A LOWER BOUND FOR THE RANK OF $J_0(q)$

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À Andrzej Schinzel pour son soixantième anniversaire

Abstract. This paper is the second of a series devoted to the study of the rank of $J_0(q)$ (the Jacobian of the modular curve $X_0(q)$), from the analytic point of view stemming from the Birch and Swinnerton-Dyer conjecture, which is tantamount to the study, on average, of the order of vanishing at the central critical point of the L-functions of primitive weight two forms f of level q (q prime). We prove that, for a large proportion of such forms, the associated L function vanishes at order exactly one at the critical point. From the work of Gross-Zagier, this implies a strong lower bound for the geometric rank of $J_0(q)$.

Contents

1. Introduction

Let q be a prime number, and consider the abelian variety $J_0(q)$, the Jacobian of the modular curve $X_0(q)$. It is defined over $\mathbf{Q}$, of dimension $\dim J_0(q) \sim q/12$. Eichler and Shimura [Sh] have shown that its Hasse-Weil L-function is given by

$$L(J_0(q), s) = \prod_{f \in S_2(q)^*} L(f, s) \tag{1.1}$$

where the product is over the finite set $S_2(q)^*$ ($|S_2(q)^*| = \dim J_0(q)$) of primitive weight 2 forms f of level q, and the L-functions are normalized so that $\mathrm{Re}(s) = 1/2$ is the critical line.

According to the Birch and Swinnerton-Dyer conjecture, one should have then

$$\mathrm{rank}\, J_0(q) = \sum_{f \in S_2(q)^*} \mathrm{ord}_{s=1/2} L(f, s)$$

and it is expected that

$$\mathrm{rank}\, J_0(q) \sim \frac{1}{2} \dim J_0(q)$$

based on heuristics concerning the zeros of L-functions.

In [KM1] we used the factorization (1.1) to obtain the upper bound

$$\mathrm{rank}\, J_0(q) \leq C \dim J_0(q)$$

for some absolute (and effectively computable, see [KM2]) constant $C > 0$, on the Birch and Swinnerton-Dyer conjecture. This was proved by bounding from above the average order of vanishing of the L-functions at $s = 1/2$.

Here we consider the dual problem of non-vanishing of $L(f, 1/2)$. More precisely we look at forms f with order of vanishing exactly one. We prove

Theorem 1. *Let $\varepsilon > 0$ be any positive real number. For q large enough (in terms of ε), we have*

$$|\{f \in S_2(q)^* \mid L(f, 1/2) = 0,\ L'(f, 1/2) \neq 0\}| \geq \Bigl(\frac{19}{54} - \varepsilon\Bigr) |S_2(q)^*|.$$

By work of Gross and Zagier [GZ], the product

$$\prod_f L(f, s)$$

over the forms f with $L(f, 1/2) = 0$, $L'(f, 1/2) \neq 0$, is the L-function of a quotient of $J_0(q)$ with rank exactly equal to its dimension. Thus we have the following corollary:

Corollary 1. *Let $\varepsilon > 0$ be any positive real number. For q large enough (in terms of ε), we have*

$$\mathrm{rank}\, J_0(q) \geq \Bigl(\frac{19}{54} - \varepsilon\Bigr) \dim J_0(q).$$

Since $19/54 = 0.35...$, this is quite close to the conjectured value.

The method used here works equally well for the non-vanishing of $L(f, 1/2)$ itself. We indicate briefly how they lead (more easily) to the

Theorem 2. *Let $\varepsilon > 0$ be any positive real number. For q large enough (in terms of ε), we have*

$$|\{f \in S_2(q)^* \mid L(f, 1/2) \neq 0\}| \geq \Bigl(\frac{1}{6} - \varepsilon\Bigr) |S_2(q)^*|.$$

This result is weaker, however, than what Iwaniec and Sarnak [IS] have obtained in the course of their work on the Landau-Siegel zero. Indeed, their more advanced techniques can be used to improve the constant 19/54 to 7/16.

Remarks. (1) Independently, and using different methods, VanderKam [VdK] has obtained the same non-vanishing results, except for a smaller numerical value of the proportion achieved.

(2) Luo, Iwaniec and Sarnak [ILS] have proved (assuming the Generalized Riemann Hypothesis) that

$$\sum_{f\in S_2(q)^*} \mathrm{ord}_{s=1/2} L(f,s) \leq (c+o(1))|S_2(q)^*|$$

for some (explicit) $c<1$; this is of great significance for the conjectures and heuristics of Katz and Sarnak [KS].

We now give the precise statement of the main result.

Theorem 3. *For any $0\leq \Delta < 1/4$ and any prime q large enough (depending on Δ only), we have*

$$\sum_{\substack{f\in S_2(q)^* \\ L(f,1/2)=0,\, L'(f,1/2)\neq 0}} 1 \geq \frac{1}{2}\Big(1-\frac{1}{(1+2\Delta)^3}\dim J_0(q). \tag{1.2}$$

In particular, letting $\Delta\to 1/4$, Theorem 1 follows.

Since the set of f such that $L(f,s)$ has a simple zero at the critical point is contained in the set of odd forms, we have proved that for at least 70 percent of the odd forms, the order of $L(f)$ at the critical point is exactly one.

Remark. Coincidentally, Soundararajan [Sou], has shown that the proportion of quadratic twists of a given quadratic Dirichlet character χ for which $L(\chi\otimes\psi,1/2)\neq 0$ satisfies the same lower bound, when the length of the mollifier is suitably parameterized. This is explained in part by the heuristics of Katz and Sarnak [KS]. Less clear is the coincidence of those proportions with that obtained by Conrey, Ghosh and Gonek [CGG] for the number of simple zeros of the Riemann ξ function on the critical line.

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We also wish to thank H. Iwaniec and P. Sarnak for showing us some of their ongoing work [IS]. Also we thank the referee for carefully reading the most delicate parts of our arguments and pointing out some inaccuracies.

Notations. For any $q\geq 1$ we will write ε_q for the trivial Dirichlet character modulo q.

All summations over f will be implicitly over $f\in S_2(q)^*$, with other conditions explicitly indicated in the summation indices.

We write $\log_2 x := \log\log x$.

Finally we make the following convention concerning the use of Vinogradov's and Landau's symbol $\ll$, $O(\ )$: the constants implied by these notations are meant to be absolute. In case there are other parameters involved, say ε, Δ, we (usually) indicate the dependency of the constants by the subscript notations $\ll_{\varepsilon,\Delta}, O_{\varepsilon,\Delta}(\ )$. The reader is encouraged to show good will towards analytic number theorists and interpret such inequalities in the most reasonable way (provided it is correct and proves the result which is sought...)

2. Non-vanishing in harmonic average

As in [KM1], we proceed by working first with the "harmonic" average

$$\sideset{}{^h}\sum_{\substack{f\in S_2(q)^* \\ L(f,1/2)=0,\, L'(f,1/2)\neq 0}} 1$$

where we write

$$\sum_f^h \alpha_f = \sum_f \frac{1}{4\pi(f,f)}\alpha_f$$

for any family α_f of complex numbers. We then derive the corresponding result for the "natural" average

$$\sum_{\substack{f\in S_2(q)^* \\ L(f,1/2)=0,\, L'(f,1/2)\neq 0}} 1.$$

2.1. **The principle.** As in previous investigations of such questions ([Du], [Iw], [KM1]. . .), the theorem will follow, by an application of Cauchy's inequality, from a comparison of a lower bound for a sum

$$M_1 := \sum_{L(f,1/2)=0}^h M(f)L'(f,1/2)$$

and an upper bound for

$$M_2 := \sum_{L(f,1/2)=0}^h |M(f)L'(f,1/2)|^2$$

for certain suitable complex numbers $M(f)$ (the "mollifier"). Indeed we have directly

$$M_1 \leq \Bigl(\sum_{L(f,1/2)=0,\, L'(f,1/2)\neq 0}^h 1\Bigr)^{1/2} M_2^{1/2}$$

so that

$$\sum_{L(f,1/2)=0,\, L'(f,1/2)\neq 0}^h 1 \geq \frac{M_1^2}{M_2}. \tag{2.1}$$

We will follow this plan, except that in order to achieve the best possible numerical proportion, we will seek asymptotics for M_1 and M_2. It will be noticed that if the mollifier is ignored (take $M(f)=1$), a factor $\log q$ is lost in the final estimate.

In the case of the special values themselves, we consider of course

$$N_1 = \sum_{f\in S_2(q)^*}^h M(f)L(f,1/2)$$

$$N_2 = \sum_{f\in S_2(q)^*}^h |M(f)L(f,1/2)|^2$$

and compare.

2.2. **The gamma factor effect.** For $f\in S_2(q)^*$ we write its Fourier expansion

$$f(z) = \sum_{n\geq 1}\lambda_f(n)n^{1/2}e(nz)$$

and its L-function

$$L(f,s) = \sum_{n\geq 1}\lambda_f n^{-s} = \prod_p (1-\lambda_f(p)p^{-s}+\epsilon(p)p^{-2s})^{-1}$$

putting, as mentioned, the center of the critical strip at 1/2.

The functional equation is written in terms of the completed L-function

$$\Lambda(f,s)=\Big(\frac{\sqrt{q}}{2\pi}\Big)^s\Gamma\big(s+\frac{1}{2}\big)L(f,s),$$

namely

$$\Lambda(f,s)=\varepsilon_f\Lambda(f,1-s) \tag{2.2}$$

and the sign ε_f of the functional equation is (see [Miy] for instance)

$$\varepsilon_f=q^{1/2}\lambda_f(q)=\pm 1. \tag{2.3}$$

A form is said to be even (resp. odd) if $\varepsilon_f=1$ (resp. $\varepsilon_f=-1$). By the functional equation, this is the same parity as that of the order of $L(f)$ at $s=1/2$. We will write

$$\varepsilon_f^+=\frac{1+\varepsilon_f}{2},\ \varepsilon_f^-=\frac{1-\varepsilon_f}{2}$$

so $f\mapsto\varepsilon_f^+f$ is the projection of the space of primitive forms onto the space of even forms, and correspondingly for the odd ones. In particular, we have

$$(\varepsilon_f^{\pm})^2=\varepsilon_f^{\pm}.$$

Since the "Gamma factor"

$$\gamma(s)=\Big(\frac{\sqrt{q}}{2\pi}\Big)^s\Gamma\big(s+\frac{1}{2}\big)$$

doesn't vanish at 1/2, the order of $L(f)$ at $s=1/2$ is the same as that of $\Lambda(f)$. If f is even, the vanishing of $\Lambda'(f,1/2)$ thus implies that

$$L(f,1/2)=0\Rightarrow L'(f,1/2)=0. \tag{2.4}$$

¿From this we deduce an easy but very important proposition.

Proposition 1. *Let (α_f) be any family of complex numbers. Then*

$$\sum_{L(f,1/2)=0}^{h}\alpha_fL'(f,1/2)=\sum_{f}^{h}\varepsilon_f^-\alpha_fL'(f,1/2). \tag{2.5}$$

The point of this formula, which applies to the sums of type M_1 and M_2 above, is that an average over f in the restricted subset where $L(f,1/2)=0$ (the "non-rank 0" set) is written as an average over all f, for which suitable analytic summation formulae may apply, at the cost of inserting ε_f which is much the same as $\lambda_f(q)$ (see (2.3)). We may notice at this point that this is special to the order 1 case: sums of the type

$$\sum_{L(f,1/2)=L'(f,1/2)=0}^{h}\alpha_fL''(f,1/2)$$

– which one would like to study for estimating the (conjectural) dimension of the quotients of $J_0(q)$ of normalized rank 2 – do not readily lend themselves to such an easy simplification.

2.3. **Computing M_1.** By the proposition, we have

$$M_1 = \sum_f^h \varepsilon_f^- M(f) L'(f, 1/2).$$

To make the sum manageable, we choose $M(f)$ of the shape

$$M(f) = \sum_{m \leq M} x_m \lambda_f(m) m^{-1/2}$$

for real numbers (x_m) (and a parameter $M > 0$) which we will try to choose to optimize the resulting bound (2.1). If $m > M$, we will write, for convenience, $x_m = 0$. Now we only impose that the x_m be supported on squarefree integers and satisfy

$$x_m \ll \big(\tau(m)(\log qm)\big)^A \tag{2.6}$$

for some absolute constant $A > 0$. We write $M = q^\Delta$, and will assume $0 \leq \Delta < 1$.

First we express $L'(f, 1/2)$ as a rapidly convergent series using contour integration and the functional equation: we consider the integral

$$I = \frac{1}{2i\pi} \int_{(2)} \Lambda(f, s + 1/2) G(s) \frac{ds}{s^2}$$

where G is a polynomial of degree N (large enough, $N = 2$ works already) satisfying

$$G(-s) = G(s), \text{ and } G(0) = 1 \tag{2.7}$$

$$G(-N) = \ldots = G(-1) = 0. \tag{2.8}$$

Notice that from the first of these, we obtain also

$$G'(0) = 0,\ G^{(3)}(0) = 0. \tag{2.9}$$

Shifting the contour to $\mathrm{Re}(s) = -1$, applying the functional equation, gives

$$2\varepsilon_f^- I = \mathrm{Res}_{s=0} \frac{\Lambda(f, s + 1/2) G(s)}{s^2}$$

and this, from (2.9), (2.7)

$$2\varepsilon_f^- I = \Lambda'(f, 1/2)$$

whence, multiplying through by ε_f^-

$$2\varepsilon_f^- I = \varepsilon_f^- \Big(\frac{\sqrt{q}}{2\pi}\Big)^{1/2} L'(f, 1/2).$$

Expanding now $L(f)$ as a Dirichlet series in I we get after some simplifications

$$\varepsilon_f^- L'(f, 1/2) = 2\varepsilon_f^- \sum_{l \geq 1} \lambda_f(l) l^{-1/2} V\Big(\frac{2\pi}{\sqrt{q}} l\Big) \tag{2.10}$$

with

$$V(y) = \frac{1}{2i\pi} \int_{(3/2)} \Gamma(s + 1) G(s) y^{-s} \frac{ds}{s^2}. \tag{2.11}$$

¿From this we obtain at once

$$M_1 = \sum_{l,m} x_m (lm)^{-1/2} V\Big(\frac{2\pi}{\sqrt{q}} l\Big) \times \Delta_-(l,m) \tag{2.12}$$

where

$$\Delta_-(l,m) = 2 \sum_f{}^h \varepsilon_f^- \lambda_f(l)\lambda_f(m).$$

As can be expected, Δ_- is a close relative to the Kronecker delta-symbol (in certain ranges).

