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EMERGENCE OF THE WITT GROUP IN THE CELLULAR LATTICE OF RATIONAL SPACES

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ABSTRACT. We give an embedding of a quotient of the Witt semigroup into the lattice of rational cellular classes represented by formal 2-cones between S^{2n} and the two-cell complex $X_n = S^{2n} \cup_{[\iota,\iota]} e^{4n}$ $(n \geq 1)$.

1. INTRODUCTION

It is now a well-established fact in unstable homotopy theory that the structure of the Dror Farjoun (cellular) lattice $(Top_*, \ll)$ [8] defined on the category of pointed spaces having the homotopy type of a CW-complex is highly nontrivial. At the time this paper is being written its full classification is still intractable, to the best of the authors' knowledge. Just recently the partial order was in fact shown to be a (complete) lattice [7]. The dream of course would be to classify it, as Hopkins and Smith [12] did in the case of the Bousfield lattice [3] of stable p-torsion finite complexes.

In a first attempt to tackle this classification problem, Chachólski, Stanley and the second author came up with a criterion to determine when $A \ll B$ rationally [6]. This criterion can be extended to the case of spaces that are Bousfield equivalent to S^n, i.e., $(n-1)$-connected spaces A such that $map_*(S^n, X) \simeq *$ whenever $map_*(A, X) \simeq *$. We immediately deduce the following strict building relation:

$$S^2 \ll \mathbb{C}P^2 \ll \mathbb{C}P^3 \ll \ldots \ll \mathbb{C}P^\infty.$$

Afterwards, Félix constructed yet another such example over the rationals, i.e.,

$$S^2 \ll \ldots \ll \underbrace{\mathbb{C}P^2\# \ldots \#\mathbb{C}P^2}_{n \text{ times}} \ll \ldots \ll \mathbb{C}P^2\#\mathbb{C}P^2 \ll \mathbb{C}P^2.$$

Furthermore, using the Chachólski-Parent-Stanley criterion together with results concerning inert cell attachments [10], the first author produced an infinite family of cellularly incomparable spaces, which all have the same rational homotopy Lie algebra and the same rational homology coalgebra [11]. Even rationally, therefore, the structure of the cellular lattice is remarkable.

In this paper we continue the quest for the holy grail of complete classification of the rational cellular lattice. We consider cellular classes generated by $(2k-1)$-connected spaces X of dimension at most $6k-1$ $(k \geq 1)$ such that $\pi_{2k}(X) \otimes \mathbb{Q} \neq 0$. Moreover, we ask that these spaces be rationally formal. In these cases, for degree reasons, the only relevant cells are in dimensions $2k$ and $4k$. The crucial remark is

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that all these cellular classes are contained between S^{2k} and the two-cell complex $X_k = S^{2k} \cup_{[\iota,\iota]} e^{4k}$ $(k \geq 1)$ (Proposition 3). Let $\mathbb{M}$ be the set of cellular classes generated by such spaces with at most one cell in dimension $4k$. In Theorem 7, we show that $\mathbb{M}$ is in one-to-one correspondence with a certain quotient $\overline{W(\mathbb{Q})}$ of the Witt semigroup $W(\mathbb{Q})$, where the operation considered is the Kronecker product of quadratic forms. The quadratic forms within this quotient have been classified in the work of Lewis and Tignol [13], where they generalize the classical invariants, i.e., Clifford algebra, signature, and discriminant, associated to isometry classes of quadratic forms (see the Appendix).

Thanks to the isomorphism between $\mathbb{M}$ and $\overline{W(\mathbb{Q})}$, there is a partial order on $\overline{W(\mathbb{Q})}$, corresponding to the building relation on cellular classes. We believe this partial order to have been unknown previously and intend to study the implications of its existence for the classification of quadratic forms.

Finally, we exhibit an action of $\mathbb{M}$ on the set of cellular classes generated by these 2-cones where the restriction on the number of $4k$-cells is lifted. To the best of the authors' knowledge, this is the first time that such algebraic operations have been defined on a subset of the rational lattice.

2. Notation and background

Throughout this paper all spaces are rationalizations of simply-connected spaces having the homotopy type of a CW-complex.

2.1. **Rational homotopy theory and Quillen models.** Rational homotopy theory has its roots in the work of Quillen [15] and Sullivan [17]. For a complete overview of these techniques, we refer the reader to [9]. We recall here some of the highlights, in particular, the equivalence of categories between the homotopy category of simply-connected rational spaces and the homotopy category of connected differential graded rational Lie algebras.

Recall that a *graded Lie algebra* over the field of rational numbers $\mathbb{Q}$ consists of a positively graded $\mathbb{Q}$-vector space L, together with a bilinear product called the *Lie bracket* that we denote $[-,-]$, such that

$$[x,y] = -(-1)^{|x||y|}[y,x]$$

and

$$[x,[y,z]] = [[x,y],z] + (-1)^{|x||y|}[y,[x,z]]$$

for all homogeneous $x, y, z \in L$, where $|\alpha|$ refers to the degree of a homogeneous element $\alpha \in L$. This last identity is known as the Jacobi identity and simply expresses the fact that the adjoint representation acts as a derivation with respect to the bracket.

