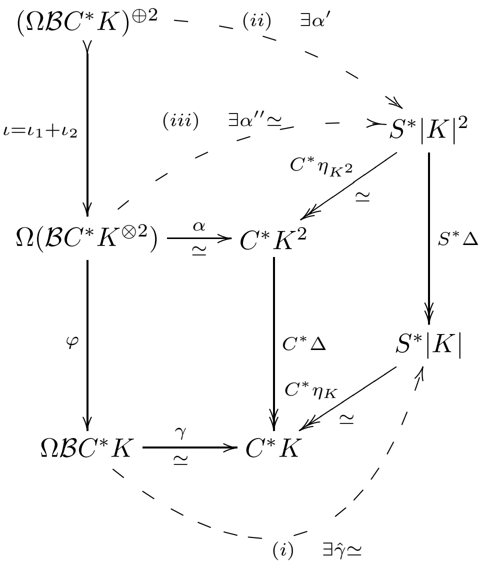
$$\iota_k : \Omega\mathcal{B}C^*K \longrightarrow \Omega(\mathcal{B}C^*K\otimes \mathcal{B}C^*K)$$

is the inclusion into the k*th tensor factor and* pr_k *is the projection onto the* k*th factor, and*

(2) $\ker\varepsilon_K\subseteq\ker\tilde\gamma$.

We do not claim here that $\tilde\gamma$ and $\tilde\alpha$ are strict algebra maps. In fact, it is probably impossible in general for $\tilde\gamma$ simultaneously to be an algebra map and to satisfy $\ker\varepsilon_K\subseteq\ker\tilde\gamma$. We show in the proof, however, that $\tilde\gamma$ and $\tilde\alpha$ are at least quasi-algebra maps.

Proof. Consider the following commutative diagram, in which the solid arrows are given and are all cochain algebra maps. We will explain step-by-step the construction and properties of the dashed arrows.



Step (i) Consider the following commutative diagram of cochain complexes.

$$\begin{array}{ccc} \ker \varepsilon_K & \xrightarrow{\quad 0 \quad} & S^*|K| \\ {\scriptstyle \text{incl.}}\downarrow & & \simeq \downarrow {\scriptstyle C^*\eta_K} \\ \Omega\mathcal{B}C^*K & \underset{\simeq}{\xrightarrow{\gamma=\varepsilon_K}} & C^*K \end{array}$$

Since $C^*\eta_K$ is a surjective quasi-isomorphism and the inclusion map on the left is a free extension of cochain complexes, there exists $\hat{\gamma} : \Omega\mathcal{B}C^*K \longrightarrow S^*|K|$ such that $\ker \varepsilon_K \subset \ker \hat{\gamma}$ and $S^*|K| \circ \hat{\gamma} = \varepsilon_K$. In particular $\hat{\gamma}$ is a quasi-isomorphism by "2-out-of-3" and $\mathrm{H}^*\,\hat{\gamma} = (\mathrm{H}^*\,\eta_K)^{-1} \circ \mathrm{H}^*\,\gamma$ is a map of algebras, i.e., $\hat{\gamma}$ is a quasi-algebra map.

Step (ii) Let $\alpha' = S^*pr_1 \circ \hat{\gamma} + S^*pr_2 \circ \hat{\gamma}$. Since $\varphi\iota_k = Id$ for $k = 1, 2$ and $S^*\Delta \circ S^*pr_k = Id$, the diagram

$$\begin{array}{ccc} (\Omega\mathcal{B}C^*K)^{\oplus 2} & \xrightarrow{\alpha'} & S^*|K|^2 \\ {\scriptstyle \varphi\iota}\downarrow & & \downarrow {\scriptstyle S^*\Delta} \\ \Omega\mathcal{B}C^*K & \xrightarrow{\hat{\gamma}} & S^*|K| \end{array}$$

commutes, by the universal property of the direct sum.

Step (iii) Consider finally the commutative diagram below of cochain complexes and maps.

$$\begin{array}{ccc} (\Omega\mathcal{B}C^*K)^{\oplus 2} & \xrightarrow{\alpha'} & S^*|K|^2 \\ \downarrow{\scriptstyle\iota} & & \simeq\downarrow{\scriptstyle C^*\eta_{K^2}} \\ \Omega(\mathcal{B}C^*K^{\otimes 2}) & \xrightarrow{\alpha} & C^*K^2 \end{array}$$

As usual, since $C^*\eta_{K^2}$ is a surjective quasi-isomorphism and ι is a free extension of cochain complexes, α lifts to $\alpha'' : \Omega(\mathcal{B}C^*K^{\otimes 2}) \longrightarrow S^*|K|^2$, which is a quasi-algebra quasi-isomorphism such that $\alpha''\iota = \alpha'$ and $C^*\eta_{K^2} \circ \alpha'' = \alpha$.

Notice that

$$\begin{aligned} C^*\eta_K \circ S^*\Delta \circ \alpha'' &= C^*\Delta \circ C^*\eta_{K^2} \circ \alpha'' \\ &= C^*\Delta \circ \alpha \\ &= \gamma \circ \varphi \\ &= C^*\eta_K \circ \hat{\gamma} \circ \varphi. \end{aligned}$$

We would like to have $S^*\Delta \circ \alpha'' = \hat{\gamma} \circ \varphi$, but, as we show next, all we can be sure of is that $S^*\Delta \circ \alpha'' \simeq \hat{\gamma} \circ \varphi$, as cochain maps. We then apply the homotopy extension property of free extensions to "fix" α'' and thus conclude the proof.

For any cochain complex, (A, d), with free underlying graded abelian group, let $I(A, d) = (A \oplus A \oplus sA, D)$ denote its canonical cylinder. Let $\mathfrak{j}_k : A \longrightarrow I(A, d)$ denote the inclusion onto the kth summand ($k = 1, 2$). Note that $\mathfrak{j}_k$ is always a quasi-isomorphism.

Let (P, d) be the cochain complex such that the following diagram is a push-out in the category of cochain complexes.

$$\begin{array}{ccc} \Omega(\mathcal{B}C^*K)^{\oplus 2} \oplus \Omega(\mathcal{B}C^*K)^{\oplus 2} & \xrightarrow{\mathfrak{j}_1 + \mathfrak{j}_2} & I(\Omega(\mathcal{B}C^*K)^{\oplus 2}) \\ \downarrow{\scriptstyle\iota\oplus\iota} & & \downarrow \\ \Omega(\mathcal{B}C^*K^{\otimes 2}) \oplus \Omega(\mathcal{B}C^*K^{\otimes 2}) & \longrightarrow & (P, d) \end{array}$$

We then have a commuting diagram

$$\begin{array}{ccc} (P, d) & \xrightarrow{(S^*\Delta\circ\alpha'' + \hat{\gamma}\varphi) + G} & S^*|K| \\ \downarrow{\scriptstyle \text{incl.}} & & \simeq\downarrow{\scriptstyle C^*\eta_K} \\ I\big(\Omega(\mathcal{B}C^*K^{\otimes 2})\big) & \xrightarrow{H} & C^*K \end{array}$$

where G is the constant homotopy (i.e., $G(s\Omega(\mathcal{B}C^*K)^{\oplus 2}) = \{0\}$) from $S^*\Delta \circ \alpha' = \hat{\gamma}\varphi\iota$ to itself and H is the constant homotopy from $C^*\Delta \circ \alpha = \gamma\varphi$ to itself. Again, since $C^*\eta_K$ is a surjective quasi-isomorphism and the inclusion on the left is a free extension, we can lift the homotopy H to

$$H' : I\big(\Omega(\mathcal{B}C^*K^{\otimes 2})\big) \longrightarrow S^*|K|$$

such that

$$H'|_{(P,d)} = (S^*\Delta \circ \alpha'' + \hat{\gamma}\varphi) + G,$$

i.e., H' is a homotopy from $S^*\Delta \circ \alpha''$ to $\hat{\gamma}\varphi$ that is constant on the subcomplex $\Omega(\mathcal{B}C^*K)^{\oplus 2}$.

Let $G' : I(\Omega(\mathcal{B}C^*K)^{\oplus 2}) \longrightarrow S^*|K|^2$ be the constant homotopy from α' to itself. In particular, $S^*\Delta \circ G' = G$. Consider the push-out diagram of cochain complexes

$$\begin{array}{ccccc}
\Omega(\mathcal{B}C^*K)^{\oplus 2} & \overset{\mathfrak{j}_1}{\underset{\simeq}{\rightarrowtail}} & I(\Omega(\mathcal{B}C^*K)^{\oplus 2}) & & \\
\downarrow \iota & & \downarrow & \searrow G' & \\
\Omega(\mathcal{B}C^*K^{\otimes 2}) & \overset{\hat{\mathfrak{j}}_1}{\underset{\simeq}{\rightarrowtail}} & (Q,d) & \overset{\alpha''+G'}{\underset{\simeq}{\dashrightarrow}} & S^*|K|^2 \\
\end{array}$$

$$\alpha'' : \Omega(\mathcal{B}C^*K^{\otimes 2}) \xrightarrow{\simeq} S^*|K|^2$$

in which all arrows are free extensions, since the push-out of a cofibration is a cofibration. Furthermore, $\hat{\mathfrak{j}}_1$ is a quasi-isomorphism because the push-out of an acyclic cofibration is an acyclic cofibration. By "2-out-of-3", $\alpha'' + G'$ is a quasi-isomorphism.

Let $\mathfrak{k} : I(\Omega(\mathcal{B}C^*K)^{\oplus 2}) \rightarrowtail I\big(\Omega(\mathcal{B}C^*K^{\otimes 2})\big)$ be the natural inclusion. The triangle

$$\Omega(\mathcal{B}C^*K^{\otimes 2}) \overset{\hat{\mathfrak{j}}_1}{\underset{\simeq}{\rightarrowtail}} (Q,d) \overset{\mathfrak{j}_1+\mathfrak{k}}{\rightarrowtail} I\big(\Omega(\mathcal{B}C^*K^{\otimes 2})\big), \qquad \mathfrak{j}_1 : \Omega(\mathcal{B}C^*K^{\otimes 2}) \xrightarrow{\simeq} I\big(\Omega(\mathcal{B}C^*K^{\otimes 2})\big)$$

then commutes, proving that $\mathfrak{j}_1 + \mathfrak{k}$ is also a quasi-isomorphism.

We now have a commuting diagram

$$\begin{array}{ccc}
(Q,d) & \overset{\alpha''+G'}{\underset{\simeq}{\longrightarrow}} & S^*|K|^2 \\
\simeq \downarrow \mathfrak{j}_1+\mathfrak{k} & & \downarrow S^*\Delta \\
I\big(\Omega(\mathcal{B}C^*K^{\otimes 2})\big) & \overset{H'}{\longrightarrow} & S^*|K|
\end{array}$$

and therefore, since $\mathfrak{j}_1 + \mathfrak{k}$ is a free extension and a quasi-isomorphism and $S^*\Delta$ is surjective, there is a homotopy $\hat{H} : I\big(\Omega(\mathcal{B}C^*K^{\otimes 2})\big) \longrightarrow S^*|K|^2$ such that $\widehat{H} \circ (\mathfrak{j}_1 + \mathfrak{k}) = \alpha'' + G'$ and $S^*\Delta \circ \widehat{H} = H'$. Furthermore, $\widehat{H}$ is a quasi-isomorphism by "2-out-of-3".