Lemma 1. *Let $\epsilon > 0$ be any positive real number. Then for $l \geq 1$ and $1 \leq m \leq q$, it holds*

$$\Delta_-(l,m) = \delta(l,m) + O\Big(\frac{(lm)^{1/2+\epsilon}}{q}\Big)$$

where δ is the Kronecker symbol.

Proof. We have, by (2.3)

$$\sum_f{}^h \varepsilon_f^- \lambda_f(l)\lambda_f(m) = \sum_f{}^h \lambda_f(l)\lambda_f(m) + q^{1/2} \sum_f{}^h \lambda_f(q)\lambda_f(l)\lambda_f(m)$$

and moreover $\lambda_f(q)\lambda_f(l) = \lambda_f(lq)$ for any l. We now apply Petersson's formula and classical bounds for Kloosterman sums and Bessel functions, supplemented in the second term by the remarks that for $m < q$ we have $lq \neq m$, and the Kloosterman sum $S(m, lq; q)$ is a Ramanujan sum, from which a factor $q^{1/2}$ is saved when estimating sums $S(m, lq; cq)$ for $(c, q) = 1$, those for $q \mid c$ being easily treated. All this is explained in more details in the next section, where a more refined analysis of the remainder term is required for the second moment. □

To conclude the analysis of M_1, we estimate V (by shifting the contour to the left, or right):

$$V(y) = -\log y - \gamma + O(y^N),\ y \to 0$$
$$\forall\ j \geq 1,\ V(y) = O_j(y^{-j}),\ y \to +\infty$$

($\gamma = -\Gamma'(1)$ being Euler's constant); then from (2.12), the lemma, and those estimates, we obtain the next proposition.

Proposition 2. *Let $M = q^\Delta$ with $\Delta < 1/2$, define $\hat{q}$ by*

$$\log \hat{q} = -\log \frac{2\pi}{\sqrt{q}} - \gamma,$$

then, for some absolute constant $c > 0$

$$M_1 = \sum_{m \leq M} \frac{x_m}{m} \log(\hat{q}/m) + O(q^{-c}). \tag{2.13}$$

In the following, when we write an error term of the form $O(q^{-c})$, it is implied that $c > 0$, and the value of c may change from line to line.

In the case of the first moment N_1 of special values, we consider similarly the integral

$$\frac{1}{2i\pi} \int_{(2)} \Lambda(f, s + 1/2) G(s) \frac{ds}{s}$$

and derive

$$N_1 = \sum_{m\leq M} \frac{x_m}{m} + O(q^{-c}) \tag{2.14}$$

for some $c = c(\Delta) > 0$ if $\Delta < 1/2$. We only need the estimate

$$\sum_{f\in S_2(q)^*}^h l(m)\lambda_f(n) = \delta(m,n) + O_\varepsilon((mn)^{1/2+\varepsilon}q^{-3/2})$$

(see below (2.21)).

2.4. **Computing M_2.** We now wish to get an expression for M_2 as a quadratic form in the x_m. A new phenomenon appears, however, at the point where we would like to appeal to lemma 1, as the remainder term in the Petersson formula (the series of Kloosterman sums) can't be ignored, and has to be analyzed to yield a contribution to the main term.

2.4.1. *Expressing $L'(f,1/2)^2$ for f odd.* We consider this time

$$J = \frac{1}{2i\pi}\int_{(2)} \Lambda(f, s+1/2)^2 G(s)\frac{ds}{s^3}$$

and proceed to evaluate it as before. From

$$L(f,s)^2 = \zeta_q(2s)\sum_{n\geq 1}\tau(n)\lambda_f(n)n^{-s}$$

it follows

$$2\times\frac{\sqrt{q}}{2\pi}\sum_{n\geq 1}\lambda_f(n)\tau(n)n^{-1/2}W\Big(\frac{4\pi^2 n}{q}\Big) = \mathrm{Res}_{s=0}\frac{\Lambda(f,s+1/2)^2G(s)}{s^3}$$

with

$$W(y) = \frac{1}{2i\pi}\int_{(1/2)} \zeta_q(1+2s)\Gamma(s)^2G(s)y^{-s}\frac{ds}{s}. \tag{2.15}$$

For our purpose, W is basically a 'cut-off' function. Indeed, we have the following

Lemma 2. *The function W satisfies*

$$y^iW^{(j)}(y) \ll_{i,j} \log(y+1/y)^3,\ \textit{for all } i\geq j\geq 0 \tag{2.16}$$

$$\forall j\geq 1,\ W(y) = O_j(y^{-j}). \tag{2.17}$$

Moreover, there exists a polynomial P, independent of q, of degree at most 2, such that for $y\to 0$

$$W(y) = -\frac{1}{12}(\log y)^3 + P(\log y) + O(q^{-1}(\log y)^2 + y^N). \tag{2.18}$$

Proof. The first two inequalities are obtained by the usual contour shifts and differentiating under the integral sign. As for the last, we write

$$W(y) = \mathrm{Res}_{s=0}\frac{G(s)\Gamma(s)^2\zeta_q(1+2s)y^{-s}}{s} + O(y^N)$$

again by shifting, and simply compute the residue. □

Remark The polynomial P can be explicitly computed. However its exact value is of no importance in what follows, the only relevant fact being that its degree is ≤ 2.

Now if f is odd, we have $\Lambda(f, 1/2) = 0$ and then we find that

$$\frac{d^2}{ds^2}\Lambda(f, s+1/2)^2\Big|_{s=0} = 2 \times \frac{\sqrt{q}}{2\pi} L'(f, 1/2)^2$$

so, evaluating the residue, we derive for f odd

$$L'(f, 1/2)^2 = 2\sum_{n\geq 1} \lambda_f(n)\tau(n)n^{-1/2}W\Big(\frac{4\pi^2 n}{q}\Big). \tag{2.19}$$

2.4.2. *Applying Petersson's formula.* Working towards incorporating the mollifier, we fix some $0 \leq \Delta < 1$, $1 \leq m \leq q^\Delta$, and consider the following average over f:

$$X(m) = \sideset{}{^h}\sum_{f\in S_2(q)^*} \varepsilon_f^- \lambda_f(m) L'(f, 1/2)^2. \tag{2.20}$$

From (2.19) and (2.3), we have

$$X(m) = \sideset{}{^h}\sum_{f} (1 - q^{1/2}\lambda_f(q))\lambda_f(m) \sum_{n\geq 1} \tau(n)\lambda_f(n)n^{-1/2}W\Big(\frac{4\pi^2 n}{q}\Big).$$

For any l_1 and l_2, Petersson's formula is

$$\sideset{}{^h}\sum_{f} \lambda_f(l_1)\lambda_f(l_2) = \delta(l_1, l_2) - \mathcal{J}(l_1, l_2)$$

where

$$\mathcal{J}(l_1, l_2) = \frac{2\pi}{q}\sum_{r\geq 1} r^{-1} S(l_1, l_2; qr) J_1\Big(\frac{4\pi\sqrt{l_1 l_2}}{qr}\Big).$$

The trivial bound for this, from Weil's bound for Kloosterman sums and $J_1(x) \ll x$, is

$$\mathcal{J}(l_1, l_2) \ll_\epsilon \frac{(l_1 l_2)^{1/2+\epsilon}}{q^{3/2}}. \tag{2.21}$$

Since q is the level, $\lambda_f(q)\lambda_f(n) = \lambda_f(nq)$ for all n, and moreover $qn \neq m$ since $(m, q) = 1$, therefore we get

$$X(m) = X^+(m) + X^-(m)$$

with

$$X^+(m) = \frac{\tau(m)}{\sqrt{m}} W\Big(\frac{4\pi^2 m}{q}\Big) - \sum_{n\geq 1} \frac{\tau(n)}{\sqrt{n}} W\Big(\frac{4\pi^2 n}{q}\Big)\mathcal{J}(n, m),$$

$$X^-(m) = q^{1/2}\sum_{n\geq 1} \frac{\tau(n)}{\sqrt{n}} W\big(\frac{4\pi^2 n}{q}\big)\mathcal{J}(qn, m).$$

2.4.3. *Treatment of $X^+(m)$.* Using the trivial bound (2.21) and (2.17) ($N=2$ is enough) the second term is seen to be

$$\ll_\epsilon m^{1/2+\epsilon}q^{-1/2}(\log q)^4$$

and by (2.18) we infer

$$X^+(m)=\frac{1}{12}\frac{\tau(m)}{\sqrt{m}}\Bigl(\log\frac{\hat{Q}}{m}\Bigr)^3+\frac{\tau(m)}{\sqrt{m}}P\Bigl(\log\frac{\hat{Q}}{m}\Bigr)+O_\varepsilon\Bigl(\frac{m^{1/2}q^\epsilon}{\sqrt{q}}\Bigr), \tag{2.22}$$

with $\hat{Q}$ defined by $\log\hat{Q}=\log\frac{q}{4\pi^2}$.

2.4.4. *Treatment of $X^-(m)$.* The contribution, in $\mathcal{J}(qn,m)$, of those r for which $(r,q)>1$ (so $q\mid r$) is also found to be $O((mn)^{1/2+\epsilon}q^{-5/2})$ and *in toto* this gives

$$\ll_\epsilon m^{1/2+\epsilon}q^{-1}(\log q)^4. \tag{2.23}$$

It remains to study

$$\frac{2\pi}{\sqrt{q}}\sum_{(r,q)=1}\frac{1}{r}\sum_{n\geq 1}\frac{\tau(n)}{\sqrt{n}}S(m,qn;qr)J_1\Bigl(\frac{4\pi}{r}\sqrt{\frac{mn}{q}}\Bigr)W\Bigl(\frac{4\pi^2n}{q}\Bigr).$$

For $(r,q)=1$, the Kloosterman sum $S(m,qn;qr)$ factorizes

$$S(m,qn;qr)=S(m\overline{q},n;r)S(0,m;q)=-S(m\overline{q},n;r)$$

since $S(0,m;q)$ is a Ramanujan sum with q prime, and $(m,q)=1$.

Fix $R>0$, to be chosen later (but such that $\log R\ll\log q$). In the previous expression we estimate the tail of the series for $r>R$:

$$-\frac{2\pi}{\sqrt{q}}\sum_{\substack{r>R\\(r,q)=1}}\frac{1}{r}\sum_{n\geq 1}\frac{\tau(n)}{\sqrt{n}}S(m\overline{q},n;r)J_1\Bigl(\frac{4\pi}{r}\sqrt{\frac{mn}{q}}\Bigr)W\Bigl(\frac{4\pi^2n}{q}\Bigr)=O(\frac{m^{1/2+\epsilon}(\log q)^4}{R^{1/2}}) \tag{2.24}$$

and reduce the study of $X^-(m)$ to that of the remaining part, say $X'(m)$.

2.4.5. *Extraction of the main term.* We denote by X_r the inner sum in (the weighted) $X'(m)$:

$$X_r=-\sum_{n\geq 1}\frac{\tau(n)}{\sqrt{n}}S(m\overline{q},n;r)J_1\Bigl(\frac{4\pi}{r}\sqrt{\frac{mn}{q}}\Bigr)W\Bigl(\frac{4\pi^2n}{q}\Bigr)\xi(n).$$

For technical reasons (which only occur because the weight is 2), we have fixed a C^∞ function $\xi:\mathbf{R}^+\to[0,1]$ which satisfies

$$\xi(x)=0,\ 0\leq x\leq 1/2,\quad \xi(x)=1,\ x\geq 1$$

and attached the weight $\xi(n)$ to the summation in n, without changing the value of X_r, of course.

Now we open the Kloosterman sum

$$S(m\overline{q},n;r)=\sum_{d\bmod r}{}^{*}\,e\Bigl(\frac{m\overline{qd}+nd}{r}\Bigr)$$

and take the summation over d outside. For each d, Jutila's extension ([Jut], theorem 1.7) of the Voronoi summation formula can be applied.

