The Non-linear Stochastic Wave Equation in High Dimensions: Existence, Hölder-continuity and Itô-Taylor Expansion

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Résumé

Le sujet principal de cette thèse est l'étude de l'équation des ondes stochastique non-linéaire, perturbée par un bruit Gaussien spatialement homogène et blanc en temps, dans le cas où la dimension de l'espace est strictement plus grande que 3. Nous nous intéressons en particulier aux questions d'existence et d'unicité de solutions, de même qu'aux propriétés de ces solutions, comme l'existence de moments d'ordre élevé et la continuité hölderienne.

L'équation des ondes stochastique est formulée sous forme intégrale. Dans celle-ci apparaissent des intégrales stochastiques par rapport à des mesures martingales (au sens de J.B. Walsh). Comme, en dimension strictement plus grande que 3, la solution fondamentale de l'équation des ondes n'est ni une fonction, ni une mesure positive, mais une distribution de Schwartz au sens large, nous développons tout d'abord une extension de l'intégrale stochastique de Dalang-Walsh qui permet d'intégrer une large classe de distributions de Schwartz. Sous une hypothèse déjà utilisée dans la littérature sur la mesure spectrale du bruit, cette classe contient la solution fondamentale de l'équation des ondes.

A l'aide de cette extension de l'intégrale stochastique, nous pouvons établir l'existence d'un champ aléatoire de carré intégrable, solution de l'équation des ondes stochastique non-linéaire, quelle que soit la dimension de l'espace. Cette solution est unique au sein d'une classe restreinte de processus.

Dans le cas d'un bruit multiplicatif affine, nous obtenons une représentation en série de la solution et des estimations sur les moments d'ordre p ($p \geqslant 1$). A partir de là, sous des hypothèses standard, nous pouvons déduire la continuité hölderienne de la solution. L'ordre de continuité que nous obtenons est optimal.

Dans le cas d'un bruit multiplicatif au sens large, nous mettons en place un cadre permettant de travailler avec des intégrales stochastiques itérées appropriées, afin de déduire un développement limité d'Itô-Taylor pour la solution de l'équation des ondes stochastique. La convergence de ce développement est un problème ouvert. C'est pourquoi nous terminons par quelques remarques qui donnent à penser qu'il peut être possible d'obtenir une série d'Itô-Taylor pour la solution.

Mots-clés : mesure martingale, intégration stochastique, équation des ondes stochastique, équation aux dérivées partielles stochastique, expression pour les moments, continuité hölderienne, intégrales stochastiques itérées, développement d'Itô-Taylor.

Abstract

The main topic of this thesis is the study of the non-linear stochastic wave equation in spatial dimension greater than 3 driven by spatially homogeneous Gaussian noise that is white in time. We are interested in questions of existence and uniqueness of solutions, as well as in properties of solutions, such as existence of high order moments and Hölder-continuity properties.

The stochastic wave equation is formulated as an integral equation in which appear stochastic integrals with respect to martingale measures (in the sense of J.B. Walsh). Since, in dimensions greater than 3, the fundamental solution of the wave equation is neither a function nor a non-negative measure, but a general Schwartz distribution, we first develop an extension of the Dalang-Walsh stochastic integral that makes it possible to integrate a wide class of Schwartz distributions. This class contains the fundamental solution of the wave equation, under a hypothesis on the spectral measure of the noise that has already been used in the literature.

With this extended stochastic integral, we establish existence of a square-integrable random-field solution to the non-linear stochastic wave equation in any dimension. Uniqueness of the solution is established within a specific class of processes.

In the case of affine multiplicative noise, we obtain a series representation of the solution and estimates on the p-th moments of the solution ($p \geqslant 1$). From this, we deduce Hölder-continuity of the solution under standard assumptions. The Hölder exponent that we obtain is optimal.

For the case of general multiplicative noise, we construct a framework for working with appropriate iterated stochastic integrals and then derive a truncated Itô-Taylor expansion for the solution of the stochastic wave equation. The convergence of this expansion remains an open problem, so we conclude with some remarks that suggest an Itô-Taylor series expansion for the solution.

Keywords : martingale measures, stochastic integration, stochastic wave equation, stochastic partial differential equations, moment formulae, Hölder continuity, iterated stochastic integrals, Itô-Taylor expansion.

Contents

Chapter 1

Introduction

The main topic of this thesis is the study of the non-linear stochastic wave equation. In particular, we are going to study existence and uniqueness of a random field solution in the case where the spatial dimension is $d \geqslant 4$. This question is a challenging problem in the sense that the techniques developped previously in the literature do not apply to this situation. Then, we will address the question of the Hölder-continuity of this solution and develop an Itô-Taylor expansion for it, with the long-time objective of obtaining a better understanding of several properties of the solution.

We begin by recalling some facts about the wave equation. The wave equation arises from Newton's classical equation of mechanics when we model the displacement of a physical entity (a solid or an incompressible fluid) subject to internal and external forces. This entity can be a piece of wire held at its endpoints or a DNA molecule moving in a fluid, for example. In these cases, the space variable is 1-dimensional. A 2-dimensional variable is needed when, we consider the membrane of a drum. A 3-dimensional variable is needed for example when modelling a pressure wave in water. The higher dimensional case is more abstract. It is natural to study this extension as the mathematical generalisation of the preceding examples, in particular because the development of new mathematical tools is needed. As a motivation for the study of problems with spatial dimension greater than three, we mention that some problems of quantum mechanics, unrelated with the wave equation, for example, need higher dimensional spaces.

Newton's equation states that the acceleration of a point of the entity is proportional to the forces acting on the entity at this point. If we denote by $u(t,x)$ the position at time t of the point x of the entity, the acceleration at this point is given by $\frac{\partial^2}{\partial t^2}u(t,x)$. Physical arguments show that the internal forces are given by $\Delta u(t,x)$, where Δ is the Laplacian operator. Hence, if we only consider internal forces, Newton's equation states that u must satisfy the homogeneous wave equation

$$\frac{\partial^2}{\partial t^2} u(t,x) - \Delta u(t,x) = 0, \qquad t \geqslant 0,\ x \in E$$

with an initial condition that gives the starting position of the entity and some boundary conditions corresponding to the problem and where E is the space for the spatial variable, namely a subset of $\mathbb{R}^d$. We generally solve this equation using a Fourier transform or Fourier series.

Now, if we consider some additional external forces $w(t,x)$ acting on the entity at time t and position x, Newton's equation becomes a non-homogeneous wave equation, namely

$$\frac{\partial^2}{\partial t^2} u(t,x) - \Delta u(t,x) = w(t,x), \qquad t \geqslant 0,\ x \in E. \tag{1.1}$$

A classical technique to solve (1.1) is to first solve the corresponding equation with $w(t,x)$ replaced by a mass point at $t = 0$, $x = 0$, $w(t,x) = \delta_0(t,x)$, where δ_0 stands for Dirac delta measure. This solution is called the *fundamental solution* or *Green kernel* of (1.1) and is denoted by $\Gamma(t,x)$. The fundamental solution models the motion of the entity if we give a single impulsion at position $x = 0$ at time $t = 0$. Letting

$$u(t,x) = \int_0^t ds \int_E dy\, \Gamma(t-s, x-y) w(s,y) \tag{1.2}$$

be the convolution of Γ with w, then the function u is a solution of (1.1) with vanishing initial conditions. The fundamental solution in the case of the wave equation is given in Section 4.2. We refer to [13] or [28] for more information about solutions to deterministic partial differential equations and, in particular, to the wave equation.

In our case, we would like to include some randomness in the model. A first step is to consider a random w. This corresponds to the case where the external forces are random. We can take for example the membrane of a drum with rain falling on it or the DNA molecule, for which we cannot precisely model all external forces and, hence, consider these as random. For example, we can consider $w(t,x) = \dot{W}(t,x)$, a white noise. In this case, Newton's equation becomes a linear stochastic partial differential equation (s.p.d.e.), namely

$$\frac{\partial^2}{\partial t^2} u(t,x) - \Delta u(t,x) = \dot{W}(t,x), \qquad t \geqslant 0,\ x \in E. \tag{1.3}$$

Using the same technique as for (1.1), one way to solve (1.3) with vanishing initial conditions is to set

$$u(t,x) = \int_0^t \int_E \Gamma(t-s, x-y) dW(s,y). \tag{1.4}$$

We must then give a precise meaning to the stochastic integral.

In this thesis, we go a step further and consider random (and deterministic) external forces that depend on the state of the entity at time t and position x. Namely, we are interested in random field solutions to the stochastic wave equation

$$\frac{\partial^2}{\partial t^2}u(t,x) - \Delta u(t,x) = \alpha(u(t,x))\dot{F}(t,x) + \beta(u(t,x)), \qquad t > 0,\ x \in \mathbb{R}^d, \quad (1.5)$$

with vanishing initial conditions. In this equation, $d \geqslant 1$, Δ denotes the Laplacian on $\mathbb{R}^d$, the functions $\alpha, \beta : \mathbb{R} \to \mathbb{R}$ are Lipschitz continuous and $\dot{F}$ is a spatially homogeneous Gaussian noise that is white in time. Informally, the covariance functional of $\dot{F}$ is given by

$$\mathbb{E}[\dot{F}(t,x)\dot{F}(s,y)] = \delta_0(t-s)f(x-y), \qquad s,t \geqslant 0,\ x,y \in \mathbb{R}^d, \quad (1.6)$$

where δ_0 denotes the Dirac delta function and $f : \mathbb{R}^d \to \mathbb{R}_+$ is continuous on $\mathbb{R}^d \setminus \{0\}$ and even.

We recall that a random field solution to (1.5) is a family of random variables $(u(t,x),\ t \in \mathbb{R}_+,\ x \in \mathbb{R}^d)$ such that $(t,x) \mapsto u(t,x)$ from $\mathbb{R}_+ \times \mathbb{R}^d$ into $L^2(\Omega)$ is continuous and solves an integral form of (1.5) inspired from (1.2) and (1.4): see Chapter 4. Having a random field solution is interesting if, for instance, one wants to study the probability density function of the random variable $u(t,x)$ for each (t,x), as in [17] for the 2-dimensional case.

A different notion is the notion of function-valued solution, which is a process $t \mapsto u(t)$ with values in a space such as $L^2(\Omega, L^2_{\mathrm{loc}}(\mathbb{R}^d, dx))$. The necessary tools needed in order to find function-valued solutions for stochastic partial differential equations have been developped by Da Prato and Zabczyk (see for instance [10]). A framework for solutions for a class of stochastic partial differential equations that includes (1.5) has been developped in [7] by Dalang and Mueller. In some cases, such as in [9] by Dalang and Sanz-Solé, a random field solution can be obtained from a function-valued solution by establishing (Hölder) continuity properties of $(t,x) \mapsto u(t,x)$, but such results are not available for the stochastic wave equation in dimensions $d \geqslant 4$. In other cases, the two notions are genuinely distinct (since the latter would correspond to $(t,x) \mapsto u(t,x)$ from $\mathbb{R}_+ \times \mathbb{R}^d$ into $L^2(\Omega)$ is merely measurable), and one type of solution may exist but not the other. This is has been shown by Dalang and Lévêque (see [6]) where they consider stochastic partial differential equations with a noise concentrated on a subset of $\mathbb{R}^d$, namely a hyperplan. They also studied the case of a noise concentrated on a sphere in [5]. Function-valued solutions to (1.5) have been obtained in all dimensions by Peszat in [19]. Random field solutions have only been shown to exist when $d \in \{1, 2, 3\}$ (see [2]).

In spatial dimension 1, a solution to the non-linear wave equation driven by space-time white noise was given in [30], using Walsh's martingale measure stochastic integral. In dimensions 2 or higher, there is no function-valued solution with space-time white noise as a random input: some spatial correlation is needed in this case. This is why we consider for (1.5) a noise $\dot{F}$ that is white in time and correlated in space (see (1.6)). In spatial dimension 2, a necessary and sufficient condition on the spatial correlation for existence of a random field solution was given by Dalang and Frangos in [4]. Study of the probability law of the solution is carried out by Millet and Sanz-Solé in [17].

In spatial dimension $d = 3$, existence of a random field solution to (1.5) is given by Dalang in [2]. Since the fundamental solution in this dimension is not a function (see Section 4.2), this required an extension of Walsh's martingale measure stochastic integral to integrands that are (Schwartz) distributions. This extension has nice properties when the integrand is a non-negative measure, as is the case for the fundamental solution of the wave equation when $d = 3$. The solution constructed in [2] had moments of all orders but no spatial sample path regularity was established. Absolute continuity and smoothness of the probability law was studied by Quer-Sardanyons and Sanz-Solé in [21] and [22]. These results have also been obtained by Nualart and Quer-Sardanyons in [18] using another approach to define the stochastic integral. Hölder continuity of the solution was only recently established by Dalang and Sanz-Solé in [9], using Sobolev spaces and embeddings in order to prove that the solution belongs to some Hölder space. Sharp exponents were also obtained.

In spatial dimension $d \geqslant 4$, random field solutions were only known to exist in the case of the linear wave equation ($\alpha \equiv 1$, $\beta \equiv 0$). The methods used in dimension 3 do not apply to higher dimensions, because for $d \geqslant 4$, the fundamental solution of the wave equation is not a measure, but a Schwartz distribution that is a derivative of some order of a measure (see Section 4.2). It was therefore not even clear that the solution to (1.5) should be Hölder continuous, even though this has been established for the linear equation by Sanz-Solé and Sarrà ([26]) under natural assumptions on the covariance function f.

In Chapter 2, we first recall the construction of Walsh's stochastic integral with respect to martingale measures. Then we formally define the noise considered in (1.6) in order that it becomes a martingale measure. Finally we present Dalang's extension of the stochastic integral to non-negative Schwartz distributions that is used to treat the 3-dimensional case.

In Chapter 3, we then extend the construction of the stochastic integral presented in Chapter 2 so as to be able to define

$$\int_0^t \int_{\mathbb{R}^d} S(s,x)Z(s,x)M(ds,dx)$$

in the case where $M(ds,dx)$ is the martingale measure associated with the Gaussian noise $\dot F$, $Z(s,x)$ is an $L^2(\Omega,\mathcal{F},\mathbb{P})$-valued random field with a spatially homogeneous covariance function, and S is a Schwartz distribution, that is not necessarily non-negative (as it was in Chapter 2 and [2]). Among other technical conditions, S must satisfy the following condition, that also appears in [19] in order to obtain function-valued solutions:

$$\int_0^t ds \sup_{\eta\in\mathbb{R}^d} \int_{\mathbb{R}^d} \mu(d\xi)\, |\mathcal{F}S(s)(\xi+\eta)|^2 < \infty,$$

where μ is the spectral measure of $\dot F$ (that is, $\mathcal{F}\mu = f$, where $\mathcal{F}$ denotes the Fourier transform). We also present an extension of the Lebesgue integral in order to consider the additive non-linearity on the right-hand side of (1.5).

With this stochastic integral, we can establish (in Chapter 4) existence of a random field solution of a wide class of stochastic partial differential equations, that contains (1.5) as a special case, in all spatial dimensions d (see Section 4.2).

However, for $d \geqslant 4$, we do not know in general if this solution has moments of all orders. We recall that higher order moments, and, in particular, estimates on high order moments of increments of a process, are needed for instance to apply Kolmogorov's continuity theorem and to obtain Hölder continuity of sample paths of the solution.

The main issue of Chapter 5 is to consider the special case where α is an affine function and $\beta \equiv 0$. This is analogous to the hyperbolic Anderson problem considered by Dalang, Mueller and Tribe in [8] for $d \leqslant 3$. In Section 5.1, we show that, in this particular case, the solution to (1.5) has moments of all orders, by using a series representation of the solution in terms of iterated stochastic integrals of the type defined in Section 3.1. Then, in Section 5.2, we use the results of Section 5.1 to establish Hölder continuity of the solution to (1.5) (Propositions 5.6 and 5.7) for α affine and $\beta \equiv 0$. In the case where the covariance function is a Riesz kernel, we obtain the optimal Hölder exponent, which turns out to be the same as that obtained in [9] for dimension 3.

After that, inspired from the series expansion obtained in the case of an affine multiplicative noise in Chapter 5 and using similar techniques to those developed by Kloeden and Platen in [14] for stochastic differential equations driven by a Brownian motion, we present an Itô-Taylor type expansion for the solution $u(t,x)$ of (1.5). Such an expansion allows to write a function of $u(t,x)$ as a sum of iterated stochastic integrals of the type defined in Chapter 3, namely

$$g(u(t,x)) = \sum_{\beta\in\bar{\mathcal{C}}_n} \pi_\beta(g) I_\beta^{(0)}(t;t,x) + \sum_{\beta\in\bar{\mathcal{C}}_{(n+1)}} J_\beta(\kappa_\beta(g))(t;t,x), \tag{1.7}$$

where $\bar{\mathcal{C}}_n$ is a set of multi-indices β, $I_\beta^{(0)}$ is an iterated stochastic integral, J_β is stochastic integral operator, $\pi_\beta(g)$ are real constants and $\kappa_\beta(g)$ are functions depending on g and α. The Itô-Taylor terminology has been chosen because this expansion is a stochastic analogue of a deterministic Taylor expansion in which we use Itô's formula and stochastic integrals instead of the fundamental theorem of calculus and classical Lebesgue integrals.

In Chapter 6, we set the definitions and notations that are needed to state the Itô-Taylor expansion. We first present the multi-indices β that are used, then we define the iterated stochastic integrals $I_\beta^{(0)}$ and operators J_β. Some properties are then presented. Among these properties, the most important is certainly the fact that a product of iterated stochastic integrals can be written as a linear combinations of such integrals.

Finally, Chapter 7 is devoted to the Itô-Taylor expansion. In Section 7.1 we establish the proof of the truncated Itô-Taylor expansion (1.7) of order n (Theorem 7.2). Then, as the convergence to 0 of the remainder term in (1.7) is still an open problem, we will assume that the first term on the right-hand side of (1.7) converges and show that in this case, it should converge to $u(t, x)$ and, as a consequence, give an explicit expression for the solution of the non-linear stochastic wave equation (1.5).

The results of Chapters 3 to 5 are presented in [1].

Chapter 2

Framework

In this chapter, we present the main tools that we will use in this document. First, we recall the definition of a martingale measure and Walsh's theory on the stochastic integral with respect to such martingale measures ([30]). Secondly, we give the construction of the noise that we will consider for the stochastic partial differential equation that we study. Finally, we present a short reminder on the extension of Walsh's stochastic integral to more irregular integrands, namely non-negative Schwartz distributions. This was developed by Dalang in [2].

2.1 Walsh's martingale measure stochastic integral

The main objective of this section is to present the notion of a martingale measure and, then, to present a survey of the construction of the stochastic integral with respect to a martingale measure. The concept of martingale measure is central in this work. We refer to [30] for more information on martingale measures and the related theory of stochastic integration.

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geqslant 0}, \mathbb{P})$ be a filtered probability space with a right-continuous filtration. We consider $L^2(\Omega)$-valued set functions on $\mathbb{R}^d$ ($d \geqslant 1$), that is, maps from $\mathcal{B}(\mathbb{R}^d)$, the Borel subsets of $\mathbb{R}^d$, into $L^2(\Omega)$.

Definition 2.1. A process $(M_t(A), t \geqslant 0, A \in \mathcal{B}(\mathbb{R}^d))$ with values in $L^2(\Omega)$ is a *martingale measure*, if

(a) $M_0(A) = 0$, for all $A \in \mathcal{B}(\mathbb{R}^d)$;

(b) for any fixed $t > 0$, M_t is a σ-finite $L^2(\Omega)$-valued measure ;

(c) for any fixed $A \in \mathcal{B}(\mathbb{R}^d)$, $t \mapsto M_t(A)$ is a martingale.

In order to define the stochastic integral with respect to a martingale measure M, we need to define the covariance functional and the notion of worthiness of a martingale measure.

Definition 2.2. Let M be a martingale measure. The *covariance functional* of M is a set function Q_M defined on the collection $\{A \times B \times [0,t] : A, B \in \mathcal{B}(\mathbb{R}^d), t \geqslant 0\}$ of rectangles in $\mathcal{B}(\mathbb{R}^d) \times \mathcal{B}(\mathbb{R}^d) \times \mathcal{B}(\mathbb{R}_+)$ by

$$Q_M(A \times B \times [0,t]) = \langle M(A), M(B)\rangle_t,$$

where $\langle \cdot, \cdot \rangle$ denotes the usual quadratic variation for a continuous-time martingale.

Now, we would like to extend the set function Q_M to a measure on $\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}_+$. We are led to the following definitions.

Definition 2.3. A signed measure $K(dx, dy, ds)$ on $\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}_+$ is *positive-definite* if for each bounded measurable function f for which the integral makes sense,

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_0^\infty f(x,s) f(y,s) K(dx, dy, ds) \geqslant 0.$$

In this case, we write

$$\langle f, g \rangle_K = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_0^\infty f(x,s) g(y,s) K(dx, dy, ds).$$

Definition 2.4. A martingale measure M is *worthy* if there exists a random σ-finite measure K on $\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}_+$ such that

(a) K is positive-definite and symmetric in x and y ;

(b) for fixed $A, B \in \mathcal{B}(\mathbb{R}^d)$, $(K(A \times B \times [0,t]))_{t \geqslant 0}$ is a predictable process ;

(c) for all $n \in \mathbb{N}$ and all $T \geqslant 0$, $\mathbb{E}[K(E_n \times E_n \times [0,T])] < +\infty$, where $(E_n)_{n \in \mathbb{N}}$ is a possible decomposition of $\mathbb{R}^d$ in the definition of the σ-finiteness of K ;

(d) for any $A, B \in \mathcal{B}(\mathbb{R}^d)$ and any $t \geqslant 0$, $|Q_M(A \times B \times [0,t])| \leqslant K(A \times B \times [0,t])$.

We call K the *dominating measure* of M.

If M is a worthy martingale measure, then the covariation functional Q_M can be extended to a signed measure on $\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}_+$. We call it the *covariation measure* of M.

Let M be a worthy martingale maeasure with covariation measure Q_M and dominating measure K_M. We are first going to define a stochastic integral with respect to M for a class of elementary functions and then extend it to a larger space by a completion argument.

Definition 2.5. A (random) function $g : \mathbb{R}_+ \times \mathbb{R}^d \times \Omega \to \mathbb{R}$ is *elementary* if it is of the form

$$g(s, x, \omega) = \mathbf{1}_{]a,b]}(s)\mathbf{1}_A(x)X(\omega),$$

where $0 \leqslant a < b$, $A \in \mathcal{B}(\mathbb{R}^d)$ and X is a bounded, $\mathcal{F}_a$-measurable random variable.

Now, let $\mathcal{E}$ be the set of predictable *simple functions*, i.e., the set of all finite linear combinations of elementary functions. We define a norm $\|\cdot\|_+$ on $\mathcal{E}$ by

$$\begin{aligned} \|g\|_+^2 &= \mathbb{E}[\langle |g|, |g|\rangle_{K_M}] \\ &= \mathbb{E}\left[\int_0^T ds \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^d} dy\, |g(s,x,\cdot)|\, |g(s,y,\cdot)| K_M(dx,dy,ds)\right]. \end{aligned}$$

and set $\mathcal{P}_+ = \{g : \mathbb{R}_+ \times \mathbb{R}^d \times \Omega \to \mathbb{R} : g \text{ is predictable and } \|g\|_+ < +\infty\}$.

The space $\mathcal{P}_+$ is the space of integrands for the stochastic integral with respect to M. Theorem 2.6 will be the central tool to define the stochastic integral.

Theorem 2.6. ([30, Proposition 2.3])

(a) The space $\mathcal{P}_+$ is a Banach space.

(b) The set $\mathcal{E}$ of simple functions is dense in $\mathcal{P}_+$.

The proof of Theorem 2.6 in [30] is partially left as an exercise for the reader. Therefore, we are going to give the details of this proof. For this, we consider a measure λ on $\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}_+ \times \Omega$ defined by

$$\lambda(dx, dy, ds, d\omega) = K_M(dx, dy, ds; \omega)\mathbb{P}(d\omega).$$

Letting $T \geqslant 0$ and $(E_n)_{n\in\mathbb{N}}$ be a possible decomposition in the definition of the σ-finiteness of K, the measure λ is finite on $E_n \times E_n \times [0,T] \times \Omega$. With these notations, we obtain

$$\|g\|_+^2 = \int_{\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}_+\times\Omega} |g(x,s,\omega)||g(y,s,\omega)|\lambda(dx,dy,ds,d\omega).$$

Fix $n \in \mathbb{N}$ and $T > 0$. We first restrict ourselves to the set $E_n \times [0,T]$ and present the following lemma, which is a generalized Chebychev's inequality.

Lemma 2.7. *Let $g \in \mathcal{P}_+$ and $\varepsilon > 0$. Let $A = \{(x,s,\omega) \in E_n \times [0,T] \times \Omega : |g(x,s,\omega)| \geqslant \varepsilon\}$. Then, let $A_\omega \subset E_n \times [0,T]$ be defined by*

$$A_\omega = \{(x,s) \in E_n \times [0,T] : (x,s,\omega) \in A\}.$$

The set A_ω is the section of A along ω. Hence, $(x,s,\omega)\in A$ if and only if $(x,s)\in A_\omega$. Then,

$$\begin{aligned}\lambda(E_n\times A) &= \mathbb{E}[K_M(E_n\times A.)]\\ &\leqslant \frac{\|g\|_+}{\varepsilon}\mathbb{E}[K_M(E_n\times E_n\times[0,T])]^{\frac{1}{2}}\\ &= \frac{\|g\|_+}{\varepsilon}\lambda(E_n\times E_n\times[0,T]\times\Omega)^{\frac{1}{2}}.\end{aligned}$$

Proof. We have

$$\begin{aligned}\varepsilon\,\mathbb{E}[K_M(E_n\times A.)] &= \int_\Omega\int_{E_n\times A_\omega}\varepsilon K_M(dx,dy,ds;\omega)\mathbb{P}(d\omega)\\ &\leqslant \int_\Omega\int_{E_n\times A_\omega}|g(y,s,\omega)|K_M(dx,dy,ds;\omega)\mathbb{P}(d\omega)\\ &\leqslant \int_{E_n\times E_n\times[0,T]\times\Omega}|g(y,s,\omega)|K_M(dx,dy,ds;\omega)\mathbb{P}(d\omega)\\ &= \mathbb{E}\left[\langle|g|,1\rangle_{K_M}\right]\\ &\leqslant \mathbb{E}\left[\langle|g|,|g|\rangle_{K_M}^{\frac{1}{2}}\langle1,1\rangle_{K_M}^{\frac{1}{2}}\right]\\ &\leqslant \mathbb{E}\left[\langle|g|,|g|\rangle_{K_M}\right]^{\frac{1}{2}}\mathbb{E}\left[\langle1,1\rangle_{K_M}\right]^{\frac{1}{2}}\\ &= \|g\|_+\,\mathbb{E}\left[K_M(E_n\times E_n\times[0,T])\right]^{\frac{1}{2}},\end{aligned}$$

and the result is established. ■

Now, we can prove the following result.

Proposition 2.8. *The space $\mathcal{P}_+$ constructed on $E_n\times[0,T]$ is complete.*

Proof. As mentionned above, on this restricted space, λ is a finite measure. This proof is analogous to the classical proof that the space $L^2(\Omega)$ is complete.

Consider a Cauchy sequence $(g_n)_{n\in\mathbb{N}}$ in $\mathcal{P}_+$. It is possible to choose an increasing sequence $(n_k)_{k\in\mathbb{N}}\subset\mathbb{N}$ such that

$$\|g_n-g_{n_k}\|_+\leqslant 3^{-k} \tag{2.1}$$

for all $n\geqslant n_k$. In particular, the subsequence $(g_{n_k})_{k\in\mathbb{N}}$ satisfies

$$\|g_{n_{k+1}}-g_{n_k}\|_+\leqslant 3^{-k}. \tag{2.2}$$

Then, let $B_k = \{(x,s,\omega) \in E_n \times [0,T] \times \Omega : |g_{n_{k+1}}(x,s,\omega) - g_{n_k}(x,s,\omega)| \geqslant 2^{-k}\}$ and $B_{k,\omega}$ the section of B_k along ω. By Lemma 2.7,

$$\lambda(E_n \times B_k) \leqslant \frac{\|g_{n_{k+1}} - g_{n_k}\|}{2^{-k}} \lambda(E_n \times E_n \times [0,T] \times \Omega)^{\frac{1}{2}} \leqslant C \left(\frac{2}{3}\right)^k,$$

where $C^2 = \lambda(E_n \times E_n \times [0,T] \times \Omega) < \infty$.

Then let $B = \limsup_{k\to\infty} B_k = \bigcap_{n=0}^{\infty} \bigcup_{j\geqslant n} B_j$. We have $\bigcup_{j\geqslant n+1} B_j \subset \bigcup_{j\geqslant n} B_j$, which implies $\bigcap_{n=0}^{m} \bigcup_{j\geqslant n} B_j = \bigcup_{j\geqslant m} B_j$ and $B = \lim_{m\to\infty} \bigcup_{j\geqslant m} B_j$. As a consequence,

$$\begin{aligned} \lambda(E_n \times B) &= \lambda\left(E_n \times \limsup_{k\to\infty} B_k\right) = \lim_{m\to\infty} \lambda\left(E_n \times \bigcup_{j\geqslant m} B_j\right) \\ &\leqslant \lim_{m\to\infty} \sum_{j\geqslant m} \lambda\left(E_n \times B_j\right) \leqslant \lim_{m\to\infty} C \sum_{j\geqslant m} \left(\frac{2}{3}\right)^j = 0. \end{aligned}$$

If $(y,s,\omega) \notin B$, then there exists $K > 0$ such that for all $k \geqslant K$, $|g_{n_{k+1}}(y,s,\omega) - g_{n_k}(y,s,\omega)| < 2^{-k}$. Hence, $(g_{n_k}(y,s,\omega))_{k\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}$. As it is convergent, let $g(y,s,\omega)$ denote its limit and set $g(y,s,\omega) = 0$ if $(y,s,\omega) \in B$.

We will show that $g \in \mathcal{P}_+$. We have $|g_{n_k}(x,s,\omega)||g_{n_k}(y,s,\omega)| \geqslant 0$ and

$$\int_{E_n\times E_n\times[0,T]\times\Omega} |g_{n_k}(x,s,\omega)||g_{n_k}(y,s,\omega)|\lambda(dx,dy,ds,d\omega) = \|g_{n_k}\|_+^2 < \infty.$$

Using (2.2), one can show that $\|g_{n_k}\|_+ \leqslant 1 + \|g_{n_0}\|_+ = G < \infty$. Moreover, $g_{n_k}(y,s,\omega) \to g(y,s,\omega)$ as $k \to \infty$ for all $(y,s,\omega) \in B^c$. As a consequence, $g_{n_k} \to g$ λ-a.e. as $k \to \infty$. Then, by Fatou's lemma,

$$\begin{aligned} &\int_{E_n\times E_n\times[0,T]\times\Omega} |g(x,s,\omega)||g(y,s,\omega)|\lambda(dx,dy,ds,d\omega) \\ &= \int_{E_n\times B^c} |g(x,s,\omega)||g(y,s,\omega)|\lambda(dx,dy,ds,d\omega) \\ &= \int_{\substack{(x,s,\omega)\in B^c \\ (y,s,\omega)\in B^c}} |g(x,s,\omega)||g(y,s,\omega)|\lambda(dx,dy,ds,d\omega) \\ &\leqslant \liminf_{k\to\infty} \int_{\substack{(x,s,\omega)\in B^c \\ (y,s,\omega)\in B^c}} |g_{n_k}(x,s,\omega)||g_{n_k}(y,s,\omega)|\lambda(dx,dy,ds,d\omega) \\ &= \liminf_{k\to\infty} \|g_{n_k}\|_+^2 \leqslant G^2 < \infty. \end{aligned}$$

Hence, $g \in \mathcal{P}_+$.

We must now show that $\lim_{k\to\infty} \|g_{n_k} - g\|_+ = 0$. This is shown directly by Fatou's lemma. Indeed, let $\varepsilon > 0$. Then, there exists ℓ such that $3^{-\ell} < \varepsilon$. Let $k \geqslant \ell$. We have

$$\begin{aligned}
&\|g - g_{n_k}\|_+^2 \\
&= \int_{\substack{(x,s,\omega)\in B^c \\ (y,s,\omega)\in B^c}} |(g - g_{n_k})(x,s,\omega)||(g - g_{n_k})(y,s,\omega)|\lambda(dx,dy,ds,d\omega) \\
&\leqslant \liminf_{m\to\infty} \int_{\substack{(x,s,\omega)\in B^c \\ (y,s,\omega)\in B^c}} |(g_{n_m} - g_{n_k})(x,s,\omega)||(g_{n_m} - g_{n_k})(y,s,\omega)|\lambda(dx,dy,ds,d\omega) \\
&\leqslant \liminf_{m\to\infty} \|g_{n_m} - g_{n_k}\|_+^2 \leqslant \left(3^{-\ell}\right)^2 < \varepsilon^2. \qquad (2.3)
\end{aligned}$$

Hence, $\lim_{k\to\infty} \|g - g_{n_k}\| = 0$.

Hence, we have constructed a convergent subsequence of $(g_n)_{n\in\mathbb{N}}$ in $\mathcal{P}_+$. It remains to prove that the sequence itself converges to the same limit g. Therefore, let $\varepsilon > 0$. There exists ℓ such that $3^{-\ell} < \frac{\varepsilon}{2}$. Let $n \geqslant n_\ell$. Then,

$$\|g - g_n\| \leqslant \|g - g_{n_\ell}\| + \|g_{n_\ell} - g_n\| \leqslant 3^{-\ell} + 3^{-\ell} < \varepsilon,$$

where the first inequality comes from (2.3) and the second inequality comes from (2.1). Hence, we have shown that $\|g - g_n\| \to 0$ as $n \to \infty$.

As a consequence, the space $\mathcal{P}_+$ constructed on $E_n \times [0,T]$ is complete. ■

We can now turn to the proof of the first part of Theorem 2.6.

Proof of Theorem 2.6 (a). Let $(g_i)_{i\in\mathbb{N}}$ be a Cauchy sequence in $\mathcal{P}_+$. Let $g_i^{n,T}$ denote the restriction of g_i to $E_n \times [0,T]$. By Proposition 2.8, the space $\mathcal{P}_M$ constructed on $E_n \times [0,T]$ is complete. Hence, the sequence $(g_i^{n,T})_{i\in\mathbb{N}}$ converges to a limit $g^{n,T}$. We then define a function g as follows : $g(x,s,\omega) = g^{n,T}(x,s,\omega)$ if $(x,s) \in E_n \times [0,T]$. This definition is independent on the choice of n and T. One can show that $\|g_i - g\|_+ \to 0$ as $i \to \infty$. As a consequence, $\mathcal{P}_+$ is complete. ■

In order to complete the proof of Theorem 2.6, it remains to prove the second part. For that purpose, we show the two following propositions.

Proposition 2.9. *Let $\mathcal{P}_+^b$ be the set of all bounded functions in $\mathcal{P}_+$. The space $\mathcal{P}_+^b$ is dense in $\mathcal{P}_+$.*

Proof. Let $g \in \mathcal{P}_+$. Let

$$g_n(x,s,\omega) = \begin{cases} g(x,s,\omega) & \text{if } |g(x,s,\omega)| < N \\ 0 & \text{else.} \end{cases}$$

Then,

$$\|g-g_n\|_+^2 = \int_{\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}_+\times\Omega} |(g-g_n)(x,s,\omega)||(g-g_n)(y,s,\omega)|\lambda(dx,dy,ds,d\omega) \underset{n\to\infty}{\longrightarrow} 0,$$

by the Dominated Convergence Theorem. Indeed, $|g-g_n| \leqslant |g|$, and g is integrable, because $g \in \mathcal{P}_+$. Hence, $\mathcal{P}_+^b$ is dense in $\mathcal{P}_+$. ■

As a next step, we will show that $\mathcal{E}$ is dense in $\mathcal{P}_+^b$.

Proposition 2.10. *The set $\mathcal{E}$ of simple functions is dense in $\mathcal{P}_+^b$.*

The proof of Proposition 2.10 is based on the Monotone Class Theorem, that we remind below.

Theorem 2.11 (Monotone Class Theorem)**.** *Let $\mathcal{A}$ be a collection of subsets of a space X, closed for intersection and that contains X. Let $\mathcal{H}$ be a vector space of real-valued functions on X such that*

- *if $A \in \mathcal{A}$, then $\mathbf{1}_A \in \mathcal{H}$;*
- *if $(g_n)_{n\in\mathbb{N}}$ is a sequence of functions in $\mathcal{H}$ with $0 \leqslant g_n \uparrow g$, where g is a bounded function, then $g \in \mathcal{H}$.*

Then, $\mathcal{H}$ contains all bounded $\sigma(\mathcal{A})$-measurable functions.

For a proof of Theorem 2.11, we refer to [12, p. 280].

Proof of Proposition 2.10. Let $\mathcal{P}_+^b(E_i \times [0,T])$ be the set of functions $g \in \mathcal{P}_+^b$ with support in $E_i \times [0,T]$. We know that λ is a finite-measure on $E_i \times E_i \times [0,T] \times \Omega$. Let $\mathcal{H}$ denote the set of functions for which there exists a sequence $(g_n)_{n\in\mathbb{N}}$ of simple functions such that $\lim_{n\to\infty} \|g_n - g\|_+ = 0$. Let $\mathcal{A}$ denote the collection of subsets of $E_i \times [0,T] \times \Omega$ of the form $B \times]t,u] \times C$, where $t < u$, $B \in \mathcal{B}(\mathbb{R}^d)$ and $C \in \mathcal{F}_t$.

The collection $\mathcal{A}$ is closed for intersection and contains $E_i \times [0,T] \times \Omega$. The space $\mathcal{H}$ is a vector space. Moreover, if $A \in \mathcal{A}$, $\mathbf{1}_A(x,s,\omega) = \mathbf{1}_B(x)\mathbf{1}_{]t,u]}(s)\mathbf{1}_C(\omega) \in \mathcal{E} \subset \mathcal{H}$. Now, let $(g_n)_{n\in\mathbb{N}}$ be a sequence such that $0 \leqslant g_n \uparrow g$. Then $|g-g_n| \to 0$ almost everywhere as $n \to \infty$. Moreover, $|g - g_n| \leqslant 2|g|$, and g is integrable, because

g is bounded and λ is a finite measure. Hence, by the Dominated Convergence Theorem,

$$\|g - g_n\|_+^2 = \int_{E_i \times E_i \times [0,T] \times \Omega} |(g - g_n)(x, s, \omega)||(g - g_n)(y, s, \omega)|\lambda(dx, dy, ds, d\omega) \underset{n\to\infty}{\longrightarrow} 0.$$

As a consequence, for all $\varepsilon > 0$, there exists n_ε such that

$$n \geqslant n_\varepsilon \Longrightarrow \|g_n - g\|_+ < \frac{\varepsilon}{2}.$$

For all n, there exists a sequence $(g_m^n)_{m\in\mathbb{N}}$ of simple functions such that $\|g_m^n - g_n\| \to 0$ as $m \to \infty$. Then, for all $\varepsilon > 0$, there exists $m_{n,\varepsilon}$ such that

$$m \geqslant m_{n,\varepsilon} \Longrightarrow \|g_m^n - g_n\|_+ < \frac{\varepsilon}{2}.$$

We now set $h_\ell = g^\ell_{m_{\ell,2^{-\ell}}}$ for all $\ell \in \mathbb{N}$. Let $\varepsilon > 0$ and $\ell_\varepsilon = \max\{n_\varepsilon, 1 - \log_2(\varepsilon)\}$. Then, if $\ell \geqslant \ell_\varepsilon$, then

$$\|h_\ell - g\|_+ = \|g^\ell_{m_{\ell,2^{-\ell}}} - g\|_+ \geqslant \|g^\ell_{m_{\ell,2^{-\ell}}} - g_\ell\|_+ + \|g_\ell - g\|_+ < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \quad (2.4)$$

Hence, $(h_\ell)_{\ell\in\mathbb{N}}$ is a sequence of simple functions that converges to g and $g \in \mathcal{H}$.

The assumptions of the Monotone Class Theorem (Theorem 2.11) are satisfied. As a consequence, $\mathcal{H}$ contains all bounded $\sigma(\mathcal{A})$-measurable functions, that is all bounded $\sigma(\mathcal{E})$-measurable functions. The functions in $\mathcal{P}_+^b(E_i \times [0, T])$ can be approximated by simple functions, that is, $\mathcal{E}$ is dense in $\mathcal{P}_+^b(E_i \times [0, T])$.

Now let $g \in \mathcal{P}_+^b$. We have to show that it can be approximated by simple functions. Let

$$g_i(x, s, \omega) = g(x, s, \omega)\mathbf{1}_{\bigcup_{j\leqslant i} E_j}(x)$$

and

$$g_{i,T}(x, s, \omega) = g(x, s, \omega)\mathbf{1}_{\bigcup_{j\leqslant i} E_j}(x)\mathbf{1}_{[0,T]}(s).$$

The function $g_{i,T} \in \mathcal{P}_+^b(E_i \times [0, T])$ and, hence, can be approximated by simple functions. Moreover, $g_{i,T} \to g_i$ almost everywhere as $T \to \infty$ for all i and $g_i \to g$ almost everywhere as $i \to \infty$. We also have $|g_{i,T} - g_i| \leqslant |g|$ and $|g_i - g| \leqslant |g|$, and g is integrable, because $g \in \mathcal{P}_+$. Hence, by the Dominated Convergence Theorem, for all i,

$$\lim_{T\to\infty} \|g_{i,T} - g_i\|_+ = 0$$

and

$$\lim_{i\to\infty} \|g_i - g\|_+ = 0.$$

Hence, an argument analogous to the one used for (2.4) shows that there exists a sequence of simple functions that converges to g with respect to $\|\cdot\|_+$. As a consequence, $\mathcal{E}$ is dense in $\mathcal{P}_+^b$. ■

Proof of Theorem 2.6 (b). It is a direct consequence of Proposition 2.9 and Proposition 2.10. ■

We can now turn to the definition of the stochastic integral with respect to a martingale measure.

Let $g(s,x,\omega) = \mathbf{1}_{]a,b]}(s)\mathbf{1}_A(x)X(\omega)$ be an elementary function. We define the stochastic integral $g \cdot M : \mathbb{R}_+ \times \mathcal{B}(\mathbb{R}^d)$ of g with respect to the martingale measure M by

$$(g \cdot M)_t(B) = X(\omega)\left(M_{t\wedge b}(A \cap B) - M_{t\wedge a}(A\cap B)\right),$$

for all $t \geqslant 0$ and $B \in \mathcal{B}(\mathbb{R}^d)$.

Proposition 2.12. ([30, Lemma 2.4]) *The process* $((g\cdot M)_t(B), t \geqslant 0, B \in \mathcal{B}(\mathbb{R}^d))$ *is a worthy martingale measure. Its covariation and dominating measures* $Q_{g\cdot M}$ *and* $K_{g\cdot M}$ *are given by*

$$Q_{g\cdot M}(dx,dy,ds) = g(s,x)g(s,y)Q_M(ds,dx,dy) \tag{2.5}$$

and

$$K_{g\cdot M}(dx,dy,ds) = |g(s,x)||g(s,y)|K_M(ds,dx,dy). \tag{2.6}$$

Moreover,

$$\mathbb{E}[(g\cdot M)_t(B)] = \mathbb{E}\left[\int_0^t \int_B \int_B g(s,x)g(s,y)Q_M(dx,dy,ds)\right] \leqslant \|g\|_+^2. \tag{2.7}$$

Proof. See [30, Lemma 2.4].

We extend the stochastic integral to all functions in $\mathcal{E}$ by linearity. Then, let $g \in \mathcal{P}_+$. By Theorem 2.6, there exists a sequence $(g_n)_{n\in\mathbb{N}} \subset \mathcal{E}$ such that $\lim_{n\to\infty} \|g_n - g\|_+ = 0$. By (2.7), for all $t \geqslant 0$ and all $B \in \mathcal{B}(\mathbb{R}^d)$,

$$\mathbb{E}[((g_n\cdot M)_t(B) - (g_m \cdot M)_t(B))^2] \leqslant \|g_n - g_m\|_+^2.$$

As a consequence $((g_n \cdot M)_t(B))_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^2(\Omega)$. We define the stochastic integral of g with respect to M by

$$(g\cdot M)_t(B) = \lim_{n\to\infty} (g_n \cdot M)_t(B),$$

for all $t \geqslant 0$ and all $B \in \mathcal{B}(\mathbb{R}^d)$. We have defined $(g \cdot M)$ for all $g \in \mathcal{P}_+$. The following result completes the construction of Walsh's stochastic integral with respect to martingale measures.

Theorem 2.13. [30, Theorem 2.5] *For $g \in \mathcal{P}_+$, $(g \cdot M)$ is a worthy martingale measure. Its covariance (resp. dominating) measure is given by (2.5) (resp. (2.6)). Moreover, for $h \in \mathcal{P}_+$,*

$$\langle (g \cdot M)(A), (h \cdot M)(B) \rangle_t = \int_0^t \int_A \int_B g(s,x)h(s,y)Q_M(dx,dy,ds) \tag{2.8}$$

and (2.7) is still valid.

Proof. See [30, Theorem 2.5].

2.2 Spatially homogeneous noise as a martingale measure

Walsh's stochastic integration with respect to martingale measures is the tool that we use in order to build and study the solution of the non-linear stochastic wave equation (1.5). We present in this section the noise that we consider in equation (1.5) and, in particular, we show how this noise can be seen as a martingale measure.

We consider a Gaussian noise $\dot{F}$, white in time and correlated in space. Its covariance function is informally given by

$$\mathbb{E}[\dot{F}(t,x)\dot{F}(s,y)] = \delta_0(t-s)f(x-y), \qquad s,t \geqslant 0,\ x,y \in \mathbb{R}^d,$$

where δ_0 stands for the Dirac delta function and $f : \mathbb{R}^d \to \mathbb{R}_+$ is continuous on $\mathbb{R}^d \setminus \{0\}$ and even. Formally, let $\mathcal{D}(\mathbb{R}^{d+1})$ be the space of C^∞-functions with compact support on $\mathbb{R}^{d+1}$. Let $F = \{F(\varphi),\ \varphi \in \mathcal{D}(\mathbb{R}^{d+1})\}$ be an $L^2(\Omega, \mathcal{F}, \mathbb{P})$-valued mean zero Gaussian process with covariance functional

$$\mathbb{E}[F(\varphi)F(\psi)] = \int_0^\infty dt \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^d} dy\, \varphi(t,x)f(x-y)\psi(t,y).$$

Since f is a covariance, by Bochner's theorem [27, Chap.VII,Theorem XVIII], there exists a non-negative tempered measure μ whose Fourier transform is f. That is, for all $\phi \in \mathcal{S}(\mathbb{R}^d)$, the Schwartz space of C^∞-functions with rapid decrease, we have

$$\int_{\mathbb{R}^d} f(x)\phi(x)dx = \int_{\mathbb{R}^d} \mathcal{F}\phi(\xi)\mu(d\xi).$$

As f is the Fourier transform of a tempered measure, it satisfies an integrability condition of the form

$$\int_{\mathbb{R}^d} \frac{f(x)}{1+|x|^p}\, dx < \infty, \tag{2.9}$$

for some $p < \infty$ (see [27, Theorem XIII, p.251]).

Following [4], we extend this process to a worthy martingale measure $M = (M_t(B),\ t \geqslant 0,\ B \in \mathcal{B}_b(\mathbb{R}^d))$, where $\mathcal{B}_b(\mathbb{R}^d)$ denotes the bounded Borel subsets of $\mathbb{R}$. More precisely, approximating $\mathbf{1}_A$ by functions in $\mathcal{D}(\mathbb{R}^{d+1})$, we first extend the function $\varphi \mapsto F(\varphi)$ to an $L^2(\Omega)$-valued measure $A \mapsto F(A) = F(\mathbf{1}_A)$, defined for all $A \in \mathcal{B}_b(\mathbb{R}^{d+1})$. Then, we set

$$M_t(B) = F([0,t] \times B), \qquad B \in \mathcal{B}_b(\mathbb{R}^d).$$

We consider the filtration $\mathcal{F}_t$ given by $\mathcal{F}_t = \mathcal{F}_t^0 \vee \mathcal{N}$, where

$$\mathcal{F}_t^0 = \sigma(M_s(B),\, s \leqslant t,\, B \in \mathcal{B}_b(\mathbb{R}^d))$$

and $\mathcal{N}$ is the σ-field generated by the $\mathbb{P}$-null sets. Then, we have the following proposition.

Proposition 2.14. *The process $(M_t(B),\ t \geqslant 0,\ B \in \mathcal{B}_b(\mathbb{R}^d))$ is a worthy martingale measure. Its covariation and dominating measure Q_M and K_M are given by*

$$\begin{aligned} Q_M([0,t] \times A \times B) &= K_M([0,t] \times A \times B) \\ &= \langle M(A), M(B) \rangle_t = t \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^d} dy\, \mathbf{1}_A(x) f(x-y) \mathbf{1}_B(y). \end{aligned}$$

Moreover, for all $\varphi \in \mathcal{S}(\mathbb{R}^{d+1})$

$$F(\varphi) = \int_0^\infty \int_{\mathbb{R}^d} \varphi(t,x) M(dt,dx),$$

where the stochastic integral is Walsh's stochastic integral with respect to the martingale measure M (see Section 2.1).

Proof. See [4, Section 2].

Hence, we see that the stochastic integral developped by Walsh ([30]) can be used in order to obtain a solution for the non-linear stochastic wave equation (1.5) in dimensions 1 and 2. In these cases, the fundamental solution of the wave equation is a function (see Section 4.2). The construction of a solution to the 1-dimensional non-linear stochastic wave equation can be found in [30, Chapter 3] and the 2-dimensional case is the subject of [4].

2.3 Extension to non-negative Schwartz distributions

If one wants to study the 3-dimensional non-linear stochastic wave equation (1.5), the stochastic integral presented in Section 2.1 is not sufficient. In this case, the fundamental solution of the wave equation is not a function, but a positive measure (see Section 4.2). Therefore, an extension of Walsh's stochastic integral is needed. This has been developed by Dalang in [2] and is the subject of this section.

Let M be the martingale measure built in Section 2.2. The main point in the construction of Dalang's stochastic integral is to consider the norm $\|\cdot\|_0$ defined by

$$\begin{aligned}\|g\|_0^2 &= \mathbb{E}\left[\int_0^T ds \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^d} dy\, g(s,x,\cdot)g(s,y,\cdot)Q_M(dx,dy,ds)\right]. \\ &= \mathbb{E}\left[\int_0^T ds \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^d} dy\, g(s,x,\cdot)f(x-y)g(s,y,\cdot)\right]. \qquad (2.10)\end{aligned}$$

Since the set of predictable functions such that $\|g\|_0 < \infty$ is not complete, let $\mathcal{P}_0$ denote the completion of the set $\mathcal{E}$ of simple predictable functions with respect to $\|\cdot\|_0$. Clearly, $\mathcal{P}_+ \subset \mathcal{P}_0$. Both $\mathcal{P}_0$ and $\mathcal{P}_+$ can be identified with subspaces of $\overline{\mathcal{P}}$, where

$$\begin{aligned}\overline{\mathcal{P}} := \big\{ & t \mapsto S(t) \text{ from } [0,T]\times\Omega \to \mathcal{S}'(\mathbb{R}^d) \text{ predictable, such that } \mathcal{F}S(t) \text{ is a.s.} \\ & \text{a function and } \|S\|_0 < \infty\big\},\end{aligned}$$

where

$$\|S\|_0^2 = \mathbb{E}\left[\int_0^T dt \int_{\mathbb{R}^d} \mu(d\xi)\, |\mathcal{F}S(t)(\xi)|^2\right]. \qquad (2.11)$$

For $S(t) \in \mathcal{S}(\mathbb{R}^d)$, elementary properties of convolution and Fourier transform show that (2.10) and (2.11) are equal. When $d \geqslant 4$, the fundamental solution of the wave equation provides an example of an element of $\mathcal{P}_0$ that is not in $\mathcal{P}_+$ (see Section 4.2).