Let $\hat{\alpha} = \widehat{H} \circ \mathfrak{j}_2$. Then $\hat{\alpha}$ is a quasi-algebra quasi-isomorphism, as it is homotopic to a quasi-algebra quasi-isomorphism. Moreover,

$$\hat{\alpha}\iota = \widehat{H}\mathfrak{k}\mathfrak{j}_2 = G'\mathfrak{j}_2 = \alpha' = S^*pr_1 \circ \hat{\gamma} + S^*pr_2 \circ \hat{\gamma},$$

while

$$S^*\Delta \circ \hat{\alpha} = H'\mathfrak{j}_2 = \hat{\gamma}\varphi,$$

i.e, the following diagram commutes exactly and is therefore a noncommutative model.

$$\begin{array}{ccc} \Omega(\mathcal{B}C^*K \otimes \mathcal{B}C^*K) & \xrightarrow{\varphi} & \Omega\mathcal{B}C^*K \\ \simeq \downarrow \hat{\alpha} & & \simeq \downarrow \hat{\gamma} \\ S^*|K|^2 & \xrightarrow{S^*\Delta} & S^*|K| \end{array}$$

To complete the proof, compose $\hat{\alpha}$ and $\hat{\gamma}$ with the natural cochain algebra quasi-isomorphisms

$$S^*|K|^2 \xrightarrow{\simeq} CU^*X^2 \quad \text{and} \quad S^*|K| \xrightarrow{\simeq} CU * X$$

to obtain $\tilde{\alpha}$ and $\tilde{\gamma}$. □

The next step in our simplification of the free loop model is to find an appropriate model for the topological path fibration

$$p : X^I \twoheadrightarrow X \times X : \ell \longmapsto (\ell(0), \ell(1)) .$$

The advantage to working directly with cubical cochains is that we are able to define an explicit model morphism, extending $\tilde{\alpha}$, from $\Omega(\mathcal{B}C^*K \otimes \mathcal{B}C^*K) \odot \mathcal{B}C^*K$ (defined exactly as in section 1.2) to CU^*X^I. This may be possible simplicially as well, but the formulas are most probably not nearly as simple.

Given $x \in X$, let ℓ_x denote the constant path at x. Let $i : X \longrightarrow X^I$ be defined by $i(x) = \ell_x$. Let $H : CU_*X^I \longrightarrow CU_{*+1}X^I$ be the chain homotopy such that for all $T \in CU_nX^I$,

$$H(T)(t_0, ..., t_n) := T(t_1, ..., t_n)(t_0 \cdot -) : I \longrightarrow X.$$

It is easy to see that $[d, H](T) = \ell_{T(-,...,-)(0)} - T$, i.e., $[d, H] = \iota\pi - Id$, where

$$\iota : \operatorname{Im} CU_*i \longrightarrow CU_*X^I$$

is the inclusion and

$$\pi : CU_*X^I \longrightarrow \operatorname{Im} CU_*i$$

is defined by $\pi(T) = \ell_{T(-,...,-)(0)}$. Observe that $H\iota = 0$, as $H(T)$ is degenerate if the path $T(t_1,...,t_n)$ is constant for all $(t_1,...,t_n)$. The homotopy H therefore induces a homotopy

$$H' : \operatorname{coker} CU_*i \longrightarrow \operatorname{coker} CU_{*+1}i : [T] \longmapsto [H(T)]$$

satisfying $[d, H'] = -Id$, i.e., H' is a contracting homotopy.

Upon dualizing, we obtain a cochain homotopy

$$J = -(H')^\sharp : \ker CU^*i \longrightarrow \ker CU^{*-1}i$$

such that $[d^\sharp, J] = Id$. Note that $[d^\sharp, J] = Id$ implies that $d^\sharp J^2 = J^2 d^\sharp$.

It is important for the constructions in section 2 to observe as well that J is a $(0, Id)$-derivation, i.e.,

$$J(fg) = J(f) \cdot g$$

for all $f, g \in \ker CU^*i$, since $\overline{\Delta}H = (\iota\pi \otimes H - H \otimes Id)\overline{\Delta}$, which implies that $\overline{\Delta}H' = -(H' \otimes Id)\overline{\Delta}$, where $\overline{\Delta}$ denotes the reduced coproduct on CU_*X^I or $\operatorname{coker} CU_*i$.

We resume the construction above in the following proposition.

Proposition 1.3.2. *Let $i : X \longrightarrow X^I$ be defined by $i(x) = \ell_x$, the constant path at x. There is a natural cochain homotopy*

$$J : \ker CU^*i \longrightarrow \ker CU^{*-1}i$$

such that

1. $[d^\sharp, J] = Id$,
2. $d^\sharp J^2 = J^2 d^\sharp$, *and*
3. $J(fg) = J(f) \cdot g$.

We now apply J to giving an explicit definition of a model morphism $\tilde{\beta}$ from $\Omega(\mathcal{B}C^*K \otimes \mathcal{B}C^*K) \odot \mathcal{B}C^*K$ to CU^*X^I.

Proposition 1.3.3. *There is a noncommutative model*

$$\begin{array}{ccc}
\Omega(\mathcal{B}C^*K \otimes \mathcal{B}C^*K) & \xrightarrow{\iota} & \Omega(\mathcal{B}C^*K \otimes \mathcal{B}C^*K) \odot \mathcal{B}C^*K \\
\simeq \downarrow \tilde{\alpha} & & \simeq \downarrow \tilde{\beta} \\
CU^*(X \times X) & \xrightarrow{CU^*p} & CU^*X^I
\end{array}$$

where $\tilde{\beta}$ is defined recursively by

$$\tilde{\beta}(w \otimes c) = (-1)^{wc}\tilde{\beta}(1 \otimes c) \cdot CU^*p \circ \tilde{\alpha}(w)$$

and

$$\tilde{\beta}(1 \otimes c) = J\tilde{\beta}D(1 \otimes c).$$

Proof. Suppose that if $\tilde{\beta}$ is defined in accordance with the formulas in the statement above on $\Omega(\mathcal{B}C^*K \otimes \mathcal{B}C^*K) \odot (\mathcal{B}C^*K)_{<n}$, then $d^\sharp\tilde{\beta} = \tilde{\beta}D$ and $CU^*i \circ \tilde{\beta} = \tilde{\gamma} \circ \bar{\varphi}$ when restricted to this same complex.

Let $c \in (\mathcal{B}C^*K)_n$. It is clear from the definition of D that $\tilde{\gamma}\bar{\varphi}D(1 \otimes c) = 0$. Thus, by the induction hypothesis, $\tilde{\beta}D(1 \otimes c) \in \ker CU^*i$, which implies that $J\tilde{\beta}D(1 \otimes c)$ is defined. Moreover,

$$\begin{aligned} d^\sharp J\tilde{\beta}D(1 \otimes c) &= \tilde{\beta}D(1 \otimes c) - Jd^\sharp\tilde{\beta}D(1 \otimes c) \\ &= \tilde{\beta}D(1 \otimes c) - J\tilde{\beta}D^2(1 \otimes c) \\ &= \tilde{\beta}D(1 \otimes c) \end{aligned}$$

and $CU^*iJ\tilde{\beta}D(1 \otimes c) = 0 = \tilde{\gamma}\varphi(1 \otimes c)$, so we can set

$$\tilde{\beta}(1 \otimes c) = J\tilde{\beta}D(1 \otimes c).$$

For the usual reasons, $\tilde{\beta}$ is then a quasi-algebra quasi-isomorphism. □

We can now twist together the models of Theorem 1.3.1 and of Proposition 1.3.3, obtaining a noncommutative model

$$\begin{array}{ccc} \Omega\mathcal{B}C^*K & \longrightarrow & \Omega\mathcal{B}C^*K \odot \mathcal{B}C^*K \\ \simeq \downarrow \tilde{\gamma} & & \simeq \downarrow \tilde{\delta} \\ CU^*X & \xrightarrow{CU^*e} & CU^*\mathcal{L}X \end{array}$$

where $\Omega\mathcal{B}C^*K \odot \mathcal{B}C^*K$ is defined exactly as in section 1.2 and

$$\tilde{\delta}(w \otimes c) := (-1)^{wc}\tilde{\delta}(1 \otimes c) \cdot CU^*e \circ \tilde{\gamma}(w)$$

and

$$\tilde{\delta}(1 \otimes c) := CU^*jJ\tilde{\beta}D(1 \otimes c).$$

Recall that $j : \mathcal{L}X \longrightarrow X^I$ is the natural inclusion and that $e : \mathcal{L}X \twoheadrightarrow X$ is the basepoint evaluation.

An easy Zeeman's comparison theorem argument shows that $\tilde{\delta}$ is a quasi-isomorphism. Furthermore, applying the Eilenberg-Moore spectral sequence of algebras, one obtains that $\tilde{\delta}$ induces an algebra isomorphism on the E_∞-terms, which is not quite as strong as saying that it is a quasi-algebra map, but is good enough for our purposes in chapter 2. By arguments similar to those in [DH4], we can show that $\tilde{\delta}$ is truly a quasi-algebra map in certain special cases, and it may perhaps be a quasi-algebra map in general.

1.4 Linearization of the free loop model. In this section we simplify even further the free loop model, making it as small as possible, to facilitate homotopy orbit space computations in section 2.

Consider the surjection

$$\varepsilon_K \otimes Id : \Omega\mathcal{B}C^*K \otimes \mathcal{B}C^*K \longrightarrow C^*K \otimes \mathcal{B}C^*K.$$

Extend the differential on C^*K to a differential $\check{D}$ on $C^*K \otimes \mathcal{B}C^*K$, which is a free right C^*K-module, by

$$\check{D}(1 \otimes c) := (\varepsilon_K \otimes Id)\overline{D}(1 \otimes c),$$

extended as a right module derivation. Consequently, $\check{D}(\varepsilon_K \otimes Id) = (\varepsilon_K \otimes Id)\overline{D}$, which implies in turn that $\check{D}^2 = 0$.

Let $\pi : \mathcal{B}C^*K \longrightarrow C^*K$ be the projection onto (desuspended) linear terms, i.e.,

$$\pi(sx_1|\cdots|sx_n) := \begin{cases} x_1 : n = 1 \\ \ 0 : n > 1. \end{cases}$$

Using the notation introduced in the definition of $\Omega(\mathcal{B}C^*K \otimes \mathcal{B}C^*K) \odot \mathcal{B}C^*K$ in section 1.2, we obtain the following explicit formula for $\check{D}$, when $c = sx_1|\cdots|sx_n$.

$$\check{D}(1 \otimes c) = 1 \otimes d_{\mathcal{B}}c + \sum_{i<j} \pi\big(\lambda_{ij,k}(c) \star \lambda^k_{ij}(c)\big) \otimes sx_i|\cdots|sx_j.$$

Notice that it is entirely possible that $\check{D}(1 \otimes c) \neq 1 \otimes d_{\mathcal{B}}c$, since the formula for ψ_K implies that if $\deg x \gg 0$, then $\psi_K(s^{-1}x)$ has a nonzero summand in $T^{>1}s^{-1}C_+K \otimes s^{-1}C_+K$.

Define now a left C^*K-action on $C^*K \otimes \mathcal{B}C^*K$ by

$$(x \otimes 1) \cdot (1 \otimes c) := (\varepsilon_K \otimes Id)\big((s^{-1}(sx) \otimes 1) \cdot (1 \otimes c)\big)$$

for all $x \in C^*K$ and $c \in \mathcal{B}C^*K$. If $\overline{D}(1 \otimes c) - 1 \otimes d_{\mathcal{B}}c = \sum_i s^{-1}(a_i) \otimes b^i$, then

$$\begin{aligned}
&\check{D}\big((x \otimes 1) \cdot (1 \otimes c)\big) \\
=&(\varepsilon_K \otimes Id)\big((s^{-1}(sdx) \otimes 1) \cdot (1 \otimes c) + (-1)^x (s^{-1}(sx) \otimes 1) \cdot \overline{D}(1 \otimes c)\big) \\
=&(dx \otimes 1) \cdot (1 \otimes c) \\
&+ (-1)^x (\varepsilon_K \otimes Id)\Bigg((s^{-1}(sx) \otimes 1) \cdot \Big(1 \otimes d_{\mathcal{B}}c \\
&\qquad\qquad + \sum_i (-1)^{(a_i+1)b^i} (1 \otimes b^i)(s^{-1}(sa_i) \otimes 1)\Big)\Bigg) \\
=&(dx \otimes 1) \cdot (1 \otimes c) \\
&+ (-1)^x \Bigg((x \otimes 1) \cdot \Big(1 \otimes d_{\mathcal{B}}c + \sum_i (-1)^{(a_i+1)b^i} (1 \otimes b^i)(a_i \otimes 1)\Big)\Bigg) \\
=&(dx \otimes 1) \cdot (1 \otimes c) + (-1)^x (x \otimes 1) \cdot \check{D}(1 \otimes c),
\end{aligned}$$

i.e., the left C^*K action commutes with the differential.