If a graded Lie algebra L is endowed with a derivation ∂ of degree -1 such that $\partial^2 = 0$, we call (L, ∂) a *differential graded Lie algebra*, abbreviated *dgL*, and ∂ is its *differential*.

Let V be a positively-graded $\mathbb{Q}$-vector space, and let TV denote the tensor algebra on V. When endowed with the commutator bracket, TV is a graded Lie algebra. The *free Lie algebra* on V, denoted $\mathbb{L}V$, is the smallest sub Lie algebra of TV containing V. An element in $\mathbb{L}V$ has *bracket length* k if it is a linear combination of iterated brackets of k elements of V, i.e., if it belongs to the intersection $\mathbb{L}V \cap T^kV$, where T^kV denotes the subspace of TV generated by the words of tensor length

k. The subspace of elements of bracket length k is denoted $\mathbb{L}^k V$. Note that any graded linear map of degree 0 from V into a graded Lie algebra L can be extended uniquely to a Lie algebra map from $\mathbb{L}V$ into L. We say that a dgL is *free* if the underlying graded Lie algebra is free. The differential ∂ of a free dgL $(\mathbb{L}V, \partial)$ can be decomposed as $\partial = \partial_1 + \partial_2 + \partial_3 + \dots$, where $\partial_k(V) \subset \mathbb{L}^k V$. The summand ∂_2 is called the *quadratic part* of the differential ∂.

Let X be a simply-connected space. The graded vector space $\pi_*(\Omega X) \otimes \mathbb{Q}$ admits a natural graded Lie algebra structure, where the bracket is given by the Samelson product. Together with the Samelson bracket, $\pi_*(\Omega X) \otimes \mathbb{Q}$ is called the *rational homotopy Lie algebra* of X, denoted $\mathcal{L}(X)$ in this article.

If X is simply-connected, then there exists a free dgL $\mathcal{M}(X) = (\mathbb{L}V, \partial)$, unique up to isomorphism, called the *Quillen (or Lie) model of* X. It completely characterizes the rational homotopy type of the space X and has, in particular, the following properties:

- $H_*(\mathcal{M}(X)) \cong \mathcal{L}(X)$ as graded Lie algebras;
- $sV \cong \widetilde{H}_*(X; \mathbb{Q})$ as graded vector spaces, where $(sV)_n = V_{n-1}$ for all $n > 0$;
- $\partial_1 = 0$; and
- ∂_2 is obtained via desuspension from the coproduct on $H_*(X; \mathbb{Q})$.

Moreover, geometrically this model corresponds to a cell decomposition where the cells are represented by the generators, and the attaching maps are encoded into the differential. This model is intimately linked to the Adams-Hilton model [1], as shown by Anick in [2]. We will thus refer to an inclusion of the type $\mathbb{L}(V_{<n}) \hookrightarrow \mathbb{L}(V)$ as the inclusion of the n^{th}-skeleton of X into X.

Dually (in the sense of Eckmann-Hilton), when the rational homology of X is of finite type, the essential property of the Quillen model of X is that there is a quasi-isomorphism of commutative cochain algebras from the Cartan-Chevalley-Eilenberg construction on $\mathcal{M}(X)$ to the de Rham algebra of piecewise-linear forms on the singular set of X.

Finally, the Quillen model is a natural construction, so that a continuous map between two simply-connected spaces induces a dgL-map between their respective Quillen models.

2.2. **Cellular spaces.** In [8] Dror Farjoun made the following definition. A full subcategory of pointed spaces, $\mathcal{C} \subset Top^*$, is called a *closed class* if it is closed under weak equivalences and arbitrary pointed homotopy colimits. We refer the reader to the work of Bousfield and Kan [4] and to Dror Farjoun's book [8] for the definitions and constructions associated with the notion of (pointed) homotopy (co)limits. A result of Chachólski in [5] shows that a class $\mathcal{C}$ of spaces is a closed class if and only if

- if $X \in \mathcal{C}$ and $Y \simeq X$, then $Y \in \mathcal{C}$; and
- if $X_i \in \mathcal{C}$, where i belongs to some small category I, then
  - $hocolim_*(X_1 \to X_2 \to \cdots) \in \mathcal{C}$,
  - $hocolim_*(X_1 \leftarrow X_2 \to X_3) \in \mathcal{C}$, and
  - $\bigvee_{i \in I} X_i \in \mathcal{C}$.

An important example of such a class is $C(A)$, the smallest closed class containing the space A (e.g., $C(S^n)$ is the class of all $(n-1)$-connected spaces). Thus we say that B is A-cellular, or that A builds B, which we denote $A \ll B$, if $B \in C(A)$. Two spaces A and B are cellularly equivalent, denoted $A \tilde{c} B$, if $A \ll B$ and $B \ll A$.

We say that two Quillen models are cellularly equivalent if their corresponding rational spaces are equivalent.

The following theorem, due to Chachólski, Stanley, and the second author, is the foundation of the research presented in this article.