Before going into more details about this extension, we are going to consider a more general setting corresponding to the situation encountered in the case of the non-linear stochastic wave equation. Consider a predictable process $(Z(t,x),\ 0 \leqslant t \leqslant T,\ x \in \mathbb{R}^d)$, such that

$$\sup_{0\leqslant t\leqslant T}\ \sup_{x\in\mathbb{R}^d} \mathbb{E}[Z(t,x)^2] < \infty. \qquad (2.12)$$

Let M^Z be the martingale measure defined by

$$M_t^Z(B) = \int_0^t \int_B Z(s,y)M(ds,dy), \qquad 0 \leqslant t \leqslant T,\ B \in \mathcal{B}_b(\mathbb{R}^d),$$

in which we again use Walsh's stochastic integral [30]. We would like to give a meaning to the stochastic integral of a large class of $S \in \overline{\mathcal{P}}$ with respect to the martingale measure M^Z. Following the same idea as before, we will consider the norms $\|\cdot\|_{+,Z}$ and $\|\cdot\|_{0,Z}$ defined by

$$\|g\|_{+,Z}^2 = \mathbb{E}\left[\int_0^T ds \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^d} dy\, |g(s,x,\cdot)Z(s,x)f(x-y)Z(s,y)g(s,y,\cdot)|\right]$$

and

$$\|g\|_{0,Z}^2 = \mathbb{E}\left[\int_0^T ds \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^d} dy\, g(s,x,\cdot)Z(s,x)f(x-y)Z(s,y)g(s,y,\cdot)\right]. \quad (2.13)$$

Let $\mathcal{P}_{+,Z}$ be the set of predictable functions g such that $\|g\|_{+,Z} < \infty$. The space $\mathcal{P}_{0,Z}$ is defined, similarly to $\mathcal{P}_0$, as the completion of the set of simple predictable functions, but taking completion with respect to $\|\cdot\|_{0,Z}$ instead of $\|\cdot\|_0$.

For $g \in \mathcal{E}$, as in Definition 2.5, the stochastic integral

$$g \cdot M^Z = ((g \cdot M^Z)_t(\mathbb{R}^d), 0 \leqslant t \leqslant T)$$

is the square-integrable martingale

$$(g \cdot M^Z)_t(\mathbb{R}^d) = X(\omega)\left(M^Z_{t\wedge b}(A) - M^Z_{t\wedge a}(A)\right) = \int_0^t \int_{\mathbb{R}^d} g(s,y,\cdot)Z(s,y)M(ds,dy),$$

defined by Proposition 2.12.

Notice that the map $g \mapsto g \cdot M^Z$, from $(\mathcal{E}, \|\cdot\|_{0,Z})$ into the Hilbert space $\mathcal{M}$ of continuous square-integrable $(\mathcal{F}_t)$-martingales $X = (X_t, 0 \leqslant t \leqslant T)$ equipped with the norm $\|X\| = \mathbb{E}[X_T^2]^{\frac{1}{2}}$, is an isometry. Therefore, this isometry can be extended to an isometry $S \mapsto S \cdot M^Z$ from $(\mathcal{P}_{0,Z}, \|\cdot\|_{0,Z})$ into $\mathcal{M}$. The square-integrable martingale $S \cdot M^Z = ((S \cdot M^Z)_t, 0 \leqslant t \leqslant T)$ is the *stochastic integral process of S with respect to M^Z*. We use the notation

$$\int_0^t \int_{\mathbb{R}^d} S(s,y)Z(s,y)M(ds,dy)$$

for $(S \cdot M^Z)_t$.

The main issue is to identify elements of $\mathcal{P}_{0,Z}$. A first step in this direction was made by Dalang ([2]). We have to restrict the class of processes Z that are taken into account.

Definition 2.15. A stochastic process $Z : \mathbb{R}_+ \times \mathbb{R}^d \mapsto \mathbb{R}$ such that $\mathbb{E}[Z(t,x)^2] < +\infty$ for all (t,x), has a *spatially homogeneous covariance function* if the function $z \mapsto \mathbb{E}[Z(t,x)Z(t,x+z)]$ does not depend on x.

Now consider the following hypothesis.

(H) The process Z has a spatially homogeneous covariance function.

Under this assumption, following [2], one can show that there exists a non-negative measure μ_s^Z such that the norm $\|\cdot\|_{0,Z}$ satisfies

$$\|\varphi\|_{0,Z}^2 = \int_0^T ds \int_{\mathbb{R}^d} \mu_s^Z(d\xi)|\mathcal{F}\varphi(s,\cdot)(\xi)|^2 \tag{2.14}$$

for any φ such that $\varphi(s,\cdot) \in \mathcal{S}(\mathbb{R}^d)$.

Then, using this expression for $\|\cdot\|_0$, one can identify that a class of non-negative Schwartz distributions belong to $\mathcal{P}_{0,Z}$.

Theorem 2.16. [2, Theorem 2] *Let $(Z(t,x), 0 \leqslant t \leqslant T, x \in \mathbb{R}^d)$ be a process satisfying (2.12) and hypothesis (H). Let $t \mapsto S(t)$ be a deterministic function with values in the space of* non-negative *distributions with rapid decrease, such that*

$$\int_0^T ds \int_{\mathbb{R}^d} \mu(d\xi)\,|\mathcal{F}S(s)(\xi)|^2 < \infty. \tag{2.15}$$

Then $S \in \mathcal{P}_{0,Z}$. In particular, the stochastic integral $(S \cdot M^Z)_t$ is well defined as a real-valued square-integrable martingale $((S \cdot M^Z)_t,\ 0 \leqslant t \leqslant T)$ and

$$\begin{aligned} \mathbb{E}[(S\cdot M^Z)_t^2] &= \int_0^t ds \int_{\mathbb{R}^d} \mu_s^Z(d\xi)\,|\mathcal{F}S(s)(\xi)|^2 \\ &\leqslant \left(\sup_{0\leqslant s\leqslant T}\sup_{x\in\mathbb{R}^d} \mathbb{E}[Z(s,x)^2]\right) \int_0^t ds \int_{\mathbb{R}^d} \mu(d\xi)\,|\mathcal{F}S(s)(\xi)|^2. \end{aligned} \tag{2.16}$$

Proof. The first step in the proof is to consider a smoothing approximation of $S(t)$ by a sequence $(S_n(t))_{n\in\mathbb{N}} \subset \mathcal{S}(\mathbb{R}^d)$. The stochastic integral $(S_n \cdot M^Z)$ is well-defined as a Walsh stochastic integral. Then, $(S \cdot M^Z)$ is defined as the limit of $(S_n \cdot M^Z)$ as $n \to \infty$ by the isometry from $\mathcal{P}_{0,Z}$ into $\mathcal{M}$. For more details, see [2, Theorem 2]. ■

Under the assumptions of Theorem 2.16, bounds on higher order moments of $(S \cdot M^Z)_t$ are available.

Theorem 2.17. [2, Theorem 5] *Suppose, in addition to the assumptions of Theorem 2.16, that for some $p \geqslant 2$,*

$$\sup_{0\leqslant t\leqslant T}\sup_{x\in\mathbb{R}^d} \mathbb{E}[|Z(t,x)|^p] < \infty.$$

Then,

$$\mathbb{E}[|(S \cdot M^Z)_t|^p] \leqslant c_p(\nu_t)^{\frac{p}{2}-1} \int_0^t ds \left(\sup_{x \in \mathbb{R}^d} \mathbb{E}[|Z(s,x)|^p] \right) \int_{\mathbb{R}^d} \mu(d\xi)\, |\mathcal{F}S(s)(\xi)|^2,$$

where c_p is a constant and

$$\nu_t = \int_0^t ds \int_{\mathbb{R}^d} \mu(d\xi)\, |\mathcal{F}S(s)(\xi)|^2.$$

Proof. See [2, Theorem 5].

In the case where $Z \equiv 1$, it is possible to enlarge the class of Schwartz distributions that belong to $\mathcal{P}_{0,Z}$ by removing the non-negativity assumption. This requires an additional assumption on S. We let $\mathcal{S}'_r(\mathbb{R}^d)$ denote the space of Schwartz distributions with rapid decrease (see [27, p.244]). We recall that for $S \in \mathcal{S}'_r(\mathbb{R}^d)$, $\mathcal{F}S$ is a function (see [27, Chapter VII, Thm. XV, p.268]).

Theorem 2.18. [2, Theorem 3] *Let $t \mapsto S(t)$ be a deterministic function with values in the space $\mathcal{S}'_r(\mathbb{R}^d)$, such that (2.15) is satisfied and*

$$\lim_{h \downarrow 0} \int_0^T ds \int_{\mathbb{R}^d} \mu(d\xi) \sup_{s<r<s+h} |\mathcal{F}S(r)(\xi) - \mathcal{F}S(s)(\xi)|^2 = 0. \tag{2.17}$$

Then, $S \in \mathcal{P}_0$ and

$$\mathbb{E}[(S \cdot M^Z)_t^2] = \int_0^t ds \int_{\mathbb{R}^d} \mu(d\xi)\, |\mathcal{F}S(s)(\xi)|^2. \tag{2.18}$$

Proof. The proof follows the same steps as the one of Theorem 3.1 under hypothesis (H2), which is itself inspired from Theorem 2.18. For more details, see [2, Theorem 3] and [3]. ■

The extension of the stochastic integral given by Theorem 2.16 allows to include non-negative Schwartz distributions as integrands and, as a consequence, to obtain existence and uniqueness for the non-linear stochastic wave equation (1.5) in dimension 3. The construction of this solution is similar to the one of the proof of Theorem 4.2. We refer to [2, Section 5] for more details. Nevertheless, for higher dimensions, the fundamental solution of the wave equation does not satisfy the non-negativity assumption. Theorem 2.18 allows us to consider such distributions as integrands but only in the case where $Z \equiv 1$. This only allows to treat the linear case of (1.5) (α constant).

Chapter 3

Stochastic integration : further extension

In this chapter, we extend Dalang's results (Theorems 2.16 and 2.18) concerning the class of Schwartz distributions for which the stochastic integral with respect to the martingale measure M^Z can be defined, by deriving a new inequality for this integral. In particular, contrary to Theorem 2.16 or Theorem 2.18, the result presented here does not require that the Schwartz distribution be non-negative, nor that $Z \equiv 1$. The content of this chapter is presented in [1, Section 3].

3.1 Extension to more general Schwartz distributions

In Theorem 3.1 below, we show that the non-negativity assumption can be removed provided the spectral measure satisfies the condition (3.6) below, which already appears in [19] and [7]. As in Theorem 2.18 ([2, Theorem 3]), an additional assumption similar to (2.17) ([2, (33), p.12]) is needed (hypothesis (H2) below). This hypothesis can be replaced by an integrability condition (hypothesis (H1) below).

Suppose Z is a process such that $\sup_{0 \leqslant s \leqslant T} \mathbb{E}[Z(s,0)^2] < +\infty$ and that satisfies hypothesis (H). Following the proof of Theorem 2.18 ([2, Theorem 3]), set $f^Z(s,x) = f(x) g_s(x)$, where $g_s(x) = \mathbb{E}[Z(s,0)Z(s,x)]$.

For s fixed, the function g_s is non-negative definite, since it is a covariance function. Hence, by Bochner's theorem ([27, Chap.VII,Theorem XVIII]), there exists a non-negative tempered measure ν_s^Z such that $g_s = \mathcal{F}\nu_s^Z$. Note that $\nu_s^Z(\mathbb{R}^d) = g_s(0) = \mathbb{E}[Z(s,0)^2]$. Using the convolution property of the Fourier transform, we have

$$f^Z(s,\cdot) = f \cdot g_s = \mathcal{F}\mu \cdot \mathcal{F}\nu_s^Z = \mathcal{F}(\mu * \nu_s^Z),$$

where $*$ denotes convolution. Looking back to the definition of $\|\cdot\|_{0,Z}$, we obtain, for a deterministic $\varphi \in \mathcal{P}_{0,Z}$ with $\varphi(t,\cdot) \in \mathcal{S}(\mathbb{R}^d)$ for all $0 \leqslant t \leqslant T$ (see [2, p.10]),

$$\begin{aligned}
\|\varphi\|_{0,Z}^2 &= \int_0^T ds \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^d} dy\, \varphi(s,x) f(x-y) g_s(x-y) \varphi(s,y) \\
&= \int_0^T ds \int_{\mathbb{R}^d} (\mu * \nu_s^Z)(d\xi)\, |\mathcal{F}\varphi(s,\cdot)(\xi)|^2 \\
&= \int_0^T ds \int_{\mathbb{R}^d} \nu_s^Z(d\eta) \int_{\mathbb{R}^d} \mu(d\xi)\, |\mathcal{F}\varphi(s,\cdot)(\xi+\eta)|^2. \qquad (3.1)
\end{aligned}$$

In particular,

$$\begin{aligned}
\|\varphi\|_{0,Z}^2 &\leqslant \int_0^T ds\, \nu_s^Z(\mathbb{R}^d) \sup_{\eta\in\mathbb{R}^d} \int_{\mathbb{R}^d} \mu(d\xi)\, |\mathcal{F}\varphi(s,\cdot)(\xi+\eta)|^2 \\
&\leqslant C \int_0^T ds \sup_{\eta\in\mathbb{R}^d} \int_{\mathbb{R}^d} \mu(d\xi)\, |\mathcal{F}\varphi(s,\cdot)(\xi+\eta)|^2, \qquad (3.2)
\end{aligned}$$

where $C = \sup_{0\leqslant s\leqslant T} \mathbb{E}[Z(s,0)^2] < \infty$ by assumption. Taking (3.1) as the definition of $\|\cdot\|_{0,Z}$, we can extend this norm to the set $\overline{\mathcal{P}}_Z$, where

$$\begin{aligned}
\overline{\mathcal{P}}_Z := \big\{ & t \mapsto S(t) \text{ from } [0,T] \to \mathcal{S}'(\mathbb{R}^d) \text{ deterministic, such that } \mathcal{F}S(t) \text{ is} \\
& \text{a function and } \|S\|_{0,Z} < \infty \big\}.
\end{aligned}$$

The spaces $\mathcal{P}_{+,Z}$ and $\mathcal{P}_{0,Z}$ will now be considered as subspaces of $\overline{\mathcal{P}}_Z$. Let $S \in \overline{\mathcal{P}}_Z$. We will need the following two hypotheses to state the next theorem. Let $B(0,1)$ denote the open ball in $\mathbb{R}^d$ that is centered at 0 with radius 1.

(H1) For all $\varphi \in \mathcal{D}(\mathbb{R}^d)$ such that $\varphi \geqslant 0$, $\text{supp}(\varphi) \subset B(0,1)$, and $\int_{\mathbb{R}^d} \varphi(x)dx = 1$, and for all $0 \leqslant a \leqslant b \leqslant T$, we have

$$\int_a^b (S(t) * \varphi)(\cdot)\, dt \in \mathcal{S}(\mathbb{R}^d), \qquad (3.3)$$

and

$$\int_{\mathbb{R}^d} dx \int_0^T ds\, |(S(s) * \varphi)(x)| < \infty. \qquad (3.4)$$

(H2) The function $\mathcal{F}S(t)$ is such that

$$\lim_{h\downarrow 0} \int_0^T ds \sup_{\eta\in\mathbb{R}^d} \int_{\mathbb{R}^d} \mu(d\xi) \sup_{s<r<s+h} |\mathcal{F}S(r)(\xi+\eta) - \mathcal{F}S(s)(\xi+\eta)|^2 = 0. \qquad (3.5)$$

This hypothesis is analogous to (2.17) ([2, (33), p.12]). We let $\mathcal{S}'_r(\mathbb{R}^d)$ denote the space of Schwartz distributions with rapid decrease (see [27, p.244]). We recall that for $S \in \mathcal{S}'_r(\mathbb{R}^d)$, $\mathcal{F}S$ is a function (see [27, Chapter VII, Thm. XV, p.268]).

Theorem 3.1. *Let $(Z(t,x),\ 0 \leqslant t \leqslant T,\ x \in \mathbb{R}^d)$ be a predictable process satisfying hypothesis (H) such that $\sup_{0\leqslant t\leqslant T}\sup_{x\in\mathbb{R}^d}\mathbb{E}[Z(t,x)^2] < \infty$. Let $t \mapsto S(t)$ be a deterministic function with values in the space $\mathcal{S}'_r(\mathbb{R}^d)$. Suppose that $(s,\xi) \mapsto \mathcal{F}S(s)(\xi)$ is measurable and*

$$\int_0^T ds \sup_{\eta\in\mathbb{R}^d} \int_{\mathbb{R}^d} \mu(d\xi)\, |\mathcal{F}S(s)(\xi+\eta)|^2 < \infty. \tag{3.6}$$

Suppose in addition that either hypothesis (H1) or (H2) is satisfied. Then $S \in \mathcal{P}_{0,Z}$. In particular, the stochastic integral $(S\cdot M^Z)_t$ is well defined as a real-valued square-integrable martingale $((S\cdot M^Z)_t,\ 0 \leqslant t \leqslant T)$ and

$$\begin{aligned}
&\mathbb{E}[(S\cdot M^Z)_t^2] \\
&\quad= \int_0^t ds \int_{\mathbb{R}^d} \nu_s^Z(d\eta) \int_{\mathbb{R}^d} \mu(d\xi)\, |\mathcal{F}S(s)(\xi+\eta)|^2 \\
&\quad\leqslant \left(\sup_{0\leqslant s\leqslant T}\sup_{x\in\mathbb{R}^d}\mathbb{E}[Z(s,x)^2]\right) \int_0^t ds \sup_{\eta\in\mathbb{R}^d} \int_{\mathbb{R}^d} \mu(d\xi)\, |\mathcal{F}S(s)(\xi+\eta)|^2.
\end{aligned} \tag{3.7}$$

Proof. We are now going to show that $S \in \mathcal{P}_{0,Z}$ and that its stochastic integral with respect to M^Z is well defined. We follow the approach of the proof of Theorem 2.18 ([2, Theorem 3]).

Take $\psi \in \mathcal{D}(\mathbb{R}^d)$ such that $\psi \geqslant 0$, $\mathrm{supp}(\psi) \subset B(0,1)$, $\int_{\mathbb{R}^d}\psi(x)dx = 1$. For all $n \geqslant 1$, take $\psi_n(x) = n^d\psi(nx)$. Then $\psi_n \to \delta_0$ in $\mathcal{S}'(\mathbb{R}^d)$ as $n \to \infty$. Moreover, $\mathcal{F}\psi_n(\xi) = \mathcal{F}\psi(\frac{\xi}{n})$ and $|\mathcal{F}\psi_n(\xi)| \leqslant 1$, for all $\xi \in \mathbb{R}^d$. Define $S_n(t) = (\psi_n * S)(t)$. As $S(t)$ is of rapid decrease, we have $S_n(t) \in \mathcal{S}(\mathbb{R}^d)$ (see [27], Chap. VII, §5, p.245).

Suppose that $S_n \in \mathcal{P}_{0,Z}$ for all n. Then

$$\begin{aligned}
\|S_n - S\|_{0,Z}^2 &= \int_0^T ds \int_{\mathbb{R}^d} \nu_s^Z(d\eta) \int_{\mathbb{R}^d} \mu(d\xi)\, |\mathcal{F}(S_n(s)-S(s))(\xi+\eta)|^2 \\
&= \int_0^T ds \int_{\mathbb{R}^d} \nu_s^Z(d\eta) \int_{\mathbb{R}^d} \mu(d\xi)\, |\mathcal{F}\psi_n(\xi+\eta)-1|^2 |\mathcal{F}S(s)(\xi+\eta)|^2.
\end{aligned} \tag{3.8}$$

The expression $|\mathcal{F}\psi_n(\xi+\eta)-1|^2$ is bounded by 4 and goes to 0 as $n \to \infty$ for every ξ and η. By (3.6), the Dominated Convergence Theorem shows that $\|S_n - S\|_{0,Z} \to 0$ as $n \to \infty$. As $\mathcal{P}_{0,Z}$ is complete, if $S_n \in \mathcal{P}_{0,Z}$ for all n, then $S \in \mathcal{P}_{0,Z}$.

To complete the proof, it remains to show that $S_n \in \mathcal{P}_{0,Z}$ for all n.

First consider assumption (H2). In this case, the proof that $S_n \in \mathcal{P}_{0,Z}$ is based on the same approximation as in [2]. For n fixed, we can write $S_n(t,x)$ because

$S_n(t) \in \mathcal{S}(\mathbb{R}^d)$ for all $0 \leqslant t \leqslant T$. The idea is to approximate S_n by a sequence of elements of $\mathcal{P}_{+,Z}$. For all $m \geqslant 1$, set

$$S_{n,m}(t,x) = \sum_{k=0}^{2^m-1} S_n(t_m^{k+1}, x) \mathbf{1}_{[t_m^k, t_m^{k+1}[}(t), \tag{3.9}$$

where $t_m^k = kT2^{-m}$. Then $S_{n,m}(t,\cdot) \in \mathcal{S}(\mathbb{R}^d)$. We now show that $S_{n,m} \in \mathcal{P}_{+,Z}$. Being a deterministic function, $S_{n,m}$ is predictable. Moreover, using the definition of $\|\cdot\|_{+,Z}$ and the fact that $|g_s(x)| \leqslant C$ for all s and x, we have

$$\begin{aligned}
&\|S_{n,m}\|_{+,Z}^2 \\
&= \int_0^T ds \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^d} dy\, |S_{n,m}(s,x)|\, f(x-y)\, |g_s(x-y)||S_{n,m}(s,y)| \\
&= \sum_{k=0}^{2^m-1} \int_{t_m^k}^{t_m^{k+1}} ds \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^d} dy\, |S_n(t_m^{k+1},x)|\, f(x-y)\, |g_s(x-y)|\, |S_n(t_m^{k+1},y)| \\
&\leqslant C \sum_{k=0}^{2^m-1} \int_{t_m^k}^{t_m^{k+1}} ds \int_{\mathbb{R}^d} dz\, f(z)(|S_n(t_m^{k+1},\cdot)| * |\tilde{S}_n(t_m^{k+1},\cdot)|)(z),
\end{aligned}$$

where $\tilde{S}_n(t_m^{k+1},x) = S_n(t_m^{k+1},-x)$. By Leibnitz' formula (see [28], Ex. 26.4, p.283), the function $z \mapsto (|S_n(t_m^{k+1},\cdot)| * |\tilde{S}_n(t_m^{k+1},\cdot)|)(z)$ decreases faster than any polynomial in $|z|^{-1}$. Therefore, by (2.9), the preceding expression is finite and $\|S_{n,m}\|_{+,Z} < \infty$, and $S_{n,m} \in \mathcal{P}_{+,Z} \subset \mathcal{P}_{0,Z}$.

The sequence of elements of $\mathcal{P}_{+,Z}$ that we have constructed converges in $\|\cdot\|_{0,Z}$ to S_n. Indeed,

$$\begin{aligned}
&\|S_{n,m} - S_n\|_{0,Z}^2 \\
&= \int_0^T ds \int_{\mathbb{R}^d} \nu_s^Z(d\eta) \int_{\mathbb{R}^d} \mu(d\xi)\, |\mathcal{F}(S_{n,m}(s,\cdot) - S_n(s,\cdot))(\xi+\eta)|^2 \\
&\leqslant \int_0^T ds \int_{\mathbb{R}^d} \nu_s^Z(d\eta) \int_{\mathbb{R}^d} \mu(d\xi) \sup_{s<r<s+T2^{-m}} |\mathcal{F}(S_n(r,\cdot) - S_n(s,\cdot))(\xi+\eta)|^2,
\end{aligned}$$

which goes to 0 as $m \to \infty$ by (H2). Therefore, $S_{n,m} \to S_n$ as $m \to \infty$ and $S_n \in \mathcal{P}_{0,Z}$. This concludes the proof under assumption (H2).

Now, we are going to consider assumption (H1) and check that $S_n \in \mathcal{P}_{0,Z}$ under this condition. We will take the same discretization of time to approximate S_n, but we will use the mean value over the time interval instead of the value at the right extremity. That is, we are going to consider

$$S_{n,m}(t,x) = \sum_{k=0}^{2^m-1} a_{n,m}^k(x) \mathbf{1}_{[t_m^k, t_m^{k+1}[}(t), \tag{3.10}$$

where $t_m^k = kT2^{-m}$ and

$$a_{n,m}^k(x) = \frac{2^m}{T} \int_{t_m^k}^{t_m^{k+1}} S_n(s,x)\, ds. \tag{3.11}$$

By (3.3) in assumption (H1), $a_{n,m}^k \in \mathcal{S}(\mathbb{R}^d)$ for all n, m and k. Moreover, using Fubini's theorem, which applies by (3.4) since $\int_{\mathbb{R}^d} dx \int_a^b ds\, |S_n(s,x)| < \infty$ for all $0 \leqslant a < b \leqslant T$, we have

$$\begin{aligned}
\mathcal{F}a_{n,m}^k(\xi) &= \frac{2^m}{T} \int_{\mathbb{R}^d} dx \int_{t_m^k}^{t_m^{k+1}} ds\, e^{-i\langle \xi, x\rangle} S_n(s,x) \\
&= \frac{2^m}{T} \int_{t_m^k}^{t_m^{k+1}} ds\, \mathcal{F}S_n(s,\cdot)(\xi).
\end{aligned}$$

We now show that $S_{n,m} \in \mathcal{P}_{+,Z}$. We only need to show that $a_{n,m}^k(x)\mathbf{1}_{[t_m^k, t_m^{k+1}[}(t) \in \mathcal{P}_{+,Z}$ for all $k = 1, \ldots, 2^m - 1$. We have

$$\|a_{n,m}^k(\cdot)\mathbf{1}_{[t_m^k, t_m^{k+1}[}(\cdot)\|_{+,Z} \leqslant C\frac{2^m}{T} \int_{\mathbb{R}^d} dz\, f(z)(|a_{n,m}^k(\cdot)| * |\widetilde{a_{n,m}^k(\cdot)}|)(z),$$

where $\widetilde{a_{n,m}^k}(x) = a_{n,m}^k(-x)$. Since $a_{n,m}^k \in \mathcal{S}(\mathbb{R}^d)$, a similar argument as above, using Leibnitz' formula, shows that this expression is finite. Hence $S_{n,m} \in \mathcal{P}_{+,Z} \subset \mathcal{P}_{0,Z}$.

It remains to show that $S_{n,m} \to S_n$ as $m \to \infty$. Indeed,

$$\begin{aligned}
&\|S_{n,m} - S_n\|_{0,Z}^2 \\
&= \int_0^T ds \int_{\mathbb{R}^d} \nu_s^Z(d\eta) \int_{\mathbb{R}^d} \mu(d\xi)\, |\mathcal{F}(S_{n,m}(s,\cdot) - S_n(s,\cdot))(\xi+\eta)|^2 \\
&= \sum_{k=0}^{2^m-1} \int_{t_m^k}^{t_m^{k+1}} ds \int_{\mathbb{R}^d} \nu_s^Z(d\eta) \int_{\mathbb{R}^d} \mu(d\xi)\, |\mathcal{F}a_{n,m}^k(\xi+\eta) - \mathcal{F}S_n(s,\cdot)(\xi+\eta)|^2 \\
&= \sum_{k=0}^{2^m-1} \int_{t_m^k}^{t_m^{k+1}} ds \int_{\mathbb{R}^d} \nu_s^Z(d\eta) \int_{\mathbb{R}^d} \mu(d\xi) \left| \frac{2^m}{T} \int_{t_m^k}^{t_m^{k+1}} \mathcal{F}S_n(u,\cdot)(\xi+\eta) du \right. \\
&\qquad \left. - \mathcal{F}S_n(s,\cdot)(\xi+\eta) \right|^2.
\end{aligned} \tag{3.12}$$

We are going to show that the preceding expression goes to 0 as $m \to \infty$ using the martingale L^2-convergence theorem (see [12, thm 4.5, p.252]). Take $\Omega = \mathbb{R}^d \times \mathbb{R}^d \times [0,T]$, endowed with the σ-field $\mathcal{F} = \mathcal{B}(\mathbb{R}^d) \times \mathcal{B}(\mathbb{R}^d) \times \mathcal{B}([0,T])$ of Borel subsets and the measure $\mu(d\xi) \times \nu_s^Z(d\eta) \times ds$. We also consider the filtration $(\mathcal{H}_m = \mathcal{B}(\mathbb{R}^d) \times \mathcal{B}(\mathbb{R}^d) \times \mathcal{G}_m)_{m \geqslant 0}$, where $\mathcal{G}_m = \sigma([t_m^k, t_m^{k+1}[, k = 0, \ldots, 2^m - 1)$. For

n fixed, we consider the function $X : \Omega \to \mathbb{R}$ given by $X(\xi, \eta, s) = \mathcal{F}S_n(s, \cdot)(\xi+\eta)$. This function is in $L^2(\Omega, \mathcal{F}, \mu(d\xi) \times \nu_s^Z(d\eta) \times ds)$. Indeed,

$$\int_0^T ds \int_{\mathbb{R}^d} \nu_s^Z(d\eta) \int_{\mathbb{R}^d} \mu(d\xi)\, |\mathcal{F}S_n(s, \cdot)(\xi+\eta)|^2 \\ \leqslant C \int_0^T ds \sup_{\eta \in \mathbb{R}^d} \int_{\mathbb{R}^d} \mu(d\xi)\, |\mathcal{F}S(s, \cdot)(\xi+\eta)|^2,$$

which is finite by assumption (3.6). Then, setting

$$X_m = \mathbb{E}_{\mu(d\xi)\times\nu_s^Z(d\eta)\times ds}[X|\mathcal{H}_m] = \sum_{k=0}^{2^m-1} \left(\frac{2^m}{T} \int_{t_m^k}^{t_m^{k+1}} \mathcal{F}S_n(u, \cdot)(\xi+\eta) du \right) \mathbf{1}_{[t_m^k, t_m^{k+1}[}(s),$$

we have that $(X_m)_{m \geqslant 0}$ is a martingale. Moreover,

$$\sup_m \mathbb{E}_{\mu(d\xi)\times\nu_s^Z(d\eta)\times ds}[X_m^2] \leqslant \mathbb{E}_{\mu(d\xi)\times\nu_s^Z(d\eta)\times ds}[X^2] < \infty.$$

The martingale L^2-convergence theorem then shows that (3.12) goes to 0 as $m \to \infty$ and hence that $S_n \in \mathcal{P}_{0,Z}$.

Now, by the isometry property of the stochastic integral between $\mathcal{P}_{0,Z}$ and the set $\mathcal{M}^2$ of square-integrable martingales, $(S \cdot M^Z)_t$ is well-defined and

$$\mathbb{E}[(S \cdot M^Z)_T^2] = \|S\|_{0,Z}^2 = \int_0^T ds \int_{\mathbb{R}^d} \nu_s^Z(d\eta) \int_{\mathbb{R}^d} \mu(d\xi)\, |\mathcal{F}S(s, \cdot)(\xi+\eta)|^2.$$

The bound in the second part of (3.7) is obtained as in (3.2). The result is proved. ■

Remark 3.2. As can be seen by inspecting the proof, Theorem 3.1 is still valid if we replace (H2) by the following assumptions :

- $t \mapsto \mathcal{F}S(t)(\xi)$ is continuous in t for all $\xi \in \mathbb{R}^d$;
- there exists a function $t \mapsto k(t)$ with values in the space $\mathcal{S}'_r(\mathbb{R}^d)$ such that, for all $0 \leqslant t \leqslant T$ and $h \in [0, \varepsilon]$,

  $$|\mathcal{F}S(t+h)(\xi) - \mathcal{F}S(t)(\xi)| \leqslant |\mathcal{F}k(t)(\xi)|,$$

  and

  $$\int_0^T ds \sup_{\eta \in \mathbb{R}^d} \int_{\mathbb{R}^d} \mu(d\xi)\, |\mathcal{F}k(s)(\xi+\eta)|^2 < +\infty.$$

Remark 3.3. There are two limitations to our construction of the stochastic integral in Theorem 3.1. The first concerns stationarity of the covariance of Z (hypothesis (H)). Under certain conditions (which, in the case where S is the fundamental solution of the wave equation, only hold for $d \leqslant 3$), Nualart and Quer-Sardanyons [18] have removed this assumption. Moreover, in Chapter 6, we will show that, in some cases, a slightly more general class of Z can be considered, namely the harmonizable processes. The second limitation concerns positivity of the covariance function f. A weaker condition appears in [19], where function-valued solutions are studied.

3.2 Extension in deterministic integrals

In addition to the stochastic integral defined above, we will have to define the integral of the product of a Schwartz distribution and a spatially homogeneous process with respect to Lebesgue measure. More precisely, we have to give a rigorous definition to the process informally given by

$$t \mapsto \int_0^t ds \int_{\mathbb{R}^d} dy\, S(s,y) Z(s,y),$$

where $t \mapsto S(t)$ is a deterministic function with values in the space of Schwartz distributions with rapid decrease and Z is a stochastic process, both satisfying the assumptions of Theorem 3.1.

In addition, suppose first that $S \in L^2([0,T], L^1(\mathbb{R}^d))$. By Hölder's inequality, we have

$$\begin{aligned}
&\mathbb{E}\left[\left(\int_0^T ds \int_{\mathbb{R}^d} dx\, |S(s,x)||Z(s,x)|\right)^2\right] \\
&\leqslant\ C\mathbb{E}\left[\int_0^T ds \left(\int_{\mathbb{R}^d} dx\, |S(s,x)||Z(s,x)|\right)^2\right] \\
&\leqslant\ C\int_0^T ds \int_{\mathbb{R}^d} dx\, |S(s,x)| \int_{\mathbb{R}^d} dy\, |S(s,y)|\, \mathbb{E}[|Z(s,x)||Z(s,y)|] \\
&\leqslant\ C\int_0^T ds \int_{\mathbb{R}^d} dx\, |S(s,x)| \int_{\mathbb{R}^d} dy\, |S(s,y)| < \infty,
\end{aligned} \tag{3.13}$$

by the assumptions on Z. Hence $\int_0^T ds \int_{\mathbb{R}^d} dx\, |S(s,x)||Z(s,x)| < \infty$ a.s. and the process

$$\int_0^t ds \int_{\mathbb{R}^d} dx\, S(s,x) Z(s,x), \qquad t \geqslant 0,$$

is a.s. well-defined as a Lebesgue-integral. Moreover,

$$\begin{aligned}
0 &\leqslant \mathbb{E}\left[\left(\int_0^T ds \int_{\mathbb{R}^d} dx\, S(s,x)Z(s,x)\right)^2\right] \\
&= \int_0^T ds \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^d} dy\, S(s,x)S(s,y)\, \mathbb{E}[Z(s,x)Z(s,y)] \\
&= \int_0^T ds \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^d} dy\, S(s,x)S(s,y) g_s(x-y) \\
&= \int_0^T ds \int_{\mathbb{R}^d} \nu_s^Z(d\eta)\, |\mathcal{F}S(s)(\eta)|^2, \qquad (3.14)
\end{aligned}$$

where ν_s^Z is the measure such that $\mathcal{F}\nu_s^Z = g_s$. Let us define a norm $\|\cdot\|_{1,Z}$ on the space $\overline{\mathcal{P}}_Z$ by

$$\|S\|_{1,Z}^2 = \int_0^T ds \int_{\mathbb{R}^d} \nu_s^Z(d\eta)\, |\mathcal{F}S(s)(\eta)|^2. \qquad (3.15)$$

This norm is similar to $\|\cdot\|_{0,Z}$, but with $\mu(d\xi) = \delta_0(d\xi)$. In order to establish the next proposition, we will need the following assumption.

(H2*) The function $\mathcal{F}S(s)$ is such that

$$\lim_{h\downarrow 0} \int_0^T ds \sup_{\eta\in\mathbb{R}^d} \sup_{s<r<s+h} |\mathcal{F}S(r)(\eta) - \mathcal{F}S(s)(\eta)|^2 = 0. \qquad (3.16)$$

This hypothesis is analogous to (H2) but with $\mu(d\xi) = \delta_0(d\xi)$.

Proposition 3.4. *Let $(Z(t,x),\ 0 \leqslant t \leqslant T,\ x \in \mathbb{R}^d)$ be a stochastic process satisfying the assumptions of Theorem 3.1. Let $t \mapsto S(t)$ be a deterministic function with values in the space $\mathcal{S}'_r(\mathbb{R}^d)$. Suppose that $(s,\xi) \mapsto \mathcal{F}S(s)(\xi)$ is measurable and*

$$\int_0^T ds \sup_{\eta\in\mathbb{R}^d} |\mathcal{F}S(s)(\eta)|^2 < \infty. \qquad (3.17)$$

Suppose in addition that either hypothesis (H1) or (H2) is satisfied. Then*

$$\begin{aligned}
&\mathbb{E}\left[\left(\int_0^T ds \int_{\mathbb{R}^d} dx\, S(s,x)Z(s,x)\right)^2\right] \\
&= \quad \|S\|_{1,Z}^2 \quad \leqslant \quad C\left(\sup_{0\leqslant s\leqslant T}\sup_{x\in\mathbb{R}^d} \mathbb{E}[Z(s,x)^2]\right) \int_0^T ds \sup_{\eta\in\mathbb{R}^d} |\mathcal{F}S(s)(\eta)|^2.
\end{aligned}$$

In particular, the process $\left(\int_0^t ds \int_{\mathbb{R}^d} dx\, S(s,x)Z(s,x),\ 0 \leqslant t \leqslant T\right)$ is well defined and takes values in $L^2(\Omega)$.

Proof. We will consider $(S_n)_{n\in\mathbb{N}}$ and $(S_{n,m})_{n,m\in\mathbb{N}}$ to be the same approximating sequences of S as in the proof of Theorem 3.1. Recall that the sequence $(S_{n,m})$ depends on which of (H1) or (H2*) is satisfied. If (H1) is satisfied, then (3.10), (3.11) and (H1) show that $S_{n,m} \in L^2([0,T], L^1(\mathbb{R}^d))$. If (H2*) is satisfied, then (3.9) and the fact that $S_n \in \mathcal{S}(\mathbb{R}^d)$ shows that $S_{n,m} \in L^2([0,T], L^1(\mathbb{R}^d))$. Hence, by (3.13), the process $t \mapsto \int_0^t ds \int_{\mathbb{R}^d} dx\, S_{n,m}(s,x)Z(s,x)$ is well-defined.

Moreover, by arguments analogous to those used in the proof of Theorem 3.1, where we just consider $\mu(d\xi) = \delta_0(d\xi)$, replace (3.6) by (3.17) and (H2) by (H2*), we can show that

$$\|S_{n,m} - S_n\|_{1,Z} \to 0, \qquad \text{as } m \to \infty,$$

in both cases. As a consequence, the sequence

$$\left(\int_0^T ds \int_{\mathbb{R}} dx\, S_{n,m}(s,x)Z(s,x)\right)_{m\in\mathbb{N}}$$

is Cauchy in $L^2(\Omega)$ by (3.14) and hence converges. We set the limit of this sequence as the definition of $\int_0^T ds \int_{\mathbb{R}^d} dx\, S_n(s,x)Z(s,x)$ for any $n \in \mathbb{N}$. Note that (3.14) is still valid for S_n.

Using the same argument as in the proof of Theorem 3.1 again, we now can show that

$$\|S_n - S\|_{1,Z} \to 0, \qquad \text{as } n \to \infty.$$

Hence, by a Cauchy sequence argument similar to the one above, we can define the random variable $\int_0^T ds \int_{\mathbb{R}^d} dx\, S(s,x)Z(s,x)$ as a limit in $L^2(\Omega)$ by

$$\int_0^T ds \int_{\mathbb{R}^d} dx\, S(s,x)Z(s,x) = \lim_{n\to\infty} \int_0^T ds \int_{\mathbb{R}^d} dx\, S_n(s,x)Z(s,x).$$

Moreover, (3.14) remains true. ■

Remark 3.5. Assumption (3.17) appears in [9] to give estimates concerning an integral of the same type as in Proposition 3.4. In this reference, $S \geqslant 0$ and the process Z is considered to be in $L^2(\mathbb{R}^d)$, which is not the case here.

Chapter 4

Application to SPDE's

In this chapter, we apply the results of Chapter 3 on stochastic integration to construct random field solutions of non-linear stochastic partial differential equations. We will be interested in equations of the form

$$Lu(t,x) = \alpha(u(t,x))\dot{F}(t,x) + \beta(u(t,x)), \tag{4.1}$$

with vanishing initial conditions, where L is a second order partial differential operator with constant coefficients, $\dot{F}$ is the noise described in Chapter 2 and α, β are real-valued functions. Let Γ be the fundamental solution of equation $Lu(t,x) = 0$. In [2], Dalang shows that (4.1) admits a unique solution $(u(t,x),\ 0 \leqslant t \leqslant T,\ x \in \mathbb{R}^d)$ when Γ is a non-negative Schwartz distribution with rapid decrease. Moreover, this solution is in $L^p(\Omega)$ for all $p \geqslant 1$. Using the extension of the stochastic integral presented in Chapter 3, we are going to show that there is still a random-field solution when Γ is a (not necessarily non-negative) Schwartz distribution with rapid decrease. However, this solution will only be in $L^2(\Omega)$. We will see in Section 5.1 that this solution is in $L^p(\Omega)$ for any $p \geqslant 1$ in the case where α is an affine function and $\beta \equiv 0$. The question of uniqueness is considered in Theorem 4.8.

4.1 Existence and uniqueness for a class of SPDE's

By a random-field solution of (4.1), we mean a jointly measurable process $(u(t,x),$ $t \geqslant 0,\ x \in \mathbb{R}^d)$ such that $(t,x) \mapsto u(t,x)$ from $\mathbb{R}_+ \times \mathbb{R}^d$ into $L^2(\Omega)$ is continuous and satisfies the assumptions needed for the right-hand side of (4.3) below to be well defined, namely $(u(t,x))$ is a predictable process such that

$$\sup_{0 \leqslant t \leqslant T} \sup_{x \in \mathbb{R}^d} \mathbb{E}[u(t,x)^2] < \infty, \tag{4.2}$$

and such that, for $t \in [0,T]$, $\alpha(u(t,\cdot))$ and $\beta(u(t,\cdot))$ have stationary covariance and such that for all $0 \leqslant t \leqslant T$ and $x \in \mathbb{R}^d$, a.s.,

$$u(t,x) = \int_0^t \int_{\mathbb{R}^d} \Gamma(t-s,x-y)\alpha(u(s,y))M(ds,dy) + \int_0^t \int_{\mathbb{R}^d} \Gamma(t-s,x-y)\beta(u(s,y))\,ds\,dy. \tag{4.3}$$

In this equation, the first (stochastic) integral is defined in Theorem 3.1 and the second (deterministic) integral is defined in Proposition 3.4.

We recall the following integration result, which will be used in the proof of Lemma 4.6.

Proposition 4.1. *Let $\mathcal{B}$ be a Banach space with norm $\|\cdot\|_{\mathcal{B}}$. Let $f : \mathbb{R} \to \mathcal{B}$ be a function such that $f \in L^2(\mathbb{R},\mathcal{B})$, i.e.*

$$\int_{\mathbb{R}} \|f(s)\|^2_{\mathcal{B}}\,ds < +\infty.$$

Then

$$\lim_{|h|\to 0} \int_{\mathbb{R}} \|f(s+h) - f(s)\|^2_{\mathcal{B}}\,ds = 0.$$

Proof. For a proof in the case where $f \in L^1(\mathbb{R},\mathcal{B})$, see [16, Chap.XIII, Theorem 1.2, p.165]. Using the fact that simple functions are dense in $L^2(\mathbb{R},\mathcal{B})$ (see [11, Corollary III.3.8, p.125]), the proof in the case where $f \in L^2(\mathbb{R},\mathcal{B})$ is analogous. ■

Theorem 4.2. *Suppose that the fundamental solution Γ of equation $Lu = 0$ is a deterministic space-time Schwartz distribution of the form $\Gamma(t)dt$, where $\Gamma(t) \in \mathcal{S}'_r(\mathbb{R}^d)$, such that $(s,\xi) \mapsto \mathcal{F}\Gamma(s)(\xi)$ is measurable,*

$$\int_0^T ds \sup_{\eta\in\mathbb{R}^d} \int_{\mathbb{R}^d} \mu(d\xi)\,|\mathcal{F}\Gamma(s)(\xi+\eta)|^2 < \infty \tag{4.4}$$

and

$$\int_0^T ds \sup_{\eta\in\mathbb{R}^d} |\mathcal{F}\Gamma(s)(\eta)|^2 < \infty. \tag{4.5}$$

Suppose in addition that either hypothesis (H1), or hypotheses (H2) and (H2), are satisfied with S replaced by Γ. Then equation (4.1), with α and β Lipschitz functions, admits a random-field solution $(u(t,x),\ 0 \leqslant t \leqslant T, x \in \mathbb{R}^d)$.*

Remark 4.3. The main example, that we will treat in the following section, is the case of equation (1.5), where $L = \frac{\partial^2}{\partial t^2} - \Delta$ is the wave operator and $d \geqslant 4$.

Proof. We are going to use a Picard iteration scheme. Suppose that α and β have Lipschitz constant K, so that $|\alpha(u)| \leqslant K(1+|u|)$ and $|\beta(u)| \leqslant K(1+|u|)$. For $n \geqslant 0$, set

$$\begin{cases} u_0(t,x) \equiv 0, \\ Z_n(t,x) = \alpha(u_n(t,x)), \\ W_n(t,x) = \beta(u_n(t,x)), \\ u_{n+1}(t,x) = \displaystyle\int_0^t \int_{\mathbb{R}^d} \Gamma(t-s, x-y) Z_n(s,y) M(ds,dy) \\ \qquad\qquad\qquad + \displaystyle\int_0^t \int_{\mathbb{R}^d} \Gamma(t-s,x-y) W_n(s,y) ds\, dy. \end{cases} \tag{4.6}$$

Now suppose by induction that, for all $T > 0$,

$$\sup_{0 \leqslant t \leqslant T} \sup_{x \in \mathbb{R}^d} \mathbb{E}[u_n(t,x)^2] < \infty. \tag{4.7}$$

Suppose also that $u_n(t,x)$ is $\mathcal{F}_t$-measurable for all x and t, and that $(t,x) \mapsto u_n(t,x)$ is L^2-continuous. These conditions are clearly satisfied for $n = 0$. The L^2-continuity ensures that $(t,x;\omega) \mapsto u_n(t,x;\omega)$ has a jointly measurable version and that the conditions of [4, Prop.2] are satisfied. Moreover, Lemma 4.5 below shows that Z_n and W_n satisfy the assumptions needed for the stochastic integral and the integral with respect to Lebesgue-measure to be well-defined. Therefore, $u_{n+1}(t,x)$ is well defined in (4.6), and is L^2-continuous by Lemma 4.6. We now show that u_{n+1} satisfies (4.7). By (4.6),

$$\begin{aligned} \mathbb{E}[u_{n+1}(t,x)^2] \;\leqslant\; & 2\mathbb{E}\left[\left(\int_0^t \int_{\mathbb{R}^d} \Gamma(t-s,x-y) Z_n(s,y) M(ds,dy)\right)^2\right] \\ & + 2\mathbb{E}\left[\left(\int_0^t \int_{\mathbb{R}^d} \Gamma(t-s,x-y) W_n(s,y) ds\, dy\right)^2\right]. \end{aligned}$$

Using the linear growth of α, (4.7) and the fact that $\Gamma(s,\cdot) \in \mathcal{P}_{0,Z_n}$, (4.4) and Theorem 3.1 imply that

$$\sup_{0 \leqslant t \leqslant T} \sup_{x \in \mathbb{R}^d} \|\Gamma(t-\cdot, x-\cdot)\|_{0,Z_n}^2 < +\infty.$$

Further, the linear growth of β, (4.5) and Proposition 3.4 imply that

$$\sup_{0 \leqslant t \leqslant T} \sup_{x \in \mathbb{R}^d} \|\Gamma(t-\cdot, x-\cdot)\|_{1,W_n}^2 < +\infty.$$

It follows that the sequence $(u_n(t,x))_{n\geqslant 0}$ is well-defined. It remains to show that it converges in $L^2(\Omega)$. For this, we are going to use the generalization of Gronwall's lemma presented in [2, Lemma 15]. We have

$$\mathbb{E}[|u_{n+1}(t,x)-u_n(t,x)|^2] \leqslant 2A_n(t,x)+2B_n(t,x),$$

where

$$A_n(t,x)=\mathbb{E}\left[\left|\int_0^t\int_{\mathbb{R}^d}\Gamma(t-s,x-y)(Z_n(s,y)-Z_{n-1}(s,y))M(ds,dy)\right|^2\right]$$

and

$$B_n(t,x)=\mathbb{E}\left[\left|\int_0^t\int_{\mathbb{R}^d}\Gamma(t-s,x-y)(W_n(s,y)-W_{n-1}(s,y))ds\,dy\right|^2\right].$$

First consider $A_n(t,x)$. Set $Y_n=Z_n-Z_{n-1}$. By the Lipschitz property of α, the process Y_n satisfies the assumptions of Theorem 3.1 on Z by Lemma 4.5 below. Hence, by Theorem 3.1,

$$\begin{aligned}
A_n(t,x) &= C\int_0^t ds\int_{\mathbb{R}^d}\nu_s^{Y_n}(d\eta)\int_{\mathbb{R}^d}\mu(d\xi)\,|\mathcal{F}\Gamma(t-s,x-\cdot)(\xi+\eta)|^2\\
&\leqslant C\int_0^t ds\,\nu_s^{Y_n}(\mathbb{R}^d)\sup_{\eta\in\mathbb{R}^d}\int_{\mathbb{R}^d}\mu(d\xi)\,|\mathcal{F}\Gamma(t-s,x-\cdot)(\xi+\eta)|^2\\
&\leqslant C\int_0^t ds\left(\sup_{z\in\mathbb{R}^d}\mathbb{E}[Y_n(s,z)^2]\right)\sup_{\eta\in\mathbb{R}^d}\int_{\mathbb{R}^d}\mu(d\xi)\,|\mathcal{F}\Gamma(t-s,x-\cdot)(\xi+\eta)|^2.
\end{aligned}$$

Then set $M_n(t)=\sup_{x\in\mathbb{R}^d}\mathbb{E}[|u_{n+1}(t,x)-u_n(t,x)|^2]$ and

$$J_1(s)=\sup_{\eta\in\mathbb{R}^d}\int_{\mathbb{R}^d}\mu(d\xi)\,|\mathcal{F}\Gamma(s,\cdot)(\xi+\eta)|^2.$$

The Lipschitz property of α implies that

$$\begin{aligned}
\sup_{z\in\mathbb{R}^d}\mathbb{E}[Y_n(s,z)^2] &= \sup_{z\in\mathbb{R}^d}\mathbb{E}[(Z_n(s,z)-Z_{n-1}(s,z))^2]\\
&\leqslant \sup_{z\in\mathbb{R}^d}K^2\,\mathbb{E}[(u_n(s,z)-u_{n-1}(s,z))^2]\\
&\leqslant K^2M_{n-1}(s),
\end{aligned}$$

and we deduce that

$$A_n(t,x)\leqslant C\int_0^t ds\,M_{n-1}(s)J_1(t-s). \tag{4.8}$$

Now consider $B_n(t,x)$. Set $V_n = W_n - W_{n-1}$. The process V_n satisfies the assumptions of Theorem 3.1 on Z by Lemma 4.5 below. Hence, by Proposition 3.4,

$$\begin{aligned} B_n(t,x) &\leqslant C \int_0^t ds \int_{\mathbb{R}^d} \nu_s^{V_n}(d\eta)\, |\mathcal{F}\Gamma(t-s,x-\cdot)(\eta)|^2 \\ &\leqslant C \int_0^t ds\, \nu_s^{V_n}(\mathbb{R}^d) \sup_{\eta\in\mathbb{R}^d} |\mathcal{F}\Gamma(t-s,x-\cdot)(\eta)|^2 \\ &\leqslant C \int_0^t ds \left(\sup_{z\in\mathbb{R}^d} \mathbb{E}[V_n(s,z)^2]\right) \sup_{\eta\in\mathbb{R}^d} |\mathcal{F}\Gamma(t-s,x-\cdot)(\eta)|^2. \end{aligned}$$

Then set

$$J_2(s) = \sup_{\eta\in\mathbb{R}^d} |\mathcal{F}\Gamma(s,\cdot)(\eta)|^2.$$

The Lipschitz property of β implies that

$$\begin{aligned} \sup_{z\in\mathbb{R}^d} \mathbb{E}[V_n(s,z)^2] &\leqslant \sup_{z\in\mathbb{R}^d} \mathbb{E}[(W_n(s,z)-W_{n-1}(s,z))^2] \\ &\leqslant \sup_{z\in\mathbb{R}^d} K^2\, \mathbb{E}[(u_n(s,z)-u_{n-1}(s,z))^2] \\ &\leqslant K^2 M_{n-1}(s), \end{aligned}$$

and we deduce that

$$B_n(t,x) \leqslant C \int_0^t ds\, M_{n-1}(s) J_2(t-s). \tag{4.9}$$

Then, setting $J(s) = J_1(s) + J_2(s)$ and putting together (4.8) and (4.9), we obtain

$$M_n(t) \leqslant \sup_{x\in\mathbb{R}^d} (A_n(t,x) + B_n(t,x)) \leqslant C \int_0^t ds\, M_{n-1}(s) J(t-s).$$

Lemma 15 in [2] implies that $(u_n(t,x))_{n\geqslant 0}$ converges uniformly in L^2, say to $u(t,x)$. As a consequence of [2, Lemma 15], u_n satisfies (4.2) for any $n \geqslant 0$. Hence, u also satisfies (4.2) as the L^2-limit of the sequence $(u_n)_{n\geqslant 0}$. As u_n is continuous in L^2 by Lemma 4.6 below, u is also continuous in L^2. Therefore, u admits a jointly measurable version, which, by Lemma 4.5 below has the property that $\alpha(u(t,\cdot))$ and $\beta(u(t,\cdot))$ have stationary covariance functions. The process u satisfies (4.3) by passing to the limit in (4.6). ■

The following definition and lemmas were used in the proof of Theorem 4.2 and will be used in Theorem 4.8.

