Contribution to the Ergodic Theory of Piecewise Monotone Continuous Maps

THÈSE N^{O} 4232 (2008)

PRÉSENTÉE LE 9 JANVIER 2009

À LA FACULTÉ SCIENCES DE BASE

GROUPE PFISTER

PROGRAMME DOCTORAL EN MATHÉMATIQUES

ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE

POUR L'OBTENTION DU GRADE DE DOCTEUR ÈS SCIENCES

PAR

Bastien FALLER

acceptée sur proposition du jury:

Prof. R. Dalang, président du jury
Prof. C. Pfister, directeur de thèse
Prof. H. Bruin, rapporteur
Prof. P. Collet, rapporteur
Prof. M. Troyanov, rapporteur

ÉCOLE POLYTECHNIQUE
FÉDÉRALE DE LAUSANNE

Suisse
2008

Résumé

Cette thèse est consacrée à la théorie ergodique des applications de l'intervalle dites monotones continues par morceaux. Le codage est une méthode classique pour étudier ces applications. Grâce au codage, on obtient un système dynamique symbolique qui est presque isomorphe au système dynamique initial. Le principe du codage est intimement lié à celui du développement des nombres réels.

Nous commençons par définir le codage dans une optique proche de celle des développements des nombres réels; cette optique est celle adoptée par Rényi et Parry dans leurs articles concernant les développements des nombres. Puis nous présentons les travaux de Hofbauer qui abordent les liens entre les propriétés ergodiques d'une application de l'intervalle monotone continue par morceaux et celles du système dynamique symbolique correspondant. Nous montrons qu'il y a une bijection entre les ensembles de mesures d'entropie maximale de ces deux systèmes dynamiques.

Nous appliquons ces résultats à l'étude de deux familles d'applications particulières: d'une part les applications $T_{\alpha,\beta}(x) := \beta x + \alpha \mod 1$, d'autre part celles que nous appellerons des β-transformations généralisées. Pour la famille d'applications $T_{\alpha,\beta}$, nous décrivons en détail la famille de systèmes dynamiques symboliques obtenus grâce au codage. Puis nous abordons la question de la normalité des orbites pour les applications $T_{\alpha,\beta}$. Finalement nous étudions les β-transformations généralisées: nous montrons que la plupart d'entre elles ont une unique mesure d'entropie maximale, puis nous étudions également la normalité des orbites pour ces applications.

Mots-clés: applications de l'intervalle, dynamique symbolique, développement des nombres, entropie topologique, diagramme de Markov, nombres normaux.

Abstract

This thesis is devoted to the ergodic theory of the piecewise monotone continuous maps of the interval. The coding is a classical approach for these maps. Thanks to the coding, we get a symbolic dynamical system which is almost isomorphic to the initial dynamical system. The principle of the coding is very similar to the one of expansion of real numbers.

We first define the coding in a perspective similar to the one of the expansions of real numbers; this perspective was already adopted by Rényi and Parry in their papers about the expansions of numbers. Then we present the theory of Hofbauer about the links between the ergodic properties of a piecewise monotone continuous map of the interval and the corresponding symbolic dynamical system. We prove that there is a bijection between the sets of measures of maximal entropy of these two dynamical systems.

We apply these results to the study of two families of maps: first the maps $T_{\alpha,\beta}(x) := \beta x + \alpha \mod 1$, then the maps we will call generalized β-transformations. For the family of maps $T_{\alpha,\beta}$, we describe in detail the family of symbolic dynamical systems obtained by the coding. Then we turn to the question of normality of the orbits for the maps $T_{\alpha,\beta}$. Finally we study the generalized β-transformations: we prove that most of them have a a unique measure of maximal entropy, then we also study the normality of the orbits for theses maps.

Key words: maps of the interval, symbolic dynamic, expansion of numbers, topological entropy, Markov diagram, normal numbers.

Remerciements

Je tiens en premier lieu à remercier Charles Pfister dont le soutien et les encouragements ont été constants tout au long de mon travail. C'est un directeur de thèse d'une rare disponibilité, prêt à me consacrer une demi-heure les jours où il disait n'avoir pas le temps de me voir. C'est évidemment ses questions toujours nouvelles qui m'ont guidé dans ma recherche. J'ai aujourd'hui du plaisir à me retourner sur le chemin parcouru ensemble: un chemin parsemé de difficultés et de changements de direction, mais fourmillant d'interrogations passionnantes.

Mes remerciements vont ensuite aux membres du jury de ma thèse, Robert Dalang, Henk Bruin, Pierre Collet et Marc Troyanov, qui se sont penchés avec attention sur mon travail. Je tiens à remercier particulièrement Henk Bruin pour sa lecture attentive de ma thèse ainsi que ses réponses enrichissantes à quelques questions précises que je lui ai posées.

Je remercie également mon épouse Rachel et mon ami Olivier: en plus d'être de très agréable compagnie à tout moment de la journée, ils ont accepté de tendre une oreille attentive chaque fois que je sentais le besoin de parler des problèmes rencontrés dans mon travail. De plus, ils ont généreusement consacré du temps à une relecture soigneuse de ma thèse. L'armée suisse qui a retenu Olivier loin de son bureau sans lui fournir d'autres activités que la relecture de mon travail mérite peut-être également des remerciements?

J'en viens à remercier mes camarades qui ont rendu plaisant mon séjour à l'EPFL. D'une part, mes anciens camarades de physique avec lesquels j'ai partagé repas et jeux de cartes presque tous les jours: Rachel, Olivier, Dominique, Lysianne, Cyril, Sébastien, Nicolas, Anne, Stefano, Aurélie et Raphaël. D'autre part, mes collègues mathématiciens qui ont animé le fond de notre couloir: Chantal, Guillaume, Virabouth, Thomas, Gilles, Simone, Frédéric et Hugo.

Enfin mes remerciements vont à ma famille qui m'a encouragé tout au long de mon travail. Leur regard sur mon travail a souvent été perplexe (ma grand-mère me demande encore de lui expliquer le titre de mon travail de diplôme), mais ils m'ont toujours soutenu du fond du coeur.

Contents

Chapter 1

Introduction

1.1 Generalities

The framework of this thesis is the ergodic theory of piecewise monotone continuous maps of the interval. We recall briefly the underlying concepts. A discrete time dynamical system is a pair (X, T). The set X is the phase space and its elements correspond to the possible states of the system. T is a map from X to X; it describes the time evolution of the system. If the system is in state $x \in X$ at time 0, then it is in state $Tx \in X$ at time 1, in state T^2x at time 2 and so on. This justifies the name of *discrete time* dynamical system. If the map T is invertible, the evolution can be extended to negative time in the same manner: the state at time n is $T^n x$ for all $n \in \mathbb{Z}$.

Often the phase space X is endowed with a particular structure and the map T preserves this structure. For example, (X, μ) is a measure space and T is a measure-preserving transformation, ie $\mu(T^{-1}A) = \mu(A)$ for all measurable sets A. The ergodic theory is the field which studies the measure-preserving transformations. Some notions in this field are known beyond the area of dynamical systems: the question of the ergodicity of an invariant measure or Poincaré's recursion Theorem. We give another example of preserved structure: $(X, \mathcal{T})$ is a topological space and T is a continuous map. The topological dynamics deals with continuous transformations.

The piecewise monotone continuous maps are particular examples of discrete time dynamical systems. The state space X is the compact interval $[0, 1] \subset \mathbb{R}$. Moreover, there exists a countable partition of $[0, 1]$ in intervals such that the map T restricted on each interval of the partition is a monotone continuous map. We do not require that T is continuous on the whole interval $[0, 1]$, thus the topological structure is not preserved. Fortunately, there is another structure on $[0, 1]$ which is very useful: the order $\leq$. The piecewise monotonicity of T preserves partially the order structure and this allows to control the orbits of points. Although this family has been extensively studied, there are still many questions to investigate. Many familiar examples belong to this family, among them:

1. The logistic maps $x \mapsto \mu x(1-x)$, where $\mu \in [0, 4]$ is a parameter. For μ large enough, the behavior of the orbits is very complex, because the map is expanding on an area of the phase space and contracting on another area. The period-doubling cascade of this map is famous.

2. The tent maps $x \mapsto \min\{\beta x, \beta(1-x)\}$, where $\beta \in [0, 2]$ is a parameter. It is a typical example of a unimodal map (a unimodal map is a continuous map with one hill);

indeed many unimodal maps are conjugated to a tent map.

3. The intervals exchange maps, where the intervals of the partition are simply permuted. They are examples of piecewise isometries.

4. The map $x \mapsto 1/x \mod 1$, which is very important for continued fractions.

5. The maps $x \mapsto \beta x \mod 1$, where $\beta \in (1, \infty)$ is a parameter. It plays a central role in expansion of numbers in non-integer basis.

6. The maps $x \mapsto x + x^\alpha \mod 1$ where $\alpha \in (1, \infty)$ is a parameter; these maps are typical examples for the intermittency.

The piecewise monotone continuous maps are one of the families of discrete time dynamical systems considered in this thesis.

The second family which we consider is the class of shift maps. Let $\mathtt{A}$ be a finite (or sometimes countable) set endowed with the discrete topology and consider the product spaces $\Sigma_{\mathtt{A}} = \mathtt{A}^{\mathbb{Z}}$ or $\Sigma_{\mathtt{A}}^{+} = \mathtt{A}^{\mathbb{Z}_+}$. We denote by σ the left shift map. Let Σ be a σ-invariant subset of $\Sigma_{\mathtt{A}}$ or $\Sigma_{\mathtt{A}}^{+}$. The pair (Σ, σ) is a discrete time dynamical system. It is reversible if Σ is a subset of $\Sigma_{\mathtt{A}}$, and irreversible if Σ is a subset of $\Sigma_{\mathtt{A}}^{+}$. Notice that the complexity of the dynamics comes from the state space Σ, the map σ being very easy to understand.

In 1957, Rényi published his fundamental paper [R], where he showed that there is a strong link between the piecewise monotone continuous maps and the shift maps: these two families of discrete time dynamical systems are linked by the expansions of real numbers. We present this link with the help of the decimal expansion of real numbers: in this case, the representation function is $\varphi(t) = t/10$. A number $x \in [0, 1)$ is represented by a sequence of digits $\mathtt{i}_0, \mathtt{i}_1, \ldots$ such that

$$x = \varphi(\mathtt{i}_0 + \varphi(\mathtt{i}_1 + \varphi(\mathtt{i}_2 + \ldots))) = \frac{\mathtt{i}_0}{10} + \frac{\mathtt{i}_1}{10^2} + \frac{\mathtt{i}_2}{10^3} + \ldots .$$

If we truncate this expansion of x, we can define a sequence of remainders $r_0, r_1, \ldots$ such that

$$x = \varphi(\mathtt{i}_0 + \varphi(\mathtt{i}_1 + \cdots + \varphi(\mathtt{i}_{n-1} + r_n))) = \frac{\mathtt{i}_0}{10} + \frac{\mathtt{i}_1}{10^2} + \cdots + \frac{\mathtt{i}_{n-1}}{10^n} + \frac{r_n}{10^n} .$$

It is easy to check that the digits are elements of $\{0, 1, \ldots, 9\}$, the remainders satisfy $r_n \in [0, 1)$ and the sequences of remainders can be constructed by $r_0 = x$ and

$$r_{n+1} = 10\, r_n \mod 1 .$$

Now the link between the piecewise monotone continuous maps and the shift maps appears. The map $T : [0, 1] \to [0, 1]$ given by $T(x) := 10\, x \mod 1$ is a piecewise monotone continuous map. The space $\Sigma := \{0, 1, \ldots, 9\}^{\mathbb{Z}_+}$ is a shift space. Moreover the two discrete time dynamical systems $([0, 1], T)$ and (Σ, σ) are almost isomorphic. The map $x \mapsto \mathtt{i}(x) \equiv (\mathtt{i}_0\mathtt{i}_1 \ldots)$ is called the coding map. The coding map is *almost* an isomorphism, because some numbers can have two decimal expansions: for example, $0.399999\ldots \equiv 0.4$. There are countably many such numbers and the coding map is an isomorphism outside this problematic set. Notice that the Lebesgue measure on $[0, 1]$ is an ergodic T-invariant measure. Using Birkhoff's Ergodic Theorem, we get Borel's Theorem about normal numbers. For Lebesgue almost all numbers in $x \in [0, 1)$, the following property holds: any finite sequence of digits appears in x with a frequency 10^{-n}, where n is the length of the sequence.

In his paper, Rényi generalized this analysis to a wide family of expansions of numbers. Given a representation function φ, we can define a piecewise monotone continuous map $T : [0,1] \to [0,1]$ by $T = \varphi^{-1} \mod 1$ and a shift space Σ as the closure of the set of codings of points $x \in [0,1)$. Under a good hypothesis about the representation map φ (roughly speaking, all numbers must have an expansion), the coding map is almost an isomorphism between the two dynamical systems $([0,1],T)$ and (Σ,σ). In this case, we say the the expansion is valid.

At the end of his paper, Rényi studied the expansion in non-integer basis. The representation function is $\varphi(t) = t/\beta$ with $\beta \in (1,\infty)$ and the piecewise monotone continuous map is $T(x) = \beta x \mod 1$; the map T is called a β-transformation. When β is an integer q, we recover the q-adic expansion, but when β is not an integer, we get a new expansion with particular properties. For example, the set Σ is no more a product space or the Lebesgue measure is no more T-invariant. In 1960, Parry published in [P1] an explicit formula of the density of a T-invariant measure equivalent to Lebesgue measure. This formula allows to compute the frequencies of sequences of digits appearing in the expansion of Lebesgue almost all numbers $x \in [0,1)$.

There are natural generalizations of the expansion in non-integer basis. One of them was introduced by Parry in [P2]: the representation map is $\varphi(t) = (t-\alpha)/\beta$ and the corresponding piecewise monotone continuous map is $T(x) = \beta x + \alpha \mod 1$. If $\beta > 1$ or if $\beta = 1$ and α is not a rational, this expansion is valid. Parry gave an expression of the density of a T-invariant measure absolutely continuous with respect to Lebesgue measure. This expansion is a non trivial generalization of the expansion in non integer basis, some properties are really different. For example, the unique T-invariant measure absolutely continuous with respect to Lebesgue measure (the one constructed by Parry) is not equivalent to Lebesgue measure for some parameters α, β. When Parry published his paper, he did not know if this measure was positive (this fact has been proved by Halfin in [Ha]). The family of piecewise monotone continuous maps $T(x) = \beta x + \alpha \mod 1$ is one of the two families of dynamical systems particularly studied in this thesis. The second one is the family of generalized β-transformations. The maps $\beta x \mod 1$ are piecewise increasing, where the generalized β-transformations may be decreasing on some intervals of continuity. This family of maps was recently studied by Góra in [G] in a point of view very similar to the one of Parry in [P2].

This thesis has two main directions of research. The first one is a good description of the sets Σ obtained by the coding for the maps $\beta x + \alpha \mod 1$ and the generalized β-transformations. The second one is the study of the normality of the expansion of number in the sense of Borel. There is an important difference between the point of view of Borel and our one: Borel fixed the expansion (the dyadic expansion) and he computed the size of the set of normal numbers; whereas we consider a family of expansions described by a parameter, we fix $x \in [0,1)$ and we consider the set of parameters for which x is normal.

1.2 Contents of this thesis

In Chapter 2, we present the general theory of φ-expansions. It is presented with the point of view adopted by Parry in [P2]. Most of the results are already known, but the proofs are personal. Moreover, Theorems 2.11 and 2.12 are original. These theorems give necessary and sufficient conditions for the φ-expansion to be valid, in term of surjectivity or continuity of the map $\overline{\varphi}_\infty$. The concepts are illustrated by two examples that will come back all along this thesis. The first one (Example A) is the map $T_{\alpha,\beta}(x) := \beta x + \alpha \mod 1$. The second one (Example B) is the generalized β-transformation; this map is obtained from a β-transformation by replacing some increasing laps by decreasing laps (a lap of T is a map obtained by restricting T on an interval of monotonicity and continuity).

Chapter 3 is a classical subject of ergodic theory. We present the measure-theoretical entropy and the topological entropy in a very common way, except for Proposition 3.6.