Theorem 1 ([6]). *Let X and A be two $(n-1)$-connected $(n>1)$ rational spaces such that $\pi_n(X)$ and $\pi_n(A)$ are nontrivial. Then $A \ll X$ if and only if there exists a continuous map $f : \vee_{i\in J} A \to X$ for some index set J, such that $\pi_n(f)$ is surjective.*

This result, as mentioned by the first author in [11], has the following algebraic translation. Let X and A be as in the previous theorem. Then $A \ll X$ if and only if there is a dgL morphism $\phi : \coprod_{i\in J} \mathcal{M}(A) \to \mathcal{M}(X)$ inducing a homology surjection in degree $n-1$, where J is an arbitrary index set, and the coproduct is taken in the dgL category. It is thus obvious that any nontrivial n^{th}-skeleton of X builds X.

Proposition 2. *Let X be an $(n-1)$-connected space such that $\pi_n(X) \neq 0$. If the differential ∂ of its Quillen model $(\mathbb{L}(V), \partial)$ is such that $\partial_2 = 0$, then $X \tilde{c} S^n$.*

Proof. This is a consequence of the Jacobi identity, which implies that all triple brackets of the type $[x,[x,x]]$ are zero. Let $(\mathbb{L}(\omega), 0)$ be the Quillen model of S^n where $|\omega| = n-1$. Choose a basis $\{x_1, ..., x_k\}$ for V_{n-1}, and define a dgL morphism $(\mathbb{L}(V_{n-1} \oplus V_{\geq n}), \partial) \to (\mathbb{L}(\omega), 0)$ by sending x_1 to ω, all x_i to zero for $i > 1$, and $V_{\geq n}$ also to zero. This is indeed a dgL morphism since any bracket of length at least three must contain a generator different from x_1. Clearly this morphism induces a homology surjection in degree $n-1$. □

Evidently the quadratic part of the differential plays a crucial role in furnishing examples of spaces cellularly distinct from the spheres. In a first attempt to classify the rational cellular lattice, we restrict our analysis primarily to models such that $\partial = \partial_2$, i.e., rational formal spaces.

Finally, recall a result from [6] (for an algebraic proof see [11]) which states that if X is a $(2k)$-connected rational space such that $\pi_{2k+1}(X) \neq 0$, then $X \tilde{c} S^{2k+1}$. Thus the relevant part of the analysis is done for $(2k-1)$-connected spaces X such that $\pi_{2k}(X) \neq 0$.

2.3. **Quadratic forms.** We refer the reader to [16] for the proof of the classical results on quadratic forms. We recall here the relevant results for this paper.

All quadratic forms that we consider are regular rational forms of finite dimension. Let Q and Q' be two forms of the same dimension n. They are said to be *isometric*, denoted $Q \simeq Q'$, if there exists an invertible $n \times n$ matrix C such that $A_Q = CA_{Q'}C^{\dagger}$, where A_Q and $A_{Q'}$ are the matrices associated to Q and Q', respectively. They are *similar*, denoted $Q \sim Q'$, if there exists a nonzero rational number r such that $rQ \simeq Q'$.

A form Q is said to be *isotropic* if there exists a nontrivial vector v such that $Q(v) = 0$. Otherwise it is said to be *anisotropic*. Note that the empty form $\phi = 0$ of dimension 0 is counted as being regular and anisotropic.

A classical result asserts that over any field of characteristic different from two, any n-dimensional form is isometric to a diagonal form. We use the notation $\langle \alpha_1, \alpha_2, \ldots, \alpha_n \rangle$ to denote a diagonal form, where the α_i are the diagonal entries.

We have two operations defined up to isometry, i.e., given $Q \simeq \langle \alpha_1, \alpha_2, \dots, \alpha_n \rangle$ and $Q' \simeq \langle \beta_1, \beta_2, \dots, \beta_m \rangle$ one has the sum and the product

$$Q \perp Q' \simeq \langle \alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_m \rangle \quad \text{and} \quad Q \otimes Q' \simeq \perp_{i=1}^n \langle \alpha_i \beta_1, \dots, \alpha_i \beta_m \rangle.$$

Note that only the product is well defined up to similarity.

Example. Let $Q = \langle 1 \rangle$ and $Q' = \langle -1 \rangle$. We have $Q \sim Q'$ (clearly $Q \not\simeq Q'$). Then $Q \perp Q \not\sim Q \perp Q'$ since $Q \perp Q$ is anisotropic while $Q \perp Q'$ is isotropic.

In general, the sum or the product of two anisotropic forms is not anisotropic as the following examples show. Consider $Q \simeq \langle 1, 1 \rangle$ and $Q' \simeq \langle -1, 2 \rangle$. They are both anisotropic, but their sum $\langle 1, 1, -1, 2 \rangle$ and their product $\langle -1, 2, -1, 2 \rangle$ are not. A result known as the Witt Decomposition Theorem says that any form ϕ is isometric to

$$\underbrace{\langle 1, -1 \rangle \perp \dots \perp \langle 1, -1 \rangle}_{i} \perp \phi_o,$$

where ϕ_o is anisotropic and uniquely determined by ϕ up to isometry. In the examples above, $(Q \perp Q')_o$ is isometric to $\langle 1, 2 \rangle$, while $(Q \otimes Q')_o$ is isometric to the empty quadratic form. We call the association $\phi \mapsto \phi_o$ the *reduction* of ϕ.