Definition 4.4 ("S" property)**.** For $z \in \mathbb{R}^d$, write $z + B = \{z + y : y \in B\}$, $M_s^{(z)}(B) = M_s(z + B)$ and $Z^{(z)}(s, x) = Z(s, x + z)$. We say that the process $(Z(s, x),\ s \geqslant 0,\ x \in \mathbb{R}^d)$ has the "S" property if, for all $z \in \mathbb{R}^d$, the finite dimensional distributions of

$$\left((Z^{(z)}(s, x),\ s \geqslant 0,\ x \in \mathbb{R}^d)), (M_s^{(z)}(B),\ s \geqslant 0,\ B \in \mathcal{B}_b(\mathbb{R}^d))\right)$$

do not depend on z.

Lemma 4.5. *For $n \geqslant 1$, the process $(u_n(s, x), u_{n-1}(s, x),\ 0 \leqslant s \leqslant T,\ x \in \mathbb{R}^d)$ admits the "S" property.*

Proof. It follows from the definition of the martingale measure M and the fact that u_0 is constant that the finite dimensional distributions of $(u_0^{(z)}(s, x), M_s^{(z)}(B),\ s \geqslant 0,\ x \in \mathbb{R}^d,\ B \in \mathcal{B}_b(\mathbb{R}^d))$ do not depend on z. Now, we can write

$$u_1(t, x) = \int_0^t \int_{\mathbb{R}^d} \Gamma(t - s, -y)\alpha(0)M^{(x)}(ds, dy) + \int_0^t \int_{\mathbb{R}^d} \Gamma(t - s, -y)\beta(0)ds\, dy,$$

so $u_1(t, x)$ is an abstract function Φ of $M^{(x)}$. As the function Φ does not depend on x, we have $u_1^{(z)}(t, x) = \Phi(M^{(x+z)})$. Then, for $(s_1, \ldots, s_k)$, $(t_1, \ldots, t_j) \in \mathbb{R}_+^k, \mathbb{R}_+^j$, $(x_1, \ldots, x_k) \in (\mathbb{R}^d)^k$, $B_1, \ldots, B_j \in \mathcal{B}_b(\mathbb{R}^d)$, the joint distribution of

$$\left(u_1^{(z)}(s_1, x_1), \ldots, u_1^{(z)}(s_k, x_k), M_{t_1}^{(z)}(B_1), \ldots, M_{t_j}^{(z)}(B_j)\right)$$

is an abstract function of the distribution of

$$\left(M_\cdot^{(z+x_1)}(\cdot), \ldots, M_\cdot^{(z+x_k)}(\cdot), M_{t_1}^{(z)}(B_1), \ldots, M_{t_j}^{(z)}(B_j)\right),$$

which, as mentioned above, does not depend on z. Hence, the conclusion holds for $n = 1$, because u_0 is constant. Now suppose that the conclusion holds for some $n \geqslant 1$ and show that it holds for $n + 1$. We can write

$$\begin{aligned} u_{n+1}(t, x) &= \int_0^t \int_{\mathbb{R}^d} \Gamma(t - s, -y)\alpha(u_n^{(x)}(s, y))M^{(x)}(ds, dy) \\ &\qquad + \int_0^t \int_{\mathbb{R}^d} \Gamma(t - s, -y)\beta(u_n^{(x)}(s, y))ds\, dy, \end{aligned}$$

so $u_{n+1}(t, x)$ is an abstract function Ψ of $u_n^{(x)}$ and $M^{(x)}$: $u_{n+1}(t, x) = \Psi(u_n^{(x)}, M^{(x)})$. The function Ψ does not depend on x and we have $u_{n+1}^{(z)}(t, x) = \Psi(u_n^{(x+z)}, M^{(x+z)})$.

Hence, for every choice of $(s_1, \ldots, s_k) \in \mathbb{R}_+^k$, $(t_1, \ldots, t_j) \in \mathbb{R}_+^j$, $(r_1, \ldots, r_\ell) \in \mathbb{R}_+^\ell$, and $(x_1, \ldots, x_k) \in (\mathbb{R}^d)^k$, $(y_1, \ldots, y_j) \in (\mathbb{R}^d)^j$, the joint distribution of

$$\left(u_{n+1}^{(z)}(s_1, x_1), \ldots, u_{n+1}^{(z)}(s_k, x_k), u_n^{(z)}(t_1, y_1), \ldots, u_n^{(z)}(t_j, y_j), M_{r_1}^{(z)}(B_1), \ldots, M_{r_\ell}^{(z)}(B_\ell)\right)$$

is an abstract function of the distribution of

$$\begin{aligned}\big(u_n^{(z+x_1)}(\cdot,\cdot),\dots,u_n^{(z+x_k)}(\cdot,\cdot),u_n^{(z)}(\cdot,\cdot),M_\cdot^{(z+x_1)}(\cdot),\dots,M_\cdot^{(z+x_k)}(\cdot),\\ M_{r_1}^{(z)}(B_1),\dots,M_{r_\ell}^{(z)}(B_\ell)\big)\,,\end{aligned}$$

which does not depend on z by the induction hypothesis. ■

Lemma 4.6. *For all $n \geqslant 0$, the process $(u_n(t,x),\ t \geqslant 0,\ x \in \mathbb{R}^d)$ defined in (4.6) is continuous in $L^2(\Omega)$.*

Proof. For $n = 0$, the result is trivial. We are going to show by induction that if $(u_n(t,x),\ t \geqslant 0,\ x \in \mathbb{R}^d)$ is continuous in L^2, then $(u_{n+1}(t,x),\ t \geqslant 0,\ x \in \mathbb{R}^d)$ is too.

We begin with time increments. We have

$$\mathbb{E}[(u_{n+1}(t,x) - u_{n+1}(t+h,x))^2] \leqslant 2A_n(t,x,h) + 2B_n(t,x,h),$$

where

$$\begin{aligned}A_n(t,x,h) &= \mathbb{E}\Bigg[\bigg(\int_0^{t+h}\int_{\mathbb{R}^d}\Gamma(t+h-s,x-y)Z_n(s,y)M(ds,dy)\\ &\qquad -\int_0^{t}\int_{\mathbb{R}^d}\Gamma(t-s,x-y)Z_n(s,y)M(ds,dy)\bigg)^2\Bigg]\end{aligned}$$

and

$$\begin{aligned}B_n(t,x,h) &= \mathbb{E}\Bigg[\bigg(\int_0^{t+h}\int_{\mathbb{R}^d}\Gamma(t+h-s,x-y)W_n(s,y)ds\,dy\\ &\qquad -\int_0^{t}\int_{\mathbb{R}^d}\Gamma(t-s,x-y)W_n(s,y)ds\,dy\bigg)^2\Bigg].\end{aligned}$$

First of all, $A_n(t,x,h) \leqslant X_1 + X_2$, where

$$\begin{aligned}X_1 &= \mathbb{E}\Bigg[\bigg(\int_0^{t}\int_{\mathbb{R}^d}(\Gamma(t+h-s,x-y)-\Gamma(t-s,x-y))Z_n(s,y)M(ds,dy)\bigg)^2\Bigg],\\ X_2 &= \mathbb{E}\Bigg[\bigg(\int_t^{t+h}\int_{\mathbb{R}^d}\Gamma(t+h-s,x-y)Z_n(s,y)M(ds,dy)\bigg)^2\Bigg].\end{aligned}$$

The term X_2 goes to 0 as $h \to 0$ because, by (3.7),

$$\begin{aligned} X_2 &\leqslant \sup_{0\leqslant s\leqslant T} \mathbb{E}[Z_n(s,0)^2] \int_t^{t+h} ds \sup_{\eta\in\mathbb{R}^d} \int_{\mathbb{R}^d} \mu(d\xi)\, |\mathcal{F}\Gamma(t+h-s, x-\cdot)(\xi+\eta)|^2 \\ &= \sup_{0\leqslant s\leqslant T} \mathbb{E}[Z_n(s,0)^2] \int_0^{h} ds \sup_{\eta\in\mathbb{R}^d} \int_{\mathbb{R}^d} \mu(d\xi)\, |\mathcal{F}\Gamma(s, x-\cdot)(\xi+\eta)|^2 \\ &\underset{h\to 0}{\longrightarrow} 0, \end{aligned}$$

by the Dominated Convergence Theorem and (4.4). Concerning X_1, we have

$$\begin{aligned} X_1 &= \int_0^t ds \int_{\mathbb{R}^d} \nu_s^{Z_n}(d\eta) \int_{\mathbb{R}^d} \mu(d\xi)\, |\mathcal{F}\Gamma(t+h-s)(\xi+\eta) - \mathcal{F}\Gamma(t-s)(\xi+\eta)|^2 \\ &= \int_0^t ds \int_{\mathbb{R}^d} \nu_{t-s}^{Z_n}(d\eta) \int_{\mathbb{R}^d} \mu(d\xi)\, |\mathcal{F}\Gamma(s+h)(\xi+\eta) - \mathcal{F}\Gamma(s)(\xi+\eta)|^2 \\ &\leqslant C \int_0^t ds \sup_{\eta\in\mathbb{R}^d} \int_{\mathbb{R}^d} \mu(d\xi)\, |\mathcal{F}\Gamma(s+h)(\xi+\eta) - \mathcal{F}\Gamma(s)(\xi+\eta)|^2 \end{aligned}$$

This integral goes to 0 as $h \to 0$, either by (4.4) and Proposition 4.1 with $\mathcal{B} = L^\infty(\mathbb{R}^d, L^2_\mu(\mathbb{R}^d))$ and $f(s;\eta,\xi) = \mathcal{F}\Gamma(s)(\xi+\eta)\mathbf{1}_{[0,T]}(s)$, or by assumption (H2).

Secondly, $B_n(t,x,h) \leqslant Y_1 + Y_2$, where

$$\begin{aligned} Y_1 &= \mathbb{E}\left[\left(\int_0^t \int_{\mathbb{R}^d} (\Gamma(t+h-s, x-y) - \Gamma(t-s, x-y)) W_n(s,y) ds\, dy\right)^2\right], \\ Y_2 &= \mathbb{E}\left[\left(\int_t^{t+h} \int_{\mathbb{R}^d} \Gamma(t+h-s, x-y) W_n(s,y) ds\, dy\right)^2\right]. \end{aligned}$$

The term Y_2 goes to 0 as $h \to 0$ because, by Proposition 3.4,

$$\begin{aligned} 0 \leqslant Y_2 &\leqslant \sup_{0\leqslant s\leqslant T} \mathbb{E}[W_n(s,0)^2] \int_t^{t+h} ds \sup_{\eta\in\mathbb{R}^d} |\mathcal{F}\Gamma(t+h-s, x-\cdot)(\eta)|^2 \\ &= \sup_{0\leqslant s\leqslant T} \mathbb{E}[W_n(s,0)^2] \int_0^{h} ds \sup_{\eta\in\mathbb{R}^d} |\mathcal{F}\Gamma(s, x-\cdot)(\eta)|^2 \\ &\underset{h\to 0}{\longrightarrow} 0, \end{aligned}$$

by the Dominated Convergence Theorem. Concerning Y_1, we have

$$\begin{aligned} Y_1 &= \int_0^t ds \int_{\mathbb{R}^d} \nu_s^{W_n}(d\eta)\, |\mathcal{F}\Gamma(t+h-s)(\eta) - \mathcal{F}\Gamma(t-s)(\eta)|^2 \\ &= \int_0^t ds \int_{\mathbb{R}^d} \nu_{t-s}^{W_n}(d\eta)\, |\mathcal{F}\Gamma(s+h)(\eta) - \mathcal{F}\Gamma(s)(\eta)|^2 \\ &\leqslant C \int_0^t ds \sup_{\eta\in\mathbb{R}^d} |\mathcal{F}\Gamma(s+h)(\eta) - \mathcal{F}\Gamma(s)(\eta)|^2 \end{aligned}$$

This integral goes to 0 as $h \to 0$ either by (4.5) and Proposition 4.1 with $\mathcal{B} = L^\infty(\mathbb{R}^d)$ and $f(s;\eta) = \mathcal{F}\Gamma(s)(\eta)\mathbf{1}_{[0,T]}(s)$, or by assumption (H2*). This establishes the L^2-continuity in time.

Turning to spatial increments, we have

$$\mathbb{E}[(u_{n+1}(t,x+z) - u_{n+1}(t,x))^2] \leqslant 2C_n(t,x,z) + 2D_n(t,x,z),$$

where

$$\begin{aligned} C_n(t,x,z) &= \mathbb{E}\Bigg[\bigg(\int_0^t \int_{\mathbb{R}^d} \Gamma(t-s,x+z-y) Z_n(s,y) M(ds,dy) \\ &\qquad - \int_0^t \int_{\mathbb{R}^d} \Gamma(t-s,x-y) Z_n(s,y) M(ds,dy)\bigg)^2\Bigg] \end{aligned}$$

and

$$\begin{aligned} D_n(t,x,z) &= \mathbb{E}\Bigg[\bigg(\int_0^t \int_{\mathbb{R}^d} \Gamma(t-s,x+z-y) W_n(s,y) ds\, dy \\ &\qquad - \int_0^t \int_{\mathbb{R}^d} \Gamma(t-s,x-y) W_n(s,y) ds\, dy\bigg)^2\Bigg]. \end{aligned}$$

First consider C_n. We have

$$\begin{aligned} &C_n(t,x,z) \\ &= \int_0^t ds \int_{\mathbb{R}^d} \nu_s^{Z_n}(d\eta) \int_{\mathbb{R}^d} \mu(d\xi)\, |\mathcal{F}\Gamma(t-s,x+z-\cdot)(\xi+\eta) \\ &\qquad\qquad - \mathcal{F}\Gamma(t-s,x-\cdot)(\xi+\eta)|^2 \\ &= \int_0^t ds \int_{\mathbb{R}^d} \nu_s^{Z_n}(d\eta) \int_{\mathbb{R}^d} \mu(d\xi)\, |1 - e^{-i\langle\xi+\eta,z\rangle}|^2\, |\mathcal{F}\Gamma(t-s,\cdot)(\xi+\eta)|^2. \end{aligned}$$

Clearly, $|1 - e^{-i\langle\xi+\eta,z\rangle}|^2 \leqslant 4$ and the integrand converges to 0 as $\|z\| \to 0$. Therefore, for n fixed, by the Dominated Convergence Theorem, $C_n(t,x,z) \to 0$ as $\|z\| \to 0$.

Moreover, considering D_n, we have

$$\begin{aligned} D_n(t,x,z) &= \int_0^t ds \int_{\mathbb{R}^d} \nu_s^{W_n}(d\eta)\, |\mathcal{F}\Gamma(t-s,x+z-\cdot)(\eta) - \mathcal{F}\Gamma(t-s,x-\cdot)(\eta)|^2 \\ &= \int_0^t ds \int_{\mathbb{R}^d} \nu_s^{W_n}(d\eta)\, |1-e^{-i\langle \eta,z\rangle}|^2\, |\mathcal{F}\Gamma(t-s,\cdot)(\eta)|^2. \end{aligned}$$

Clearly, $|1-e^{-i\langle \eta,z\rangle}|^2 \leqslant 4$ and the integrand converges to 0 as $\|z\| \to 0$. Therefore, for n fixed, by the Dominated Convergence Theorem, $D_n(t,x,z) \to 0$ as $\|z\| \to 0$. This establishes the L^2-continuity in the spatial variable. ■

Remark 4.7. The induction assumption on the L^2-continuity of u_n is stronger than needed to show the L^2-continuity of u_{n+1}. In order that the stochastic integral process $\Gamma(t-\cdot, x-\cdot)\cdot M^Z$ be L^2-continuous, it suffices that the process Z satisfy the assumptions of Theorem 3.1.

We can now state the following theorem, which ensures uniqueness of the solution constructed in Theorem 4.2 within a more specific class of processes.

Theorem 4.8. *Under the assumptions of Theorem 4.2, let $u(t,x)$ be the solution of equation (4.3) constructed in the proof of Theorem 4.2. Let $(v(t,x),\ t\in[0,T],\ x\in\mathbb{R}^d)$ be a jointly measurable, predictable process such that*

$$\sup_{0\leqslant t\leqslant T}\ \sup_{x\in\mathbb{R}^d} \mathbb{E}[v(t,x)^2] < \infty,$$

that satisfies property "S" and (4.3). Then, for all $0 \leqslant t \leqslant T$ and $x \in \mathbb{R}^d$, $v(t,x) = u(t,x)$ a.s.

Proof. We are going to show that $\mathbb{E}[(u(t,x)-v(t,x))^2] = 0$. In the case where Γ is a non-negative distribution, we consider the sequence $(u_n)_{n\in\mathbb{N}}$ used to construct u, defined in (4.6). The approximating sequence $(\Gamma_m)_{m\geqslant 0}$ built in [2, Theorem 2] to define the stochastic integral is a positive function. Hence the stochastic integral below is a Walsh stochastic integral and using the Lipschitz property of α, we have (in the case $\beta \equiv 0$):

$$\begin{aligned}
&\mathbb{E}[(u_{n+1}(t,x) - v(t,x))^2] \\
&= \lim_{m\to\infty} \mathbb{E}\left[\left(\int_0^t \int_{\mathbb{R}^d} \Gamma_m(t-s,x-y)(\alpha(u_n(s,y)) - \alpha(v(s,y)))M(ds,dy)\right)^2\right] \\
&= \lim_{m\to\infty} \mathbb{E}\Bigg[\int_0^t ds \int_{\mathbb{R}^d} dy \int_{\mathbb{R}^d} dz\, \Gamma_m(t-s,x-y)(\alpha(u_n(s,y)) - \alpha(v(s,y))) \\
&\qquad \times f(x-y)(\alpha(u_n(s,z)) - \alpha(v(s,z)))\Gamma_m(t-s,x-z)\Bigg] \\
&\leqslant \lim_{m\to\infty} \int_0^t ds \sup_{y\in\mathbb{R}^d} \mathbb{E}[(u_n(s,y) - v(s,y))^2] \int_{\mathbb{R}^d} \mu(d\xi)|\mathcal{F}\Gamma_m(t-s,x-\cdot)(\xi)|^2.
\end{aligned}$$

Using a Gronwall-type argument ([2, Lemma 15]), uniqueness follows.

In the case considered here, the sequence $(\Gamma_m)_{m\geqslant 0}$ is not necessarily positive and the argument above does not apply. We need to know a priori that the processes $Z(t,x) = \alpha(u_n(t,x)) - \alpha(v(t,x))$ and $W(t,x) = \beta(u_n(t,x)) - \beta(v(t,x))$ have a spatially homogeneous covariance. This is why we consider the restricted class of processes satisfying property "S".

As $u_0 \equiv 0$, it is clear that the joint process $(u_0(t,x), v(t,x),\, t \geqslant 0,\, x \in \mathbb{R}^d)$ satisfies the "S" property. A proof analogous to that of Lemma 4.5 with u_{n-1} replaced by v shows that the process $(u_n(t,x), v(t,x),\, t \geqslant 0,\, x \in \mathbb{R}^d)$ also satisfies the "S" property. Then $\alpha(u_n(t,\cdot)) - \alpha(v(t,\cdot))$ and $\beta(u_n(t,\cdot)) - \beta(v(t,\cdot))$ have spatially homogeneous covariances. This ensures that the stochastic integrals below are well defined. We have

$$\mathbb{E}[(u_n(t,x) - v(t,x))^2] \leqslant 2A(t,x) + 2B(t,x),$$

where

$$A_n(t,x) = \mathbb{E}\left[\left(\int_0^t \int_{\mathbb{R}^d} \Gamma(t-s,x-y)(\alpha(u_n(t,x)) - \alpha(v(t,x)))M(ds,dy)\right)^2\right]$$

and

$$B_n(t,x) = \mathbb{E}\left[\left(\int_0^t \int_{\mathbb{R}^d} \Gamma(t-s,x-y)(\beta(u_n(t,x)) - \beta(v(t,x)))ds\,dy\right)^2\right].$$

Clearly,

$$\begin{aligned}
&A_n(t,x) \\
&\leqslant C \int_0^t ds \sup_{x\in\mathbb{R}^d} \mathbb{E}[(u_{n-1}(t,x) - v(t,x))^2] \sup_{\eta\in\mathbb{R}^d} \int_{\mathbb{R}^d} \mu(d\xi)\,|\mathcal{F}\Gamma(t-s,\cdot)(\xi+\eta)|^2.
\end{aligned} \tag{4.10}$$

Setting

$$\tilde{M}_n(t) = \sup_{x\in\mathbb{R}^d} \mathbb{E}[(u_n(t,x) - v(t,x))^2]$$

and using the notations in the proof of Theorem 4.2 we obtain, by (4.10),

$$A_n(t,x) \leqslant \int_0^t \tilde{M}_{n-1}(s) J_1(t-s) ds.$$

Moreover,

$$B_n(t,x) \leqslant C \int_0^t ds \sup_{x\in\mathbb{R}^d} \mathbb{E}[(u_{n-1}(t,x) - v(t,x))^2] \sup_{\eta\in\mathbb{R}^d} |\mathcal{F}\Gamma(t-s,\cdot)(\eta)|^2, \tag{4.11}$$

so

$$B_n(t,x) \leqslant \int_0^t \tilde{M}_{n-1}(s) J_2(t-s) ds.$$

Hence,

$$\tilde{M}_n(t) \leqslant \int_0^t \tilde{M}_{n-1}(s) J(t-s) ds.$$

By [2, Lemma 15], this implies that

$$\tilde{M}_n(t) \leqslant \left(\sup_{0\leqslant s\leqslant t} \sup_{x\in\mathbb{R}^d} \mathbb{E}[v(s,x)^2] \right) a_n,$$

where $(a_n)_{n\in\mathbb{N}}$ is a sequence such that $\sum_{n=0}^{\infty} a_n < \infty$. This shows that $\tilde{M}_n(t) \to 0$ as $n \to \infty$. Finally, we conclude that

$$\mathbb{E}[(u(t,x) - v(t,x))^2] \leqslant 2\mathbb{E}[(u(t,x) - u_n(t,x))^2] + 2\mathbb{E}[(u_n(t,x) - v(t,x))^2] \to 0, \tag{4.12}$$

as $n \to \infty$. This establishes the theorem. ■

4.2 The case of the non-linear wave equation

As an application of Theorem 4.2, we check the different assumptions in the case of the non-linear stochastic wave equation in dimensions greater than 3. The cases of dimensions 1, 2 and 3 have been treated in [30], [4] and [2] as mentionned in Chapter 2. We are interested in the equation

$$\frac{\partial^2 u}{\partial t^2} - \Delta u = \alpha(u)\dot{F} + \beta(u), \tag{4.13}$$

with vanishing initial conditions, where $t \geqslant 0$, $x \in \mathbb{R}^d$ with $d > 3$ and $\dot{F}$ is the noise presented in Chapter 2. In the case of the wave operator, the fundamental solution (see [13, Chap.5]) is

$$\Gamma(t) = \frac{2\pi^{\frac{d}{2}}}{\gamma(\frac{d}{2})} \mathbf{1}_{\{t>0\}} \left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{d-3}{2}} \frac{\sigma_t^d}{t}, \qquad \text{if } d \text{ is odd,} \tag{4.14}$$

$$\Gamma(t) = \frac{2\pi^{\frac{d}{2}}}{\gamma(\frac{d}{2})} \mathbf{1}_{\{t>0\}} \left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{d-2}{2}} (t^2 - |x|^2)_+^{-\frac{1}{2}}, \qquad \text{if } d \text{ is even,} \tag{4.15}$$

where σ_t^d is the Hausdorff surface measure on the d-dimensional sphere of radius t and γ is Euler's gamma function. The action of $\Gamma(t)$ on a test function is explained in (4.18) and (4.19) below. It is also well-known (see [29, §7]) that

$$\mathcal{F}\Gamma(t)(\xi) = \frac{\sin(2\pi t|\xi|)}{2\pi|\xi|},$$

in all dimensions. Hence, there exist constants C_1 and C_2, depending on T, such that for all $s \in [0, T]$ and $\xi \in \mathbb{R}^d$,

$$\frac{C_1}{1+|\xi|^2} \leqslant \frac{\sin^2(2\pi s|\xi|)}{4\pi^2|\xi|^2} \leqslant \frac{C_2}{1+|\xi|^2}. \tag{4.16}$$

Theorem 4.9. *Let $d \geqslant 1$, and suppose that*

$$\sup_{\eta\in\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{\mu(d\xi)}{1+|\xi+\eta|^2} < \infty. \tag{4.17}$$

Then equation (4.13), with α and β Lipschitz functions, admits a random-field solution $(u(t,x),\ 0 \leqslant t \leqslant T,\ x \in \mathbb{R}^d)$. In addition, the uniqueness statement of Theorem 4.8 holds.

Proof. We are going to check that the assumptions of Theorem 4.2 are satisfied. The estimates in (4.16) show that Γ satisfies (4.4) since (4.17) holds. This condition can be shown to be equivalent to the condition (40) of Dalang [2], namely $\int_{\mathbb{R}^d} \frac{\mu(d\xi)}{1+|\xi|^2} < \infty$ since $f \geqslant 0$ (see [7, Lemma 8] and [19]). Moreover, taking the supremum over ξ in (4.16) shows that (4.5) is satisfied.

To check (H1), and in particular, (3.3) and (3.4), fix $\varphi \in \mathcal{D}(\mathbb{R}^d)$ such that $\varphi \geqslant 0$, $\operatorname{supp}\varphi \subset B(0,1)$ and $\int_{\mathbb{R}^d} \varphi(x)\,dx = 1$. From formulas (4.14) and (4.15), if d is odd, then

$$(\Gamma(t-s) * \varphi)(x) = c_d \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{\frac{d-3}{2}} \left[r^{d-2} \int_{\partial B_d(0,1)} \varphi(x+ry)\ \sigma_1^{(d)}(dy) \right]\Bigg|_{r=t-s}, \tag{4.18}$$

where $\sigma_1^{(d)}$ is the Hausdorff surface measure on $\partial B_d(0,1)$, and when d is even,

$$(\Gamma(t-s)*\varphi)(x) = c_d \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{\frac{d-2}{2}} \left[r^{d-2} \int_{B_d(0,1)} \frac{dy}{\sqrt{1-|y|^2}}\, \varphi(x+ry)\right]\Bigg|_{r=t-s}. \quad (4.19)$$

For $0 \leqslant a \leqslant b \leqslant T$ and $a \leqslant t \leqslant b$, this is a uniformly bounded C^∞-function of x, with support contained in $B(0, T+1)$, and (3.3) and (3.4) clearly hold. Indeed, $(\Gamma(t-s)*\varphi)(x)$ is always a sum of products of a positive power of r and an integral of the same form as above but with respect to the derivatives of φ, evaluated at $r = t-s$. This proves Theorem 4.9. ■

Remark 4.10. When $f(x) = \|x\|^{-\beta}$, with $0 < \beta < d$, then (4.17) holds if and only if $0 < \beta < 2$.

Chapter 5

The case of affine multiplicative noise

In the previous chapters, we have seen that the stochastic integral constructed in Chapter 3 can be used to obtain a random field solution to the non-linear stochastic wave equation in dimensions greater than 3 (Sections 4.1 and 4.2). As for the stochastic integral proposed in [2] (see Chapter 2), this stochastic integral is square-integrable if the process Z used as integrand is square-integrable. This property makes it possible to show that the solution $u(t,x)$ of the non-linear stochastic wave equation is in $L^2(\Omega)$ in any dimension.

Theorem 2.17 ([2, Theorem 5]) states that Dalang's stochastic integral is $L^p(\Omega)$-integrable if the process Z is. We would like to extend this result to our generalization of the stochastic integral, even though the approach used in the proof of Theorem 2.17 fails in our case. Indeed, that approach is strongly based on Hölder's inequality which can be used when the Schwartz distribution S is non-negative.

The main interest of a result concerning $L^p(\Omega)$-integrability of the stochastic integral is to show that the solution of an s.p.d.e. admits moments of any order and to deduce Hölder-continuity properties. The first question is whether the solution of the non-linear stochastic wave equation admits moments of any order, in any dimension ? We are going to prove that this is indeed the case for a particular form of the non-linear stochastic wave equation, where α is an affine function and $\beta \equiv 0$. This will not be obtained via a result on the $L^p(\Omega)$-integrability of the stochastic integral. However, a slightly stronger assumption on the integrability of the Fourier transform of the fundamental solution of the equation is required ((5.1) below instead of (4.4)). The proof is based mainly on the specific form of the process that appears in the Picard iteration scheme when α is affine. Indeed, we will be able to use the fact that the approximating random variable $u_n(t,x)$ is an n-fold iterated stochastic integral. The results of this chapter are presented in [1, Section 6].

5.1 Moments of order p of the solution ($p > 2$)

Theorem 5.1. *Suppose that the fundamental solution Γ of the equation $Lu = 0$ is a space-time Schwartz distribution of the form $\Gamma(t)dt$, where $\Gamma(t) \in \mathcal{S}'(\mathbb{R}^d)$ satisfies*

$$\sup_{0 \leqslant s \leqslant T} \sup_{\eta \in \mathbb{R}^d} \int_{\mathbb{R}^d} \mu(d\xi)\, |\mathcal{F}\Gamma(s)(\xi + \eta)|^2 < \infty, \tag{5.1}$$

as well as the assumptions of Theorem 4.2. Let $\alpha : \mathbb{R} \to \mathbb{R}$ be an affine function given by $\alpha(u) = au + b$, $a, b \in \mathbb{R}$, and let $\beta \equiv 0$. Then equation (4.1) admits a random-field solution $(u(t, x),\ 0 \leqslant t \leqslant T,\ x \in \mathbb{R}^d)$ that is unique in the sense of Theorem 4.8, given by

$$u(t, x) = \sum_{n=1}^{\infty} v_n(t, x), \tag{5.2}$$

where

$$v_1(t, x) = b \int_0^t \int_{\mathbb{R}^d} \Gamma(t - s, x - y) M(ds, dy) \tag{5.3}$$

and v_n is defined recursively for $n \geqslant 1$ by

$$v_{n+1}(t, x) = a \int_0^t \int_{\mathbb{R}^d} \Gamma(t - s, x - y) v_n(s, y) M(ds, dy). \tag{5.4}$$

Moreover, for all $p \geqslant 1$ and all $T > 0$, this solution satisfies,

$$\sup_{0 \leqslant t \leqslant T} \sup_{x \in \mathbb{R}^d} \mathbb{E}[|u(t, x)|^p] < \infty.$$

Proof. The existence and uniqueness are a consequence of Theorems 4.2 and 4.8. Multiplying the covariance function f by a, we can suppose, without loss of generality, that the affine function is $\alpha(u) = u + b$ ($b \in \mathbb{R}$), that is, $a = 1$. In this case, the Picard iteration scheme defining the sequence $(u_n)_{n \in \mathbb{N}}$ is given by $u_0 \equiv 0$ and

$$\begin{aligned} u_{n+1}(t, x) \quad = \quad & \int_0^t \int_{\mathbb{R}^d} \Gamma(t - s, x - y) u_n(s, y) M(ds, dy) \\ & + b \int_0^t \int_{\mathbb{R}^d} \Gamma(t - s, x - y) M(ds, dy), \end{aligned} \tag{5.5}$$

where the stochastic integrals are well defined by Theorem 3.1. Set $v_n(t, x) = u_n(t, x) - u_{n-1}(t, x)$ for all $n \geqslant 1$. Then

$$v_1(t, x) = u_1(t, x) = b \int_0^t \int_{\mathbb{R}^d} \Gamma(t - s, x - y) M(ds, dy).$$

Hence, $u(t,x) = \lim_{m\to\infty} u_m(t,x) = \lim_{m\to\infty} \sum_{n=1}^m v_n(t,x) = \sum_{n=1}^\infty v_n(t,x)$ and (5.2) is proved.

By Theorem 3.1 and because $v_1(t,x)$ is a Gaussian random variable, $v_1(t,x)$ admits finite moments of order p for all $p \geqslant 1$. Suppose by induction that for some $n \geqslant 1$, v_n satisfies, for all $p \geqslant 1$,

$$\sup_{0\leqslant t\leqslant T} \sup_{x\in\mathbb{R}^d} \mathbb{E}[|v_n(t,x)|^p] < \infty. \tag{5.6}$$

We are going to show that v_{n+1} also satisfies (5.6).

By its definition and (5.5), v_{n+1} satisfies the recurrence relation

$$v_{n+1}(t,x) = \int_0^t \int_{\mathbb{R}^d} \Gamma(t-s, x-y) v_n(s,y) M(ds,dy), \tag{5.7}$$

for all $n \geqslant 1$. The stochastic integral above is defined by Theorem 3.1 using the approximating sequence $\Gamma_{m,k} \in \mathcal{P}_+$, denoted $S_{n,m}$ in the proof of Theorem 3.1 (whose definition depends on which of (H1) or (H2) is satisfied). For $s \leqslant t \leqslant T$, we set

$$\begin{aligned} M_1(s;t,x) &= \int_0^s \int_{\mathbb{R}^d} \Gamma(t-\rho, x-y) M(d\rho,dy), \\ M_1^{(m,k)}(s;t,x) &= \int_0^s \int_{\mathbb{R}^d} \Gamma_{m,k}(t-\rho, x-y) M(d\rho,dy), \end{aligned}$$

and, for $n \geqslant 1$,

$$M_{n+1}(s;t,x) = \int_0^s \int_{\mathbb{R}^d} \Gamma(t-\rho, x-y) v_n(\rho,y) M(d\rho,dy)$$

and

$$M_{n+1}^{(m,k)}(s;t,x) = \int_0^s \int_{\mathbb{R}^d} \Gamma_{m,k}(t-\rho, x-y) v_n(\rho,y) M(d\rho,dy). \tag{5.8}$$

For all $n \geqslant 1$, set also $v_n^{(m,k)}(t,x) = M_n^{(m,k)}(t;t,x)$.

Fix an even integer p and set $q = \frac{p}{2}$. We know that $s \mapsto M_n^{(m,k)}(s;t,x)$ is a continuous martingale and so, by Burkholder's inequality (see [20, Chap. IV, Theorem 73]),

$$\mathbb{E}[|v_{n+1}^{(m,k)}(t,x)|^p] = \mathbb{E}[|M_{n+1}^{(m,k)}(t;t,x)|^p] \leqslant C\,\mathbb{E}[\langle M_{n+1}^{(m,k)}(\cdot;t,x)\rangle_t^q],$$

and by Theorem 2.13 and Hölder's inequality, the last expectation above is bounded by

$$\begin{aligned}
&\mathbb{E}\left[\left(\int_0^t ds \int_{\mathbb{R}^d} dy \int_{\mathbb{R}^d} dz\, \Gamma_{m,k}(t-s,x-y)f(y-z)\Gamma_{m,k}(t-s,x-z)\right.\right.\\
&\qquad\qquad\left.\left.\times\, v_n(s,y)v_n(s,z)\right)^q\right]\\
&\leqslant t^{q-1}\mathbb{E}\left[\int_0^t ds \left(\int_{\mathbb{R}^d} dy \int_{\mathbb{R}^d} dz\, \Gamma_{m,k}(t-s,x-y)f(y-z)\Gamma_{m,k}(t-s,x-z)\right.\right.\\
&\qquad\qquad\left.\left.\times\, v_n(s,y)v_n(s,z)\right)^q\right]\\
&= t^{q-1}\int_0^t ds \int_{\mathbb{R}^d} dy_1 \int_{\mathbb{R}^d} dz_1\, \Gamma_{m,k}(t-s,x-y_1)f(y_1-z_1)\Gamma_{m,k}(t-s,x-z_1)\\
&\qquad \times\cdots\times \int_{\mathbb{R}^d} dy_q \int_{\mathbb{R}^d} dz_q\, \Gamma_{m,k}(t-s,x-y_q)f(y_q-z_q)\Gamma_{m,k}(t-s,x-z_q)\\
&\qquad\qquad \times\, \mathbb{E}[v_n(s,y_1)v_n(s,z_1)\cdots v_n(s,y_q)v_n(s,z_q)]. \qquad (5.9)
\end{aligned}$$

The last step uses Fubini's theorem, the assumptions of which are satisfied because $\Gamma_{m,k} \in \mathcal{P}_+$ and is deterministic for all m, k, and $v_n(t,x)$ has finite moments of any order by the induction assumption. In particular, the right-hand side of (5.9) is finite.

We are going to study the expression $\mathbb{E}[v_n(s,y_1)v_n(s,z_1)\cdots v_n(s,y_q)v_n(s,z_q)]$ and come back to (5.9) later on. More generally, we consider a term of the form

$$\mathbb{E}\left[\prod_{i=1}^p M_{n_i}(s;t_i,x_i)\right],$$

where p is a fixed even integer, $s \in [0,T]$ and for all i, $1 \leqslant n_i \leqslant n$, $x_i \in \mathbb{R}$, and $t_i \in [s,T]$. In the next lemma, we provide an explicit expression for this expectation.

Lemma 5.2. *Let p be a fixed even integer, $(n_i)_{i=1}^p$ be a sequence of integers such that $1 \leqslant n_i \leqslant n$ for all i, let $s \in [0,T]$, $(t_i)_{i=1}^p \subset [s,T]$ and $(x_i)_{i=1}^p \subset \mathbb{R}^d$. Suppose moreover that n is such that for all $m \leqslant n$ and all $q \geqslant 1$,*

$$\sup_{0\leqslant s\leqslant t\leqslant T}\ \sup_{x\in\mathbb{R}^d} \mathbb{E}[|M_m(s;t,x)|^q] < \infty.$$

If the sequence (n_i) is such that each term in this sequence appears an even number *of times, then*

$$\begin{aligned} &\mathbb{E}\left[\prod_{i=1}^{p} M_{n_i}(s;t_i,x_i)\right] \qquad (5.10)\\ &\overset{S}{=} \int_0^s d\rho_1 \cdots \int_0^{\rho_{N-1}} d\rho_N \prod_{j=1}^{N} \int_{\mathbb{R}^d} \mu(d\xi_j) \overline{\mathcal{F}\Gamma(\sigma_j-\rho_j)(\xi_j+\eta_j)} \mathcal{F}\Gamma(\sigma'_j-\rho_j)(\xi_j+\eta'_j)\\ &\quad \times \left(\prod_{k=1}^{p} e^{i\langle x_k,\delta_k\rangle}\right), \end{aligned}$$

where

(a) $\overset{S}{=}$ *means "is a sum of terms of the form" (a bound on the number of terms is given in Lemma 5.4 below);*

(b) $N = \frac{1}{2}\sum_{i=1}^{p} n_i$;

(c) σ_j *and* σ'_j *are linear combinations of* $\rho_1,\dots,\rho_N, t_1,\dots,t_p$ $(j = 1,\dots,N)$;

(d) η_j *and* η'_j *are linear combinations of* $\xi_1,\dots,\xi_{j-1}$ $(j = 1,\dots,N)$;

(e) δ_k *is a linear combination of* $\xi_1,\dots,\xi_N$ $(k = 1,\dots,p)$.

(f) *In (c)-(e), the linear combinations only admit* 0, $+1$ *and* -1 *as coefficients.*

Remark 5.3. (a) We will see in the proof of Lemma 5.2 that if the elements of the sequence (n_i) do not appear an even number of times, then the expectation vanishes.

(b) It is possible to give an exact expression for the linear combinations in (c)-(e). The exact expression is not needed to prove Theorem 5.1.

Proof. We want to calculate $\mathbb{E}[\prod_{i=1}^{p} M_{n_i}(s;t_i,x_i)]$. We say that we are interested in the expectation with respect to a *configuration* $(n_i)_{i=1}^{p}$. The *order* of this configuration (n_i) is defined to be the number $N = \frac{1}{2}\sum_{i=1}^{p} n_i$.

The proof of the lemma will be based on Itô's formula (see [24, Theorem 3.3, p.147]), by induction on the order of the configuration considered. Suppose first that we have a configuration of order $N = 1$. The only case for which the expectation does not vanish is $p = 2$, $n_1 = n_2 = 1$ in which the term 1 appears an even number of times. In this case, by Theorem 2.13 and properties of the Fourier transform,

$$\begin{aligned} &\mathbb{E}[M_1^{(m,k)}(s;t_1,x_1)M_1^{(m,k)}(s;t_2,x_2)]\\ &= \int_0^s d\rho_1 \int_{\mathbb{R}^d} dy \int_{\mathbb{R}^d} dz\, \Gamma_{m,k}(t_1-\rho_1,x_1-y) f(y-z) \Gamma_{m,k}(t_2-\rho_1,x_2-z)\\ &= \int_0^s d\rho_1 \int_{\mathbb{R}^d} \mu(d\xi)\, \overline{\mathcal{F}\Gamma_{m,k}(t_1-\rho_1)(\xi_1)} \mathcal{F}\Gamma_{m,k}(t_2-\rho_1)(\xi_1) e^{i\langle \xi_1, x_1-x_2\rangle}. \end{aligned}$$

Taking limits as k, then m tend to infinity, we obtain

$$\begin{aligned}
&\mathbb{E}[M_1(s;t_1,x_1)M_1(s;t_2,x_2)] \\
&\quad= \int_0^s d\rho_1 \int_{\mathbb{R}^d} \mu(d\xi_1)\, \overline{\mathcal{F}\Gamma(t_1-\rho_1)(\xi_1)}\mathcal{F}\Gamma(t_2-\rho_1)(\xi_1)e^{i\langle \xi_1, x_1-x_2\rangle}.
\end{aligned}$$

This expression satisfies (5.10) with $N=1$, $\sigma_1=t_1$, $\sigma_1'=t_2$, $\eta_1=\eta_1'=0$, $\delta_1=\xi_1$, $\delta_2=-\xi_1$.

Now suppose that (5.10) is true for all configurations of order not greater than N and consider a configuration $(n_i)_{i=1}^p$ of order $N+1$. For all $i=1,\dots,p$, the process $s\mapsto M_{n_i}(s;t_i,x_i)$ is a continuous martingale. We want to find the expectation of $h(M_{n_1},\dots,M_{n_p})$, where $h(x_1,\dots,x_p)=x_1\cdots x_p$. To evalute this expectation, we first use Itô's formula with the function h and the processes $M_{n_i}^{(m_i,k_i)}$ $(i=1,\dots,p)$. We obtain

$$\begin{aligned}
&\mathbb{E}\left[\prod_{i=1}^p M_{n_i}^{(m_i,k_i)}(s;t_i,x_i)\right] \\
&\quad= \sum_{i=1}^p \mathbb{E}\left[\int_0^s \prod_{\substack{j=1\\ j\neq i}}^p M_{n_j}^{(m_j,k_j)}(\rho;t_j,x_j)\, dM_{n_i}^{(m_i,k_i)}(\rho;t_i,x_i)\right] \qquad (5.11)\\
&\qquad+\frac{1}{2}\sum_{\substack{i,j=1\\ i\neq j}}^p \mathbb{E}\left[\int_0^s \prod_{\substack{\ell=1\\ \ell\neq i,j}}^p M_{n_\ell}^{(m_\ell,k_\ell)}(\rho;t_\ell,x_\ell)\, d\left\langle M_{n_i}^{(m_i,k_i)}(\cdot;t_i,x_i); M_{n_j}^{(m_j,k_j)}(\cdot;t_j,x_j)\right\rangle_\rho\right].
\end{aligned}$$

As the processes $M_{n_i}^{(m_i,k_i)}$ admit finite moments for all $i=1,\dots,p$, the process in the expectation in the first sum of the right-hand side of (5.11) is a martingale that vanishes at time zero. Hence, this expectation is zero. In the second sum on the right-hand side of (5.11), all terms are similar. For the sake of simplicity, we will only consider here the term for $i=1$, $j=2$: the right-hand side of (5.10) is a sum of terms similar to this one. In the case where $n_1\neq n_2$, the cross-variation is zero. Indeed, the two processes are multiple stochastic integrals of different orders and hence do not belong to the same Wiener chaos. Otherwise, using Theorem 2.13 and Fubini's theorem (which is valid because $M_{n_i}^{(m_i,k_i)}$ has finite moments of any order for all i and $\Gamma_{m,k} \in \mathcal{P}_+$), we have

$$\begin{aligned}
&\mathbb{E}\left[\prod_{i=1}^{p} M_{n_i}^{(m_i,k_i)}(s;t_i,x_i)\right] \qquad (5.12)\\
&\overset{S}{=} \int_0^s d\rho \int_{\mathbb{R}^d} dy \int_{\mathbb{R}^d} dz\, \Gamma_{m_1,k_1}(t_1-\rho, x_1-y) f(y-z) \Gamma_{m_2,k_2}(t_2-\rho, x_2-z)\\
&\quad \times \mathbb{E}\left[M_{n_1-1}(\rho;\rho,y) M_{n_2-1}(\rho;\rho,z) \prod_{j=3}^{p} M_{n_j}^{(m_j,k_j)}(\rho;t_j,x_j)\right].
\end{aligned}$$

(We set $M_0 \equiv 1$ when $n_1 = n_2 = 1$.) Because $M_{n_j}^{(m_j,k_j)}$ have finite moments of any order and $M_{n_j}^{(m_j,k_j)} \to M_{n_j}$ in $L^2(\Omega)$ by the definition of the stochastic integral (see the proof of Theorem 3.1), we know that $M_{n_j}^{(m_j,k_j)} \to M_{n_j}$ in $L^p(\Omega)$. As $\Gamma_{m,k} \in \mathcal{P}_+$, taking limits as $k_3, \ldots, k_p$ tend to $+\infty$ and then as $m_3, \ldots, m_p$ tend to $+\infty$, we obtain

$$\begin{aligned}
&\mathbb{E}\left[M_{n_1}^{(m_1,k_1)}(s;t_1,x_1) M_{n_2}^{(m_2,k_2)}(s;t_2,x_2) \prod_{i=3}^{p} M_{n_i}(s;t_i,x_i)\right] \qquad (5.13)\\
&\overset{S}{=} \int_0^s d\rho \int_{\mathbb{R}^d} dy \int_{\mathbb{R}^d} dz\, \Gamma_{m_1,k_1}(t_1-\rho, x_1-y) f(y-z) \Gamma_{m_2,k_2}(t_2-\rho, x_2-z)\\
&\quad \times \mathbb{E}\left[M_{n_1-1}(\rho;\rho,y) M_{n_2-1}(\rho;\rho,z) \prod_{j=3}^{p} M_{n_j}(\rho;t_j,x_j)\right].
\end{aligned}$$

At this point in the proof, we can see why the terms of (n_i) have to appear an even number of times. Indeed, if we consider $n_1 \neq n_2$, we have seen that the expectation is zero. When $n_1 = n_2$, the product in the expectation on the right-hand side of (5.13) is of order N. Hence, we can use the induction assumption to express it as in (5.10). By the induction assumption, if the terms of (n_i) do not appear an even number of times, the expectation on the right-hand side of (5.13) vanishes and hence the one on the left-hand side does too. If these terms do appear an even number of times, then setting $t_1 = s = \rho$, $t_2 = \rho$, $x_1 = y$, $x_2 = z$ in (5.10) and substituting into (5.13), we obtain

$$\begin{aligned}
&\mathbb{E}\left[M_{n_1}^{(m_1,k_1)}(s;t_1,x_1)M_{n_2}^{(m_2,k_2)}(s;t_2,x_2)\prod_{i=3}^{p} M_{n_i}(s,t_i,x_i)\right] \qquad (5.14)\\
&\overset{S}{=} \int_0^s d\rho \int_{\mathbb{R}^d} dy \int_{\mathbb{R}^d} dz\, \Gamma_{m_1,k_1}(t_1-\rho,x_1-y) f(y-z) \Gamma_{m_2,k_2}(t_2-\rho,x_2-z)\\
&\quad \times \int_0^{\rho} d\rho_1 \cdots \int_0^{\rho_{N-1}} d\rho_N \prod_{j=1}^{N} \int_{\mathbb{R}^d} \mu(d\xi_j) \overline{\mathcal{F}\Gamma(\sigma_j-\rho_j)(\xi_j+\eta_j)}\\
&\quad \times \mathcal{F}\Gamma(\sigma_j'-\rho_j)(\xi_j+\eta_j')\left(e^{i\langle y,\delta_1\rangle}\cdot e^{i\langle z,\delta_2\rangle}\cdot\prod_{k=3}^{p} e^{i\langle x_k,\delta_k\rangle}\right),
\end{aligned}$$

where

(i) σ_j and σ_j' are linear combinations of $\rho_1,\dots,\rho_N$, ρ, $t_3,\dots,t_p$ $(j=1,\dots,N)$;

(ii) η_j and η_j' are linear combinations of $\xi_1,\dots,\xi_{j-1}$ $(j=1,\dots,N)$;

(iii) δ_k is a linear combination of $\xi_1,\dots,\xi_N$ $(k=1,\dots,p)$.