Chapter 4 introduces a sometimes tedious, but necessary tool for the study of the shift spaces obtained by coding: the Markov diagram. This graph was constructed by Hofbauer in [H3] following an idea of Takahashi in [T]. It allows one to construct an almost isomorphism between the shift space obtained by coding and a countable Markov shift. The aim of Hofbauer was the study of the measure of maximal entropy, however the Markov diagram is also used for other purposes. There are many variations in the definition of this graph, because the authors often adapt the graph to the situation considered. We do our own choice with its advantages and its disadvantages: first we present a very general definition suitable for the abstract works (see in particular Chapter 5), then we modify a bit the graph to be more suitable for the particular situations considered. This adaptation is the aim of Section 4.2.

In Chapter 5, we present the works of Hofbauer in [H3]. The aim is to show that the set of measures of maximal entropy for a piecewise monotone continuous map is in 1-to-1 correspondence with the set of measures of maximal entropy of a countable Markov shift. This proof is based on the Markov diagram. The presentation follows strictly the one of Hofbauer, but the proofs are brought up to date. We emphasize that we found no recent presentation of these results, although they are 30 years old.

Almost all new results of this thesis are presented in Chapters 6 and 7. The contents of the first part of Chapter 6 is an article submitted in April 2008 to the journal *Nonlinearity.* The second part of this chapter is at the moment not submitted. The contents of Chapter 7 has been submitted in May 2008 to the journal *Ergodic Theory & Dynamical Systems.*

In Chapter 6, we study in detail two particular piecewise monotone continuous maps. The first one is $T_{\alpha,\beta}(x) := \beta x + \alpha \mod 1$. Let $\overline{\varphi}^{\alpha,\beta}$ denote the corresponding φ-expansion. We first study the set of expansions of 0 and 1, the main results are summarized in Proposition 6.4. Then we present an algorithm that allows to compute the topological entropy of shift spaces $\Sigma(\underline{u}, \underline{v})$, which have the same structure as the shift spaces obtained by coding with the map $T_{\alpha,\beta}$. The algorithm is defined in Proposition 6.10 and we show in Theorem 6.16 that the algorithm gives the topological entropy of the shift $\Sigma(\underline{u}, \underline{v})$. Finally we consider the inverse problem: given the coding of 0 and 1, can we find parameters (α, β) such that the coding of 0 and 1 are the given sequences? This problem is easy for the β-transformations, but far from being trivial in our case. Theorems 6.20 and 6.21 summarize our results.

The second part of Chapter 6 is devoted to the generalized β-transformations. In Theorem 6.23, we give sufficient conditions (that are fulfilled by most of the generalized β-transformations) for the uniqueness of the measure of maximal entropy.

In Chapter 7, we discuss the question of normality. A point x is normal with respect to μ, if the sequence of empirical measures $\{\frac{1}{n}\sum_{i=0}^{n-1} \delta_x \circ T^{-i}\}_n$ converges (in the weak*-topology) to μ as the time tends to ∞. First we consider the normality for the map $T_{\alpha,\beta}$. We work with $x \in [0,1]$ fixed and we look to the normality with respect to the parameters (α, β). In Theorem 7.8, we prove that, for all $x \in [0,1]$, the orbit is normal with respect to the measure of maximal entropy for almost all parameters (α, β) (with respect to 2-dimensional Lebesgue measure). However we construct a family of disjoint analytic curves that fill the plane (α, β) such that, along a given curve, the point x is normal with respect to the measure of maximal entropy for at most one pair of parameters (α, β). This amazing result is Theorem 7.10. Finally we consider the normality for the generalized β-transformations. As before, we fix $x \in [0, 1]$ and study the normality with respect to the parameter β. In Theorem 7.13, we prove that the orbit of x is normal with respect to the measure of maximal entropy for all parameters β, except for β too small in a particular class of generalized β-transformations.

Chapter 2

Coding and φ-expansion

In 1957, Rényi published his fundamental paper [R] about the representation of real numbers. In this work, he showed that the sequence of remainders is governed by a discrete time dynamical system on the interval $[0,1]$, namely a piecewise monotone continuous map. Using this remark, Rényi was able to study some ergodic properties for a given expansion of numbers. He also gave sufficient conditions for a map to generate valid representations of numbers. Finally he studied the particular example of expansions in non integer basis. These last two questions make the main lines of this chapter, where we present general results about the representations of real numbers. The representations of real numbers are generalizations of the decimal expansion. Some representations are especially studied, like the representation in continued fractions, the q-adic expansions or the expansions in non integer basis. Our presentation does not include the continued fractions, because we restrict ourself to the representation with a finite set of digits. On the other hand, we include the theory of expansions with non monotone maps. The structure of the chapter is inspired by the paper of Parry [P2], where he developed the ideas of Rényi in a very general point of view.

The chapter begins with a section where we introduce notations and properties of shift spaces that we need in this thesis. Then we turn to the theory of expansion of numbers. A representation function φ is a bijective map from a compact subinterval of $\mathbb{R}$ to $[0,1]$. The idea of Rényi is to represent $r_0 \in [0,1)$ as

$$r_0 = \varphi(\mathtt{i}_0 + r_1)\,,$$

where $\mathtt{i}_0$ and r_1 are uniquely determined by the prescriptions $\mathtt{i}_0$ is an integer and $r_1 \in [0,1)$. Then r_1 is again represented as

$$r_1 = \varphi(\mathtt{i}_1 + r_2)\,.$$

Continuing in this way, we construct two sequences $\{\mathtt{i}_n\}_{n\geq 0}$ and $\{r_n\}_{n\geq 0}$. For all $n \geq 0$, $\mathtt{i}_n$ is an integer and $r_n \in [0,1)$. The sequence $\{\mathtt{i}_n\}_{n\geq 0}$ is called the coding of r_0. The sequence $\{r_n\}_{n\geq 0}$ is governed by the formula

$$r_{n+1} = \varphi^{-1}(r_n) \mod 1\,.$$

Let us denote $T := \varphi^{-1} \mod 1$. The map T is a piecewise monotone continuous map of the interval. We want to define a map φ_∞ on the sequences of digits which is the inverse of the coding map. In view of the construction of the coding map, an idea to construct φ_∞ is

$$\varphi_\infty(w_0, w_1, w_2, \ldots) = \varphi(w_0 + \varphi(w_1 + \varphi(w_2 + \ldots)))\,.$$

This map sends a sequence of digits on a real number in $[0,1]$. If $x = \varphi_\infty(w_0, w_1, \ldots)$, then we call $\{w_n\}_{n\geq 0}$ a φ-expansion of x. The formal construction of the map φ_∞ is more complicated, because one has to take a limit $n \to \infty$; an important part of Section 2.2 is devoted to the existence of this limit. The φ-expansion is valid, if for all $r_0 \in [0,1)$, the coding of r_0 is a φ-expansion of x. In the last part of Section 2.2, we are looking for conditions for the validity of the φ-expansion. One of the main results presented in [P2] is that the φ-expansion is valid, if and only if the coding is injective. In Theorem 2.11, we prove that the φ-expansion is valid, if and only if the φ-expansion is surjective. In Theorem 2.12, we link the validity of the φ-expansion with the continuity of the map φ_∞.

Although most of the results of Section 2.2 are known, some of them are original, in particular Theorems 2.11 and 2.12. The presentation is illustrated with two examples which are generalizations of expansions in non integer basis. The first one was introduced by Parry in [P2], the representation function φ is

$$\varphi(x) = \frac{x-\alpha}{\beta}\,.$$

The second one, called generalized β-expansion, was recently described by Góra in [G] and it includes non-monotone representation functions.

2.1 Shift spaces

We introduce the basic notations and definitions that will be used in the whole thesis. We follow the book of Lind and Marcus [LM] as much as possible. The sets $\mathbb{N}$, $\mathbb{Z}_+$ and $\mathbb{Z}$ denote

$$\mathbb{N} = \{1,2,\ldots\}\,, \quad \mathbb{Z}_+ = \{0\} \cup \mathbb{N} \quad \text{and} \quad \mathbb{Z} = \{\ldots,-1,0,1,\ldots\}\,.$$

Consider a finite or countable **alphabet** $\mathtt{A}$ endowed with the discrete topology. The elements of $\mathtt{A}$ are called **symbols** and we often choose the digits 0,1,2.... In particular, we write $\mathtt{A}_k := \{0,1,\ldots,k-1\}$ for the alphabet of k-symbols. We define the **full semi-infinite $\mathtt{A}$-shift** $\Sigma_\mathtt{A}^+ := \mathtt{A}^{\mathbb{Z}_+}$ and the **full bi-infinite $\mathtt{A}$-shift** $\Sigma_\mathtt{A} := \mathtt{A}^{\mathbb{Z}}$. These two shifts are endowed with the product topology. We write $\Sigma_k^+ := \Sigma_{\mathtt{A}_k}^+$ for the full semi-infinite k-shift and $\Sigma_k := \Sigma_{\mathtt{A}_k}$ for the full bi-infinite k-shift; by Tychonoff's theorem, these two sets are compact. An element of $\Sigma_\mathtt{A}$ or $\Sigma_\mathtt{A}^+$ is a **sequence**. We will use underlined letters like $\underline{u}, \underline{v}, \underline{x}, \ldots$ to denote sequences. A sequence $\underline{z} \in \Sigma_\mathtt{A}^+$ (resp. $\underline{z} \in \Sigma_\mathtt{A}$) will be defined by all of its **coordinates** z_i with $i \in \mathbb{Z}_+$ (resp. $i \in \mathbb{Z}$). If we need an explicit notation, we will write the coordinates without separation, for example

$$\underline{z} = \ldots z_{-1} z_0 z_1 z_2 \cdots \in \Sigma_\mathtt{A}\,.$$

A **word** over $\mathtt{A}$ is a *finite* string of elements of $\mathtt{A}$, including the string with no symbol, called the **empty word** and denoted by ε. The set of all words over $\mathtt{A}$ is denoted by $\mathtt{A}^*$. As for sequences, we use underlined letters for words and we write a word as a string of symbols without separation. The **length** of a word $\underline{u}$ is the number of symbols it contains, it is denoted $|\underline{u}|$. We have $|\varepsilon| = 0$. The **concatenation** of two words $\underline{u}$ and $\underline{v}$ is the word $\underline{z} := \underline{u}\ \underline{v}$ formed by the string of symbols of $\underline{u}$ followed by the string of symbols of $\underline{v}$. If $n \geq 1$, let $\underline{u}^n$ denote the concatenation of n copies of $\underline{u}$ and set $\underline{u}^0 := \varepsilon$. In $\Sigma_\mathtt{A}^+$, a word $\underline{u}$ can be concatenated with a sequence $\underline{x}$ and they form the sequence $\underline{z} := \underline{u}\ \underline{x}$. Likewise, if $\underline{u}$ is a word, we define the sequence $\underline{z} := \underline{u}^\infty \equiv \underline{u}\ \underline{u}\ \underline{u} \cdots \in \Sigma_\mathtt{A}^+$. For $\underline{z} \in \Sigma_\mathtt{A}^+$ and $i \leq j \in \mathbb{Z}_+$, we will denote the word formed by the coordinates of $\underline{z}$ from i to j by $\underline{z}_{[i,j]} := z_i z_{i+1} \ldots z_j$.

If $i > j$, we define $\underline{z}_{[i,j]}$ to be ε. Similarly, if $i < j \in \mathbb{Z}_+$, then $\underline{z}_{[i,j)} = z_i z_{i+1} \dots z_{j-1}$ and if $i \geq j$, then $\underline{z}_{[i,j)} = \varepsilon$. We define similary $\underline{z}_{(i,j]}$ and $\underline{z}_{(i,j)}$. We use also the notation $\underline{z}_{[i,\infty)} = z_i z_{i+1} z_{i+2} \dots$. Of course, we have analogous definitions for $\underline{z} \in \Sigma_{\mathtt{A}}$ with $i, j \in \mathbb{Z}$.

For $\underline{w}$ a word on $\mathtt{A}$ of length n and $m \geq 0$, we define the **cylinder** in $\Sigma_{\mathtt{A}}^+$

$$
{}_m[\underline{w}] := \{\underline{z} \in \Sigma_{\mathtt{A}}^+ : \underline{z}_{[m,m+n)} = \underline{w}\} .
$$

The family $\{{}_m[\underline{w}] : \underline{w} \in \mathtt{A}^*, m \in \mathbb{Z}_+\}$ is a base for the topology on $\Sigma_{\mathtt{A}}^+$. The topology is metrizable, the following metric is compatible with the topology: let $\beta > 1$, then

$$
d_\beta(\underline{w}, \underline{w}') := \begin{cases} 0 & \text{if } \underline{w} = \underline{w}' \\ \beta^{-\min\{n \geq 0:\ w_n \neq w'_n\}} & \text{otherwise.} \end{cases} \tag{2.1}
$$

Similarly, for $\underline{w}$ a word on $\mathtt{A}$ of length n and $m \in \mathbb{Z}$, we define the cylinder in $\Sigma_{\mathtt{A}}$ by

$$
{}_m[\underline{w}] := \{\underline{z} \in \Sigma_{\mathtt{A}} : \underline{z}_{[m,m+n)} = \underline{w}\} .
$$

As before, the family $\{{}_m[\underline{w}] : \underline{w} \in \mathtt{A}^*, m \in \mathbb{Z}\}$ is a base for the topology on $\Sigma_{\mathtt{A}}$ and this topology is metrizable. We define the **left shift map** $\sigma : \Sigma_{\mathtt{A}}^+ \to \Sigma_{\mathtt{A}}^+$ by $\underline{z} := \sigma(\underline{w})$ with $z_i = w_{i+1}$ for all $i \in \mathbb{Z}_+$. The left shift on $\Sigma_{\mathtt{A}}^+$ is continuous and surjective. Similarly, we define the left shift map $\sigma : \Sigma_{\mathtt{A}} \to \Sigma_{\mathtt{A}}$ by $\underline{z} := \sigma(\underline{w})$ with $z_i = w_{i+1}$ for all $i \in \mathbb{Z}$. The left shift on $\Sigma_{\mathtt{A}}$ is continuous and bijective. We use the same notations for the left shift or the cylinders on $\Sigma_{\mathtt{A}}^+$ and on $\Sigma_{\mathtt{A}}$, but the context will always clarify which object is used.

On Σ_k^+, we define a total order denoted by $\prec$. Given a map $s : \mathtt{A}_k \to \{+1, -1\}$, define a map $\delta : \mathtt{A}_k^* \to \{+1, -1\}$ by

$$
\delta(\underline{w}) := \begin{cases} s(w_0)s(w_1)\cdots s(w_{n-1}) & \text{if } \underline{w} = w_0 w_1 \dots w_{n-1} \\ +1 & \text{if } \underline{w} = \varepsilon\,. \end{cases}
$$

Let $\underline{x} \neq \underline{x}' \in \Sigma_k^+$ and $n = \min\{j \geq 0 : x_j \neq x'_j\}$. Then we define the total order by

$$
\underline{x} \prec \underline{x}' \quad \iff \quad \begin{cases} x_n < x'_n & \text{if } \delta(\underline{x}_{[0,n)}) = +1 \\ x_n > x'_n & \text{if } \delta(\underline{x}_{[0,n)}) = -1\,. \end{cases} \tag{2.2}
$$

As usual we write $\underline{x} \preceq \underline{x}'$, if and only if $\underline{x} \prec \underline{x}'$ or $\underline{x} = \underline{x}'$. When $s(j) = +1$ for all $j \in \mathtt{A}_k$, this order is the lexicographic order.

A **shift space** is a closed subset $\Sigma \subset \Sigma_{\mathtt{A}}^+$ (or $\Sigma \subset \Sigma_{\mathtt{A}}$) such that $\sigma(\Sigma) \subset \Sigma$. It is endowed with the induced topology. In particular, for all $k \in \mathbb{N}$, a shift space $\Sigma \subset \Sigma_k^+$ (or $\Sigma \subset \Sigma_k$) is compact. Let Σ be a shift space, the **language** of Σ is

$$
\mathcal{L}(\Sigma) := \{\underline{w} \in \mathtt{A}^* : \exists \underline{z} \in \Sigma, i, j \in \mathbb{Z} \text{ s.t. } \underline{w} = \underline{z}_{[i,j]}\} .
$$

2.2 Piecewise monotone continuous maps

We turn to the question of the expansion of real numbers. We first define a family of piecewise monotone continuous maps, then we define the representation function and the coding map. Notice that, in the papers of Rényi [R] or Parry [P2], the authors first choose a representation function, then they define the corresponding coding map and piecewise monotone continuous map; this choice is justified, because the underlying idea of Rényi and Parry is the expansion of numbers. We are mainly interested by the piecewise monotone continuous map; thus we first choose this map, then we define the corresponding representation function and the coding map. This approach is also the one of Hofbauer in [H3]. We consider the class of piecewise monotone continuous maps of the interval in the following setting.