Proposition 3. *(Jutila). Let $t : \mathbf{R}^+ \to \mathbf{C}$ be a C^∞ function which vanishes in the neighborhood of 0 and is rapidly decreasing at infinity. Then for $c \geq 1$ and d coprime with c, we have*

$$\begin{aligned}\sum_{m\geq 1}\tau(m)e\Big(\frac{dm}{c}\Big)t(m) &= \frac{2}{c}\int_0^{+\infty}(\log\frac{\sqrt{x}}{c}+\gamma)t(x)dx\\ &-\frac{2\pi}{c}\sum_{h\geq 1}\tau(h)e\Big(-\frac{\bar{d}h}{c}\Big)\int_0^{+\infty}Y_0\Big(\frac{4\pi\sqrt{hx}}{c}\Big)t(x)dx\\ &+\frac{4}{c}\sum_{h\geq 1}\tau(h)e\Big(\frac{\bar{d}h}{c}\Big)\int_0^{+\infty}K_0\Big(\frac{4\pi\sqrt{hx}}{c}\Big)t(x)dx\end{aligned}$$

This yields

$$X_2 = -\frac{2}{r}S(m,0;r)\int_0^{\infty}(\log\frac{\sqrt{x}}{r}+\gamma)J_1\Big(\frac{4\pi}{r}\sqrt{\frac{mx}{q}}\Big)W\Big(\frac{4\pi^2 x}{q}\Big)\xi(x)\frac{dx}{\sqrt{x}} \tag{2.25}$$

$$+\frac{2\pi}{r}\sum_{h\geq 1}\tau(h)S(hq-m,0;r)\int_0^{+\infty}Y_0\Big(\frac{4\pi\sqrt{hx}}{r}\Big)J_1\Big(\frac{4\pi}{r}\sqrt{\frac{mx}{q}}\Big)W\Big(\frac{4\pi^2 x}{q}\Big)\xi(x)\frac{dx}{\sqrt{x}} \tag{2.26}$$

$$-\frac{4}{r}\sum_{h\geq 1}\tau(h)S(hq+m,0;r)\int_0^{+\infty}K_0\Big(\frac{4\pi\sqrt{hx}}{r}\Big)J_1\Big(\frac{4\pi}{r}\sqrt{\frac{mx}{q}}\Big)W\Big(\frac{4\pi^2 x}{q}\Big)\xi(x)\frac{dx}{\sqrt{x}} \tag{2.27}$$

We reserve for later consideration the last two sums (see section 2.4.6), and proceed to immediately remove ξ from the first, which we can do with an error which is at most

$$\frac{1}{\sqrt{q}}\sum_{r\leq R}\frac{1}{r^2}|S(m,0;r)|\int_0^1\Big|(\log\frac{\sqrt{x}}{r}+\gamma)J_1\Big(\frac{4\pi}{r}\sqrt{\frac{mx}{q}}\Big)W\Big(\frac{4\pi^2 x}{q}\Big)\Big|\frac{dx}{\sqrt{x}} \ll \frac{1}{\sqrt{q}}(\log q)^4$$

by (2.16) and simply $J_1(x) \ll 1$.

We are therefore studying

$$\begin{aligned}&-\frac{4\pi}{\sqrt{q}}\sum_{\substack{r\leq R\\(r,q)=1}}\frac{1}{r^2}S(m,0;r)\int_0^{\infty}(\log\frac{\sqrt{x}}{r}+\gamma)J_1\Big(\frac{4\pi}{r}\sqrt{\frac{mx}{q}}\Big)W\Big(\frac{4\pi^2 x}{q}\Big)\frac{dx}{\sqrt{x}}\\ =\;& -2\sum_{\substack{r\leq R\\(r,q)=1}}\frac{1}{r}S(m,0;r)\int_0^{\infty}(\log\frac{\sqrt{qx}}{2\pi}+\gamma)J_1(2\sqrt{mx})W(r^2x)\frac{dx}{\sqrt{x}}\end{aligned}$$

by the change of variable $x \mapsto \dfrac{r^2}{4\pi^2}qy$.

Using (2.15), this is equal to

$$\frac{1}{2i\pi}\int_{(1/2)}(-2)Z_m^R(1+2s)\zeta_q(1+2s)s^{-1}\Gamma(s)^2G(s)L(s)ds, \tag{2.28}$$

with

$$Z_m^R(s) = \sum_{\substack{r\leq R\\(r,q)=1}}S(m,0;r)r^{-s},$$

$$L(s)=\int_0^{+\infty}(\log\frac{\sqrt{qx}}{2\pi}+\gamma)J_1(2\sqrt{mx})x^{-s-1/2}dx.$$

Both Z_m^R and L can be computed.

Lemma 3. *We have for* $\mathrm{Re}(s)=\sigma>1$

$$Z_m^R(s)=\zeta_q(s)^{-1}\sum_{d|m}d^{1-s}+O_\sigma(\tau(m)R^{1-\sigma}).$$

Proof. By the formula giving the Ramanujan sum (the star meaning 'prime to q')

$$\begin{aligned}
Z_m^R(s) &= \sum_{r\leq R}{}^{*} r^{-s}\sum_{d|(m,r)}d\mu\Big(\frac{r}{d}\Big)\\
&= \sum_{d|m}d\sum_{fd\leq R}{}^{*}\mu(f)(fd)^{-s}\\
&= \sum_{d|m}d^{1-s}\Big\{\zeta_q(s)^{-1}+O\Big(\Big(\frac{R}{d}\Big)^{1-\sigma}\Big)\Big\}\\
&= \zeta_q(s)^{-1}\sum_{d|m}d^{1-s}+O(\tau(m)R^{1-\sigma})
\end{aligned}$$

□

Lemma 4. *Recall that* $\log\hat{Q}=\log\frac{q}{4\pi^2}$. *For all* s *with* $1/4<\mathrm{Re}(s)<1$, *we have*

$$L(s)=-\frac{1}{2}m^{s-1/2}\Gamma(-s)\Gamma(s)^{-1}\Big(\log\frac{\hat{Q}}{m}+2\gamma+\psi(1+s)+\psi(1-s)\Big)$$

where $\psi=\Gamma'/\Gamma$.

Proof. The following formula is valid for $-2<\mathrm{Re}(s)<-1/2$ (see [G-R] 6.561.14):

$$\ell(s):=\int_0^{+\infty}J_1(x)x^s dx=2^s\Gamma\Big(1+\frac{s}{2}\Big)\Gamma\Big(1-\frac{s}{2}\Big)^{-1} \tag{2.29}$$

and putting $y=2\sqrt{mx}$ in $L(s)$ gives

$$L(s)=4^s m^{s-1/2}\Big((\frac{1}{2}\log\frac{\hat{Q}}{m}+\gamma)\ell(-2s)+\ell'(-2s)\Big).$$

From (2.29) we deduce

$$\ell'(s)=2^s\Gamma\Big(1+\frac{s}{2}\Big)\Gamma\Big(1-\frac{s}{2}\Big)^{-1}\Big(\log 2+\frac{1}{2}\psi\Big(1+\frac{s}{2}\Big)+\frac{1}{2}\psi\Big(1-\frac{s}{2}\Big)\Big)$$

and the result follows. □

This allows us to replace $Z_m^R(1+2s)$ in (2.28) by $\sigma_{-2s}(m)\zeta_q(1+2s)^{-1}$, up to an error which is bounded by $O(\tau(m)(\log q)R^{-1})$. Denote by $X''(m)$ the resulting expression.

The lemmas show that the integrand in $X''(m)$ is

$$F(s)=m^{-1/2}s^{-1}G(s)\eta_s(m)\Gamma(s)\Gamma(-s)\Big(\log\frac{\hat{Q}}{m}+2\gamma+\psi(1+s)+\psi(1-s)\Big)$$

where η_s is the arithmetic function defined by

$$\eta_s(m)=\sum_{ab=m}\Big(\frac{a}{b}\Big)^s.$$

Thus, the integrand is seen to be an *odd* function of s, which is moreover holomorphic in the strip $|\mathrm{Re}(s)| < 1$, except for a triple pole at $s = 0$, and decreases exponentially in vertical strips. Shifting the contour to $\mathrm{Re}(s) = -1/2$ and changing then s into $-s$ allows us to conclude that

$$X''(m) = \frac{1}{2}\mathrm{Res}_{s=0}F(s).$$

Around $s = 0$, the following expansions hold:

$$\eta_s(m) = \tau(m) + \frac{1}{2}T(m)s^2 + O(s^3)$$

$$G(s) = 1 + \frac{1}{2}G''(0)s^2 + O(s^3)$$

$$s^{-1}\Gamma(s)\Gamma(-s) = -\frac{1}{s^3} + \frac{\gamma^2 - \Gamma''(1)}{s} + O(s)$$

$$\log\frac{\hat{Q}}{m} + 2\gamma + \psi(1+s) + \psi(1-s) = \log\frac{\hat{Q}}{m} + \psi''(0)s^2 + O(s^4)$$

where T is the arithmetic function defined by

$$T(m) = \sum_{ab=m}\Big(\log\frac{a}{b}\Big)^2.$$

Combining those, we obtain

$$\frac{1}{2}\mathrm{Res}_{s=0}F(s) = -\frac{1}{4}\frac{T(m)}{\sqrt{m}}\Big(\log\frac{\hat{Q}}{m}\Big) + \alpha\frac{\tau(m)}{\sqrt{m}}\Big(\log\frac{\hat{Q}}{m}\Big),$$

where we have set

$$\alpha = \frac{1}{2}\Big(\gamma^2 - \Gamma''(1) - \frac{G''(0)}{2} - \psi''(0)\Big).$$

If we now take $R = q^2$, we infer from (2.23), (2.24), and lemmas 6 and 8 of section 2.4.6 an approximate formula for $X^-(m)$.

Proposition 4. *Let $0 \le \Delta < 1$ and $1 \le m \le q^\Delta$. For any $\epsilon > 0$*

$$X^-(m) = -\frac{1}{4}\frac{T(m)}{\sqrt{m}}\Big(\log\frac{\hat{Q}}{m}\Big) + \alpha\frac{\tau(m)}{\sqrt{m}}\Big(\log\frac{\hat{Q}}{m}\Big) + O_{\Delta,\epsilon}\Big(q^\epsilon\Big(\frac{m^{1/2}}{q} + \frac{1}{\sqrt{q}}\Big)\Big).$$

This together with (2.22) yields an approximate formula for $X(m)$.

Proposition 5. *Set $P_1 = P + \alpha X$. Then for $0 \le \Delta < 1/2$, and $1 \le m \le q^\Delta$, we have for any $\epsilon > 0$,*

$$\begin{aligned} X(m) &= \frac{1}{12}\frac{\tau(m)}{\sqrt{m}}\Big(\log\frac{\hat{Q}}{m}\Big)^3 - \frac{1}{4}\frac{T(m)}{\sqrt{m}}\Big(\log\frac{\hat{Q}}{m}\Big) + \frac{\tau(m)}{\sqrt{m}}P_1\Big(\log\frac{\hat{Q}}{m}\Big) \\ &\quad + O_{\Delta,\epsilon}\Big(\frac{m^{1/2}q^\epsilon}{\sqrt{q}}\Big). \end{aligned}$$

For later use, we record a few properties of the function T.

Lemma 5. *Let $\tau^{(i)}$ be defined for $i \ge 0$ by*

$$\tau^{(i)}(m) = \sum_{d|m}(\log d)^i.$$

Then we have

$$T(m) = 4\tau^{(2)}(m) - 2(\log m)\tau^{(1)}(m). \tag{2.30}$$

Moreover, T satisfies

$$T(m_1m_2)=\tau(m_1)T(m_2)+\tau(m_2)T(m_1) \tag{2.31}$$

for $(m_1,m_2)=1$.

Proof. The first formula is immediate, and the second follows from

$$\sum_{ab=m}\Bigl(\log\frac{a}{b}\Bigr)=0.$$

□

2.4.6. *Estimation of the integrals.* We still have to vindicate our contention that the two expressions involving the Bessel functions Y_0 and K_0 in (2.26) and (2.27) are of smaller order of magnitude (in our situation) than the main term isolated in the previous section. We will denote by $Y(m)$ and $K(m)$ their respective contributions to $X(m)$.

Lemma 6. *For all $\epsilon>0$, we have*

$$K(m)\ll_\epsilon\frac{q^\epsilon m^{1/2}}{q}.$$

Proof. Because K_0 has exponential decay at infinity and ξ cuts off the small values of x, this is easy. We have

$$K(m)=-\frac{8\pi}{\sqrt{q}}\sideset{}{^*}\sum_{r\leq R}\frac{1}{r^2}\sum_{h\geq 1}\tau(h)S(hq+m,0;r)k(h)$$

and $k(h)$ is the integral involving the K_0 function, for which we have, employing the bound

$$K_0(y)\ll y^{-1/2}e^{-y}$$

$$\begin{aligned}
k(h) &= \int_0^{+\infty}K_0\Bigl(\frac{4\pi\sqrt{hx}}{r}\Bigr)J_1\Bigl(\frac{4\pi}{r}\sqrt{\frac{mx}{q}}\Bigr)W\Bigl(\frac{4\pi^2x}{q}\Bigr)\xi(x)\frac{dx}{\sqrt{x}}\\
&= \frac{r}{2\pi\sqrt{h}}\int_0^{+\infty}K_0(y)J_1\Bigl(\sqrt{\frac{m}{hq}}y\Bigr)W\Bigl(\frac{r^2y^2}{4qh}\Bigr)\xi\Bigl(\frac{r^2y^2}{16\pi^2h}\Bigr)dy\\
&\ll \frac{r}{h}\sqrt{\frac{m}{q}}(\log q)^3\int_{\sqrt{h}r^{-1}}^{+\infty}y^{1/2}e^{-y}dy\\
&\ll \frac{r}{h}\sqrt{\frac{m}{q}}(\log q)^3e^{-\frac{\sqrt{h}}{2r}}
\end{aligned}$$

so that

$$\begin{aligned}
K(m) &\ll \frac{\sqrt{m}}{q}(\log q)^3\sum_{h\geq 1}\frac{\tau(h)}{h}e^{-\frac{\sqrt{h}}{2R}}\sideset{}{^*}\sum_{r\leq R}\frac{(r,hq+m)}{r}\\
&\ll \frac{\sqrt{m}}{q}(\log q)^4\sum_{h\geq 1}\frac{\tau(h)\tau(hq+m)}{h}e^{-\frac{\sqrt{h}}{2R}}\ll_\epsilon\frac{q^\epsilon\sqrt{m}}{q}.
\end{aligned}$$

□

The case of $Y(m)$ is slightly more complicated because Y_0 is an oscillating function. We will use the following lemma which is quite standard.