Since the modulus of the exponentials is 1, by (ii), (5.1) and because $\Gamma_{m,k}\in\mathcal{P}_+$, we see that the right-hand side of (5.14) is finite. So, by Fubini's theorem, we permute the integrals in dy and dz first with the $d\rho_i$-integrals, then with the $\mu(d\xi_j)$-integrals, to obtain

$$\begin{aligned}
&\mathbb{E}\left[M_{n_1}^{(m_1,k_1)}(s;t_1,x_1)M_{n_2}^{(m_2,k_2)}(s;t_2,x_2)\prod_{i=3}^{p} M_{n_i}(s;t_i,x_i)\right]\\
&\overset{S}{=} \int_0^s d\rho \int_0^{\rho} d\rho_1 \cdots \int_0^{\rho_{N-1}} d\rho_N \prod_{j=1}^{N} \int_{\mathbb{R}^d} \mu(d\xi_j) \overline{\mathcal{F}\Gamma(\sigma_j-\rho_j)(\xi_j+\eta_j)}\\
&\quad \times \mathcal{F}\Gamma(\sigma_j'-\rho_j)(\xi_j+\eta_j')\left(\prod_{k=3}^{p} e^{i\langle x_k,\delta_k\rangle}\right)\int_{\mathbb{R}^d} dy \int_{\mathbb{R}^d} dz\, \Gamma_{m_1,k_1}(t_1-\rho,x_1-y)\\
&\quad \times e^{i\langle y,\delta_1\rangle} f(y-z)\Gamma_{m_2,k_2}(t_2-\rho,x_2-z)e^{i\langle z,\delta_2\rangle}.
\end{aligned}$$

Rewriting the last two integrals with the Fourier transforms, we have

$$\begin{aligned}
&\mathbb{E}\left[M_{n_1}^{(m_1,k_1)}(s;t_1,x_1)M_{n_2}^{(m_2,k_2)}(s;t_2,x_2)\prod_{i=3}^{p}M_{n_i}(s;t_i,x_i)\right]\\
&\stackrel{S}{=}\int_0^s d\rho\int_0^{\rho}d\rho_1\cdots\int_0^{\rho_{N-1}}d\rho_N\\
&\quad\times\prod_{j=1}^{N}\int_{\mathbb{R}^d}\mu(d\xi_j)\overline{\mathcal{F}\Gamma(\sigma_j-\rho_j)(\xi_j+\eta_j)}\mathcal{F}\Gamma(\sigma_j'-\rho_j)(\xi_j+\eta_j')\\
&\quad\times\left(\prod_{k=3}^{p}e^{i\langle x_k,\delta_k\rangle}\right)\int_{\mathbb{R}^d}\mu(d\xi)\,\overline{\mathcal{F}\Gamma_{m_1,k_1}(t_1-\rho)(\xi+\delta_1)}\mathcal{F}\Gamma_{m_2,k_2}(t_2-\rho)(\xi+\delta_2)\\
&\qquad\qquad\qquad\qquad\times\, e^{i\langle x_1,\xi+\delta_1\rangle}\cdot e^{i\langle x_2,\xi+\delta_2\rangle}.
\end{aligned}\tag{5.15}$$

Setting $\xi_{n+1}=\xi$, $\sigma_{N+1}=t_1$, $\sigma_{N+1}'=t_2$, $\eta_{N+1}=\delta_1$, $\eta_{N+1}'=\delta_2$, $\tilde{\delta}_1=\xi+\delta_1$, $\tilde{\delta}_2=\xi+\delta_2$, the assumptions needed on these linear combinations are satisfied and (5.15) is of the desired form. It remains to take limits as k_1,k_2 and then m_1,m_2 tend to infinity.

The left-hand side has the desired limit because M_{n_i} has finite moments of any order and $\lim_{m_i\to\infty}\lim_{k_i\to\infty}M_{n_i}^{(m_i,k_i)}(s;t_i,x_i)=M_{n_i}(s;t_i,x_i)$ in $L^2(\Omega,\mathcal{F},\mathbb{P})$, $i=1,2$. For the right-hand side, first consider the limit with respect to k_1 and k_2. To show convergence, we consider the left-hand side of (5.15) as the inner product of $\mathcal{F}\Gamma_{m_1,k_1}(t_1-\rho)(\xi+\delta_1)$ and $\mathcal{F}\Gamma_{m_2,k_2}(t_2-\rho)(\xi+\delta_2)$ in the L^2-space with respect to the measure

$$ds\times\cdots\times d\rho_N\times\left(\times_{j=1}^{N}\overline{\mathcal{F}\Gamma(\sigma_j-\rho_j)(\xi_j+\eta_j)}\mathcal{F}\Gamma(\sigma_j'-\rho_j)(\xi_j+\eta_j')\,\mu(d\xi_j)\right)\times\mu(d\xi).\tag{5.16}$$

Note that the exponentials are of modulus one and hence do not play any role in the convergence. Therefore, it is sufficient to consider $i=1$ and to show that

$$\begin{aligned}
&\int_0^s d\rho\int_0^{\rho}d\rho_1\cdots\int_0^{\rho_{N-1}}d\rho_N\\
&\times\prod_{j=1}^{N}\int_{\mathbb{R}^d}\mu(d\xi_j)\overline{\mathcal{F}\Gamma(\sigma_j-\rho_j)(\xi_j+\eta_j)}\mathcal{F}\Gamma(\sigma_j'-\rho_j)(\xi_j+\eta_j')\\
&\times\left(\prod_{k=3}^{p}e^{i\langle x_k,\delta_k\rangle}\right)\int_{\mathbb{R}^d}\mu(d\xi)\,|\mathcal{F}\Gamma_{m,k}(t_1-\rho)(\xi+\delta_1)-\mathcal{F}\Gamma_m(t_1-\rho)(\xi+\delta_1)|^2
\end{aligned}$$

goes to 0 as k tends to infinity. This limit has to be treated differently according to which assumption (H1) or (H2) in Theorem 3.1 is satisfied.

In the case where assumption (H1) is satisfied, the proof of convergence is based on the martingale convergence theorem in a way analogous to the approach used in the proof of Theorem 3.1 with the measure $ds \times \nu_s(d\eta) \times \mu(d\xi)$ replaced by the one in (5.16). Assumption (5.1) allows to bound the $\mu(d\xi_j)$-integrals $(1 \leqslant j \leqslant N)$ when we check the L^2-boundedness of $\mathcal{F}\Gamma_m(t_1 - \rho)(\xi + \delta_1)$.

In the case where (H2) is satisfied, we bound the $\mu(d\xi_j)$-integrals by (5.1) again, compute the time-integrals (except the one with respect to ρ) and finally the continuity assumption (H2) shows the desired convergence.

Finally, the limit with respect to m_1 and m_2 is treated as in the proof of Theorem 3.1 by the Dominated Convergence Theorem. Lemma 5.2 is proved. ■

Proof of Theorem 5.1 (continued)

We use (5.10) with $n_i = n$, $t_i = s$ for all $i = 1, \dots, p$, to express the expectation in (5.9). Using the same idea as in the proof of Lemma 5.2, we can permute the integrals to obtain

$$\begin{aligned} &\mathbb{E}[|v_{n+1}^{(m,k)}(t,x)|^p] \\ &\overset{S}{\leqslant} \quad t^{q-1} \int_0^t ds \int_0^s d\rho_1 \cdots \int_0^{\rho_{N-1}} d\rho_N \\ &\quad \times \prod_{j=1}^N \int_{\mathbb{R}^d} \mu(d\xi_j) \overline{\mathcal{F}\Gamma(\sigma_j - \rho_j)(\xi_j - \eta_j)} \mathcal{F}\Gamma(\sigma'_j - \rho_j)(\xi_j - \eta'_j) \\ &\quad \times \prod_{\ell=1}^q \int_{\mathbb{R}^d} \mu(d\beta_\ell) \overline{\mathcal{F}\Gamma_{m,k}(t-s)(\beta_\ell - \gamma_\ell)} \mathcal{F}\Gamma_{m,k}(t-s)(\beta_\ell - \gamma'_\ell) e^{i\langle x, \delta\rangle}, \end{aligned} \tag{5.17}$$

where $\overset{S}{\leqslant}$ means "is bounded by a sum of terms of the form" and $N = nq$ is the order of the particular configuration considered in that case. The variables $\sigma_j, \sigma'_j, \eta_j, \eta'_j$ $(j = 1, \dots, N)$ satisfy the same assumptions as in Lemma 5.2, the variables $\gamma_\ell, \gamma'_\ell$ $(\ell = 1, \dots, q)$ are linear combinations of $\xi_1, \dots, \xi_N$ and δ is a linear combination of $\xi_1, \dots, \xi_N, \beta_1, \dots, \beta_q$. When using (5.10) in (5.9), exponentials of the form $e^{i\langle y_j, \delta_j\rangle}$ and $e^{i\langle z_j, \tilde{\delta}_j\rangle}$ appear. When writing the y_ℓ, z_ℓ-integrals as a $\mu(d\beta_\ell)$-integral, these exponentials become shifts. This explains why the variables $\gamma_\ell, \gamma'_\ell$ $(\ell = 1, \dots, q)$ and δ appear.

Now, using the Cauchy-Schwartz inequality and setting

$$I = \sup_{0 \leqslant s \leqslant T} \sup_{\eta \in \mathbb{R}^d} \int_{\mathbb{R}^d} \mu(d\xi)\, |\mathcal{F}\Gamma(s)(\xi + \eta)|^2,$$

which is finite by (5.1), and taking limits as k and m tend to $+\infty$, we obtain

$$\begin{aligned}
\mathbb{E}[|v_{n+1}(t,x)|^p] &\overset{S}{\leqslant} t^{q-1}\int_0^t ds \int_0^s d\rho_1 \cdots \int_0^{\rho_{N-1}} d\rho_N \, I^{N+q} \\
&= \frac{t^{N+q}}{(N+1)!} I^{N+q} = \frac{t^{(n+1)q}}{(nq+1)!} I^{(n+1)q}, \qquad (5.18)
\end{aligned}$$

where $q = \frac{p}{2}$. We have obtained an expression that bounds the moment of order p of v_{n+1} as a finite sum of finite terms. In order to have a bound for this moment, it remains to estimate the number of terms in the sum. This is the goal of Lemma 5.4.

Lemma 5.4. *In the case where $n_i = n$, for all $i = 1, \dots, p$ and $q = \frac{p}{2}$, then the number of terms in the sum implied by $\overset{S}{=}$ in (5.18) is bounded by $R = (q(p-1))^{nq}$.*

Proof. We have to estimate the number of terms appearing in the sum when we use Itô's formula. For each application of Itô's formula, we have to sum over all choices of pairs in $(n_i)_{i=1}^p$. Hence, we have at most $\frac{1}{2}p(p-1)$ choices. Moreover, Itô's formula has to be iterated at most $N = nq$ times to completely develop the expectation. Hence, the number of terms in the sum implied by $\overset{S}{=}$ is bounded by $R = (q(p-1))^{nq}$. ■

Proof of Theorem 5.1 (continued)

We return to the proof of Theorem 5.1. Using Lemma 5.4 together with (5.18), we obtain

$$\mathbb{E}[|v_{n+1}(t,x)|^p] \leqslant (q(p-1))^{nq} \frac{t^{(n+1)q}}{(nq+1)!} I^{(n+1)q}. \qquad (5.19)$$

Clearly, the series $\sum_{n=0}^{\infty} \|v_{n+1}(t,x)\|_p$ converges, where $\|\cdot\|_p$ stands for the norm in $L^p(\Omega)$. Hence,

$$\|u_n(t,x)\|_p = \|v_1(t,x) + \cdots + v_n(t,x)\|_p \leqslant \sum_{i=0}^{n-1} \|v_{i+1}(t,x)\|_p.$$

As the bound on the series does not depend on x and as $t \leqslant T$, we have

$$\sup_{n\in\mathbb{N}} \sup_{0\leqslant t\leqslant T} \sup_{x\in\mathbb{R}^d} \mathbb{E}[|u_n(t,x)|^p] < \infty, \qquad (5.20)$$

for all even integers p. Jensen's inequality then shows that (5.20) is true for all $p \geqslant 1$. As the sequence $(u_n(t,x))_{n\in\mathbb{N}}$ converges in $L^2(\Omega)$ to $u(t,x)$ by Theorem 3.1, (5.20) ensures the convergence in $L^p(\Omega)$ and we have

$$\sup_{0\leqslant t\leqslant T} \sup_{x\in\mathbb{R}^d} \mathbb{E}[|u(t,x)|^p] < \infty,$$

for all $p \geqslant 1$. Theorem 5.1 is proved. ■

Remark 5.5. The fact that α is an affine function is strongly used in this proof. The key fact is that its derivative is constant and so Itô's formula can be applied iteratively. This is not the case for a general Lipschitz function α, as we will see in Chapter 7.

5.2 Hölder continuity

In this section, we are going to study the regularity of the solution of the non-linear wave equation (4.1) in the specific case considered in Theorem 5.1 : let $u(t,x)$ be the random field solution of the equation

$$Lu = (u+b)\dot{F}, \tag{5.21}$$

with vanishing initial conditions, where $b \in \mathbb{R}$ and the spatial dimension is $d \geqslant 1$. We will need the following hypotheses, which are analogous to those that appear in [26], in order to guarantee the regularity of the solution.

(H3) For all $T > 0$, $h \geqslant 0$, there exist constants C, $\gamma_1 \in]0,+\infty[$ such that

$$\sup_{0 \leqslant s \leqslant T} \sup_{\eta \in \mathbb{R}^d} \int_{\mathbb{R}^d} \mu(d\xi)\, |\mathcal{F}\Gamma(s+h)(\xi+\eta) - \mathcal{F}\Gamma(s)(\xi+\eta)|^2 \leqslant Ch^{2\gamma_1}.$$

(H4) For all $T > 0$, $t \in [0,T]$, there exist constants C, $\gamma_2 \in]0,+\infty[$ such that

$$\sup_{\eta \in \mathbb{R}^d} \int_{\mathbb{R}^d} \mu(d\xi)\, |\mathcal{F}\Gamma(t)(\xi+\eta)|^2 \leqslant Ct^{2\gamma_2}.$$

(H5) For all $T > 0$ and compact sets $K \subset \mathbb{R}^d$, there exist constants C, $\gamma_3 \in]0,+\infty[$ such that for any $z \in K$,

$$\sup_{0 \leqslant s \leqslant T} \sup_{\eta \in \mathbb{R}^d} \int_{\mathbb{R}^d} \mu(d\xi)\, |\mathcal{F}\Gamma(s, z-\cdot)(\xi+\eta) - \mathcal{F}\Gamma(s,\cdot)(\xi+\eta)|^2 \leqslant C|z|^{2\gamma_3}.$$

The next result concerns the regularity in time of the solution of (5.21).

Proposition 5.6. *Suppose that the fundamental solution of $Lu = 0$ satisfies the assumptions of Theorem 5.1, (H3) and (H4), and u is the solution of (5.21) given by Theorem 5.1. Then for any $x \in \mathbb{R}^d$, $t \mapsto u(t,x)$ is a.s. γ-Hölder-continuous, for any $\gamma \in]0, \gamma_1 \wedge (\gamma_2 + \frac{1}{2})[$.*

Proof. Following Theorem 5.1, the solution $u(t,x)$ to (5.21) is given recursively by (5.2)-(5.4). Hence, for any $h \geqslant 0$ and $t \in [0, T-h]$, we have

$$u(t+h,x) - u(t,x) = \sum_{n=1}^{\infty} (v_n(t+h,x) - v_n(t,x)). \tag{5.22}$$

The Gaussian process v_1 is given by (5.3). Hence,

$$v_1(t+h,x) - v_1(t,x) = A_1(t,x;h) + B_1(t,x;h),$$

where

$$A_1(t,x;h) = \int_0^t \int_{\mathbb{R}^d} (\Gamma(t+h-s, x-y) - \Gamma(t-s,x-y)) M(ds,dy) \tag{5.23}$$

and

$$B_1(t,x;h) = \int_t^{t+h} \int_{\mathbb{R}^d} \Gamma(t+h-s,x-y) M(ds,dy). \tag{5.24}$$

Fix p an even integer. By Burkholder's inequality (see [20, Chap. IV, Theorem 73]),

$$\begin{aligned}
&\mathbb{E}[|A_1(t,x;h)|^p] \\
&\leqslant \; C \left(\int_0^t ds \int_{\mathbb{R}^d} \nu_s(d\eta) \int_{\mathbb{R}^d} \mu(d\xi) \, |\mathcal{F}\Gamma(t+h-s)(\xi+\eta) - \mathcal{F}\Gamma(t-s)(\xi+\eta)| \right)^{\frac{p}{2}} \\
&\leqslant \; C \left(\int_0^t ds \sup_{\eta \in \mathbb{R}^d} \int_{\mathbb{R}^d} \mu(d\xi) \, |\mathcal{F}\Gamma(t+h-s)(\xi+\eta) - \mathcal{F}\Gamma(t-s)(\xi+\eta)| \right)^{\frac{p}{2}} \\
&\leqslant \; C h^{p\gamma_1}
\end{aligned} \tag{5.25}$$

by (H3). On another hand, using again Burkholder's inequality, we see that

$$\begin{aligned}
\mathbb{E}[|B(t,x;h)|^p] &\leqslant \left(\int_t^{t+h} ds \int_{\mathbb{R}^d} \nu_s(d\eta) \int_{\mathbb{R}^d} \mu(d\xi) \, |\mathcal{F}\Gamma(t+h-s)(\xi+\eta)|^2 \right)^{\frac{p}{2}} \\
&\leqslant \left(\int_t^{t+h} ds \sup_{\eta \in \mathbb{R}^d} \int_{\mathbb{R}^d} \mu(d\xi) \, |\mathcal{F}\Gamma(t+h-s)(\xi+\eta)|^2 \right)^{\frac{p}{2}} \\
&\leqslant C \left(\int_t^{t+h} ds \, (t+h-s)^{2\gamma_2} \right) \\
&\leqslant C h^{p(\gamma_2 + \frac{1}{2})},
\end{aligned} \tag{5.26}$$

by (H4). Hence, putting together (5.25) and (5.26), we see that there exists a constant C_0 such that

$$\mathbb{E}[|v_1(t+h,x) - v_1(t,x)|^p] \leqslant C_0 h^{p(\gamma_1 \wedge (\gamma_2 + \frac{1}{2}))}. \tag{5.27}$$

For $n \geqslant 2$, set $w_n(t,x;h) = v_n(t+h,x) - v_n(t,x)$, where v_n is defined by (5.4). Then

$$w_{n+1}(t,x;h) = A_n(t,x;h) + B_n(t,x;h),$$

where

$$A_n(t,x;h) = \int_0^t \int_{\mathbb{R}^d} (\Gamma(t+h-s,x-y) - \Gamma(t-s,x-y))v_n(s,y)M(ds,dy) \quad (5.28)$$

and

$$B_n(t,x;h) = \int_t^{t+h} \int_{\mathbb{R}^d} \Gamma(t+h-s,x-y)v_n(s,y)M(ds,dy). \quad (5.29)$$

Setting $\tilde{\Gamma}(s,y) = \Gamma(t+h-s,x-y) - \Gamma(t-s,x-y)$ and letting $A_n^{(m,k)}$ be the approximation of A_n with Γ replaced by $\Gamma_{m,k}$ in (5.28), we can use the same argument as in (5.9) to see that

$$\begin{aligned}\mathbb{E}[|A_n^{(m,k)}(t,x;h)|^p] \quad &\leqslant \quad C\int_0^t ds \prod_{j=1}^q \int_{\mathbb{R}^d} dy_j \int_{\mathbb{R}^d} dz_j\, \tilde{\Gamma}_{m,k}(s,y_j) f(y_j - z_j)\tilde{\Gamma}_{m,k}(s,z_j) \\ &\qquad \times \mathbb{E}[v_n(s,y_1)v_n(s,z_1)\cdots v_n(s,y_q)v_n(s,z_q)], \qquad (5.30)\end{aligned}$$

where p is an even integer and $q = \frac{p}{2}$. Using Lemma 5.2 to express the expectation and using the same argument as used to reach (5.17), we obtain

$$\begin{aligned}&\mathbb{E}[|A_n^{(m,k)}(t,x;h)|^p] \\ &\overset{S}{\leqslant} \int_0^t ds \int_0^s d\rho_1 \cdots \int_0^{\rho_{N-1}} d\rho_N \\ &\quad \times \prod_{j=1}^N \int_{\mathbb{R}^d} \mu(d\xi_j)\overline{\mathcal{F}\Gamma(\sigma_j - \rho_j)(\xi_j - \eta_j)}\mathcal{F}\Gamma(\sigma'_j - \rho_j)(\xi_j - \eta'_j) \\ &\quad \times \prod_{\ell=1}^q \int_{\mathbb{R}^d} \mu(d\beta_\ell)\overline{\mathcal{F}\tilde{\Gamma}_{m,k}(s)(\beta_\ell - \gamma_\ell)}\mathcal{F}\tilde{\Gamma}_{m,k}(s)(\beta_\ell - \gamma'_\ell)e^{i\langle x,\delta\rangle}, \qquad (5.31)\end{aligned}$$

where $\overset{S}{\leqslant}$ means "is bounded by a sum of terms of the form", $N = nq$ and $\sigma_j, \sigma'_j, \eta_j, \eta'_j, \gamma_\ell, \gamma'_\ell$ and δ ($1 \leqslant j \leqslant N$, $1 \leqslant \ell \leqslant q$) satisfy the same assumptions as in (5.17). Notice that Γ appears in the first N integrals and $\tilde{\Gamma}$ in the last q integrals.

We take limits in (5.31) as k and m tend to $+\infty$. Then, using the Cauchy-Schwartz inequality, we bound the first N spatial integrals in (5.31) using (5.1), bound the other q spatial integrals by hypothesis (H3), compute the time integrals and bound the number of terms in the sum by Lemma 5.4 and, similarly to (5.19), we obtain

$$\mathbb{E}[|A_n(t,x;h)|^p] \leqslant (q(p-1))^{nq} \frac{T^{(n+1)q}}{(nq+1)!} I^{nq} h^{p\gamma_1} = C_n^{(1)} h^{p\gamma_1}, \tag{5.32}$$

where $C_n^{(1)} = (q(p-1))^{nq} \frac{T^{(n+1)q}}{(nq+1)!} I^{nq}$.

On another hand, let $B_n^{(m,k)}$ be the corresponding approximation of B_n. The same arguments as those used to obtain (5.9) show that

$$\begin{aligned} &\mathbb{E}[|B_n^{(m,k)}(t,x;h)|^p] \\ &\quad \leqslant \; C h^{q-1} \int_t^{t+h} ds \prod_{j=1}^q \int_{\mathbb{R}^d} dy_j \int_{\mathbb{R}^d} dz_j \, \Gamma_{m,k}(t+h-s, y_j) f(y_j - z_j) \\ &\qquad \times \Gamma_{m,k}(t+h-s, z_j)\, \mathbb{E}[v_n(s,y_1) v_n(s,z_1) \cdots v_n(s,y_q) v_n(s,z_q)]. \end{aligned} \tag{5.33}$$

Note that the factor h^{q-1} appears because Hölder's inequality is used on the interval $[t, t+h]$ instead of $[0,t]$. Using Lemma 5.2 and the argument used to reach (5.17), we obtain

$$\begin{aligned} &\mathbb{E}[|B_n^{(m,k)}(t,x;h)|^p] \\ &\overset{S}{\leqslant} \; C h^{q-1} \int_t^{t+h} ds \int_0^s d\rho_1 \cdots \int_0^{\rho_{N-1}} d\rho_N \\ &\quad \times \prod_{j=1}^N \int_{\mathbb{R}^d} \mu(d\xi_j)\, \overline{\mathcal{F}\Gamma(\sigma_j - \rho_j)(\xi_j - \eta_j)} \mathcal{F}\Gamma(\sigma_j' - \rho_j)(\xi_j - \eta_j') \\ &\quad \times \prod_{\ell=1}^q \int_{\mathbb{R}^d} \mu(d\beta_\ell)\, \overline{\mathcal{F}\Gamma_{m,k}(t+h-s)(\beta_\ell - \gamma_\ell)} \mathcal{F}\Gamma_{m,k}(t+h-s)(\beta_\ell - \gamma_\ell') e^{i\langle x, \delta\rangle}, \end{aligned} \tag{5.34}$$

where $\overset{S}{\leqslant}$ means "is bounded by a sum of terms of the form", $N = nq$ and $\sigma_j, \sigma_j', \eta_j, \eta_j', \gamma_\ell, \gamma_\ell'$ and δ $(1 \leqslant j \leqslant N,\ 1 \leqslant \ell \leqslant q)$ satisfy the same assumptions as in (5.17).

We take limits in (5.34) as k and m tend to $+\infty$. Then, using the Cauchy-Schwartz inequality, we bound the first N spatial integrals in (5.34) using (5.1), bound the other q spatial integrals by hypothesis (H4) and bound the number of terms in the sum by Lemma 5.4. Then

$$\mathbb{E}[|B_n(t,x;h)|^p] \leqslant C h^{q-1} (q(p-1))^{nq} I^{nq} \int_t^{t+h} ds \int_0^s d\rho_1 \cdots \int_0^{\rho_{N-1}} d\rho_N \, (t+h-s)^{p\gamma_2}.$$

The n-fold integral is bounded by

$$\int_t^{t+h} ds \, \frac{s^{nq}}{(nq)!} (t+h-s)^{p\gamma_2} \leqslant \frac{T^{nq}}{(nq)!} \int_t^{t+h} ds \, (t+h-s)^{p\gamma_2} = \frac{T^{nq}}{(nq)!} h^{p\gamma_2+1},$$

Therefore,

$$\mathbb{E}[|B_n(t,x;h)|^p] \leqslant C_n^{(2)} h^{p(\gamma_2+\frac{1}{2})}, \tag{5.35}$$

where $C_n^{(2)} = C(q(p-1))^{nq} I^{nq} \frac{T^{nq}}{(nq)!}$.

Finally, putting (5.32) and (5.35) together, we have for any $n \geqslant 2$,

$$\mathbb{E}[|w_{n+1}(t,x;h)|^p] \leqslant (C_n^{(1)} + C_n^{(2)}) h^{p(\gamma_1 \wedge (\gamma_2+\frac{1}{2}))} \tag{5.36}$$

and, by (5.27) and (5.36),

$$\mathbb{E}[|u(t+h,x) - u(t,x)|^p] \leqslant \left(\sum_{n=1}^{\infty} (C_n^{(1)} + C_n^{(2)}) \right) h^{p(\gamma_1 \wedge (\gamma_2+\frac{1}{2}))}, \tag{5.37}$$

for any even integer p and $h \geqslant 0$. The series $\sum_{n=1}^{\infty}(C_n^{(1)} + C_n^{(2)})$ converges, as in (5.19). Jensen's inequality establishes that (5.37) holds for an arbitrary $p \geqslant 1$, which shows γ-Hölder-continuity of $t \mapsto u(t,x)$ for any $\gamma \in]0, \gamma_1 \wedge (\gamma_2 + \frac{1}{2})[$ by Kolmogorov's continuity theorem (see [24, Theorem 2.1, p.26]). ■

The next result concerns the spatial regularity of the solution.

Proposition 5.7. *Suppose that the fundamental solution of $Lu = 0$ satisfies the assumptions of Theorem 5.1 and (H5) and u is the solution of (5.21) built in Theorem 5.1. Then for any $t \in [0,T]$, $x \mapsto u(t,x)$ is a.s. γ-Hölder-continuous, for any $\gamma \in]0, \gamma_3[$.*

Proof. The proof is similar to that of Proposition 5.6. We know that $u(t,x)$ is given by (5.2)-(5.4). Hence, for any compact set $K \subset \mathbb{R}^d$ and for any $z \in K$,

$$u(t,x+z) - u(t,x) = \sum_{n=1}^{\infty} (v_n(t,x+z) - v_n(t,x)).$$

The Gaussian process v_1 is given by (5.3). Hence,

$$v_1(t,x+z) - v_1(t,x) = \int_0^t \int_{\mathbb{R}^d} (\Gamma(t-s,x+z-y) - \Gamma(t-s,x-y)) M(ds,dy).$$

By Burkholder's inequality,

$$
\begin{aligned}
&\mathbb{E}[|v_1(t,x+z)-v_1(t,x)|^p] \\
\leqslant\ & \left(\int_0^t ds \int_{\mathbb{R}^d} \nu_s(d\eta) \int_{\mathbb{R}^d} \mu(d\xi) \right. \\
&\qquad \left. \times\ |\mathcal{F}\Gamma(t-s,x+z-\cdot)(\xi+\eta) - \mathcal{F}\Gamma(t-s,x-\cdot)(\xi+\eta)|^2 \right)^{\frac{p}{2}} \\
\leqslant\ & \left(\int_0^t ds \sup_{\eta\in\mathbb{R}^d} \int_{\mathbb{R}^d} \mu(d\xi)\, |\mathcal{F}\Gamma(t-s,z-\cdot)(\xi+\eta) - \mathcal{F}\Gamma(t-s,\cdot)(\xi+\eta)|^2 \right)^{\frac{p}{2}} \\
\leqslant\ & C|z|^{p\gamma_3}, \qquad (5.38)
\end{aligned}
$$

by (H5). Therefore, there exists a constant C_0 such that

$$\mathbb{E}[|v_1(t,x+z)-v_1(t,x)|^p] \leqslant C_0|z|^{p\gamma_3}. \qquad (5.39)$$

For $n \geqslant 2$, set $w_n(t,x;z) = v_n(t,x+z) - v_n(t,x)$, where v_n is defined by (5.4). Then

$$w_{n+1}(t,x;z) = \int_0^t \int_{\mathbb{R}^d} (\Gamma(t-s,x+z-y) - \Gamma(t-s,x-y)) v_n(s,y) M(ds,dy). \quad (5.40)$$

Setting $\check{\Gamma}(s,y) = \Gamma(t-s,z+y) - \Gamma(t-s,y)$ and letting $w_n^{(m,k)}$ be the approximation of w_n with Γ replaced by $\Gamma_{m,k}$ in (5.40), we can use the same argument as in (5.9) to see that

$$
\begin{aligned}
&\mathbb{E}[|w_{n+1}^{(m,k)}(t,x;z)|^p] \\
\leqslant\ & t^{q-1} \int_0^t ds \prod_{j=1}^q \int_{\mathbb{R}^d} dy_j \int_{\mathbb{R}^d} dz_j \check{\Gamma}_{m,k}(s,x-y_j) f(y_j-z_j) \check{\Gamma}_{m,k}(s,x-z_j) \\
&\qquad \times \mathbb{E}[v_n(s,y_1)v_n(s,z_1)\cdots v_n(s,y_q)v_n(s,z_q)], \qquad (5.41)
\end{aligned}
$$

where p is an even integer and $q = \frac{p}{2}$. Using Lemma 5.2 to express the expectation and using the same argument as used to reach (5.17), we obtain

$$
\begin{aligned}
&\mathbb{E}[|w_{n+1}^{(m,k)}(t,x;z)|^p] \\
\overset{S}{\leqslant}\ & t^{q-1} \int_0^t ds \int_0^s d\rho_1 \cdots \int_0^{\rho_{N-1}} d\rho_N \\
&\times \prod_{j=1}^N \int_{\mathbb{R}^d} \mu(d\xi_j) \overline{\mathcal{F}\Gamma(\sigma_j-\rho_j)(\xi_j-\eta_j)} \mathcal{F}\Gamma(\sigma_j'-\rho_j)(\xi_j-\eta_j') \\
&\qquad \times \prod_{l=1}^q \int_{\mathbb{R}^d} \mu(d\beta_l) \overline{\mathcal{F}\check{\Gamma}_{m,k}(s)(\beta_k-\gamma_k)} \mathcal{F}\check{\Gamma}_{m,k}(s)(\beta_k-\gamma_k') e^{i\langle x,\delta\rangle}, \quad (5.42)
\end{aligned}
$$

where $\overset{S}{\leqslant}$ means "is bounded by a sum of terms of the form", $N = nq$ and $\sigma_j, \sigma'_j, \eta_j, \eta'_j, \gamma_k, \gamma'_k$ and δ $(1 \leqslant j \leqslant N,\ 1 \leqslant k \leqslant q)$ satisfy the same assumptions as in (5.17). Notice that Γ appears in the first N integrals and $\check{\Gamma}$ in the last q integrals.

We take limits in (5.42) as k and m tend to $+\infty$, then bound the first N spatial integrals in (5.42) using (5.1), bound the other q spatial integrals by hypothesis (H5), compute the time integrals and bound the number of terms in the sum by Lemma 5.4 and we finally reach

$$\mathbb{E}[|w_{n+1}(t,x;z)|^p] \leqslant (q(p-1))^{nq} \frac{T^{(n+1)q}}{(nq+1)!} I^{nq} |z|^{p\gamma_3} = C_n^{(3)} |z|^{p\gamma_3}, \tag{5.43}$$

where $C_n^{(3)} = (q(p-1))^{nq} \frac{T^{(n+1)q}}{(nq+1)!} I^{nq}$. Finally, by (5.39) and (5.43), we have

$$\mathbb{E}[|u(t,x+z) - u(t,x)|^p] \leqslant \sum_{n=1}^{\infty} C_n^{(3)} |z|^{p\gamma_3}, \tag{5.44}$$

for any even integer p and $z \in K$. The series $\sum_{n=1}^{\infty} C_n^{(3)}$ converges, as in (5.19). Jensen's inequality establishes (5.44) for an arbitrary $p \geqslant 1$, , which shows γ-Hölder-continuity of $x \mapsto u(t,x)$ for any $\gamma \in]0, \gamma_3[$ by Kolmogorov's continuity theorem (see [24, Theorem 2.1, p.26]). ■

As a consequence of Propositions 5.6 and 5.7, we easily obtain the following corollary.

Corollary 5.8. *Suppose that the fundamental solution of $Lu = 0$ satisfies the assumptions of Theorem 5.1 as well as (H3) to (H5), and u is the solution of (5.21) given by Theorem 5.1. Then $(t,x) \mapsto u(t,x)$ is a.s. jointly γ-Hölder-continuous in time and space for any $\gamma \in]0, \gamma_1 \wedge (\gamma_2 + \frac{1}{2}) \wedge \gamma_3[$.*

Proof. By (5.37) and (5.44),

$$\mathbb{E}[|u(t,x) - u(s,y)|^p] \leqslant C \left(|t-s|^{\gamma_1 \wedge (\gamma_2 + \frac{1}{2})} + |x-y|^{\gamma_3} \right)^p,$$

so the conclusion follows from Kolmogorov's continuity theorem (see [24, Theorem 2.1, p.26]). ■

Now, we are going to check that the fundamental solution of the wave equation satisfies hypotheses (H3) to (H5). This requires an integrability condition on the covariance function f (or the spectral measure μ) of $\dot{F}$: we suppose that there exists $\alpha \in]0,1[$ such that

$$\int_{\mathbb{R}^d} \frac{\mu(d\xi)}{(1+|\xi|^2)^\alpha} < \infty. \tag{5.45}$$

This assumption is the same as condition (40) in [2]. Since $f \geqslant 0$, it is equivalent (see [7, Lemma 8] and [19]) to the property

$$\sup_{\eta \in \mathbb{R}^d} \int_{\mathbb{R}^d} \frac{\mu(d\xi)}{(1+|\xi+\eta|^2)^\alpha} < \infty. \tag{5.46}$$

Proposition 5.9. *Suppose (5.46) is satisfied for some $\alpha \in]0,1[$. Then the fundamental solution of the wave equation satisfies hypotheses (H3) to (H5) for any $\gamma_i \in]0, 1-\alpha]$, $i = 1,2,3$.*

Proof. Omitting the factors 2π, which do not play any role, we recall that the fundamental solution Γ of the wave equation satisfies

$$\mathcal{F}\Gamma(s)(\xi) = \frac{\sin(s|\xi|)}{|\xi|}$$

in any spatial dimension $d \geqslant 1$. Consider first hypothesis (H3). Fix Q sufficiently large. For any $s \in [0,T]$ and $h \geqslant 0$, we have

$$\begin{aligned}
&\int_{\mathbb{R}^d} \mu(d\xi)\, |\mathcal{F}\Gamma(s+h)(\xi+\eta) - \mathcal{F}\Gamma(s)(\xi+\eta)|^2 \\
&\qquad = \int_{\mathbb{R}^d} \mu(d\xi)\, \frac{|\sin((s+h)|\xi+\eta|) - \sin(s|\xi+\eta|)|^2}{|\xi+\eta|^2} \\
&\qquad = \int_{|\xi+\eta| \leqslant Q} \mu(d\xi)\, \frac{|\sin((s+h)|\xi+\eta|) - \sin(s|\xi+\eta|)|^2}{|\xi+\eta|^2} \\
&\qquad\quad + \int_{|\xi+\eta| > Q} \mu(d\xi)\, \frac{|\sin((s+h)|\xi+\eta|) - \sin(s|\xi+\eta|)|^2}{|\xi+\eta|^2}.
\end{aligned}$$

Using elementary properties of trigonometric functions and the fact that $|\sin(x)| \leqslant x$ for all $x \geqslant 0$ in the first integral and using the same on the $2(1-\alpha)$ power in the second integral, the previous expression is bounded by

$$\begin{aligned}
&\int_{|\xi+\eta| \leqslant Q} \mu(d\xi)\, 4\,h^2 \cos^2((2s+h)|\xi+\eta|) \\
&\quad + \int_{|\xi+\eta| > Q} \mu(d\xi)\, \frac{|\sin((s+h)|\xi+\eta|) - \sin(s|\xi+\eta|)|^{2\alpha}}{|\xi+\eta|^{2\alpha}} \\
&\qquad\qquad \times\ (2h|\cos((2s+h)|\xi+\eta|)|)^{2(1-\alpha)}.
\end{aligned}$$

Bounding the trigonometric functions by 1 and using properties of the domain of integration of each integral, the previous expression is not greater than

$$\left(\int_{|\xi+\eta|\leqslant Q} \mu(d\xi)\frac{4(1+Q^2)}{1+|\xi+\eta|^2}\right) h^2 + \left(\int_{|\xi+\eta|>Q} \mu(d\xi)\frac{4(1+\frac{1}{Q^2})^\alpha}{(1+|\xi+\eta|^2)^\alpha}\right) h^{2(1-\alpha)}$$
$$\leqslant C\left(\int_{\mathbb{R}^d} \frac{\mu(d\xi)}{(1+|\xi+\eta|^2)^\alpha}\right) h^{2(1-\alpha)}.$$

Hence,

$$\sup_{0\leqslant s\leqslant T}\sup_{\eta\in\mathbb{R}^d}\int_{\mathbb{R}^d} \mu(d\xi)\,|\mathcal{F}\Gamma(s+h)(\xi+\eta) - \mathcal{F}\Gamma(s)(\xi+\eta)|^2$$
$$\leqslant C\left(\sup_{\eta\in\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{\mu(d\xi)}{(1+|\xi+\eta|^2)^\alpha}\right) h^{2(1-\alpha)},$$

and hypothesis (H3) is satisfied for any $\gamma_1 \in]0, 1-\alpha]$.

For hypothesis (H4), for any $s \in [0,T]$,

$$\int_{\mathbb{R}^d} \mu(d\xi)\,|\mathcal{F}\Gamma(s)(\xi+\eta)|^2 = \int_{\mathbb{R}^d} \mu(d\xi)\,\frac{\sin^2(s|\xi+\eta|)}{|\xi+\eta|^2}$$
$$\leqslant \int_{|\xi+\eta|<1} \mu(d\xi)\,\frac{\sin^2(s|\xi+\eta|)}{|\xi+\eta|^2} + \int_{|\xi+\eta|\geqslant 1} \mu(d\xi)\,\frac{\sin^2(s|\xi+\eta|)}{|\xi+\eta|^2}.$$

Using the fact that $|\sin(x)| \leqslant x$ for all $x \geqslant 0$ in the first integral and the same on the $2(1-\alpha)$ power in the second integral, the previous expression is bounded by

$$s^2 \int_{|\xi+\eta|<1} \mu(d\xi) + \int_{|\xi+\eta|\geqslant 1} \mu(d\xi)\, s^{2(1-\alpha)}\frac{|\sin(s|\xi+\eta|)|^{2\alpha}}{|\xi+\eta|^{2\alpha}}.$$

Bounding the trigonometric function by 1 and using properties of the domain of integration of each integral, the previous expression is not greater than

$$\begin{aligned}
&\leqslant \quad s^2 \int_{|\xi+\eta|<1} \mu(d\xi)\,\frac{2}{1+|\xi+\eta|^2} + s^{2(1-\alpha)}\int_{|\xi+\eta|\geqslant 1} \mu(d\xi)\,\frac{2^\alpha}{(1+|\xi+\eta|^2)^\alpha} \\
&\leqslant \quad C\left(\int_{\mathbb{R}^d} \frac{\mu(d\xi)}{(1+|\xi+\eta|^2)^\alpha}\right) |s|^{2(1-\alpha)}.
\end{aligned}$$

Hence,

$$\sup_{\eta\in\mathbb{R}^d}\int_{\mathbb{R}^d} \mu(d\xi)\,|\mathcal{F}\Gamma(s)(\xi+\eta)|^2 \leqslant C\left(\sup_{\eta\in\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{\mu(d\xi)}{(1+|\xi+\eta|^2)^\alpha}\right) s^{2(1-\alpha)},$$

and hypothesis (H4) is satisfied for any $\gamma_2 \in]0, 1-\alpha]$.

Finally, for hypothesis (H5), for any $x \in \mathbb{R}$ and $z \in K$, K a compact subset of $\mathbb{R}^d$,

$$\begin{aligned}
&\int_{\mathbb{R}^d} \mu(d\xi)\, |\mathcal{F}\Gamma(t-s, z-\cdot)(\xi+\eta) - \mathcal{F}\Gamma(t-s,\cdot)(\xi+\eta)|^2 \\
&= \int_{|\xi+\eta|<1} \mu(d\xi)\, |e^{-i\langle \xi+\eta, z\rangle} - 1|^2 \, \frac{\sin^2((t-s)|\xi+\eta|)}{|\xi+\eta|^2} \\
&\quad + \int_{|\xi+\eta|\geqslant 1} \mu(d\xi)\, |e^{-i\langle \xi+\eta, z\rangle} - 1|^2 \, \frac{\sin^2((t-s)|\xi+\eta|)}{|\xi+\eta|^2}.
\end{aligned}$$

Bounding the trigonometric functions by 1, using properties of the domain of integration in the first integral and bounding the 2α power of the second factor by 2 in the second integral, the previous expression is not greater than

$$\begin{aligned}
&\int_{|\xi+\eta|<1} \mu(d\xi)\, |e^{-i\langle \xi+\eta, z\rangle} - 1|^2 \frac{2}{1+|\xi+\eta|^2} \\
&+ \int_{|\xi+\eta|\geqslant 1} \mu(d\xi)\, |e^{-i\langle \xi+\eta, z\rangle} - 1|^{2(1-\alpha)} 2^{2\alpha} \frac{1}{|\xi+\eta|^2}.
\end{aligned}$$

Using the fact that $|e^{-i\langle \xi+\eta, z\rangle} - 1| \leqslant |\xi+\eta||z|$ and properties of the domain of integration of each integral, the previous expression is bounded by

$$\begin{aligned}
&|z|^2 \int_{|\xi+\eta|<1} \mu(d\xi) \frac{2}{1+|\xi+\eta|^2} + |z|^{2(1-\alpha)} \int_{|\xi+\eta|\geqslant 1} \mu(d\xi) \frac{4^{2\alpha}}{(1+|\xi+\eta|^2)^\alpha} \\
&\qquad \leqslant C \left(\int_{\mathbb{R}^d} \frac{\mu(d\xi)}{(1+|\xi+\eta|^2)^\alpha} \right) |z|^{2(1-\alpha)}.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\sup_{0\leqslant s\leqslant T} \sup_{\eta\in\mathbb{R}^d} \int_{\mathbb{R}^d} \mu(d\xi)\, |\mathcal{F}\Gamma(t-s, x+z-\cdot)(\xi+\eta) - \mathcal{F}\Gamma(t-s, x-\cdot)(\xi+\eta)|^2 \\
&\qquad \leqslant C \left(\sup_{\eta\in\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{\mu(d\xi)}{(1+|\xi+\eta|^2)^\alpha} \right) |z|^{2(1-\alpha)},
\end{aligned}$$

and hypothesis (H5) is satisfied for any $\gamma_3 \in]0, 1-\alpha]$. ∎

We recall the following result for the covariance function $f(x) = \frac{1}{|x|^\beta}$, with $0 < \beta < d$. For a proof, see [26, Prop.5.3].

Proposition 5.10. *If $f(x) = \frac{1}{|x|^\beta}$, where $0 < \beta < d$, then $\mu(dx) = \frac{dx}{|x|^{d-\beta}}$ and (5.45) (hence (5.46)) is satisfied for any $\alpha \in]\frac{\beta}{2}, +\infty[$.*

Putting together Propositions 5.6-5.9, Corollary 5.8 and Proposition 5.10, we have the following.

Theorem 5.11. *If $f(x) = \frac{1}{|x|^\beta}$, with $0 < \beta < 2$, then the random-field solution $u(t,x)$ of the non-linear wave equation with spatial dimension $d > 3$ built in Theorem 5.1 is jointly γ-Hölder-continuous in time and space for any exponent $\gamma \in]0, \frac{2-\beta}{2}[$.*

Remark 5.12. (a) Note that Theorem 5.11 and its proof are still valid when the spatial dimension is less than or equal to 3. In these cases, the regularity of the solution has already been obtained for a more general class of non-linear functions α, namely Lipschitz continuous functions. For more details, see [30] for $d = 1$, [17] for $d = 2$ and [9] for $d = 3$.

(b) The exponent $\frac{2-\beta}{2}$ in Theorem 5.11 is the optimal exponent. Indeed, $u(t,x)$ is not γ-Hölder-continuous for any exponent $\gamma > \frac{2-\beta}{2}$ as is shown in [9, Theorem 5.1]. Their proof applies to the general d-dimensional case, essentially without change.

Chapter 6

Iterated Stochastic Integrals

In [14, Chapter 5], Kloeden and Platen iteratively use Itô's formula in order to study the process $(X_t)_{t\geqslant 0}$, solution of the stochastic differential equation

$$dX_t = a(t, X_t)dt + b(t, X_t)dW_t \tag{6.1}$$

with $X_0 = x_0$, where a, b are C^∞-functions and W_t is a standard Brownian motion. They first define iterated stochastic integrals, which they use in order to establish a stochastic Taylor expansion of order n for any process of the form $g(t, X_t)$, where g is a C^∞-function. Namely, $g(t, X_t)$ is equal to a finite sum of iterated stochastic integrals of maximal order n, in addition to a remainder term, expressed in terms of iterated stochastic integrals, see [14, Theorem 5.5.1]. Then, under certain assumptions on the functions a, b and g, they can prove the convergence to 0 of the remainder term in the case where $g(t, x) = x$, see [14, Proposition 5.9.1].

From now on, we are going to use the same machinery as above in order to study the solution $u(t, x)$ of (4.1), the existence and uniqueness of which were studied in Section 4.1. The main idea is to state an Itô-Taylor type expansion for $u(t, x)$, that is, an expansion of the solution $u(t, x)$ as a sum of iterated stochastic integrals obtained by a repeated use of Itô's formula and the fundamental theorem of calculus.

In this chapter, we are going to define the iterated stochastic integrals that will be used in Chapter 7 to state the Itô-Taylor type expansion for the non-linear stochastic wave equation (4.13). First of all, we set up the notation that we are going to use.

6.1 Multi-indices

We set $\mathcal{A} = \mathbb{Z} \cup \mathbb{Z}^2$ and, for $j \in \mathbb{N}$, $\mathcal{A}_j = (\mathbb{Z} \cap]-\infty, j]) \cup (\mathbb{Z}^2 \cap]-\infty, j]^2)$. These sets will be the sets of values of the components of our multi-indices. We note that if $\beta_1 \in \mathcal{A}_0$, then $\beta_1 \leqslant 0$ if $\beta_1 \in \mathbb{Z}$, and both components of β_1 are non-positive if $\beta_1 \in \mathbb{Z}^2$.

For $\beta \in \mathcal{A}^n$, we write $\beta = (\beta_1, \dots, \beta_n)$, with $\beta_j \in \mathcal{A}$. We now set $\mathcal{A}^{(0)} = \{\emptyset\}$ and, for all $n \in \mathbb{N}^*$,

$$\mathcal{A}^{(n)} = \{\beta \in \mathcal{A}^n : \beta_j \in \mathcal{A}_{\sum_{\ell=1}^{j-1} |\beta_\ell|}, \text{ for all } j = 1, \dots, n\}, \tag{6.2}$$

with the convention that $\sum_{\ell=1}^{0} |\beta_\ell| = 0$, where

$$|\beta_\ell| = \begin{cases} 1 & \text{if } \beta_\ell \in \mathbb{Z}, \\ 2 & \text{if } \beta_\ell \in \mathbb{Z}^2. \end{cases}$$

We call $|\beta_\ell|$ the *size* of β_ℓ. The set $\mathcal{A}^{(n)}$ is the set of multi-indices of length n. The conditions on the components of β in the definition of $\mathcal{A}^{(n)}$ will be useful to define an iterated stochastic integral associated with this multi-index. Let

$$\mathcal{A}^* = \bigcup_{i=0}^{\infty} \mathcal{A}^{(i)}$$

be the set of multi-indices and for $\beta \in \mathcal{A}^*$, we set

$$\ell(\beta) = n \qquad \text{if } \beta \in \mathcal{A}^{(n)}$$

and call this the *length* of β : this is the number of *components* of β. We also let

$$\|\beta\| = \sum_{\ell=1}^{n} |\beta_\ell| \qquad \text{if } \beta \in \mathcal{A}^{(n)},$$

and $\|\emptyset\| = 0$; this is the number of integers, or *values*, that define β. We extend these definitions naturally to any element of $\bigcup_{n=0}^{\infty} \mathcal{A}^n$.

The set $\mathcal{A}^*$ allows negative values as components of the multi-indices, but only for technical reasons. The multi-indices that will appear in the Itô-Taylor type series are those of the set

$$\bar{\mathcal{A}} = \{\beta \in \bigcup_{n=0}^{\infty} \mathcal{A}^n : \beta_j \in \mathbb{N} \cup \mathbb{N}^2, \text{ for all } j = 1, \dots, \ell(\beta)\} \cup \{\emptyset\}.$$

We also write $\bar{\mathcal{A}}^{(n)} = \bar{\mathcal{A}} \cap \mathcal{A}^{(n)}$ and $\bar{\mathcal{A}}^* = \bar{\mathcal{A}} \cap \mathcal{A}^*$.