The set of isometry classes of anisotropic forms $W(\mathbb{Q})$ equipped with the two operations $\perp$ and $\otimes$ is known as the *Witt ring* where the appropriate reductions are applied. We denote the set of similarity classes of anisotropic forms by $\overline{W(\mathbb{Q})}$. Thus $(\overline{W(\mathbb{Q})}, \otimes)$ becomes a semigroup with unity $\langle 1 \rangle$, the unique 1-dimensional similarity class. Moreover, we have a natural semigroup epimorphism

$$W(\mathbb{Q}) \longrightarrow \overline{W(\mathbb{Q})}.$$

3. Main results

Let k be a positive integer. Let $M = (\mathbb{L}(X \oplus W \oplus Y \oplus Z), \partial = \partial_2)$ be a Quillen model, where X, W, Y, and Z are graded vector spaces such that

- X is of dimension $n > 0$ and is concentrated in degree $2k - 1$;
- W is nontrivial in at most degrees $2k$ through $4k - 2$;
- Y is of dimension m and is concentrated in degree $4k - 1$; and
- Z is nontrivial in at most degrees $4k$ through $6k - 2$.

In other words, M is the Quillen model of a $(2k - 1)$-connected CW-complex of dimension at most $6k - 1$, with n cells of dimension $2k$ and m cells of dimension $4k$. For degree reasons $\partial(X \oplus W) = 0$, while $\partial(Y)$ corresponds to m choices of n-dimensional quadratic forms on X. Indeed, given bases $\{x_1, \dots, x_n\}$ and $\{y_1, \dots, y_m\}$ for X and Y, respectively, $\partial(y_i)$ is a rational homogeneous polynomial of degree 2 in the variables x_j, i.e.,

$$\partial(y_i) = \sum_{j=1}^n \alpha_j [x_j, x_j] + \sum_{l<j} \beta_{l,j} [x_l, x_j] \quad \text{with} \quad \alpha_j, \beta_{l,j} \in \mathbb{Q}.$$

Notice that $[x_l, x_j] = [x_j, x_l]$ since the degree of the x_j is odd.

Moreover, $\partial(Z) \subset \mathbb{L}^2(X \oplus W)$ with at least one nontrivial $w \in W$ in each bracket. Note that if $Z_{6k-2} = 0$, then the formality condition is automatically satisfied, i.e., $\partial = \partial_2$. Clearly, from the inclusion and the projection we get

$$M \tilde{c} (\mathbb{L}(X \oplus Y), \partial = \partial_2).$$

Moreover, the following relations hold among these models.

Proposition 3.

$$\mathcal{M}(S^{2k}) \cong (\mathbb{L}(x), 0) \ll (\mathbb{L}(X \oplus Y), \partial = \partial_2) \ll (\mathbb{L}(x, y), \partial(y) = [x, x]) \cong \mathcal{M}(X_k).$$

Proof. The only thing to show is the right-hand building relation. Choose bases $\{x_1, \dots, x_n\}$, and $\{y_1, \dots, y_m\}$ for X and Y, respectively. Denote by Q_j the j^{th} quadratic form $\partial(y_j)$ on X $(1 \leq j \leq m)$. Define a Lie algebra map

$$\phi : (\mathbb{L}(X \oplus Y), \partial = \partial_2) \to (\mathbb{L}(x, y), \partial(y) = [x, x])$$

by sending x_i to x $(1 \leq i \leq n)$ and y_j to $Q_j(v)y$, where v is the n-tuple $(1, \dots, 1)$. A simple computation shows that ϕ is a dgL-morphism that induces a homology surjection in degree $2k - 1$. □

As a first step, we consider spaces having $M_n = (\mathbb{L}(X \oplus Y), \partial = \partial_2)$ as Lie models where X and $Y = \mathbb{Q}y$ are respectively n- and 1-dimensional. We will denote by Q_M the quadratic form ∂y on X. Thus we can speak of an isotropic or anisotropic model M_n if its associated quadratic form has the respective characteristic. We denote by $\mathbb{M}$ the set of closed classes generated by the anisotropic models. The fact that the set of quadratic forms is partitioned in two, i.e., the isotropic forms versus the anisotropic ones, together with the following proposition and the initial remarks of this section show that this definition is equivalent to the one given in the introduction.

Proposition 4. *A model M_n is isotropic if and only if $M_n \tilde{c} S^{2k}$.*

Proof. Let $M_n = (\mathbb{L}(x_1, ..., x_n, y), \partial y = Q_M)$. If Q_M is isotropic, then there is a nontrivial $V = (v_1, ..., v_n)$ such that $Q_M(V) = 0$. But then, it is easy to check that the map $\phi : M_n \to (\mathbb{L}(\omega), 0)$ defined by $\phi(x_i) = v_i \omega$ for $1 \leq i \leq n$, and $\phi(y) = 0$ is a dgL-morphism that is an H_{2k-1}-surjection. Conversely, if $M_n \tilde{c} S^{2k}$, then there is a dgL-morphism $\phi : M_n \to (\mathbb{L}(\omega), 0)$ that is an H_{2k-1}-surjection. Let V be the $(1 \times n)$-matrix associated to the linear map ϕ_{2k-1}. Again, the requirement that ϕ be a dgL-morphism implies that $Q_M(V) = 0$. □

Lemma 5. *If a morphism $\phi : M_n \to M'_m$ between anisotropic models is such that $n \geq m$ is nontrivial, then it is an H_{2k-1}-surjection. Moreover, there are no nontrivial maps when $n < m$.*