Lemma 7. *Let $\nu \geq 0$ be a real number, $J \geq 0$ an integer. If f is a compactly supported C^∞ function, and $\beta > 0$ is a real number such that f is supported on $[Y, 2Y]$ and satisfies*

$$y^j f^{(j)}(y) \ll_j (1+\beta Y)^j$$

for $0 \leq j \leq J$, then for any $\alpha > 1$

$$\int_0^{+\infty} Y_\nu(\alpha y) f(y) dy \ll \Big(\frac{1+\beta Y}{1+\alpha Y}\Big)^J Y.$$

Proof. One could write the asymptotic development of Y_0 to show the oscillating behavior and integrate by parts, but it is cleaner (and amounts to the same thing) to make use of the recurrence formula

$$(y^\nu Y_\nu(y))' = y^\nu Y_{\nu-1}(y)$$

to get, integrating by part also

$$\int_0^{+\infty} Y_\nu(\alpha y) f(y) dy = \frac{1}{\alpha}\int_0^\infty Y_{\nu+1}(\alpha y)\Big(-f'(y) + (\nu+1)\frac{f(y)}{y}\Big)dy.$$

Let $g(y) = -f'(y) + (\nu+1)f(y)/y$; it is immediate that g satisfies

$$y^{j+1} g^{(j)}(y) \ll (1+\beta Y)^{j+1}, \text{ for } 0 \leq j \leq J-1$$

so that by iterating this procedure we obtain

$$\int_0^{+\infty} Y_\nu(\alpha y) f(y) dy = \frac{1}{\alpha^J}\int_0^{+\infty} Y_{\nu+J}(\alpha y) h(y) dy,$$

where the function h is such that

$$y^J h(y) \ll_J (1+\beta Y)^J$$

and therefore the result follows by using $Y_{\nu+J}(y) \ll_{J+\nu} 1$. □

Lemma 8. *For $\Delta < 1$, $m \leq q^\Delta$, and any $\epsilon > 0$, we have*

$$Y(m) \ll_{\Delta,\epsilon} q^\epsilon\Big(\frac{m^{1/2}}{q} + \frac{1}{\sqrt{q}}\Big)$$

Proof. We write

$$Y(m) = \frac{4\pi^2}{\sqrt{q}} \sideset{}{^*}\sum_{r \leq R} \frac{1}{r^2} \sum_h \tau(h) S(hq-m, 0; r) y(h) \tag{2.32}$$

with

$$y(h) = \int_0^{+\infty} Y_0\Big(\frac{4\pi\sqrt{hx}}{r}\Big) J_1\Big(\frac{4\pi}{r}\sqrt{\frac{mx}{q}}\Big) W\Big(\frac{4\pi^2 x}{q}\Big)\xi(x)\frac{dx}{\sqrt{x}}. \tag{2.33}$$

Note that $hq \neq m$ since $m < q$, so the Ramanujan sum never degenerates to the trivial sum $S(0,0;r) = r-1$ but is always much smaller.

We make a smooth dyadic partition of unity, so

$$\xi = \sum_{k \geq 1} \xi_k$$

where each ξ_k is a C^∞ function with compact support in a dyadic interval $[X_k, 2X_k]$ that satisfies

$$x^j \xi_k^{(j)}(x) \ll 1, \text{ for all } j \geq 0, \tag{2.34}$$

the implied constants depending on j alone (in particular, they are uniform in k).

We study each ξ_k individually, but we keep writing ξ instead of ξ_k, and accordingly we use X rather than X_k.

By the change of variable $2r^{-1}\sqrt{x} = y$, the integral is

$$y(h) = r\int_0^{+\infty} Y_0(2\pi\sqrt{h}x)J_1\Bigl(2\pi\sqrt{\frac{m}{q}}x\Bigr)W\Bigl(\frac{\pi^2r^2x^2}{q}\Bigr)\xi\Bigl(\frac{r^2x^2}{4}\Bigr)dx, \tag{2.35}$$

so we define the function f by

$$f(x) = J_1\Bigl(2\pi\sqrt{\frac{m}{q}}x\Bigr)W\Bigl(\frac{\pi^2r^2x^2}{q}\Bigr)\xi\Bigl(\frac{r^2x^2}{4}\Bigr).$$

This is a C^∞ function compactly supported in the dyadic interval $[\rho, 2\rho]$, with

$$\rho = 2\frac{\sqrt{X}}{r}. \tag{2.36}$$

We first treat the case

$$1/2 \leq X \leq q^2,$$

(which involves $\ll \log q$ terms) and for this quote from (2.16) the bound

$$x^jW^{(j)}(x) \ll_j (\log q)^3, \text{ for all } j \geq 0,$$

valid for $1/q \ll x \ll 2q^2$. This, together with (2.33), the recurrence relation

$$(x^\nu J_\nu(x))' = x^\nu J_{\nu-1}(x)$$

and some elementary manipulations with inequalities, yields

$$x^jf^{(j)}(x) \ll_j \Bigl(1+\sqrt{\frac{m}{q}}x\Bigr)^j(\log q)^3, \text{ for all } j \geq 0.$$

Thus, we are in a position to apply the preceding lemma to f with $\alpha = 2\pi\sqrt{h}$, $\beta = 2\pi\sqrt{\frac{m}{q}}$ and $Y = \rho$. Unfortunately, this is inefficient for certain ranges of X, r and/or h, and it will be necessary to split into other cases.

What the lemma implies is, for any integer $J \geq 0$

$$y(h) \ll_J r\rho\frac{\Bigl(1+\sqrt{\frac{m}{q}}\rho\Bigr)^J}{(1+\sqrt{h}\rho)^J}(\log q)^3. \tag{2.37}$$

Consider first the case $\rho > 2$, or $r < \sqrt{X}$: applying (2.37) with $J \geq 3$ (to win convergence in h) yields a contribution in (2.32) which is therefore

$$\begin{aligned}
&\ll_J \frac{(\log q)^3}{\sqrt{q}}\sideset{}{^*}\sum_{r<\sqrt{X}}\frac{1}{r^2}r\rho^{-(J-1)}\Bigl(1+\sqrt{\frac{m}{q}}\rho\Bigr)^J\tau(r)\\
&\ll_J \frac{(\log q)^3}{\sqrt{q}}\Bigl(\sideset{}{^*}\sum_{r<\sqrt{\frac{mX}{q}}}\rho\frac{\tau(r)}{r}\Bigl(\sqrt{\frac{m}{q}}\Bigr)^J + \sideset{}{^*}\sum_{\sqrt{\frac{mX}{q}}\leq r<\sqrt{X}}\frac{\tau(r)}{r}\Bigr)\\
&\ll_J \frac{(\log q)^{5+J}}{\sqrt{q}}(1+q^{1+J(\Delta-1)/2}), \text{ since } m/q \leq q^{\Delta-1}
\end{aligned}$$

at which point, since $\Delta < 1$, we can choose J large enough so that $1 + J(\Delta - 1)/2 \le 0$ to conclude that this part is

$$\ll_{\Delta,\epsilon} \frac{q^\epsilon}{\sqrt{q}}. \tag{2.38}$$

On the other hand, for $\rho \le 1$, we split the summation in h in the following way

$$\sum_{h\ge 1} = \sum_{h\le \rho^{-2(1+\kappa)}} + \sum_{h>\rho^{-2(1+\kappa)}}$$

where $\kappa > 0$ will be chosen (sufficiently small) a little later.

For the first sum, we come back to (2.33), using again $J_1(x) \ll x$, $Y_0(x) \ll 1 + |\log x|$ to derive first the bound

$$y(h) \ll \sqrt{\frac{m}{q}}\frac{X}{r}(\log q)^3.$$

Then, since $|S(hq-m,0;r)| \le \sum_{d|(hq-m,r)} d$

$$\begin{aligned}
&\frac{4\pi^2}{\sqrt{q}} \sideset{}{^*}\sum_{\sqrt{X}\le r\le R} \frac{1}{r^2} \sum_{h\le \rho^{-2(1+\kappa)}} \tau(h)S(hq-m,0;r)y(h) \\
&\ll \frac{(\log q)^3}{\sqrt{q}} \sideset{}{^*}\sum_{\sqrt{X}\le r\le R} \frac{1}{r^2} \sum_{h\le \rho^{-2(1+\kappa)}} \tau(h) \sum_{d|(hq-m,r)} d\frac{X}{r}\sqrt{\frac{m}{q}} \\
&\ll \frac{\sqrt{m}}{q}(\log q)^3 X \sum_{h\le (R^2/X)^{1+\kappa}} \tau(h) \sideset{}{^*}\sum_{\sqrt{X}h^\theta\le r\le R} \sum_{d|(hq-m,r)} \frac{d}{r^3} \qquad (2.39)
\end{aligned}$$

(exchanging the order of summation), where $\theta = (2+2\kappa)^{-1}$. We transform the inner sum over d and r and estimate

$$\begin{aligned}
\sum_{d|hq-m} \frac{1}{d^2} \sideset{}{^*}\sum_{\sqrt{X}h^\theta\le dr\le R} \frac{1}{r^3} &\ll X^{-1}h^{-2\theta} \sum_{d|hq-m} 1 \\
&\ll \tau(hq-m)X^{-1}h^{-1+\kappa/(1+\kappa)}.
\end{aligned}$$

Then (2.39) is estimated to be

$$\begin{aligned}
&\ll \frac{\sqrt{m}}{q}(\log q)^3 X \sum_{h\le (R^2/X)^{1+\kappa}} \tau(h) \sum_{d|hq-m} \frac{1}{d^2} \sideset{}{^*}\sum_{\sqrt{X}h^\theta\le dr\le R} \frac{1}{r^3} \\
&\ll \frac{\sqrt{m}}{q}(\log q)^3 X \sum_{h\le (R^2/X)^{1+\kappa}} \tau(h)\tau(hq-m)X^{-1}h^{-1+\kappa/(1+\kappa)} \\
&\ll_\epsilon \frac{\sqrt{m}}{q} q^\epsilon R^{2\kappa}, \text{ for all } \epsilon > 0. \qquad (2.40)
\end{aligned}$$

For the second sum, applying (2.37) for $J \ge 3$ entails

$$y(h) \ll_J \sqrt{X}\rho^{-J}h^{-J/2}(\log q)^3$$

and so as above

$$\frac{4\pi^2}{\sqrt{q}} \sideset{}{^*}\sum_{\sqrt{X}\le r\le R} \frac{1}{r^2} \sum_{h>\rho^{-2(1+\kappa)}} \tau(h)S(hq-m,0;r)y(h)$$

$$\ll_J \frac{\sqrt{X}}{\sqrt{q}}(\log q)^3 \sideset{}{^*}\sum_{\sqrt{X}\leq r\leq R} \frac{1}{r^2} \sum_{h>\rho^{-2(1+\kappa)}} \tau(h) \sum_{d|(hq-m,r)} d\rho^{-J}h^{-J/2}$$
$$\ll_J \frac{X^{(1-J)/2}}{\sqrt{q}}(\log q)^3 \sum_{h\geq 1}\tau(h)h^{-J/2} \sum_{d|hq-m} d^{J-1} \sum_{rd\leq\sqrt{X}h^{\theta}} r^{J-2}$$

(where $\theta = (2+2\kappa)^{-1}$ as before)

$$\ll_J \frac{(\log q)^3}{\sqrt{q}} \sum_{h\geq 1}\tau(h)\tau(hq-m)h^{-J/2+\theta(J-1)} \tag{2.41}$$

We choose $\kappa = \epsilon/4$, then J large enough so that $J(\theta - 1/2) - \theta > 1$ (in addition to the previous condition that $1 + J(\Delta - 1)/2 \leq 0$), so that the series over h in (2.41) converges absolutely. Then (2.40) and (2.41) together are

$$\ll_{\Delta,\epsilon} q^{\epsilon}\Bigl(\frac{m^{1/2}}{q} + \frac{1}{\sqrt{q}}\Bigr).$$

Finally, we return to the case $X > q^2$ which remains. We appeal to (2.17) (for $j = 2$), and again use elementary estimations to prove that for $X > q$ the function f satisfies the better bound

$$x^j f^{(j)}(x) \ll \Bigl(1 + x\sqrt{\frac{m}{q}}\Bigr)^j q^2(rx)^{-4}.$$

The lemma admits then an immediate generalization to the effect that

$$y(h) \ll r\rho\frac{\Bigl(1+\sqrt{\frac{m}{q}}\rho\Bigr)^J}{(1+\sqrt{h}\rho)^J}\frac{q^2}{X^2}(\log q)^3$$

in addition to the bound in (2.37).