Definition 2.1. *A map $T : [0,1] \to [0,1]$ is* ***piecewise monotone continuous****, if there exist $k \geq 2$ and $0 = a_0 < a_1 < \cdots < a_k = 1$ such that, for all $j \in \mathtt{A}_k$, the map $f_j := T|_{(a_j,a_{j+1})}$ is monotone and continuous. We set $I_j := (a_j, a_{j+1})$ and $J_j := f_j(I_j) \subset [0,1]$. The maps $f_j : I_j \to J_j$ are the* ***laps*** *of T. For a piecewise monotone continuous map T, define $X_0 := [0,1]$, $S_0 := \{a_0, \ldots, a_k\}$ and for all $n \geq 1$*

$$X_n := X_{n-1} \backslash S_{n-1}, \quad S_n := \{x \in X_n : T(x) \in S_{n-1}\}.$$

Finally we set $S := \bigcup_{n \geq 0} S_n$ and $X := X_0 \backslash S$.

In all this thesis, we reserve the letter k to denote the number of laps of the piecewise monotone continuous map considered or the number of symbol of a shift space. The values of T on S_0 are not very important and we do not need to define T on this set for this chapter. However, if $\lim_{x \uparrow a_j} f_{j-1}(x) = \lim_{x \downarrow a_j} f_j(x)$ for some $j = 1, \ldots, k-1$, then we define $T(a_j)$ by continuity. We will see later that when all maps f_j are increasing, there is a convenient way to define T on S_0. When necessary, we also denote by f_j the continuous extension of the map on the closure $\overline{I}_j$ of I_j.

Henceforth Σ_k^+ is endowed with the total order defined by (2.2), where the map $s : \mathtt{A}_k \to \{+1, -1\}$ is given by

$$s(j) := \begin{cases} +1 & \text{if } f_j \text{ is increasing} \\ -1 & \text{if } f_j \text{ is decreasing.} \end{cases}$$

We define a map φ on the disjoint union

$$\operatorname{dom} \varphi := \bigcup_{j=0}^{k-1} j + J_j \subset \mathbb{R},$$

by setting

$$\varphi(x) := f_j^{-1}(t) \quad \text{if } x = j + t \text{ and } t \in J_j\,. \tag{2.3}$$

The map φ is continuous, injective and has range X_1. On X_1, the inverse map is

$$\varphi^{-1}(x) = j + T(x) \quad \text{if } x \in I_j\,.$$

For each $j \in \mathtt{A}_k$ such that f_j is increasing, we define $\overline{\varphi}^j$ on $j + [0,1]$ (using the extension of f_j on $[a_j, a_{j+1}]$) by

$$\overline{\varphi}^j(x) := \begin{cases} a_j & \text{if } x = j + t \text{ and } t \leq f_j(a_j) \\ f_j^{-1}(t) & \text{if } x = j + t \text{ and } t \in J_j \\ a_{j+1} & \text{if } x = j + t \text{ and } t \geq f_j(a_j)\,. \end{cases} \tag{2.4}$$

For each $j \in \mathtt{A}_k$ such that f_j is decreasing, we define $\overline{\varphi}^j$ on $j + [0,1]$ by

$$\overline{\varphi}^j(x) := \begin{cases} a_{j+1} & \text{if } x = j + t \text{ and } t \leq f_j(a_j) \\ f_j^{-1}(t) & \text{if } x = j + t \text{ and } t \in J_j \\ a_j & \text{if } x = j + t \text{ and } t \geq f_j(a_j)\,. \end{cases} \tag{2.5}$$

It is convenient below to consider the family of maps $\overline{\varphi}^j$ as a single map defined on $[0,k]$, which is denoted by $\overline{\varphi}$ and called the **representation function**. In order to avoid ambiguities at integers, where the map may be multi-valued, we always write a point of $[j,j+1]$ as $x=j+t$, $t\in[0,1]$, so that

$$\overline{\varphi}(x)\equiv\overline{\varphi}(j+t):=\overline{\varphi}^j(t)\,.$$

We define the **coding map** $\mathtt{i}:X\to\Sigma_k^+$ by

$$\mathtt{i}(x):=\mathtt{i}_0(x)\mathtt{i}_1(x)\dots\quad\text{with } \mathtt{i}_n(x)=j\iff T^n(x)\in I_j\,.$$

For any $x\in X$ and any $n\geq 0$,

$$\varphi^{-1}(T^n(x))=\mathtt{i}_n(x)+T^{n+1}(x)\quad\text{and}\quad \mathtt{i}(T^n(x))=\sigma^n\mathtt{i}(x)\,. \tag{2.6}$$

Let $\underline{z}\in\Sigma_k^+$ and $t\in[0,1]$; we set

$$\overline{\varphi}_1(z_0+t)=\overline{\varphi}(z_0+t)$$

and for all $n\geq 2$

$$\overline{\varphi}_n(z_0,\dots,z_{n-1}+t):=\overline{\varphi}_{n-1}(z_0,\dots,z_{n-2}+\overline{\varphi}(z_{n-1}+t))\,.$$

For all $n\geq 1$ and all $m\geq 1$, we have

$$\overline{\varphi}_{n+m}(z_0,\dots,z_{n+m-1}+t)=\overline{\varphi}_n(z_0,\dots,z_{n-1}+\overline{\varphi}_m(z_n,\dots,z_{n+m-1}+t))\,. \tag{2.7}$$

The map $t\mapsto\overline{\varphi}_n(z_0,\dots,z_{n-1}+t)$ is increasing if $\delta(\underline{z}_{[0,n)})=+1$ and decreasing if $\delta(\underline{z}_{[0,n)})=-1$. It is convenient to write $\overline{\varphi}_n(\underline{z})$ for $\overline{\varphi}_n(z_0,\dots,z_{n-1})$.

Example A. Let $\beta\geq 1$ and $\alpha\in[0,1)$ and set $k=\lceil\beta+\alpha\rceil$. The set $S_0=\{a_0,\dots,a_k\}$ is defined by

$$a_0=0\,,\qquad a_j=\frac{j-\alpha}{\beta}\quad\text{for } 1\leq j\leq k-1\,,\qquad a_k=1\,.$$

For all $j\in\mathtt{A}_k$, the map f_j is given by $f_j(x)=\beta x+\alpha-j$. The corresponding piecewise monotone continuous map is denoted by $T_{\alpha,\beta}$. It was introduced by Parry in [P2]; henceforth we denote simply this map by $T_{\alpha,\beta}(x)=\beta x+\alpha \mod 1$. The corresponding coding map is denoted by $\mathtt{i}^{\alpha,\beta}$ and the representation function $\overline{\varphi}^{\alpha,\beta}:[0,k]\to[0,1]$ is given by

$$\overline{\varphi}^{\alpha,\beta}(x)=\begin{cases}0 & \text{if } 0\leq x\leq\alpha\\ \frac{x-\alpha}{\beta} & \text{if } \alpha\leq x\leq\beta+\alpha\\ 1 & \text{if } \beta+\alpha\leq x\leq 1\,.\end{cases}$$

Example B. Let $k\geq 2$, fix a map $s:\mathtt{A}_k\to\{+1,-1\}$ and choose $\beta\in(k-1,k]$. The set S_0 is given by

$$a_j=\frac{j}{\beta}\quad\text{for } j\in\mathtt{A}_k\,,\qquad a_k=1\,.$$

For all $j\in\mathtt{A}_k$, the map f_j is defined by

$$f_j(x)=\begin{cases}\beta x-j & \text{if } s(j)=+1\\ 1-(\beta x-j) & \text{if } s(j)=-1\,.\end{cases} \tag{2.8}$$

The corresponding piecewise monotone continuous map is denoted T_β and called a generalized β-transformation; it was introduced by Góra in [G]. The corresponding coding map is denoted by $\mathtt{i}^\beta$. In Figure 2.1, we plot T_β and the corresponding map $\overline{\varphi}$ with $k=3$, $\{s(j)\}_j=(1,-1,-1)$ and $\beta=2.6$. Remark that T_β and $\mathtt{i}^\beta$ depend on the map s, however we do not write this dependence explicitly.

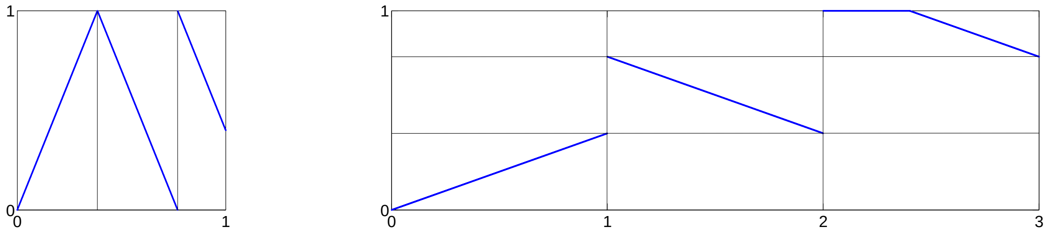

Figure 2.1: On the left, the map T_β; on the right the corresponding $\overline{\varphi}$.

2.2.1 Validity of the φ-expansion

We have defined the piecewise monotone continuous map T, the corresponding representation function $\overline{\varphi}$ and the coding map $\mathtt{i}$. We now discuss about the validity of the representation of numbers with the map $\overline{\varphi}$.

Definition 2.2. *The real number $s \in [0,1]$ has a φ-**expansion** $\underline{x} \in \Sigma_k^+$ if the following limit exists,*

$$s = \lim_{n\to\infty} \overline{\varphi}_n(\underline{x}) \equiv \overline{\varphi}\big(x_0 + \overline{\varphi}(x_1 + \dots)\big) \equiv \overline{\varphi}_\infty(\underline{x})\,.$$

*The φ-expansion is **well-defined** if for all $\underline{x} \in \Sigma_k^+$, $\lim_{n\to\infty} \overline{\varphi}_n(\underline{x}) \equiv \overline{\varphi}_\infty(\underline{x})$ exists.*
*The φ-expansion is **valid** if for all $x \in X$, $\mathtt{i}(x)$ is a φ-expansion of x.*

If the φ-expansion is valid, then for $x \in X$, using (2.6), (2.7) and the continuity of the maps $\overline{\varphi}^j$,

$$\begin{aligned} x &= \lim_{n\to\infty} \overline{\varphi}_n\big(\mathtt{i}(x)\big) \\ &= \lim_{m\to\infty} \overline{\varphi}_n\big(\mathtt{i}_0(x),\dots,\mathtt{i}_{n-1}(x) + \overline{\varphi}_m\big(\mathtt{i}_n(x),\dots,\mathtt{i}_{n+m-1}(x)\big)\big) \\ &= \overline{\varphi}_n\big(\mathtt{i}_0(x),\dots,\mathtt{i}_{n-1}(x) + \overline{\varphi}_\infty(\mathtt{i}(T^n x)\big)\,. \end{aligned} \tag{2.9}$$

The elementary fact of the φ-expansion is

$$a, b \in [0,1] \text{ and } x_0 < x_0' \in \mathtt{A}_k \implies \overline{\varphi}(x_0 + a) \le \overline{\varphi}(x_0' + b)\,. \tag{2.10}$$

We begin with two lemmas on the φ-code.

Lemma 2.3. *The map $\mathtt{i}$ is $\preceq$-order-preserving on X: $x \le y \in X$ implies $\mathtt{i}(x) \preceq \mathtt{i}(y)$.*

Proof: Let $x < y$. Either $\mathtt{i}_0(x) < \mathtt{i}_0(y)$, or $\mathtt{i}_0(x) = \mathtt{i}_0(y)$; in the latter case, the strict monotonicity of $f_{\mathtt{i}_0(x)}$ implies

$$\begin{aligned} \varphi^{-1}(x) &= \mathtt{i}_0(x) + T(x) < \varphi^{-1}(y) = \mathtt{i}_0(y) + T(y) \quad \text{if } \delta(\mathtt{i}_0(x)) = +1 \\ \varphi^{-1}(x) &= \mathtt{i}_0(x) + T(x) > \varphi^{-1}(y) = \mathtt{i}_0(y) + T(y) \quad \text{if } \delta(\mathtt{i}_0(x)) = -1. \end{aligned}$$

Repeating this argument we get $\mathtt{i}(x) \preceq \mathtt{i}(y)$. $\square$

Lemma 2.4. *The φ-code $\mathtt{i} : X \to \Sigma_k^+$ is continuous.*

Proof: Let $x \in X$ and $\{x^n\} \subset X$ such that $\lim_n x^n = x$. Let $j_0 = \mathtt{i}_0(x)$. There is n_0 such that for $n \geq n_0$, $x^n \in I_{j_0}$ and $\mathtt{i}_0(x^n) = \mathtt{i}_0(x) = j_0$. Let $j_1 := \mathtt{i}_1(x)$; we can choose n_1 so large that for $n \geq n_1$ $T(x_n) \in I_{j_1}$. Hence $\mathtt{i}_0(x^n) = j_0$ and $\mathtt{i}_1(x^n) = j_1$ for all $n \geq n_1$. By induction we can find an increasing sequence $\{n_m\}$ such that $n \geq n_m$ implies $\mathtt{i}_j(x) = \mathtt{i}_j(x^n)$ for all $j = 0, \ldots, m$. □

The next lemmas give the essential properties of the map $\overline{\varphi}_\infty$.

Lemma 2.5. *Let $\underline{x} \in \Sigma_k^+$. Then there exist $y_\uparrow(\underline{x})$ and $y_\downarrow(\underline{x})$ in $[0,1]$, such that $y_\uparrow(\underline{x}) \leq y_\downarrow(\underline{x})$; $y_\uparrow(\underline{x})$ and $y_\downarrow(\underline{x})$ are the only possible cluster points of the sequence $\{\overline{\varphi}_n(\underline{x})\}_n$.*
Let $x \in X$ and set $\underline{x} := \mathtt{i}(x)$. Then

$$a_{x_0} \leq y_\uparrow(\underline{x}) \leq x \leq y_\downarrow(\underline{x}) \leq a_{x_0+1}\,.$$

If the φ-expansion is valid, then each $y \in X$ has a unique φ-expansion,

$$y = \overline{\varphi}_\infty(\underline{x}) \in X \iff \underline{x} = \mathtt{i}(y)\,.$$

Proof: Consider the map

$$t \mapsto \overline{\varphi}_n(x_0, \ldots, x_{n-1} + t)\,.$$

Suppose that $\delta(\underline{x}_{[0,n)}) = -1$. Then it is decreasing, and for any m

$$\begin{aligned} \overline{\varphi}_{n+m}(x_0, \ldots, x_{n+m-1}) &= \overline{\varphi}_n(x_0, \ldots, x_{n-1} + \overline{\varphi}_m(x_n, \ldots, x_{n+m-1})) \\ &\leq \overline{\varphi}_n(x_0, \ldots, x_{n-1})\,. \end{aligned}$$

In particular the subsequence $\{\overline{\varphi}_n(\underline{x})\}_n$ of all n such that $\delta(\underline{x}_{[0,n)}) = -1$ is decreasing with limit $y_\downarrow(\underline{x})$. If the subsequence is finite, then $y_\downarrow(\underline{x})$ is the last point of the subsequence. If there is no n such that $\delta(\underline{x}_{[0mn)}) = -1$, we set $y_\downarrow(\underline{x}) := a_{x_0+1}$. Similarly, the subsequence $\{\overline{\varphi}_n(\underline{x})\}_n$ of all n such that $\delta(\underline{x}_{[0,n)}) = 1$ is increasing with limit $y_\uparrow(\underline{x}) \leq y_\downarrow(\underline{x})$. When there is no n such that $\delta(\underline{x}_{[0,n)}) = 1$, we set $y_\uparrow(\underline{x}) := a_{x_0}$. Since any $\overline{\varphi}_n(\underline{x})$ appears in one of these sequences, there are at most two cluster points for $\{\overline{\varphi}_n(\underline{x})\}_n$.
Let $x \in X$; $x = \varphi(\varphi^{-1}(x))$ and by (2.6)

$$\begin{aligned} x &= \varphi(\mathtt{i}_0(x) + T(x)) = \varphi(\mathtt{i}_0(x) + \varphi(\varphi^{-1}(Tx))) = \varphi(\mathtt{i}_0(x) + \varphi(\mathtt{i}_1(x) + T^2x)) = \cdots \\ &= \varphi\big(\mathtt{i}_0(x) + \varphi(\mathtt{i}_1(x) + \cdots + \varphi(\mathtt{i}_{n-1}(x) + T^n x))\big)\,. \end{aligned} \tag{2.11}$$

By monotonicity

$$\big(x \in X \text{ and } \delta(\mathtt{i}_{[0,n)}(x)) = -1\big) \implies \overline{\varphi}_n(\mathtt{i}_{[0,n)}(x)) \geq x\,,$$

and

$$\big(x \in X \text{ and } \delta(\mathtt{i}_{[0,n)}(x)) = 1\big) \implies \overline{\varphi}_n(\mathtt{i}_{[0,n)}(x)) \leq x\,.$$

The inequalities of Lemma 2.5 follow from these equations and $\overline{\varphi}(\mathtt{i}_0(x) + t) \in [a_{x_0}, a_{x_0+1}]$.