Now, we are going to define some operators on multi-indices that will be useful in the following definitions and proofs. First of all, let us define the *cut* operators. For $\beta \in \mathcal{A}^* \setminus \mathcal{A}^{(0)}$, we set

$$-\beta = \begin{cases} \emptyset & \text{if } \beta \in \mathcal{A}^{(1)}, \\ (\beta_2, \dots, \beta_n) & \text{if } \beta \in \mathcal{A}^{(n)} \text{ and } n \geqslant 2, \end{cases}$$

($-\beta$ corresponds to deletion of the first component) and

$$\beta- = \begin{cases} \emptyset & \text{if } \beta \in \mathcal{A}^{(1)}, \\ (\beta_1, \ldots, \beta_{n-1}) & \text{if } \beta \in \mathcal{A}^{(n)} \text{ and } n \geqslant 2. \end{cases}$$

In order to define iterated integrals, we will need the following shift operators on multi-indices. For $k \in \mathbb{N}$, we define $\theta_k : \bigcup_{n=1}^{\infty} \mathcal{A}^n \to \bigcup_{n=1}^{\infty} \mathcal{A}^n$ by

$$\theta_k(\beta)_j = \begin{cases} \beta_j - k & \text{if } \beta_j \in \mathbb{Z}, \\ \beta_j - \binom{k}{k} & \text{if } \beta_j \in \mathbb{Z}^2, \end{cases}$$

for all $\beta \in \mathcal{A}^n$, and all $j = 1, \ldots, \ell(\beta)$, where, in the second case, the subtraction is the usual operation in $\mathbb{Z}^2$. Then, we set

$$\mathcal{A}_{(k)} = \theta_k^{-1}(\mathcal{A}^*)$$

and

$$\bar{\mathcal{A}}_{(k)} = \mathcal{A}_{(k)} \cap \bar{\mathcal{A}}.$$

A multi-index β belongs to $\mathcal{A}_{(k)}$ if the constraints of (6.2) are still satisfied when we substract k from each value in β and $\beta \in \bar{\mathcal{A}}$ if its values are non-negative. As a consequence, $\beta \in \bar{\mathcal{A}}_{(k)}$ if both of these conditions are satisfied. Moreover, Lemma 6.1 below gives an explicit description of the elements of $\bar{\mathcal{A}}_{(k)}$. Notice that θ_0 is the identity operator and hence $\mathcal{A}_{(0)} = \mathcal{A}^* \supset \bar{\mathcal{A}}^*$. All of this notation will be used in Section 6.2 to define the iterated stochastic integrals recursively.

Lemma 6.1. *The set $\bar{\mathcal{A}}_{(k)}$ corresponds to the set of all multi-indices γ for which there exists $\beta \in \bar{\mathcal{A}}^*$ such that γ is obtained from β by deleting the first k values that determine β.*

Example 6.2. In Lemma 6.1, let $\gamma = (2, \binom{2}{3}, 4) \in \bar{\mathcal{A}}_{(2)}$. We can see that γ is obtained by deleting the first two variables of $\beta = (0, 0, 2, \binom{2}{3}, 4) \in \bar{\mathcal{A}}^*$. Hence, Lemma 6.1 is satisfied in that case.

Proof. First of all consider $\gamma \in \bar{\mathcal{A}}_{(k)}$. We know that $\theta_k(\gamma) \in \mathcal{A}^*$ and, as a consequence, $\theta_k(\gamma) \in \mathcal{A}^{(n)}$ for some $n = \ell(\gamma)$. Hence, we know that

$$\theta_k(\gamma)_j \in \mathcal{A}_{\sum_{\ell=1}^{j-1} |\gamma_\ell|}$$

for any $j = 1, \ldots, \ell(\gamma)$. This is equivalent to

$$\theta_k(\gamma)_j \leqslant \sum_{\ell=1}^{j-1} |\gamma_\ell|$$

if $\gamma_j \in \mathbb{Z}$ and to

$$\theta_k(\gamma)_j \leqslant \begin{pmatrix} \sum_{\ell=1}^{j-1} |\gamma_\ell| \\ \sum_{\ell=1}^{j-1} |\gamma_\ell| \end{pmatrix}$$

if $\gamma_j \in \mathbb{Z}^2$, where the inequality must be taken component by component. Finally, by the definition of θ_k, this is equivalent to

$$\gamma_j \leqslant k + \sum_{\ell=1}^{j-1} |\gamma_\ell| \tag{6.3}$$

if $\gamma_j \in \mathbb{Z}$ and to

$$\gamma_j \leqslant \begin{pmatrix} k + \sum_{\ell=1}^{j-1} |\gamma_\ell| \\ k + \sum_{\ell=1}^{j-1} |\gamma_\ell| \end{pmatrix} \tag{6.4}$$

if $\gamma_j \in \mathbb{Z}^2$. Now, setting $\beta = (\beta_1, \dots, \beta_{k+\ell(\gamma)})$ with

$$\beta_j = \begin{cases} 0 & \text{if } j \leqslant k, \\ \gamma_{k-j} & \text{if } j \geqslant k+1, \end{cases}$$

we see that the components number $k+1$ to $\ell(\beta)$ of the multi-index β are those of γ and $\beta \in \bar{\mathcal{A}}^*$ by the inequalities (6.3) and (6.4). Hence, γ is obtained by deleting the first $|\beta_1| + \cdots + |\beta_k| = k$ values that determine β and the first inclusion is proved.

Now consider a multi-index $\beta \in \bar{\mathcal{A}}^*$ and $k \in \mathbb{N}$ such that deleting the first k values of β yields a multi-index γ. (For example $\beta = \left(\binom{0}{0}, 1\right)$ and $k = 1$ is excluded). Then there is a j such that $\gamma_i = \beta_{j+i}$ for all $i = 1, \dots, \ell(\beta) - j$. This j satisifies $\sum_{\ell=1}^{j-1} |\beta_\ell| = k$. Now, let $1 \leqslant i \leqslant \ell(\gamma) = \ell(\beta) - j$ and suppose that $\gamma_i \in \mathbb{Z}$. Then, as $\beta \in \bar{\mathcal{A}}$,

$$\gamma_i = \beta_{j+i} \leqslant \sum_{\ell=1}^{j+i-1} |\beta_\ell| = \sum_{\ell=1}^{j-1} |\beta_\ell| + \sum_{\ell=j}^{j+i-1} |\beta_\ell| = k + \sum_{\ell=j}^{j+i-1} |\gamma_{\ell-j}| = k + \sum_{\ell=1}^{i-1} |\gamma_\ell|.$$

Hence,

$$\theta_k(\gamma)_i \leqslant \sum_{\ell=1}^{i-1} |\gamma_\ell|,$$

for all $i = 1, \dots, \ell(\gamma)$ such that $\gamma_i \in \mathbb{Z}$. The analogous result is also true in the case where $\gamma_i \in \mathbb{Z}^2$. As a consequence, $\theta_k(\gamma) \in \mathcal{A}^*$ and the result is proved. ■

6.2 Iterated integral processes

Now we would like to define iterated integral processes $I_\beta^{(k)}$ for all $k \in \mathbb{N}$ and all $\beta \in \mathcal{A}_{(k)}$. As the set of multi-indices $\bar{\mathcal{A}}$ is contained in $\mathcal{A}_{(0)}$, our final goal is to define $I_\beta^{(0)}$ for $\beta \in \mathcal{A}_0$. First of all, we need to define integral operators. We are going to use the stochastic integral with respect to martingale measures first defined for real-valued integrands in [30] (see Section 2.1), then extended to non-negative measures in [2] (see Section 2.3) and finally to general Schwartz distributions in Chapter 3.

Let us assume that the fundamental solution Γ of the partial differential operator L in (4.1) satisfies the assumptions of Theorem 4.2. Moreover, suppose that Γ is such that

$$\sup_{0\leqslant s\leqslant T}\sup_{\eta\in\mathbb{R}^d}\int_{\mathbb{R}^d}\mu(d\xi)|\mathcal{F}\Gamma(s)(\xi+\eta)|^2<+\infty. \tag{6.5}$$

This is the same additional assumption as used in Theorem 5.1 to obtain finite moments of any order in the case of an affine multiplicative noise.

Let $j,k \in \mathbb{N}$ be such that $j \leqslant k$. Let $(r_k)_{k\in\mathbb{N}}$ be a sequence of time variables in $\mathbb{R}_+$ and $(y_k)_{k\in\mathbb{N}}$ be a sequence of spatial variables in $\mathbb{R}^d$. We denote by $\boldsymbol{r}_k = (r_0,\dots,r_k)$ (resp. $\boldsymbol{y}_k = (y_0,\dots,y_k)$) the vector composed by the first $k+1$ time (resp. spatial) variables. For a (random) function $g : \mathbb{R}_+^{k+2}\times(\mathbb{R}^d)^{k+2}\times\Omega\to\mathbb{R}$, we define $I_j^{(k)}(g) : \mathbb{R}_+\times\mathbb{R}_+^{k+1}\times(\mathbb{R}^d)^{k+1}\times\Omega\to\mathbb{R}$ by

$$\begin{aligned} I_j^{(k)}&(g)(s;\boldsymbol{r}_k;\boldsymbol{y}_k)\\ &= \int_0^s\int_{\mathbb{R}^d}\Gamma(r_j-r_{k+1},y_j-y_{k+1})g(\boldsymbol{r}_k,r_{k+1};\boldsymbol{y}_k,y_{k+1})M(dr_{k+1}\,dy_{k+1}), \end{aligned}\tag{6.6}$$

where the stochastic integral is defined according to Theorem 2.13 ([30]), Theorem 2.16 ([2]), Theorem 3.1 or by Proposition 6.8 below depending on the nature of Γ and g.

Moreover, let $j, j', k \in \mathbb{N}$ such that $j \leqslant k$ and $j' \leqslant k$. For a (random) function

$g : \mathbb{R}_+^{k+3} \times (\mathbb{R}^d)^{k+3} \times \Omega \to \mathbb{R}$, we define $I^{(k)}_{\binom{j}{j'}}(g) : \mathbb{R}_+ \times \mathbb{R}_+^{k+1} \times (\mathbb{R}^d)^{k+1} \times \Omega \to \mathbb{R}$ by

$$\begin{aligned}
I^{(k)}_{\binom{j}{j'}}&(g)(s; \boldsymbol{r}_k; \boldsymbol{y}_k) \\
= &\int_0^s dr_{k+1} \int_0^s dr_{k+2} \int_{\mathbb{R}^d} dy_{k+1} \int_{\mathbb{R}^d} dy_{k+2}\, \delta_0(r_{k+1} - r_{k+2}) \\
&\qquad \times \Gamma(r_j - r_{k+1}, y_j - y_{k+1}) f(y_{k+1} - y_{k+2}) \Gamma(r_{j'} - r_{k+2}, y_{j'} - y_{k+2}) \\
&\qquad \times g(\boldsymbol{r}_k, r_{k+1}, r_{k+2}; \boldsymbol{y}_k, y_{k+1}, y_{k+2}) \qquad (6.7) \\
= &\int_0^s dr \int_{\mathbb{R}^d} dy \int_{\mathbb{R}^d} dz\, \Gamma(r_j - r, y_j - y) f(y - z) \Gamma(r_{j'} - r, y_{j'} - z) \\
&\qquad \times g(\boldsymbol{r}_k, r, r; \boldsymbol{y}_k, y, z), \qquad (6.8)
\end{aligned}$$

where the integral is defined according to Proposition 6.11 below. In the case where Γ is a distribution, $I^{(k)}_{\binom{j}{j'}}(g)$ is first defined with Γ replaced by a smooth function Γ_n and then by taking limits in $L^2(\Omega)$ (see Proposition 6.11). Notice that in the two definitions above, certain additional conditions on g must be satisfied in order that the integrals be well defined.

Now that we have integral operators, let $\beta \in \mathcal{A}^*$ and let $k \in \mathbb{N}$ be such that $\beta \in \mathcal{A}_{(k)} \cup \{\emptyset\}$. First of all, we set $I^{(k)}_{\emptyset}(s; \boldsymbol{r}_k, \boldsymbol{y}_k) = 1$ for all $s \in [0,T]$, $\boldsymbol{r}_k \in [0,T]^{k+1}$ and $\boldsymbol{y}_{k+1} \in (\mathbb{R}^d)^{k+1}$. Then, for $\beta \in \mathcal{A}_{(k)}$, we define $I^{(k)}_{\beta} : [0,T] \times [0,T]^{k+1} \times (\mathbb{R}^d)^{k+1}$ recursively on $\ell(\beta)$ by

$$I^{(k)}_{\beta}(s; \boldsymbol{r}_k; \boldsymbol{y}_k) = I^{(k)}_{\beta_1}\left(\tilde{I}^{(k+|\beta_1|)}_{-\beta}\right)(s; \boldsymbol{r}_k, \boldsymbol{y}_k), \qquad (6.9)$$

for $s \in [0,T]$, $\boldsymbol{r}_k \in [0,T]^{k+1}$, $\boldsymbol{y}_k \in (\mathbb{R}^d)^{k+1}$, where

$$\tilde{I}^{(k+|\beta_1|)}_{-\beta}(\boldsymbol{r}_{k+|\beta_1|}; \boldsymbol{y}_{k+|\beta_1|}) = I^{(k+|\beta_1|)}_{-\beta}(r_{k+|\beta_1|}; \boldsymbol{r}_{k+|\beta_1|}; \boldsymbol{y}_{k+|\beta_1|})$$

and $I^{(k)}_{\beta_1}$ is one of the two operators defined in (6.6) or (6.8), depending on the size of β_1.

Remark 6.3. When considering the iterated integral with respect to a multi-index $\beta \in \bar{\mathcal{A}}_{(k)}$, the variables $\boldsymbol{r}_k$ and $\boldsymbol{y}_k$ are called *final variables.* The variables appearing in the successive integrals are called *integration variables.* They are numbered in an increasing order so that $r_0, \ldots, r_k$ are the final variables and $r_{k+1}, \ldots, r_m$ ($m = k + \ell(\beta)$) are the integration variables. For an integration variable r_{k+j} ($j = 1, \ldots, \ell(\beta)$), the variable to which r_{k+j} is substracted when integrated in

(6.6) or (6.7) is called the *antecedent* of r_{k+j}. The index of the antecedent is equal to the value in the multi-index β which corresponds to the position of the integral considered. An antecedent variable can be a final variable or an integration variable of lower index. This is implied by the conditions given in the definition of the multi-indices (6.2). A final variable has no antecedent as it is not integrated. (See Example 6.31 for an illustration.)

Remark 6.4. The processes $I_\beta^{(k)}$ are defined as functions of $\boldsymbol{r}_k$ and $\boldsymbol{y}_k$. Nevertheless they often only depend on a few components of those variables. Indeed, if all values determining β are different from j ($j \leqslant k$), then the process $I_\beta^{(k)}$ does not depend on r_j nor y_j. In all cases, for $j > k$, the process $I_\beta^{(k)}$ does not depend on r_j nor y_j.

In order to illustrate the definitions and remarks above, we now present two examples of multi-indices and the iterated stochastic integrals to which they correspond.

Example 6.5. Let us consider $\beta = \big(0, 1, \binom{1}{2}\big)$. Then $\beta \in \bar{\mathcal{A}}^*$ and, in particular, $\beta \in \bar{\mathcal{A}}_{(0)}$. Hence, the process $I_\beta^{(0)}$ is defined by (6.9). Namely, for $s, r_0 \in [0, T]$, $y_0 \in \mathbb{R}^d$,

$$
\begin{aligned}
&I_\beta^{(0)}(s; r_0, y_0)\\
&\quad= \int_0^s \int_{\mathbb{R}^d} \Gamma(r_0 - r_1, y_0 - y_1) M(dr_1, dy_1) \int_0^{r_1} \int_{\mathbb{R}^d} \Gamma(r_1 - r_2, y_1 - y_2) M(dr_2, dy_2)\\
&\qquad\times \int_0^{r_2} dr_3 \int_0^{r_2} dr_4 \int_{\mathbb{R}^d} dy_3 \int_{\mathbb{R}^d} dy_4\, \delta_0(r_3 - r_4)\\
&\qquad\qquad\qquad\qquad \times \Gamma(r_1 - r_3, y_1 - y_3) f(y_3 - y_4) \Gamma(r_2 - r_4, y_2 - y_4)\\
&\quad= \int_0^s \int_{\mathbb{R}^d} \Gamma(r_0 - r_1, y_0 - y_1) M(dr_1, dy_1) \int_0^{r_1} \int_{\mathbb{R}^d} \Gamma(r_1 - r_2, y_1 - y_2) M(dr_2, dy_2)\\
&\qquad\times \int_0^{r_2} dr_3 \int_{\mathbb{R}^d} dy_3 \int_{\mathbb{R}^d} dy_4\, \Gamma(r_1 - r_3, y_1 - y_3) f(y_3 - y_4) \Gamma(r_2 - r_3, y_2 - y_4).
\end{aligned}
$$

In this example, r_0 (resp. y_0) is the only final time variable, whereas $r_1, \dots, r_4$ (resp. $y_1, \dots, y_4$) are integration variables. The sequences of antecedent relations for time variables (a variable is the antecedent of the previous one in the sequence) are r_4, r_2, r_1, r_0 and r_3, r_1, r_0. This example will be considered again in Example 6.16.

Example 6.6. Let us consider $\delta = \big(1, 0, 3, \binom{3}{4}\big)$. Then $\delta \notin \bar{\mathcal{A}}^*$ and the process $I_\delta^{(0)}$ is not defined. Nevertheless, this multi-index can appear when defining iterated integrals for other multi-indices. For example, we have $\delta \in \bar{\mathcal{A}}_{(1)}$ and the process $I_\delta^{(1)}$ is defined by (6.9). Namely, for $s, r_0, r_1 \in [0,T]$, $y_0, y_1 \in \mathbb{R}^d$,

$$\begin{aligned}
& I_\delta^{(1)}(s; r_0, r_1; y_0, y_1) \\
&= \int_0^s \int_{\mathbb{R}^d} \Gamma(r_1 - r_2, y_1 - y_2) M(dr_2, dy_2) \int_0^{r_2} \int_{\mathbb{R}^d} \Gamma(r_0 - r_3, y_0 - y_3) M(dr_3, dy_3) \\
&\quad \times \int_0^{r_3} \int_{\mathbb{R}^d} \Gamma(r_3 - r_4, y_3 - y_4) M(dr_4, dy_4) \int_0^{r_4} dr_5 \int_0^{r_4} dr_6 \int_{\mathbb{R}^d} dy_5 \int_{\mathbb{R}^d} dy_6 \\
&\qquad \times \delta_0(r_5 - r_6) \Gamma(r_3 - r_5, y_3 - y_5) f(y_5 - y_6) \Gamma(r_4 - r_6, y_4 - y_6) \\
&= \int_0^s \int_{\mathbb{R}^d} \Gamma(r_1 - r_2, y_1 - y_2) M(dr_2, dy_2) \int_0^{r_2} \int_{\mathbb{R}^d} \Gamma(r_0 - r_3, y_0 - y_3) M(dr_3, dy_3) \\
&\quad \times \int_0^{r_3} \int_{\mathbb{R}^d} \Gamma(r_3 - r_4, y_3 - y_4) M(dr_4, dy_4) \int_0^{r_4} dr_5 \int_{\mathbb{R}^d} dy_5 \int_{\mathbb{R}^d} dy_6 \\
&\qquad \times \Gamma(r_3 - r_5, y_3 - y_5) f(y_5 - y_6) \Gamma(r_4 - r_5, y_4 - y_6).
\end{aligned}$$

In this example, r_0 and r_1 (resp. y_0 and y_1) are the final time variables, whereas $r_2, \dots, r_6$ (resp. $y_2, \dots, y_6$) are integration variables. The sequences of antecedent relations for time variables (a variable is the antecedent of the previous one in the sequence) are r_6, r_4, r_3, r_0, r_5, r_3, r_0 and r_2, r_1. This example will be considered again in Example 6.31.

We would like to show that the iterated integrals defined above are well-defined. For this, we first have to prove the following result, that extends Theorem 3.1. This is necessary because the iterated integrals do not always have a spatially homogeneous covariance. Nevertheless, since we have exact expressions for their second moments, we can show that they are *harmonizable* processes. In that case, Proposition 6.8 shows that it remains possible to define the needed stochastic integrals. We recall the definition of a harmonizable process.

Definition 6.7. A stochastic process $(Z(x), x \in \mathbb{R}^d)$ is *harmonizable* if there exists a bimeasure ν on $\mathbb{R}^d \times \mathbb{R}^d$ (i.e. a mapping $\nu : \mathcal{B}(\mathbb{R}^d) \times \mathcal{B}(\mathbb{R}^d) \to \mathbb{R}$ such that for any fixed $A \subset \mathbb{R}^d$, $\nu(A, \cdot)$ and $\nu(\cdot, A)$ are measures on $\mathbb{R}^d$) such that

$$\mathbb{E}[Z(x)Z(y)] = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \nu(d\xi, d\eta) e^{i\langle \xi, x\rangle} e^{-i\langle \eta, y\rangle}.$$

We refer to [15, Note by M. Loève] and [23] for more information about harmonizable processes. Notice that if the bimeasure ν concentrates on the diagonal, then Z has a spatially homogeneous covariance function.

Proposition 6.8. *Let $(Z(t,x),\ 0 \leqslant t \leqslant T,\ x \in \mathbb{R}^d)$ be a predictable process such that there exists a signed measure $\nu_{s,t}$ defined on $(\mathbb{R}^d)^k$ for some $k \geqslant 1$, depending on time parameters s, t such that*

$$\sup_{s,t\in[0,T]} \int_{(\mathbb{R}^d)^k} |\nu_{s,t}|(d\eta_1,\dots,d\eta_k) < +\infty \tag{6.10}$$

and

$$\mathbb{E}[Z(s,x)Z(t,y)] = \int_{(\mathbb{R}^d)^k} \nu_{s,t}(d\eta_1,\dots,d\eta_k) e^{i\langle\delta,x\rangle} e^{i\langle\delta',y\rangle}, \tag{6.11}$$

where δ and δ' are linear combinations of $\eta_1,\dots,\eta_k$. In particular,

$$\sup_{0\leqslant t\leqslant T} \sup_{x\in\mathbb{R}^d} \mathbb{E}[Z(t,x)^2] < \infty$$

and $x \mapsto Z(t,x)$ is harmonizable for any fixed t.

Let $t \mapsto S(t)$ be a deterministic function with values in the space $\mathcal{S}'_r(\mathbb{R}^d)$. Suppose that $(s,\xi) \mapsto \mathcal{F}S(s)(\xi)$ is measurable and

$$\int_0^T ds \sup_{\eta\in\mathbb{R}^d} \int_{\mathbb{R}^d} \mu(d\xi)\, |\mathcal{F}S(s)(\xi+\eta)|^2 < \infty. \tag{6.12}$$

Suppose in addition that either hypothesis (H1) or (H2) is satisfied. Then $S \in \mathcal{P}_{0,Z}$. In particular, the stochastic integral $(S\cdot M^Z)_t$ is well defined as a real-valued square-integrable martingale $((S\cdot M^Z)_t,\ 0 \leqslant t \leqslant T)$ and

$$\begin{aligned}
&\mathbb{E}[(S\cdot M^Z)_t^2] \\
&\quad = \int_0^t ds \int_{(\mathbb{R}^d)^k} \nu_{s,s}(d\eta_1,\dots,d\eta_k) \int_{\mathbb{R}^d} \mu(d\xi)\, \mathcal{F}S(s)(\xi+\delta)\overline{\mathcal{F}S(s)(\xi-\delta')} \\
&\quad \leqslant \left(\sup_{0\leqslant s\leqslant T} |\nu_{s,s}|((\mathbb{R}^d)^k)\right) \int_0^t ds \sup_{\eta\in\mathbb{R}^d} \int_{\mathbb{R}^d} \mu(d\xi)\, |\mathcal{F}S(s)(\xi+\eta)|^2.
\end{aligned} \tag{6.13}$$

Remark 6.9. If $\delta = -\delta'$ in (6.11), then the process Z has a spatially homogeneous covariance function and satisfies the assumptions of Theorem 3.1, in particular hypothesis (H).

Remark 6.10. Remark 6.9 shows that Theorem 3.1 is a particular case of Proposition 6.8. The form of the covariance function of the process Z in Proposition 6.8 is only slightly different from a spatially homogeneous covariance function. The main point in order to obtain a well-defined stochastic integral is to have a Fourier-type expression for the covariance function. Sharper bounds can be obtained when this function only depends on the difference $x - y$, but this property is not essential.

Proof. Let us consider the definition (2.5) of $\|\cdot\|_{0,Z}$ and consider a deterministic $\phi \in \mathcal{P}_{0,Z}$ with $\phi(s,\cdot) \in \mathcal{S}(\mathbb{R}^d)$. We have

$$
\begin{aligned}
&\|\phi\|_{0,Z}^2 \\
&= \int_0^T ds \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^d} dy\, \phi(s,x) f(x-y) \phi(s,y) \mathbb{E}[Z(s,x)Z(s,y)] \\
&= \int_0^T ds \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^d} dy\, \phi(s,x) f(x-y) \phi(s,y) \int_{(\mathbb{R}^d)^k} \nu_{s,s}(d\eta_1,\dots,d\eta_k) e^{i\langle \delta, x\rangle} e^{i\langle \delta', y\rangle} \\
&= \int_0^T ds \int_{(\mathbb{R}^d)^k} \nu_{s,s}(d\eta_1,\dots,d\eta_k) \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^d} dy\, e^{i\langle \delta, x\rangle} \phi(s,x) f(x-y) \phi(s,y) e^{i\langle \delta', y\rangle} \\
&= \int_0^T ds \int_{(\mathbb{R}^d)^k} \nu_{s,s}(d\eta_1,\dots,d\eta_k) \int_{\mathbb{R}^d} \mu(d\xi) \mathcal{F}\phi(s)(\xi+\delta) \overline{\mathcal{F}\phi(s)(\xi-\delta')}. \qquad (6.14)
\end{aligned}
$$

Notice that if Z is spatially homogeneous (i.e. $\delta = -\delta'$), then this expression simply reduces to (3.1). In particular, using Hölder's inequality, we see that

$$
\begin{aligned}
\|\phi\|_{0,Z}^2 &\leqslant \int_0^T ds \int_{(\mathbb{R}^d)^k} |\nu_{s,s}|(d\eta_1,\dots,d\eta_k) \sup_{\eta_1,\dots,\eta_k \in \mathbb{R}^d} \left(\int_{\mathbb{R}^d} \mu(d\xi)\, |\mathcal{F}\phi(s)(\xi+\delta)|^2 \right) \\
&\leqslant \int_0^T ds \int_{(\mathbb{R}^d)^k} |\nu_{s,s}|(d\eta_1,\dots,d\eta_k) \sup_{\eta \in \mathbb{R}^d} \int_{\mathbb{R}^d} \mu(d\xi)\, |\mathcal{F}\phi(s)(\xi+\eta)|^2 \\
&\leqslant \left(\sup_{0 \leqslant s \leqslant T} |\nu_{s,s}|((\mathbb{R}^d)^k) \right) \int_0^T ds \sup_{\eta \in \mathbb{R}^d} \int_{\mathbb{R}^d} \mu(d\xi)\, |\mathcal{F}\phi(s)(\xi+\eta)|^2 \qquad (6.15)
\end{aligned}
$$

and (6.13) is proved for a deterministic and smooth ϕ.

From now on, we use the same approximations for S as in the proof of Theorem 3.1. Using (6.14) instead of (3.1), the proofs are similar and use the same techniques depending on which of (H1) or (H2) is satisfied. Note that in the case where Z has spatially homogeneous covariance, the two constructions lead to the same stochastic integrals. Indeed, the definition of the integral for the approximated integrand $S_{n,m}$ is a Walsh stochastic integral and, hence, is the same in the two cases. Then, by uniqueness of the limit, the two integrals must be the same. ■

Proposition 6.11 gives conditions for existence of deterministic integrals involving a random integrand. These integrals are those appearing in the definition (6.8) of the integral operator, where the index belongs to $\mathbb{Z}^2$.

Proposition 6.11. *Let $(Z(t,x,y),\ 0 \leqslant t \leqslant T,\ x \in \mathbb{R}^d,\ y \in \mathbb{R}^d)$ be a predictable process such that there exists a measure $\nu_{s,t}$ defined on $(\mathbb{R}^d)^k$ for some $k \geqslant 1$, depending on time parameters s and t, such that*

$$
\sup_{s,t \in [0,T]} \int_{(\mathbb{R}^d)^k} |\nu_{s,t}|(d\eta_1,\dots,d\eta_k) < +\infty \qquad (6.16)
$$

and

$$\mathbb{E}[Z(s,x,y)Z(t,\tilde{x},\tilde{y})] = \int_{(\mathbb{R}^d)^k} \nu_{s,t}(d\eta_1,\dots,d\eta_k) e^{i\langle \delta, x\rangle} e^{i\langle \delta', y\rangle} e^{i\langle \tilde{\delta}, \tilde{x}\rangle} e^{i\langle \tilde{\delta}', \tilde{y}\rangle}, \tag{6.17}$$

where δ, δ', $\tilde{\delta}$, $\tilde{\delta}'$ *are linear combinations of* $\eta_1,\dots,\eta_k$. *In particular,*

$$\sup_{0\leqslant t\leqslant T} \sup_{x,y\in\mathbb{R}^d} \mathbb{E}[Z(t,x,y)^2] < \infty. \tag{6.18}$$

Let $t \mapsto S_i(t)$ $(i=1,2)$ *be deterministic functions with values in the space* $\mathcal{S}'_r(\mathbb{R}^d)$. *Suppose that* $(s,\xi) \mapsto \mathcal{F}S_i(s)(\xi)$ $(i=1,2)$ *are measurable and*

$$\sup_{0\leqslant s\leqslant T} \sup_{\eta\in\mathbb{R}^d} \int_{\mathbb{R}^d} \mu(d\xi)\, |\mathcal{F}S_i(s)(\xi+\eta)|^2 < \infty \tag{6.19}$$

for $i=1,2$. *Suppose in addition that hypothesis (H2) is satisfied for* S_1 *and* S_2. *Then, the integral process* $J : [0,T]\times[0,T]^2\times(\mathbb{R}^d)^2 \to \mathbb{R}$ *defined by*

$$\begin{aligned} & J^{(Z)}(S_1,S_2)(s;r_1,r_2;y_1,y_2) \\ &= \int_0^s dr \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^d} dy\, S_1(r_1-r,y_1-x) f(x-y) S_2(r_2-r,y_2-y) Z(r,x,y) \end{aligned} \tag{6.20}$$

for $s,r_1,r_2\in[0,T]$, $y_1,y_2\in\mathbb{R}^d$, *is well defined as a stochastic process with values in* $L^2(\Omega)$.

Proof. As in the proof of Theorem 3.1, we consider the approximating sequence $(S_{n_1,m_1})_{n_1,m_1\in\mathbb{N}}$ of S_1 and $(S_{n_2,m_2})_{n_2,m_2\in\mathbb{N}}$ of S_2. By the proof of Theorem 3.1, the functions S_{n_1,m_1} and S_{n_2,m_2} satisfy

$$\int_0^T dr \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^d} dy |S_{n_1,m_1}(r,x)| f(x-y) |S_{n_2,m_2}(r,y)| < +\infty, \tag{6.21}$$

for any $n_1, m_1, n_2, m_2 \in \mathbb{N}$. Hence,

$$
\begin{aligned}
&\mathbb{E}[|J^{(Z)}(S_{n_1,m_1}, S_{n_2,m_2})|^2] \qquad (6.22)\\
&= \mathbb{E}\Bigg[\Bigg(\int_0^s dr \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^d} dy\, S_{n_1,m_1}(r_1 - r, y_1 - x) f(x-y) S_{n_2,m_2}(r_2 - r, y_2 - y) \\
&\qquad \times Z(r,x,y)\Bigg)^2\Bigg] \\
&= \int_0^s dr \int_0^s d\tilde{r} \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^d} dy\, S_{n_1,m_1}(r_1 - r, y_1 - x) f(x-y) S_{n_2,m_2}(r_2 - r, y_2 - y) \\
&\qquad \times \int_{\mathbb{R}^d} d\tilde{x} \int_{\mathbb{R}^d} d\tilde{y}\, S_{n_1,m_1}(r_1 - \tilde{r}, y_1 - \tilde{x}) f(\tilde{x}-\tilde{y}) S_{n_2,m_2}(r_2 - \tilde{r}, y_2 - \tilde{y}) \\
&\qquad \times \mathbb{E}[Z(r,x,y) Z(\tilde{r},\tilde{x},\tilde{y})] \\
&= \int_0^s dr \int_0^s d\tilde{r} \int_{(\mathbb{R}^d)^k} \nu_{r,\tilde{r}}(d\eta_1, \dots, d\eta_k) \\
&\quad \times \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^d} dy\, S_{n_1,m_1}(r_1 - r, y_1 - x) f(x-y) S_{n_2,m_2}(r_2 - r, y_2 - y) e^{i\langle \delta, x\rangle} e^{i\langle \delta', y\rangle} \\
&\quad \times \int_{\mathbb{R}^d} d\tilde{x} \int_{\mathbb{R}^d} d\tilde{y}\, S_{n_1,m_1}(r_1 - \tilde{r}, y_1 - \tilde{x}) f(\tilde{x}-\tilde{y}) S_{n_2,m_2}(r_2 - \tilde{r}, y_2 - \tilde{y}) e^{i\langle \tilde{\delta}, \tilde{x}\rangle} e^{i\langle \tilde{\delta}', \tilde{y}\rangle} \\
&= \int_0^s dr \int_0^s d\tilde{r} \int_{(\mathbb{R}^d)^k} \nu_{r,\tilde{r}}(d\eta_1, \dots, d\eta_k) \\
&\quad \times \int_{\mathbb{R}^d} \mu(d\xi) \overline{\mathcal{F}S_{n_1,m_1}(r_1 - r)(\xi + \delta)} \mathcal{F}S_{n_2,m_2}(r_2 - r)(\xi - \delta') e^{i\langle \xi + \delta, y_1\rangle} e^{-i\langle \xi - \delta', y_2\rangle} \\
&\quad \times \int_{\mathbb{R}^d} \mu(d\tilde{\xi}) \overline{\mathcal{F}S_{n_1,m_1}(r_1 - \tilde{r})(\tilde{\xi} + \tilde{\delta})} \mathcal{F}S_{n_2,m_2}(r_2 - \tilde{r})(\tilde{\xi} - \tilde{\delta}') e^{i\langle \tilde{\xi} + \tilde{\delta}, y_1\rangle} e^{-i\langle \tilde{\xi} - \tilde{\delta}', y_2\rangle}.
\end{aligned}
$$

By (6.16) and Hölder's inequality,

$$
\begin{aligned}
\mathbb{E}[|J^{(Z)}(S_{n_1,m_1}, S_{n_2,m_2})|^2] &\leqslant \left(\sup_{s,t\in[0,T]} \int_{(\mathbb{R}^d)^k} |\nu_{s,t}|(d\eta_1, \dots, d\eta_k)\right) \\
&\quad \times \prod_{i=1}^2 \int_0^s dr \sup_{\eta \in \mathbb{R}^d} \int_{\mathbb{R}^d} \mu(d\xi) |\mathcal{F}S_{n_i,m_i}(r_i - r)(\xi + \eta)|^2 \\
&< +\infty \qquad (6.23)
\end{aligned}
$$

by (6.16) and (6.19). As a consequence, the process $J^{(Z)}(S_{n_1,m_1}, S_{n_2,m_2})$ is well-defined.

For all $n \in \mathbb{N}$ fixed,

$$
\int_0^s dr \sup_{\eta \in \mathbb{R}^d} \int_{\mathbb{R}^d} \mu(d\xi) |\mathcal{F}S_{n,m}(r_0 - r)(\xi + \eta) - \mathcal{F}S_n(r_0 - r)(\xi + \eta)|^2 \underset{m\to\infty}{\longrightarrow} 0 \qquad (6.24)
$$

by assumption (H2) (see proof of Theorem 3.1). Hence, as

$$J^{(Z)}(S_{n_1,m_1}, S_{n_2,m_2}) - J^{(Z)}(S_{n_1,m_1'}, S_{n_2,m_2}) = J^{(Z)}(S_{n_1,m_1} - S_{n_1,m_1'}, S_{n_2,m_2}),$$

and by (6.23) with S_{n_1,m_1} replaced by $S_{n_1,m_1} - S_{n_1,m_1'}$, the sequence

$$\left(J^{(Z)}(S_{n_1,m_1}, S_{n_2,m_2})\right)_{m_1\in\mathbb{N}}$$

is Cauchy in $L^2(\Omega)$. Let us set

$$J^{(Z)}(S_{n_1}, S_{n_2,m_2}) = \lim_{m_1\to\infty} J^{(Z)}(S_{n_1,m_1}, S_{n_2,m_2})$$

and, by the same argument, let us set

$$J^{(Z)}(S_{n_1}, S_{n_2}) = \lim_{m_2\to\infty} J^{(Z)}(S_{n_1}, S_{n_2,m_2}).$$

Taking limits as m_1, m_2 tend to ∞ in (6.22), we obtain

$$\begin{aligned}
&\mathbb{E}[|J^{(Z)}(S_{n_1}, S_{n_2})|^2] \\
&= \int_0^s dr \int_0^s d\tilde{r} \int_{(\mathbb{R}^d)^k} \nu_{r,\tilde{r}}(d\eta_1, \dots, d\eta_k) \\
&\qquad \times \int_{\mathbb{R}^d} \mu(d\xi)\overline{\mathcal{F}S_{n_1}(r_1 - r)(\xi + \delta)}\mathcal{F}S_{n_2}(r_2 - r)(\xi - \delta')e^{i\langle \xi+\delta, y_1\rangle} e^{-i\langle \xi - \delta', y_2\rangle} \\
&\qquad \times \int_{\mathbb{R}^d} \mu(d\tilde{\xi})\overline{\mathcal{F}S_{n_1}(r_1 - \tilde{r})(\tilde{\xi} + \tilde{\delta})}\mathcal{F}S_{n_2}(r_2 - \tilde{r})(\tilde{\xi} - \tilde{\delta}')e^{i\langle \tilde{\xi}+\tilde{\delta}, y_1\rangle} e^{-i\langle \tilde{\xi} - \tilde{\delta}', y_2\rangle}
\end{aligned} \tag{6.25}$$

and, by Hölder's inequality,

$$\begin{aligned}
\mathbb{E}[|J^{(Z)}(S_{n_1}, S_{n_2})|^2] &\leqslant \left(\sup_{s,t\in[0,T]} \int_{(\mathbb{R}^d)^k} |\nu_{s,t}|(d\eta_1, \dots, d\eta_k)\right) \\
&\quad \times \prod_{i=1}^2 \int_0^s dr \sup_{\eta\in\mathbb{R}^d} \int_{\mathbb{R}^d} \mu(d\xi)|\mathcal{F}S_{n_i}(r_i - r)(\xi + \eta)|^2 \\
&< +\infty
\end{aligned} \tag{6.26}$$

by (6.16) and (6.19). Again by the proof of Theorem 3.1,

$$\int_0^s dr \sup_{\eta\in\mathbb{R}^d} \int_{\mathbb{R}^d} \mu(d\xi)|\mathcal{F}S_n(r_0 - r)(\xi + \eta) - \mathcal{F}S(r_0 - r)(\xi + \eta)|^2 \underset{n\to\infty}{\longrightarrow} 0. \tag{6.27}$$

Hence, by (6.26) with S_{n_1} replaced by $S_{n_1} - S_{n_1'}$, $\left(J^{(Z)}(S_{n_1}, S_{n_2})\right)_{n_1\in\mathbb{N}}$ is Cauchy in $L^2(\Omega)$. Let us set

$$J^{(Z)}(S_1, S_{n_2}) = \lim_{n_1\to\infty} J^{(Z)}(S_{n_1}, S_{n_2})$$

and, by the same argument, let us set

$$J^{(Z)}(S_1, S_2) = \lim_{n_2\to\infty} J^{(Z)}(S_1, S_{n_2}).$$

Taking limits as n_1, n_2 tend to ∞ in (6.25), we obtain

$$\begin{aligned}
&\mathbb{E}[|J^{(Z)}(S_1, S_2)|^2] \\
&= \int_0^s dr \int_0^s d\tilde{r} \int_{(\mathbb{R}^d)^k} \nu_{r,\tilde{r}}(d\eta_1, \dots, d\eta_k) \\
&\qquad \times \int_{\mathbb{R}^d} \mu(d\xi) \overline{\mathcal{F}S_1(r_1 - r)(\xi + \delta)} \mathcal{F}S_2(r_2 - r)(\xi - \delta') e^{i\langle \xi+\delta, y_1\rangle} e^{-i\langle \xi - \delta', y_2\rangle} \\
&\qquad \times \int_{\mathbb{R}^d} \mu(d\tilde{\xi}) \overline{\mathcal{F}S_1(r_1 - \tilde{r})(\tilde{\xi} + \tilde{\delta})} \mathcal{F}S_2(r_2 - \tilde{r})(\tilde{\xi} - \tilde{\delta}') e^{i\langle \tilde{\xi}+\tilde{\delta}, \tilde{y}_1\rangle} e^{-i\langle \tilde{\xi} - \tilde{\delta}', \tilde{y}_2\rangle}
\end{aligned} \tag{6.28}$$

and, by Hölder's inequality,

$$\begin{aligned}
\mathbb{E}[|J^{(Z)}(S_1, S_2)|^2] &\leqslant \left(\sup_{s,t\in[0,T]} \int_{(\mathbb{R}^d)^k} |\nu_{s,t}|(d\eta_1, \dots, d\eta_k) \right) \\
&\quad \times \prod_{i=1}^{2} \int_0^s dr \sup_{\eta\in\mathbb{R}^d} \int_{\mathbb{R}^d} \mu(d\xi) |\mathcal{F}S_i(r_i - r)(\xi + \eta)|^2 \\
&< +\infty
\end{aligned} \tag{6.29}$$

by (6.16) and (6.19). Hence, the process $J^{(Z)}(S_1, S_2)$ is well-defined and satisfies (6.28). ■

The following corollary of Proposition 6.11 will be used in Section 6.4.

Corollary 6.12. *In the case where the process Z is such that $\mathbb{E}[Z(t, x, y)] = 0$ for all $t \in [0, T]$ and all $x, y \in \mathbb{R}^d$, then $\mathbb{E}[J^{(Z)}(S_1, S_2)(s; r_1, r_2; y_1, y_2)] = 0$ for all $s, r_1, r_2 \in [0, T]$ and all $y_1, y_2 \in \mathbb{R}^d$.*

Proof. By (6.19) and (6.21), we can permute the integrals and the expectation in $J^{(Z)}_{S_{n_1,m_1}, S_{n_2,m_2}}(s; r_1, r_2; y_1, y_2)$ and

$$\mathbb{E}[J^{(Z)}_{S_{n_1,m_1}, S_{n_2,m_2}}(s; r_1, r_2; y_1, y_2)] = 0$$

for all $n_1, m_1, n_2, m_2 \in \mathbb{N}$, all $s, r_1, r_2 \in [0, T]$ and all $y_1, y_2 \in \mathbb{R}^d$. By Proposition 6.11, $J^{(Z)}_{S_{n_1,m_1}, S_{n_2,m_2}}(s; r_1, r_2; y_1, y_2)$ converges in $L^2(\Omega)$ to $J^{(Z)}_{S_1,S_2}(s; r_1, r_2; y_1, y_2)$ as $m_1.m_2, n_1, n_2$ successively tend to $+\infty$. As a consequence,

$$\mathbb{E}[J^{(Z)}_{S_1,S_2}(s; r_1, r_2; y_1, y_2)] = 0$$

for all $s, r_1, r_2 \in [0,T]$ and all $y_1, y_2 \in \mathbb{R}^d$. ■

As a consequence of Propositions 6.8 and 6.11, we can show that the iterated integrals $I_\beta^{(k)}$ are well-defined.

Proposition 6.13. *The iterated stochastic integrals defined by (6.9) for all $k \in \mathbb{N}$ and all $\beta \in \mathcal{A}_{(k)}$ are well-defined. Moreover, for any $s, \tilde{s} \in [0,T]$, any $\boldsymbol{r}_k, \tilde{\boldsymbol{r}}_k \in [0,T]^k$ and any $\boldsymbol{y}_k, \tilde{\boldsymbol{y}}_k \in (\mathbb{R}^d)^k$,*

$$\begin{aligned} &\mathbb{E}[I_\beta^{(k)}(s; \boldsymbol{r}_k; \boldsymbol{y}_k) I_\beta^{(k)}(\tilde{s}; \tilde{\boldsymbol{r}}_k; \tilde{\boldsymbol{y}}_k)] \\ &\quad = \prod_{i=1}^{\ell(\beta)} \int_0^T \int_0^T \tau_i(ds_i, d\tilde{s}_i) \prod_{j=1}^{\ell(\beta)} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \chi_j(d\xi_j, d\tilde{\xi}_j) \prod_{\ell=1}^{k} e^{i\langle \delta_\ell, y_\ell \rangle} e^{i\langle \tilde{\delta}_\ell, \tilde{y}_\ell \rangle}, \end{aligned} \tag{6.30}$$

where we set $s_0 = s$, $\tilde{s}_0 = \tilde{s}$, and for $i = 1, \ldots, \ell(\beta)$

$$\tau_i(ds_i, d\tilde{s}_i) = \begin{cases} \delta_0(s_i - \tilde{s}_i) \mathbf{1}_{\{s_i < s_{i-1} \wedge \tilde{s}_{i-1}\}} ds_i & \text{if } \beta_i \in \mathbb{Z}, \\ \mathbf{1}_{\{s_i < s_{i-1}\}} \mathbf{1}_{\{\tilde{s}_i < \tilde{s}_{i-1}\}} ds_i d\tilde{s}_i & \text{if } \beta_i \in \mathbb{Z}^2, \end{cases}$$

and, for $j = 1, \ldots, \ell(\beta)$,

$$\begin{aligned} &\chi_j(d\xi_j, d\tilde{\xi}_j) \\ &= \begin{cases} \delta_0(\xi_j - \tilde{\xi}_j) \overline{\mathcal{F}\Gamma(\sigma_j - s_j)(\xi_j + \eta_j)} \mathcal{F}\Gamma(\sigma_j' - s_j)(\xi_j + \eta_j') \mu(d\xi_j) & \text{if } \beta_j \in \mathbb{Z}, \\ \\ \overline{\mathcal{F}\Gamma(\sigma_j - s_j)(\xi_j + \eta_j)} \mathcal{F}\Gamma(\sigma_j' - s_j)(\xi_j + \eta_j') \mu(d\xi_j) & \\ \times \overline{\mathcal{F}\Gamma(\tilde{\sigma}_j - \tilde{s}_j)(\tilde{\xi}_j + \tilde{\eta}_j)} \mathcal{F}\Gamma(\tilde{\sigma}_j' - \tilde{s}_j)(\tilde{\xi}_j + \tilde{\eta}_j') \mu(d\tilde{\xi}_j) & \text{if } \beta_j \in \mathbb{Z}^2, \end{cases} \end{aligned}$$

where

- *$\sigma_j, \sigma_j', \tilde{\sigma}_j, \tilde{\sigma}_j'$ are linear combinations of $s, \tilde{s}, r_0, \tilde{r}_0, \ldots, r_k, \tilde{r}_k$, $s_1, \tilde{s}_1, \ldots, s_{j-1}, \tilde{s}_{j-1}$ ($j = 1, \ldots, \ell(\beta)$);*

- *$\eta_j, \eta_j', \tilde{\eta}_j, \tilde{\eta}_j'$ are linear combinations of $\xi_1, \tilde{\xi}_1, \ldots, \xi_{j-1}, \tilde{\xi}_{j-1}$ ($j = 1, \ldots, \ell(\beta)$);*

- *$\delta_\ell, \tilde{\delta}_\ell$ are linear combinations of $\xi_1, \tilde{\xi}_1, \ldots, \xi_{\ell(\beta)}, \tilde{\xi}_{\ell(\beta)}$ ($\ell = 1, \ldots, k$).*

In particular $I_\beta^{(k)}$ has finite second order moments for all $k \in \mathbb{N}$ and $\beta \in \mathcal{A}_{(k)}$.

Proof. Let $k \in \mathbb{N}$. First consider $\beta \in \mathcal{A}_{(k)}$ such that $\ell(\beta) = 1$, i.e. $\beta = (\beta_1)$. In this case, we have $-\beta=\emptyset$ and $\tilde{I}^{(k+|\beta_1|)}_{-\beta}\equiv 1$. Hence

$$\begin{aligned}
&I^{(k)}_{\beta}(s;\boldsymbol{r}_k;\boldsymbol{y}_k)\\
&\quad= I^{(k)}_{\beta_1}(1)(s;\boldsymbol{r}_k;\boldsymbol{y}_k)\\
&\quad= \begin{cases} \displaystyle\int_0^s\int_{\mathbb{R}^d}\Gamma(r_j-r,y_j-y)M(dr,dy) & \text{if } \beta_1=j,\\ \displaystyle\int_0^s dr\int_{\mathbb{R}^d}\mu(d\xi)\,\overline{\mathcal{F}\Gamma(r_j-r)(\xi)}\mathcal{F}\Gamma(r_{j'}-r)(\xi)e^{i\langle\xi,y_j-y_{j'}\rangle} & \text{if } \beta_1=\binom{j}{j'},\end{cases}
\end{aligned}$$

where $j,j'\leqslant k$. These processes are well-defined according to (6.5) for all $k\in\mathbb{N}$. Moreover, if $\beta_1=j$,

$$\begin{aligned}
&\mathbb{E}[I^{(k)}_{\beta}(s;\boldsymbol{r}_k;\boldsymbol{y}_k)I^{(k)}_{\beta}(\tilde{s};\tilde{\boldsymbol{r}}_k;\tilde{\boldsymbol{y}}_k)]\\
&\quad=\int_0^{s\wedge\tilde{s}}dr\int_{\mathbb{R}^d}\mu(d\xi)\overline{\mathcal{F}\Gamma(r_j-r)(\xi)}\mathcal{F}\Gamma(\tilde{r}_j-r)(\xi)e^{i\langle\xi,y_j-\tilde{y}_j\rangle}
\end{aligned}$$

and (6.30) is satisfied with $\sigma_1=r_j,\sigma_1'=\tilde{r}_j,\eta_1=\eta_1'=0$, $\delta_j=-\tilde{\delta}_j=\xi_1$, and $\delta_\ell=\tilde{\delta}_\ell=0$ for $\ell\neq j$. Then, if $\beta=\binom{j}{j'}$, the process is deterministic and

$$\begin{aligned}
&\mathbb{E}[I^{(k)}_{\beta}(s;\boldsymbol{r}_k;\boldsymbol{y}_k)I^{(k)}_{\beta}(\tilde{s};\tilde{\boldsymbol{r}}_k;\tilde{\boldsymbol{y}}_k)]\\
&\quad=\int_0^s dr\int_0^{\tilde{s}}d\tilde{r}\int_{\mathbb{R}^d}\mu(d\xi)\,\overline{\mathcal{F}\Gamma(r_j-r)(\xi)}\mathcal{F}\Gamma(r_{j'}-r)(\xi)\\
&\qquad\times\int_{\mathbb{R}^d}\mu(d\tilde{\xi})\,\overline{\mathcal{F}\Gamma(\tilde{r}_j-\tilde{r})(\tilde{\xi})}\mathcal{F}\Gamma(\tilde{r}_{j'}-\tilde{r})(\tilde{\xi})e^{i\langle\xi,y_j-y_{j'}\rangle}e^{i\langle\tilde{\xi},\tilde{y}_j-\tilde{y}_{j'}\rangle}.
\end{aligned}$$

Hence, (6.30) is satisfied with $\sigma_1=r_j,\sigma_1'=r_{j'},\tilde{\sigma}_1=\tilde{r}_j,\tilde{\sigma}_1'=\tilde{r}_{j'},\eta_1=\eta_1'=\tilde{\eta}_1=\tilde{\eta}_1'=0$, $\delta_j=-\delta_{j'}=\xi$, $\tilde{\delta}_j=-\tilde{\delta}_{j'}=\tilde{\xi}$ and $\delta_\ell=\tilde{\delta}_\ell=0$ for $\ell\neq j,j'$.

Notice that we consider the process $I^{(k)}_\beta$ to depend on $k+1$ time variables and $k+1$ spatial variables for technical reasons, but it often happens that the process only depends on few of those variables (see Remark 6.4).

Now we are going to show that the iterated integrals are well-defined by induction on $\ell(\beta)$. Suppose those integrals are well-defined for all multi-indices β with $\ell(\beta)\leqslant n$. The result is true for $n=1$. Consider $k\in\mathbb{N}$ and $\beta\in\mathcal{A}_{(k)}$ such that $\ell(\beta)=n+1$. We want to show that $I^{(k)}_\beta$ is well-defined and satisfies (6.30).