Since $X > q^2$, the quantity saved is

$$\frac{q^2}{X^2} \ll X^{-1}$$

which is more than sufficient to allow for the sum over the dyadic values of X involved to converge, and proves that all the previous bounds where (2.37) was used remain valid. The only place where this is not the case is the inequality (2.40), but this part of the sum is void for $\sqrt{X} > R$ and the former estimate works in the larger interval $X \leq R^2$. □

2.4.7. *A formula for the second moment.* The definition of M_2 yields

$$M_2 = \sum_b \frac{1}{b} \sum_{m_1,m_2\leq M} \frac{X(m_1m_2)}{\sqrt{m_1m_2}} x_{bm_1}x_{bm_2}.$$

Proposition 6. *Assume $M = q^{\Delta}$ with $\Delta < 1/4$. Then*

$$M_2 = \frac{1}{12}M_{21} - \frac{1}{4}M_{22} + M_3 + O(q^{-c}) \tag{2.42}$$

where

$$M_{21} = \sum_b \frac{1}{b} \sum_{m_1,m_2} \frac{\tau(m_1m_2)}{m_1m_2} x_{bm_1}x_{bm_2}\Bigl(\log\frac{\hat{Q}}{m_1m_2}\Bigr)^3 \tag{2.43}$$

(2.44) $$M_{22} = \sum_b \frac{1}{b} \sum_{m_1,m_2} \frac{T(m_1m_2)}{m_1m_2} x_{bm_1} x_{bm_2} \Bigl(\log \frac{\hat{Q}}{m_1m_2}\Bigr)$$

(2.45) $$M_3 = \sum_b \frac{1}{b} \sum_{m_1,m_2} \frac{\tau(m_1m_2)}{m_1m_2} P_1\Bigl(\log \frac{\hat{Q}}{m_1m_2}\Bigr).$$

Proof. We apply proposition 5 with $R = q$. The first three terms give exactly the three quadratic forms M_{21}, M_{22} and M_3. Moreover, using (2.6), the error term is dominated, for any $\epsilon > 0$, by

$$q^{-1/2+\epsilon}\Bigl|\sum_{m\leq M} x_m\Bigr|^2 \ll M^2 q^{-1/2+2\epsilon}.$$

If $\Delta < 1/4$, we can take ϵ small enough so that this is $O(q^{-c})$ for some $c > 0$. □

Our strategy is now to write M_{21} as a linear combination of easily diagonalized quadratic forms; the simplest in shape, say Π, is chosen and we are able to select x_m to optimize the value of Π with respect to M_1. Then the remaining terms in M_{21} are evaluated, and so is M_{22}. Both are of the same order of magnitude, so our choice may not be perfectly optimal. On the other hand, with our specific choice of x_m, we finally prove that M_3 gives a smaller contribution, namely that

(2.46) $$M_3 = O\Bigl(\frac{M_{21}}{\log q}\Bigr).$$

The second moment N_2 of central values $L(f, 1/2)$ is much simpler to handle: no detailed analysis of the remainder term in the Petersson formula is needed, (2.21) being sufficient to evaluate N_2 asymptotically for $M = q^\Delta$, $\Delta < 1/4$. This is because the sign of the functional equation is always $+1$ for $L(f, s)^2$ and no contamination by ε_f^- occurs. Considering the integral

$$\frac{1}{2i\pi}\int_{(2)} L(f, s + 1/2)^2 G(s)^2 \frac{ds}{s}$$

one finds that

$$L(f, 1/2)^2 = 2\sum_{n\geq 1} \frac{\lambda_f(n)}{\sqrt{n}} \tau(n) U\Bigl(\frac{4\pi^2 n}{q}\Bigr)$$

with

$$U(y) = 2\frac{1}{2i\pi}\int_{(2)} \zeta_q(1 + 2s)G(s)^2\Gamma(s + 1)^2 y^{-s}\frac{ds}{s}.$$

This test function decays faster than any polynomial as $y \to +\infty$ and satisfies

$$U(y) = -\frac{\varphi(q)}{q}\log y + C_q + O(y^{1/2})$$

as $y \to 0$. Here $C_q = c_0 + O(q^{-1}\log q)$ for some explicitly computable, but unimportant, absolute constant c_0. Then computations similar to that leading to the main term in M_2

(but simpler) yield the expression

$$N_2 = \sum_b \frac{1}{b} \sum_{m_1,m_2} \frac{\tau(m_1m_2)}{m_1m_2} x_{bm_1}x_{bm_2}\Bigl(\log \frac{\hat{Q}}{m_1m_2}\Bigr) + O(q^{-c}) \tag{2.47}$$

for some $c = c(\Delta) > 0$, for $\Delta < 1/4$.

The optimization of N_2 proceeds in a way similar as that of M_2. We let N_{21} denote the quadratic form which is the main term of N_2.

2.5. **The preferred quadratic form I.** Separating m_1 and m_2 in (2.48) by means of the formula

$$\tau(m_1m_2) = \sum_{a|(m_1,m_2)} \mu(a)\tau\Bigl(\frac{m_1}{a}\Bigr)\tau\Bigl(\frac{m_2}{a}\Bigr)$$

we get

$$M_{21} = \sum_b \frac{1}{b} \sum_a \frac{\mu(a)}{a^2} \sum_{m_1,m_2} \frac{\tau(m_1)\tau(m_2)}{m_1m_2} x_{abm_1}x_{abm_2}\Bigl(\log \frac{\hat{Q}}{a^2m_1m_2}\Bigr)^3 \tag{2.48}$$

We define the following arithmetic functions

$$\nu_t(k) = \frac{1}{k}\sum_{ab=k} \frac{\mu(a)(\log a)^t}{a}, \text{ for } t = 1,2,3; \tag{2.49}$$

$$h(m) = \frac{\tau(m)}{m}. \tag{2.50}$$

Then expanding the logarithm in (2.48) and rearranging, we see that M_{21} is a linear combination of the quadratic forms $\Pi(t,u,v,w)$ in the x_m's defined by

$$\Pi(t,u,v,w) = (\log \hat{Q})^u \sum_k \nu_t(k) \sum_{m_1,m_2} h(m_1)h(m_2)(\log m_1)^v(\log m_2)^w x_{km_1}x_{km_2} \tag{2.51}$$

where t, u, v and w are non-negative integers such that $t+u+v+w = 3$.[1]

We further restrict our attention to $\Pi(u,v,w) := \Pi(0,u,v,w)$; again it will be seen that for the chosen (x_m)

$$\Pi(t,u,v,w) = O\Bigl(\Pi(0,u,v,w)\frac{(\log_2 q)^t}{\log q}\Bigr) \tag{2.52}$$

which justifies this restriction. Accordingly we write ν for ν_0, for which we have the formula

$$\nu(k) = \frac{\varphi(k)}{k^2}, \text{ for } k \leq M. \tag{2.53}$$

The part of the expansion of M_{21} involving those $\Pi(u,v,w)$ is then (using the obvious symmetry $\Pi(u,v,w) = \Pi(u,w,v)$) denoted by m_{21}:

$$m_{21} = \Pi(3,0,0) - 6\Pi(2,1,0) + 6\Pi(1,1,1) + 6\Pi(1,2,0) - 6\Pi(0,1,2) - 2\Pi(0,0,3). \tag{2.54}$$

Finally, we choose the one quadratic form $\Pi := \Pi(3,0,0)$ as reference: we will choose (x_m) to optimize Π and evaluate afterwards the other $\Pi(u,v,w)$, for this choice, before doing the same with M_{22}.

[1] Actually, M_3 is also such a linear combination with the difference that either $t+u+v+w \leq 2$. This will explain (2.46).

Similarly, for N_2, we have by (2.47)

$$N_{21} = \Pi(1,0,0) - 2\Pi(0,1,0) + 2\Pi(1,0,0,0) \tag{2.55}$$

(and the last term will be of smaller order of magnitude).

2.5.1. *Optimizing* Π. Making the linear change of variable

$$y_k = \sum_m h(m) x_{km}$$

we have the immediate diagonalization

$$\Pi = (\log \hat{Q})^3 \sum_k \nu(k) y_k^2. \tag{2.56}$$

Conversely, let g be the Dirichlet convolution inverse of h, then

$$x_m = \sum_k g(k) y_{km}. \tag{2.57}$$

¿From this we express the linear form[2] in (2.13) in terms of y_k

$$M_1 = \sum_m \frac{x_m}{m} \log \frac{\hat{q}}{m} = \sum_k j(k) y_k \tag{2.58}$$

where

$$j(k) = \sum_{ab=k} g(a) \frac{\log \hat{q}/b}{b}.$$

Lemma 9. *For any integer $k \geq 1$ we have*

$$j(k) = \frac{\mu(k)}{k} (\log \hat{q} k).$$

Proof. We have

$$\sum_{k \geq 1} g(k) k^{-s} = \zeta(s+1)^{-2}$$

and therefore

$$\begin{aligned} \sum_{k \geq 1} j(k) k^{-s} &= \zeta(s+1)^{-2} \times \Big((\log \hat{q}) \zeta(s+1) + \zeta'(s+1) \Big) \\ &= (\log \hat{q}) \zeta(s+1)^{-1} - (\zeta^{-1})'(s+1) \end{aligned}$$

whence the result. □

By Cauchy's inequality, the best choice to optimize Π with respect to M_1 is

$$y_k = \begin{cases} \dfrac{j(k)}{\nu(k)} = \dfrac{k\mu(k)}{\varphi(k)} (\log \hat{q} k), & \text{if } k \leq M \\ 0, & \text{if } k > M \end{cases} \tag{2.59}$$

and x_m is given by (2.57), from which (and the lemma) the conditions required in section 2.3 are immediately verified.

We now compute the various terms in (2.54) to apply the estimate (2.1). [3]

[2] Strictly speaking, the main term of the linear form, but we will keep the same notation.

[3] Since $j(k)$ is about $(\log k)/k$ and ν is about k^{-1}, it is already quite clear that we will get a positive (harmonic) proportion if $M = q^{\Delta}$ with $\Delta > 0$.

Lemma 10. *With the previous notations and hypothesis, with $M = q^{\Delta}$, we have*

$$M_1 = (\log q)^3 \times \Delta\Big(\frac{\Delta^2}{3} + \frac{\Delta}{2} + \frac{1}{4}\Big) + O\big((\log q)^2\big);$$

$$\Pi = (\log q)^6 \times \Delta\Big(\frac{\Delta^2}{3} + \frac{\Delta}{2} + \frac{1}{4}\Big) + O\big((\log q)^5\big).$$

Proof. By the choice of (y_k), we have

$$(\log \hat{Q})^{-3}\Pi = M_1 = \sum_k \frac{j(k)^2}{\nu(k)} = \sum_k \frac{\mu(k)^2}{\varphi(k)}(\log \hat{q}k)$$

whence the result follows, by partial summation, from

$$\sum_{k \leq K} \frac{\mu(k)^2}{\varphi(k)} = \log K + O(1).$$

□

2.5.2. *Estimation of* $\Pi(u,v,w)$. For the other quadratic forms, we write

$$\Pi(u,v,w) = (\log \hat{Q})^u \sum_k \nu(k) y_k^{(v)} y_k^{(w)}$$

where

$$y_k^{(i)} = \sum_m h(m)(\log m)^i x_{km}.$$

We can express $y_k^{(i)}$ in terms of (y_k) using the higher Von Mangoldt function Λ_i, which is defined by the Dirichlet convolution

$$\Lambda_i = \mu * (\log)^i,$$

so that $(\log m)^i = \sum_{ab=m} \Lambda_i(a)$. ¿From this, and the fact that the x_m's are supported on squarefree integers, we derive

$$y_k^{(i)} = \sum_{\ell \leq M/k} h(\ell)\Lambda_i(\ell) y_{k\ell}. \tag{2.60}$$

We state the properties of Λ_i which we will use.

- $\Lambda_1 = \Lambda$, the usual Van-Mangoldt function.
- Λ_i is supported on integers having at most i distinct prime factors.
- If $m = p_1 \ldots p_i$, for distinct primes $p_1, \ldots, p_i$, then

$$\Lambda_i(m) = i!(\log p_1)\ldots(\log p_i).$$

- If p_1 and p_2 are distinct primes, then

$$\Lambda_i(p_1) = (\log p_1)^i$$

$$\Lambda_3(p_1p_2) = 3(\log p_1)(\log p_2)(\log p_1p_2).$$

All of these are well known and (or) easy to prove from the recurrence relation

$$\Lambda_{i+1} = (\log)\Lambda_i + \Lambda * \Lambda_i.$$

In (2.60) we are thus actually dealing with a sum over squarefree ℓ having at most i prime factors, and $i \leq 3$. We separate the sum into the parts with a fixed number of prime factors, which produces multiple (at most triple) sums over primes (of Mertens type since $h(\ell) = 2^j \ell^{-1}$ for such ℓ with $\omega(\ell) = j$ prime factors).

The subsum with i distinct prime factors is, by the above

$$2^i i! \sum_{\substack{\ell \leq M/k \\ \omega(\ell)=i}} \frac{\Lambda_i(\ell)}{\ell} \mu(k\ell)(\log \hat{q}k\ell) \frac{k\ell}{\varphi(k\ell)}$$

$$= (-2)^i i! \frac{k\mu(k)}{\varphi(k)} \sum_{\substack{p_1 < \ldots < p_i \\ p_1 \ldots p_i \leq M/k \\ (p_1 \ldots p_i, k)=1}} \frac{(\log p_1)\ldots(\log p_i)}{p_1 \ldots p_i} (\log \hat{q}kp_1 \ldots p_i) + O\big((\log q)^i \frac{k}{\phi(k)}\big)$$

$$= (-2)^i i! \frac{k\mu(k)}{\varphi(k)} \sum_{\substack{p_1 < \ldots < p_i \\ p_1 \ldots p_i \leq M/k}} \frac{(\log p_1)\ldots(\log p_i)}{p_1 \ldots p_i} (\log \hat{q}kp_1 \ldots p_i) + O\big((\log q)^i (\log_2 q) \frac{k}{\phi(k)}\big)$$

$$= (-2)^i \frac{k\mu(k)}{\varphi(k)} \sum_{\substack{p_1, \ldots, p_i \\ p_1 \ldots p_i \leq M/k}} \frac{(\log p_1)\ldots(\log p_i)}{p_1 \ldots p_i} (\log \hat{q}kp_1 \ldots p_i) + O\big((\log q)^i (\log_2 q) \frac{k}{\phi(k)}\big)$$

the error term arising from neglecting the smaller contribution from the primes dividing k and replacing $\varphi(p)^{-1}$ by p^{-1} using the fact that

$$\sum_p \frac{(\log p)^A}{p(p-1)} < +\infty.$$

¿From Mertens's formula, the last sum is equal, up to $O\big((\log q)^i\big)$, to the integral

$$\int_{\substack{y_1 \geq 0, \ldots, y_i \geq 0 \\ y_1 + \ldots + y_i \leq (\log M/k)}} (\log \hat{q}k + y_1 + \ldots + y_i) dy = (\log \hat{q}k)\Big(\log \frac{M}{k}\Big)^i \int_{S_i} dx + i\Big(\log \frac{M}{k}\Big)^{i+1} \int_{S_i} x_1 dx$$