Proof. Let $M_n = (\mathbb{L}(x_1, ..., x_n, y), \partial y = Q_M)$ and $M'_m = (\mathbb{L}(x'_1, ..., x'_m, y'), \partial' y' = Q_{M'})$. Let C be the $(m \times n)$-matrix representing the linear map ϕ_{2k-1} with respect to the bases $\{x_1, ..., x_n\}$ and $\{x'_1, ..., x'_m\}$, and let $\alpha \in \mathbb{Q}$ be the scalar such that $\phi(y) = \alpha \cdot y'$. A simple but tedious computation shows that $\alpha \cdot A' = CAC^{\dagger}$, where A and A' are the matrix representations of Q_M and $Q_{M'}$, respectively. If $\alpha = 0$, then each row vector of C is a zero of Q_M, and since ϕ is not trivial, there must be one such row that is nonzero. This would contradict the fact that Q_M is anisotropic, and hence $\alpha \neq 0$. Now, if ϕ_{2k-1} is not onto, then $rk(C) < m$, i.e., there is a nontrivial solution to the homogeneous system $vC = 0$. The result follows since nontrivial solutions would imply $Q_{M'}$ isotropic, a contradiction. Finally, if $n < m$, then there are no nontrivial maps since the homogeneous system $vC = 0$ always has a nontrivial solution. □

Proposition 6. *Two anisotropic models M_n and M'_n generate the same cellular class, i.e., $M_n \tilde{c} M'_n$, if and only if their associated quadratic forms are similar, i.e., $Q_M \sim Q_{M'}$.*

Proof. Let $M_n = (\mathbb{L}(x_1, ..., x_n, y), \partial y = Q_M)$ and $M'_n = (\mathbb{L}(x_1, ..., x_n, y), \partial' y = Q_{M'})$. Let A and A' be the matrix representations of Q_M and $Q_{M'}$, respectively. If $M_n \tilde{c} M'_n$, then, by Lemma 5 there is a dgL-morphism $\phi : M_n \to M'_n$ that is an H_{2k-1}-surjection. Let C be the matrix representing the linear map ϕ_{2k-1} with respect to the basis $\{x_1, ..., x_n\}$, and let α be the nonzero rational such that $\phi(y) = \alpha \cdot y$. But then, C is invertible, and $\alpha \cdot A' = CAC^\dagger$, i.e., $Q_M \sim Q_{M'}$. Conversely, if $Q_M \sim Q_{M'}$, then there is an invertible matrix $C = (c_{ij})$ and a nonzero rational α such that $\alpha \cdot A' = CAC^\dagger$. Define a map $\phi : M_n \to M'_n$ by $\phi(x_i) = \sum_{j=1}^n c_{ij} x_j$ for $1 \le i \le n$, and $\phi(y) = \alpha \cdot y$. Clearly it is an H_{2k-1}-surjection, and a routine verification shows that ϕ is a dgL-morphism, and thus $M_n \ll M'_n$. Since C is invertible and $\alpha \neq 0$, one can construct from C^{-1} and α^{-1} an H_{2k-1}-surjection in the other direction using the same scheme, i.e., $M'_n \ll M_n$. □

Clearly, from the proof of Proposition 6, we deduce that $\mathbb{M}$ is also classified by the set of isomorphism classes of anisotropic models, i.e., by the set of homotopy types of spaces having anisotropic models as Lie models.

Theorem 7. *There is a natural semigroup isomorphism between $\overline{W(\mathbb{Q})}$ and $\mathbb{M}$.*

Proof. The only thing left to do is to define the multiplication on $\mathbb{M}$. Because of Proposition 6, the following binary operation is well defined on $\mathbb{M}$. Given two anisotropic models $M_n = (\mathbb{L}(X \oplus \mathbb{Q}y), \partial y = Q_M)$ and $M'_m = (\mathbb{L}(X' \oplus \mathbb{Q}y), \partial y = Q_{M'})$ one defines the operation $M_n \otimes_o M'_m$ by

$$M_n \otimes_o M'_m = (\mathbb{L}((X \otimes X')_o \oplus \mathbb{Q}y), \partial y = (Q_M \otimes Q_{M'})_o).$$

Note that we have to apply the appropriate reduction to the model $M_n \otimes M'_m$ since it is not in general anisotropic. But the reduction process is well defined because of the Witt Decomposition Theorem. The dimension of the rational vector space $(X \otimes X')_o$ is, in general, less than the dimension of $X \otimes X'$ and corresponds to the dimension of the reduced quadratic form $(Q_M \otimes Q_{M'})_o$. Finally, the Quillen model of $X_k = S^{2k} \cup_{[\iota,\iota]} e^{4k}$, i.e., $(\mathbb{L}(x, y), \partial y = [x, x])$, for which the corresponding quadratic form is $\langle 1 \rangle$, clearly acts as the unity with respect to this product. □

Figure 1 summarizes the results above.