First consider the case where $\beta_1=j\in\mathbb{Z}$. For $s\in[0,T]$, $\boldsymbol{r}_k\in[0,T]^k$ and $\boldsymbol{y}_k\in(\mathbb{R}^d)^k$,

$$I^{(k)}_\beta(s;\boldsymbol{r}_k;\boldsymbol{y}_k)=I^{(k)}_j(\tilde{I}^{(k+1)}_{-\beta})(s;\boldsymbol{r}_k;\boldsymbol{y}_k).$$

As $\ell(-\beta)=n$, by the induction assumption, $\tilde{I}^{(k+1)}_{-\beta}$ satisifies (6.30). Introducing absolute values, we can bound the exponentials by 1, the spatial integrals using

Hölder's inequality and (6.5) and the time integrals by $T^{\|-\beta\|} = T^{\|\beta\|-1}$. Hence, $\tilde{I}_{-\beta}^{(k+1)}$ has a covariance function given by (6.11) with measure

$$\nu_{s,\tilde{s}}(d\xi_1, \dots, d\xi_n, d\tilde{\xi}_1, \dots, d\tilde{\xi}_n) = \prod_{i=1}^{n} \int_0^T \int_0^T \tau_i(ds_i, d\tilde{s}_i) \prod_{j=1}^{n} \chi_j(d\xi_j, d\tilde{\xi}_j) \qquad (6.31)$$

where the notations are defined in (6.30). Hence, it satisfies the assumptions of Proposition 6.8 and the stochastic integral $I_\beta^{(k)}$ is well-defined. Now, for $s \in [0,T]$, $\boldsymbol{r}_k \in [0,T]^k$, $\boldsymbol{y}_k \in (\mathbb{R}^d)^k$, $\tilde{s} \in [0,T]$, $\tilde{\boldsymbol{r}}_k \in [0,T]^k$ and $\tilde{\boldsymbol{y}}_k \in \mathbb{R}^d$, a slight modification of (6.13) leads to

$$\begin{aligned}
&\mathbb{E}[I_\beta^{(k)}(s; \boldsymbol{r}_k; \boldsymbol{y}_k) I_\beta^{(k)}(\tilde{s}; \tilde{\boldsymbol{r}}_k; \tilde{\boldsymbol{y}}_k)] \\
&\quad = \int_0^{s\wedge\tilde{s}} dr \int_{(\mathbb{R}^d)^n} \nu_{r,r}(d\xi_1, \dots, d\xi_n, d\tilde{\xi}_1, \dots, d\tilde{\xi}_n) \\
&\qquad\qquad \times \int_{\mathbb{R}^d} \mu(d\xi) \mathcal{F}(\Gamma(r_j - r, y_j - \cdot))(\xi + \delta_{k+1}) \overline{\mathcal{F}(\Gamma(\tilde{r}_j - r, \tilde{y}_j - \cdot)(\xi - \tilde{\delta}_{k+1}))} \\
&\quad = \int_0^{s\wedge\tilde{s}} dr \int_{(\mathbb{R}^d)^n} \nu_{r,r}(d\xi_1, \dots, d\xi_n, d\tilde{\xi}_1, \dots, d\tilde{\xi}_n) \\
&\qquad\qquad \times \int_{\mathbb{R}^d} \mu(d\xi) \overline{\mathcal{F}\Gamma(r_j - r)(\xi + \delta_{k+1})} \mathcal{F}\Gamma(\tilde{r}_j - r)(\xi - \tilde{\delta}_{k+1}) \\
&\qquad\qquad\qquad\qquad \times\ e^{i\langle \xi + \delta_{k+1}, y_j\rangle} e^{-i\langle \xi - \tilde{\delta}_{k+1}, \tilde{y}_j\rangle}.
\end{aligned}$$

Replacing the measure $\nu_{r,r}$ by (6.31), this shows that (6.30) is satisfied.

Now consider the case where $\beta_1 = \binom{j}{j'} \in \mathbb{Z}^2$. For $s \in [0,T], \boldsymbol{r}_k \in [0,T]^k$ and $\boldsymbol{y}_k \in (\mathbb{R}^d)^k$,

$$I_\beta^{(k)}(s; \boldsymbol{r}_k; \boldsymbol{y}_k) = I_{\binom{j}{j'}}^{(k)}(\tilde{I}_{-\beta}^{(k+2)})(s; \boldsymbol{r}_k; \boldsymbol{y}_k).$$

As $\ell(-\beta) = n$, by the induction assumption, $\tilde{I}_{-\beta}^{(k+2)}$ satisifies (6.30). Hence, it has finite second order moments. Then, if we set

$$Z_{\boldsymbol{r}_k;\boldsymbol{y}_k}(r; x, y) = \tilde{I}_{-\beta}^{(k+2)}(\boldsymbol{r}_k, r, r; \boldsymbol{y}_k, x, y)$$

for any fixed parameters $\boldsymbol{r}_k \in [0,T]$, $\boldsymbol{y}_k \in \mathbb{R}^d$, $Z_{\boldsymbol{r}_k;\boldsymbol{y}_k}$ satisfies the assumptions of Proposition 6.11. Hence, the process

$$I_\beta^{(k)}(s; \boldsymbol{r}_k; \boldsymbol{y}_k) = J^{(Z_{\boldsymbol{r}_k;\boldsymbol{y}_k})}(\Gamma, \Gamma)(s; r_j, r_{j'}; y_j, y_{j'})$$

taking values in $L^2(\Omega)$ is well-defined according to Proposition 6.11. Moreover, the process $Z_{\boldsymbol{r}_k;\boldsymbol{y}_k}(r; x, y) = \tilde{I}_{-\beta}^{(k+2)}(\boldsymbol{r}_k, r, r; \boldsymbol{y}_k, x, y)$ satisifies (6.30) and hence (6.17) with

$$\begin{aligned}&\nu_{s,\tilde{s}}(d\xi_1,\dots,d\xi_n,d\tilde{\xi}_1,\dots,d\tilde{\xi}_n)\\&\quad=\prod_{i=1}^{n}\int_0^T\int_0^T\tau_i(ds_i,d\tilde{s}_i)\prod_{j=1}^{n}\chi_j(d\xi_j,d\tilde{\xi}_j)\prod_{\ell=1}^{k}e^{i\langle\delta_\ell,y_\ell\rangle}e^{i\langle\tilde{\delta}_\ell,\tilde{y}_\ell\rangle}.\end{aligned}\tag{6.32}$$

Then, slight modifications of (6.22), (6.25) and (6.28) lead to

$$\begin{aligned}&\mathbb{E}[I_\beta^{(k)}(s;\boldsymbol{r}_k;\boldsymbol{y}_k)I_\beta^{(k)}(\tilde{s};\tilde{\boldsymbol{r}}_k;\tilde{\boldsymbol{y}}_k)]\\&\quad=\mathbb{E}[J^{(Z_{\boldsymbol{r}_k;\boldsymbol{y}_k})}(\Gamma,\Gamma)(s;r_j,r_{j'};y_j,y_{j'})J^{(Z_{\tilde{\boldsymbol{r}}_k;\tilde{\boldsymbol{y}}_k})}(\Gamma,\Gamma)(\tilde{s};\tilde{r}_j,\tilde{r}_{j'};\tilde{y}_j,\tilde{y}_{j'})]\\&\quad=\int_0^s dr\int_0^{\tilde{s}}d\tilde{r}\int_{(\mathbb{R}^d)^k}\nu_{r,\tilde{r}}(d\xi_1,\dots,d\xi_n,d\tilde{\xi}_1,\dots,d\tilde{\xi}_n)\\&\qquad\times\int_{\mathbb{R}^d}\mu(d\xi)\overline{\mathcal{F}\Gamma(r_j-r)(\xi+\delta_{k+1})}\mathcal{F}\Gamma(r_{j'}-r)(\xi-\delta_{k+2})e^{i\langle\xi+\delta_{k+1},y_j\rangle}e^{i\langle\xi-\delta_{k+2},y_{j'}\rangle}\\&\qquad\times\int_{\mathbb{R}^d}\mu(d\tilde{\xi})\overline{\mathcal{F}\Gamma(\tilde{r}_j-\tilde{r})(\tilde{\xi}+\tilde{\delta}_{k+1})}\mathcal{F}\Gamma(\tilde{r}_{j'}-\tilde{r})(\tilde{\xi}-\tilde{\delta}_{k+2})e^{i\langle\tilde{\xi}+\tilde{\delta}_{k+1},\tilde{y}_j\rangle}e^{i\langle\tilde{\xi}-\tilde{\delta}_{k+2},\tilde{y}_{j'}\rangle},\end{aligned}\tag{6.33}$$

Replacing the measure $\nu_{r,\tilde{r}}$ by (6.32) in (6.33) shows that $I_\beta^{(k)}$ satisfies (6.30) in the case where $\beta_1\in\mathbb{Z}^2$. The result is proved. ■

As a direct consequence of Proposition 6.13, we have the following Corollary.

Corollary 6.14. *The iterated stochastic integrals $I_\beta^{(0)}$, $\beta\in\bar{\mathcal{A}}^*$, are well-defined as processes taking values in $L^2(\Omega)$.*

The processes $I_\beta^{(0)}$, $\beta\in\bar{\mathcal{A}}^*$, are the processes that will appear in the Itô-Taylor expansion in Chapter 7.

Remark 6.15. The processes $I_\beta^{(0)}$, $\beta\in\mathcal{A}^*$, have been defined recursively on $\ell(\beta)$ according to

$$I_\beta^{(0)}(s;r_0,y_0)=I_{\beta_1}^{(0)}\left(\tilde{I}_{-\beta}^{(|\beta_1|)}\right)(s;r_0,y_0),\tag{6.34}$$

for all $r_0\in[0,T]$ and $y_0\in\mathbb{R}^d$ (see (6.9)). Notice that another way of defining those processes is to define operators J_β recursively on $\ell(\beta)$. The operator J_β, $\beta\in\mathcal{A}^*$ is defined for $g:[0,T]^{\|\beta\|+1}\times(\mathbb{R}^d)^{\|\beta\|+1}$ by

$$J_\beta(g)(s;r_0,y_0)=I_\beta^{(0)}(g)(s;r_0,y_0)\tag{6.35}$$

if $\ell(\beta)=1$ and

$$J_\beta(g)(s;r_0,y_0)=J_{\beta-}(\tilde{I}_{\beta_{\ell(\beta)}}^{(\|\beta\|-|\beta_{\ell(\beta)}|)}(g))(s;r_0,y_0),\tag{6.36}$$

if $\ell(\beta) > 1$, where $\tilde{I}^{(k)}_{\beta_{\ell(\beta)}}(\boldsymbol{r}_k;\boldsymbol{y}_k) = I^{(k)}_{\beta_{\ell(\beta)}}(r_k;\boldsymbol{r}_k;\boldsymbol{y}_k)$ and $I^{(k)}_{\beta_{\ell(\beta)}}$ is one of the two operators defined in (6.6) or (6.8). Then, setting $I^{(0)}_\beta(s;r_0,y_0) = J_\beta(1)(s;r_0,y_0)$, it is clear that we obtain the same iterated stochastic integrals as those defined in (6.9). As an illustration, we have the following example.

Example 6.16. As in Example 6.5, let us consider $\beta = \left(0,1,\binom{1}{2}\right)$, $\ell(\beta) = 3$, $\|\beta\| = 4$. On the one hand, taking (6.9), we have

$$I^{(0)}_{0,1,\binom{1}{2}}(s;r_0,y_0) = I^{(0)}_0\left(\tilde{I}^{(1)}_{1,\binom{1}{2}}\right)(s;r_0,y_0)$$

where

$$\tilde{I}^{(1)}_{1,\binom{1}{2}}(r_0,r_1;y_0,y_1) = I^{(1)}_{1,\binom{1}{2}}(r_1;r_0,r_1;y_0,y_1) = I^{(1)}_1\left(\tilde{I}^{(2)}_{\binom{1}{2}}\right)(r_1;r_0,r_1;y_0,y_1)$$

and

$$\tilde{I}^{(2)}_{\binom{1}{2}}(\boldsymbol{r}_2;\boldsymbol{y}_2) = I^{(2)}_{\binom{1}{2}}(r_2;\boldsymbol{r}_2,\boldsymbol{y}_2) = I^{(2)}_{\binom{1}{2}}\left(\tilde{I}^{(4)}_\emptyset\right)(r_2;\boldsymbol{r}_2,\boldsymbol{y}_2) = I^{(2)}_{\binom{1}{2}}(1)\,(r_2;\boldsymbol{r}_2,\boldsymbol{y}_2).$$

On the other hand, taking the definition of Remark 6.15, we have

$$\begin{aligned} I^{(0)}_{0,1,\binom{1}{2}}(s;r_0,y_0) &= J_{0,1,\binom{1}{2}}(1)(s;r_0,y_0) = J_{0,1}\left(\tilde{I}^{(2)}_{\binom{1}{2}}(1)\right)(s;r_0,y_0) \\ &= J_0\left(\tilde{I}^{(1)}_1\left(\tilde{I}^{(2)}_{\binom{1}{2}}(1)\right)\right)(s;r_0,y_0) \\ &= I^{(0)}_0\left(\tilde{I}^{(1)}_1\left(\tilde{I}^{(2)}_{\binom{1}{2}}(1)\right)\right)(s;r_0,y_0). \end{aligned}$$

We see that the two definitions lead to the same integrals. This recursive procedure can be used for any $\beta \in \mathcal{A}^*$. Hence, the two definitions are equivalent.

The next proposition gives conditions under which it is possible to permute a limit and a stochastic integral with a limit. It will be used in Chapter 7.

Proposition 6.17. *Let $(Z_n(t,x))_{n\in\mathbb{N}}$ be a sequence of processes such that*

$$\sup_{0\leqslant t\leqslant T}\sup_{x\in\mathbb{R}^d}\mathbb{E}[Z_n(t,x)^2] < +\infty,$$

for all $n \in \mathbb{N}$ and such that $S \cdot M^{Z_n}$ is well-defined for all $n \in \mathbb{N}$ by [30], [2], Theorem 3.1 or Proposition 6.8. Suppose in addition that $Z_n(t,x)$ converges uniformly over $(t,x) \in [0,T]\times\mathbb{R}^d$ in $L^2(\Omega)$ to a process $Z(t,x)$ such that $S \cdot M^Z$ is well-defined. If $Z_{n_1} - Z_{n_2}$ satisfies hypothesis (H) for all $n_1,n_2 \in \mathbb{N}$, then the sequence $\left((S\cdot M^{Z_n})_t\right)_{n\in\mathbb{N}}$ converges in $L^2(\Omega)$ to $(S\cdot M^Z)_t$.

Proof. Let $n \in \mathbb{N}$. By the linearity of the stochastic integral, $(S \cdot M^{Z_n})_t - (S \cdot M^Z)_t = (S \cdot M^{Z_n - Z})_t$. Hence, as $Z_{n_1} - Z_{n_2}$ has a spatially homogeneous covariance function for all $n_1, n_2 \in \mathbb{N}$, the same is true for $Z_n - Z$. Therefore, Theorem 3.1 applies and

$$\begin{aligned}&\mathbb{E}[(S \cdot M^{Z_n - Z})_t^2]\\ &\quad\leqslant \sup_{0\leqslant t\leqslant T} \sup_{x\in\mathbb{R}^d} \mathbb{E}[(Z_n(t,x) - Z(t,x))^2] \int_0^t ds \sup_{\eta\in\mathbb{R}^d} \int_{\mathbb{R}^d} \mu(d\xi) |\mathcal{F}S(s)(\xi+\eta)|^2.\end{aligned}$$

By (4.4) and the fact that $(Z_n(t,x))_{n\in\mathbb{N}}$ converges uniformly in $L^2(\Omega)$ to $Z(t,x)$, we see that $(S \cdot M^{Z_n})_t$ converges in $L^2(\Omega)$ to $(S \cdot M^Z)_t$. ■

Remark 6.18. If we replace $S(s,y)$ by $\Gamma(r_0 - s, y_0 - y)$ for $r_0 \in [0,T]$ and $y_0 \in \mathbb{R}^d$ fixed, then $(S \cdot M^Z)_t = I_0^{(0)}(Z)(t; r_0, y_0)$. This will be used in Chapter 7.

6.3 Products of iterated stochastic integrals

Now that the iterated stochastic integrals are well-defined as processes taking values in $L^2(\Omega)$, we would like to study products of these objects. Understanding the behavior of the product of iterated stochastic integrals is needed in order to discuss convergence of the Itô-Taylor expansion in Chapter 7. We are going to establish some formula (Corollary 6.29) to express a product of iterated integrals as a sum of iterated integrals of different orders. This formula will be used in Chapter 7, when discussing the convergence of the truncated Itô-Taylor expansion to the solution $u(t,x)$.

First of all, we recall the integration by parts formula for stochastic integrals. Let $g, h : \mathbb{R}_+^{k+2} \times (\mathbb{R}^d)^{k+2} \to \mathbb{R}$ be random functions and, for $i, j, k \in \mathbb{N}$, $i, j \leqslant k$, let $I_i^{(k)}$ and $I_j^{(k)}$ denote the integral operators defined in (6.6). As soon as $I_i^{(k)}(g)$ and $I_j^{(k)}(h)$ are well-defined (cf. Proposition 6.8), for $\boldsymbol{r}_k \in [0,T]^{k+1}$ and $\boldsymbol{y}_k \in (\mathbb{R}^d)^{k+1}$ fixed, the processes $s \mapsto I_i^{(k)}(g)(s; \boldsymbol{r}_k; \boldsymbol{y}_k)$, $s \mapsto I_j^{(k)}(h)(s; \boldsymbol{r}_k; \boldsymbol{y}_k)$ are continuous martingales. Hence, the integration by parts formula ([24, Chapter IV, Theorem 3.1])

$$X_s Y_s = \int_0^s Y_r \, dX_r + \int_0^s X_r \, dY_r + \langle X, Y \rangle_s$$

with

$$X_s = I_i^{(k)}(g)(s; \boldsymbol{r}_k; \boldsymbol{y}_k) \text{ and } Y_s = I_j^{(k)}(h)(s; \boldsymbol{r}_k; \boldsymbol{y}_k)$$

gives

$$\begin{aligned}
&\left(I_i^{(k)}(g)\cdot I_j^{(k)}(h)\right)(s;\boldsymbol{r}_k;\boldsymbol{y}_k)\\
&= \int_0^s\int_{\mathbb{R}^d} I_i^{(k)}(g)(u;\boldsymbol{r}_k;\boldsymbol{y}_k)\Gamma(r_j-u,y_j-y)h(\boldsymbol{r}_k,u;\boldsymbol{y}_k,y)M(du,dy)\\
&\quad+\int_0^s\int_{\mathbb{R}^d} I_j^{(k)}(h)(u;\boldsymbol{r}_k;\boldsymbol{y}_k)\Gamma(r_i-u,y_i-y)g(\boldsymbol{r}_k,u;\boldsymbol{y}_k,y)M(du,dy)\\
&\quad+\int_0^s du\int_{\mathbb{R}^d}dy\int_{\mathbb{R}^d}dz\,\Gamma(r_i-u,y_i-y)f(y-z)\Gamma(r_j-u,y_j-z)\\
&\qquad\qquad\times g(\boldsymbol{r}_k,u;\boldsymbol{y}_k,y)h(\boldsymbol{r}_k,u;\boldsymbol{y}_k,z).
\end{aligned}\tag{6.37}$$

If we consider a product involving one or two operators with index in $\mathbb{Z}^2$ (defined in (6.8)), an integration by parts formula of the type (6.37) holds without the last integral on the right-hand side. Indeed, the process $s\mapsto I^{(k)}_{\binom{j}{j'}}(g)(s;\boldsymbol{r}_k;\boldsymbol{y}_k)$ is of finite variation and the quadratic variation term vanishes.

We would like to use (6.37) to obtain an integration by parts formula for the product of two iterated stochastic integrals of the type $I_\beta^{(k)}$, $k\in\mathbb{N}$, $\beta\in\mathcal{A}_{(k)}$. First of all, we have the trivial formulas

$$\left(I_\emptyset^{(k)}\cdot I_\emptyset^{(k)}\right)(s;\boldsymbol{r}_k;\boldsymbol{y}_k)=1$$

and

$$\left(I_\emptyset^{(k)}\cdot I_\beta^{(k)}\right)(s;\boldsymbol{r}_k;\boldsymbol{y}_k)=I_\beta^{(k)}(s;\boldsymbol{r}_k;\boldsymbol{y}_k),$$

for all $\beta\in\mathcal{A}^{(k)}$, $s\in[0,T]$, $\boldsymbol{r}_k\in[0,T]^{k+1}$ and $\boldsymbol{y}_k\in(\mathbb{R}^d)^{k+1}$. In order to state and prove the integration by parts formula (Proposition 6.24 below), we need the following technical lemmas.

For $k\in\mathbb{N}$, let us define the operator $A_k:\bar{\mathcal{A}}_{(k)}\to\bar{\mathcal{A}}_{(k+1)}$ by

$$A_k(\beta)_j=\begin{cases}\beta_j+\mathbf{1}_{\{\beta_j>k\}} & \text{if } \beta_j\in\mathbb{Z},\\[4pt] \begin{pmatrix}\beta_j+\mathbf{1}_{\{\beta_j>k\}}\\ \beta'_j+\mathbf{1}_{\{\beta'_j>k\}}\end{pmatrix} & \text{if } \beta_j\in\mathbb{Z}^2.\end{cases}$$

Similarly and because all values of a multi-index in $\bar{\mathcal{A}}^*$ are non-negative, let us define the operator $A_{-1}:\bar{\mathcal{A}}^*\to\bar{\mathcal{A}}^*$ by

$$A_{-1}(\beta)_j=\begin{cases}\beta_j+1 & \text{if } \beta_j\in\mathbb{Z},\\[4pt] \begin{pmatrix}\beta_j+1\\ \beta'_j+1\end{pmatrix} & \text{if } \beta_j\in\mathbb{Z}^2.\end{cases}$$

Moreover, for $k \in \mathbb{N}$, we also define the operator $T_k : \bar{\mathcal{A}}_{(k)} \to \bar{\mathcal{A}}_{(k+1)}$ by

$$T_k(\beta)_j = \begin{cases} \beta_j & \text{if } \beta_j \notin \{k, k+1\} \\ k+1 & \text{if } \beta_j = k \\ k & \text{if } \beta_j = k+1 \end{cases} \tag{6.38}$$

if $\beta_j \in \mathbb{Z}$ and $T_k(\beta)_j$ is obtained by applying the same transformation to each two components if $\beta_j \in \mathbb{Z}^2$. The effect of the operator T_k on β is to switch the values k and $k+1$.

Lemma 6.19. *Let $k \in \mathbb{N}$. For any multi-index $\beta \in \bar{\mathcal{A}}_{(k)} \cap \bar{\mathcal{A}}_{(k+1)}$,*

$$A_k(\beta) = T_{k+1}(A_{k+1}(\beta)).$$

Proof. We consider the case where $\beta_j \in \mathbb{Z}$. If $\beta_j \leqslant k$, then A_k as well as $T_{k+1} \circ A_{k+1}$ leave β_j unchanged. If $\beta_j \geqslant k+2$, $A_{k+1}(\beta_j) \geqslant k+3$ and T_{k+1} leaves it unchanged, equal to $A_k(\beta_j)$. Finally, the only non-trivial equality occurs when $\beta_j = k+1$. In this case, we have

$$A_k(\beta_j) = A_k(k+1) = k+2$$

and

$$T_{k+1}(A_{k+1}(\beta_j)) = T_{k+1}(A_{k+1}(k+1)) = T_{k+1}(k+1) = k+2.$$

The same argument works when $\beta_j \in \mathbb{Z}^2$. ■

Remark 6.20. Notice that by an argument similar to Lemma 6.19, for any $\beta \in \bar{\mathcal{A}}_{(0)}$,

$$A_{-1}(\beta) = T_0(A_0(\beta)).$$

Lemma 6.21. *For all $k \in \mathbb{N}$ and all $\beta \in \bar{\mathcal{A}}_{(k)}$,*

$$I_\beta^{(k)}(s; \boldsymbol{r}_k; \boldsymbol{y}_k) = I_{A_k(\beta)}^{(k+1)}(s; \boldsymbol{r}_k, 0; \boldsymbol{y}_k, 0), \tag{6.39}$$

for all $s \in [0,T], \boldsymbol{r}_k \in [0,T]^{k+1}$, $\boldsymbol{y}_k \in (\mathbb{R}^d)^{k+1}$.

Proof. First of all, we check that $A_k(\beta) \in \bar{\mathcal{A}}_{(k+1)}$. As the sizes of the components of β are not changed by the operator A_k, $\sum_{\ell=1}^{j-1} |\beta_\ell| = \sum_{\ell=1}^{j-1} |A_k(\beta)_\ell|$ for all $j = 1, \ldots, \ell(\beta)$ and $\ell(\beta) = \ell(A_k(\beta))$. Hence, using the fact that $\beta \in \bar{\mathcal{A}}_{(k)}$, we have, for all $j = 1, \ldots, \ell(\beta)$ for which $\beta_j \in \mathbb{Z}$,

$$A_k(\beta)_j \leqslant 1 + \beta_j \leqslant 1 + k + \sum_{\ell=1}^{j-1} |\beta_\ell| \leqslant k + 1 + \sum_{\ell=1}^{j-1} |A_k(\beta)_j|.$$

A similar result is true in the case where $\beta_j \in \mathbb{Z}^2$ and $A_k(\beta) \in \bar{\mathcal{A}}_{(k+1)}$.

One can check that the multi-index $A_k(\beta)$ does not contain the value $k+1$, by construction. Hence, by Remark 6.4, $I^{(k+1)}_{A_k(\beta)}$ does not depend on the variables r_{k+1} and y_{k+1}, i.e.

$$I^{(k+1)}_{A_k(\beta)}(s;\boldsymbol{r}_k, r_{k+1};\boldsymbol{y}_k, y_{k+1}) = I^{(k+1)}_{A_k(\beta)}(s;\boldsymbol{r}_k, 0;\boldsymbol{y}_k, 0),$$

for all $s, r_{k+1} \in [0,T]$, $\boldsymbol{r}_k \in [0,T]^{k+1}$, $\boldsymbol{y}_k \in (\mathbb{R}^d)^{k+1}$, $y_{k+1} \in \mathbb{R}^d$.

We are going to prove the lemma by induction on $\ell(\beta)$. First suppose that $\ell(\beta) = 1$. Let $k \in \mathbb{N}$. If $\beta \in \mathbb{Z}$, we have

$$A_k(\beta) = \begin{cases} \beta + 1 & \text{if } \beta > k, \\ \beta & \text{otherwise.} \end{cases}$$

Hence, if $\beta \in \bar{\mathcal{A}}_{(k)}$, then $A_k(\beta) = \beta \in \{0, \ldots, k\}$ and

$$\begin{aligned} I^{(k+1)}_{A_k(\beta)}(s;\boldsymbol{r}_k, 0;\boldsymbol{y}_k, 0) &= \int_0^s \int_{\mathbb{R}^d} \Gamma(r_\beta - r_{k+2}, y_\beta - y_{k+2}) M(dr_{k+2}, dy_{k+2}) \\ &= \int_0^s \int_{\mathbb{R}^d} \Gamma(r_\beta - r_{k+1}, y_\beta - y_{k+1}) M(dr_{k+1}, dy_{k+1}) \\ &= I^{(k)}_\beta(s;\boldsymbol{r}_k;\boldsymbol{y}_k). \end{aligned}$$

If $\beta \in \mathbb{Z}^2$, by the same argument as above, the components of $A_k(\beta)$ are all in $\{0, \ldots, k\}$ and (6.39) is satisfied. The lemma is proved in the case where $\ell(\beta) = 1$.

Now suppose that the lemma is satisfied for all $k \in \mathbb{N}$ and for all β such that $\ell(\beta) \leqslant n-1$. We would like to prove that it is satisfied for $\ell(\beta) = n$. Consider $\beta \in \bar{\mathcal{A}}_{(k)}$ such that $\ell(\beta) = n$ and $\beta_1 = j \in \mathbb{Z}$. Since $\beta \in \mathcal{A}_{(k)}$, we have $j \leqslant k$, $A_k(\beta)_1 = j$ and $-A_k(\beta) = A_k(-\beta) = T_{k+1}(A_{k+1}(-\beta))$ by Lemma 6.19, because $-\beta \in \bar{\mathcal{A}}_{(k)} \cap \bar{\mathcal{A}}_{(k+1)}$ if $\beta \in \bar{\mathcal{A}}_k$.

Hence, by the definition of $I_\beta^{(k)}$,

$$\begin{aligned}
&I^{(k+1)}_{A_k(\beta)}(s;\boldsymbol{r}_k,0;\boldsymbol{y}_k,0)\\
&\quad= I_j^{(k+1)}\left(\tilde{I}^{(k+2)}_{-A_k(\beta)}\right)(s;\boldsymbol{r}_k,0;\boldsymbol{y}_k,0)\\
&\quad= \int_0^s\int_{\mathbb{R}^d}\Gamma(r_j-r_{k+2},y_j-y_{k+2})\tilde{I}^{(k+2)}_{-A_k(\beta)}(\boldsymbol{r}_k,0,r_{k+2};\boldsymbol{y}_k,0,y_{k+2})M(dr_{k+2},dy_{k+2})\\
&\quad= \int_0^s\int_{\mathbb{R}^d}\Gamma(r_j-r_{k+2},y_j-y_{k+2})\\
&\qquad\qquad\times I^{(k+2)}_{-A_k(\beta)}(r_{k+2};\boldsymbol{r}_k,0,r_{k+2};\boldsymbol{y}_k,0,y_{k+2})M(dr_{k+2},dy_{k+2})\\
&\quad= \int_0^s\int_{\mathbb{R}^d}\Gamma(r_j-r_{k+2},y_j-y_{k+2})\\
&\qquad\qquad\times I^{(k+2)}_{T_{k+1}(A_{k+1}(-\beta))}(r_{k+2};\boldsymbol{r}_k,0,r_{k+2};\boldsymbol{y}_k,0,y_{k+2})M(dr_{k+2},dy_{k+2})
\end{aligned}$$

Considering that the operator T_{k+1} switches the variables $k+1$ and $k+2$, removing T_{k+1} and switching the roles of $r_{k+1}=0$ and r_{k+2} does not change the value of the integral. Then, by the induction assumption,

$$\begin{aligned}
&I^{(k+1)}_{A_k(\beta)}(s;\boldsymbol{r}_k,0;\boldsymbol{y}_k,0)\\
&\quad= \int_0^s\int_{\mathbb{R}^d}\Gamma(r_j-r_{k+2},y_j-y_{k+2})\\
&\qquad\qquad\times I^{(k+2)}_{A_{k+1}(-\beta)}(r_{k+2};\boldsymbol{r}_k,r_{k+2},0;\boldsymbol{y}_k,y_{k+2},0)M(dr_{k+2},dy_{k+2})\\
&\quad= \int_0^s\int_{\mathbb{R}^d}\Gamma(r_j-r_{k+1},y_j-y_{k+1})\\
&\qquad\qquad\times I^{(k+2)}_{A_{k+1}(-\beta)}(r_{k+1};\boldsymbol{r}_k,r_{k+1},0;\boldsymbol{y}_k,y_{k+1},0)M(dr_{k+1},dy_{k+1})\\
&\quad= \int_0^s\int_{\mathbb{R}^d}\Gamma(r_j-r_{k+1},y_j-y_{k+1})I^{(k+1)}_{-\beta}(r_{k+1};\boldsymbol{r}_k,r_{k+1};\boldsymbol{y}_k,y_{k+1})M(dr_{k+1},dy_{k+1})\\
&\quad= \int_0^s\int_{\mathbb{R}^d}\Gamma(r_j-r_{k+1},y_j-y_{k+1})\tilde{I}^{(k+1)}_{-\beta}(\boldsymbol{r}_k,r_{k+1};\boldsymbol{y}_k,y_{k+1})M(dr_{k+1},dy_{k+1})\\
&\quad= I_j^{(k)}\left(\tilde{I}^{(k+1)}_{-\beta}\right)(s;\boldsymbol{r}_k;\boldsymbol{y}_k)\\
&\quad= I_\beta^{(k)}(s;\boldsymbol{r}_k;\boldsymbol{y}_k).
\end{aligned}$$

Hence (6.39) is satisfied in the case where $\beta_1\in\mathbb{Z}$. The case where $\beta_1\in\mathbb{Z}^2$ is established similarly. The lemma is proved. ■

More generally, let us define a sequence of operators $A_k^\ell(\beta) : \bar{\mathcal{A}}_{(k)} \to \bar{\mathcal{A}}_{(k+\ell)}$ by

$$A_k^\ell(\beta)_j = \begin{cases} \beta_j + \ell\,\mathbf{1}_{\{\beta_j > k\}} & \text{if } \beta_j \in \mathbb{Z}, \\ \begin{pmatrix} \beta_j + \ell\,\mathbf{1}_{\{\beta_j > k\}} \\ \beta'_j + \ell\,\mathbf{1}_{\{\beta'_j > k\}} \end{pmatrix} & \text{if } \beta_j \in \mathbb{Z}^2, \end{cases}$$

It is not difficult to see that $A_k^\ell = A_{k+\ell-1} \circ \cdots \circ A_{k+1} \circ A_k = A_k \circ \cdots \circ A_k$ (ℓ times).

Lemma 6.22. *For all $k \in \mathbb{N}$ and all $\beta \in \bar{\mathcal{A}}_{(k)}$,*

$$I_\beta^{(k)}(s; \boldsymbol{r}_k; \boldsymbol{y}_k) = I_{A_k^\ell(\beta)}^{(k+l)}(s; \boldsymbol{r}_k, 0, \dots, 0; \boldsymbol{y}_k, 0, \dots, 0), \tag{6.40}$$

for all $s \in [0, T]$, $\boldsymbol{r}_k \in [0, T]^{k+1}$, $\boldsymbol{y}_k \in (\mathbb{R}^d)^{k+1}$, where the integral on the right-hand side has ℓ zeros for the time variables and for the spatial variables.

Proof. As $A_k^\ell = A_{k+\ell-1} \circ \cdots \circ A_{k+1} \circ A_k$, this statement is a direct consequence of an iterative use of Lemma 6.21. ■

Remark 6.23. In this document, only the cases $\ell = 1$ (corresponding to Lemma 6.21) and $\ell = 2$ are used. In the latter case, we will write B_k instead of A_k^2.

Proposition 6.24 (Integration by parts formula for iterated stochastic integrals)**.** *Let $k \in \mathbb{N}$ and $\beta, \gamma \in \bar{\mathcal{A}}_{(k)}$. Let A_k and B_k be the operators on multi-indices defined in Lemmas 6.21 and 6.22. If $\beta_1, \gamma_1 \in \mathbb{Z}$, then*

$$\begin{aligned} & I_\beta^{(k)} \cdot I_\gamma^{(k)} \\ = \;& I_{\beta_1}^{(k)}\left(\tilde{I}_{-\beta}^{(k+1)} \cdot \tilde{I}_{A_k(\gamma)}^{(k+1)}\right) + I_{\gamma_1}^{(k)}\left(\tilde{I}_{A_k(\beta)}^{(k+1)} \cdot \tilde{I}_{-\gamma}^{(k+1)}\right) + I_{\binom{\beta_1}{\gamma_1}}^{(k)}\left(\tilde{I}_{A_{k+1}(-\beta)}^{(k+2)} \cdot \tilde{I}_{A_k(-\gamma)}^{(k+2)}\right) \;\; a.s. \end{aligned} \tag{6.41}$$

If $\beta_1 \in \mathbb{Z}$, $\gamma_1 \in \mathbb{Z}^2$, then

$$I_\beta^{(k)} \cdot I_\gamma^{(k)} = I_{\beta_1}^{(k)}\left(\tilde{I}_{-\beta}^{(k+1)} \cdot \tilde{I}_{A_k(\gamma)}^{(k+1)}\right) + I_{\gamma_1}^{(k)}\left(\tilde{I}_{B_k(\beta)}^{(k+2)} \cdot \tilde{I}_{-\gamma}^{(k+2)}\right) \;\; a.s. \tag{6.42}$$

Finally, if $\beta_1, \gamma_1 \in \mathbb{Z}^2$, then

$$I_\beta^{(k)} \cdot I_\gamma^{(k)} = I_{\beta_1}^{(k)}\left(\tilde{I}_{-\beta}^{(k+2)} \cdot \tilde{I}_{B_k(\gamma)}^{(k+2)}\right) + I_{\gamma_1}^{(k)}\left(\tilde{I}_{B_k(\beta)}^{(k+2)} \cdot \tilde{I}_{-\gamma}^{(k+2)}\right) \;\; a.s. \tag{6.43}$$

Remark 6.25. Due to the symmetry of $I_{\binom{j}{j'}}^{(k)}$, (6.41) can be written as

$$\begin{aligned} I_\beta^{(k)} \cdot I_\gamma^{(k)} \;=\;& I_{\beta_1}^{(k)}\left(\tilde{I}_{-\beta}^{(k+1)} \cdot \tilde{I}_{A_k(\gamma)}^{(k+1)}\right) + I_{\gamma_1}^{(k)}\left(\tilde{I}_{A_k(\beta)}^{(k+1)} \cdot \tilde{I}_{-\gamma}^{(k+1)}\right) \\ & + \frac{1}{2} I_{\binom{\beta_1}{\gamma_1}}^{(k)}\left(\tilde{I}_{A_{k+1}(-\beta)}^{(k+2)} \cdot \tilde{I}_{A_k(-\gamma)}^{(k+2)}\right) + \frac{1}{2} I_{\binom{\gamma_1}{\beta_1}}^{(k)}\left(\tilde{I}_{A_k(-\beta)}^{(k+2)} \cdot \tilde{I}_{A_{k+1}(-\gamma)}^{(k+2)}\right) \quad \text{a.s.} \end{aligned} \tag{6.44}$$

Proof. First consider the case $\beta_1, \gamma_1 \in \mathbb{Z}$. We recall that

$$I_\beta^{(k)}(s;\boldsymbol{r}_k;\boldsymbol{y}_k) = I_{\beta_1}^{(k)}\left(\tilde{I}_{-\beta}^{(k+1)}\right)(s;\boldsymbol{r}_k;\boldsymbol{y}_k).$$

Hence, by (6.37),

$$\begin{aligned}
&\left(I_\beta^{(k)} \cdot I_\gamma^{(k)}\right)(s;\boldsymbol{r}_k;\boldsymbol{y}_k) \\
&= \int_0^s \int_{\mathbb{R}^d} I_\beta^{(k)}(u;\boldsymbol{r}_k;\boldsymbol{y}_k)\Gamma(r_{\gamma_1}-u, y_{\gamma_1}-y_{k+1}) \\
&\qquad \times \quad \tilde{I}_{-\gamma}^{(k+1)}(\boldsymbol{r}_k, u;\boldsymbol{y}_k, y_{k+1})M(du, dy_{k+1}) \\
&\quad + \int_0^s \int_{\mathbb{R}^d} I_\gamma^{(k)}(u;\boldsymbol{r}_k;\boldsymbol{y}_k)\Gamma(r_{\beta_1}-u, y_{\beta_1}-y_{k+1}) \\
&\qquad \times \quad \tilde{I}_{-\beta}^{(k+1)}(\boldsymbol{r}_k, u;\boldsymbol{y}_k, y_{k+1})M(du, dy_{k+1}) \\
&\quad + \int_0^s dr_{k+1} \int_{\mathbb{R}^d} dy_{k+1} \int_{\mathbb{R}^d} dz_{k+1}\, \Gamma(r_{\beta_1}-r_{k+1}, y_{\beta_1}-y_{k+1}) \\
&\qquad \times \quad f(y_{k+1}-z_{k+1})\Gamma(r_{\gamma_1}-r_{k+1}, y_{\gamma_1}-z_{k+1}) \\
&\qquad \times \quad \tilde{I}_{-\beta}^{(k+1)}(\boldsymbol{r}_k, r_{k+1};\boldsymbol{y}_k, y_{k+1})\tilde{I}_{-\gamma}^{(k+1)}(\boldsymbol{r}_k, r_{k+1};\boldsymbol{y}_k, z_{k+1}). \qquad (6.45)
\end{aligned}$$

Moreover, by Lemma 6.21, we have

$$\begin{aligned}
I_\beta^{(k)}(u;\boldsymbol{r}_k;\boldsymbol{y}_k) &= I_{A_k(\beta)}^{(k+1)}(u;\boldsymbol{r}_k, 0;\boldsymbol{y}_k, 0) \\
&= I_{A_k(\beta)}^{(k+1)}(u;\boldsymbol{r}_k, u;\boldsymbol{y}_k, y_{k+1}) \\
&= \tilde{I}_{A_k(\beta)}^{(k+1)}(\boldsymbol{r}_k, u;\boldsymbol{y}_k, y_{k+1}),
\end{aligned}$$

and by the same argument

$$I_\gamma^{(k)}(u;\boldsymbol{r}_k;\boldsymbol{y}_k) = \tilde{I}_{A_k(\gamma)}^{(k+1)}(\boldsymbol{r}_k, u;\boldsymbol{y}_k, y_{k+1}).$$

Again by Lemma 6.21, we have

$$\begin{aligned}
\tilde{I}_{-\beta}^{(k+1)}(\boldsymbol{r}_k, r_{k+1};\boldsymbol{y}_k, y_{k+1}) &= I_{-\beta}^{(k+1)}(r_{k+1};\boldsymbol{r}_k, r_{k+1};\boldsymbol{y}_k, y_{k+1}) \\
&= I_{A_{k+1}(-\beta)}^{(k+2)}(r_{k+1};\boldsymbol{r}_k, r_{k+1}, 0;\boldsymbol{y}_k, y_{k+1}, 0) \\
&= I_{A_{k+1}(-\beta)}^{(k+2)}(r_{k+1};\boldsymbol{r}_k, r_{k+1}, r_{k+1};\boldsymbol{y}_k, y_{k+1}, z_{k+1}) \\
&= \tilde{I}_{A_{k+1}(-\beta)}^{(k+2)}(\boldsymbol{r}_k, r_{k+1}, r_{k+1};\boldsymbol{y}_k, y_{k+1}, z_{k+1})
\end{aligned}$$

and, by the same reasoning, then by using the operator T_{k+1} and exchanging the last two variables, we have

$$\begin{aligned}
\tilde{I}_{-\gamma}^{(k+1)}(\boldsymbol{r}_k, r_{k+1};\boldsymbol{y}_k, z_{k+1}) &= \tilde{I}_{A_{k+1}(-\gamma)}^{(k+2)}(\boldsymbol{r}_k, r_{k+1}, r_{k+1};\boldsymbol{y}_k, z_{k+1}, y_{k+1}) \\
&= \tilde{I}_{T_{k+1}(A_{k+1}(-\gamma))}^{(k+2)}(\boldsymbol{r}_k, r_{k+1}, r_{k+1};\boldsymbol{y}_k, y_{k+1}, z_{k+1}) \\
&= \tilde{I}_{A_k(-\gamma)}^{(k+2)}(\boldsymbol{r}_k, r_{k+1}, r_{k+1};\boldsymbol{y}_k, y_{k+1}, z_{k+1}),
\end{aligned}$$

by Lemma 6.19. Replacing in (6.45), we have

$$\begin{aligned}\left(I^{(k)}_{\beta}\cdot I^{(k)}_{\gamma}\right)(s;\boldsymbol{r}_k;\boldsymbol{y}_k) &= I^{(k)}_{\gamma_1}\left(\tilde{I}^{(k+1)}_{A_k(\beta)}\cdot\tilde{I}^{(k+1)}_{-\gamma}\right)(s;\boldsymbol{r}_k;\boldsymbol{y}_k)\\ &\quad+I^{(k)}_{\beta_1}\left(\tilde{I}^{(k+1)}_{-\beta}\cdot\tilde{I}^{(k+1)}_{A_k(\gamma)}\right)(s;\boldsymbol{r}_k;\boldsymbol{y}_k)\\ &\quad+I^{(k)}_{\binom{\beta_1}{\gamma_1}}\left(\tilde{I}^{(k+2)}_{A_{k+1}(-\beta)}\cdot\tilde{I}^{(k+2)}_{A_k(-\gamma)}\right)(s;\boldsymbol{r}_k;\boldsymbol{y}_k).\end{aligned}$$

The case where $\beta_1,\gamma_1\in\mathbb{Z}$ is established. The proof for the two other cases is the same, using (6.37) without the third term on the right-hand side and Lemma 6.22 where needed. ■

The next step in our construction is to use the integration by parts formula for iterated stochastic integrals to show that a product of iterated stochastic integrals can be written as a linear combination of such iterated integrals. We need the following notations.

Let $\beta\in\bar{\mathcal{A}}_{(k)}$ and , for $v\in\mathbb{N}^{k+1}$, let $|v|=\sum_{i=0}^{k}v_i$. We set $i_k(\beta)=(i_0,\dots,i_k)$, where i_j is the number of times the integer j appears in β. In determining the values i_j, we count all integers that appear in the components of β (i.e. one for $\beta_j\in\mathbb{Z}$ and one for each of the two integers for $\beta_j\in\mathbb{Z}^2$) in order to have $|i_{\|\beta\|}(\beta)|=\|\beta\|$. The vector $i_k(\beta)$ is useful to list the values in β which are not changed by the operators A_k and B_k. Then, for $v\in\mathbb{N}^{k+1}$, let

$$\mathcal{B}_v=\{\beta\in\bar{\mathcal{A}}_{(k)}:i_k(\beta)=v\}.$$

Notice that the set $\mathcal{B}_v$ depends on the number $k+1$ of components of the vector v.

Proposition 6.26 (Product of iterated stochastic integrals)**.** *Let $k\in\mathbb{N}$ and let $\beta,\gamma\in\bar{\mathcal{A}}_{(k)}$. Then, there exist a finite subset $\mathcal{D}(\beta,\gamma)$ of $\mathcal{D}=\mathcal{B}_{i_k(\beta)+i_k(\gamma)}$ (the sum is the usual sum in $\mathbb{N}^{k+1}$) and constants $(c_\delta)_{\delta\in\mathcal{D}(\beta,\gamma)}$ such that*

$$\left(I^{(k)}_{\beta}\cdot I^{(k)}_{\gamma}\right)(s;\boldsymbol{r}_k;\boldsymbol{y}_k)=\sum_{\delta\in\mathcal{D}(\beta,\gamma)}c_\delta\,I^{(k)}_{\delta}(s;\boldsymbol{r}_k;\boldsymbol{y}_k),\qquad(6.46)$$

for all $s\in[0,T]$, $\boldsymbol{r}_k\in[0,T]^{k+1}$, $\boldsymbol{y}_k\in(\mathbb{R}^d)^{k+1}$.

Remark 6.27. As a consequence of Proposition 6.26, a product of iterated stochastic integrals can be written as a finite linear combination of such integrals. Notice that the set and the constants in (6.46) is not unique, as we can split or gather some of the integrals and change the constants. For example, as $I^{(2)}_{\binom{1}{2}}=I^{(2)}_{\binom{2}{1}}$, we can replace $I^{(2)}_{\binom{1}{2}}$ by $\frac{1}{2}I^{(2)}_{\binom{1}{2}}+\frac{1}{2}I^{(2)}_{\binom{2}{1}}$ without changing the sum.

Proof. We are going to prove Proposition 6.26 by induction on $\ell(\beta)$ and $\ell(\gamma)$. First consider the case where $\ell(\beta) = \ell(\gamma) = 1$. In order to simplify the notation, we omit the variables in the proof. First of all, if $\beta_1, \gamma_1 \in \mathbb{Z}$, by the integration by parts formula ((6.41) in Proposition 6.24) and the fact that $-\beta = -\gamma = \emptyset$,

$$\begin{aligned} I_\beta^{(k)} \cdot I_\gamma^{(k)} &= I_{\beta_1}^{(k)}(\tilde{I}_{A_k(\gamma)}^{(k+1)}) + I_{\gamma_1}^{(k)}(\tilde{I}_{A_k(\beta)}^{(k+1)}) + I_{\binom{\beta_1}{\gamma_1}}^{(k)} \\ &= I_{\beta_1, A_k(\gamma)}^{(k)} + I_{\gamma_1, A_k(\beta)}^{(k)} + I_{\binom{\beta_1}{\gamma_1}}^{(k)}. \end{aligned} \tag{6.47}$$

Clearly, since $\beta, \gamma \in \bar{\mathcal{A}}_{(k)}$, $\beta_1, \gamma_1 \in \{0, \ldots, k\}$, therefore $A_k(\beta) = \beta$ and $A_k(\gamma) = \gamma$ in this case. Indeed, the values less than or equal to k are not changed by the operator A_k. In particular $i_k(\beta_1, A_k(\gamma)) = i_k(\beta, \gamma) = i_k(\beta) + i_k(\gamma)$. Hence, $\left(\binom{\beta_1}{\gamma_1}\right)$, $(\beta_1, A_k(\gamma))$ and $(\gamma_1, A_k(\beta)) \in \mathcal{B}_{i_k(\beta)+i_k(\gamma)}$ and setting

$$\mathcal{D}(\beta, \gamma) = \left\{ \left(\binom{\beta_1}{\gamma_1}\right); (\beta_1, A_k(\gamma)); (\gamma_1, A_k(\beta)) \right\},$$

and all the constants equal to 1, Proposition 6.26 is satisfied in this first case.

Next, if $\beta \in \mathbb{Z}$, $\gamma \in \mathbb{Z}^2$, then (6.42) in Proposition 6.24 gives

$$\begin{aligned} I_\beta^{(k)} \cdot I_\gamma^{(k)} &= I_{\beta_1}^{(k)}(\tilde{I}_{A_k(\gamma)}^{(k+1)}) + I_{\gamma_1}^{(k)}(\tilde{I}_{B_k(\beta)}^{(k+2)}) \\ &= I_{\beta_1, A_k(\gamma)}^{(k)} + I_{\gamma_1, B_k(\beta)}^{(k)}. \end{aligned} \tag{6.48}$$

Again, $A_k(\beta) = \beta$ and $A_k(\gamma) = \gamma$. In particular, $i_k(\beta_1, A_k(\gamma)) = i_k(\beta) + i_k(\gamma)$ and $i_k(\gamma_1, B_k(\beta)) = i_k(\gamma) + i_k(\beta)$. Setting

$$\mathcal{D}(\beta, \gamma) = \{(\beta_1, A_k(\gamma)); (\gamma_1, B_k(\beta))\}$$

and the constants equal to 1, Proposition 6.26 is satisfied.

The same argument, but using (6.43) works in the case where $\beta, \gamma \in \mathbb{Z}^2$. Proposition 6.26 is proved for all $k \in \mathbb{N}$ in the case where $\ell(\beta) = \ell(\gamma) = 1$.