Here $S_i = \{(x_1, \ldots, x_i) \mid x_j \geq 0, x_1 + \ldots + x_i \leq 1\}$ is the standard i-simplex. By induction, one gets immediately

$$\int_{S_i} dx = \frac{1}{i!}, \quad \int_{S_i} x_1 dx = \frac{1}{(i+1)!}$$

so this contribution to the sum (2.60) can be written as

$$\frac{(-2)^i \mu(k)}{(i+1)!} \Big(\log \frac{M}{k}\Big)^i (\log \hat{q}^{i+1} M^i k) + O\big((\log q)^i (\log_2 q) \frac{k}{\phi(k)}\big). \tag{2.61}$$

This is enough to give $y_k^{(1)}$; for $y_k^{(2)}$ there is an additional sum over primes which, by similar computations, is

$$-2 \frac{k\mu(k)}{\varphi(k)} \sum_{p \leq M/K} \frac{(\log p)^2}{p} (\log \hat{q}kp) + O\big((\log q)(\log_2 q)^2 \frac{k}{\phi(k)}\big)$$

$$= -\frac{1}{3}\frac{k\mu(k)}{\varphi(k)}\Big(\log\frac{M}{k}\Big)^2(\log\hat{q}^3M^2k) + O\big((\log q)^2\frac{k}{\phi(k)}\big);$$

and for $y_k^{(3)}$ there are two other sums, first

$$-2\frac{k\mu(k)}{\varphi(k)}\sum_{p\leq M/K}\frac{(\log p)^3}{p}(\log\hat{q}kp) + O\big((\log q)(\log_2 q)^3\frac{k}{\phi(k)}\big)$$

$$= -\frac{1}{6}\frac{k\mu(k)}{\varphi(k)}\Big(\log\frac{M}{k}\Big)^3(\log\hat{q}^4M^3k) + O\big((\log q)^3\frac{k}{\phi(k)}\big);$$

and finally

$$12\frac{k\mu(k)}{\varphi(k)}\sum_{\substack{p_1<p_2\\ p_1p_2\leq M/k}}\frac{(\log p_1p_2)(\log p_1)(\log p_2)}{p_1p_2}(\log\hat{q}kp_1p_2) + O\big((\log q)^2(\log_2 q)^2\frac{k}{\phi(k)}\big)$$

$$= 12\frac{k\mu(k)}{\varphi(k)}\sum_{p_1p_2\leq M/k}\frac{(\log p_1)^2(\log p_2)}{p_1p_2}(\log\hat{q}kp_1p_2) + O\big((\log q)^2(\log_2 q)^2\frac{k}{\phi(k)}\big)$$

$$= \frac{1}{2}\frac{k\mu(k)}{\varphi(k)}\Big(\log\frac{M}{k}\Big)^3(\log\hat{q}^4M^3k) + O\big((\log q)^3\frac{k}{\phi(k)}\big).$$

¿From all this we conclude:

Lemma 11. *For $i = 1, 2, 3$, we have*

$$y_k^{(i)} = c_i\frac{k\mu(k)}{\varphi(k)}\Big(\log\frac{M}{k}\Big)^i(\log\hat{q}^{i+1}M^ik) + O\big((\log q)^i(\log_2 q)\frac{k}{\phi(k)}\big) \tag{2.62}$$

with

$$c_1 = -1,\ c_2 = \frac{1}{3},\ c_3 = 0. \tag{2.63}$$

It is now easy to finish the computation of the quadratic form m_{21} for our choice of y_k.

Lemma 12. *With notations as in lemma 10*

$$\Pi(2,1,0) = -(\log q)^6\times\Delta^2\Big(\big(\frac{1}{2}+\Delta\big)^2 - \Delta\big(\frac{1}{2}+\Delta\big) + \frac{\Delta^2}{4}\Big) + O\big((\log q)^5\log_2 q\big)$$

$$\Pi(1,1,1) = (\log q)^6\times\Delta^3\Big(\frac{4}{3}\big(\frac{1}{2}+\Delta\big)^2 - \Delta\big(\frac{1}{2}+\Delta\big) + \frac{\Delta^2}{5}\Big) + O\big((\log q)^5\log_2 q\big)$$

$$\Pi(1,2,0) = \frac{1}{3}(\log q)^6\times\Delta^3\Big(\big(\frac{1}{2}+\Delta\big)^2 - \Delta\big(\frac{1}{2}+\Delta\big) + \frac{\Delta^2}{5}\Big) + O\big((\log q)^5\log_2 q\big)$$

$$\Pi(0,1,2) = -\frac{1}{3}(\log q)^6\times\Delta^4\Big(\frac{3}{2}\big(\frac{1}{2}+\Delta\big)^2 - \Delta\big(\frac{1}{2}+\Delta\big) + \frac{\Delta^2}{6}\Big) + O\big((\log q)^5\log_2^3 q\big)$$

$$\Pi(0,0,3) = O\big((\log q)^5\log_2^3 q\big)$$

Proof. All are similar, so take for instance $\Pi(0,1,2)$; from the previous lemma

$$\Pi(0,1,2) = -\frac{1}{3}\sum_{k\le M}\frac{\mu(k)^2}{\varphi(k)}\Big(\log\frac{M}{k}\Big)^3(\log \hat{q}^3M^2k)(\log\hat{q}^2Mk) + O\big((\log q)^5(\log_2 q)^3\big)$$

and the sum, by summation by parts again, is – up to $O\big((\log q)^5\big)$ – the same as the integral

$$\int_1^M \Big(\log\frac{M}{x}\Big)^3(\log\hat{q}^3M^2x)(\log\hat{q}^2Mx)\frac{dx}{x} = \int_0^{\log M} y^3(3\log\hat{q}M - y)(2\log\hat{q}M-y)dy$$

from which the result follows, since moreover $\log\hat{q} = \log\sqrt{q} + O(1)$. □

2.5.3. *Diagonalization of M_{22}.* Recall that

$$M_{22} = \sum_b \frac{1}{b}\sum_{m_1,m_2}\frac{T(m_1m_2)}{m_1m_2}x_{bm_1}x_{bm_2}\Big(\log\frac{\hat{Q}}{m_1m_2}\Big).$$

Using the multiplicative property of T (see lemma 5), and the fact that (x_m) is supported on squarefree integers, we compute

$$\begin{aligned}
M_{22} &= \sum_b\frac{1}{b}\sum_a\frac{\tau(a^2)}{a^2}\sum_{(m_1,m_2)=1}\frac{T(m_1m_2)}{m_1m_2}x_{abm_1}x_{abm_2}\Big(\log\frac{\hat{Q}}{a^2m_1m_2}\Big)\\
&\quad+\sum_b\frac{1}{b}\sum_a\frac{T(a^2)}{a^2}\sum_{(m_1,m_2)=1}\frac{\tau(m_1)\tau(m_2)}{m_1m_2}x_{abm_1}x_{abm_2}\Big(\log\frac{\hat{Q}}{a^2m_1m_2}\Big)\\
&= 2\sum_b\frac{1}{b}\sum_a\frac{\tau(a^2)}{a^2}\sum_{(m_1,m_2)=1}\frac{\tau(m_1)T(m_2)}{m_1m_2}x_{abm_1}x_{abm_2}\Big(\log\frac{\hat{Q}}{a^2m_1m_2}\Big)\\
&\quad+\sum_b\frac{1}{b}\sum_a\frac{T(a^2)}{a^2}\sum_{(m_1,m_2)=1}\frac{\tau(m_1)\tau(m_2)}{m_1m_2}x_{abm_1}x_{abm_2}\Big(\log\frac{\hat{Q}}{a^2m_1m_2}\Big)\\
&= 2\sum_b\frac{1}{b}\sum_a\frac{\tau(a^2)}{a^2}\sum_\delta\frac{\mu(\delta)\tau(\delta)^2}{\delta^2}\sum_{m_1,m_2}\frac{\tau(m_1)T(m_2)}{m_1m_2}x_{ab\delta m_1}x_{ab\delta m_2}\Big(\log\frac{\hat{Q}}{a^2\delta^2m_1m_2}\Big)\\
&\quad+2\sum_b\frac{1}{b}\sum_a\frac{\tau(a^2)}{a^2}\sum_\delta\frac{\mu(\delta)\tau(\delta)T(\delta)}{\delta^2}\sum_{m_1,m_2}\frac{\tau(m_1)\tau(m_2)}{m_1m_2}x_{ab\delta m_1}x_{ab\delta m_2}\Big(\log\frac{\hat{Q}}{a^2\delta^2m_1m_2}\Big)\\
&\quad+\sum_b\frac{1}{b}\sum_a\frac{T(a^2)}{a^2}\sum_\delta\frac{\mu(\delta)\tau(\delta)^2}{\delta^2}\sum_{m_1,m_2}\frac{\tau(m_1)\tau(m_2)}{m_1m_2}x_{ab\delta m_1}x_{ab\delta m_2}\Big(\log\frac{\hat{Q}}{a^2\delta^2m_1m_2}\Big).
\end{aligned}$$

Let m_{22} denote the part of the first term arising by using

$$\log\frac{\hat{Q}}{a^2\delta^2m_1m_2} = \log\frac{\hat{Q}}{m_1m_2} - 2\log a\delta;$$

this will be the main contribution: all the other terms can be directly estimated and shown to be of order of magnitude at most $(\log q)^5\log_2 q$.

We have

$$m_{22} = 2\sum_k \nu(k)\sum_{m_1,m_2}\frac{\tau(m_1)T(m_2)}{m_1m_2}x_{km_1}x_{km_2}\Big(\log\frac{\hat{Q}}{m_1m_2}\Big)$$

since, for squarefree k

$$\frac{1}{k}\sum_{ab\delta=k}\frac{\mu(\delta)\tau(\delta)^2\tau(a^2)}{a\delta}=\frac{1}{k}\prod_p\Bigl(1-\frac{4}{p}+\frac{3}{p}\Bigr)=\nu(k).$$

The treatment is now similar to that of m_{21}: define

$$z_k := z_k^{(0)} := \sum_m \frac{T(m)}{m}x_{km}$$

$$z_k^{(1)} := \sum_m \frac{T(m)}{m}(\log m)x_{km}$$

and

$$\tilde{\Pi}(a,b,c)=(\log\hat{Q})^a\sum_k \nu(k)y_k^{(b)}z_k^{(c)};$$

then

$$m_{22}=2\Bigl(\tilde{\Pi}(1,0,0)-\tilde{\Pi}(0,1,0)-\tilde{\Pi}(0,0,1)\Bigr). \tag{2.64}$$

Lemma 13. *We have*

$$z_k=2\sum_{\ell\leq M/k}\frac{(\log\ell)\Lambda(\ell)}{\ell}y_{k\ell}$$

$$z_k^{(1)}=\sum_{\ell\leq M/k}\frac{\tau(\ell)\Lambda(\ell)}{\ell}z_{k\ell}+\sum_{\ell\leq M/k}\frac{T(\ell)\Lambda(\ell)}{\ell}y_{k\ell}.$$

Proof. For the first one, (2.57) implies

$$z_k=\sum_\ell\Bigl(\sum_{mn=\ell}\frac{T(m)}{m}g(n)\Bigr)y_{k\ell}$$

and the Dirichlet generating series for the coefficient of ℓ is $L(s+1)$ where

$$L(s)=\zeta(s)^{-2}\sum_n T(n)n^{-s}.$$

¿From the first part of lemma 5, we get

$$\sum_n T(n)n^{-s}=4\zeta\zeta''-2(\zeta\zeta')'=2(\zeta\zeta''-(\zeta')^2)$$

so

$$L(s)=2(\zeta'\zeta^{-1})'.$$

As to $z_k^{(1)}$, write

$$\log m=\sum_{ab=m}\Lambda(a)$$

and use again the multiplicative property of T.

□

2.5.4. *Evaluation of* m_{22}. The mollifier was defined by (2.59).