Again using the notation of section 1.2, we can write

$$(x \otimes 1) \cdot (1 \otimes c) = x \otimes c - \sum_{i<j} \pi\big(x \star \lambda_{ij,k}(c) \star \lambda_{ij}^{k}(c)\big) \otimes sx_i | \cdots | sx_j.$$

The following proposition summarizes the observations above. Let $C^*K \widetilde{\otimes} \mathcal{B}C^*K$ denote $C^*K \otimes \mathcal{B}C^*K$ endowed with the differential $\check{D}$ and the C^*K-bimodule structure defined above.

Proposition 1.4.1. *There is a twisted bimodule extension*

$$C^*K \longrightarrow C^*K \widetilde{\otimes} \mathcal{B}C^*K \longrightarrow \mathcal{B}C^*K$$

such that $\varepsilon_K \otimes Id : \Omega\mathcal{B}C^*K \widetilde{\otimes} \mathcal{B}C^*K \longrightarrow C^*K \widetilde{\otimes} \mathcal{B}C^*K$ *is a map of differential right* C^*K*-modules. In particular,* $\varepsilon_K \otimes Id$ *is a quasi-isomorphism.*

Note that $\varepsilon_K \otimes Id$ is not a bimodule map itself, since it is possible that for some $w \in \perp^{>1} sC^+K$ and $c \in \mathcal{B}C^*K$, the product $(s^{-1}(w) \otimes 1)(1 \otimes c)$ has a nonzero summand in $s^{-1}(sC^*K) \otimes \mathcal{B}C^*K$, i.e., that $(\varepsilon_K \otimes Id)\big((s^{-1}(w) \otimes 1)(1 \otimes c)\big) \neq 0$, even though $s^{-1}(w) \in \ker \varepsilon_K$. On the other hand if we filter both $\Omega\mathcal{B}C^*K \odot \mathcal{B}C^*K$ and $C^*K \widetilde{\otimes} \mathcal{B}C^*K$ by degree in the left tensor factor, then $\varepsilon_K \otimes Id$ induces an isomorphism of bigraded bimodules on the E_∞-terms of the associated spectral sequences, so $\varepsilon_K \otimes Id$ is almost a quasi-bimodule map. It may even be possible to define explicitly a cochain homotopy ensuring that $\varepsilon_K \otimes Id$ truly is a quasi-bimodule map.

For the constructions in sections 2 and 3, we need a quasi-isomorphism

$$\Upsilon : C^*K \widetilde{\otimes} \mathcal{B}C^*K \xrightarrow{\simeq} CU^*\mathcal{L}X\,,$$

which we obtain as follows. Recall that ε_K has a differential, though not multiplicative, section

$$\sigma_K : C^*K \longrightarrow \Omega\mathcal{B}C^*K : x \longmapsto s^{-1}(sx).$$

Consider the following commutative diagram of cochain complexes and maps.

$$\begin{array}{ccccc}
C^*K \widetilde{\otimes} \perp^{\leq 2} sC^+K & \xrightarrow{\sigma_K \otimes Id} & \Omega\mathcal{B}C^*K \odot \perp^{\leq 2} sC^+K & \xrightarrow{\text{incl.}} & \Omega\mathcal{B}C^*K \odot \mathcal{B}C^*K \\
\Big\downarrow \text{incl.} & & & & \simeq \Big\downarrow \varepsilon_K \otimes Id \\
C^*K \widetilde{\otimes} \mathcal{B}C^*K & & \xrightarrow{\qquad\qquad Id \qquad\qquad} & & C^*K \widetilde{\otimes} \mathcal{B}C^*K
\end{array}$$

Since the inclusion map on the left is a free extension of cochain complexes and $\varepsilon_K \otimes Id$ is a surjective quasi-isomorphism, we can extend $\sigma_K \otimes Id$ to a cochain map

$$\hat{\sigma} : C^*K\widetilde{\otimes}\mathcal{B}C^*K \longrightarrow \Omega\mathcal{B}C^*K \odot \mathcal{B}C^*K$$

such that $(\varepsilon_K \otimes Id)\hat{\sigma}_K = Id$, i.e., $\hat{\sigma}$ is a section of $\varepsilon_K \otimes Id$. In particular, $\hat{\sigma}$ is a quasi-isomorphism and for all $x \otimes c \in C^*K \otimes \mathcal{B}C^*K$

$$s^{-1}(sx) \otimes c - \hat{\sigma}(x \otimes c) \in \ker \varepsilon_K \otimes \mathcal{B}C^*K. \tag{1.4.1}$$

Furthermore, like $\varepsilon_K \otimes Id$, $\hat{\sigma}$ induces an isomorphism of bigraded bimodules on the E_∞-terms of the usual spectral sequences and is a quasi-bimodule map if and only if $\varepsilon_K \otimes Id$ is.

We now define Υ to be the composition below.

$$\underbrace{C^*K\widetilde{\otimes}\mathcal{B}C^*K \xrightarrow{\hat{\sigma}} \Omega\mathcal{B}C^*K \odot \mathcal{B}C^*K \xrightarrow{\tilde{\delta}} CU^*\mathcal{L}X}_{\Upsilon}$$

Observe that (1.4.1) implies that for all $x \otimes c \in C^*K\widetilde{\otimes}\mathcal{B}C^*K$,

$$\Upsilon(x \otimes c) = \tilde{\delta}(s^{-1}(sx) \otimes c) \tag{1.4.2}$$

since $\ker \varepsilon_K \subseteq \ker \tilde{\gamma}$.

Definition. Let X be a 1-connected space with the homotopy type of a finite-type CW-complex, and let K be a finite-type, 1-reduced simplicial set such that $|K| \simeq X$. The twisted C^*K-bimodule extension

$$fls^*(X) := C^*K\widetilde{\otimes}\mathcal{B}C^*K$$

together with the quasi-isomorphism

$$\Upsilon : fls^*(X) \xrightarrow{\simeq} CU^*\mathcal{L}X$$

is a *thin free loop model* for X.

We conclude this section with an important observation concerning the relation between Υ and the product $\psi_K^\sharp$.

Proposition 1.4.2. *Suppose that $\psi_K^\sharp$, the multiplication on $\mathcal{B}C^*K$, is commutative. If $x \in C^*K$ is a cycle, then $\Upsilon(1 \otimes c \star sx) = \Upsilon(1 \otimes c) \cdot \Upsilon(1 \otimes sx)$. In particular*

$$\Upsilon(1 \otimes sx_1 \star \cdots \star sx_n) = \Upsilon(1 \otimes sx_1) \cdots \Upsilon(1 \otimes sx_n)$$

if $dx_i = 0$ for some i.

Consequently, if $\psi_K^\sharp$ is commutative, then the commutator $[\Upsilon(1\otimes sx), \Upsilon(1\otimes sy)] = 0$, for all cycles $x, y \in C^*K$.

Proof. Recall from section 1.2 that if $\psi_K^\sharp$ is commutative, then we can choose the multiplication in the model $\Omega(\mathcal{B}C^*K\otimes\mathcal{B}C^*K)\odot\mathcal{B}C^*K$ so that $(1\otimes c)\cdot(1\otimes c') = 1\otimes c\star c'$ for all $c, c' \in C^*K$. According to Proposition 1.3.3 we then have that

$$\begin{aligned}\tilde\beta(1\otimes c\star c') =& J\tilde\beta D(1\otimes c\star c')\\ =& J\tilde\beta D\big((1\otimes c)(1\otimes c')\big)\\ =& J\tilde\beta\big(D(1\otimes c)\cdot(1\otimes c') + (-1)^c(1\otimes c)\cdot D(1\otimes c')\big)\\ =& J\tilde\beta D(1\otimes c)\cdot\tilde\beta(1\otimes c') + (-1)^c J\tilde\beta(1\otimes c)\cdot\tilde\beta D(1\otimes c')\\ =& \tilde\beta(1\otimes c)\cdot\tilde\beta(1\otimes c') + (-1)^c J^2\tilde\beta D(1\otimes c)\cdot d^\sharp\tilde\beta(1\otimes c').\end{aligned}$$

Thus, since $d^\sharp J^2 = J^2 d^\sharp$,

$$\begin{aligned}\Upsilon(1\otimes c\star c') =& CU^*j\tilde\beta(1\otimes c\star c')\\ =& \Upsilon(1\otimes c)\cdot\Upsilon(1\otimes c') + (-1)^c d^\sharp CU^*jJ^2\tilde\beta(1\otimes c)\cdot d^\sharp\Upsilon(1\otimes c)\\ =& \Upsilon(1\otimes c)\cdot\Upsilon(1\otimes c') + (-1)^c d^\sharp CU^*jJ^2\tilde\beta(1\otimes c)\cdot\Upsilon\check D(1\otimes c).\end{aligned}$$

Thus, if $x \in C^*K$ is a cycle, then $\check D(1\otimes sx) = 0$ and so $\Upsilon(1\otimes c\star sx) = \Upsilon(1\otimes c)\cdot\Upsilon(1\otimes sx)$. The second part of the statement follows by induction. □

2. Homotopy orbit spaces

In this section we construct a noncommutative model $hos^*(X)$ for the homotopy orbit space $(\mathcal{L}X)_{hS^1}$ of the natural S^1-action on the free loop space $\mathcal{L}X$. The form of $hos^*(X)$ is, not surprisingly, similar to that of the complex that gives the cyclic cohomology of an algebra. The author is grateful to Nicolas Dupont for the ideas he contributed during our discussions of $(\mathcal{L}X)_{hS^1}$ over the years.

We begin by proving the existence of a very special family of primitive elements in the reduced cubical chains on S^1 and then studying its properties. We then introduce a particularly useful resolution of the cubical chains on ES^1 as a module over the cubical chains on S^1, which we apply to constructing a model of the homotopy orbit space of any S^1-action. In the final part of this section we specialize to the case of $\mathcal{L}X$, obtaining a small, noncommutative model for $(\mathcal{L}X)_{hS^1}$ as an extension of the thin free loop model $fls^*(X)$.

2.1 A special family of primitives.

Let $CU_*(X)$ denote the reduced cubical chains on a topological space X. We begin by defining a suspension-type degree +1 operation on CU_*S^1.

Definition. Given any continuous map $f : I^n \longrightarrow \mathbb{R}^+$, let

$$\hat{f} : I^{n+1} \longrightarrow \mathbb{R}^+ : (t_0, ..., t_n) \longmapsto t_0 \cdot f(t_1, ..., t_n) .$$

If $T : I^n \longrightarrow S^1$ is an n-cube such that $T(t_1, ..., t_n) = e^{i2\pi f(t_1,...,t_n)}$, let $\sigma(T)$ be the $(n+1)$-cube defined by

$$\sigma(T)(t_0, ..., t_n) := e^{i2\pi \hat{f}(t_0,...,t_n)},$$

where we are considering S^1 as the unit circle in the complex plane, i.e.,

$$S^1 = \{e^{i\theta} \mid 0 \leq \theta \leq 2\pi\}.$$

Remark. It is clear that $\sigma(T)$ is degenerate if T is degenerate. The operation σ can therefore be extended linearly to all of CU_*S^1.