4. Examples

Example 1. (Unreduced tensor product) Let $M_n = (\mathbb{L}(x_1, ..., x_n, y), \partial y = Q_M)$ and $M'_m = (\mathbb{L}(x'_1, ..., x'_m, y'), \partial' y' = Q_{M'})$ be two anisotropic models. Then

$$M_n \otimes M'_m \ll M_n \quad \text{and} \quad M'_m.$$

Since the quadratic forms are (regular) anisotropic, there are diagonal representations $\langle 1, \alpha_2, \dots, \alpha_n \rangle$ and $\langle 1, \beta_2, \dots, \beta_m \rangle$ of Q_M and $Q_{M'}$, respectively, up to similarity. Now

$$Q_M \otimes Q_{M'} \sim \langle 1, \beta_2, \dots, \beta_m, \alpha_2, \alpha_2\beta_2, \dots, \alpha_2\beta_m, \dots\dots, \alpha_n, \alpha_n\beta_2, \dots, \alpha_n\beta_m \rangle.$$

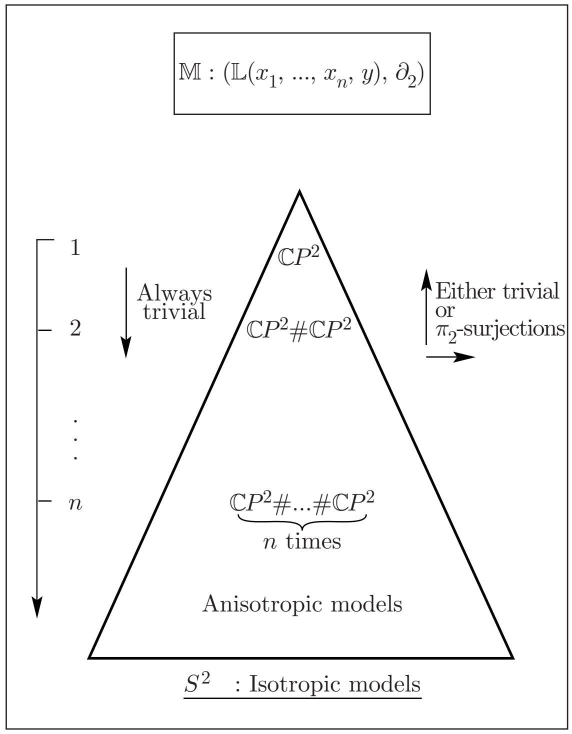

FIGURE 1.

Clearly the required H_{2k-1}-surjections exist, and the statement follows. Moreover, we have

$$M_n \otimes M'_m \tilde{c} M'_m \otimes M_n,$$

since $Q_M \otimes Q_{M'} \sim Q_{M'} \otimes Q_M$ (a simple permutation does the trick). Thus $\overline{W(\mathbb{Q})}$ and $\mathbb{M}$ are abelian semigroups.

Note that we did not apply the reduction to the resulting model. In general there is no building relationship between the two original models and their reduced product. Consider the two anisotropic models $M_3 = (\mathbb{L}(x_1, x_2, x_3, y), \partial y = [x_1, x_1] + [x_2, x_2] + [x_3, x_3])$ and $M'_3 = (\mathbb{L}(x_1, x_2, x_3, y), \partial y = [x_1, x_1] + [x_2, x_2] - 3[x_3, x_3])$. Then the model of $M_3 \otimes M'_3$ has nine generators in degree $2k-1$ together with

$$\partial y \sim \langle 1, 1, 1, 1, 1, 1, -3, -3, -3 \rangle,$$

which is isotropic. The reduced product is

$$(\mathbb{L}(x_1, x_2, x_3, y), \partial y = 2[x_1, x_1] + 2[x_2, x_2] + 3[x_3, x_3]).$$

Clearly no building relations exist among M_3, M'_3, and their reduced product.

Example 2 (The Félix example)**.** Félix showed by an argument involving the rational cohomology algebra that

$$S^2 \ll \ldots \ll \underbrace{\mathbb{C}P^2 \# \ldots \# \mathbb{C}P^2}_{n \text{ times}} \ll \ldots \ll \mathbb{C}P^2 \# \mathbb{C}P^2 \ll \mathbb{C}P^2.$$

We can also obtain his result using Lie models. Consider

$$\mathcal{M}(\underbrace{\mathbb{C}P^2\#\dots\#\mathbb{C}P^2}_{n\text{ times}}) \cong (\mathbb{L}(x_1,\dots,x_n,y),\partial)$$

where $\partial(y)=[x_1,x_1]+\cdots+[x_n,x_n]$, $\partial(x_i)=0$, $|x_i|=1$, and $|y|=3$ for $1\le i\le n$. The projections give the required H_1-surjections, while Lemma 5 makes it strict. Notice that the associated quadratic form is anisotropic.