Lemma 14. *We have*

$$z_k = -\frac{1}{3}\frac{k\mu(k)}{\varphi(k)}\Big(\log\frac{M}{k}\Big)^2(\log\hat{q}^3M^2k) + O((\log q)^2\frac{k}{\phi(k)}) = -y_k^{(2)} + O\big(\frac{k}{\varphi(k)}(\log q)^2\log_2 q\big)$$

and

$$z_k^{(1)} = O\big(\frac{k}{\varphi(k)}(\log q)^3\big)$$

Proof. We will be brief : on the one hand

$$\begin{aligned} z_k &= -2\frac{k\mu(k)}{\varphi(k)}\sum_{p\le M/k}\frac{(\log p)^2}{p}\log\hat{q}kp + O(\frac{k}{\varphi(k)}(\log_2 q)^3) \\ &= -2\frac{k\mu(k)}{\varphi(k)}\int_0^{\log M/k} y(y+\log\hat{q}k)dy + O\big(\frac{k}{\varphi(k)}(\log q)^2\big) \\ &= -\frac{1}{3}\frac{k\mu(k)}{\varphi(k)}\Big(\log\frac{M}{k}\Big)^2(\log\hat{q}^3M^2k) + O\big(\frac{k}{\varphi(k)}(\log q)^2\big) \end{aligned}$$

and on the other hand the two contributions to $z_k^{(1)}$ are respectively (using the previous computation)

$$\frac{1}{3}\frac{k\mu(k)}{\varphi(k)}\sum_{p\le M/k}\frac{2\log p}{p}\Big(\log\frac{M}{p}\Big)^2(\log\hat{q}^3M^2p) = \frac{1}{6}\frac{k\mu(k)}{\varphi(k)}\Big(\log\frac{M}{k}\Big)^3(\log\hat{q}^4M^3k) + O\big(\frac{k}{\varphi(k)}(\log q)^3\big)$$

and (this is the same as one of the sums considered in $y_k^{(3)}$)

$$-\frac{k\mu(k)}{\varphi(k)}\sum_{p\le M/k}\frac{2(\log p)^3}{p}(\log\hat{q}kp) = -\frac{1}{6}\frac{k\mu(k)}{\varphi(K)}\Big(\log\frac{M}{k}\Big)^3(\log\hat{q}^4M^3k) + O\big(\frac{k}{\varphi(k)}(\log q)^3\big)$$

□

¿From this (referring to lemma 12), we obtain

$$\begin{aligned} \tilde{\Pi}(1,0,0) &= -(\log\hat{Q})\sum_k \nu(k)y_k y_k^{(2)} + O\big((\log q)^5\big) \\ &= -\Pi(1,2,0) + O\big((\log q)^5\big) \qquad (2.65) \\ \tilde{\Pi}(0,1,0) &= -\sum_k \nu(k)y_k^{(1)}y_k^{(2)} + O\big((\log q)^5\big) \\ &= -\Pi(0,1,2) + O\big((\log q)^5\big) \qquad (2.66) \\ \tilde{\Pi}(0,0,1) &= O\big((\log q)^5\big). \end{aligned}$$

2.5.5. *The case of* N_2. For N_2 and N_1, the situation is much simpler. Recall the decomposition (2.55). We have

$$\Pi(1,0,0) = \sum_k \nu(k)y_k^2$$

where y_k is as before, and

$$N_1 = \sum_k j(k)y_k$$

with $j(k)=\mu(k)/k$. Hence we select

$$y_k=\frac{j(k)}{\nu(k)}=\frac{k\mu(k)}{\varphi(k)}$$

for $k\leq M$, to optimize $\Pi(1,0,0)$ with respect to N_1. We then have

$$\Pi(1,0,0)=(\log M)(\log\hat{Q})+O(1),\quad N_1=\log M+O(1)=\Delta\log q+O(1).$$

Moreover $\Pi(0,1,0)=\sum\nu(k)y_ky_k^{(1)}$ and proceeding as before we evaluate $y_k^{(1)}$, namely

$$y_k^{(1)}=-2\frac{k\mu(k)}{\varphi(k)}\Bigl(\log\frac{M}{k}\Bigr)+O(k/\varphi(k)\log_2 q).$$

Finally we find $\Pi(0,1,0)=2(\log M)^2$ by summation by parts, and

$$N_2=\Delta(1+2\Delta)(\log q)^2+O((\log q)\log_2 q).$$

Hence

$$\frac{N_1^2}{N_2}=\frac{\Delta}{1+2\Delta}+O\Bigl(\frac{\log_2 q}{\log q}\Bigr).$$

Letting $\Delta\rightarrow 1/4$ we obtain the harmonic analogue of Theorem 2.

2.6. **Conclusion.** To summarize our computations, consider the two polynomials in the variable Δ:

$$M_1(\Delta):=\Delta\Bigl(\frac{\Delta^2}{3}+\frac{\Delta}{2}+\frac{1}{4}\Bigr)\tag{2.67}$$

$$\begin{aligned}M_2(\Delta)\ :=\ &\frac{1}{12}M_1(\Delta)+\frac{1}{2}\Delta^2\Bigl(\bigl(\frac{1}{2}+\Delta\bigr)^2-\Delta\bigl(\frac{1}{2}+\Delta\bigr)+\frac{\Delta^2}{4}\Bigr)\\&+\frac{1}{2}\Delta^3\Bigl(\frac{4}{3}\bigl(\frac{1}{2}+\Delta\bigr)^2-\Delta\bigl(\frac{1}{2}+\Delta\bigr)+\frac{\Delta^2}{5}\Bigr)\\&+\frac{1}{3}\Delta^3\Bigl(\bigl(\frac{1}{2}+\Delta\bigr)^2-\Delta\bigl(\frac{1}{2}+\Delta\bigr)+\frac{\Delta^2}{5}\Bigr)\\&+\frac{1}{3}\Delta^4\Bigl(\frac{3}{2}\bigl(\frac{1}{2}+\Delta\bigr)^2-\Delta\bigl(\frac{1}{2}+\Delta\bigr)+\frac{\Delta^2}{6}\Bigr)\end{aligned}\tag{2.68}$$

$$=\ \frac{2}{9}x^6+\frac{2}{3}x^5+\frac{5}{6}x^4+\frac{19}{36}x^3+\frac{1}{6}x^2+\frac{1}{48}x\tag{2.69}$$

then it follows from lemma 10, that for $\Delta<1/2$

$$M_1=M_1(\Delta)(\log q)^3+O\bigl((\log q)^2\bigr)$$

and from (2.42), (2.48), (2.44), (2.54), lemma 12, (2.64), (2.65), (2.66) (and the computations to check (2.46) and consequences of the previous estimations [4]) that for $\Delta<1/4$, we have

$$M_2=M_2(\Delta)(\log q)^6+O\bigl((\log q)^5\log_2 q\bigr).$$

Now, partial fraction decomposition yields

$$\frac{M_1(\Delta)^2}{M_2(\Delta)}=\frac{1}{2}\Bigl(1-\frac{1}{(1+2\Delta)^3}\Bigr).$$

[4] Count the number of logarithms and estimate directly.

Hence the harmonic analogue of Theorem 3 follows, in the more precise form

$$\sum_{L(f,1/2)=0,\, L'(f,1/2)\neq 0}^{h} 1 \geq \frac{1}{2}\Big(1 - \frac{1}{(1+2\Delta)^3}\Big) + O\Big(\frac{\log_2 q}{\log q}\Big).$$

3. Non-vanishing in natural average

We consider now the first and second moments for the natural average

$$M_1^n := \sum_{L(f,1/2)=0} M(f)L'(f,1/2),$$

$$M_2^n := \sum_{L(f,1/2)=0} |M(f)L'(f,1/2)|^2.$$

To get from the harmonic averages to the natural average, we use the method of [KM1]. For $x \geq 1$, let

$$\omega_f(x) := \sum_{dl^2 \leq x} \frac{\lambda_f(d^2)}{dl^2}$$

(a partial sum of the value of the symmetric square L-function of f at $s = 1$). Applying Proposition 2 of [KM1], it follows that for $x = q^{\varepsilon}$, $\varepsilon > 0$ being small enough, we have

$$M_1^n = \frac{q}{2\pi^2} \sum_{L(f,1/2)=0}^{h} \omega_f(x)M(f)L'(f,1/2) + O(q^{-c}),$$

$$M_2^n = \frac{q}{2\pi^2} \sum_{L(f,1/2)=0}^{h} \omega_f(x)|M(f)L'(f,1/2)|^2 + O(q^{-c}),$$

for some $c = c(\varepsilon)$.

To check the conditions of [KM1, Prop. 2], we use the growth condition

$$x_m \ll (\tau(m)\log qm)^A$$

together with the estimates

$$\begin{aligned} \frac{1}{(f,f)} &\ll (\log q)q^{-1} \\ L'(f,1/2) &\ll q^{1/4}(\log q)^2 \end{aligned}$$

to obtain the individual bounds, and with Propositions 2 and 6 for the average bounds (see [Kow, Lemma 43] for more details).

We write

$$\tilde{M}_1 = \sum_{L(f,1/2)=0}^{h} \omega_f(x)M(f)L'(f,1/2)$$

$$\tilde{M}_2 = \sum_{L(f,1/2)=0}^{h} \omega_f(x)|M(f)L'(f,1/2)|^2$$

3.1. **Computing the first moment.** Using the formula

$$\lambda_f(m)\lambda_f(d^2)=\sum_{r|(m,d^2)}\varepsilon_q(r)\lambda_f\Bigl(\frac{md^2}{r^2}\Bigr),$$

we get, for $M=q^{\Delta}$ with $\Delta<1/2$,

$$\tilde{M}_1=\sum_{m\leq M}\frac{x_m}{m}\sum_{dl^2\leq x}\frac{1}{d^2l^2}\sum_{r|(d^2,m)}r\log(\hat{q}r^2/d^2m)+O(q^{-c}).$$

We remove the constraint $dl^2\leq x$ at the cost of an admissible error term (that is, $O(q^{-c})$). Since m is square-free, $r|d^2\Leftrightarrow r|d$ and we get, setting $d'=d/r$,

$$\begin{aligned}\tilde{M}_1&=\zeta(2)\sum_{m\leq M}\frac{x_m d_{-1}(m)}{m}\sum_{d'}\frac{1}{d'^2}\bigl(\log\hat{q}m)-2\log d'\bigr)+O(q^{-c})\\ &=\zeta(2)^2\sum_{m\leq M}\frac{x_m d_{-1}(m)}{m}(\log\hat{q}m)+2\zeta(2)\zeta'(2)\sum_{m\leq M}\frac{x_m d_{-1}(m)}{m}+O(q^{-c}).\end{aligned}\tag{3.1}$$

3.2. **Computing $\tilde{M}_2$.** We have

$$\tilde{M}_2=\sum_b\frac{1}{b}\sum_{m_1,m_2\leq M}\frac{x_{bm_1}x_{bm_2}}{\sqrt{m_1m_2}}\sum_{dl^2\leq x}\sum_{r|(d^2,m_1m_2)}X\Bigl(\frac{d^2m_1m_2}{r^2}\Bigr).$$

By proposition 5, the second moment decomposes in way similar to (2.42): for $M=q^{\Delta}$, $\Delta<1/4$

$$\tilde{M}_2=\frac{1}{12}\tilde{M}_{21}-\frac{1}{4}\tilde{M}_{22}+\tilde{M}_3+O(q^{-c})\tag{3.2}$$

where

$$\tilde{M}_{21}=\sum_b\frac{1}{b}\sum_{dl^2\leq x}\frac{1}{d^2l^2}\sum_{m_1,m_2\leq M}\sum_{r|(d^2,m_1m_2)}r\frac{\tau(m_1m_2d^2/r^2)}{m_1m_2}x_{bm_1}x_{bm_2}\Bigl(\log\frac{\hat{Q}r^2}{m_1m_2d^2}\Bigr)^3\tag{3.3}$$

$$\tilde{M}_{22}=\sum_b\frac{1}{b}\sum_{dl^2\leq x}\frac{1}{d^2l^2}\sum_{m_1,m_2}\sum_{r|(d^2,m_1m_2)}r\frac{T(m_1m_2d^2/r^2)}{m_1m_2}x_{bm_1}x_{bm_2}\Bigl(\log\frac{\hat{Q}r^2}{m_1m_2d^2}\Bigr)\tag{3.4}$$

$$\tilde{M}_3=\sum_b\frac{1}{b}\sum_{dl^2\leq x}\frac{1}{d^2l^2}\sum_{m_1,m_2}\sum_{r|(d^2,m_1m_2)}r\frac{\tau(m_1m_2d^2/r^2)}{m_1m_2}P_1\Bigl(\log\frac{\hat{Q}r^2}{m_1m_2d^2}\Bigr)\tag{3.5}$$

(compare with (2.44), (2.45)). As in [KM1, 4.5], we drop the constraint $dl^2\leq x$ in (3.3),(3.4), (3.5) at the cost of an error term which is $O(q^{-c})$.

As before, the strategy is now to optimize the quadratic form $\tilde{M}_{21}$ with respect to the linear form (3.1). For this, we shall need properties of some "quasi-multiplicative" arithmetic functions. For a more detailed treatment, see [Kow, 6.2].

3.2.1. *Some quasi-multiplicative functions.* During the transformation process of $\tilde{M}_{21}$ we will meet expressions of the following type, where r stands for a divisor of m_1m_2:

$$\sum_{\substack{d\geq 1\\ d^2\equiv 0 \bmod r}} \frac{\tau(d^2/r)}{d^2}(\log d^2)^k = (-1)^k (g(\delta,r))^{(k)}_{\delta=0}, \text{ with } g(\delta,r) := \sum_{\substack{d\geq 1\\ d^2\equiv 0 \bmod r}} \frac{\tau(d^2/r)}{d^{2+2\delta}}$$

(k is an integer).

This Dirichlet series is computed in [KM1, Lemma 13] (it is the case $\delta = 0$), and is the product of a constant and a multiplicative function $\kappa(\delta, r)$:

$$g(\delta,r) = \frac{\zeta(2+2\delta)^3}{\zeta(4+4\delta)}\kappa(\delta,r),\ \kappa(\delta,r) = \prod_{p|r}\frac{1}{p^{2+2\delta}}\prod_{p||r}\frac{2}{1+p^{-2(1+\delta)}}.$$

For an integer $k \geq 0$, let $\kappa^{(k)}(r) := (\kappa(\delta,r))^{(k)}_{\delta=0}$ be the k-th derivative of $\delta \mapsto \kappa(\delta, r)$ at $\delta = 0$. By Leibniz's rule, the series $(g(\delta,r))^{(k)}_{\delta=0}$ is a linear combination of the $\kappa^{(k')}(r)$, for $k' \leq k$.