As the next lemma states, σ is a contracting homotopy in degrees greater than one and is a $(0, Id)$-coderivation.

Lemma 2.1.1. *Let $T \in CU_*S^1$.*

(1) *If $\deg T \geq 2$, then $d\sigma(T) = T - \sigma(dT)$ where d is the usual differential on CU_*S^1.*
(2) *$\overline{\Delta}(\sigma(T)) = \sigma(T_i) \otimes T^i$, where $\overline{\Delta}$ is the usual reduced coproduct on CU_*S^1 and $\Delta(T) = T_i \otimes T^i$ (using the Einstein summation convention).*

Simple calculations, applying the definitions of the cubical differential and the cubical coproduct, as given for example in [Mas] and [An], suffice to prove this lemma.

We now apply the σ operation to the recursive construction of an important family of elements in CU_*S^1.

Definition. Let $T_0 : I \longrightarrow S^1$ be the 1-cube defined by $T_0(t) = e^{i2\pi t}$. Given $T_k \in CU_{2k+1}S^1$ for all $k < n$, let T_n be the $(2n+1)$-cubical chain defined by

$$T_n := \sigma\left(\sum_{i=1}^{n} T_{i-1} \cdot T_{n-i}\right) \in CU_{2n+1}S^1.$$

Let $\mathcal{T} := \{T_n \mid n \geq 0\}$.

Examples. It is easy to see that

$$T_1(t_0, t_1, t_2) = e^{i2\pi t_0(t_1+t_2)}$$

and that $T_2 = U + V$ where

$$U(t_0, ...t_4) = e^{i2\pi t_0(t_1+(t_2+t_3)t_4)} \text{ and } V(t_0, ...t_4) = e^{i2\pi t_0((t_1+t_2)t_3+t_4)}.$$

Proposition 2.1.2. *The family $\mathcal{T}$ satisfies the following properties.*

(1) $dT_0 = 0$ *and* $0 \neq [T_0]$ *in* $H_1 S^1$.
(2) $dT_n = \sum_{i=1}^{n} T_{i-1} T_{n-i}$ *for all* $n > 0$.
(3) *Every* T_n *is primitive in* $CU_* S^1$

Proof. Points (1) and (2) are easy consequences of Lemma 2.1.1. It is well known that T_0 represents the unique nonzero homology generator of $H_* S^1$.

An easy inductive argument applying Lemma 2.1.1(2) proves point (3), since if T_k is primitive for all $k < n$, then the sum $\sum_{i=1}^{n} T_{i-1} \cdot T_{n-i}$ is also primitive, as it is symmetric and all factors are of odd degree. □

Let $< \mathcal{T} >$ denote the subalgebra of $CU_* S^1$ generated by the family $\mathcal{T}$. Since all the T_n's are primitive, $< \mathcal{T} >$ is a sub Hopf algebra of $CU_* S^1$. Proposition 2.1.2(1) and (2) imply that $< \mathcal{T} >$ is closed under the differential, and that the inclusion

$$< \mathcal{T} > \xhookrightarrow{\simeq} CU_* S^1$$

is a quasi-isomorphism.

2.2 A useful resolution of $CU_* ES^1$.

We now put the family $\mathcal{T}$ to work in constructing a simple, neat resolution of $CU_* ES^1$ as a $CU_* S^1$-module. To understand why this is important, recall that a special case of Moore's theorem (cf., [Mc], Thm. 7.27) states that for any left S^1-space X and any free $CU^* S^1$-resolution (Q, d) of $CU_* ES^1$, there is a diagram of quasi-isomorphisms of chain complexes

$$\begin{array}{ccc}
(Q,d) \otimes_{CU_* S^1} CU_* X & \xrightarrow{\simeq} & CU_*(X_{hS^1}) \\
\downarrow \pi & & \downarrow \pi \\
(Q,d) \otimes_{CU_* S^1} \mathbb{Z} & \xrightarrow{\simeq} & CU_*(BS^1)
\end{array}$$

where the projection maps π are induced by the map $X \longrightarrow *$. Hence,

$$\mathrm{Tor}_*^{CU_* S^1}(CU_* ES^1, CU_* X) \cong H_*(X_{hS^1}),$$

and so a resolution of $CU_* ES^1$ provides us with a general tool for computing $H_*(X_{hS^1})$ for an arbitrary S^1-space X.

Let Γ denote the divided powers algebra functor. Recall that if w is in even degree, then

$$\Gamma w = \bigoplus_{k \geq 0} \mathbb{Z} \cdot w(k),$$

where $\deg w(k) = k \cdot |w|$, $w(0) = 1$, $w(1) = w$ and $w(k)w(l) = \binom{k+l}{k} w(k+l)$. Furthermore, Γw is in fact a Hopf algebra, where the coproduct is specified by $\Delta(w) = w \otimes 1 + 1 \otimes w$.

Consider $(\Lambda u, 0) = (\mathbb{Z} \cdot u, 0)$, where u is in degree 1, and its acyclic extension $(\Gamma v \otimes \Lambda u, \partial)$, where v is in degree 2 and $\partial v(k) := v(k-1) \otimes u$ for all $k \geq 1$. There is a chain algebra quasi-isomorphism

$$\xi : CU_*S^1 \xrightarrow{\simeq} (\Lambda u, 0)$$

defined by $\xi(T_0) = u$ and $\xi(T) = 0$ for all other cubes T.

Define a semifree extension of right CU_*S^1-modules

$$\iota : CU_*S^1 \longrightarrow \big(\Gamma v \otimes CU_*S^1, \tilde{\partial}\big)$$

by

$$\tilde{\partial}(v(n) \otimes 1) := \sum_{k=0}^{n-1} v(n-k-1) \otimes T_k.$$

It is an immediate consequence of Proposition 2.1.2(2) that $\tilde{\partial}^2 = 0$. Furthermore, for all $n \geq 1$,

$$(Id \otimes \xi)\tilde{\partial}(v(n) \otimes 1) = v(n-1) \otimes u = \partial(Id \otimes \xi)(v(n) \otimes 1),$$

which implies that the CU_*S^1-module map

$$Id \otimes \xi : (\Gamma v \otimes CU_*S^1, \tilde{\partial}) \longrightarrow (\Gamma v \otimes \Lambda u, \partial)$$

is a differential map. A quick Zeeman's Comparison Theorem argument then shows that $Id \otimes \xi$ is a quasi-isomorphism, so that $(\Gamma v \otimes CU_*S^1, \tilde{\partial})$ is acyclic.

We claim that $(\Gamma v \otimes CU_*S^1, \tilde{\partial})$ is a CU_*S^1-resolution of CU_*ES^1. To verify this, observe that there is a commutative diagram of CU_*S^1-modules

$$\begin{array}{ccc} CU_*S^1 & \xrightarrow{CU_*j} & CU_*ES^1 \\ \downarrow{\scriptstyle \iota} & & \downarrow{\scriptstyle \simeq} \\ (\Gamma v \otimes CU_*S^1, \tilde{\partial}) & \xrightarrow{\simeq} & \mathbb{Z} \end{array}$$

where j is the inclusion of S^1 as the base of the construction of ES^1, which is an S^1-equivariant map. Since $(\Gamma v \otimes CU_*S^1, \tilde{\partial})$ is a semifree extension and the map from CU_*ES^1 to $\mathbb{Z}$ is a surjective quasi-isomorphism, we can extend CU_*j to a CU_*S^1-module map

$$\varepsilon : (\Gamma v \otimes CU_*S^1, \tilde{\partial}) \longrightarrow CU_*ES^1$$

which is a quasi-isomorphism by "2-out-of-3".

2.3 Modeling S^1-homotopy orbits.
Let X be any (left) S^1-space, where $g : S^1 \times X \longrightarrow X$ is the action map. There is then a natural CU_*S^1 module structure on CU_*X, given by the composition

$$\kappa : CU_*S^1 \otimes CU_*X \xrightarrow[\simeq]{EZ} CU_*(S^1 \times X) \xrightarrow{CU_*g} CU_*X$$

where EZ denotes the Eilenberg-Zilber equivalence. Observe that κ is a coalgebra map, as it is the composition of two coalgebra maps. Since Moore's Theorem implies that

$$H_*(X_{hS^1}) \cong H_*\big((\Gamma v \otimes CU_*S^1, \tilde{\partial}) \underset{CU_*S^1}{\otimes} CU_*X\big),$$

we need to try to understand better the complex $(\Gamma v \otimes CU_*S^1, \tilde{\partial}) \underset{CU_*S^1}{\otimes} CU_*X$.

Define an extension $(\Gamma v \otimes CU_*X, D)$ of Γv by

$$D(v(n) \otimes U) := v(n) \otimes dU + \sum_{k=1}^{n-1} v(n-k-1) \otimes \kappa(T_k \otimes U).$$

We again use Proposition 2.1.2(2) to verify that $D^2 = 0$. Observe that $(\Gamma v \otimes CU_*X, D)$ is naturally a chain coalgebra that is a cofree left Γv-comodule, since κ is a coalgebra map and each T_k is primitive.

It is then easy to show that the following two maps are chain isomorphisms, one inverse to the other.

$$(\Gamma v \otimes CU_*S^1, \tilde{\partial}) \underset{CU_*S^1}{\otimes} CU_*X \longrightarrow (\Gamma v \otimes CU_*X, D)$$

$$v(n) \otimes U \otimes V \longmapsto v(n) \otimes \kappa(U \otimes V)$$

$$(\Gamma v \otimes CU_*X, D) \longrightarrow (\Gamma v \otimes CU_*S^1, \tilde{\partial}) \underset{CU_*S^1}{\otimes} CU_*X$$

$$v(n) \otimes V' \longmapsto v(n) \otimes 1 \otimes V'$$

Thus, $H_*(X_{hS^1}) \cong H_*(\Gamma v \otimes CU_*X, D)$.

In this chapter we are interested in cohomology calculations and so must dualize this model. Dualizing κ directly poses a problem, however, since

$$(CU_*S^1 \otimes CU_*X)^\sharp \ncong CU^*S^1 \otimes CU^*X$$

because the cubical chain complex on a space is not of finite type. We can avoid this problem by observing that it is enough to dualize the composition

$$< \mathcal{T} > \otimes CU_*X \underset{\simeq}{\overset{\iota}{\hookrightarrow}} CU_*S^1 \otimes CU_*X \xrightarrow{\kappa} CU_*X.$$

Let $j_n : \mathbb{Z} \cdot T_n \otimes CU_*X \longrightarrow < \mathcal{T} > \otimes CU_*X$ denote the natural inclusion of graded modules. Let $T_n^\sharp \in CU^{2n+1}S^1$ denote the cochain such that $T_n^\sharp(T_n) = 1$ and $T_n^\sharp(T) = 0$ if T is any other $(2n+1)$-cube. Let $< \mathcal{T} >^\sharp = \mathrm{Hom}(< \mathcal{T} >, \mathbb{Z})$

For each $n \geq 0$, define a linear map $\omega_n : CU^*X \longrightarrow CU^{*-(2n+1)}X$ of degree $-(2n+1)$ by

$$j_n^\sharp \circ (\kappa\iota)^\sharp(f) := T_n^\sharp \otimes \omega_n(f),$$

where

$$CU^*X \xrightarrow{(\kappa\iota)^\sharp} < \mathcal{T} >^\sharp \otimes CU^*X \xrightarrow{j_n^\sharp} \mathbb{Z} \cdot T_n^\sharp \otimes CU^*X.$$

In other words, ω_n is the dual of $\kappa(T_n \otimes -)$.