Suppose that the φ-expansion is valid and that $\overline{\varphi}_\infty(\underline{x}) = y \in X$. We prove that $\underline{x} = \mathtt{i}(y)$. By hypothesis $y \in I_{x_0}$; using (2.11) and the fact that I_{x_0} is open, we can write

$$y = \overline{\varphi}\big(x_0 + \overline{\varphi}(x_1 + \overline{\varphi}(x_2 + \ldots))\big) = \varphi\big(x_0 + \overline{\varphi}(x_1 + \overline{\varphi}(x_2 + \ldots))\big)\,.$$

This implies that

$$\varphi^{-1}(y) = \mathtt{i}_0(y) + Ty = x_0 + \overline{\varphi}(x_1 + \overline{\varphi}(x_2 + \ldots))\,.$$

Since $Ty \in X$, we can iterate this argument. □

Lemma 2.6. *Let $\underline{x}, \underline{x}' \in \Sigma_k^+$ and $\underline{x} \preceq \underline{x}'$. Then any cluster point of $\{\overline{\varphi}_n(\underline{x})\}_n$ is smaller then any cluster point of $\{\overline{\varphi}_n(\underline{x}'\}_n$. In particular, if the φ-expansion is well-defined, then $\overline{\varphi}_\infty$ is order-preserving.*

Proof: Let $\underline{x} \prec \underline{x}'$ with $x_j = x'_j$, $j = 0, \dots, m-1$ and $x_m \neq x'_m$. We have

$$\overline{\varphi}_{m+n}(\underline{x}) = \overline{\varphi}_m(x_0, \dots, x_{m-1} + \overline{\varphi}_n(\sigma^m \underline{x}))\,.$$

By (2.10), if $\delta(\underline{x}_{[0,m)}) = 1$, then $x_m < x'_m$ and for any $n \geq 1$, $\ell \geq 1$,

$$\overline{\varphi}_n(\sigma^m \underline{x}) = \overline{\varphi}_1(x_m + \overline{\varphi}_{n-1}(\sigma^{m+1}\underline{x})) \leq \overline{\varphi}_\ell(\sigma^m \underline{x}') = \overline{\varphi}_1(x'_m + \overline{\varphi}_{\ell-1}(\sigma^{m+1}\underline{x}'))\,;$$

if $\delta(\underline{x}_{[0,m)}) = -1$, then $x_m > x'_m$ and

$$\overline{\varphi}_n(\sigma^m \underline{x}) = \overline{\varphi}_1(x_m + \overline{\varphi}_{n-1}(\sigma^{m+1}\underline{x})) \geq \overline{\varphi}_\ell(\sigma^m \underline{x}') = \overline{\varphi}_1(x'_m + \overline{\varphi}_{\ell-1}(\sigma^{m+1}\underline{x}'))\,.$$

Therefore, in both cases, for any $n \geq 1$, $\ell \geq 1$,

$$\overline{\varphi}_{m+n}(\underline{x}) \leq \overline{\varphi}_{m+\ell}(\underline{x}')\,.$$

□

Lemma 2.7. *Let $\underline{x} \in \Sigma_k^+$ and $x_0 = j$.*
1) Let $\delta(j) = 1$ and $y_\uparrow(\underline{x}) \in \overline{I}_j$ be a cluster point of $\{\overline{\varphi}_n(\underline{x})\}$. Then $f_j\big(y_\uparrow(\underline{x})\big) \geq y_\uparrow(\sigma\underline{x})$ if $y_\uparrow(\underline{x}) = a_j$, $f_j\big(y_\uparrow(\underline{x})\big) \leq y_\uparrow(\sigma\underline{x})$ if $y_\uparrow(\underline{x}) = a_{j+1}$ and $f_j\big(y_\uparrow(\underline{x})\big) = y_\uparrow(\sigma\underline{x})$ otherwise. The same conclusions hold when $y_\downarrow(\underline{x})$ is a cluster point of $\{\overline{\varphi}_n(\underline{x})\}$.
2) Let $\delta(j) = -1$ and $y_\uparrow(\underline{x}) \in \overline{I}_j$ be a cluster point of $\{\overline{\varphi}_n(\underline{x})\}$. Then $f_j\big(y_\uparrow(\underline{x})\big) \leq y_\downarrow(\sigma\underline{x})$ if $y_\uparrow(\underline{x}) = a_j$, $f_j\big(y_\uparrow(\underline{x})\big) \geq y_\downarrow(\sigma\underline{x})$ if $y_\uparrow(\underline{x}) = a_{j+1}$ and $f_j\big(y_\uparrow(\underline{x})\big) = y_\downarrow(\sigma\underline{x})$ otherwise. The same conclusions hold when $y_\downarrow(\underline{x})$ is a cluster point of $\{\overline{\varphi}_n(\underline{x})\}$.

Proof: Set $f_j(\overline{I}_j) := [\alpha_j, \beta_j]$. Suppose for example that $\delta(j) = -1$ and that n_k is the subsequence of all m such that $\delta(\underline{x}_{[0,m)}) = 1$. Since $\delta(j) = -1$ the sequence $\{\overline{\varphi}_{n_k-1}(\sigma\underline{x})\}_k$ is decreasing. Hence by continuity

$$y_\uparrow(\underline{x}) = \lim_k \overline{\varphi}_{n_k}(\underline{x}) = \overline{\varphi}(j + \lim_k \overline{\varphi}_{n_k-1}(\sigma\underline{x})) = \overline{\varphi}(j + y_\downarrow(\sigma\underline{x}))\,.$$

If $y_\uparrow(\underline{x}) = a_j$, then $f_j(a_j) = \beta_j \leq y_\downarrow(\sigma\underline{x})$; if $y_\uparrow(\underline{x}) = a_{j+1}$, then $f_j(a_{j+1}) = \alpha_j \geq y_\downarrow(\sigma\underline{x})$; if $a_j < y_\uparrow(\underline{x}) < a_{j+1}$, then

$$j + f_j\big(y_\uparrow(\underline{x})\big) = \varphi^{-1}\big(\varphi(j + \lim_k \overline{\varphi}_{n_k-1}(\sigma\underline{x}))\big) = j + y_\downarrow(\sigma\underline{x})\,.$$

Similar proofs for the other cases. □

Lemma 2.8. *Let $\underline{x} \in \Sigma_k^+$.*
1) If $\{\overline{\varphi}_n(\underline{x})\}$ has two cluster points $y_\uparrow(\underline{x})$ and $y_\downarrow(\underline{x})$, and if $y \in \big(y_\uparrow(\underline{x}), y_\downarrow(\underline{x})\big)$, then $y \in X$, $\mathtt{i}(y) = \underline{x}$ and y has no φ-expansion.
Let $x \in X$ and set $\underline{x} := \mathtt{i}(x)$.
2) If $\lim_n \overline{\varphi}_n(\underline{x}) = y_\uparrow(\underline{x})$ and if $y \in (y_\uparrow(\underline{x}), x)$, then $y \in X$, $\mathtt{i}(y) = \underline{x}$ and y has no φ-expansion.
3) If $\lim_n \overline{\varphi}_n(\underline{x}) = y_\downarrow(\underline{x})$ and if $y \in (x, y_\downarrow(\underline{x}))$, then $y \in X$, $\mathtt{i}(y) = \underline{x}$ and y has no φ-expansion.

Proof: Suppose that $y_\uparrow(\underline{x}) < y < y_\downarrow(\underline{x})$. Then $y \in I_{x_0}$ and $\mathtt{i}_0(y) = x_0$. From Lemma 2.3

$$y_\uparrow(\sigma\underline{x}) < Ty < y_\downarrow(\sigma\underline{x}) \quad \text{if } \delta(x_0) = 1\,,$$

and

$$y_\downarrow(\sigma\underline{x}) > Ty > y_\uparrow(\sigma\underline{x}) \quad \text{if } \delta(x_0) = -1\,.$$

Iterating this argument we prove that $T^n y \in I_{x_n}$ and $\mathtt{i}_n(y) = x_n$ for all $n \geq 1$. Suppose that y has a φ-expansion, $y = \overline{\varphi}_\infty(\underline{x}')$. If $\underline{x}' \prec \underline{x}$, then by Lemma 2.6 $\overline{\varphi}(\underline{x}') \leq y_\uparrow(\underline{x})$ and if $\underline{x} \prec \underline{x}'$, then by Lemma 2.6 $y_\downarrow(\underline{x}) \leq \overline{\varphi}(\underline{x}')$, which leads to a contradiction. Similar proofs in cases 2 and 3. $\square$

Lemma 2.9. *Let $\underline{x}' \in \Sigma_k^+$ and $x \in X$. Then*

$$y_\downarrow(\underline{x}') < x \implies \underline{x}' \preceq \mathtt{i}(x) \quad \textit{and} \quad x < y_\uparrow(\underline{x}') \implies \mathtt{i}(x) \preceq \underline{x}'.$$

Proof: Suppose that $y_\downarrow(\underline{x}') < x$ and $y_\downarrow(\underline{x}')$ is a cluster point. Either $x_0' < \mathtt{i}_0(x)$ or $x_0' = \mathtt{i}_0(x)$ and by Lemma 2.7

$$y_\downarrow(\sigma\underline{x}') < Tx \quad \text{if } \delta(x_0') = 1,$$

or

$$y_\uparrow(\sigma\underline{x}') > Tx \quad \text{if } \delta(x_0') = -1.$$

Since $y_\downarrow(\sigma\underline{x}')$ or $y_\uparrow(\sigma\underline{x}')$ is a cluster point we can repeat the argument and conclude that $\underline{x}' \preceq \mathtt{i}(x)$. If $y_\downarrow(\underline{x}')$ is not a cluster point, then we use the cluster point $y_\uparrow(\underline{x}') < y_\downarrow(\underline{x}')$ for the argument. $\square$

The next theorem is due to Parry in [P2].

Theorem 2.10. *A φ-expansion is valid if and only if the φ-code $\mathtt{i}$ is injective on X.*

Proof: Suppose that the φ-expansion is valid. If $x \neq z$, then

$$x = \overline{\varphi}\big(\mathtt{i}_0(x) + \overline{\varphi}(\mathtt{i}_1(x) + \ldots)\big) \neq \overline{\varphi}\big(\mathtt{i}_0(z) + \overline{\varphi}(\mathtt{i}_1(z) + \ldots)\big) = z\,,$$

and therefore $\mathtt{i}(x) \neq \mathtt{i}(z)$. Conversely, assume that $x \neq z$ implies $\mathtt{i}(x) \neq \mathtt{i}(z)$. Let $x \in X$, $\underline{x} = \mathtt{i}(x)$, and suppose for example that $y_\uparrow(\underline{x}) < y_\downarrow(\underline{x})$ are two cluster points. Then by Lemma 2.8 any y such that $y_\uparrow(\underline{x}) < y < y_\downarrow(\underline{x})$ is in X and $\mathtt{i}(y) = \underline{x}$, contradicting the hypothesis. Therefore $z := \lim_n \overline{\varphi}_n(\underline{x})$ exists. If $z \neq x$, then we get again a contradiction using Lemma 2.8. $\square$

Theorem 2.10 states that the validity of the φ-expansion is equivalent to the injectivity of the map $\mathtt{i}$ defined on X. One can also state that the validity of the φ-expansion is equivalent to the surjectivity of the map $\overline{\varphi}_\infty$.

Theorem 2.11. *A φ-expansion is valid if and only if $\overline{\varphi}_\infty : \Sigma_k^+ \to [0,1]$ is well-defined on Σ_k^+ and surjective.*

Proof: Suppose that the φ-expansion is valid. Let $\underline{x} \in \Sigma_k^+$ and suppose that $\{\overline{\varphi}_n(\underline{x})\}_n$ has two different accumulation points $y_\uparrow < y_\downarrow$. By Lemma 2.8 we get a contradiction. Thus $\overline{\varphi}_\infty(\underline{x})$ is well-defined for any $\underline{x} \in \Sigma_k^+$.

To prove the surjectivity of $\overline{\varphi}_\infty$ it is sufficient to consider $s \in S$. The argument is a variant of the proof of Lemma 2.8. Let $\underline{x}'$ be a string such that for any $n \geq 1$

$$f_{x_{n-1}'} \circ \cdots \circ f_{x_0'}(s) \in \overline{I}_{x_n'}\,.$$

We use here the extension of f_j to $\overline{I}_j$; we have a choice for x'_n whenever $f_{x'_{n-1}} \circ \cdots \circ f_{x'_0}(s) \in S_0$. Suppose that $\overline{\varphi}_\infty(\underline{x}') < s$ and that $\overline{\varphi}_\infty(\underline{x}') < z < s$. Since $s, \overline{\varphi}_\infty(\underline{x}') \in \overline{I}_{x'_0}$, we have $z \in I_{x'_0}$ and therefore $\mathtt{i}(z) = x'_0$. Moreover,

$$\overline{\varphi}_\infty(\sigma\underline{x}') < Tz < f_{x'_0}(s) \quad \text{if } \delta(x'_0) = 1$$

or

$$f_{x'_0}(s) < Tz < \overline{\varphi}_\infty(\sigma\underline{x}') \quad \text{if } \delta(x'_0) = -1\,.$$

Iterating the argument we get $z \in X$ and $\mathtt{i}(z) = \underline{x}'$, contradicting the validity of the φ-expansion. Similarly we exclude the possibility that $\overline{\varphi}_\infty(\underline{x}') > s$, thus proving the surjectivity of the map $\overline{\varphi}_\infty$.

Suppose that $\overline{\varphi}_\infty : \Sigma_k^+ \to [0,1]$ is well-defined and surjective. Let $x \in X$ and $\underline{x} = \mathtt{i}(x)$. Suppose that $x < \overline{\varphi}_\infty(\underline{x})$. By Lemma 2.8 any z, such that $x < z < \overline{\varphi}_\infty(\underline{x})$, does not have a φ-expansion. This contradicts the hypothesis that $\overline{\varphi}_\infty$ is surjective. Similarly we exclude the possibility that $x > \overline{\varphi}_\infty(\underline{x})$. □

In the next theorem, we present the links between the validity of the φ-expansion and the continuity of the map $\overline{\varphi}_\infty$.