In the next steps, we shall use the following two properties of $\kappa^{(k)}(r)$: the bound

$$\kappa^{(k)}(r) \ll_k \frac{\tau(r)(\log r)^k}{\prod_{p|r} p^2}; \tag{3.6}$$

and the quasi-multiplicativity

$$\text{for } (r_1,r_2)=1,\ \kappa^{(k)}(r_1r_2) = \sum_{k'+k''=l}\binom{k}{k'}\kappa^{(k')}(r_1)\kappa^{(k'')}(r_2). \tag{3.7}$$

We will also encounter the arithmetic convolution $f^{(j)} = \mu * (Id \times \log^j)$. This last function also enjoys quasi-multiplicative properties:

$$\text{for } (r_1,r_2)=1,\ f^{(j)}(r_1r_2) = \sum_{j'+j''=j}\binom{j}{j'}f^{(j')}(r_1)f^{(j'')}(r_2).$$

The following formula will be used to separate m_1m_2/r and d^2/r.

$$\sum_{r|(m,n)} r(\log r)^j \tau\Big(\frac{mn}{r^2}\Big) = \sum_{r|(m,n)} f^{(j)}(r)\tau\Big(\frac{m}{r}\Big)\tau\Big(\frac{n}{r}\Big). \tag{3.8}$$

3.3. **The preferred quadratic form II.** Decompose $\log \hat{Q}/m_1m_2d^2 = \log(\hat{Q}/m_1m_2) + \log r^2 - \log d^2 := U + V + W$, say. We have $(U+V+W)^3 = \sum_{i+j+k=3} c_{i,j,k}U^iV^jW^k$, and $\tilde{M}_{21}$ decomposes accordingly

$$\tilde{M}_{21} = \sum_{i+j+k=3} c_{i,j,k}\tilde{M}_{21}^{i,j,k}.$$

3.3.1. *Decomposing* $\tilde{M}_{21}$. Using the results of the preceding section and (3.8), we decompose $\tilde{M}_{21}$ into pieces of the form

$$\sum_b \frac{1}{b}\sum_{m_1,m_2}\frac{x_{bm_1}x_{bm_2}}{m_1m_2}\Big(\log\frac{\hat{Q}}{m_1m_2}\Big)^i \sum_{r|m_1m_2} f^{(j)}\kappa^{(k)}(r)\tau\Big(\frac{m_1m_2}{r}\Big)$$

where i,j,k are integers satisfying $0 \leq i,j,k,\ i+j+k \leq 3$.

Note at this point (use the quasi-multiplicativity of $f^{(j)}$, $\kappa^{(k)}$, and the multiplicativity of τ) that the convolution $\tau * f^{(j)}\kappa^{(k)}$ is quasi-multiplicative, in the following sense: for $(m,n)=1$, $\tau * f^{(j)}\kappa^{(k)}(mn)$ is a linear combination (with at most jk terms) of products $\tau * f^{(j')}\kappa^{(k')}(m) \times \tau * f^{(j'')}\kappa^{(k'')}(n)$ with $j'+j''\leq j$ and $k'+k''\leq k$.

Then m_1 and m_2 decompose uniquely into $m_1=m_1'm_3$, $m_2=m_2'm_3$ with $(m_1',m_2')=1$, and we further have $(m_1',m_2',m_3)=1$ since m_1,m_2 were assumed to be square-free. Define $h^{j,k}$ by

$$h^{j,k}(m)=\frac{1}{m}\tau * f^{(j)}\kappa^{(k)}(m);$$

then we see by quasi-multiplicativity that $\tilde{M}_{21}$ is a linear combination of the quadratic forms

$$\sum_b \frac{1}{b}\sum_{\substack{m_1,m_2,m_3\\(m_1,m_2)=1}} x_{bm_3m_1}x_{bm_3m_2}h^{j_1,k_1}(m_1)h^{j_2,k_2}(m_2)h^{j_3,k_3}(m_3^2)\Big(\log\frac{\hat{Q}}{m_1m_2m_3^2}\Big)^i,$$

with $i+j_1+k_1+j_2+k_2+j_3+k_3\leq 3$.

Finally, we detect the remaining condition $(m_1,m_2)=1$ using the Möbius function, and obtain, after expanding the factor $(\log\hat{Q}/m_1m_2m_3^2)^i$, the decomposition of $\tilde{M}_{21}$ as a linear combination of the quadratic forms

$$\tilde{M}_{21}^{i,\mathbf{i},\mathbf{j},\mathbf{k}} := \sum_k \nu^{\mathbf{i},\mathbf{j},\mathbf{k}}(k)\sum_{m_1,m_2} x_{km_1}x_{km_2}h^{j_4,k_4}(m_1)h^{j_5,k_5}(m_2)\Big(\log\frac{\hat{Q}}{m_1m_2}\Big)^i \tag{3.9}$$

with

$$\mathbf{i}=(i_1,i_3),\ \mathbf{j}=(j_1,j_2,j_3,j_4,j_5),\ \mathbf{k}=(k_1,k_2,k_3,k_4,k_5),$$
$$i+i_1+i_3+j_1+j_2+j_3+j_4+j_5+k_1+k_2+k_3+k_4+k_5\leq 3$$

and

$$i,i_1,\ldots,k_5\geq 0.$$

More precisely, we have

$$\tilde{M}_{21}=\frac{\zeta(2)^4}{\zeta(4)}\tilde{M}_{21}^{3,(0,0),(0,0,0),(0,0,0)}+\sum_{i<3}\sum_{\mathbf{i},\mathbf{j},\mathbf{k}} c_{i,\mathbf{i},\mathbf{j},\mathbf{k}}\tilde{M}_{21}^{i,\mathbf{i},\mathbf{j},\mathbf{k}}.$$

To ease the notations, set

$$\tilde{M} := \tilde{M}_{21}^{3,(0,0),(0,0,0),(0,0,0)}.$$

For the optimal choice of the vector (x_m), the main contribution to $\tilde{M}_{21}$ will be seen to come from $\tilde{M}$: for $i<3$ we will check that

$$\tilde{M}_{21}^{i,\mathbf{i},\mathbf{j},\mathbf{k}} = O\Big(\tilde{M}\frac{\log_2 q}{\log q}\Big). \tag{3.10}$$

Let's now concentrate on $\tilde{M}$.

3.4. **Diagonalization of $\tilde{M}$.** Opening the factor $\log(\hat{Q}/m_1m_2)^3$, we have the decomposition

$$\tilde{M}=\tilde{\Pi}(3,0,0)-6\tilde{\Pi}(2,1,0)+6\tilde{\Pi}(1,1,1)+6\tilde{\Pi}(1,2,0)-6\tilde{\Pi}(0,1,2)-2\tilde{\Pi}(0,0,3). \tag{3.11}$$

3.4.1. *Optimizing* $\tilde\Pi(3,0,0)$. Set $\tilde h(m) = h^{0,0}(m)$, $\tilde\nu(k) := \nu^{(0,0),(0,0,0),(0,0,0)}(k)$, write $\tilde g$ for the convolution inverse of $\tilde h$, and set

$$\tilde\jmath(k) := \tilde g * \frac{d_{-1}\times\log(\hat q/Id)}{Id}(k) = \sum_{ab=k}\tilde g(a)\log(\hat q/b)\frac{d_{-1}(b)}{b}$$

$$j_0(k) := \tilde g * \frac{d_{-1}}{Id}(k) = \sum_{ab=k}\tilde g(a)\frac{d_{-1}(b)}{b}$$

After some computations, it is found that for p prime we have

$$\tilde\jmath(p) = -A(p^{-1}), \quad \text{with } A = X\frac{(1-X)(1+X)^2}{1+X^2} \tag{3.12}$$

$$\tilde\nu(p) = B(p^{-1}), \quad \text{with } B = \frac{(1-X^2)^3}{(1+X^2)^2} \tag{3.13}$$

The next lemma is the analogue of Lemma 9:

Lemma 15. *For all square-free integer k, we have*

$$\tilde\jmath(k) = j_0(k)(\log\hat q k + O(1)).$$

Proof. We have

$$\begin{aligned}
\tilde\jmath(k) &= j_0(k)\log\hat q - \sum_{ab=k} g(a)\frac{d_{-1}(b)}{b}\log b\\
&= j_0(k)\log\hat q - \sum_{p|k}\frac{\log p}{p}d_{-1}(p)j_0(k/p)\\
&= j_0(k)\left(\log\hat q - \sum_{p|k}\frac{\log p d_{-1}(p)}{p j_0(p)}\right)\\
&= j_0(k)\left(\log\hat q + \sum_{p|k}\log p\frac{(1+1/p^2)}{(1-1/p^2)}\right).
\end{aligned}$$

□

With our notations we have

$$\tilde\Pi(3,0,0) = (\log\hat Q)^3\sum_k\tilde\nu(k)\sum_{m_1,m_2\le M}\tilde h(m_1)\tilde h(m_2)x_{km_1}x_{km_2}.$$

To diagonalize $\tilde\Pi(3,0,0)$ we make the – now classical – change of variable

$$y_k = \sum_m \tilde h(m)x_{km}, \text{ so that } x_m = \sum_k \tilde g(k)y_{km} \text{ and } \tilde\Pi(3,0,0) = (\log\hat Q)^3\sum_k\tilde\nu(k)y_k^2. \tag{3.14}$$

We now choose (y_k) optimally to optimize $\tilde\Pi(3,0,0)$ with respect to $\tilde M_1$:

$$y_k = \mu(k)^2\frac{\tilde\jmath(k)}{\tilde\nu(k)}, \text{ for } k\le M. \tag{3.15}$$

We immediately see that the corresponding (x_m) satisfies condition (2.6), and (3.1) gives

$$\begin{aligned}\tilde{M}_1 &= \zeta(2)^2\sum_{k\leq M}\tilde{\jmath}(k)y_k + 2\zeta(2)\zeta'(2)\sum_{k\leq M} j_0(k)y_k\\ &= \zeta(2)^2\sum_{k}\mu(k)^2\frac{\tilde{\jmath}(k)^2}{\tilde{\nu}(k)} + 2\zeta(2)\zeta'(2)\sum_{k}\mu(k)^2\frac{\tilde{\jmath}(k)j_0(k)}{\tilde{\nu}(k)}\\ &= \zeta(2)^2\sum_{k\leq M}\mu(k)^2\frac{\tilde{\jmath}_0(k)^2}{\tilde{\nu}(k)}\big(\log^2(\hat{q}k)+O(\log(\hat{q}k)\big)\\ &= \frac{\zeta(2)^3}{\zeta(4)}\Delta\Big(\frac{\Delta^2}{3}+\frac{\Delta}{2}+\frac{1}{4}\Big)(\log q)^3 + O\big((\log q)^2\big).\end{aligned}\tag{3.16}$$

In the last two lines, we have used lemma 15, and a partial summation exactly similar to that performed in lemma 10 with the following variant

$$\sum_{k\leq M}\mu(k)^2\frac{\tilde{\jmath}_0(k)^2}{\tilde{\nu}(k)} = \frac{\zeta(2)}{\zeta(4)}\log M + O(1)\tag{3.17}$$

which follows by computing the residue at $s=0$ of the Dirichlet series

$$\sum_{k\geq 1}\frac{\mu(k)^2\tilde{\jmath}_0(k)^2}{\tilde{\nu}(k)}k^{-s} = \prod_p\Big(1+\frac{2p^{-(s+1)}}{(1+p^{-(s+1)})(p-1)}\Big)$$

which has analytic continuation to $\mathrm{Re}(s)>-1$. Similarly, we have

$$\tilde{\Pi}(3,0,0) = \frac{\zeta(2)}{\zeta(4)}\Delta\Big(\frac{\Delta^2}{3}+\frac{\Delta}{2}+\frac{1}{4}\Big)(\log q)^6 + O\big((\log q)^5\big).$$

3.4.2. *Estimation of* $\tilde{\Pi}(u,v,w)$. We set

$$y_k^{(i)} = \sum_m \tilde{h}(m)(\log m)^i x_{km}.$$

The next lemma is the analogue of lemma 11, and its proof is exactly the same, using lemma 15, and the equality $\tilde{h}(p) = \frac{\tau(p)}{p} + O(p^{-2})$.

Lemma 16. *For $i=1$, 2, 3, we have*

$$y_k^{(i)} = c_i\frac{j_0(k)}{\nu(k)}\Big(\log\frac{M}{k}\Big)^i(\log\hat{q}^{i+1}M^ik) + O\Big(\frac{j_0(k)}{\nu(k)}(\log q)^i\log_2 q\Big)\tag{3.18}$$

with

$$c_1=-1,\ c_2=\frac{1}{3},\ c_3=0.$$

Then, the computations of section 2.5.2 hold *verbatim* and we get

$$\tilde{M}_{21} = \frac{\zeta(2)^5}{\zeta(4)^2}M_{21} + O\big((\log q)^5\log_2 q\big)\tag{3.19}$$

where by $\tilde{M}_{21}$ we mean the *value* of the quadratic form at the vector (x_m) in (3.14) and (3.15), and similarly for M_{21} and the vector in (2.57) and (2.59). Similarly, one can show, with the same abuse of notation, that

$$\tilde{M}_{22} = \frac{\zeta(2)^5}{\zeta(4)^2}M_{22} + O\big((\log q)^5\log_2 q\big)\tag{3.20}$$

and

$$\tilde{M}_3 = O\big((\log q)^5\big).$$

3.5. **Contribution of the residual quadratic forms.** We have to show is that the quadratic forms $\tilde{M}_{21}^{i,\mathbf{j},\mathbf{k}}$ for $i < 3$ do not contribute to the main term. All the necessary arguments were given the preceding sections. After having computed, in terms of the y_k's, new variables of the form

$$y_k^{i,j_4,k_4} := \sum_m h^{j_4,k_4}(m)(\log m)^i x_{km},$$

one can show by cumbersome but easy calculations that for the mollifier chosen in (3.14) and (3.15), (3.10) holds for $i < 3$.

3.6. **Conclusion.** From (3.19), (3.20), (3.16), and (3.2), we see that for $\Delta < 1/4$ one has

$$\sum_{\substack{f\in S_2(q)^* \\ L(f,1/2)=0,\, L'(f,1/2)\neq 0}} 1 \geq \frac{(M_1^n)^2}{M_2^n} = q\frac{\zeta(2)}{2\pi^2}\frac{M_1(\Delta)^2}{M_2(\Delta)} + O\Big(\frac{\log_2 q}{\log q}\Big),$$

and Theorem 3 follows (in a more precise form) since we know that

$$q\frac{\zeta(2)}{2\pi^2} = \frac{q}{12} = \dim J_0(q) + O(1).$$

In the case of the central critical values, we go from harmonic averages N_1 and N_2 to the natural averages, and prove Theorem 2 in a similar way. All computations being simpler, we omit the details.

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