Let $(\Lambda\upsilon \otimes CU^*X, D^\sharp)$ denote the $\mathbb{Z}$-dual of $(\Gamma v \otimes CU_*X, D)$. In particular $\upsilon(v) = 1$. Since it is the dual of a cofree comodule, $(\Lambda\upsilon \otimes CU^*X, D^\sharp)$ is a free, right $\Lambda\upsilon$-module. We need to identify $D^\sharp$ as precisely as possible, since

$$\mathrm{H}^*(\Lambda\upsilon \otimes CU^*X, D^\sharp) \cong \mathrm{H}^*(X_{hS^1}).$$

A simple dualization calculation gives us the following result.

Lemma 2.3.1. *If $f \in CU^mX$, then*

$$D^\sharp(\upsilon^n \otimes f) = \upsilon^n \otimes d^\sharp f + \sum_{k=0}^{\lceil \frac{m-1}{2} \rceil} \upsilon^{n+k+1} \otimes \omega_k(f)$$

where $d^\sharp$ denotes the differential of CU^*X.

As a consequence of this description of $D^\sharp$ we obtain the following useful properties of the operators ω_k.

Corollary 2.3.2. *The operators ω_n satisfy the following properties.*

1. *For all $n \geq 1$, $[d^\sharp, \omega_n] = -\sum_{k=0}^{n-1} \omega_k \circ \omega_{n-k-1}$, while $d^\sharp\omega_0 = -\omega_0 d^\sharp$.*
2. *Each ω_n is a derivation, i.e., $\omega_n(f \cdot g) = \omega_n(f) \cdot g + (-1)^f f \cdot \omega_n(g)$.*

Proof. The proof of (1) proceeds by expansion of the equation $0 = (D^\sharp)^2(1 \otimes f)$. To prove (2), expand the equation

$$D^\sharp(1 \otimes f \cdot g) = D^\sharp(1 \otimes f) \cdot (1 \otimes g) + (-1)^f (1 \otimes f) \cdot D^\sharp(1 \otimes g).$$

The differential $D^\sharp$ is a derivation, since it is the dual of the differential of a chain coalgebra. $\square$

Remark. This corollary implies that ω_0 induces a derivation of degree -1

$$\varpi : \mathrm{H}^* X \longrightarrow \mathrm{H}^{*-1} X$$

such that $\varpi^2 = 0$

2.4 The case of the free loop space.
Let K be a finite-type, 1-reduced simplicial set such that $|K|$ has the same homotopy type as X. As we saw in section 1.4, there is commutative diagram

$$\begin{array}{ccc} C^*K & \xrightarrow{\iota} & C^*K\widetilde{\otimes}\mathcal{B}C^*K \\ \downarrow{\scriptstyle \widetilde{\gamma}\sigma_K} & & \simeq\downarrow{\scriptstyle \Upsilon} \\ CU^*X & \xrightarrow{CU^*e} & CU^*\mathcal{L}X \end{array}$$

in which $\widetilde{\gamma}\sigma_K$ is a quasi-algebra quasi-isomorphism, ι is a twisted bimodule extension and Υ is a quasi-isomorphism inducing an isomorphism on the E_∞-terms of the Eilenberg-Moore spectral sequence.

Our goal here is to combine this thin free loop space model with the general homotopy orbit space model of the section 2.3, obtaining an extension $(\Lambda\upsilon \otimes C^*K\widetilde{\otimes}\mathcal{B}C^*K, \widetilde{D})$ of $(\Lambda\upsilon, 0)$ by $fls^*(X)$, together with a quasi-isomorphism

$$\widetilde{\Upsilon} : (\Lambda\upsilon \otimes C^*K\widetilde{\otimes}\mathcal{B}C^*K, \widetilde{D}) \longrightarrow (\Lambda\upsilon \otimes CU^*\mathcal{L}X, D^\sharp)$$

such that

(2.4.1)
$$\begin{array}{ccccc} (\Lambda\upsilon, 0) & \xrightarrow{incl.} & (\Lambda\upsilon \otimes C^*K\widetilde{\otimes}\mathcal{B}C^*K, \widetilde{D}) & \xrightarrow{\pi} & C^*K\widetilde{\otimes}\mathcal{B}C^*K \\ \| & & \downarrow{\scriptstyle \widetilde{\Upsilon}} & & \downarrow{\scriptstyle \Upsilon} \\ (\Lambda\upsilon, 0) & \xrightarrow{incl.} & (\Lambda\upsilon \otimes CU^*\mathcal{L}X, D^\sharp) & \xrightarrow{\pi} & CU^*\mathcal{L}X \end{array}$$

commutes, where π denotes the obvious projections.

We begin by an easy, though crucial, observation concerning the relations between Υ and the operations ω_k.

Lemma 2.4.1. *For all cocycles $x \in C^*K$, $\Upsilon(1 \otimes sx) = \omega_0 \Upsilon(x \otimes 1)$.*

Proof. From the definition of $\tilde{\beta}$ from Proposition 1.3.3 and of D from section 1.2, we can show that

$$\begin{aligned}\Upsilon(1 \otimes sx) =& CU^* j\tilde{\beta}(1 \otimes sx)\\ =& CU^* jJ\tilde{\beta}D(1 \otimes sx)\\ =& CU^* jJ\tilde{\beta}(s^{-1}(sx \otimes 1) - s^{-1}(1 \otimes sx))\\ =& CU^* jJCU^* p\big(C^*pr_1 - C^*pr_2\big)\tilde{\gamma}(s^{-1}(sx)).\end{aligned}$$

A straightforward computation suffices to establish that

$$CU^* jJCU^* p\big(C^*pr_1 - C^*pr_2\big) = \omega_0 CU^* e,$$

implying that

$$\Upsilon(1 \otimes sx) = \omega_0 CU^* e\tilde{\gamma}(s^{-1}(sx)) = \omega_0 \Upsilon(x \otimes 1). \quad \square$$

Restriction. Henceforth, to simplify the presentation, we assume that $\psi_K^\sharp$, the multiplication on $\mathcal{B}C^*K$, is such that the primitives of $\mathcal{B}C^*K$, i.e., the elements of sC^+K, are all indecomposable.

This is certainly a strong hypothesis, but it still allows us to treat a number of interesting cases, such as wedges of spheres. More general cases are treated in [H2].

The special properties of the free loop space model in the restricted case we consider are summarized in the following lemma.

Lemma 2.4.2. *If the primitives of $\mathcal{B}C^*K$ are all indecomposable, then the following properties hold.*

1. *The multiplication on $\mathcal{B}C^*K$ is the shuffle product, which is commutative.*
2. *The graded algebra C^*K is commutative.*
3. *For all $y, x_1, ..., x_n \in C^*K$,*

$$\begin{aligned}\check{D}(y \otimes sx_1|\cdots|sx_n) =& dy \otimes sx_1|\cdots|sx_n + (-1)^y y \otimes d_{\mathcal{B}}(sx_1|\cdots|sx_n)\\ &+ (-1)^y\big[yx_1 \otimes sx_2|\cdots|sx_n - (-1)^N yx_n \otimes sx_1|\cdots|sx_{n-1}\big],\end{aligned}$$

where $N = (1 + \deg x_n)(n - 1 + \sum_{1 \le j < n} \deg x_j)$, and

$$(y \otimes 1)(1 \otimes sx_1|\cdots|sx_n) = y \otimes sx_1|\cdots|sx_n.$$

In other words, the differential of $C^*K\widetilde{\otimes}\mathcal{B}C^*K$ is exactly that of the usual Hochschild complex on C^*K, while the left C^*K-action is untwisted, when the primitives of $\mathcal{B}C^*K$ are all indecomposable.

Proof. (1) This is obvious.

(2) Recall from section 1.2 that if $\psi_K^\sharp$ is commutative, then

$$(1\otimes c)(1\otimes c')=1\otimes c\star c'.$$

Thus, if $x\in C^{m+1}K, y\in C^{n+1}K$, then

$$\begin{aligned}\check{D}(1\otimes sx\star sy)=&\check{D}(1\otimes sx)\cdot(1\otimes sy)+(-1)^m(1\otimes sx)\cdot\check{D}(1\otimes sy)\\ =&-(1\otimes s(dx))\cdot(1\otimes sy)-(-1)^m(1\otimes sx)\cdot(1\otimes s(dy)),\end{aligned}$$

whenever $\psi_K^\sharp$ is commutative. If, moreover, all primitives of $\mathcal{B}C^*K$ are indecomposable, then

$$\begin{aligned}\check{D}(1\otimes sx\star sy)=&\check{D}(1\otimes sx|sy+(-1)^{mn}1\otimes sy|sx)\\ =&x\otimes sy-(-1)^{mn}y\otimes sx+(-1)^m1\otimes s(xy)\\ &-1\otimes s(dx)|sy-(-1)^m1\otimes sx|s(dy)\\ &+(-1)^{mn}\big[y\otimes sx-(-1)^{mn}x\otimes sy+(-1)^n1\otimes s(yx)\\ &\qquad-1\otimes s(dy)|sx-(-1)^n1\otimes sy|s(dx)\big]\\ =&(-1)^m1\otimes s\big([x,y]\big)-1\otimes s(dx)\star sy-(-1)^m1\otimes sx\star s(dy)\end{aligned}$$

and so $[x,y]=0$. Hence, C^*K is commutative if all primitives of $\mathcal{B}C^*K$ are indecomposable.

(3) When all primitives of $\mathcal{B}C^*K$ are indecomposable, the formulas of section 1.4 obviously reduce to those given in the statement. □

We now define the desired extension

$$(\Lambda\upsilon,0)\longrightarrow(\Lambda\upsilon\otimes C^*K\otimes\mathcal{B}C^*K,\widetilde{D})\longrightarrow C^*K\odot\mathcal{B}C^*K$$

and show that

$$\mathrm{H}^*(\Lambda\upsilon\otimes C^*K\otimes\mathcal{B}C^*K,\widetilde{D})\cong\mathrm{H}^*(X_{hS^1}).$$

We define the extension by

$$\widetilde{D}=Id\otimes\check{D}+\upsilon\cdot-\otimes S$$

where

1. $S(y\otimes 1)=1\otimes sy$ for all $y\in C^+K$;
2. $S(1\otimes c)=0$ for all $c\in\mathcal{B}C^*K$; and
3. $S(y\otimes sx_1|\cdots|sx_n)=\sum_{j=1}^{n+1}\pm1\otimes sx_j|\cdots|sx_n|sy|sx_1|\cdot|sx_{j-1}$, where the sign is chosen in accord with the Koszul convention (cf., section 0.1).

Thus, for example, if $x, y, z \in C^*K$ are of degrees $l+1$, $m+1$ and $n+1$, respectively, then $S(x \otimes sy) = 1 \otimes (sx|sy + (-1)^{lm} sy|sx) = 1 \otimes sx \star sy$ and

$$S(x \otimes sy|sz) = 1 \otimes (sx|sy|sz + (-1)^{(m+l)n} sz|sx|sy + (-1)^{l(m+n)} sy|sz|sx).$$

It is obvious that $S^2 = 0$. A tedious, though not difficult, combinatorial calculation, shows that $\check{D}S = -S\check{D}$. The proof of this equality depends strongly on the fact that C^*K is commutative; in the general case we need to add terms to $S(x \otimes c)$ to kill certain commutators [H2]. Thus $\widetilde{D}^2 = 0$, i.e., $(\Lambda\upsilon \otimes C^*K \odot \mathcal{B}C^*K, \widetilde{D})$ is a cochain complex. Indeed this is exactly the negative cyclic complex of the commutative algebra C^*K, looked at as a cochain complex in positive degrees, rather than as a chain complex in negative degrees.