Example 3. As we have seen in section 2, the sum $\perp$ is not well defined up to similarity. Thus one cannot define such an operation on the isomorphism classes of models of the type M_n as we did for $\otimes$. Geometrically, this reflects the fact that one has to choose an orientation before making the connected sum. Clearly, homotopy invariants are blind to this choice. But given two explicit models $M_n = (\mathbb{L}(x_1,...,x_n,y),\partial y = Q_M)$ and $M'_m = (\mathbb{L}(x'_1,...,x'_m,y'),\partial' y' = Q_{M'})$, one can define the (formal) connected sum as

$$M_n\#M'_m = (\mathbb{L}(x_1,...,x_n,x'_1,...,x'_m,y),\partial y = Q_M\perp Q_{M'}).$$

This operation makes sense only on the (entire) set of models of the type M_n. Moreover, without lost of generality, one can assume that both Q_M and $Q_{M'}$ are diagonal since $\perp$ is well defined on the isometry classes. It is now clear that

$$M_n\#M'_m \ll M_n \quad\text{and}\quad M'_m.$$

There are no simple building relations between the two operations $\#$ and $\otimes$ as the following examples show. Let $M_1=(\mathbb{L}(x,y),\partial y=[x,x])$. Then

$$M_1\#M_1 \ll M_1\otimes M_1 \cong M_1,$$

which is strict. On the other hand, we have

$$(M_1\#M_1\#M_1)\otimes(M_1\#M_1) \ll (M_1\#M_1\#M_1)\#(M_1\#M_1),$$

which is strict. Moreover, if $M_2=(\mathbb{L}(x_1,x_2,y),\partial y=[x_1,x_1]+2[x_2,x_2])$, then the two models

$$(M_1\#M_1)\#M_2 \quad\text{and}\quad (M_1\#M_1)\otimes M_2$$

are not even comparable.

Again there is no building compatibility between the original models and their reduced sum. Consider the two anisotropic models $M'_2=(\mathbb{L}(x_1,x_2,y),\partial y = [x_1,x_1]+[x_2,x_2])$ and $M''_2=(\mathbb{L}(x_1,x_2,y),\partial y=-[x_1,x_1]+2[x_2,x_2])$. Then

$$M'_2\#M''_2=(\mathbb{L}(x_1,x_2,x_3,x_4,y),\partial y=[x_1,x_1]+[x_2,x_2]-[x_3,x_3]+2[x_4,x_4]),$$

which is isotropic. The reduced sum is

$$(\mathbb{L}(x_1,x_2,y),\partial y=[x_1,x_1]+2[x_2,x_2]).$$

Clearly no building relations exist among M'_2, M''_2, and their reduced sum.

5. Final remarks and open problems

5.1. **Minimal number of generators.** We have seen that one of the invariants that classify $\mathbb{M}$ is the dimension of the quadratic form. Given two anisotropic models $M_n = (\mathbb{L}(x_1, ..., x_n, y), \partial y = Q_M)$ and $M'_m = (\mathbb{L}(x'_1, ..., x'_m, y'), \partial' y' = Q_{M'})$ such that neither builds the other, one could ask what is the smallest number of w_i in a model of the type $M''_p = (\mathbb{L}(w_1, ..., w_p, y), \partial y = Q_{M''})$ that builds M_n and M'_m. We have seen that $p = n \cdot m$ for $M_n \otimes M'_m$, while $p = n + m$ for $M_n \# M'_m$. One could do it with one less, i.e., if $Q_M = \langle 1, \alpha_2, \ldots , \alpha_n\rangle$ and $Q_{M'} = \langle 1, \beta_2, \ldots , \beta_m\rangle$, then we can set $p = n + m - 1$ together with

$$Q_{M''} = \langle 1, \alpha_2, \ldots , \alpha_n, \beta_2, \ldots , \beta_m\rangle.$$

It is still unknown if one could do better in general. Note that both $M_n \otimes M'_m$ and $M_n \# M'_m$ build M''_p.

5.2. **Geometric interpretation of $M_n \otimes M'_m$ (unreduced).** If one allows for a shift in the degrees of the generators, then the smash product $M_n \wedge M'_m$ has all the required characteristics of $M_n \otimes M'_m$. Let

$$M_n = (\mathbb{L}(x_1, ..., x_n, y), \partial y = Q_M) \text{ and } M'_m = (\mathbb{L}(x'_1, ..., x'_m, y'), \partial' y' = Q_{M'})$$

be two models. Recall that the smash product is the homotopy cofibre of the following inclusion:

$$M_n \vee M'_m \hookrightarrow M_n \times M'_m.$$

Let $\partial y = \sum_{i=1}^n \alpha_i[x_i, x_i]$ and $\partial' y' = \sum_{j=1}^m \beta_j[x'_j, x'_j]$ be two diagonal representations of Q_M and $Q_{M'}$, respectively. A model for the product is given by (see [18])

$$(\mathbb{L}(x_i, x'_j, c_{ij}, y, y', d_i, e_j, z), \delta),$$

where $1 \le i \le n$, $1 \le j \le m$, $|c_{ij}| = 4k - 1$, $|d_i| = |e_j| = 6k - 1$ and $|z| = 8k - 1$. Moreover, after a simple computation ensuring that $\delta^2 = 0$ we get

1. $\delta x_i = \delta x'_j = 0$;
2. $\delta y = Q_M$, and $\delta y' = Q_{M'}$;
3. $\delta c_{ij} = [x_i, x'_j]$;
4. $\delta d_i = [x_i, y'] - 2\sum_{j=1}^m \beta_j[x'_j, c_{ij}]$;
5. $\delta e_j = [y, x'_j] + 2\sum_{i=1}^n \alpha_i[c_{ij}, x_i]$; and
6.

$$\begin{aligned}\delta z = [y, y'] \quad &- \quad 2\left(\sum_{i=1}^n \alpha_i[d_i, x_i] + \sum_{j=1}^m \beta_j[x'_j, e_j]\right) \\ &+ \quad 2\sum_{i=1}^n\sum_{j=1}^m \alpha_i\beta_j[c_{ij}, c_{ij}].\end{aligned}$$