Theorem 2.12. *A φ-expansion is valid if and only if $\overline{\varphi}_\infty : \Sigma_k^+ \to [0,1]$ is well-defined, continuous and there exist $\underline{x}^+$ with $\overline{\varphi}_\infty(\underline{x}^+) = 1$ and $\underline{x}^-$ with $\overline{\varphi}_\infty(\underline{x}^-) = 0$.*

Proof: Suppose that the φ-expansion is valid. By Theorem 2.11 $\overline{\varphi}_\infty$ is well-defined and surjective so that there exist $\underline{x}^+$ and $\underline{x}^-$ with $\overline{\varphi}_\infty(\underline{x}^+) = 1$ and $\overline{\varphi}_\infty(\underline{x}^-) = 0$. Suppose that $\underline{x}^n \downarrow \underline{x}$ and set $y := \overline{\varphi}_\infty(\underline{x})$, $x_n := \overline{\varphi}_\infty(\underline{x}^n)$. By Lemma 2.6 the sequence $\{x_n\}$ is monotone decreasing; let $x := \lim_n x_n$. Suppose that $y < x$ and $y < z < x$. Since $y < z < x_n$ for any $n \geq 1$ and $\lim_n \underline{x}^n = \underline{x}$, we prove, as in the beginning of the proof of Lemma 2.8, that $z \in X$. The validity of the φ-expansion implies that $z = \overline{\varphi}_\infty(\mathtt{i}(z))$. By Lemma 2.9

$$\underline{x} \preceq \mathtt{i}(z) \preceq \underline{x}^n\,.$$

Since these inequalities are valid for any z, with $y < z < x$, the validity of φ-expansion implies that we have strict inequalities, $\underline{x} \prec \mathtt{i}(z) \prec \underline{x}^n$. This contradicts the hypothesis that $\lim_{n\to\infty} \underline{x}^n = \underline{x}$. A similar argument holds in the case $\underline{x}^n \uparrow \underline{x}$. Hence

$$\lim_{n\to\infty} \underline{x}^n = \underline{x} \implies \lim_{n\to\infty} \overline{\varphi}_\infty(\underline{x}^n) = \overline{\varphi}_\infty(\underline{x})\,.$$

Conversely, suppose that $\overline{\varphi}_\infty : \Sigma_k^+ \to [0,1]$ is well-defined and continuous. Then, given $\delta > 0$ and $\underline{x} \in \Sigma_k^+$, $\exists n$ so that

$$0 \leq \sup\{\overline{\varphi}_\infty(\underline{x}') : x'_j = x_j \ j = 0,\ldots,n-1\} - \inf\{\overline{\varphi}_\infty(\underline{x}') : x'_j = x_j \ j = 0,\ldots,n-1\} \leq \delta\,.$$

We set

$$\underline{x}^{n,-} := x_0 \cdots x_{n-1}\underline{x}^- \quad \text{and} \quad \underline{x}^{n,+} := x_0 \cdots x_{n-1}\underline{x}^+\,.$$

For any $x \in X$ we have the identity (2.11),

$$x = \varphi\big(\mathtt{i}_0(x) + \varphi(\mathtt{i}_1(x) + \ldots + \varphi(\mathtt{i}_{n-1} + T^n x))\big) = \overline{\varphi}_n(\mathtt{i}_0(x), \ldots, \mathtt{i}_{n-1}(x) + T^n x)\,.$$

If $\delta(\mathtt{i}_0(x)\cdots\mathtt{i}_{n-1}(x))=1$, then

$$\begin{aligned}
\overline{\varphi}_\infty(\underline{x}^{n,-}) &:= \overline{\varphi}_n(\mathtt{i}_0(x),\ldots,\mathtt{i}_{n-1}(x)+\overline{\varphi}_\infty(\underline{x}^-))\\
&= \overline{\varphi}_n(\mathtt{i}_0(x),\ldots,\mathtt{i}_{n-1}(x))\\
&\leq \overline{\varphi}_n(\mathtt{i}_0(x),\ldots,\mathtt{i}_{n-1}(x)+T^n x)\\
&\leq \overline{\varphi}_n(\mathtt{i}_0(x),\ldots,\mathtt{i}_{n-1}(x)+1)\\
&= \overline{\varphi}_n(\mathtt{i}_0(x),\ldots,\mathtt{i}_{n-1}(x)+\overline{\varphi}_\infty(\underline{x}^+)) =: \overline{\varphi}_\infty(\underline{x}^{n,+})\,.
\end{aligned}$$

If $\delta(\mathtt{i}_0(x)\cdots\mathtt{i}_{n-1}(x))=-1$, then the inequalities are reversed. Letting n going to infinity, we get $\overline{\varphi}_\infty(\mathtt{i}(x))=x$. □

Notice that when the map f_0 is increasing, then we can take $\underline{x}^-=(0,0,\ldots)$.

Theorem 2.13. *A necessary and sufficient condition for a φ-expansion to be valid is that S is dense in $[0,1]$. A sufficient condition is that the derivative of $\varphi(t)$ exists and $\sup_t|\varphi'(t)|<1$.*

This is a theorem of Parry in [P2]; it is a corollary of Theorem 2.10.

2.2.2 The structure of $\mathtt{i}(X)$

In this section, we study the structure of the space $\mathtt{i}(X)\subset\Sigma_k^+$. The main result is Theorem 2.15. In particular, it shows, under the hypothesis of the validity of the φ-expansion, that the coding map is an isomorphism between (X,T) and $(\mathtt{i}(X),\sigma)$.

For each $j\in\mathtt{A}_k$ we define (the limits are taken with $x\in X$ and they exist by the monotonicity of the coding map)

$$\underline{u}^j := \lim_{x\downarrow a_j}\mathtt{i}(x) \quad\text{and}\quad \underline{v}^j := \lim_{x\uparrow a_{j+1}}\mathtt{i}(x)\,. \tag{2.12}$$

The strings $\underline{u}^j$ and $\underline{v}^j$ are called **virtual itineraries**. Notice that $\underline{v}^j\prec\underline{u}^{j+1}$ since $v_0^j<u_0^{j+1}$.

$$\sigma^n\underline{u}^j=\sigma^n(\lim_{x\downarrow a_j}\mathtt{i}(x))=\lim_{x\downarrow a_j}\sigma^n\mathtt{i}(x)=\lim_{x\downarrow a_j}\mathtt{i}(T^n x)\qquad(x\in X)\,. \tag{2.13}$$

The next proposition give a sufficient condition for a sequence to be the coding of some $x\in X$. When $k=2$ and the map T is continuous, it is a classical result (see for example Theorem II.3.8 of Collet-Eckmann in [CE]). Notice that the proof of Collet-Eckmann requires the continuity of T on $[0,1]$. Our proof is different and does not need the continuity.

Proposition 2.14. *Suppose that $\underline{x}'\in\Sigma_k^+$ verifies $\underline{u}^{x'_n}\prec\sigma^n\underline{x}'\prec\underline{v}^{x'_n}$ for all $n\geq 0$. Then there exists $x\in X$ such that $\mathtt{i}(x)=\underline{x}'$.*

Notice that we do not assume that the φ-expansion is valid or that the map $\overline{\varphi}_\infty$ is well-defined.

Proof: If $y_\uparrow(\underline{x}')<y_\downarrow(\underline{x}')$ are two cluster points, then this follows from Lemma 2.8. Therefore, assume that $\lim_n\overline{\varphi}_n(\underline{x}')$ exists. Either there exists $m>1$ so that $y_\uparrow(\sigma^m\underline{x}')<y_\downarrow(\sigma^m\underline{x}')$ are two cluster points, or $\lim_n\overline{\varphi}_n(\sigma^m\underline{x}')$ exists for all $m\geq 1$.

In the first case, there exists $z_m\in X$,

$$y_\uparrow(\sigma^m\underline{x}')<z_m<y_\downarrow(\sigma^m\underline{x}')\quad\text{and}\quad \mathtt{i}(z_m)=\sigma^m\underline{x}'\,.$$

Let

$$z_{m-1} := \overline{\varphi}(x'_{m-1} + z_m) \, .$$

We show that $a_{x'_{m-1}} < z_{m-1} < a_{x'_{m-1}+1}$. This implies that $z_m \in \mathrm{int}(\mathrm{dom}\ \varphi)$ so that

$$\varphi^{-1}(z_{m-1}) = x'_{m-1} + Tz_{m-1} = x'_{m-1} + z_m \, .$$

Suppose that $\delta(x'_{m-1}) = 1$ and $a_{x'_{m-1}} = z_{m-1}$. Then for any $y \in X$, $y > a_{x'_{m-1}}$, we have $Ty > z_m$. Therefore, by Lemma 2.3, $\mathtt{i}(Ty) \succeq \mathtt{i}(z_m) = \sigma^m \underline{x}'$; $\mathtt{i}_0(y) = x'_{m-1}$ when y is close to $a_{x'_{m-1}}$, so that

$$\lim_{y \downarrow a_{x'_{m-1}}} \mathtt{i}(y) = \underline{u}^{x'_{m-1}} \succeq \sigma^{m-1}\underline{x}' \, ,$$

which is a contradiction. Similarly we exclude the cases $\delta(x'_{m-1}) = 1$ and $a_{x'_{m-1}+1} = z_{m-1}$, $\delta(x'_{m-1}) = -1$ and $a_{x'_{m-1}} = z_{m-1}$, $\delta(x'_{m-1}) = -1$ and $a_{x'_{m-1}+1} = z_{m-1}$. Iterating this argument we get the existence of $z_0 \in X$ with $\mathtt{i}(z_0) = \underline{x}'$.

In the second case, $\lim_n \overline{\varphi}_n(\sigma^m \underline{x}')$ exists for all $m \geq 1$. Let $x := \lim_n \overline{\varphi}_n(\underline{x}')$. Suppose that $x'_0 = j$, so that $\underline{u}^j \prec \underline{x}' \prec \underline{v}^j$. By Lemma 2.3 and definition of $\underline{u}^j$ and $\underline{v}^j$ there exist $z_1, z_2 \in I_j$ such that

$$z_1 < x < z_2 \quad \text{and} \quad \underline{u}^j \preceq \mathtt{i}(z_1) \prec \underline{x}' \prec \mathtt{i}(z_2) \preceq \underline{v}^j \, .$$

Therefore $a_j < x < a_{j+1}$, $\mathtt{i}_0(x) = x'_0$ and $Tx = \overline{\varphi}_\infty(\sigma \underline{x}')$ (Lemma 2.7). Iterating this argument we get $\underline{x}' = \mathtt{i}(x)$. □

The next theorem collects well known results (see for example the publication of Bowen [B3] for the point 1 and the paper of Hofbauer [H3]).

Theorem 2.15. *Suppose that the φ-expansion is valid. Then*

1. $\Sigma := \{\mathtt{i}(x) \in \Sigma_k^+ : x \in X\} = \{\underline{x} \in \Sigma_k^+ : \underline{u}^{x_n} \prec \sigma^n \underline{x} \prec \underline{v}^{x_n} \quad \forall\, n \geq 0\}$.
2. *The map* $\mathtt{i} : X \to \Sigma$ *is bijective,* $\overline{\varphi}_\infty \circ \mathtt{i} = \mathrm{id}$ *and* $\mathtt{i} \circ \overline{\varphi}_\infty = \mathrm{id}$.
   Both maps $\mathtt{i}$ *and* $\overline{\varphi}_\infty$ *are order-preserving.*
3. $\sigma(\Sigma) \subset \Sigma$ *and* $\overline{\varphi}_\infty(\sigma \underline{x}) = T\overline{\varphi}_\infty(\underline{x})$ *if* $\underline{x} \in \Sigma$.
4. *If* $\underline{x} \in \Sigma_k^+ \backslash \Sigma$, *then there exist* $m \in \mathbb{Z}_+$ *and* $j \in \mathtt{A}_k$ *such that* $\overline{\varphi}_\infty(\sigma^m \underline{x}) = a_j$.
5. $\forall n \geq 0\, , \forall j \in \mathtt{A}_k :\quad \underline{u}^{u_n^j} \preceq \sigma^n \underline{u}^j \prec \underline{v}^{u_n^j}$ *if* $\delta(\underline{u}_0^j \cdots \underline{u}_{n-1}^j) = 1$ *and* $\underline{u}^{u_n^j} \prec \sigma^n \underline{u}^j \preceq \underline{v}^{v_n^j}$ *if* $\delta(\underline{u}_0^j \cdots \underline{u}_{n-1}^j) = -1$.
6. $\forall n \geq 0\, , \forall j \in \mathtt{A}_k :\quad \underline{u}^{u_n^j} \preceq \sigma^n \underline{v}^j \prec \underline{v}^{u_n^j}$ *if* $\delta(\underline{v}_0^j \cdots \underline{v}_{n-1}^j) = -1$ *and* $\underline{u}^{u_n^j} \prec \sigma^n \underline{v}^j \preceq \underline{v}^{v_n^j}$ *if* $\delta(\underline{u}_0^j \cdots \underline{u}_{n-1}^j) = 1$.

Proof: Let $x \in X$. Clearly, by monotonicity,

$$\underline{u}^{\mathtt{i}_n(x)} \preceq \sigma^n \mathtt{i}(x) \preceq \underline{v}^{\mathtt{i}_n(x)} \quad \forall\, n \in \mathbb{Z}_+ \, .$$

Suppose that there exist $x \in X$ and n such that $\sigma^n \mathtt{i}(x) = \underline{v}^{\mathtt{i}_n(x)}$. Since $\mathtt{i}_n(x) = \mathtt{i}_0(T^n x)$, we can assume, without restricting the generality, that $n = 0$ and $\mathtt{i}_0(x) = j$. Therefore $x \in (a_j, a_{j+1})$, and for all $y \in X$, such that $x \leq y < a_{j+1}$, we have by Lemma 2.3 that $\mathtt{i}(y) = \mathtt{i}(x) = \underline{v}^j$. By Theorem 2.10 this contradicts the hypothesis that the φ-expansion is valid. The other case, $\sigma^n \mathtt{i}(x) = \underline{u}^{\mathtt{i}_n(x)}$, is treated similarly. This proves half of the first statement. The second half is a consequence of Proposition 2.14. The second statement also follows, as well as the third, since $T(X) \subset X$.

Let $\underline{x} \in \Sigma_k^+ \backslash \Sigma$ and $m \in \mathbb{Z}_+$ be the smallest integer such that one of the conditions defining Σ is not verified. Then either $\sigma^m \underline{x} \preceq \underline{u}^{x_m}$, or $\sigma^m \underline{x} \succeq \underline{v}^{x_m}$. The map $\overline{\varphi}_\infty$ is continuous (Theorem 2.12). Hence, for any $j \in \mathtt{A}_k$,

$$\overline{\varphi}_\infty(\underline{u}^j) = a_j \quad \text{and} \quad \overline{\varphi}_\infty(\underline{v}^j) = a_{j+1} \, .$$

Let $\sigma^m \underline{x} \preceq \underline{u}^{x_m}$. Since $\underline{v}^{x_m - 1} \prec \sigma^m \underline{x}$,

$$a_{x_m} = \overline{\varphi}_\infty(\underline{v}^{x_m-1}) \leq \overline{\varphi}_\infty(\sigma^m \underline{x}) \leq \overline{\varphi}_\infty(\underline{u}^{x_m}) = a_{x_m} \, .$$

The other case is treated in the same way. From definition (2.12) $\underline{u}^{u_n^j} \preceq \sigma^n \underline{u}^j \preceq \underline{v}^{u_n^j}$. Suppose that $\delta(\underline{u}_0^j \cdots \underline{u}_{n-1}^j) = 1$ and $\sigma^n \underline{u}^j = \underline{v}^{u_n^j}$. By continuity of the φ-code there exists $x \in X$ such that $x > a_j$ and $\mathtt{i}_k(x) = \underline{u}_k^j$, $k = 0, \ldots, n$. Let $a_j < y < x$. Since $\delta(\underline{u}_0^j \cdots \underline{u}_{n-1}^j) = 1$, $T^n y < T^n x$ and consequently

$$\lim_{y \downarrow a_j} \mathtt{i}(T^n y) = \sigma^n \underline{u}^j \preceq \mathtt{i}(T^n x) \preceq \underline{v}^{u_n^j} \, .$$

Hence $\sigma^n \mathtt{i}(x) = \underline{v}^{x_n}$, which is a contradiction. The other cases are treated similarly. □

Example A (continuation). We consider the piecewise monotone continuous map $T_{\alpha,\beta}(x) := \beta x + \alpha \mod 1$ with $\beta \geq 1$ and $\alpha \in [0,1)$. By Theorem 2.13, the corresponding φ-expansion is valid, if and only if $\beta > 1$ or $\beta = 1$ and α is an irrational number. Henceforth we assume that the φ-expansion is valid. Since all laps of $T_{\alpha,\beta}$ are increasing, the order $\preceq$ is the lexicographic order. From the $2k$ virtual itineraries, only two of them are really important. Indeed setting $\underline{u}^{\alpha,\beta} := \underline{u}^0$ and $\underline{v}^{\alpha,\beta} := \underline{v}^{k-1}$, we have

$$\underline{u}^j = j\underline{u}^{\alpha,\beta} \quad \text{for } j = 1, \ldots, k-1, \qquad \underline{v}^j = j\underline{v}^{\alpha,\beta} \quad \text{for } j = 0, \ldots, k-2. \tag{2.14}$$