As Jones proved in [J], $\mathrm{H}^*(\mathcal{L}X_{hS^1})$ is isomorphic to the negative cyclic homology of the algebra S^*X, and therefore to that of C^*K, if $|K| \simeq X$. Thus

$$\mathrm{H}^*(\Lambda\upsilon \otimes C^*K \widetilde{\otimes} \mathcal{B}C^*K, \widetilde{D}) \cong \mathrm{H}^*(\Lambda\upsilon \otimes C^*\mathcal{L}X, D^\sharp) \cong \mathrm{H}^*(\mathcal{L}X_{hS^1}),$$

as desired. To build our model for topological cyclic homology, however, we need a cochain quasi-isomorphism $\widetilde{\Upsilon}$ lifting Υ and inducing this cohomology isomorphism. In the next theorem, we prove the existence of $\widetilde{\Upsilon}$ when K is an odd-dimensional sphere. Using results from [P], we can generalize this theorem to wedges of odd spheres and, when working over $\mathbb{F}_2$, to wedges of even spheres. The essential ideas of the general proof are already present in the proof for a single odd sphere, so we restrict to this case, to simplify the presentation. In [H2], we prove the existence of $\widetilde{\Upsilon}$ for a somewhat larger class of spaces.

Before stating and proving the theorem, we analyze carefully the S^1- action on $\mathcal{L}S^{2n+1}$. It is well known that $\mathrm{H}^*(\mathcal{L}S^{2n+1})$ is isomorphic to the tensor product of an exterior algebra Λx on a generator of degree $2n+1$ with the divided powers algebra Γy on a generator of degree $2n$ (cf., e.g., [Sm]). The generator x is represented by $CU^*e(\zeta)$, where $\zeta \in CU^{2n+1}S^{2n+1}$ represents the fundamental class of S^{2n+1}. More explicitly, if $U : I^{2n+1} \longrightarrow S^{2n+1}$ is a $(2n+1)$-cube collapsing ∂I^{2n+1} to a point, then

$$\zeta : CU_{2n+1}S^{2n+1} \longrightarrow \mathbb{Z}$$

is specified by $\zeta(U) = 1$ and $\zeta(V) = 0$ if V is any other $(2n+1)$-cube. Consider the transpose of U

$$U^\flat : I^{2n} \longrightarrow (S^{2n+1})^I : (t_1, ..., t_{2n}) \longmapsto U(-, t_1, ..., t_{2n}) \ .$$

Since U collapses the boundary of the cube, $U^\flat(t_1, .., t_{2n})$ is always a (based) loop. Let $\xi \in CU^{2n}\mathcal{L}X$ be the cochain such that $\xi(U^\flat) = 1$ and $\xi(V) = 0$ for any other $2n$-cube V. The generator y is represented by ξ.

Recall the definition of ϖ at the end of section 2.3. A simple, explicit calculation shows that $\varpi(x) = y$, which implies that $\varpi(y) = 0$, whence

$$\varpi(x \otimes y(m)) = (m+1) \cdot y(m+1) \quad \text{and} \quad \varpi(1 \otimes y(m)) = 0$$

for all $m \geq 0$, since ϖ is a derivation. In particular, $\varpi(\mathrm{H}^{\mathrm{even}}\, \mathcal{L}S^{2n+1}) = 0$, and $\varpi : \mathrm{H}^{\mathrm{odd}}\, \mathcal{L}S^{2n+1} \longrightarrow \mathrm{H}^{\mathrm{even}}\, \mathcal{L}S^{2n+1}$ is an isomorphism.

Theorem 2.4.3. *Let K be the simplicial model of S^{2n+1} with exactly two non-degenerate simplices, in degrees 0 and $2n+1$, where $n > 0$. There is a quasi-isomorphism $\widetilde{\Upsilon} : (\Lambda\upsilon \otimes C^*K \widetilde{\otimes} \mathcal{B}C^*K, \widetilde{D}) \longrightarrow (\Lambda\upsilon \otimes CU^*\mathcal{L}X, D^\sharp)$ such that*

$$\begin{array}{ccccc}
(\Lambda\upsilon, 0) & \xrightarrow{incl.} & (\Lambda\upsilon \otimes C^*K \widetilde{\otimes} \mathcal{B}C^*K, \widetilde{D}) & \xrightarrow{\pi} & C^*K \widetilde{\otimes} \mathcal{B}C^*K \\
\Vert & & \downarrow{\scriptstyle \widetilde{\Upsilon}} & & \downarrow{\scriptstyle \Upsilon} \\
(\Lambda\upsilon, 0) & \xrightarrow{incl.} & (\Lambda\upsilon \otimes CU^*\mathcal{L}S^{2n+1}, D^\sharp) & \xrightarrow{\pi} & CU^*\mathcal{L}S^{2n+1}
\end{array}$$

commutes, where π denotes the obvious projection maps.

Proof. In this case, $C^*K = \Lambda z$, an exterior algebra on an odd generator of degree $2n+1$ and $\mathcal{B}C^*K = \Gamma(sz)$, the divided powers algebra on an even generator of degree $2n$. Furthermore, $\check{D} = 0$, i.e., $(C^*K \widetilde{\otimes} \mathcal{B}C^*K, \check{D}) = (\Lambda z \otimes \Gamma sz, 0)$.

We need to define a cochain map $\widetilde{\Upsilon} = \sum_{k \geq 0} \upsilon^k \otimes \Upsilon_k$, where $\Upsilon_0 = \Upsilon$ and, for all k, $\Upsilon_k : C^*K \widetilde{\otimes} \mathcal{B}C^*K \longrightarrow CU^{*-2k}\mathcal{L}S^{2n+1}$ is a linear map of degree $-2k$. Since $\widetilde{D} = 1 \otimes \check{D} + \upsilon \otimes S$ and $D^\sharp = 1 \otimes d^\sharp + \sum_{k \geq 0} \upsilon^{k+1} \otimes \omega_k$, the equation $D^\sharp \widetilde{\Upsilon} = \widetilde{\Upsilon}\widetilde{D}$ is equivalent to the set of equations

$$(2.4.2)_k \qquad d^\sharp \Upsilon_k + \sum_{i+j=k-1} \omega_i \Upsilon_j = \Upsilon_k \check{D} + \Upsilon_{k-1} S$$

for $k \geq 0$. Thus, to define $\widetilde{\Upsilon}$, we can build up the family of Υ_k's by induction on both k and wordlength in $\mathcal{B}C^*K = \Gamma sx$.

For $k = 0$, the equation above becomes simply $d^\sharp \Upsilon_0 = \Upsilon_0 \check{D}$, which holds because Υ is a cochain map.

When $k = 1$, the appropriate equation is

$$(2.4.2)_1 \qquad d^\sharp \Upsilon_1 = \Upsilon_1 \check{D} + \Upsilon_0 S - \omega_0 \Upsilon_0$$

Applied to $z \otimes 1 \in C^*K \widetilde{\otimes} \mathcal{B}C^*K$, the right-hand side of this equation becomes

$$0 + \Upsilon(1 \otimes sz) - \omega_0 \Upsilon(z \otimes 1),$$

which is 0, by Lemma 2.4.1. Thus, we can set

$$\Upsilon_1(z\otimes 1)=0.$$

When we apply the right-hand side of $(2.4.2)_1$ to $1\otimes sz$, we obtain

$$0+0-\omega_0\Upsilon(1\otimes sz),$$

which is equal to $-\omega_0^2\Upsilon(z\otimes 1)$, by Lemma 2.4.1. We can therefore choose

$$\Upsilon_1(1\otimes sz)=\omega_1\Upsilon(z\otimes 1),$$

by Corollary 2.3.2 (1). By a similar argument, we can choose

$$\Upsilon_1(z\otimes sz)=\omega_1\Upsilon(z\otimes 1)\cdot\Upsilon(z\otimes 1).$$

Suppose that Υ_1 has been defined on $\Lambda z\otimes\Gamma^{\leq l-1}(sz)$ satisfying equation $(2.4.2)_1$, where $l\geq 2$. Applying the right-hand side of $(2.4.2)_1$ to $1\otimes sz(l)$, we obtain

$$0+0-\omega_0\Upsilon(1\otimes sz(l)),$$

which is a cycle of odd degree. Since ϖ is injective on odd cohomology, either $\omega_0(1\otimes sz(l))$ is a boundary or $\omega_0^2(1\otimes sz(l))$ is not a boundary. The second option is impossible, since $\omega_0^2(1\otimes sz(l))=-d^\sharp\omega_1(1\otimes sz(l))$, and so $\omega_0(1\otimes sz(l))$ must be a boundary, i.e., there is possible choice of $\Upsilon_1(1\otimes sz(l))$ satisfying $(2.4.2)_1$. If we then set

$$\Upsilon_1(z\otimes sz(l))=\Upsilon_1(1\otimes sz(l))\cdot\Upsilon(z\otimes 1),$$

then equation $(2.4.2)_1$ is satisfied on $\Lambda z\otimes\Gamma^{\leq l}(sz)$.

Suppose now that for all $k<m$, there is a linear map $\Upsilon_k:\Lambda z\otimes\Gamma sz\longrightarrow CU^{*-2k}\mathcal{L}S^{2n+1}$ satisfying $(2.4.2)_k$ and such that

$$\Upsilon_k(z\otimes 1)=0\quad\text{and}\quad\Upsilon_k(1\otimes sz)=\omega_k\Upsilon(z\otimes 1).$$

For $k=m$, the equation we must satisfy is

$$(2.4.2)_k\qquad d^\sharp\Upsilon_m=\Upsilon_m\check{D}+\Upsilon_{m-1}S-\sum_{i+j=m-1}\omega_i\Upsilon_j.$$

Applied to $z\otimes 1$, the right-hand side of the equation becomes

$$0+\Upsilon_{m-1}(1\otimes sz)-\sum_{i+j=m-1}\omega_i\Upsilon_j(z\otimes 1),$$

which is zero, by the induction hypotheses. We can therefore set $\Upsilon_m(z\otimes 1)=0$.

If the right-hand side of the equation $(2.4.2)_m$ is evaluated on $1\otimes sz$, it becomes

$$\begin{aligned} 0+0-\sum_{i+j=m-1}\omega_i\Upsilon_j(1\otimes sz) &= -\sum_{i+j=m-1}\omega_i\omega_j\Upsilon(z\otimes 1)\\ &= d^\sharp\omega_m\Upsilon(z\otimes 1), \end{aligned}$$

implying that we may set $\Upsilon(1\otimes sz)=\omega_m\Upsilon(z\otimes 1)$.