Thus $M_n \wedge M'_m$ has as Lie model $(\mathbb{L}(c_{ij}, d_i, e_j, z), \overline{\delta})$, where

$$\overline{\delta} z = 2\sum_{i=1}^n\sum_{j=1}^m \alpha_i\beta_j[c_{ij}, c_{ij}]$$

is the only nontrivial differential. Clearly this quadratic form is similar to $Q_M \otimes Q_{M'}$. From the initial remarks of section 3, we obviously have

$$M_n \wedge M'_m \tilde{c} (\mathbb{L}(c_{ij}, z), \overline{\delta}),$$

i.e., we fall back on a model of $M_n \otimes M'_m$ modulo the aforementioned shift in the degrees of the generators.

Finally, notice that the fact that $Q_M \otimes Q_{M'}$ is well defined up to similarity shows that $M_n \wedge M'_m$ is independent of the chosen homotopy representative for M_n and M'_m, respectively, a fact that does not hold for the connected sum.

5.3. **Models** $(\mathbb{L}(X \oplus Y), \partial = \partial_2)$ **with** $dim(Y) = n > 1$**.** This case, in which each model can be identified with a system of n quadratic forms, is far from being classified. As mentioned in [14], Chapter 9, there is no natural way of defining a Witt group on such systems. Nonetheless, one can define the notion of an isotropic system if that system has a common nontrivial zero. Proposition 4 generalizes in this context, but Lemma 5 has no such generalization, as the following examples show. Let $M_1 = (\mathbb{L}(x, y), \partial y = [x, x])$, $M_2 = (\mathbb{L}(x_1, x_2, y), \partial y = [x_1, x_1] + [x_2, x_2])$, and $M'_2 = (\mathbb{L}(x_1, x_2, y), \partial y = [x_1, x_1] + 2[x_2, x_2])$, and consider the spaces

$$M_1 \vee M_1 \cong (\mathbb{L}(x_1, x_2, y_1, y_2), \partial y_1 = [x_1, x_1], \partial y_2 = [x_2, x_2]),$$

and

$$\begin{aligned} M_2 \vee M'_2 \cong (\mathbb{L}(x_1, x_2, x_3, x_4, y_1, y_2), &\partial y_1 = [x_1, x_1] + [x_2, x_2], \\ &\partial y_2 = [x_3, x_3] + 2[x_4, x_4]). \end{aligned}$$

Clearly the inclusion $M_1 \hookrightarrow M_1 \vee M_1$ into either the left or right component is nontrivial, while the projection of $M_2 \vee M'_2$ onto either M_2 or M'_2 is also nontrivial.

Finally, all is not lost, since there is an obvious action of $\mathbb{M}$ on the set of such models. Namely, given two models $M = (\mathbb{L}(X \oplus \mathbb{Q}y), \partial y = Q)$ and $N = (\mathbb{L}(X' \oplus \mathbb{Q}y_1 \oplus \ldots \oplus \mathbb{Q}y_n), \partial y_1 = Q_1, \ldots, \partial y_n = Q_n)$ we have

$$M \cdot N = (\mathbb{L}(X \otimes X' \oplus \mathbb{Q}y_1 \oplus \ldots \oplus \mathbb{Q}y_n), \partial y_1 = Q \otimes Q_1, \ldots, \partial y_n = Q \otimes Q_n).$$

Note that we cannot hope to diagonalize simultaneously a system of quadratic forms. Thus the product is the Kronecker product of two matrices.

6. Appendix : Classification of similarity classes

We have to distinguish the odd-dimensional case from the even-dimensional one. The odd case is easily disposed of.

Proposition 8. *Let Q_1 and Q_2 be two $(2k+1)$-dimensional regular quadratic forms. Then*

$$Q_1 \sim Q_2 \quad \textit{if and only if} \quad (det(Q_1))Q_1 \simeq (det(Q_2))Q_2.$$

Proof. Suppose we have $r \cdot Q_1 \simeq Q_2$, i.e., there exists an invertible matrix C such that $r \cdot Q_1 = CQ_2C^{\dagger}$. Applying the determinant to each side yields

$$r^{(2k+1)} \cdot det(Q_1) = (det(C))^2 \cdot det(Q_2).$$

Thus $r \equiv (det(Q_1) \cdot det(Q_2))$ modulo $(\mathbb{Q}^*)^2$. The result follows. ∎

The similarity classes of even-dimensional quadratic forms have only recently been classified, by Lewis and Tignol.

Proposition 9 ([13]). *Let Q_1 and Q_2 be two even-dimensional regular quadratic forms. Then $Q_1 \sim Q_2$ if and only if*

- $dim(Q_1) = dim(Q_2)$;
- $det(Q_1) \equiv det(Q_2)$ *modulo* $(\mathbb{Q}^*)^2$;
- $C_o(Q_1) \cong C_o(Q_2)$ *as algebras over* $\mathbb{Q}$; *and*
- $|sign(Q_1)| = |sign(Q_2)|$.

Recall that $C_o(Q)$ refers to the even Clifford algebra associated to Q (see [16]).

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