As a second step, fix $n \in \mathbb{N}$ and suppose by induction that (6.46) is proved for all $k \in \mathbb{N}$ and $\beta, \gamma \in \bar{\mathcal{A}}_{(k)}$ such that $\ell(\beta) = 1$, $\ell(\gamma) \leqslant n$. Consider $\beta, \gamma \in \bar{\mathcal{A}}_{(k)}$ such that $\ell(\beta) = 1$, $\ell(\gamma) = n + 1$ and $\beta_1, \gamma_1 \in \mathbb{Z}$. Then, by (6.41) in Proposition 6.24,

$$\begin{aligned} I_\beta^{(k)} \cdot I_\gamma^{(k)} &= I_{\beta_1}^{(k)}(\tilde{I}_{A_k(\gamma)}^{(k+1)}) + I_{\gamma_1}^{(k)}(\tilde{I}_{-\gamma}^{(k+1)} \cdot \tilde{I}_{A_k(\beta)}^{(k+1)}) + I_{\binom{\beta_1}{\gamma_1}}^{(k)}(\tilde{I}_{A_k(-\gamma)}^{(k+2)}) \\ &= I_{\beta_1, A_k(\gamma)}^{(k)} + I_{\gamma_1}^{(k)}(\tilde{I}_{-\gamma}^{(k+1)} \cdot \tilde{I}_{A_k(\beta)}^{(k+1)}) + I_{\binom{\beta_1}{\gamma_1}, A_k(-\gamma)}^{(k)}. \end{aligned}$$

Clearly the multi-indices $(\beta_1, A_k(\gamma))$ and $\left(\binom{\beta_1}{\gamma_1}, A_k(-\gamma)\right)$ belong to $\mathcal{B}_{i_k(\beta)+i_k(\gamma)}$ because the operator A_k does not change values less than or equal to k. Moreover,

$\ell(-\gamma) = n$ and $\ell(A_k(\beta)) = \ell(\beta) = 1$ so that, by the induction assumption,

$$\begin{aligned}
&\tilde{I}^{(k+1)}_{-\gamma}(\boldsymbol{r}_{k+1}; \boldsymbol{y}_{k+1}) \cdot \tilde{I}^{(k+1)}_{A_k(\beta)}(\boldsymbol{r}_{k+1}; \boldsymbol{y}_{k+1}) \\
&= \; I^{(k+1)}_{-\gamma}(r_{k+1}; \boldsymbol{r}_{k+1}; \boldsymbol{y}_{k+1}) \cdot I^{(k+1)}_{A_k(\beta)}(r_{k+1}; \boldsymbol{r}_{k+1}; \boldsymbol{y}_{k+1}) \\
&= \sum_{\delta \in \mathcal{D}(-\gamma, A_k(\beta))} c_\delta I^{(k+1)}_\delta (r_{k+1}; \boldsymbol{r}_{k+1}; \boldsymbol{y}_{k+1}) \\
&= \sum_{\delta \in \mathcal{D}(-\gamma, A_k(\beta))} c_\delta \tilde{I}^{(k+1)}_\delta (\boldsymbol{r}_{k+1}; \boldsymbol{y}_{k+1}),
\end{aligned}$$

where the set $\mathcal{D}(-\gamma, A_k(\beta))$ is a finite subset of $\mathcal{B}_{i_{k+1}(-\gamma)+i_{k+1}(A_k(\beta))}$. Hence, by the linearity of integral operators (see (6.6) and (6.8)),

$$\begin{aligned}
I^{(k)}_{\gamma_1}(\tilde{I}^{(k+1)}_{-\gamma} \cdot \tilde{I}^{(k+1)}_{A_k(\beta)}) &= \sum_{\delta \in \mathcal{D}(-\gamma, A_k(\beta))} c_\delta \, I^{(k)}_{\gamma_1}(\tilde{I}^{(k+1)}_\delta) \\
&= \sum_{\delta \in \mathcal{D}(-\gamma, A_k(\beta))} c_\delta \, I^{(k)}_{\gamma_1, \delta}.
\end{aligned}$$

Finally,

$$i_k(\gamma_1, \delta) = i_k(\gamma_1) + i_k(\delta) = i_k(\gamma_1) + i_k(-\gamma) + i_k(A_k(\beta)) = i_k(\gamma) + i_k(\beta),$$

shows that (6.46) is satisfied for $\ell(\beta) = 1$, $\ell(\gamma) = n+1$ in the case where $\beta_1, \gamma_1 \in \mathbb{Z}$. The cases where $\beta_1 \in \mathbb{Z}^2$ or $\gamma_1 \in \mathbb{Z}^2$ are treated the same way using the corresponding formulas in Proposition 6.24.

As a last step, fix $n \in \mathbb{N}$ and suppose by induction that (6.46) is proved for all $k \in \mathbb{N}$ and all $\beta, \gamma \in \bar{\mathcal{A}}_{(k)}$ such that $\ell(\beta) \leqslant n$. The case where $\ell(\beta) = 1$ was established above. Consider $\beta \in \bar{\mathcal{A}}_{(k)}$ such that $\ell(\beta) = n+1$. Consider $m \in \mathbb{N}$ and, as a second induction assumption, that (6.46) is satisfied for β and $\gamma \in \bar{\mathcal{A}}_{(k)}$ such that $\ell(\gamma) \leqslant m$ (for $m = 1$, this is indeed the case). Consider $\gamma \in \bar{\mathcal{A}}_{(k)}$ such that $\ell(\gamma) = m+1$. Suppose moreover that $\beta_1, \gamma_1 \in \mathbb{Z}$. By (6.41) in Proposition 6.24,

$$I^{(k)}_\beta \cdot I^{(k)}_\gamma \;=\; I^{(k)}_{\beta_1}(\tilde{I}^{(k+1)}_{-\beta} \cdot \tilde{I}^{(k+1)}_{A_k(\gamma)}) + I^{(k)}_{\gamma_1}(\tilde{I}^{(k+1)}_{-\gamma} \cdot \tilde{I}^{(k+1)}_{A_k(\beta)}) + I^{(k)}_{\binom{\beta_1}{\gamma_1}}(\tilde{I}^{k+2}_{A_{k+1}(-\beta)} \cdot \tilde{I}^{(k+2)}_{A_k(-\gamma)})$$

Then, as $\ell(-\beta) = n$, by the induction assumption on n,

$$\begin{aligned}
I^{(k)}_{\beta_1}(\tilde{I}^{(k+1)}_{-\beta} \cdot \tilde{I}^{(k+1)}_{A_k(\gamma)}) &= \sum_{\delta \in \mathcal{D}(-\beta, A_k(\gamma))} c_\delta \, I^{(k)}_{\beta_1}(\tilde{I}^{(k+1)}_\delta) \\
&= \sum_{\delta \in \mathcal{D}(-\beta, A_k(\gamma))} c_\delta \, I^{(k)}_{\beta_1, \delta}
\end{aligned}$$

where $\mathcal{D}(-\beta, A_k(\gamma))$ is a finite subset of $\mathcal{B}_{i_{k+1}(-\beta)+i_{k+1}(A_k(\gamma))}$. For δ in this set, $i_k(\beta_1, \delta) = i_k(\beta) + i_k(\gamma)$. As $\ell(A_k(\beta)) = n+1$ and $\ell(-\gamma) = m$, by the induction assumption on m, a similar argument gives

$$I^{(k)}_{\gamma_1}(\tilde{I}^{(k+1)}_{-\gamma} \cdot \tilde{I}^{(k+1)}_{A_k(\beta)}) = \sum_{\delta \in \mathcal{D}(-\gamma, A_k(\beta))} c_\delta \, I^{(k)}_{\gamma_1,\delta},$$

where $\mathcal{D}(-\gamma, A_k(\beta)) \subset \mathcal{B}_{i_{k+1}(A_k(\beta))+i_{k+1}(-\gamma)}$. For $\delta \in \mathcal{D}(-\gamma, A_k(\beta))$, $i_k(\gamma_1, \delta) = i_k(\beta) + i_k(\gamma)$. As $\ell(A_{k+1}(-\beta)) = \ell(-\beta) = n$, by the induction assumption on n, a similar argument again gives

$$I^{(k)}_{\binom{\beta_1}{\gamma_1}}(\tilde{I}^{k+2}_{A_{k+1}(-\beta)} \cdot \tilde{I}^{(k+2)}_{A_k(-\gamma)}) = \sum_{\delta \in \mathcal{D}(A_{k+1}(-\beta), A_k(-\gamma))} c_\delta \, I^{(k)}_{\binom{\beta_1}{\gamma_1},\delta}$$

where $\mathcal{D}(A_{k+1}(-\beta), A_k(-\gamma))$ is a finite subset of $\mathcal{B}_{i_{k+2}(A_{k+1}(-\beta))+i_{k+2}(A_k(-\gamma))}$ Then,

$$i_k(\begin{pmatrix}\beta_1\\ \gamma_1\end{pmatrix}, \delta) = i_k(\beta) + i_k(\gamma).$$

Hence, (6.46) is satisfied in the case where $\beta_1, \gamma_1 \in \mathbb{Z}$. The cases where $\beta_1 \in \mathbb{Z}^2$ or $\gamma_1 \in \mathbb{Z}^2$ are treated analogously.

Property (6.46) is therefore satisfied in the case where $\ell(\beta) = m+1$ and $\ell(\gamma) = m+1$. This completes the induction and establishes Proposition 6.26. ■

We can extend the result of Proposition 6.26 to a product of more than two iterated stochastic integrals.

Corollary 6.28. *Let $n, k \in \mathbb{N}$ and let $\beta^{(1)}, \ldots, \beta^{(n)} \in \bar{\mathcal{A}}_{(k)}$. Let $v = \sum_{j=1}^{n} i_k(\beta^{(j)})$. Then, there exist a finite subset*

$$\mathcal{D}(\beta^{(1)}, \ldots, \beta^{(n)}) \subset \mathcal{B}_v$$

and constants $(c_\delta)_{\delta \in \mathcal{D}(\beta^{(1)},\ldots,\beta^{(n)})}$ such that

$$\left(\prod_{j=1}^{n} I^{(k)}_{\beta^{(j)}}\right)(s; \boldsymbol{r}_k; \boldsymbol{y}_k) = \sum_{\delta \in \mathcal{D}(\beta^{(1)},\ldots,\beta^{(n)})} c_\delta \, I^{(k)}_\delta(s; \boldsymbol{r}_k; \boldsymbol{y}_k), \tag{6.49}$$

for all $s \in [0, T]$, $\boldsymbol{r}_k \in [0, T]^{k+1}$, $\boldsymbol{y}_k \in (\mathbb{R}^d)^{k+1}$.

Proof. It is sufficient to apply Proposition 6.26 repeatedly. ■

The result that we are going to use to discuss the convergence of the Itô-Taylor expansion in Chapter 7 is Proposition 6.26 in the case $k = 0$, which we isolate as a corollary.

Corollary 6.29. *Let $\beta, \gamma \in \mathcal{A}_{(0)} \cap \bar{\mathcal{A}}$. Then there exist a finite subset $\mathcal{D}(\beta,\gamma)$ of $\mathcal{B}_{i_0(\beta)+i_0(\gamma)}$ and constants $(c_\delta)_{\delta\in\mathcal{D}(\beta,\gamma)}$ such that*

$$\left(I_\beta^{(0)} \cdot I_\gamma^{(0)}\right)(s;t,x) = \sum_{\delta\in\mathcal{D}(\beta,\gamma)} c_\delta\, I_\delta^{(0)}(s;t,x), \tag{6.50}$$

for all $s,t \in [0,T]$, $x \in \mathbb{R}^d$.

Proposition 6.26 and Corollary 6.28 show that the product of two or more iterated stochastic integrals can be written as a finite sum of iterated stochastic integrals. We know that the multi-indices which appear in the sum form a finite subset of $\mathcal{B}_v$, where $v = \sum_{i=1}^n i_k(\beta^{(j)})$. We can explicitly give this finite subset for fixed β and γ by identifying the terms that appear in the successive uses of the integration by parts formula. Now, we would like to fix a multi-index δ and try to find multi-indices $\beta^{(1)},\dots,\beta^{(n)}$ such that $\delta \in \mathcal{D}(\beta^{(1)},\dots,\beta^{(n)})$, that is, we would like to identify iterated stochastic integrals such that $I_\delta^{(k)}$ appears in the linear combination written in Proposition 6.26 for the product of these multiple stochastic integrals.

For $\beta,\gamma \in \bar{\mathcal{A}}_{(k)}$, we let $\mathcal{D}(\beta,\gamma)$ be the finite subset of $\mathcal{B}_{i_k(\beta)+i_k(\gamma)}$ which contains all multi-indices δ that appear in some decomposition of $I_\beta^{(k)} \cdot I_\gamma^{(k)}$ into a linear combination of iterated stochastic integrals as in (6.46) (recall that this decomposition is not unique). It is therefore sufficient that there exists one decomposition of $I_\beta^{(k)} \cdot I_\gamma^{(k)}$ in which δ appears in order that $\delta \in \mathcal{D}(\beta,\gamma)$. In other words, if $\delta \notin \mathcal{D}(\beta,\gamma)$, then δ will never appear in any decomposition of $I_\beta^{(k)} \cdot I_\gamma^{(k)}$.

Following the same idea, for $\beta^{(1)},\dots,\beta^{(n)} \in \bar{\mathcal{A}}_{(k)}$, we set $v = \sum_{j=1}^n i_k(\beta^{(j)})$ and let $\mathcal{D}(\beta^{(1)},\dots,\beta^{(n)})$ be the finite subset of $\mathcal{B}_v$ such that $\delta \in \mathcal{D}(\beta^{(1)},\dots,\beta^{(n)})$ if there exists a decomposition of $\prod_{j=1}^n I_{\beta^{(j)}}^{(k)}$ in a linear combination of iterated integrals such that δ appears in this decomposition. In order to be complete, for $\beta \in \bar{\mathcal{A}}_{(k)}$, let $\mathcal{D}(\beta) = \{\beta\}$.

Proposition 6.30. *Fix $k \in \mathbb{N}$. Let $\delta \in \bar{\mathcal{A}}_{(k)}$. Suppose δ is such that $|i_k(\delta)| = n \in \mathbb{N}^*$ (the notations are defined on page 95). Then there exists a unique (up to a permutation) sequence $\beta^{(1)},\dots,\beta^{(n)} \in \bar{\mathcal{A}}_{(k)}$ such that $|i_k(\beta^{(j)})| = 1$ for all $j = 1,\dots,n$ (that is, multi-indices with exactly one final variable) and such that $\delta \in \mathcal{D}(\beta^{(1)},\dots,\beta^{(n)})$.*

Proof. We use the terminology and notation of Remark 6.3. The proof of Proposition 6.30 in the case $n = 1$ is obvious : we just consider $\beta^{(1)} = \delta$ and the result is proved. Let us consider the case $n = 2$. This means that $I_\beta^{(k)}$ only depends on at most two of the $k+1$ final variables $r_0,\dots,r_k$ (in some cases, a single variable appear in two positions). Each of these two variables (either two different final variables or two copies of the same final variable) begins a chain of antecedent variables corresponding to the values of the multi-index δ. These two chains contain all values appearing in δ (there are $\|\delta\|$ values defining δ) and split δ into two groups of values (see Example 6.31 below). This splitting can be described with two disjoint sets $\mathcal{N}_1$ and $\mathcal{N}_2$, with $\mathcal{N}_1 \cup \mathcal{N}_2 = \{k+1, \ldots, k+\|\delta\|\}$ and such that $\mathcal{N}_1$ contains the indices of the variables for which the related value in δ belongs to the first chain, and $\mathcal{N}_2$ the indices for which it belongs to the second chain. The numbers in $\mathcal{N}_1$ and $\mathcal{N}_2$ are the indices of the corresponding variables r_j, y_j. Let $\mathcal{N}_1$ and $\mathcal{N}_2$ be ordered in increasing order.

Then, we would like to define two new multi-indices $\beta^{(1)}$ (resp. $\beta^{(2)}$) with the values in δ corresponding to the variables in $\mathcal{N}_1$ (resp. $\mathcal{N}_2$). For this purpose, before setting the values in the $\beta^{(j)}$'s, we must modify them in order to account for the fact that some variables have been attributed to the other $\beta^{(j)}$. Namely, we apply the following algorithm. First set $\beta^{(1)} = \beta^{(2)} = \emptyset$. For $i = 1, \ldots, \ell(\delta)$, we consider the component δ_i. If $\delta_i \in \mathbb{Z}$, there is $j \in \{k+1, \ldots, k+\|\delta\|\}$ such that r_j is substracted to r_{δ_i} in Γ in the integral with respect to r_j, y_j (in other words, r_{δ_i} is the antecedent of r_j). If $j \in \mathcal{N}_1$, and if r_{δ_i} is a final variable, then δ_i is appended to $\beta^{(1)}$. If r_{δ_i} is not a final variable, then it has to be in $\mathcal{N}_1$ because r_{δ_i} is the antecedent of r_j. Let x be the number corresponding to the position in $\mathcal{N}_1$ of δ_i (i.e. $x = \sum_{\ell=k+1}^{\delta_1} \mathbf{1}_{\mathcal{N}_1}(\ell)$). Then, $k+x$ is appended to $\beta^{(1)}$. If $j \in \mathcal{N}_2$, the same procedure applies, but the numbers are appended to $\beta^{(2)}$. Next, if $\delta_i \in \mathbb{Z}^2$, then there is $j \in \{k+1, \ldots, k+\|\delta\|\}$ such that δ_i corresponds to the indices of the variables to which r_j and r_{j+1} are substracted in Γ in the integrals with respect to r_j, y_j and r_{j+1}, y_{j+1} (in other words, if $\delta_i = \binom{\ell}{\ell'}$, r_ℓ (resp. $r_{\ell'}$) is the antecedent of r_j (resp. r_{j+1})). If $j, j+1 \in \mathcal{N}_1$, then a component in $\mathbb{Z}^2$ is appended to $\beta^{(1)}$, the values of which are each modified as in the case $\delta_i \in \mathbb{Z}$. If $j, j+1 \in \mathcal{N}_2$, the same applies for $\beta^{(2)}$. Finally, if $j \in \mathcal{N}_1$, $j+1 \in \mathcal{N}_2$, a component in $\mathbb{Z}$ is added to $\beta^{(1)}$ (corresponding to r_j) and to $\beta^{(2)}$ (corresponding to r_{j+1}). The two appended components are each modified as in the case $\delta_i \in \mathbb{Z}$. The symmetric case $j \in \mathcal{N}_2$, $j+1 \in \mathcal{N}_1$ is handled similarly. We have thus built two multi-indices by gathering the values within each of the two groups of values that split δ. The modifications appearing in the algorithm renumber the variables in $\beta^{(j)}$ so as to obtain a multi-index which uses the values $k+1, \ldots, k+\|\beta^{(j)}\|$ as indices for the integration variables instead of $\|\beta^{(j)}\|$ values chosen in $k+1, \ldots, k+\|\delta\|$. The couple $(\beta^{(1)}, \beta^{(2)})$ built by this algorithm is the sequence of Proposition 6.30.

Indeed, the two multi-indices $\beta^{(1)}$ and $\beta^{(2)}$ are in $\bar{\mathcal{A}}_{(k)}$ by construction, as each component of each of the $\beta^{(j)}$'s refers to a variable of smaller index (final variable or preceding integration variable). This implies that the conditions for the $\beta^{(j)}$'s to be in $\bar{\mathcal{A}}_{(k)}$ are satisfied (see Remark 6.3). Moreover, they satisfy $i_k(\beta^{(j)}) = 1$ ($j = 1, 2$) because they only depend on one final variable. Moreover, we can see that the successive use of the integration by parts formula to express $I^{(k)}_{\beta^{(1)}} \cdot I^{(k)}_{\beta^{(2)}}$ as a linear combination will make $I^{(k)}_{\delta}$ appear at some point. It suffices to consider the successive terms in the integration by parts formula, which build the multi-index δ from the values of $\beta^{(1)}, \beta^{(2)}$ (see Example 6.31 below). This procedure consists of choosing the terms in the integration by parts formula corresponding to the components of $\beta^{(1)}$ and $\beta^{(2)}$ in the same order as they appear in the algorithm stated above to build $\beta^{(1)}$ and $\beta^{(2)}$. This shows that $\delta \in \mathcal{D}(\beta^{(1)}, \beta^{(2)})$.

We check uniqueness as follows. If we consider $\gamma^{(1)}, \gamma^{(2)} \in \bar{\mathcal{A}}^{(k)}$ two multi-indices such that $i_k(\gamma^{(1)}) = i_k(\gamma^{(2)}) = 1$ and $\delta \in \mathcal{D}(\gamma^{(1)}, \gamma^{(2)})$. When expanding the product of $I^{(k)}_{\gamma^{(1)}}$ with $I^{(k)}_{\gamma^{(2)}}$ with a successive application of the integration by parts formula, we obtain terms among which $I^{(k)}_{\delta}$ will appear. It is easily seen that the values in δ are the union of those of $\gamma^{(1)}$ and of $\gamma^{(2)}$ (sometimes modified). Nevertheless, the successive application of the integration by parts formula does not create any *mixing* between variables coming from $\gamma^{(1)}$ and $\gamma^{(2)}$ in the sense that a variable coming from $\gamma^{(1)}$ will always be a final variable or have another variable coming from $\gamma^{(1)}$ as antecedent. Indeed, in the case where $\gamma^{(1)}_1, \gamma^{(2)}_1 \in \mathbb{Z}$, by (6.41)

$$\begin{aligned} I^{(k)}_{\gamma^{(1)}} \cdot I^{(k)}_{\gamma^{(2)}} &= I^{(k)}_{\gamma^{(1)}_1} \left(\tilde{I}^{(k+1)}_{-\gamma^{(1)}} \cdot \tilde{I}^{(k+1)}_{A_k(\gamma^{(2)})} \right) + I^{(k)}_{\gamma^{(2)}_1} \left(\tilde{I}^{(k+1)}_{A_k(\gamma^{(1)})} \cdot \tilde{I}^{(k+1)}_{-\gamma^{(2)}} \right) \\ &\quad + I^{(k)}_{\binom{\gamma^{(1)}_1}{\gamma^{(2)}_1}} \left(\tilde{I}^{(k+2)}_{A_{k+1}(-\gamma^{(1)})} \cdot \tilde{I}^{(k+2)}_{A_k(-\gamma^{(2)})} \right) \quad \text{a.s.} \end{aligned} \tag{6.51}$$

The first term on the right-hand side of (6.51) is the integral with respect to r_{k+1} which has the final variable $r_{\gamma^{(1)}_1}$ as antecedent. Moreover, each variable which will have r_{k+1} as antecedent will come from γ_1 as $A_k(\gamma^{(2)})$ cannot contain the value $k+1$. The same applies with the second term in a symmetric way. Finally, the third term on the right-hand side of (6.51) is the double integral with respect to r_{k+1} having the final variable $r_{\gamma^{(1)}_1}$ as antecedent and r_{k+2} having the final variable $r_{\gamma^{(2)}_1}$ as antecedent. But each variable which will have r_{k+1} as antecedent will come from $\gamma^{(1)}$, because $A_k(-\gamma^{(2)})$ cannot contain the value $k+1$. Each variable which will have r_{k+2} as antecedent will come from $\gamma^{(2)}$, because $A_{k+1}(-\gamma^{(1)})$ cannot contain the value $k+2$. This reasonning applies to the integration by parts formulas (6.42) and (6.43) as well. Then, iterating this result for each use of an integration by parts formula in the development of $I^{(k)}_{\gamma^{(1)}} \cdot I^{(k)}_{\gamma^{(2)}}$ in a sum of iterated integrals shows that no *mixing* appears in the sense defined above. If $\delta \in \mathcal{D}(\gamma^{(1)}, \gamma^{(2)})$, then we can recover $\gamma^{(1)}$ and $\gamma^{(2)}$ with the algorithm described above for the construction of $\beta^{(1)}$ and $\beta^{(2)}$. Hence, the two couples $(\gamma^{(1)}, \gamma^{(2)})$ and $(\beta^{(1)}, \beta^{(2)})$ are equal up to a permutation and the sequence is unique.

Now consider the case where $n > 2$. Choose one of the n final variables that appear in δ and split the values in δ into the set of values for which the last antecedent is this variable and the set of other variables. As for the case where $n=2$, gathering and renumbering correctly, this gives two multindices $\beta^{(1)}$ and δ' for which $i_k(\beta^{(1)})=1$ and $i_k(\delta')=n-1$ and $\delta\in\mathcal{D}(\beta^{(1)},\delta')$. Using this method succesively n times gives the sequence $(\beta^{(1)},\ldots,\beta^{(n)})$. Apart from the order, this sequence is unique. ■

Example 6.31. Suppose $k=1$, $\beta=(1)$ and $\gamma=\left(0,2,\binom{2}{3}\right)$. Clearly, $\beta,\gamma\in\bar{\mathcal{A}}_{(1)}$. We would like to write $I^{(1)}_\beta\cdot I^{(1)}_\gamma$ as a linear combination of iterated integrals. By the integration by parts formula, we have

$$\begin{aligned} I^{(1)}_1\cdot I^{(1)}_{0,2,\binom{2}{3}} &= I^{(1)}_1(\tilde I^{(2)}_{0,3,\binom{3}{4}})+I^{(1)}_0(\tilde I^{(2)}_1\cdot\tilde I^{(2)}_{2,\binom{2}{3}})+I^{(1)}_{\binom{1}{0}}(\tilde I^{(2)}_{3,\binom{3}{4}})\\ &= I^{(1)}_{1,0,3,\binom{3}{4}}+I^{(1)}_0(\tilde I^{(2)}_1\cdot\tilde I^{(2)}_{2,\binom{2}{3}})+I^{(1)}_{\binom{1}{0},3,\binom{3}{4}}. \end{aligned}\tag{6.52}$$

But, by the integration by parts formula again,

$$\begin{aligned} I^{(2)}_1\cdot I^{(2)}_{2,\binom{2}{3}} &= I^{(2)}_1(\tilde I^{(3)}_{2,\binom{2}{4}})+I^{(2)}_2(\tilde I^{(3)}_1\cdot\tilde I^{(3)}_{\binom{2}{3}})+I^{(2)}_{\binom{1}{2}}(\tilde I^{(4)}_{\binom{2}{4}})\\ &= I^{(2)}_{1,2,\binom{2}{4}}+I^{(2)}_2(\tilde I^{(3)}_{1,\binom{2}{3}}+\tilde I^{(3)}_{\binom{2}{3},1})+I^{(2)}_{\binom{1}{2},\binom{2}{4}}\\ &= I^{(2)}_{1,2,\binom{2}{4}}+I^{(2)}_{2,1,\binom{2}{3}}+I^{(2)}_{2,\binom{2}{3},1}+I^{(2)}_{\binom{1}{2},\binom{2}{4}}. \end{aligned}\tag{6.53}$$

Replacing (6.53) in (6.52),

$$I^{(1)}_1\cdot I^{(1)}_{0,2,\binom{2}{3}} = I^{(1)}_{1,0,3,\binom{3}{4}}+I^{(1)}_{0,1,2,\binom{2}{4}}+I^{(1)}_{0,2,1,\binom{2}{3}}+I^{(1)}_{0,2,\binom{2}{3},1}+I^{(1)}_{0,\binom{1}{2},\binom{2}{4}}+I^{(1)}_{\binom{1}{0},3,\binom{3}{4}}.$$

Hence, $\mathcal{D}\left(1\,;\,0,2,\binom{2}{3}\right)$ is the set consisting of the multi-indices

$$\left(1,0,3,\tbinom{3}{4}\right)\ ;\ \left(0,1,2,\tbinom{2}{4}\right)\ ;\ \left(0,2,1,\tbinom{2}{3}\right)\ ;\ \left(0,2,\tbinom{2}{3},1\right)\ ;\ \left(0,\tbinom{1}{2},\tbinom{2}{4}\right)\ ;\ \left(\tbinom{1}{0},3,\tbinom{3}{4}\right)$$

and all multi-indices obtained by permutations in the $\mathbb{Z}^2$ components, which correspond to the same integrals. We can check that all those multi-indices belong to $\bar{\mathcal{A}}_{(1)}$, satisfy $i_1(\delta)=(1,1)$ and, hence, belong to the set $\mathcal{B}_{(1,1)}$.

Now we would like to illustrate Proposition 6.30. Set $\delta=\left(1,0,3,\binom{3}{4}\right)\in\bar{\mathcal{A}}_{(1)}$. We have $i_1(\delta)=(1,1)$. The two values corresponding to final variables are 0 (δ_2) and 1 (δ_1). Finding antecedents for each variable, we have :

- r_0 and r_1 are final variables ;
- $\delta_1=1$ shows that r_2 has r_1 as antecedent ;

- $\delta_2 = 0$ shows that r_3 has r_0 as antecedent ;
- $\delta_3 = 3$ shows that r_4 has r_3 as antecedent ;
- $\delta_4 = \binom{3}{4}$ shows that r_5, (resp. r_6) have r_3, (resp. r_4) as antecedent.

Hence, r_2 has the final variable r_1 as antecedent and $r_3, \dots, r_6$ have r_0 has final antecedent. This makes it possible to split δ into two groups of values. Using the notation of the proof of Proposition 6.30, we have

$$\mathcal{N}_1 = \{2\} \qquad \text{and} \qquad \mathcal{N}_2 = \{3, 4, 5, 6\}.$$

Now, set $\beta^{(1)} = \beta^{(2)} = \emptyset$. Then, the different steps of the algorithm are

- $\delta_1 = 1$ corresponds to $j = 2 \in \mathcal{N}_1$, r_1 is a final variable. Hence, we set $\beta^{(1)} = (1)$;
- $\delta_2 = 0$ corresponds to $j = 3 \in \mathcal{N}_2$, r_0 is a final variable. Hence, we set $\beta^{(2)} = (0)$;
- $\delta_3 = 3$ corresponds to $j = 4 \in \mathcal{N}_2$, r_3 is an integration variable in first position in $\mathcal{N}_2$ and $x = 1$. Then, appending $k + x = 2$, $\beta^{(2)}$ becomes $\beta^{(2)} = (0, 2)$;
- $\delta_4 = \binom{3}{4}$ corresponds to $j = 5, j + 1 = 6$, both in $\mathcal{N}_2$. The variable r_3 (resp. r_4) is an integration variable in first (resp. second) position in $\mathcal{N}_2$ and $x = 1$ (resp. $x = 2$). Then, appending $k + x = 2$ (resp. $k + x = 3$), $\beta^{(2)}$ becomes $\beta^{(2)} = \left(0, 2, \binom{2}{3}\right)$.

This finally leads to the two multi-indices

$$\beta^{(1)} = (1) \in \bar{\mathcal{A}}_{(1)} \quad ; \quad \beta^{(2)} = \left(0, 2, \binom{2}{3}\right) \in \bar{\mathcal{A}}_{(1)}.$$

We recover the two multi-indices that we started with in (6.52). Notice that no other two multi-indices β and γ in $\bar{\mathcal{A}}_{(1)}$ are such that $I_\beta^{(1)} \cdot I_\gamma^{(1)}$ splits into a linear combination in which δ appears and hence the decomposition is unique.

6.4 Expectation and moments of iterated stochastic integrals

In this section, we establish several results concerning the expectation and higher order moments of iterated stochastic integrals. The main tools are Proposition 6.13 and the results of Section 6.3. The results of this section will be useful in Chapter 7, in order to discuss the convergence of the truncated Itô-Taylor expansion for the solution of the stochastic partial differential equation (4.1).

Proposition 6.32. *Let $k \in \mathbb{N}$ and $\beta \in \bar{\mathcal{A}}_{(k)}$. Then, for all $s \in [0,T]$, $\boldsymbol{r}_k \in [0,T]^k$ and all $\boldsymbol{y}_k \in (\mathbb{R}^d)^k$,*

$$I_\beta^{(k)}(s;\boldsymbol{r}_k;\boldsymbol{y}_k) \in L^p(\Omega),$$

for all $p \geqslant 1$.

Proof. Using the notations defined just before Proposition 6.26, let p be an even integer and $\upsilon = \frac{p}{2} \cdot i_k(\beta)$. For $n \in \mathbb{N}$, set $\mathcal{D}_n(\beta) = \mathcal{D}(\beta,\ldots,\beta)$ (n times). Proposition 6.26 shows that

$$\left(I_\beta^{(k)}(s;\boldsymbol{r}_k;\boldsymbol{y}_k)\right)^{\frac{p}{2}} = \sum_{\delta \in \mathcal{D}_{\frac{p}{2}}} c_\delta I_\delta^{(k)}(s;\boldsymbol{r}_k;\boldsymbol{y}_k),$$

where the sum is finite. Hence, taking the L^2-norm on both sides, using the triangle inequality and Proposition 6.13 shows that $I_\beta^{(k)}(s;\boldsymbol{r}_k;\boldsymbol{y}_k) \in L^p(\Omega)$ for all even integers p. Jensen's inequality implies that $I_\beta^{(k)}(s;\boldsymbol{r}_k;\boldsymbol{y}_k) \in L^p(\Omega)$ for all $p \geqslant 1$, so Proposition 6.32 is proved. ∎

From Proposition 6.32, we know that the iterated integral processes have finite moments of any order. We would like to know how to compute those moments. The following result concerns the first order moment of an iterated integral of the form $I_\beta^{(k)}$, $k \in \mathbb{N}$, $\beta \in \bar{\mathcal{A}}_{(k)}$. Let $\mathcal{Z}$ denote the set of multi-indices for which $\beta_i \in \mathbb{Z}^2$, for all $i = 1,\ldots,\ell(\beta)$.

Lemma 6.33. *Let $k \in \mathbb{N}$ and $\beta \in \bar{\mathcal{A}}_{(k)}$. Then, the expectation of $I_\beta^{(k)}(s;\boldsymbol{r}_k;\boldsymbol{x}_k)$ vanishes if $\beta \notin \mathcal{Z}$.*

Remark 6.34. In the case where $\beta \in \mathcal{Z}$, the expectation of $I_\beta^{(k)}(s;\boldsymbol{r}_k;\boldsymbol{x}_k)$ is equal to $I_\beta^{(k)}(s;\boldsymbol{r}_k;\boldsymbol{x}_k)$ itself, as this random variable is deterministic.

Proof. We proceed by induction on $\ell(\beta)$. If $\ell(\beta) = 1$, we must have $\beta_1 \in \mathbb{Z}$ and the process $s \mapsto I_\beta^{(k)}(s;\boldsymbol{r}_k;\boldsymbol{x}_k)$ is a martingale and

$$\mathbb{E}[I_\beta^{(k)}(s;\boldsymbol{r}_k;\boldsymbol{x}_k)] = \mathbb{E}[I_\beta^{(k)}(0;\boldsymbol{r}_k;\boldsymbol{x}_k)] = 0.$$

Then, we suppose that $\mathbb{E}[I_\beta^{(k)}(s;\boldsymbol{r}_k;\boldsymbol{x}_k)] = 0$ for all $\beta \notin \mathcal{Z}$ such that $\ell(\beta) < n$. Fix $\beta \notin \mathcal{Z}$ such that $\ell(\beta) = n$. If $\beta_1 \in \mathbb{Z}$, then by the same argument as in the case $\ell(\beta) = 1$, $\mathbb{E}[I_\beta^{(k)}(s;\boldsymbol{r}_k;\boldsymbol{x}_k)] = 0$. On the other hand, if $\beta_1 = \binom{j}{j'} \in \mathbb{Z}^2$, we have

$$\begin{aligned}
I_\beta^{(k)}(s;\boldsymbol{r}_k;\boldsymbol{x}_k) &= I_{\beta_1}^{(k)}(I_{-\beta}^{(k+2)})(s;\boldsymbol{r}_k;\boldsymbol{x}_k). \\
&= \int_0^s dr \int_{\mathbb{R}^d} dy \int_{\mathbb{R}^d} dz\, \Gamma(r_j - r, x_j - y) f(y-z) \Gamma(r_{j'} - r, x_{j'} - z) \\
&\qquad \times I_{-\beta}^{(k+2)}(r;\boldsymbol{r}_k, r, r;\boldsymbol{x}_k, y, z).
\end{aligned}$$

As $\ell(-\beta) = n-1$ and $-\beta \notin \mathcal{Z}$, $\mathbb{E}[I^{(k+2)}_{-\beta}(s;\boldsymbol{r}_{k+2};\boldsymbol{x}_{k+2})] = 0$, by the induction assumption. Then, by Corollary 6.12, $\mathbb{E}[I^{(k)}_{\beta}(s;\boldsymbol{r}_k;\boldsymbol{x}_k)] = 0$ and the result is proved. ■

Now, due to Lemma 6.33 and Remark 6.34, we will be able to give an expression for the moments of an iterated stochastic integral.

Proposition 6.35. *Let $k \in \mathbb{N}$, $\beta \in \bar{\mathcal{A}}_{(k)}$ and $p \in \mathbb{N}$. Then there exist a finite subset $\mathcal{D}_p(\beta)$ of $\mathcal{B}_{p\cdot i_k(\beta)}$ and constants $(c_\delta)_{\delta\in\mathcal{D}_p(\beta)}$ such that, for all $s \in [0,T]$, $\boldsymbol{r}_k \in [0,T]^{k+1}$, and all $\boldsymbol{y}_k \in (\mathbb{R}^d)^{k+1}$,*

$$\mathbb{E}[I^{(k)}_{\beta}(s;\boldsymbol{r}_k;\boldsymbol{y}_k)^p] = \sum_{\delta\in\mathcal{D}_p(\beta)\cap\mathcal{Z}} c_\delta I^{(k)}_{\delta}(s;\boldsymbol{r}_k;\boldsymbol{y}_k). \tag{6.54}$$

Proof. Fix $p \in \mathbb{N}$. By Corollary 6.28 with $n = p$ and $\beta^{(i)} = \beta$ for all $i = 1, \dots, p$, there exist a finite subset $\mathcal{D}_p(\beta) := \mathcal{D}(\beta, \dots, \beta)$ (p times) of $\mathcal{B}_{p\cdot i_k(\beta)}$ and constants $(c_\delta)_{\delta\in\mathcal{D}_p(\beta)}$ such that

$$I^{(k)}_{\beta}(s;\boldsymbol{r}_k;\boldsymbol{y}_k)^p = \sum_{\delta\in\mathcal{D}_p(\beta)} c_\delta I^{(k)}_{\delta}(s;\boldsymbol{r}_k;\boldsymbol{y}_k). \tag{6.55}$$

Then, taking expectations on both sides of (6.55), we obtain by the linearity of the expectation,

$$\mathbb{E}[I^{(k)}_{\beta}(s;\boldsymbol{r}_k;\boldsymbol{y}_k)^p] = \sum_{\delta\in\mathcal{D}_p(\beta)} c_\delta \mathbb{E}[I^{(k)}_{\delta}(s;\boldsymbol{r}_k;\boldsymbol{y}_k)]. \tag{6.56}$$

By Lemma 6.33, $\mathbb{E}[I^{(k)}_{\delta}(s;\boldsymbol{r}_k;\boldsymbol{y}_k)] = 0$ if $\delta \notin \mathcal{Z}$. Moreover, for $\delta \in \mathcal{Z}$, by Remark 6.34, $\mathbb{E}[I^{(k)}_{\delta}(s;\boldsymbol{r}_k;\boldsymbol{y}_k)] = I^{(k)}_{\delta}(s;\boldsymbol{r}_k;\boldsymbol{y}_k)$. Substituting in (6.56), the result is proved. ■

We now want to prove the following result about iterated integrals with respect to multi-indices in $\mathcal{Z}$.

Lemma 6.36. *Let $k \in \mathbb{N}$ and $\beta \in \bar{\mathcal{A}}_{(k)} \cap \mathcal{Z}$. Let $n = \|\beta\| = 2\ell(\beta)$. Then, for all*

$s \in [0,T]$, $\boldsymbol{r}_k \in [0,T]^{k+1}$, $\boldsymbol{y}_k \in (\mathbb{R}^d)^{k+1}$,

$$
\begin{aligned}
&I_\beta^{(k)}(s; \boldsymbol{r}_k; \boldsymbol{y}_k) \\
&\quad = \int_0^s dr_{k+1} \cdots \int_0^{r_{k+n-1}} dr_{k+n} \prod_{i=1}^{\frac{n}{2}} \delta_0(r_{k+2i-1} - r_{k+2i}) \\
&\qquad\quad \times \prod_{j=1}^{\frac{n}{2}} \int_{\mathbb{R}^d} \mu(d\xi_j)\, \overline{\mathcal{F}\Gamma(r_{\beta_{j,1}} - r_{k+2j})(\xi_j + \eta_j)}\, \mathcal{F}\Gamma(r_{\beta_{j,2}} - r_{k+2j})(\xi_j + \eta_j') \\
&\qquad\qquad \times \prod_{\ell=0}^{k} e^{i\langle y_\ell, \delta_\ell\rangle},
\end{aligned}
\tag{6.57}
$$

where η_j, η_j' are linear combinations of $\xi_{j+1}, \dots, \xi_{\frac{n}{2}}$ ($j = 1, \dots, \frac{n}{2}$), δ_ℓ is a linear combination of $\xi_1, \dots, \xi_{\frac{n}{2}}$ ($\ell = 0, \dots, k$) and $\beta_{j,1}$ (resp. $\beta_{j,2}$) denotes the first (resp. second) component of $\beta_j \in \mathbb{Z}^2$.

Moreover, we have

$$
\sum_{\ell=0}^{k} \delta_\ell = 0. \tag{6.58}
$$

Proof. First consider the case $\|\beta\| = n = 2$. Then, $\beta_{1,1}, \beta_{1,2} \in \{0, \dots, k\}$. By the definition of the iterated integrals (see (6.9)) and the definition of the integral operator when the index is in $\mathbb{Z}^2$ (see (6.8)), we have

$$
\begin{aligned}
&I_\beta^{(k)}(s; \boldsymbol{r}_k; \boldsymbol{y}_k) \\
&\quad = \int_0^s dr_{k+1} \int_{\mathbb{R}^d} dy_{k+1} \int_{\mathbb{R}^d} dy_{k+2}\, \Gamma(r_{\beta_{1,1}} - r_{k+1}, y_{\beta_{1,1}} - y_{k+1}) f(y_{k+1} - y_{k+2}) \\
&\qquad\qquad \times \Gamma(r_{\beta_{1,2}} - r_{k+1}, y_{\beta_{1,2}} - y_{k+2}) \\
&\quad = \int_0^s dr_{k+1} \int_{\mathbb{R}^d} \mu(d\xi) \overline{\mathcal{F}\Gamma(r_{\beta_{1,1}} - r_{k+1})(\xi)} \mathcal{F}\Gamma(r_{\beta_{1,2}} - r_{k+1})(\xi) e^{i\langle \xi, y_{\beta_{1,1}} - y_{\beta_{1,2}}\rangle}.
\end{aligned}
$$

Hence, (6.57) is satisfied with $\eta_1 = \eta_1' = 0$, $\delta_{\beta_{1,1}} = -\delta_{\beta_{1,2}} = \xi$ and $\delta_\ell = 0$ for $\ell \notin \{\beta_{1,1}, \beta_{1,2}\}$. As a consequence, (6.58) is satisfied.

Now, suppose that (6.57) and (6.58) are satisfied for all $k \in \mathbb{N}$ and all $\beta \in \bar{\mathcal{A}}_{(k)}$ such that $\|\beta\| \leqslant n$. We are going to show that they are true for β with $\|\beta\| = n+2$.

According to (6.9) and (6.8), we have

$$
\begin{aligned}
&I_\beta^{(k)}(s;\boldsymbol{r}_k;\boldsymbol{y}_k)\\
&= \int_0^s dr_{k+1}\int_0^{r_{k+1}} dr_{k+2}\,\delta_0(r_{k+1}-r_{k+2})\\
&\quad\times\int_{\mathbb{R}^d} dy_{k+1}\int_{\mathbb{R}^d} dy_{k+2}\,\Gamma(r_{\beta_{1,1}}-r_{k+1},y_{\beta_{1,1}}-y_{k+1})f(y_{k+1}-y_{k+2})\\
&\quad\times\Gamma(r_{\beta_{1,2}}-r_{k+2},y_{\beta_{1,2}}-y_{k+2})I_{-\beta}^{(k+2)}(r_{k+2};\boldsymbol{r}_{k+2};\boldsymbol{y}_{k+2}).
\end{aligned}
$$

As $\|-\beta\|=n$, by the induction assumption, we can write $I_{-\beta}^{(k+2)}$ using (6.57) and we obtain

$$
\begin{aligned}
&I_\beta^{(k)}(s;\boldsymbol{r}_k;\boldsymbol{y}_k)\\
&= \int_0^s dr_{k+1}\int_0^{r_{k+1}} dr_{k+2}\,\delta_0(r_{k+1}-r_{k+2})\int_{\mathbb{R}^d} dy_{k+1}\int_{\mathbb{R}^d} dy_{k+2}\\
&\quad\times\Gamma(r_{\beta_{1,1}}-r_{k+1},y_{\beta_{1,1}}-y_{k+1})f(y_{k+1}-y_{k+2})\Gamma(r_{\beta_{1,2}}-r_{k+2},y_{\beta_{1,2}}-y_{k+2})\\
&\quad\times\int_0^{r_{k+2}} dr_{k+3}\cdots\int_0^{r_{k+n+1}} dr_{k+n+2}\prod_{i=2}^{\frac{n}{2}+1}\delta_0(r_{k+2i-1}-r_{k+2i})\\
&\qquad\times\prod_{j=1}^{\frac{n}{2}}\int_{\mathbb{R}^d}\mu(d\tilde{\xi}_j)\,\overline{\mathcal{F}\Gamma(r_{(-\beta)_{j,1}}-r_{k+2j+2})(\tilde{\xi}_j+\tilde{\eta}_j)}\\
&\qquad\qquad\times\mathcal{F}\Gamma(r_{(-\beta)_{j,2}}-r_{k+2j+2})(\tilde{\xi}_j+\tilde{\eta}'_j)\cdot\left(\prod_{\ell=0}^{k+2}e^{i\langle y_\ell,\tilde{\delta}_\ell\rangle}\right),
\end{aligned}
$$

for some $\tilde{\xi}_j,\tilde{\eta}_j,\tilde{\eta}'_j$ ($j=1,\dots,\frac{n}{2}$) and $\tilde{\delta}_\ell$ ($\ell=1,\dots,k+2$) satisfying the assumptions of (6.57). By (3.6), we can permute the integrals and we obtain

$$
\begin{aligned}
&I_\beta^{(k)}(s;\boldsymbol{r}_k;\boldsymbol{y}_k)\\
&= \int_0^s dr_{k+1}\int_0^{r_{k+1}} dr_{k+2}\int_0^{r_{k+2}} dr_{k+3}\cdots\int_0^{r_{k+n+1}} dr_{k+n+2}\prod_{i=1}^{\frac{n}{2}+1}\delta_0(r_{k+2i-1}-r_{k+2i})\\
&\quad\times\prod_{j=1}^{\frac{n}{2}}\int_{\mathbb{R}^d}\mu(d\tilde{\xi}_j)\overline{\mathcal{F}\Gamma(r_{(-\beta)_{j,1}}-r_{k+2j+2})(\tilde{\xi}_j+\tilde{\eta}_j)}\mathcal{F}\Gamma(r_{(-\beta)_{j,2}}-r_{k+2j+2})(\tilde{\xi}_j+\tilde{\eta}'_j)\\
&\qquad\times\int_{\mathbb{R}^d} dy_{k+1}\int_{\mathbb{R}^d} dy_{k+2}\,\Gamma(r_{\beta_{1,1}}-r_{k+1},y_{\beta_{1,1}}-y_{k+1})f(y_{k+1}-y_{k+2})\\
&\qquad\quad\times\Gamma(r_{\beta_{1,2}}-r_{k+2},y_{\beta_{1,2}}-y_{k+2})\left(\prod_{\ell=0}^{k}e^{i\langle y_\ell,\tilde{\delta}_\ell\rangle}\right)e^{i\langle y_{k+1},\tilde{\delta}_{k+1}\rangle}e^{i\langle y_{k+2},\tilde{\delta}_{k+2}\rangle},\\
&= \int_0^s dr_{k+1}\int_0^{r_{k+1}} dr_{k+2}\int_0^{r_{k+2}} dr_{k+3}\cdots\int_0^{r_{k+n+1}} dr_{k+n+2}\prod_{i=1}^{\frac{n}{2}+1}\delta_0(r_{k+2i-1}-r_{k+2i})\\
&\quad\times\prod_{j=1}^{\frac{n}{2}}\int_{\mathbb{R}^d}\mu(d\tilde{\xi}_j)\overline{\mathcal{F}\Gamma(r_{\beta_{j+1,1}}-r_{k+2j+2})(\tilde{\xi}_j+\tilde{\eta}_j)}\mathcal{F}\Gamma(r_{\beta_{j+1,2}}-r_{k+2j+2})(\tilde{\xi}_j+\tilde{\eta}'_j)\\
&\qquad\times\int_{\mathbb{R}^d}\mu(d\xi)\overline{\mathcal{F}\Gamma(r_{\beta_{1,1}}-r_{k+1})(\xi+\tilde{\delta}_{k+1})}\mathcal{F}\Gamma(r_{\beta_{1,2}}-r_{k+2})(\xi-\tilde{\delta}_{k+2})\\
&\qquad\times\left(\prod_{\ell=0}^{k}e^{i\langle y_\ell,\tilde{\delta}_\ell\rangle}\right)e^{i\langle y_{\beta_{1,1}},\xi+\tilde{\delta}_{k+1}\rangle}e^{i\langle y_{\beta_{1,2}},-\xi+\tilde{\delta}_{k+2}\rangle}.
\end{aligned}
$$

Hence, setting $\xi_1=\xi$, $\xi_j=\tilde{\xi}_{j-1}$ ($j=2,\dots,\frac{n}{2}+1$), $\eta_1=\xi+\tilde{\delta}_{k+1}$, $\eta'_1=\xi-\tilde{\delta}_{k+2}$, $\eta_j=\tilde{\eta}_{j-1}$, $\eta'_j=\tilde{\eta}'_{j-1}$ ($j=2,\dots,\frac{n}{2}+1$) and $\delta_{\beta_{1,1}}=\tilde{\delta}_{\beta_1,1}+\xi+\tilde{\delta}_{k+1}$, $\delta_{\beta_{1,2}}=\tilde{\delta}_{\beta_{1,2}}-\xi+\tilde{\delta}_{k+2}$, $\delta_\ell=\tilde{\delta}_\ell$ ($\ell=0,\dots,k$, $\ell\notin\{\beta_{1,1},\beta_{1,2}\}$), (6.57) is satisfied. Moreover, we have

$$\sum_{\ell=0}^{k}\delta_\ell=\sum_{\ell=0}^{k+2}\tilde{\delta}_\ell+\xi-\xi=0$$

and (6.58) is also satisfied. As a consequence, Lemma 6.36 is established. ∎

The next results will be used in Chapter 7 to discuss the convergence of the truncated Itô-Taylor expansion of the solution of the stochastic partial differential equation (4.1).

Proposition 6.37. *Let $\beta,\gamma\in\bar{\mathcal{A}}_{(0)}$. Then, for all $s,t_1,t_2\in[0,T]$, the function $\rho_{\beta,\gamma}:\mathbb{R}^d\times\mathbb{R}^d\to\mathbb{R}$ defined by*

$$\rho_{\beta,\gamma}(x,y)=\mathbb{E}[I_\beta^{(0)}(s;t_1,x)I_\gamma^{(0)}(s;t_2,y)]$$

is a function of $x-y$.