This is because $\lim_{x \downarrow a_j} T_{\alpha,\beta}(x) = 0$ for $j = 1, \ldots, k-1$ and $\lim_{x \uparrow a_{j+1}} T_{\alpha,\beta}(x) = 1$ for $j = 0, \ldots, k-2$. Together with point 1 of Theorem 2.15, this implies

$$\Sigma_{\alpha,\beta} := \overline{\mathtt{i}^{\alpha,\beta}(X)} = \{\underline{x} \in \Sigma_k^+ : \underline{u}^{\alpha,\beta} \preceq \sigma^n \underline{x} \preceq \underline{v}^{\alpha,\beta} \; \forall n \geq 0\} \, . \tag{2.15}$$

Indeed take $\underline{x} \in \Sigma_{\alpha,\beta}$ and $m \geq 0$. If $x_m = 0$, then $\underline{u}^0 \equiv \underline{u}^{\alpha,\beta} \preceq \sigma^m \underline{x}$. If $x_m > 0$, then $\underline{u}^{x_m} = x_m \underline{u}^{\alpha,\beta} \preceq \sigma^m \underline{x}$, since $\underline{u}^{\alpha,\beta} \preceq \sigma^{m+1} \underline{x}$. We work similarly with the $\underline{v}^j$. Likewise by points 5 and 6 of Theorem 2.15, we have

$$\underline{u}^{\alpha,\beta} \preceq \sigma^n \underline{u}^{\alpha,\beta} \prec \underline{v}^{\alpha,\beta} \quad \text{and} \quad \underline{u}^{\alpha,\beta} \prec \sigma^n \underline{v}^{\alpha,\beta} \preceq \underline{v}^{\alpha,\beta} \quad \forall n \geq 0 \, . \tag{2.16}$$

The map $\overline{\varphi}^{\alpha,\beta}$ being particulary simple, there is an explicit expression for $\overline{\varphi}_\infty^{\alpha,\beta}$ on $\Sigma_{\alpha,\beta}$. Suppose $\beta > 1$ and let $x \in X$ and $\underline{x} = \mathtt{i}^{\alpha,\beta}(x)$; by (2.10), we have for all $n \geq 1$,

$$\begin{aligned}
x = \varphi^{\alpha,\beta}\big(x_0 + T_{\alpha,\beta}(x)\big) &= \frac{x_0 - \alpha}{\beta} + \frac{1}{\beta}\varphi^{\alpha,\beta}\big(x_1 + T^2_{\alpha,\beta}(x)\big) \\
&= \frac{x_0 - \alpha}{\beta} + \frac{x_1 - \alpha}{\beta^2} + \frac{1}{\beta^2}\varphi^{\alpha,\beta}\big(x_2 + T^3_{\alpha,\beta}(x)\big) = \ldots \\
&= \sum_{j=0}^{n-1} \frac{x_j - \alpha}{\beta^{j+1}} + \frac{1}{\beta^n} T^n_{\alpha,\beta}(x) \, .
\end{aligned}$$

Since $T^n_{\alpha,\beta}(x) \in [0,1]$ and $\beta > 1$, we find taking the limit $n \to \infty$

$$\overline{\varphi}^{\alpha,\beta}_{\infty}(\underline{x}) = x = \sum_{j\geq 0} \frac{\mathtt{i}^{\alpha,\beta}_j(x) - \alpha}{\beta^{j+1}}\,.$$

Since the map $\overline{\varphi}^{\alpha,\beta}_{\infty}$ is continuous, we can extend this formula on $\Sigma_{\alpha,\beta}$. Thus

$$\overline{\varphi}^{\alpha,\beta}_{\infty}(\underline{x}) = \sum_{j\geq 0} \frac{x_j - \alpha}{\beta^{j+1}} \quad \forall \underline{x} \in \Sigma_{\alpha,\beta}\,. \tag{2.17}$$

Example B (continuation). Let $k \geq 2$ and fix the map $s : \mathtt{A}_k \to \{-1,1\}$ and $\beta \in (k-1,k]$. Define T as before in Example B (see (2.8)). By Theorem 2.13, the corresponding φ-expansion is valid. From the $2k$ virtual itineraries, only one of them is really important. We set $\underline{\eta}^\beta \equiv \underline{v}^{k-1} := \lim_{x\uparrow 1} \mathtt{i}(x)$ with $x \in X$. Then for all $j \in \mathtt{A}_k$

$$\underline{u}^j = \begin{cases} j\underline{\eta}^\beta & \text{if } \lim_{x\downarrow a_j} T(x) = 1\,, \\ j0\underline{\eta}^\beta & \text{if } \lim_{x\downarrow a_j} T(x) = 0 \text{ and } \lim_{x\downarrow 0} T(x) = 1\,, \\ j0^\infty & \text{if } \lim_{x\downarrow a_j} T(x) = 0 \text{ and } \lim_{x\downarrow 0} T(x) = 0\,. \end{cases} \tag{2.18}$$

Similarly for all $j = 0,\ldots,k-2$

$$\underline{v}^j = \begin{cases} j\underline{\eta}^\beta & \text{if } \lim_{x\uparrow a_{j+1}} T(x) = 1\,, \\ j0\underline{\eta}^\beta & \text{if } \lim_{x\uparrow a_{j+1}} T(x) = 0 \text{ and } \lim_{x\downarrow 0} T(x) = 1\,, \\ j0^\infty & \text{if } \lim_{x\uparrow a_{j+1}} T(x) = 0 \text{ and } \lim_{x\downarrow 0} T(x) = 0\,. \end{cases} \tag{2.19}$$

As in Example A, we can prove that

$$\Sigma_\beta := \overline{\mathtt{i}^\beta(X)} = \{\underline{x} \in \Sigma_k^+ : \sigma^n \underline{x} \preceq \underline{\eta}^\beta \ \forall n \geq 0\}\,. \tag{2.20}$$

Moreover the virtual itinerary satisfies

$$\sigma^n \underline{\eta}^\beta \preceq \underline{\eta}^\beta \ \forall n \geq 0\,. \tag{2.21}$$

Chapter 3

Entropy

In this chapter, we define the notion of measure-theoretical entropy and topological entropy. The measure-theoretical entropy was introduced by Kolmogorov in 1958. It is the most used invariant under measure preserving isomorphism between dynamical systems. The topological entropy was introduced by Adler, Konheim and McAndrew [AKM] in 1965 as an invariant of topological conjugacy. Later Bowen gave in [B1] another equivalent definition in the framework of metric spaces. This definition is historically important, because it leads to a very elegant proof of the variational principle by Misiurewicz. The variational principle is a strong result, linking the measure-theoretical entropy (defined using only a measurable space) and the topological entropy (defined using only a topological space): the supremum of measure-theoretical entropy over all T-invariant measures is equal to the topological entropy of T.

This chapter has two parts: first we define the measure-theoretical entropy, then the topological entropy and we present the variational principle. The definitions are given in a general framework, but we discuss particulary the dynamical systems considered in this thesis: the shift maps and the piecewise monotone continuous maps. As general references, we use the book of Bauer [Ba] for the measure theory and the book of Walters [W] for the ergodic theory.

3.1 Measure theoretical entropy

3.1.1 Measurable dynamical systems

Consider a compact metric space (X,d) and denote by $\mathcal{B}$ the Borel σ-algebra on X. Let $C(X)$ be the set of continuous functions $f : X \to \mathbb{R}$ and $M(X)$ be the set of Borel probability measures on X. The set $M(X)$ is convex. For $x \in X$, we denote by $\delta_x \in M(X)$ the Dirac mass at x. The set $M(X)$ is endowed with the **weak*-topology**, ie the coarsest topology on $M(X)$ such that the mappings

$$\mu \to \int f d\mu$$

are continuous for all $f \in C(X)$. This topology is metrizable and has a countable basis. The set $M(X)$ endowed with the weak*-topology is compact.

Suppose that $T : X \to X$ is a measurable map. The triple $(X, \mathcal{B}, T)$ is called a **compact measurable dynamical system**. Let $M(X,T) \subset M(X)$ denote the subset of

T-invariant measures

$$\mu \in M(X,T) \iff \mu \in M(X) \text{ and } \mu(T^{-1}A) = \mu(A)\ \forall A \in \mathcal{B}\,.$$

$M(X,T)$ is a convex closed subset of $M(X)$, the extremal points of $M(X,T)$ are the **ergodic** measures.

Lemma 3.1. *Let $(X,\mathcal{B},T)$ be a measurable dynamical system and $\mu \in M(X,T)$. Suppose $A \subset X$ and $B = X\backslash A$ are non empty subsets of X such that $T(B) \subset B$ (or equivalently $T^{-1}A \subset A$). Then $\mu\big((\bigcap_{n\geq 0} T^{-n}A) \cup B\big) = 1$. Suppose further that μ is ergodic, then $\mu(\bigcap_{n\geq 0} T^{-n}A) = 1$ or $\mu(B) = 1$.*

Proof: Define $C := A \cap T^{-1}B$. Then $T^{-n}C \cap T^{-m}C = \emptyset$ for all $n \neq m \in \mathbb{Z}_+$. Indeed, suppose that $T^{-n}C \cap C \neq$ for some $n \geq 1$. For $x \in T^{-n}C \cap C$, we have $T^n x \in C \subset A$ and $T^n x = T^{n-1}(Tx) \in T^{n-1}B \subset B$, but $A \cap B = \emptyset$. Thus $T^{-n}C \cap C = \emptyset$ and we prove the claim, applying T^{-m} on this equality. Now let $\mu \in M(X,T)$. We prove that $\mu(C) = 0$. Suppose that $\mu(T^{-n}C) = \mu(C) > 0$. Since all $T^{-n}C$ are pairwise disjoint, we conclude that $\mu(X) = \infty$, a contraction. Thus $\mu\big((\bigcap_{n\geq 0} T^{-n}A) \cup B\big) = 1$ for all $\mu \in M(X,T)$. Finally $T^{-1}A \subset A$ implies that $T^{-1}(\bigcap_{n\geq 0} T^{-n}A) = \bigcap_{n\geq 0} T^{-n}A$. If μ is ergodic, then $\mu(\bigcap_{n\geq 0} T^{-n}A) = 0$ or 1. □

3.1.2 The entropy map $h_T(\mu)$

Suppose that $(X,\mathcal{B},T)$ is a compact measurable dynamical system. Let α, β be two finite Borel partitions of X and define their **join** (which is also a finite Borel partition of X) by

$$\alpha \vee \beta := \{A \cap B : A \in \alpha, B \in \beta\}\,.$$

For $n \geq 0$ and α a finite Borel partition, define the finite Borel partition

$$T^{-n}\alpha := \{T^{-n}A : A \in \alpha\}$$

and

$$\bigvee_{j=0}^{n-1} T^{-j}\alpha := \alpha \vee T^{-1}\alpha \vee \cdots \vee T^{-n+1}\alpha\,.$$

Definition 3.2. *Let $\mu \in M(X,T)$. For a finite Borel partition α, define*

$$H(\mu,\alpha) := -\sum_{A\in\alpha} \mu(A)\log\mu(A)\,.$$

The base of the logarithms does not matter, we will use natural logarithms. Moreover $0\log 0$ must be interpreted as 0.
*The **entropy of μ with respect to α** is*

$$h_T(\mu,\alpha) := \lim_{n\to\infty} \frac{1}{n}\, H(\mu, \bigvee_{j=0}^{n-1} T^{-j}\alpha)\,,$$

*and the **measure-theoretical entropy** of μ is*

$$h_T(\mu) := \sup_{\alpha} h_T(\mu,\alpha)\,,$$

*where the supremum ranges over all finite Borel partitions. A **measure of maximal entropy** or a **maximal measure** is a measure $\mu \in M(X,T)$ such that*

$$h_T(\mu) = \sup\{h_T(\nu) : \nu \in M(X,T)\}\,.$$

Notice that the limit involved in the definition of $h_T(\mu)$ exists by a classical argument of subadditivity. We recall a well known property of the measure-theoretical entropy: the entropy map $\mu \mapsto h_T(\mu)$ is affine, ie for $\mu, \nu \in M(X,T)$ and $p \in [0,1]$, we have

$$h_T(p\mu + (1-p)\nu) = ph_T(\mu) + (1-p)h_T(\nu)\,.$$

Therefore, if the set of measures of maximal entropy is not empty, then it is a convex subset of $M(X,T)$.

In concrete cases, there is a difficulty in the computation of the measure-theoretical entropy: we must calculate $h_T(\mu,\alpha)$ for all finite partitions α and then take the supremum. Fortunately there exist finite partitions α such that $h_T(\mu,\alpha) = h_T(\alpha)$. The next theorem treats this point. The hypothesis are not minimal but sufficient for our work (see Theorem 4.17 and 4.18 in [W]).

Theorem 3.3. *Suppose that $(X, \mathcal{B}, T)$ is a compact measurable dynamical system. Let $\mu \in M(X,T)$ and α be a finite Borel partition of X. If $\mathcal{B}$ is the smallest σ-algebra containing $\bigvee_{i=0}^{n} T^{-i}\alpha$ for all $n \geq 1$, then $h_T(\mu,\alpha) = h_T(\mu)$.*
Suppose further that T is invertible and T^{-1} is measurable. If $\mathcal{B}$ is the smallest σ-algebra containing $\bigvee_{i=-n}^{n} T^{-i}\alpha$ for all $n \geq 1$, then $h_T(\mu,\alpha) = h_T(\mu)$.

We apply this theorem to shift spaces. Let $\Sigma \subset \Sigma_k^+$ be a shift space, $\mu \in M(\Sigma,\sigma)$ and consider the partition $\alpha = \{{}_0[j] : j \in \mathtt{A}_k\}$, then $h_\sigma(\mu,\alpha) = h_\sigma(\mu)$. A similar statement is true for bi-infinite shifts. The proof is obvious, a cylinder of length n being an element of $\bigvee_{i=0}^{n-1} T^{-i}\alpha$. We can also consider the case of piecewise monotone continuous maps. Let $T : [0,1] \to [0,1]$ be a piecewise monotone continuous map such that the φ-expansion is valid. If $\mu \in M([0,1],T)$ and $\alpha = \{I_j : j \in \mathtt{A}_k\}$, then $h_T(\mu) = h_T(\mu,\alpha)$. Indeed by Theorem 2.13, the validity of the φ-expansion implies that S is dense in $[0,1]$. Thus any open I interval is the union of all intervals J such that $J \subset I$ and $J \in \bigvee_{i=0}^{n-1} T^{-i}\alpha$ for some $n \geq 1$. So a σ-algebra containing $\bigvee_{i=0}^{n-1} T^{-i}\alpha$ for all $n \geq 1$ contains all open intervals. The Borel σ-algebra is the smallest such σ-algebra.

3.2 Topological entropy

There are several definitions of the topological entropy. The original one dates from 1965 and is due to Adler, Konheim and McAndrew in [AKM], it is formulated for a continuous map on a compact topological space. In 1971, Bowen formulated in [B1] a new, but equivalent definition, for a continuous map on a compact *metric* space; we denote this topological entropy by $\overline{h_{\text{top}}}(E,T)$. In 1973, he gave in [B2] a third definition of the topological entropy for a continuous map on a metric space; we denote this topological entropy by $h_{\text{top}}(E,T)$. This last definition is important, because it applies for non compact subsets of a compact metric space; but $h_{\text{top}}(E,T) = \overline{h_{\text{top}}}(E,T)$, if E is a compact metric space and $T : E \to E$ is a continuous map. The nuance between $h_{\text{top}}(E,T)$ and $\overline{h_{\text{top}}}(E,T)$ is detailed by Pesin in [Pe]: in that book, $h_{\text{top}}(E,T)$ is called *topological entropy* and $\overline{h_{\text{top}}}(E,T)$ is called *upper capacity topological entropy.* There are similarities in the definition of $h_{\text{top}}(E,T)$ and the one of the Hausdorff dimension. Likewise, there are similarities between $\overline{h_{\text{top}}}(E,T)$ and the box dimension. We will use $h_{\text{top}}(E,T)$ in Chapter 7 to obtain upper bound of the Hausdorff dimension of some non-compact subsets of X. $\overline{h_{\text{top}}}(E,T)$ is mainly used to compute the topological entropy of $E = X$, in particular for piecewise monotone continuous maps.

Let (X,d) be a compact metric space and $T : X \to X$ a continuous map. Let $B_n(x,\varepsilon)$ denote Bowen's ball defined by

$$B_n(x,\varepsilon) := \{y \in X : d(T^j x, T^j y) \le \varepsilon,\ j = 0,1,\dots,n-1\} \equiv \bigcap_{j=0}^{n-1} T^{-j}\left(\overline{B(T^j x,\varepsilon)}\right),$$

where $B(y,r)$ is the open ball in (X,d). We give Bowen's definition of topological entropy.