Suppose that Υ_m has been defined on $\Lambda z\otimes\Gamma^{\leq l-1}(sz)$ satisfying equation $(2.4.2)_m$, where $l\geq 2$. Applying the right-hand side of $(2.4.2)_m$ to $1\otimes sz(l)$, we obtain

$$0+0-\sum_{i+j=m-1}\omega_i\Upsilon_(1\otimes sz(l)),$$

which is a cycle of odd degree. Since ϖ is injective on odd cohomology, either $\sum_{i+j=m-1}\omega_i\Upsilon_j(1\otimes sx(l))$ is a boundary or its image under ω_0 is not a boundary. Observe however that

$$\begin{aligned} &d^\sharp\Big(\sum_{i+j=m-1}\omega_{i+1}\Upsilon_j(1\otimes sz(l))\Big)\\ &\quad=\sum_{\substack{s+t=i\\ i+j=m-1}}-\omega_s\omega_t\Upsilon_j(1\otimes sz(l))+\sum_{\substack{p+q=j-1\\ i+j=m-1}}\omega_{i+1}\omega_p\Upsilon_q(1\otimes sz(l)\\ &\quad=\sum_{i+j+k=m-1}-\omega_i\omega_j\Upsilon_k(1\otimes sz(l))+\sum_{\substack{i+j+k=m-1\\ i\geq 1}}\omega_i\omega_j\Upsilon_k(1\otimes sz(l))\\ &\quad=-\sum_{i+j=m-1}\omega_0\omega_i\Upsilon_j(1\otimes sz(l))\\ &\quad=-\,\omega_0\Big(\sum_{i+j=m-1}\omega_i\Upsilon_j(1\otimes sz(l))\Big), \end{aligned}$$

and so $\sum_{i+j=m-1}\omega_i\Upsilon_j(1\otimes sz(l))$ must be a boundary. Hence, there is possible choice of $\Upsilon_m(1\otimes sz(l))$ satisfying $(2.4.2)_m$. If we then set

$$\Upsilon_m(x\otimes sz(l))=\Upsilon_m(1\otimes sz(l))\cdot\Upsilon(z\otimes 1),$$

then equation $(2.4.2)_m$ is satisfied on $\Lambda z\otimes\Gamma^{\leq l}(sz)$. □

Definition. Let X be a 1-connected space with the homotopy type of a finite-type CW-complex, and let $\Upsilon: C^*K\widetilde{\otimes}\mathcal{B}C^*K\xrightarrow{\simeq}CU^*\mathcal{L}X$ be a thin free loop model for X such that all primitives of $\mathcal{B}C^*K$ are indecomposable. The twisted bimodule extension

$$hos^*(X)=(\Lambda\upsilon\otimes C^*K\widetilde{\otimes}\mathcal{B}C^*K,\widetilde{D})$$

together with the quasi-isomorphism

$$\widetilde{\Upsilon} : hos^*(X)) \longrightarrow (\Lambda v \otimes CU^*\mathcal{L}X, D^\sharp)$$

such that diagram (2.4.1) commutes, when it exists, is a *thin model* of $\mathcal{L}X_{hS^1}$.

Related work. Bökstedt and Ottosen have recently developed an approach to Borel cohomologly calculations for free loop spaces that is Eckmann-Hilton dual to the approach considered here and thus complementary to our methods [BO2]. They have constructed a Bousfield-type spectral sequence that converges to the cohomology of $(\mathcal{L}X)_{hS^1}$. While our model is easiest to deal with for spaces with few cells, the elementary cases for their model are Eilenberg-MacLane spaces.

3. A model for mod 2 topological cyclic homology

We begin this section by supplying the final piece of the machine with which we build a model of $TC(X;2)$: a model of the p^{th}-power map, for $p = 2$. We then use the machine to obtain an explicit and precise description of $tc^*(X)$. To conclude we illustrate the power of both the $tc^*(X)$ and the $hos^*(X)$ models, by applying them to computing $\mathrm{H}^*(\mathcal{L}S^{2n+1}_{hS^1})$ and $\mathrm{H}^*(TC(S^{2n+1};2);\mathbb{F}_2)$.

3.1 The p^{th}- power map.

The p^{th}-power map, λ^p, on a free loop space $\mathcal{L}X$ sends any loop to the loop that covers the same image p times, turning p times as fast, i.e., for all $\ell \in \mathcal{L}X$ and for all $z \in S^1$

$$\lambda^p(\ell)(z) := \ell(z^p),$$

where we see S^1 as the set of complex numbers of norm 1.

There is another useful way to define λ^p. Let $\mathcal{L}X^{(p)}$ denote the pullback of the iterated diagonal $\Delta^{(p-1)} : X \longrightarrow X^p$ and of $e^p : (\mathcal{L}X)^p \longrightarrow X^p$, i.e., the elements of $\mathcal{L}X^{(p)}$ are sequences of loops $(\ell_1, ..., \ell_p)$ such that $\ell_i(1) = \ell_j(1)$ for all i, j. Let $e^{(p)} : \mathcal{L}X^{(p)} \longrightarrow X$ denote the map sending a sequence of loops to their common basepoint.

The iterated diagonal map on $\mathcal{L}X$ corestricts to $\Delta^{(p-1)} : \mathcal{L}X \longrightarrow \mathcal{L}X^{(p)}$, while concatenation of loops defines a map $\mu^{(p-1)} : \mathcal{L}X^{(p)} \longrightarrow \mathcal{L}X$, restricting to the usual iterated multiplication on ΩX. It is clear that the p^{th}-power map factors through $\mathcal{L}X^{(p)}$, as $\lambda^p = \mu^{(p-1)}\Delta^{(p-1)}$. Furthermore, the following diagram of fibrations commutes.

$$\begin{array}{ccccc}
\Omega X & \xrightarrow{\Delta^{(p-1)}} & (\Omega X)^p & \xrightarrow{\mu^{(p-1)}} & \Omega X \\
\downarrow{\scriptstyle i} & & \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle i} \\
\mathcal{L}X & \xrightarrow{\Delta^{(p-1)}} & \mathcal{L}X^{(p)} & \xrightarrow{\mu^{(p-1)}} & \mathcal{L}X \\
\downarrow{\scriptstyle e} & & \downarrow{\scriptstyle e^{(p)}} & & \downarrow{\scriptstyle e} \\
X & \xrightarrow{=} & X & \xrightarrow{=} & X
\end{array}$$

Using techniques similar to those applied in section 2, it is possible to show that there is a twisted bimodule extension $C^*K\widetilde{\otimes}(\mathcal{B}C^*K)^{\otimes p}$ and a quasi-isomorphism

$$\Upsilon' : C^*K\widetilde{\otimes}(\mathcal{B}C^*K)^{\otimes p} \xrightarrow{\simeq} CU^*(\mathcal{L}X^{(p)}).$$

This construction can be performed with sufficient naturality to ensure that the diagram

$$\begin{array}{ccc}
C^*K\widetilde{\otimes}(\mathcal{B}C^*K)^{\otimes p} & \xrightarrow{Id\otimes(\psi_K^\sharp)^{(p-1)}} & C^*K\otimes\mathcal{B}C^*K \\
\downarrow{\scriptstyle \Upsilon'} & & \downarrow{\scriptstyle \Upsilon} \\
CU^*(\mathcal{L}X^{(p)}) & \xrightarrow{\Delta^{(p-1)}} & CU^*(\mathcal{L}X)
\end{array}$$

commutes.

To complete the construction of a model of the p^{th}-power map, we need only to find a model of $\mu^{(p-1)}$. Modeling $\mu^{(p-1)}$ is very technical in the general case, however, requiring a fine analysis of the images of Υ and Υ'. For certain spaces, we can nevertheless show relatively easily that an acceptable model of $\mu^{(p-1)}$ is $Id\otimes\chi^{(p-1)}$, where χ denotes the usual coproduct on $\mathcal{B}C^*K$, an esthetically pleasing result.

We show below that $Id\otimes(\psi_K^\sharp)^{(p-1)}\chi^{(p-1)}$ is a model of λ^p, at least when K is a simplicial model of an odd sphere. To simplify calculations somewhat, we consider here only the case $p=2$; the case of arbitrary p, for a larger class of spaces, can be found in [HR].

We verify first that our candidate to be a model of λ^2 is in fact a cochain map.

Proposition 3.1.1. *If K be a finite-type, 1-reduced simplicial set such that $\psi_K^\sharp$ is commutative, then $Id\otimes(\psi_K^\sharp\chi) : C^*K\widetilde{\otimes}\mathcal{B}C^*K \longrightarrow C^*K\widetilde{\otimes}\mathcal{B}C^*K$ is a cochain map.*

Proof. We need first to show that

$$(Id\otimes(\psi_K^\sharp\chi))\check{D} = \check{D}(Id\otimes(\psi_K^\sharp\chi)).$$

The computation, while combinatorially technical, is not subtle. The formulas in section 1.4 tell us that if $\psi_K^\sharp$ is commutative and $dx_i = 0$ for all i, then (up to signs)

$$\begin{aligned}\check{D}(1\otimes sx_1|\cdots|sx_n) = \sum_{i=1}^{n-1}\Big[&\pm\pi(sx_1\star\cdots\star sx_{i-1}\star sx_n)\otimes sx_i|\cdots|sx_{n-1}\\ &\pm\pi(sx_1\star\cdots\star sx_i)\otimes sx_{i+1}|\cdots|sx_n\\ &\pm 1\otimes sx_1|\cdots|s(x_ix_{i+1})|\cdots|sx_n\Big]\end{aligned}$$

and

$$\begin{aligned}(a\otimes 1)\cdot(1\otimes sx_1|\cdots|sx_n) =& a\otimes sx_1|\cdots|sx_n\\ &+\sum_{i=1}^{n-1}\Big[\pm\pi(sa\star sx_1\star\cdots\star sx_{i-1}\star sx_n)\otimes sx_i|\cdots|sx_{n-1}\\ &\qquad\pm\pi(sa\star sx_1\star\cdots\star sx_i)\otimes sx_{i+1}|\cdots|sx_n\Big],\end{aligned}$$

while for all $a\in C^*K$

$$\begin{aligned}(Id\otimes(\psi_K^\sharp\chi))(a\otimes sx_1|\cdots|sx_n) =& 2a\otimes sx_1|\cdots|sx_n\\ &+a\otimes\sum_{i=1}^{n-1}(sx_1|\cdots|sx_i)\star(sx_{i+1}|\cdots|sx_n).\end{aligned}$$

A bit of elementary algebra and careful counting enable us to show that $Id\otimes(\psi_K^\sharp\chi)$ is indeed differential, using the formulas above. □

Theorem 3.1.2. *Let K be the simplicial model of S^{2n+1} with exactly two nondegenerate simplices, in degrees 0 and $2n+1$, where $n>0$. The diagram*

(3.1.1)
$$\begin{array}{ccc} C^*K\widetilde{\otimes}\mathcal{B}C^*K & \xrightarrow{Id\otimes(\psi_K^\sharp\chi)} & C^*K\widetilde{\otimes}\mathcal{B}C^*K\\ \simeq\big\downarrow\Upsilon & & \simeq\big\downarrow\Upsilon\\ CU^*\mathcal{L}S^{2n+1} & \xrightarrow{CU^*\lambda^2} & CU^*\mathcal{L}S^{2n+1}\end{array}$$

commutes up to cochain homotopy.