Proof. By Lemma 6.21, we have

$$I_\beta^{(0)}(s;t_1,x) = I_{A_0(\beta)}^{(1)}(s;t_1,t_2;x,y)$$

and, using the operator T_k defined in (6.38) and Remark 6.20,

$$I_\gamma^{(0)}(s;t_2,y) = I_{A_0(\gamma)}^{(1)}(s;t_2,t_1;y,x) = I_{T_0(A_0(\gamma))}^{(1)}(s;t_1,t_2;x,y) = I_{A_{-1}(\gamma)}^{(1)}(s;t_1,t_2,x,y).$$

Notice that $A_0(\beta), A_{-1}(\gamma) \in \bar{\mathcal{A}}_{(1)}$. Hence,

$$\begin{aligned}
\rho_{\beta,\gamma}(x,y) &= \mathbb{E}[I_\beta^{(0)}(s;t_1,x) I_\gamma^{(0)}(s;t_2,y)] \\
&= \mathbb{E}[I_{A_0(\beta)}^{(1)}(s;t_1,t_2;x,y) I_{A_{-1}(\gamma)}^{(1)}(s;t_1,t_2;x,y)].
\end{aligned}$$

Then, by Proposition 6.26 and Lemma 6.33, there exist a finite subset

$$\mathcal{D}(A_0(\beta), A_{-1}(\gamma)) \subset \mathcal{B}_{i_1(A_0(\beta))+i_1(A_{-1}(\gamma))}$$

and constants $(c_\delta)_{\delta\in\mathcal{D}(A_0(\beta),A_{-1}(\gamma))}$, such that

$$\mathbb{E}[I_{A_0(\beta)}^{(1)}(s;t_1,t_2;x,y) I_{A_{-1}(\gamma)}^{(1)}(s;t_1,t_2;x,y)] = \sum_{\delta\in\mathcal{D}(A_0(\beta),A_{-1}(\gamma))\cap\mathcal{Z}} c_\delta I_\delta^{(1)}(s;t_1,t_2;x,y). \tag{6.59}$$

Now, by Lemma 6.36, for $\delta \in \bar{\mathcal{A}}_{(1)} \cap \mathcal{Z}$, (6.57) is satisfied for $k=1$. But, by (6.58), $\delta_0 = -\delta_1$ and the product of exponentials on the right-hand side of (6.57) can be written as $e^{i\langle\delta_0, x-y\rangle}$. As a consequence, $I_\delta^{(1)}(s;t_1,t_2;x,y)$ is a function of $x-y$ for all $\delta \in \bar{\mathcal{A}}_{(1)} \cap \mathcal{Z}$. As the sum in (6.59) is finite, $\rho_{\beta,\gamma}(x,y)$ is also a function of $x-y$. ■

Corollary 6.38. *Let $\beta \in \bar{\mathcal{A}}_{(0)}$, then, for all $t \in [0,T]$, the process $s \mapsto I_\beta^{(0)}(s;t,x)$ has a spatially homogeneous covariance function. Hence, it satisfies hypothesis (H).*

Proof. Just consider Proposition 6.37 with $\gamma = \beta$ and $t_1 = t_2 = t$. ■

Corollary 6.39. *Let $\beta^{(1)}, \ldots, \beta^{(n)} \in \bar{\mathcal{A}}_{(0)}$ and $c_1, \ldots, c_n \in \mathbb{R}$. Then, for all $t \in [0,T]$, the process $s \mapsto \sum_{j=1}^n c_j I_{\beta^{(j)}}^{(0)}(s;t,x)$ has a spatially homogeneous covariance function. Hence, it satisfies hypothesis (H).*

Proof. We have

$$
\begin{aligned}
&\mathbb{E}\left[\left(\sum_{i=1}^{n} c_i I^{(0)}_{\beta(i)}(s;t,x)\right)\left(\sum_{\ell=1}^{n} c_j I^{(0)}_{\beta(j)}(s;t,y)\right)\right] \\
&\quad = \sum_{i=1}^{n}\sum_{j=1}^{n} c_i c_j \mathbb{E}[I^{(0)}_{\beta(i)}(s;t,x) I^{(0)}_{\beta(j)}(s;t,y)]. \qquad (6.60)
\end{aligned}
$$

By Proposition 6.37, each term in the finite sum on the right-hand side of (6.60) is a function of $x - y$. The result is established. ■

Chapter 7

Itô-Taylor expansion

In this chapter, we are going to establish an expansion for the process $u(t,x)$ as a series of iterated integrals. This expansion is said to be of Itô-Taylor type, because it is based on a recursive use of Itô's formula in order to obtain a series expansion. A similar method was already used in [14] to develop Itô-Taylor type expansions for solutions of stochastic differential equations driven by a Brownian motion. The situation here is more involved. First, we have to deal with a stochastic partial differential equation instead of a stochastic differential equation and, hence, the processes are multi-parameter processes. Secondly, the integral form of the equation which gives rise to the expression for the solution is in a convolution form. As a consequence, the time parameter t of the process appears in the bounds of the integral (as in the case of stochastic differential equations) but also in the integrand. This explains why the multi-indices and the variables of the iterated integrals defined in Chapter 6 are more complex in our case.

We will begin by stating an n-th order expansion with a remainder term (truncated Itô-Taylor expansion), corresponding to [14, Theorem 5.5.1] for an SDE. After that, assuming that the series converges in a suitable manner, we will explain why it should be equal to the solution $u(t,x)$ of the stochastic partial differential equation (4.1) constructed in Theorem 4.2. Nevertheless, the convergence of the series or convergence to 0 of the remainder term is still not established, and therefore, this last statement is still presented as a conjecture. Our intention is to explain where the main difficulties are and why the question of convergence of the series is still an open problem.

7.1 Truncated Itô-Taylor expansion

In order to state the Itô-Taylor expansion, we need some differential operators. Let $\alpha : \mathbb{R} \to \mathbb{R}$ be the Lipschitz function appearing in the non-linearity of the stochastic partial differential equation (4.1). Suppose in addition that $\alpha \in C^\infty(\mathbb{R},\mathbb{R})$ and that, for all $k \in \mathbb{N}$, there exists a constant C_k such that $\sup_{x\in\mathbb{R}^d} |\alpha^{(k)}(x)| \leqslant C_k$. (In particular, α is bounded.) For example, we can take $\alpha(x) = \cos(x)$ or $\alpha(x) = e^{-x^2}$.

For $i \in \mathbb{N}$, let $D_i : \bigcup_{n\geqslant i} \left(C^\infty(\mathbb{R},\mathbb{R})\right)^{n+1} \longrightarrow \bigcup_{n\geqslant i} \left(C^\infty(\mathbb{R},\mathbb{R})\right)^{n+2}$ be defined by

$$D_i((g_0,\dots,g_n)) = (g_0,\dots,g_{i-1},g_i',g_{i+1},\dots,g_n,\alpha).$$

For $i,j \in \mathbb{N}$, $i \leqslant j$, let $D_{\binom{i}{j}} : \bigcup_{n\geqslant j} \left(C^\infty(\mathbb{R},\mathbb{R})\right)^{n+1} \longrightarrow \bigcup_{n\geqslant j} \left(C^\infty(\mathbb{R},\mathbb{R})\right)^{n+3}$ be defined by

$$D_{\binom{i}{j}}((g_0,\dots,g_n)) = \begin{cases} (g_0,\dots,g_{i-1},g_i',g_{i+1},\dots,g_{j-1},g_j',g_{j+1},\dots,g_n,\alpha,\alpha) & \text{if } i<j, \\ (g_0,\dots,g_{i-1},g_i'',g_{i+1},\dots,g_n,\alpha,\alpha) & \text{if } i=j. \end{cases}$$

If $i > j$, we set $D_{\binom{i}{j}} = D_{\binom{j}{i}}$.

Then, for $\beta \in \bar{\mathcal{A}}^*$ and $g \in C^\infty(\mathbb{R},\mathbb{R})$, we set $D_\emptyset(g) = g$ and

$$D_\beta(g) = D_{\beta_{\ell(\beta)}} \circ D_{\beta_{\ell(\beta)-1}} \circ \cdots \circ D_{\beta_1}((g)),$$

if $\ell(\beta) \geqslant 1$. The composition of the differential operators is well-defined as soon as α is a C^∞ function.

Moreover, as $D_\beta(g)$ has $\|\beta\|+1$ components, let

$$\pi_\beta(g) = \frac{1}{2^{\|\beta\|-\ell(\beta)}} \prod_{j=0}^{\|\beta\|} \left(D_\beta(g)\right)_j(0).$$

The effect of the constant in front of the product is just to divide it by 2 for each component of β that belongs in $\mathbb{Z}^2$. This takes into account the fact that a factor $\frac{1}{2}$ appears in Itô's formula for each second derivative term (see (7.10)). A way to avoid these powers of 2 would be to consider only components of the form $\binom{i}{j}$ with $i \leqslant j$ in the multi-indices.

Example 7.1. Let us consider $\beta = \left(0,1,\binom{1}{2}\right)$ and g a C^∞ function. Then

$$\begin{aligned} D_\beta(g) &= D_{\binom{1}{2}}(D_1(D_0((g)))) = D_{\binom{1}{2}}(D_1((g',\alpha))) \\ &= D_{\binom{1}{2}}((g',\alpha',\alpha)) = (g',\alpha'',\alpha',\alpha,\alpha) \end{aligned}$$

and

$$\pi_\beta(g) = \frac{1}{2} g'(0)\alpha''(0)\alpha'(0)\alpha(0)^2.$$

We write

$$M(s;t,x) = \int_0^s \int_{\mathbb{R}^d} \Gamma(t-r,x-y)\alpha(u(r,y))M(dr,dy),$$

where $u(t,x)$ is the solution of the stochastic partial differential equation (4.1)with the function $\beta \equiv 0$. Then, we have $u(t,x) = M(t;t,x)$. Those notations are similar to those defined in Section 5.1.

Finally, for $\beta \in \bar{\mathcal{A}}^*$ such that $\|\beta\| = n$, let $\kappa_\beta(g) : [0,T]^{n+1} \times (\mathbb{R}^d)^{n+1} \to \mathbb{R}$ be a random function defined by

$$\kappa_\beta(g)(\boldsymbol{r}_n; \boldsymbol{x}_n) = \frac{1}{2^{\|\beta\|-\ell(\beta)}} \prod_{j=0}^{n} \left(D_\beta(g)\right)_j \left(M(r_n; r_j, x_j)\right). \tag{7.1}$$

Notice that $\kappa_\beta(g)(\boldsymbol{r}_{n-1}, 0; \boldsymbol{x}_n) = \pi_\beta(g)$.

We are now ready to state the main result of this chapter.

Theorem 7.2 (Itô-Taylor expansion of order n)**.** *Let $u(t,x)$ be the solution of the stochastic partial differential equation (4.1) with α a Lipschitz, C^∞ function and $\beta \equiv 0$ such that for all $k \in \mathbb{N}$, there exists a constant C_k such that $\sup_{x\in\mathbb{R}^d} |\alpha^{(k)}(x)| \leqslant C_k$. (In particular α is bounded.) Let $g : \mathbb{R} \to \mathbb{R}$ be a C^∞ function such that for all $k \in \mathbb{N}^*$, there exists a constant C'_k such that $\sup_{x\in\mathbb{R}^d} |g^{(k)}(x)| < C'_k$. Then, for all $n \geqslant 0$,*

$$g(M(s;t,x)) = \sum_{i=0}^{n} \sum_{\beta \in \bar{\mathcal{A}}^{(i)}} \pi_\beta(g) I_\beta^{(0)}(s;t,x) + \sum_{\beta \in \bar{\mathcal{A}}^{(n+1)}} J_\beta(\kappa_\beta(g))(s;t,x), \tag{7.2}$$

where $I_\beta^{(0)}$ are the iterated stochastic integrals (see Section 6.2) and J_β are the operators defined in Remark 6.15. We recall that $\bar{\mathcal{A}}^{(i)} \subset \bar{\mathcal{A}}_{(0)}$.

Remark 7.3. The assumptions on α and g in Theorem 7.2 are not optimal. Indeed, we must choose the functions α and g so that (4.1) admits a solution and such that the integrals $J_\beta(\kappa_\beta(g))$ are well-defined.

Proof. As in (5.8), let

$$M^{(m,k)}(s;t,x) = \int_0^s \int_{\mathbb{R}^d} \Gamma_{m,k}(t-r, x-y)\alpha(u(r,y))M(dr,dy)$$

denote the process in which we have replaced Γ by its approximation $\Gamma_{m,k}$ (see the proof of Theorem 3.1). By the definition of the stochastic integral, we know that $M^{(m,k)}(s;t,x)$ converges in $L^2(\Omega)$ to $M(s;t,x)$ as $k \to \infty$ and $m \to \infty$. Moreover, if $(X(r))_{r\in[0,T]}$ is an almost-surely bounded real-valued stochastic process, then

$$\int_0^s X(\rho) M^{(m,k)}(d\rho;t,x) \xrightarrow[m,k\to\infty]{} \int_0^s X(\rho) M(d\rho;t,x).$$

in $L^2(\Omega)$. To see this, just bound $X(\rho)$ in the second order moment of the difference. This leads to the same expression as for $\mathbb{E}[|M^{(m,k)}(s;t,x) - M(s;t,x)|^2]$ and, hence, converges to 0 as $m, k \to \infty$. The main point is that the process X does not depend on the spatial variable x. Finally, by Theorem 2.13, we know that

$$\begin{aligned}
&\langle M^{(m,k)}(\cdot; t, x)\rangle_s \\
&\quad = \int_0^s dr \int_{\mathbb{R}^d} dy \int_{\mathbb{R}^d} dz \, \Gamma_{m,k}(t-r, x-y) f(x-y) \Gamma_{m,k}(t-r, x-z) \\
&\qquad\qquad \times \alpha(M(r; r, y))\alpha(M(r; r, z)),
\end{aligned} \tag{7.3}$$

which converges in $L^1(\Omega)$ to

$$\int_0^s dr \int_{\mathbb{R}^d} dy \int_{\mathbb{R}^d} dz \, \Gamma(t-r, x-y) f(x-y) \Gamma(t-r, x-z) \alpha(M(r; r, y))\alpha(M(r; r, z)). \tag{7.4}$$

Indeed, the $L^1(\Omega)$-limit of the left-hand side of (7.3) is taken (in (6.20)) as the definition of the integral (7.4). Hence,

$$\begin{aligned}
\langle M(\cdot; t, x)\rangle_s &= \int_0^s dr \int_{\mathbb{R}^d} dy \int_{\mathbb{R}^d} dz \, \Gamma(t-r, x-y) f(x-y) \Gamma(t-r, x-z) \\
&\qquad \times \alpha(M(r; r, y))\alpha(M(r; r, z)).
\end{aligned} \tag{7.5}$$

Fix $t \in [0, T]$ and $x \in \mathbb{R}^d$. As $s \mapsto M(s; t, x)$ is a continuous martingale, we are going to use Itô's formula to express $g(M(s; t, x))$. As $M(0; t, x) = 0$, we have

$$\begin{aligned}
&g(M(s; t, x)) \\
&\quad = g(0) + \int_0^s \int_{\mathbb{R}^d} \Gamma(t-r, x-y) g'(M(r; t, x))\alpha(M(r; r, y)) M(dr, dy) \\
&\qquad + \frac{1}{2} \int_0^s dr \int_{\mathbb{R}^d} dy \int_{\mathbb{R}^d} dz \, \Gamma(t-r, x-y) f(y-z) \Gamma(t-r, x-z) \\
&\qquad\qquad \times g''(M(r; t, x))\alpha(M(r; r, y))\alpha(M(r; r, z)).
\end{aligned} \tag{7.6}$$

As g' ang g'' are bounded, the two integrals are well-defined by the arguments above. As $\bar{\mathcal{A}}^{(0)} = \{\emptyset\}$, $\pi_\emptyset(g) = g(0)$ and $I_\emptyset^{(0)}(s; t, x) \equiv 1$,

$$g(0) = \sum_{\beta \in \bar{\mathcal{A}}^{(0)}} \pi_\beta(g) I_\beta^{(0)}(s; t, x),$$

corresponding to the first term on the right-hand side of (7.2) for $n = 0$. Moreover, $\bar{\mathcal{A}}^{(1)} = \left\{0; \binom{0}{0}\right\}$ and so we consider the two values of $\beta \in \bar{\mathcal{A}}^{(1)} : \beta = (0)$ and $\beta = \left(\binom{0}{0}\right)$. As $\ell(\beta) = 1$ for $\beta \in \bar{\mathcal{A}}^{(1)}$,

$$\kappa_0(g)(r_0, r_1; y_0, y_1) = g'(M(r_1; r_0, y_0))\alpha(M(r_1; r_1, y_1))$$

and

$$
\begin{aligned}
&J_0(\kappa_0(g))(s; r_0, y_0)\\
&= I_0^{(0)}(\kappa_0(g))(s; r_0, y_0)\\
&= \int_0^s \int_{\mathbb{R}^d} \Gamma(r_0 - r_1, y_0 - y_1) g'(M(r_1; r_0, y_0))\alpha(M(r_1; r_1, y_1))M(dr_1, dy_1).
\end{aligned}
$$

Further,

$$
\kappa_{\binom{0}{0}}(g)(r_0, r_1, r_2; y_0, y_1, y_2) = \frac{1}{2} g''(M(r_2; r_0, y_0)\alpha(M(r_2; r_1, y_1))\alpha(M(r_2; r_2, y_2))
$$

and

$$
\begin{aligned}
&J_{\binom{0}{0}}(\kappa_{\binom{0}{0}}(g))(s; r_0, y_0)\\
&= I^{(0)}_{\binom{0}{0}}(\kappa_{\binom{0}{0}}(g))(s; r_0, y_0)\\
&= \frac{1}{2} \int_0^s dr_1 \int_0^s dr_2\, \delta_0(r_1 - r_2) \int_{\mathbb{R}^d} dy_1 \int_{\mathbb{R}^d} dy_2\, \Gamma(r_0 - r_1, y_0 - y_1) f(y_1 - y_2)\\
&\qquad \times \Gamma(r_0 - r_2, y_0 - y_2) g''(M(r_2; r_0, y_0))\alpha(M(r_2; r_1, y_1))\alpha(M(r_2; r_2, y_2)).
\end{aligned}
$$

Hence, the second term on the right-hand side of (7.2) for $n = 0$, namely

$$
\sum_{\beta \in \bar{\mathcal{A}}^{(1)}} J_\beta(\kappa_\beta(g))(s; r_0, x_0)
$$

corresponds to the sum of the two integrals above. Together with (7.6), this shows that (7.2) is valid for $n = 0$.

We would like to show by induction that (7.2) is valid for all $n \in \mathbb{N}$. Having shown that the case $n = 0$ is valid, suppose (7.2) is true for all integers less than or equal to $n - 1$ for some $n \in \mathbb{N}$. We are going to prove that it is then true for n. By the induction assumption, we know that

$$
g(M(s; t, x)) = \sum_{i=0}^{n-1} \sum_{\beta \in \bar{\mathcal{A}}^{(i)}} \pi_\beta(g) I_\beta^{(0)}(s; t, x) + \sum_{\beta \in \bar{\mathcal{A}}^{(n)}} J_\beta(\kappa_\beta(g))(s; t, x). \tag{7.7}
$$

Hence, we have to show that the remainder term can be written as

$$
\sum_{\beta \in \bar{\mathcal{A}}^{(n)}} J_\beta(\kappa_\beta(g))(s; t, x) = \sum_{\beta \in \bar{\mathcal{A}}^{(n)}} \pi_\beta(g) I_\beta^{(0)}(s; t, x) + \sum_{\beta \in \bar{\mathcal{A}}^{(n+1)}} J_\beta(\kappa_\beta(g))(s; t, x). \tag{7.8}
$$

For $\beta \in \bar{\mathcal{A}}^{(n)}$,

$$\kappa_\beta(g)(\boldsymbol{r}_{\|\beta\|}; \boldsymbol{y}_{\|\beta\|}) = \frac{1}{2^{\|\beta\| - \ell(\beta)}} \prod_{j=0}^{\|\beta\|} (D_\beta(g))_j \, (M(r_{\|\beta\|}; r_j, y_j)).$$

Due to the particular form of the function $\kappa_\beta(g)$, we are first going to write Itô's formula for a function of the form

$$\prod_{j=1}^{m} h_i(M(s; r_i, y_i)), \tag{7.9}$$

where $m \geqslant 1$ and h_i $(i = 1, \ldots, m)$ are bounded C^∞ functions. We have

$$\begin{aligned}
\prod_{i=0}^{m} h_i(M(s; r_i, y_i)) &= \prod_{i=0}^{m} h_i(0) \\
&\quad + \sum_{i=0}^{m} \int_0^s \int_{\mathbb{R}^d} \left(\prod_{\substack{j=0 \\ j \neq i}}^{m} h_j(M(r; r_j, y_j)) \right) h_i'(M(r; r_i, y_i)) \alpha(M(r, r, y)) \\
&\qquad \times \Gamma(r_i - r, y_i - y) M(dr, dy) \\
&\quad + \frac{1}{2} \sum_{i=0}^{m} \sum_{\substack{j=0 \\ j \neq i}}^{m} \int_0^s dr \int_{\mathbb{R}^d} dy \int_{\mathbb{R}^d} dz \left(\prod_{\substack{k=0 \\ k \notin \{i,j\}}}^{m} h_k(M(r; r_k, y_k)) \right) \\
&\qquad \times h_i'(M(r; r_i, y_i)) h_j'(M(r; r_j, y_j)) \alpha(M(r; r, y)) \alpha(M(r; r, z)) \\
&\qquad \times \Gamma(r_i - r, y_i - y) f(y - z) \Gamma(r_j - r, y_j - z) \\
&\quad + \frac{1}{2} \sum_{i=0}^{m} \int_0^s dr \int_{\mathbb{R}^d} dy \int_{\mathbb{R}^d} dz \left(\prod_{\substack{k=0 \\ k \neq i}}^{m} h_k(M(r; r_k, y_k)) \right) h_i''(M(r; r_i, y_i)) \\
&\qquad \times \alpha(M(r; r, y)) \alpha(M(r; r, z)) \Gamma(r_i - r, y_i - y) f(y - z) \Gamma(r_i - r, y_i - z) \\
&= \prod_{i=0}^{m} h_i(0) + \sum_{i=0}^{m} I_i^{(m)}(\kappa_i((h_0, \ldots, h_m)))(s; \boldsymbol{r}_m; \boldsymbol{y}_m) \\
&\quad + \sum_{i=0}^{m} \sum_{j=0}^{m} I_{\binom{i}{j}}^{(m)}(\kappa_{\binom{i}{j}}((h_0, \ldots, h_m)))(s; \boldsymbol{r}_m; \boldsymbol{y}_m),
\end{aligned} \tag{7.10}$$

where

$$\kappa_i((h_0,\ldots,h_m))(\boldsymbol{r}_{m+1};\boldsymbol{y}_{m+1}) = \prod_{j=0}^{m+1}\left(D_i((h_0,\ldots,h_m))\right)_j\left(M(r_{m+1};r_j,y_j)\right)$$

and

$$\kappa_{\binom{i}{j}}((h_0,\ldots,h_m))(\boldsymbol{r}_{m+2};\boldsymbol{y}_{m+2}) = \frac{1}{2}\prod_{k=0}^{m+2}\Big(D_{\binom{i}{j}}((h_0,\ldots,h_m))\Big)_k\left(M(r_{m+2};r_k,y_k)\right).$$

Now, fix $\beta \in \bar{\mathcal{A}}^{(n)}$. We apply (7.10) to the function $\kappa_\beta(r_0,\ldots,r_{\|\beta\|};y_0,\ldots,y_{\|\beta\|})$, which, except for the constant $2^{-\|\beta\|+\ell(\beta)}$, is of the corresponding product form (7.9) above, with $h_i = D_\beta(g)_i$ ($i=1,\ldots,\|\beta\|$). Moreover, as g and α are C^∞ functions with bounded derivatives, these h_i ($i=1,\ldots,n$) are bounded C^∞ functions. Letting $m=\|\beta\|$, we obtain

$$\begin{aligned}
&\kappa_\beta(g)(r_0,\ldots,r_m;y_0,\ldots,y_m)\\
&= \frac{1}{2^{\|\beta\|-\ell(\beta)}}\prod_{i=0}^{m}\left(D_\beta(g)\right)_j\left(M(r_m;r_i,y_i)\right)\\
&= \frac{1}{2^{\|\beta\|-\ell(\beta)}}\prod_{i=0}^{m}\left(D_\beta(g)\right)_i(0) + \frac{1}{2^{\|\beta\|-\ell(\beta)}}\sum_{i=0}^{m} I_i^{(m)}(\kappa_i(D_\beta(g)))(r_m;\boldsymbol{r}_m;\boldsymbol{y}_m)\\
&\qquad + \frac{1}{2^{\|\beta\|-\ell(\beta)}}\sum_{i=0}^{m}\sum_{j=0}^{m} I_{\binom{i}{j}}^{(m)}(\kappa_{\binom{i}{j}}(D_\beta(g)))(r_m;\boldsymbol{r}_m;\boldsymbol{y}_m).
\end{aligned}\tag{7.11}$$

Replacing $\kappa_\beta(g)$ from (7.11) in the left-hand side of (7.8) and using the linearity of the operators J_β, we have

$$\begin{aligned}
&J_\beta(\kappa_\beta(g))(s;t,x)\\
&= \frac{1}{2^{\|\beta\|-\ell(\beta)}}\left(\prod_{i=0}^{m}\left(D_\beta(g)\right)_i(0)\right)J_\beta(1)(s;t,x)\\
&\qquad + \frac{1}{2^{\|\beta\|-\ell(\beta)}}\sum_{i=0}^{m} J_\beta(\tilde{I}_i^{(m)}(\kappa_i(D_\beta(g))))(s;t,x)\\
&\qquad + \frac{1}{2^{\|\beta\|-\ell(\beta)}}\sum_{i=0}^{m}\sum_{j=0}^{m} J_\beta(\tilde{I}_{\binom{i}{j}}^{(m)}(\kappa_{\binom{i}{j}}(D_\beta(g))))(s;t,x)\\
&= \pi_\beta(g)I_\beta^{(0)}(s;t,x) + \sum_{i=0}^{m} J_{\beta,i}(\kappa_{\beta,i}(g))(s;t,x)\\
&\qquad + \sum_{i=0}^{m}\sum_{j=0}^{m} J_{\beta,\binom{i}{j}}(\kappa_{\beta,\binom{i}{j}}(g))(s;t,x),
\end{aligned}$$

by the definition of the operators J_β and the definitions of κ_j and $\kappa_{\binom{i}{j}}$.

Then, summing on $\beta \in \bar{\mathcal{A}}^{(n)}$, we cover all indices of the form (β, i) $(\beta \in \bar{\mathcal{A}}^{(n)}, i \leqslant \|\beta\|)$ or of the form $\left(\beta, \binom{i}{j}\right)$ $(\beta \in \bar{\mathcal{A}}^{(n)}, i, j \leqslant \|\beta\|)$. These cover all elements of $\bar{\mathcal{A}}^{(n+1)}$. Hence,

$$\sum_{\beta \in \bar{\mathcal{A}}^{(n)}} J_\beta(\kappa_\beta(g))(s;t,x) = \sum_{\beta \in \bar{\mathcal{A}}^{(n)}} \pi_\beta(g) I_\beta^{(0)}(s;t,x) + \sum_{\beta \in \bar{\mathcal{A}}^{(n+1)}} J_\beta(\kappa_\beta(g))(s;t,x). \tag{7.12}$$

Replacing (7.12) in (7.7), we see that (7.2) is valid for n. By induction, Theorem 7.2 is proved. ■

Remark 7.4. Notice that the definitions and results of Chapter 6 are still valid if the spatial dimension is less or equal to 3. Some of the proofs can be simplified, but the arguments are essentially the same. As a consequence, Theorem 7.2 is also valid in this case.

7.2 Remarks on the asymptotic behavior of the Itô-Taylor expansion

A natural question that arises from the result of Theorem 7.2 concerns the behavior of (7.2) as n goes to $+\infty$. A standard way of studying this is to study the second order moment of the remainder term $\sum_{\beta \in \bar{\mathcal{A}}^{(n)}} J_\beta(\kappa_\beta(g))(s;t,x)$ and to show that this term goes to 0. However, in the case of a general Schwartz distribution Γ, the Lebesgue integrals appearing in the remainder term of order n are quadratic variations of stochastic integrals. Hence, by Burkholder's inequality, their second order moments are bounded above and below by the fourth order moments of the stochastic integrals, for which we have no estimates, except in the case where α is affine. As a consequence, it is not possible to estimate directly second order moments of the remainder term in the general case. Moreover, we are not able to prove the convergence as $n \to +\infty$ of the first term of the right-hand side of (7.2) by a direct computation using the triangle inequality and available bounds on $I_\beta^{(0)}$. Indeed, we know that the second order moment of $I_\beta^{(0)}$ is of order $\frac{s^{\|\beta\|}}{\|\beta\|!}$ and the number of multi-indices with fixed $\|\beta\|$ in the sum is of order $\|\beta\|!$. Hence, summing over β, the series does not converge in the general case. In the case of a spatial dimension less or equal to 3, a direct estimation of the moments of the remainder term is possible, but we can not show that this term goes to 0. Indeed, we come back to the convergence of the same series as mentionned above.

Nevertheless, despite the absence of a proof of the convergence of the series, some results suggest that it should converge to the solution of the stochastic partial differential equation (4.1) for some specific functions α, namely in the case where α is analytic and bounded, for example $\alpha(x) = \cos(x)$ or $\alpha(x) = e^{-x^2}$. In this case, the order $\|\beta\|!$ above is far from optimal. We are still not able to establish the optimal order for the series. However, we are able to state an argument that leads us to think that the series should coverge to the solution $u(t,x)$ of (4.1) and we expect that it will be possible to give a meaning to the expression

$$\sum_{i=0}^{\infty} \sum_{\beta \in \mathcal{A}^{(i)}} \pi_\beta(g) I_\beta^{(0)}(t;t,x). \tag{7.13}$$

Consider $g(x) = x$ and suppose that the series (7.13) converges in some sense (namely, in $L^2(\Omega)$) in order that, for each $t \in [0,T]$ and each $x \in \mathbb{R}^d$, it defines a random variable $v(t,x)$ given by

$$v(t,x) = \sum_{i=0}^{\infty} \sum_{\beta \in \bar{\mathcal{A}}^{(i)}} \pi_\beta(\mathrm{id}) I_\beta^{(0)}(t;t,x).$$

We know that $\bigcup_{i=0}^{\infty} \bar{\mathcal{A}}^{(i)} = \bar{\mathcal{A}}^* = \bar{\mathcal{A}}_{(0)}$. Moreover, as $g^{(k)}(x) \equiv 0$ for any $k \geqslant 2$, $\pi_\beta(\mathrm{id}) \neq 0$ only if $\beta \in \mathcal{B}_1$. Hence, setting $\mathcal{C} := \bar{\mathcal{A}}_{(0)} \cap \mathcal{B}_1$, we have

$$v(t,x) = \sum_{\beta \in \mathcal{C}} \pi_\beta(\mathrm{id}) I_\beta^{(0)}(t;t,x). \tag{7.14}$$

If we suppose that $\alpha : \mathbb{R} \to \mathbb{R}$ is an analytic function, then

$$\alpha(x) = \sum_{k=0}^{\infty} \frac{\alpha^{(k)}(0)}{k!} x^k,$$

for all $x \in \mathbb{R}$. Hence,

$$\alpha(v(t,x)) = \sum_{k=0}^{\infty} \frac{\alpha^{(k)}(0)}{k!} \left(\sum_{\beta \in \mathcal{C}} \pi_\beta(\mathrm{id})\, I_\beta^{(0)}(t;t,x) \right)^k. \tag{7.15}$$

The set $\mathcal{C}$ is countable and, using the fact that

$$\left(\sum_{i=0}^{\infty} x_i \right)^k = \sum_{i_1=0}^{\infty} \cdots \sum_{i_k=0}^{\infty} x_{i_1} \cdots x_{i_k},$$

we have

$$\alpha(v(t,x)) = \sum_{k=0}^{\infty} \frac{\alpha^{(k)}(0)}{k!} \left(\sum_{\beta^{(1)} \in \mathcal{C}} \cdots \sum_{\beta^{(k)} \in \mathcal{C}} \prod_{i=1}^{k} \pi_{\beta^{(i)}}(\mathrm{id}) I_{\beta^{(i)}}^{(0)}(t;t,x) \right). \tag{7.16}$$

By Corollary 6.29, each product of iterated stochastic integrals in (7.16) can be written as a sum of iterated stochastic integrals. As $\beta^{(i)} \in \mathcal{B}_1$ for $i = 1, \dots, k$ (that is, they only have one occurence of a '0' value), the multi-indices appearing in the sum in Proposition 6.26 are all in $\mathcal{B}_k$ (that is, they have exactly k occurences of '0' values). Hence, let $\tilde{\mathcal{C}}$ be the set of multi-indices β such that there exist $k \in \mathbb{N}^*$ and a sequence $\beta^{(1)}, \dots, \beta^{(k)} \in \mathcal{C}$ with $\beta \in \mathcal{D}(\beta^{(1)}, \dots, \beta^{(k)})$, namely

$$\tilde{\mathcal{C}} = \bigcup_{k=0}^{\infty} \bigcup_{\beta^{(1)},\dots,\beta^{(k)} \in \mathcal{C}} \mathcal{D}(\beta^{(1)}, \dots, \beta^{(k)}). \tag{7.17}$$

The case $k = 0$ in (7.17) just correpsonds to the multi-index $\emptyset$. Notice that, by Proposition 6.30, the integer k is given by $i_0(\beta)$ and the sequence $\beta^{(1)}, \dots, \beta^{(k)}$ is unique up to a permutation. The case $k = 1$ shows that $\mathcal{C} \subset \tilde{\mathcal{C}}$. The set $\tilde{\mathcal{C}}$ is the set of multi-indices appearing in the sums replacing the products in (7.16). Hence, we can write

$$\alpha(v(t,x))) = \sum_{\beta \in \tilde{\mathcal{C}}} \rho_\beta \, I^{(0)}_\beta(t; t, x), \tag{7.18}$$

for some constants $(\rho_\beta)_{\beta \in \tilde{\mathcal{C}}}$. Now, let $\beta \in \tilde{\mathcal{C}} \subset \bar{\mathcal{A}}_{(0)}$. Suppose $i_0(\beta) = k$. Let $\beta^{(1)}, \dots, \beta^{(k)} \in \mathcal{C}$ be the sequence of indices from which β appears when we write the products of the integrals as a sum. This sequence is unique according to Proposition 6.30. By Lemma 6.21 and Lemma 6.19, we can write

$$I^{(0)}_\beta(s; t, x) = I^{(1)}_{A_0(\beta)}(s; t, 0; x, 0) = I^{(1)}_{A_0(\beta)}(s; t, r_1; x, x_1) = I^{(1)}_{T_0(A_0(\beta))}(s; r_1, t; x_1, x)$$

as the integral does not depend on r_1, x_1. The effect of $T_0 \circ A_0$ on β is just to add 1 to each value, the same effect as the one of the operator A_{-1}, that satisfies $A_{-1} = \theta_1^{-1}$. Now, for $t \in [0, T]$ and $x \in \mathbb{R}^d$ set

$$w(t,x) = \int_0^t \int_{\mathbb{R}^d} \Gamma(t - r, x - y)\alpha(v(r, y))M(dr, dy). \tag{7.19}$$

This stochastic integral is well-defined according to Theorem 3.1 as soon as the process v is well-defined. Indeed, Corollary 6.39 states that a linear combination of iterated stochastic integrals has a spatially homogeneous covariance function. We also suppose that the series in (7.18) converges in a sense that allows to permute the sum with the stochastic integral. For example, we can use Proposition 6.17 if the series converges in $L^2(\Omega)$. In that case, we obtain

$$\begin{aligned}
w(t,x) &= \int_0^t \int_{\mathbb{R}^d} \Gamma(t-r,x-y)\alpha(v(r,y))M(dr,dy) \\
&= \int_0^t \int_{\mathbb{R}^d} \Gamma(t-r,x-y)\left(\sum_{\beta\in\tilde{\mathcal{C}}} \rho_\beta I_\beta^{(0)}(r;r,y)\right) M(dr,dy) \\
&= \sum_{\beta\in\tilde{\mathcal{C}}} \rho_\beta \int_0^t \int_{\mathbb{R}^d} \Gamma(t-r,x-y) I_\beta^{(0)}(r;r,y)M(dr,dy) \\
&= \sum_{\beta\in\tilde{\mathcal{C}}} \rho_\beta \int_0^t \int_{\mathbb{R}^d} \Gamma(t-r,x-y) I_{A_{-1}(\beta)}^{(1)}(r;t,r;x,y)M(dr,dy) \\
&= \sum_{\beta\in\tilde{\mathcal{C}}} \rho_\beta I_{(0,A_{-1}(\beta))}^{(0)}(t;t,x). \qquad (7.20)
\end{aligned}$$

As a multi-index $(0, A_{-1}(\beta))$ is always in the set $\bar{\mathcal{A}}_{(0)} = \mathcal{C}$, setting

$$\mathcal{X} = \left\{\delta \in \bar{\mathcal{A}}_{(0)} \cap \mathcal{B}_1 : \theta_1(-\delta) \in \tilde{\mathcal{C}}\right\} \qquad (7.21)$$

and replacing in (7.20), we can write

$$w(t,x) = \sum_{\delta\in\mathcal{X}} \rho_{\theta_1(-\delta)} I_\delta^{(0)}(t;t,x). \qquad (7.22)$$

Now, we will give the two following results (Lemmas 7.5 and 7.6), which will allow us to conclude our argument.

Lemma 7.5. *The set of multi-indices $\mathcal{X}$ is equal to $\mathcal{C}$.*

Proof. By the construction of the set $\mathcal{X}$ in (7.21), it is clear that $\mathcal{X} \subset \bar{\mathcal{A}}_{(0)} \cap \mathcal{B}_1 = \mathcal{C}$.

Now, let us consider $\delta \in \mathcal{C} = \bar{\mathcal{A}}_{(0)} \cap \mathcal{B}_1$. We want to show that $\delta \in \mathcal{X}$. First of all, consider the case $\delta = (0)$. As $\emptyset \in \tilde{\mathcal{C}}$ ($k = 0$ in (7.17)), $\theta_1(-(0)) = \emptyset \in \tilde{\mathcal{C}}$ and $(0) \in \mathcal{X}$.

Now, let $\delta \neq (0)$ be a multi-index in $\mathcal{C}$. This implies that $\beta = \theta_1(-\delta)$ has at least one component 0 (the second component of δ may only have '1' as value). Suppose $i_0(\beta) = k \geqslant 1$. Hence, by Proposition 6.30, there exists a sequence $\beta^{(1)}, \ldots, \beta^{(k)} \in \bar{\mathcal{A}}_{(0)} \cap \mathcal{B}_1 = \mathcal{C}$ such that $\beta \in \mathcal{D}(\beta^{(1)}, \ldots, \beta^{(k)})$. Then, by the definition, $\beta \in \tilde{\mathcal{C}}$ and, as a consequence, $\delta \in \mathcal{X}$. The result is proved. ■

Lemma 7.6. *Let $\delta \in \mathcal{C} = \mathcal{X}$, then $\rho_{\theta_1(-\delta)} = \pi_\delta(\mathrm{id})$.*

Proof. Let $\beta = \theta_1(-\delta)$. By the definition of $\mathcal{X}$, $\beta \in \tilde{\mathcal{C}}$. We would like to identify the constant ρ_β. Let $i_0(\beta) = k$ and let $\beta^{(1)}, \dots, \beta^{(k)} \in \mathcal{C}$ be the sequence such that $\beta \in \mathcal{D}(\beta^{(1)}, \dots, \beta^{(k)})$. The constant ρ_β can be written as

$$\rho_\beta = \frac{\alpha^{(k)}(0)}{k!} \left(\prod_{i=1}^{k} \pi_{\beta^{(i)}}(\mathrm{id}) \right) c_\beta n_\beta,$$

where c_β is the constant appearing in the decomposition given by Proposition 6.26 and n_β is the number of times the sequence $\beta^{(1)}, \dots, \beta^{(k)}$ appears in the development of the k-th power of the series in (7.15). By the definition of $\pi_\beta(\mathrm{id})$, we have

$$\rho_\beta = c_\beta n_\beta \frac{\alpha^{(k)}(0)}{k!} \prod_{i=1}^{k} \left(\prod_{j_i=0}^{\|\beta^{(i)}\|} \alpha^{(\nu^i_{j_i})}(0) \right) 2^{-(\|\beta^{(i)}\| - \ell(\beta^{(i)}))}, \tag{7.23}$$

where $\nu^i_{j_i}$ is the number of times the value j_i appears in the multi-index $\beta^{(i)}$. Then, considering $\pi_\delta(\mathrm{id})$, we have

$$\pi_\delta(\mathrm{id}) = \left(\prod_{j=0}^{\|\delta\|} \alpha^{(\nu_j)}(0) \right) 2^{-(\|\delta\| - \ell(\delta))}, \tag{7.24}$$

where ν_j is the number of times the value j appears in the multi-index δ. First of all, as the number of variables remains constant when we use the integration by parts formula, we have

$$\|\delta\| = \|\beta\| + 1 = 1 + \sum_{i=1}^{k} \|\beta^{(i)}\|. \tag{7.25}$$

But

$$\ell(\delta) = 1 + \ell(\beta) \leqslant 1 + \sum_{i=1}^{k} \ell(\beta^{(i)}).$$

Indeed, two one-dimensional components are sometimes gathered to form a two-dimensional component when we use the integration by parts formula. Then, as $i_0(\delta) = 1$, the term $j = 0$ of the product in (7.24) is 1. Then, the number of 1's in δ is equal to the number of 0's in β, i.e. k. Hence,

$$\pi_\delta(\mathrm{id}) = \alpha^{(k)}(0) \left(\prod_{j=2}^{\|\delta\|} \alpha^{(\nu_j)}(0) \right) 2^{-(\|\delta\| - \ell(\delta))}. \tag{7.26}$$

Then, for each $j = 2, \dots, \|\delta\|$, the number of j's in δ is equal to the number of $j-1$'s in β. But, as there is no *mixing* when using the integration by parts formula (see the proof of Proposition 6.24), there exist an unique integer i and an unique value j_i such that the number of j's in δ is equal to the number of j_i's in $\beta^{(i)}$. This shows that

$$\prod_{j=2}^{\|\delta\|} \alpha^{(\nu_j)}(0) = \prod_{i=1}^{k} \prod_{j_i=0}^{\|\beta^{(i)}\|} \alpha^{(\nu^i_{j_i})}(0). \tag{7.27}$$

Then, suppose that in the successive use of the integration by parts formula to express β from $\beta^{(1)}, \ldots, \beta^{(k)}$, q gatherings of one-dimensional components into two-dimensional components are necessary. In this case,

$$\ell(\delta) = 1 + \sum_{i=1}^{k} \ell(\beta^{(i)}) - q. \tag{7.28}$$

It remains to compute c_β and n_β. Suppose that the sequence $\beta^{(1)}, \ldots, \beta^{(k)}$ is made of m distinct multi-indices. Let k_i ($i = 1, \ldots, m$) denote the number of multi-indices of each type appearing in the sequence, with $k_1 + \cdots + k_m = k$ (e.g. $k_1 = \cdots = k_m = 1$ if all multi-indices are distinct and $k_1 = k$ if they are all identical). Then, a combinatorial argument shows that the number of times the sequence $\beta^{(1)}, \ldots, \beta^{(k)}$ appears in the development of the k-th power in (7.15) is given by

$$n_\beta = \frac{k!}{k_1! \cdots k_m!}. \tag{7.29}$$

Moreover, the constant c_δ is given as the number of different ways (due to equal multi-indices) to obtain β when applying the integration by parts formula times the power of $\frac{1}{2}$ appearing if we consider Remark 6.27. Namely, c_δ is given by

$$c_\beta = \frac{k_1! \cdots k_m!}{2^q}. \tag{7.30}$$

The numerator corresponds to the number of permutations of identical multi-indices in the sequence $\beta^{(1)}, \ldots, \beta^{(k)}$. The denominator corresponds to the factor $\frac{1}{2}$ appearing for each of the q gatherings of one-dimensional components. Notice that a two-dimensional component who directly comes from $\beta^{(i)}$ does not correspond to a factor $\frac{1}{2}$, because the multi-index with the symmetrical component will come from a multi-index which does not appear in $\beta^{(1)}, \ldots, \beta^{(k)}$.

Replacing (7.25),(7.27),(7.28),(7.29) and (7.30) in (7.23) and using (7.26), we have shown that

$$\rho_\beta = \pi_\delta(\mathrm{id}). \tag{7.31}$$

■

As a consequence, using Lemma 7.5 and Lemma 7.6 in (7.22), we have shown that $w(t, x) = v(t, x)$ for all $t \in [0, T]$ and $x \in \mathbb{R}^d$. In other words, this means that $v(t,x)$ satisfies the integral form (4.3) of (4.1). Moreover, if the series defining $v(t,x)$ converges in $L^2(\Omega)$, then it would have a spatially homogeneous covariance function, due to Corollary 6.39. This suggests that $v(t,x)$ will satisfy the "S" property (Definition 4.4) and, as a consequence, would be equal to $u(t,x)$ by Theorem 4.8. Hence, we have the following conjecture.

Conjecture 7.7 (Itô-Taylor expansion)**.** Suppose that the assumptions of Theorem 4.2 are satisfied. If α is a Lipschitz, analytic and bounded function and $\beta \equiv 0$, then the solution of the stochastic partial differential equation (4.1) is given by

$$u(t,x) = \sum_{\beta \in \mathcal{C}} \pi_\beta(\mathrm{id}) I_\beta^{(0)}(t;t,x). \tag{7.32}$$

The convergence of the series in (7.32) is still an open problem. Nevertheless, this convergence would yield an explicit expression for the solution of the stochastic partial differential equation (4.1). It may be possible to establish this convergence under some additional assumptions on the non-linear function α. For example, one can consider $\alpha(x) = \cos(x)$, $\alpha(x) = a + \sin(x)$ or $\alpha(x) = e^{-x^2}$. In these cases, all derivatives of odd (or even) order vanish, which strongly limits the number of multi-indices in the sum.

Moreover, $\alpha(x) = ax + b$ can also be chosen even though this function is not bounded. This case corresponds to the one treated in Theorem 5.1, and Conjecture 7.7 can be established in this particular case. Indeed, $\alpha^{(k)}(0) = 0$ for $k \geqslant 2$ and only a few multi-indices have to be taken into account.

Notice that the arguments given above in the sense of Conjecture 7.7 are independent of the spatial dimension. Hence, conditionnaly on the convergence of the series in (7.14), the conjecture would be satisfied for any spatial dimension.

We expect that Itô-Taylor expansion, in addition to giving an exact expression for the solution $u(t,x)$, will lead to estimates on higher order moments of $u(t,x)$ or to study the Hölder-continuity of $u(t,x)$. In the case of a spatial dimension less or equal to 3, such results already exist and we would be able to check if they can also be established this way and if this leads to an optimal result.

Bibliography

[1] Conus D. & Dalang R.C. *The non-linear stochastic wave equation in high dimensions.* Electronic Journal of Probability Vol. 13, 2008.

[2] Dalang R.C. *Extending martingale measure stochastic integral with applications to spatially homogeneous spde's.* Electronic Journal of Probability, Vol. 4, Paper n°6, 1999.

[3] Dalang R.C. *Corrections to : Extending martingale measure stochastic integral with applications to spatially homogeneous spde's.* Electronic Journal of Probability, Vol. 4, Paper n°6 (Corrections), 1999

[4] Dalang R.C. & Frangos N.E. *The stochastic wave equation in two spatial dimensions.* Annals of Probability, Vol. 26, n°1 (1998) 187-212.

[5] Dalang R.C & Lévêque O. *Second-order linear hyperbolic SPDEs driven by isotropic Gaussian noise on a sphere.* Annals of Probability, Vol. 32, n°1B (2004), 1068-1099.

[6] Dalang R.C & Lévêque O. *Second-order hyperbolic SPDE's driven by homogeneous gaussian noise on a hyperplan.* Transactions of the AMS, Vol. 358, n°5 (2005) 2123-2159.

[7] Dalang R.C. & Mueller C. *Some non-linear s.p.d.e's that are second order in time.* Electronic Journal of Probability, Vol. 8, 2003.

[8] Dalang R.C. & Mueller C. & Tribe R. *A Feynman-Kac-type formula for the deterministic and stochastic wave equations and other p.d.e.'s.* Transactions of the AMS (2008, to appear).

[9] Dalang R.C. & Sanz-Solé M. *Hölder-Sobolev regularity of the solution to the stochastic wave equation in dimension 3.* Memoirs of the AMS (2008, to appear).

[10] Da Prato G. & Zabczyk J. *Stochastic equations in infinite dimensions.* Encyclopedia of Mathematics and its Applications, 44. Cambridge University Press, 1992.

[11] Dunford N. & Schwartz J.T. *Linear Operators, Part I, General Theory.* Wiley Classics Library, 1988.

[12] Durrett R. *Probability : Theory and Examples.* Second Edition. Duxbury Press, 1995.

[13] Folland G.B. *Introduction to Partial Differential Equations.* Princeton University Press (1976).

[14] Kloeden P.E. & Platen E. *Numerical solution of stochastic differential equations.* Applications of Mathematics (New York), 23. Springer-Verlag, Berlin, 1992.

[15] Lévy P. *Processus stochastiques et mouvement brownien.* Deuxième édition. Gauthier-Villars, 1965.

[16] Mikusinski J. *The Bochner Integral.* Birkhäuser (1978).

[17] Millet A. & Sanz-Solé M. *A stochastic wave equation in two space dimension : smoothness of the law.* Annals of Probability, Vol. 27, n°2 (1999) 803-844.

[18] Nualart, D. & Quer-Sardanyons, L. *Existence and smoothness of the density for spatially homogeneous spde's.* Potential Analysis, Vol. 27, n°3 (2007) 281-299.

[19] Peszat S. *The Cauchy problem for a non-linear stochastic wave equation in any dimension.* Journal of Evolution Equations, Vol. 2, n°3 (2002), 383-394.

[20] Protter P. *Stochastic Integration and Differential Equations.* Second Edition. Springer, 2004.

[21] Quer-Sardanyons L. & Sanz-Solé M. *Absolute continuity of the law of the solution to the 3-dimensional stochastic wave equation.* Journal of Functional Analysis, 206, n°1 (2004) 1-32.

[22] Quer-Sardanyons L. & Sanz-Solé M. *A stochastic wave equation in dimension 3 : smoothness of the law.* Bernoulli 10, n°1 (2004) 165-186.

[23] Rao M.M. *Bimeasures and harmonizable processes (analysis, classification, and representation).* Probability measures on groups, IX (Oberwolfach, 1988), 254-298, Lecture Notes in Math., 1379, Springer, Berlin, 1989.

[24] Revuz D. & Yor M. *Continuous Martingales and Brownian Motion.* Third Edition. Springer, 1999.

[25] Saks S. *Theory of the integral.* Dover Publications (1964).

[26] Sanz-Solé M. & Sarrà M. *Path properties of a class of Gaussian processes with applications to spde's.* Canadian Mathematical Society Conference Proceedings. Vol. 28 (2000).

[27] Schwartz L. *Théorie des distributions.* Hermann, Paris (1966).

[28] Treves F. *Topological Vector Spaces, Distributions and Kernels.* Academic Press, New York (1967).

[29] Treves F. *Basic Linear Partial Differential Equations.* Academic Press, New York (1975).

[30] Walsh J.B. An Introduction to Stochastic Partial Differential Equations. In : *Ecole d'Eté de Probabilités de St-Flour*, XIV, 1984. Lecture Notes in Mathematics 1180. Springer-Verlag, Berlin, Heidelberg, New-York (1986), 265-439.

Curriculum Vitae

I was born on June 12$^{\text{th}}$, 1980 in Morges, Switzerland. I attended primary school in Ecublens, secondary school at the Collège de la Planta in Chavannes and finally obtained the Swiss high school degree ("maturité fédérale") in June 1998 at the Gymnase Auguste Piccard in Lausanne. Then, I undertook to study mathematics at the Ecole Polytechnique Fédérale de Lausanne (EPFL). For my Master's thesis, I studied a problem in probability theory with applications in finance, under the supervision of Prof. Robert C. Dalang and finally obtained a Master degree in Mathematical Sciences ("diplôme d'ingénieur mathématicien") in April 2003. In September 2003, I was hired as a research and teaching assistant in the Chair of Probability directed by Prof. Robert C. Dalang and I began to work on the present Ph.D. thesis. During my doctoral studies, I had the opportunity to attend several international conferences. In particular in September 2007, I spent a month at the Mittag-Leffler Institute in Stockholm (Sweden).