Definition 3.4. *Let $E \subset X$ be such that $T(E) \subset E$, $n \ge 1$, $\varepsilon > 0$ and $t \ge 0$. Let $\mathcal{G}_n(E,T,\varepsilon)$ denotes the family of finite or countable covers of E by sets of the form $B_m(x,\varepsilon)$ with $x \in X$ and $m \ge n$. Set*

$$C_n(E,T,\varepsilon,t) := \inf\left\{\sum_{B_m(x,\varepsilon)\in\mathcal{C}} e^{-tm} : \mathcal{C} \in \mathcal{G}_n(E,T,\varepsilon)\right\}.$$

We define an outer measure by

$$C(E,T,\varepsilon,t) := \lim_{n\to\infty} C_n(E,T,\varepsilon,t).$$

The topological entropy of E is defined by

$$\begin{aligned} h_{\text{top}}(E,T,\varepsilon) &:= \inf\{t \ge 0 : C(E,T,\varepsilon,t) = 0\}, \\ h_{\text{top}}(E,T) &:= \lim_{\varepsilon\to 0} h_{\text{top}}(E,T,\varepsilon). \end{aligned}$$

It is easy to check that the limits involved in the definition exist. Indeed $\mathcal{G}_n(E,T,\varepsilon) \subset \mathcal{G}_m(E,T,\varepsilon)$ for all $n \ge m$, so the first sequence is increasing. Moreover if $\{B_{m_j}(x_j,\varepsilon) : j \in J\} \in G_n(E,T,\varepsilon)$, then for all $\delta \ge \varepsilon$, the set $\{B_{m_j}(x_j,\delta) : j \in J\}$ is a cover of E. So the last sequence is increasing. We now give another definition of the topological entropy that we denote by $\overline{h_{\text{top}}}(E,T)$.

Definition 3.5. *Let $E \subset X$ be such that $T(E) \subset E$, $n \ge 1$ and $\varepsilon > 0$. A set $K \subset X$ is an (n,ε)-spanning set for E, if*

$$E \subset \bigcup_{x\in K} B_n(x,\varepsilon).$$

Let $r_n(E,T,\varepsilon)$ denote the smallest cardinality of an (n,ε)-spanning set for E. The topological entropy of E is the growth rate of $r_n(E,T,\varepsilon)$.

$$\begin{aligned} \overline{h_{\text{top}}}(E,T,\varepsilon) &:= \limsup_{n\to\infty} \frac{1}{n} \log r_n(E,T,\varepsilon), \\ \overline{h_{\text{top}}}(E,T) &:= \lim_{\varepsilon\to 0} \overline{h_{\text{top}}}(E,T,\varepsilon). \end{aligned}$$

First, remark that $r_n(E,T,\varepsilon)$ is well defined. Indeed,

$$\{\bigcap_{i=0}^{n-1} T^{-i} B(T^i x,\varepsilon) : x \in X\}$$

is an open cover of X, thus there exists a finite subset $K \subset X$ such that

$$\{\bigcap_{i=0}^{n-1} T^{-i} B(T^i x,\varepsilon) : x \in K\}$$

is a cover of X. It is easy to see that K is an (n,ε)-cover of X and of any $E \subset X$. Secondly, since an (n,ε)-spanning set for E is an (n,δ)-spanning set for E for all $\delta \geq \varepsilon$, $r_n(E,T,\varepsilon)$ and $\overline{h_{\text{top}}}(E,T,\varepsilon)$ are non-increasing functions of ε and the last limit exists, however it may be infinite.

We see immediately the similarity of $h_{\text{top}}(E)$ with the Hausdorff dimension and $\overline{h_{\text{top}}}(E)$ with the upper box dimension. As proved in Proposition 7.1 of [Ma], using lim inf instead of lim sup, we recover the same value of topological entropy after the limit $\varepsilon \to 0$. As in case of the Hausdorff dimension, we see immediately that, for all $\varepsilon > 0$, we have $h_{\text{top}}(E,\varepsilon) \leq \overline{h_{\text{top}}}(E,\varepsilon)$, since $\{B_n(x,\varepsilon) : x \in K\} \in \mathcal{G}_n(E,\varepsilon)$ if K is an (n,ε)-spanning set. When E is compact and T continuous, $h_{\text{top}}(E,T) = \overline{h_{\text{top}}}(E,T)$. Indeed, Bowen proved in [B2] that his definition of $h_{\text{top}}(E,T)$ is equivalent to the original definition using open covers. The equivalence of $\overline{h_{\text{top}}}(E,T)$ with the definition of Adler, Konheim and McAndrew can be found in chapter 7 of [W]. As remarked by Mané in [Ma], the definition of $\overline{h_{\text{top}}}(E,T)$ does not require the continuity of T, but only the compactness of (X,d). In particular, we will use the second definition of the topological entropy when T is a piecewise monotone continuous map.

We discuss the particular case of shift spaces. Let $\Sigma \subset \Sigma_k^+$ be a shift space and $\beta > 1$. $(\Sigma, d_\beta, \sigma)$ is a compact continuous dynamical system. For $\varepsilon \in (0,1)$, choose $m \geq 0$ such that $\beta^{-m} \leq \varepsilon < \beta^{-m+1}$, then for all $\underline{w} \in \Sigma$ and $n \geq 1$

$$B_n(\underline{w},\varepsilon) = {}_0[\underline{w}_{[0,n+m)}]\,.$$

If $\underline{z} \neq \underline{z}'$ are two words of length $n+m$ in $\mathcal{L}(\Sigma)$, then $[\underline{z}] \cap [\underline{z}'] = \emptyset$. Thus $r_n(\Sigma,\sigma,\varepsilon)$ is the cardinality of words of length $n+m$ in $\mathcal{L}(\Sigma)$; in particular $r_n(\Sigma,\sigma,\varepsilon)$ is independent of ε for ε small enough. Finally

$$h_{\text{top}}(\Sigma,\sigma) = \overline{h_{\text{top}}}(\Sigma,\sigma) = \lim_{n\to\infty} \frac{1}{n} \log |\{\underline{w} \in \mathcal{L}(\Sigma) : |\underline{w}| = n\}|\,. \tag{3.1}$$

We apply this remark to compute the topological entropy of a shift space obtained by the coding of a piecewise monotone continuous map with constant slope. A proof of this result is given by Misiurewicz and Szlenk in [MS], but our proof is different and inspired by Theorem 2 in [R].

Proposition 3.6. *Let T be a piecewise monotone continuous map. Suppose further that T is differentiable on X_1 and that there exists $\beta > 1$ such that $|T'(x)| = \beta$ for all $x \in [0,1]\backslash S_0$. Then $h_{\text{top}}(\Sigma_T^+,\sigma) = \log\beta$ where $\Sigma_T^+ = \overline{\mathtt{i}(X)}$ is the shift space obtained by coding.*

Proof: Set $S_n = |\{\underline{w} \in \mathcal{L}(\Sigma_T^+) : |\underline{w}| = n\}|$. S_n is also the number of laps of T^n (the map T^0 is interpreted as the identity and it has one lap). Clearly $|\frac{d}{dx}T^n(x)| = \beta^n$, thus a lap of T^n has length at most β^{-n} and $S_n \geq \beta^n$. We now want an upper estimate of S_n. Suppose first that $\beta > 2$ and set $\delta = \min\{a_{j+1} - a_j : j = 0,\ldots,k-1\}$ where the a_j are the boundaries of laps of T. Take a lap I of T^{n-1} and denote by $J_1,\ldots,J_m$ the laps of T^n contained in I. The laps $J_2,\ldots,J_{m-1}$ (when they exist) have length at least $\delta\beta^{-n}$. Since all laps of T^n are disjoint two by two,

$$(S_n - 2S_{n-1})\delta\beta^{-n} \leq 1\,.$$

This is true for all $n \geq 1$ and we deduce that

$$\begin{aligned} S_n &\leq \delta^{-1}\beta^n + 2S_{n-1} \leq \delta^{-1}\beta^n + 2(\delta^{-1}\beta^{n-1} + 2S_{n-2}) \leq \cdots \leq \\ &\leq \delta^{-1}(\beta^n + 2\beta^{n-1} + \cdots + 2^n) = \frac{2^n}{\delta}\,\frac{(\beta/2)^{n+1} - 1}{\beta/2 - 1} \leq \frac{1}{\delta}\,\frac{\beta^{n+1}}{\beta - 2}\,. \end{aligned}$$

Thus if $\beta > 2$, then

$$h_{\text{top}}(\Sigma_T^+, \sigma) = \lim_{n\to\infty} \frac{1}{n} \log S_n = \log \beta \,.$$

If $\beta \in (1, 2]$, then choose m such that $\beta^m > 2$ and consider T^m instead of T. We have

$$h_{\text{top}}(\Sigma_T^+, \sigma) = \lim_{n\to\infty} \frac{1}{mn} \log |\{\text{laps of } (T^m)^n\}| = \frac{1}{m} \log \beta^m = \log \beta \,.$$

$\square$

Finally we recall the variational principle. It is a fundamental relationship between the measure-theoretical entropy and the topological entropy (see [W]).

Theorem 3.7. *Let (X, d, T) be a compact continuous dynamical system, then*

$$\sup\{h_T(\mu) : \mu \in M(X, T)\} = h_{\text{top}}(X, T) \,.$$

Chapter 4

Graph representations of shift spaces

In this chapter, we want to study compact shift spaces using labeled oriented graphs. Such a graph represents a shift space Σ, if the set of labels of finite paths on this graph is exactly $\mathcal{L}(\Sigma)$. For a given shift space Σ, there are many graphs representing Σ. A general theory about representation of shift by graphs can be found in Chapters 2 and 3 of [LM]. Notice that the construction of a graph representing a shift Σ is essentially based on the set $\mathcal{L}(\Sigma)$, so it works with semi-infinite shifts as well as with bi-infinite shifts.

First we define the follower set graph of a shift space Σ; this graph is the minimal graph representing Σ and having a vertex v such that, for all words $\underline{w}$ in $\mathcal{L}(\Sigma)$, there is a path labeled by $\underline{w}$ starting at v. This graph is interesting, because it is the smallest graph having these properties. It has a disadvantage: its structure depends heavily on Σ. Let T be a piecewise monotone continuous map and consider the shift space obtained by coding $\Sigma_T^+ := \overline{\mathtt{i}(X)}$. Due to the specific structure of this shift space (see Theorem 2.15), we construct a graph representing Σ_T^+ with particular properties: the Markov diagram. This graph is convenient, because it has general properties independent of the map T, but it is in general too complex for a detailed analysis. Finally we introduce the notion of simplification of a graph. This idea is useful for the study of shift spaces Σ_T^+ obtained with particular maps T. We get a not too complex graph which inherits the main properties of the Markov diagram. The follower set graph is defined in [LM]. The definition of the Markov diagram is due to Hofbauer in [H3], a more recent presentation can be found in [BB].

Definition 4.1. *A labeled oriented graph is a triple $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{L})$ where*

1. *$\mathcal{V}$ is the set of* ***vertices****.*
2. *$\mathcal{E}$ is the set of* ***edges****. An edge $e \in \mathcal{E}$ starts at a vertex $i(e) \in \mathcal{V}$ and terminates at a vertex $t(e) \in \mathcal{V}$.*
3. *$\mathcal{L} : \mathcal{E} \to \mathtt{A}$ is the* ***labeling*** *function.*

A ***path*** *on $\mathcal{G}$ is a finite sequence $E = e_0 e_1 \dots e_{m-1}$ of edges such that $t(e_i) = i(e_{i+1})$ for all $0 \le i \le m-2$. A* ***walk*** *on $\mathcal{G}$ is semi-infinite (or bi-infinite) sequence $E = e_0 e_1 \dots$ (or $E = \dots e_{-1} e_0 e_1 \dots$) such that $t(e_i) = i(e_{i+1})$ for all $i \in \mathbb{Z}_+$ (or all $i \in \mathbb{Z}$). The labeling function induces a map on paths or walks: let $E = e_0 e_1 \dots e_{m-1}$ be a path, then the labels* *of E are*

$$\mathcal{L}(E) = \mathcal{L}(e_0)\mathcal{L}(e_1)\dots\mathcal{L}(e_{m-1}) \in \mathtt{A}^*\,,$$

*and similarly for walks. A labeled oriented graph $\mathcal{G}$ is a **presentation** of Σ if*

$$\mathcal{L}(\Sigma) = \mathcal{L}\big(\{E : E \text{ is a path on } \mathcal{G}\}\big)\,.$$

*The graph $\mathcal{G}$ is **simple**, if for all $c, c' \in \mathcal{V}$, there is at most one edge $e \in \mathcal{E}$ such that $i(e) = c$ and $t(e) = c'$. The graph $\mathcal{G}$ is **right-resolving**, if for all $c \in \mathcal{V}$ and all $j \in \mathtt{A}$, there is at most one edge $e \in \mathcal{E}$ such that $i(e) = c$ and $\mathcal{L}(e) = j$.*

If $\mathcal{G}$ is simple, then a path or a walk can be defined by the sequence of visited vertices: for example, given a path $\{e_j\}_{0\leq j<m}$, the sequence $\{c_j\}_{0\leq j\leq m}$, defined by $c_j = i(e_j)$ for $0 \leq j < m$ and $c_m = t(e_{m-1})$, represents the same path. If $\mathcal{G}$ is right-resolving, a path or a semi-infinite walk can be defined by the initial vertex and the labels of the path or the walk: for example, given a path $\{e_j\}_{0\leq j<m}$, then $(i(e_0), \{\mathcal{L}(e_j)\}_{0\leq j<m})$ represents the same path.

We begin by the definition of the follower set graph (see [LM]).

Definition 4.2. *Let $\Sigma \subset \Sigma_k^+$ be a shift space. For all $\underline{w} \in \mathcal{L}(\Sigma)$, we define the **follower set**[1] of $\underline{w}$ as*

$$\mathcal{F}_{\underline{w}} = \{\underline{x} \in \Sigma : \underline{w}\ \underline{x} \in \Sigma\}\,.$$

*The **follower set graph** is the labeled oriented graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{L})$ whose set of vertices is*

$$\mathcal{V} := \{\mathcal{F}_{\underline{w}} : \underline{w} \in \mathcal{L}(\Sigma)\}\,.$$

*For all $\underline{w} \in \mathcal{L}(\Sigma)$ and all $j \in \mathtt{A}_k$ such that $\underline{w}\,j \in \mathcal{L}(\Sigma)$, there is an edge labeled by j from $\mathcal{F}_{\underline{w}}$ to $\mathcal{F}_{\underline{w}\,j}$. The vertex $\mathcal{F}_\varepsilon = \Sigma$ is called the **root**.*

Notice that the set of vertices $\mathcal{V}$ can be a finite as well as infinite (but countable) depending on Σ. By construction, for all $\underline{w} \in \mathcal{L}(\Sigma)$ and all $\underline{x} \in \mathcal{F}_{\underline{w}}$, there is a walk starting at the vertex $\mathcal{F}_{\underline{w}}$ labeled by $\underline{x}$. In particular, for all $\underline{x} \in \Sigma$, there is walk labeled by $\underline{x}$ starting at the root.

4.1 The Markov diagram

The Markov diagram is a variant of the follower set graph for shift spaces Σ having a particular structure (typically the shift spaces $\overline{\mathtt{i}(X)}$ obtained by coding). The construction of this graph is more complicated, but its global structure does not depend on Σ. Consider two different words $\underline{w}, \underline{w}' \in \Sigma$ such that $\mathcal{F}_{\underline{w}} = \mathcal{F}_{\underline{w}'}$. In the follower set graph, the paths starting at the root and labeled by $\underline{w}$ or $\underline{w}'$ end at the same vertex. In the Markov diagram, this does not always hold; the end vertices of the two paths are determined systematically according to Lemma 4.3.

We consider a shift space in the following settings. Let $k \geq 2$ and fix a map $s : \mathtt{A}_k \to \{-1, +1\}$. Define an order $\preceq$ on Σ_k^+ by (2.2). Choose $2k$ virtual itineraries $\underline{u}^j, \underline{v}^j \in \Sigma_k^+$, $j \in \mathtt{A}_k$. We suppose further that, for all $j \in \mathtt{A}_k$, we have $u_0^j = v_0^j = j$ and

$$\begin{aligned} \underline{u}^{u_n^j} \preceq \sigma^n \underline{u}^j \preceq \underline{v}^{u_n^j} \quad \forall n \geq 0\,, \\ \underline{u}^{v_n^j} \preceq \sigma^n \underline{v}^j \preceq \underline{v}^{v_n^j} \quad \forall n \geq 0\,. \end{aligned} \tag{4.1}$$

[1] Our definition slightly differs from the one of Lind and Marcus. Our definition is suitable for semi-infinite shifts, whereas the one of Lind and Marcus is better for bi-infinite shifts.