Proof. Recall that $\mathrm{H}^*(\mathcal{L}S^{2n+1}) = \Lambda x\otimes\Gamma y$, where $|x| = 2n+1$ and $|y| = 2n$. Furthermore $\mathrm{H}^*(\Omega S^{2n+1}) = \Gamma y$, and the restriction of λ^2 to ΩS^{2n+1} induces an endomorphism of Γy specified by

$$\begin{aligned}\mathrm{H}^*(\lambda^2|_{\Omega S^{2n+1}})(y(m)) &= \mathrm{H}^*\,\chi\,\mathrm{H}^*\,\mu(y(m))\\ &= \mathrm{H}^*\,\chi(\sum_{i=0}^{m} y(i)\otimes y(m-i))\\ &= \sum_{i=0}^{m} y(i)\star y(m-i)\\ &= 2^m y(m).\end{aligned}$$

Consequently, since $\lambda^2\circ i = i\circ\lambda^2|_{\Omega S^{2n+1}}$, the endomorphism $\mathrm{H}^*\,\lambda^2$ of $\Lambda x\otimes\Gamma y$ induced by λ^2 must satisfy $\mathrm{H}^*\,\lambda^2(1\otimes y(m)) = 2^m\otimes y(m)$, for degree reasons. Because $\mathrm{H}^*\,\lambda^2$ is a map of algebras, it is therefore true that

$$\mathrm{H}^*\,\lambda^2(x\otimes y(m)) = 2^m\cdot x\otimes y(m).$$

Let z denote the unique nondegenerate simplex of K in degree $2n+1$. As seen in section 2.4, the quasi-isomorphism

$$\Upsilon : C^*K\widetilde{\otimes}\mathcal{B}C^*K = (\Lambda z\otimes\Gamma sz, 0)\xrightarrow{\simeq} CU^*(\mathcal{L}S^{2n+1})$$

sends z to a representative ζ of x, sz to $\xi_1 = \omega_0(\zeta)$, which represents y, and $sz(m)$ to some representative ξ_m of $y(m)$. Furthermore, calculations identical to those above show that

$$(Id\otimes\psi_K^\sharp\chi)(1\otimes sz(m)) = 2^m\otimes sz(m)\quad\text{and}\quad(Id\otimes\psi_K^\sharp\chi)(z\otimes sz(m)) = 2^m\cdot z\otimes sz(m).$$

Hence,

$$\Upsilon(Id\otimes\psi_K^\sharp\chi)(1\otimes sz(m)) = 2^m\Upsilon(1\otimes sz(m)) = 2^m\cdot\xi_m,$$

which is a representative of $2^m y(m)$, as is

$$CU^*\lambda^2\Upsilon(1\otimes sz(m)) = CU^*\lambda^2(\xi_m).$$

Since $y(m)$ is the unique class of degree $2nm$, there exists $\varsigma_m\in CU^{2nm-1}\mathcal{L}X$ such that

$$d^\sharp\varsigma_m = \Upsilon(Id\otimes\psi_K^\sharp\chi)(1\otimes sz(m)) - CU^*\lambda^2\Upsilon(1\otimes sz(m)),$$

which implies that

$$d^\sharp(\varsigma_m\cdot\zeta) = \Upsilon(Id\otimes\psi_K^\sharp\chi)(z\otimes sz(m)) - CU^*\lambda^2\Upsilon(z\otimes sz(m)).$$

Thus, the diagram (3.1.1) commutes up to a cochain homotopy G defined by $G(1\otimes sz(m)) = \varsigma_m$ and $G(z\otimes sz(m)) = \varsigma_m\cdot\zeta$. $\square$

3.2 Topological cyclic homology.

As explained in the Preface, we can now construct a cochain complex $tc^*(X)$ such that $\mathrm{H}^*(tc^*(X) \otimes \mathbb{F}_2) \cong \mathrm{H}^*((TC(X;2);\mathbb{F}_2)$, the mod 2 spectrum cohomology of $TC(X;2)$, at least for certain spaces X. The model $tc^*(X)$ is the mapping cone of the following composition, where π denotes the obvious projection map.

$$\begin{array}{ccc} (\Lambda v \otimes C^*K\widetilde{\otimes}\mathcal{B}C^*K, \widetilde{D}) & \xrightarrow{\pi} & C^*K\widetilde{\otimes}\mathcal{B}C^*K \\ & \searrow^{\tilde{\pi}} & \downarrow{\scriptstyle Id\otimes(Id-\psi^\sharp_K\chi)} \\ & & C^*K\widetilde{\otimes}\mathcal{B}C^*K \end{array}$$

Recall that the mapping cone of a cochain map $f:(V,d)\longrightarrow(V',d')$ is a cochain complex $C_f := (V'\oplus sV, D_f)$, where $D_f(v') = d'v'$ for all $v' \in V'$ and $D_f(sv) = f(v) - s(dv)$ for all $sv \in sV$. It is an easy exercise to show that if f is cochain homotopic to g, then C_f and C_g are cochain equivalent. Theorem 3.1.1 suffices therefore to ensure that the mapping cone of the composition above has the right cohomology.

The next theorem now follows immediately from the results of the preceding chapters and section, according the justfication in [HR] of our method of construction of $tc^*(X)$.

Theorem 3.2.1. *Let X be a 1-connected space with the homotopy type of a finite-type CW-complex, and let $\Upsilon : C^*K\widetilde{\otimes}\mathcal{B}C^*K \xrightarrow{\simeq} CU^*\mathcal{L}X$ be a thin free loop model for X such that all primitives of $\mathcal{B}C^*K$ are indecomposable and such that*

$$\begin{array}{ccc} C^*K\widetilde{\otimes}\mathcal{B}C^*K & \xrightarrow{Id\otimes(\psi^\sharp_K\chi)} & C^*K\widetilde{\otimes}\mathcal{B}C^*K \\ \simeq\downarrow\Upsilon & & \simeq\downarrow\Upsilon \\ CU^*\mathcal{L}X & \xrightarrow{CU^*\lambda^2} & CU^*\mathcal{L}X \end{array}$$

commutes. Suppose that $\mathcal{L}X_{hS^1}$ has a thin model

$$\widetilde{\Upsilon} : (\Lambda v \otimes C^*K\widetilde{\otimes}\mathcal{B}C^*K, \widetilde{D}) \xrightarrow{\simeq} (\Lambda v \otimes CU^*\mathcal{L}X, D^\sharp)\ .$$

Let

$$tc^*(X) = (C^*K\widetilde{\otimes}\mathcal{B}C^*K \oplus s(\Lambda v \otimes C^*K\widetilde{\otimes}\mathcal{B}C^*K), D_{\tilde{\pi}})$$

where for all $x \otimes c \in C^*K \otimes \mathcal{B}C^*K$,

$$D_{\tilde{\pi}}(x\otimes c) = \check{D}(x\otimes c)$$

while

$$D_{\tilde\pi}\big(s(1\otimes x\otimes c)\big) = x\otimes c - x\otimes \psi_K^{\sharp}\chi(c) \\ - s\big(1\otimes \check D(x\otimes c) + \upsilon\otimes S(x\otimes c)\big)$$

and for $k>0$,

$$D_{\tilde\pi}\big(s(\upsilon^k\otimes x\otimes c)\big) = -s\big(\upsilon^k\otimes \check D(x\otimes c) + \upsilon^{k+1}\otimes S(x\otimes c)\big).$$

Then $\mathrm{H}^*(tc^*(X)\otimes\mathbb{F}_2)$ *is isomorphic to the mod 2 spectrum cohomology of* $TC(X;2)$

A more general version of this theorem will appear in [HR].

We conclude this chapter and this article with an example illustrating the use of the models we have built.

Example. Let $n>0$, and let K be the model of S^{2n+1} with exactly one nondegenerate simplex z of positive dimension, in dimension $2n+1$. As explained in the proof of Theorem 2.4.3, $C^*K=\Lambda z$, with trivial differential. Furthermore, $\mathcal{B}C^*K$ is isomorphic as an algebra to Γsz, for degree reasons, and

$$fls^*(S^{2n+1}) = (\Lambda z\otimes \Gamma sz, 0).$$

Thus, $\mathrm{H}^*(\mathcal{L}S^{2n+1})\cong \Lambda z\otimes\Gamma sz$, as has long been known.

Let $sz(m)=sz|\cdots|sz\in\perp^m sz$. We then have

$$hos^*(S^{2n+1}) = (\Lambda\upsilon\otimes\Lambda z\otimes\Gamma sz, \widetilde D)$$

where $\widetilde D(\upsilon^k\otimes 1\otimes sz(m))=0$ for all k and m, while

$$\widetilde D(\upsilon^k\otimes z\otimes sz(m)) = (m+1)\upsilon^{k+1}\otimes 1\otimes sz(m+1).$$

The integral cohomology of the homotopy orbit space is therefore

$$\mathrm{H}^*(\mathcal{L}S^{2n+1}_{hS^1})\cong \Lambda\upsilon\oplus\Gamma sz\oplus\bigoplus_{k,m\geq 1}\mathbb{Z}/m\mathbb{Z}\cdot(\upsilon^k\otimes sz(m))$$

as graded modules, while its mod p cohomology is of the form

$$\mathrm{H}^*(\mathcal{L}S^{2n+1}_{hS^1};\mathbb{F}_p)\cong \Lambda\upsilon\otimes\mathbb{F}_p\oplus\Gamma sz\otimes\mathbb{F}_p \\ \oplus\bigoplus_{\substack{k,m\geq 1\\ p|m}}\mathbb{F}_p\cdot(\upsilon^k\otimes sz(m))\oplus\bigoplus_{\substack{k\geq 0\\ p|m+1}}\mathbb{F}_p\cdot(\upsilon^k\otimes z\otimes sz(m))$$

where a Bockstein sends the class of $\upsilon^k\otimes z\otimes sz(m)$ to the class of $\upsilon^{k+1}\otimes sz(m+1)$.

Finally

$$tc^*(S^{2n+1}) = (\Lambda z \otimes \Gamma sz \oplus s(\Lambda \upsilon \otimes \Lambda z \otimes \Gamma sz), D_{\tilde{\pi}})$$

where for all m,

$$D_{\tilde{\pi}}(z \otimes sz(m)) = 0 = D_{\tilde{\pi}}(1 \otimes sz(m))$$

while

$$\begin{aligned} D_{\tilde{\pi}}\big(s(1 \otimes z \otimes sz(m))\big) =& z \otimes sz(m) - z \otimes \psi_K^\sharp \chi(sz(m)) \\ & - s\big(\upsilon \otimes S(z \otimes sz(m))\big) \\ =& (1-2^m) z \otimes sz(m) - (m+1) s\big(\upsilon \otimes 1 \otimes sz(m+1)\big) \end{aligned}$$

and

$$\begin{aligned} D_{\tilde{\pi}}\big(s(1 \otimes 1 \otimes sz(m))\big) =& 1 \otimes sz(m) - 1 \otimes \psi_K^\sharp \chi(sz(m)) \\ =& (1-2^m) \otimes sz(m) \end{aligned}$$

and for $k > 0$,

$$D_{\tilde{\pi}}\big(s(\upsilon^k \otimes z \otimes sz(m))\big) = -(m+1) s\big(\upsilon^{k+1} \otimes 1 \otimes sz(m+1)\big)$$

and

$$D_{\tilde{\pi}}\big(s(\upsilon^k \otimes 1 \otimes sz(m))\big) = 0.$$

Note that we use that

$$\psi_K^\sharp \chi(sz(m)) = \psi_K^\sharp (\sum_{i=0}^{m} sz(i) \otimes sz(m-i)) = \sum_{i=0}^{m} \binom{m}{i} sz(m) = 2^m sz(m).$$

Modulo 2, these formulas become

$$D_{\tilde{\pi}}(z \otimes sz(m)) = 0 = D_{\tilde{\pi}}(1 \otimes sz(m))$$

while

$$D_{\tilde{\pi}}\big(s(1 \otimes z \otimes sz(m))\big) = \begin{cases} z \otimes sz(m) & : m \text{ odd} \\ z \otimes sz(m) + s(\upsilon \otimes 1 \otimes sz(m+1)) & : m \text{ even} \end{cases}$$

and

$$D_{\tilde{\pi}}\big(s(1 \otimes 1 \otimes sz(m))\big) = 1 \otimes sz(m)$$

and for $k > 0$,

$$D_{\tilde{\pi}}\big(s(v^k \otimes z \otimes sz(m))\big) = \begin{cases} 0 & : m \text{ odd} \\ s(v^{k+1} \otimes 1 \otimes sz(m+1)) & : m \text{ even} \end{cases}$$

and

$$D_{\tilde{\pi}}\big(s(v^k \otimes 1 \otimes sz(m))\big) = 0.$$

We can now compute easily that

$$\begin{aligned} \mathrm{H}^*(TC(S^{2n+1};2);\mathbb{F}_2) \cong & \bigoplus_{\substack{k\geq 1\\ m\geq 0}} \mathbb{F}_2 \cdot s(v^k \otimes sz(2m)) \oplus \mathbb{F}_2 \cdot s(v^k \otimes x \otimes sx(2m+1)) \\ & \oplus \bigoplus_{m\geq 0} \mathbb{F}_2 \cdot s(v \otimes sz(2m+1)) \end{aligned}$$

as graded vector spaces.

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