We define a shift space $\Sigma(\{\underline{u}^j, \underline{v}^j\})$ by

$$\Sigma(\{\underline{u}^j, \underline{v}^j\}) = \{\underline{x} \in \Sigma_k^+ : \underline{u}^{x_n} \preceq \sigma^n \underline{x} \preceq \underline{v}^{x_n} \quad \forall n \geq 0\} . \tag{4.2}$$

Of course, this construction is motivated by the structure of the space $\Sigma = \mathtt{i}(X)$ (see Theorem 2.15). Notice that in points 5 and 6 of Theorem 2.15, there are some strict inequalities. For this chapter, we replace them by non-strict inequalities; this will be useful later. We define $U := \bigcup_{j \in \mathtt{A}_k} \{\underline{u}^j\}$ and $V := \bigcup_{j \in \mathtt{A}_k} \{\underline{v}^j\}$. We use the classical notation for the interval in $\Sigma(\{\underline{u}^j, \underline{v}^j\})$ with respect to the order $\preceq$

$$[\underline{a}, \underline{b}] := \{\underline{x} \in \Sigma(\{\underline{u}^j, \underline{v}^j\}) : \underline{a} \preceq \underline{x} \preceq \underline{b}] .$$

When the boundaries are not ordered, we use the notation $\langle \underline{a}, \underline{b} \rangle$

$$\langle \underline{a}, \underline{b} \rangle := [\min\{\underline{a}, \underline{b}\}, \max\{\underline{a}, \underline{b}\}] .$$

Lemma 4.3. *Let $p, q \in \mathbb{Z}_+$ and $\underline{a}, \underline{b} \in U \cup V$ be such that $\sigma^p \underline{a} \preceq \sigma^q \underline{b}$ and consider the interval $c_1 = [\sigma^p \underline{a}, \sigma^q \underline{b}] \subset \Sigma(\{\underline{u}^j, \underline{v}^j\})$. For all $j \in \mathtt{A}_k$ such that $a_p \leq j \leq b_q$, the set $\{\underline{w} \in c_1 : w_0 = j\}$ is not empty and $c_2 := \sigma(\{\underline{w} \in c_1 : w_0 = j\})$ is given by*

$$c_2 = \begin{cases} \langle \sigma^{p+1} \underline{a}, \sigma^{q+1} \underline{b} \rangle & \text{if } a_p = j = b_q \\ \langle \sigma^{p+1} \underline{a}, \sigma \underline{v}^j \rangle & \text{if } a_p = j < b_q \\ \langle \sigma \underline{u}^j, \sigma \underline{v}^j \rangle & \text{if } a_p < j < b_q \\ \langle \sigma \underline{u}^j, \sigma^{q+1} \underline{b} \rangle & \text{if } a_p < j = b_q . \end{cases}$$

Proof: There exists $\underline{w} \in c_1$ with $w_0 = j$ for all $a_p \leq j \leq b_q$, take for instance $\sigma^p \underline{a}$ for $j = a_p$ and $\underline{u}^j$ for $a_p < j \leq b_q$. Then the four cases are very similar, we prove only the second one. If $a_p = j < b_q$, then

$$c_2 = \sigma\big(\{\underline{w} \in c_1 : w_0 = j\}\big) = \sigma([\sigma^p \underline{a}, \sigma^q \underline{b}] \cap [\underline{u}^j, \underline{v}^j]) = \sigma([\sigma^p \underline{a}, \underline{v}^j]) = \langle \sigma^p \underline{a}, \sigma \underline{v}^j \rangle .$$

Indeed, we have $\underline{u}^j = \underline{u}^{a_p} \preceq \sigma^p \underline{a}$ and $\underline{v}^j = j \ldots \prec b_q \ldots = \sigma^q \underline{b}$. The last step is true, because all sequences in $[\sigma^p \underline{a}, \underline{v}^j]$ begin with the same symbol. $\square$

This lemma reflects the structure given by the order $\leq$ on the interval $[0, 1]$. First notice that in all cases, the set c_2 is again an interval of the type $[\sigma^{p'} \underline{a}', \sigma^{q'} \underline{b}']$. This remark is the key of Definition 4.4. Secondly notice that we write c_2 as an unordered interval for simplicity, but the order of the boundaries is easily deduced from $s(j)$. Finally notice that, when $s(j) = +1$ for all $j \in \mathtt{A}_k$ ($\preceq$ is the lexicographic order), then the map σ never inverses the order and we can use ordered intervals $[\cdot, \cdot]$ instead of $\langle \cdot, \cdot \rangle$ in Lemma 4.3. In particular, if $c_1 = [\sigma^p \underline{a}, \sigma^q \underline{b}]$ is such that $\underline{a} \in U$ and $\underline{b} \in V$ and $c_2 = [\sigma^{p'} \underline{a}', \sigma^{q'} \underline{b}']$, then $\underline{a}' \in U$ and $\underline{b}' \in V$.

In the next definition, we use quadruples $(p, \underline{a}; q, \underline{b})$ with $p, q \in \mathbb{Z}_+$ and $\underline{a}, \underline{b} \in U \cup V$. To better understand the definition, it is useful to know that the quadruple $(p, \underline{a}; q, \underline{b})$ represents the interval $[\sigma^p \underline{a}, \sigma^q \underline{b}]$. We also use an unordered notation for the quadruples

$$\langle p, \underline{a}; q, \underline{b} \rangle := \begin{cases} (p, \underline{a}; q, \underline{b}) & \text{if } \sigma^p \underline{a} \preceq \sigma^q \underline{b}, \\ (q, \underline{b}; p, \underline{a}) & \text{if } \sigma^p \underline{a} \succ \sigma^q \underline{b}. \end{cases}$$

In particular, the notation $\langle p, \underline{a}; q, \underline{b} \rangle$ represents the interval $\langle \sigma^p \underline{a}, \sigma^q \underline{b} \rangle$.

Definition 4.4. *Consider a quadruple $c_1 = (p, \underline{a}; q, \underline{b})$ where $p, q \in \mathbb{Z}_+$ and $\underline{a}, \underline{b} \in U \cup V$ are such that $\sigma^p \underline{a} \preceq \sigma^q \underline{b}$. Then for all $j \in \mathtt{A}_k$ such that $a_p \leq j \leq b_q$, c_2 is a j-**successor** (or simply a **successor**) of c_1, where c_2 is the quadruple defined by*

$$c_2 = \begin{cases} \langle p+1, \underline{a}; q+1, \underline{b} \rangle & \text{if } a_p = j = b_q, \\ \langle p+1, \underline{a}; 1, \underline{v}^j \rangle & \text{if } a_p = j < b_q, \\ \langle 1, \underline{u}^j; 1, \underline{v}^j \rangle & \text{if } a_p < j < b_q, \\ \langle 1, \underline{u}^j; q+1, \underline{b} \rangle & \text{if } a_p < j = b_q. \end{cases}$$

*The **Markov diagram** is a labeled oriented graph $\mathcal{G}(\Sigma(\{\underline{u}^j, \underline{v}^j\})) = (\mathcal{V}, \mathcal{E}, \mathcal{L})$ whose vertices are quadruples $(p, \underline{a}; q, \underline{b})$. It is the smallest graph such that $(0, \underline{u}^0; 0, \underline{v}^{k-1}) \in \mathcal{V}$ and if $c_1 \in \mathcal{V}$ and c_2 is a successor of c_1, then $c_2 \in \mathcal{V}$. Moreover there is an edge labeled by j from c_1 to c_2, if and only if c_2 is a j-successor of c_1. The vertex $(0, \underline{u}^0; 0, \underline{v}^{k-1})$ is called the **root**.*
*If $c = (p, \underline{a}; q, \underline{b}) \in \mathcal{V}$, then the **upper level** of c is $\max\{p, q\}$ and the **lower level** of c is $\min\{p, q\}$. A vertex $c \in \mathcal{V}$ is **indexed** by the labels of the shortest path from the root to c.*

We begin with several remarks about the Markov diagram:

1. The construction of the Markov diagram might seem very abstract. As said before, it is better understood when we know that the quadruple $(p, \underline{a}; q, \underline{v})$ represents the intervals $[\sigma^p \underline{a}; \sigma^q \underline{b}]$. The definition of the j-successor of a vertex is justified by Lemma 4.3. Moreover by Lemma 4.3, the following property holds: let $\underline{w} \in \mathcal{L}(\Sigma)$ and define $c = (p, \underline{a}; q, \underline{b})$ as the vertex where terminates the path starting at the root and labeled by $\underline{w}$; then $\mathcal{F}_{\underline{w}} = [\sigma^p \underline{a}, \sigma^q \underline{b}]$. We choose the notation as quadruple, instead of interval, to avoid unexpected identifications of the vertices: suppose that $\sigma^p \underline{a} = \sigma^{p'} \underline{a}'$ for some $p \neq p'$ or $\underline{a} \neq \underline{a}'$; then $[\sigma^p \underline{a}, \sigma^q \underline{b}] = [\sigma^{p'} \underline{a}', \sigma^q \underline{b}]$ as subintervals of $\Sigma(\{\underline{u}^j, \underline{v}^j\})$, but $(p, \underline{a}; q, \underline{b}) \neq (p', \underline{a}'; q, \underline{b})$ as quadruples.

2. By the definition of successors, at a given vertex c_1, there are at most k outgoing edges; each of these edges carries a different label and terminates at a different vertex. Thus the Markov diagram is simple and right-resolving. In particular, we will often choose the alternative representations of a path or walk by a sequence of vertices or by the vertex where we start and a sequence of labels.

3. The successors of a vertex of upper level n have upper level less than or equal to $n+1$. As we will show in Lemma 4.5, the upper level of a vertex is the length of the shortest path from the root to this vertex. On the other hand, for a vertex c_2 of upper level $n \geq 1$, there exists a unique vertex c_1 of upper level $n-1$ such that c_2 is a successor of c_1. In particular, for all $c \in \mathcal{V}$, there is a unique shortest path from the root to c, thus there is a unique index for each vertex.

4. Given a vertex $c \in \mathcal{V}$, all edges terminating at c carry the same label. Indeed given a vertex $c = (p, \underline{a}; q, \underline{b}) \in \mathcal{V}$ (except for the root where there is no incoming edge), then by the definition of successors, $a_{p-1} = b_{q-1}$ and all edges ending at c carry the label a_{p-1}.

5. If $c = (p, \underline{a}; q, \underline{b})$ is a vertex with lower level greater than or equal to 2, then there is only one incoming edge at c, it starts at the vertex $\langle p-1, \underline{a}; q-1, \underline{b} \rangle$.

Lemma 4.5. *Let $\underline{a} \in U \cup V$ and $p \geq 1$. The vertex c indexed by $\underline{a}_{[0,p)}$ is the quadruple $c = \langle p, \underline{a}; q, \underline{b}\rangle$ for some $\underline{b} \in U \cup V$ and $1 \leq q \leq p$.*
Suppose further that $q < p$ and c has $n \geq 2$ successors. Then the vertex c' indexed by $\underline{b}_{[0,q)}$ is the quadruple $c' = \langle p', \underline{a}'; q, \underline{b}\rangle$ for some $\underline{a}' \in U \cup V$ and $1 \leq p' \leq q$. Moreover c' has at least n successors and the b_q-successor of c and the b_q-successor of c' is the same vertex.

Proof: The word $\underline{a}_{[0,p)}$ is the prefix of length p of a virtual itinerary. By the definition of successor, we see immediately that the vertex indexed by such a prefix is of the type $c = \langle p, \underline{a}; q, \underline{b}\rangle$ for some $\underline{b} \in U \cup V$ and $1 \leq q \leq p$.
For the second statement, $\underline{b}_{[0,q)}$ is again a prefix of length q of a virtual itinerary, thus $c' = \langle p', \underline{a}'; q, \underline{b}\rangle$ for some $\underline{a}' \in U \cup V$ and $1 \leq p' \leq q$. If $q \geq 2$, then by remark 5 following Definition 4.4, there is only one ingoing edge at c; moreover this edge starts at vertex $c_{q-1} = \langle p-1, \underline{a}; q-1, \underline{b}\rangle$ and is labeled $a_{p-1} = b_{q-1}$. Again if $q - 1 \geq 2$, there is only one ingoing edge at c_{q-1}; this edge starts at vertex $c_{q-2} = \langle p-2, \underline{a}; q-2, \underline{b}\rangle$ and is labeled by $a_{p-2} = b_{q-2}$. We continue in this manner until we get a vertex $c_1 = \langle p-q+1, \underline{a}; 1, \underline{b}\rangle$. We define also c_0 as the vertex indexed by $\underline{a}_{[0,p-q)}$ and $c_q = c$. We have constructed a sequence of vertices $\{c_j\}_{0\leq j\leq q}$ such that c_j is indexed by $\underline{a}_{[0,p-q+j)}$, c_{j+1} is a b_j-successor of c_j and, for $0 < j < q$, the vertex $c_j = \langle p-q+j, \underline{a}; j, \underline{b}\rangle$ has only one successor. Define c'_0 as the root. By the definition of successors and (4.1), there is an edge labeled by b_0 starting at c'_0. Define c'_1 as the b_0-successor of c'_0. Using $c_1 = \langle p-q+1, \underline{a}; 1, \underline{b}\rangle$,$c'_1 = \langle 1, \underline{u}^{b_0}; 1, \underline{v}^{b_0}$, $a_{p-q+1} = b_1$ and (4.1), we see that there is an edge labeled by b_1 starting at c'_1. Define c'_2 as the b_1-successor of c'_1. Continuing in this manner, we define a sequence of vertices $\{c'_j\}_{0\leq j\leq q}$ such that c'_j is indexed by $\underline{b}_{[0,j)}$, there is an edge labeled by b_j starting at c'_j and $c'_q = c'$. Since $\underline{b}_{[0,q)}$ is the prefix of some virtual itinerary, we see that $c' = \langle p', \underline{a}'; q, \underline{b}\rangle$ for some $\underline{a}' \in U \cup V$ and some $1 \leq p' \leq q$. Moreover, using always (4.1), we get that c' has j-successor for $j \in \mathtt{A}_k$ such that c has a j-successor. By hypothesis, c has at least two successors, thus c' has at least two successors and

$$c = \langle p, \underline{a}; q, \underline{b}\rangle \xrightarrow{b_q} \langle 1, \underline{a}''; q+1, \underline{b}\rangle$$
$$c' = \langle p', \underline{a}'; q, \underline{b}\rangle \xrightarrow{b_q} \langle 1, \underline{a}''; q+1, \underline{b}\rangle\,,$$

where $\underline{a}'' \in \{\underline{u}^{b_q}, \underline{v}^{b_q}\}$ according to the situation. We see that the b_q-successors of c and c' are the same quadruple. $\square$

We are now able to describe the structure of the Markov diagram. For all $n \geq 0$, let $\mathcal{V}_n \subset \mathcal{V}$ be the subset of vertices having upper level n. Since a vertex in $\mathcal{V}_n$ has successors of upper level smaller than or equal to $n+1$, we construct the Markov diagram starting with $\mathcal{V}_0$; we add all vertices of $\mathcal{V}_1$, then all vertices of $\mathcal{V}_2$ and so on. The only vertex in $\mathcal{V}_0$ is the root $(0, \underline{u}^0; 0, \underline{v}^{k-1})$. There are k vertices in $\mathcal{V}_1$ given by $\langle 1, \underline{u}^j; 1, \underline{v}^j\rangle$ for all $j \in \mathtt{A}_k$. Take a vertex $c \in \mathcal{V}_n$ for $n \geq 1$.

1. If the upper level of c is equal to its lower level and c has one successor c', then c' has upper and lower level $n+1$.

2. If the upper level of c is equal to its lower level and c has at least two successors, then two of the successors have upper level $n+1$ and lower level strictly smaller, all others are in $\mathcal{V}_1$.

3. If c has upper level n and lower level m with $m < n$ and c has one successor c', then c' has upper level $n+1$ and lower level $m+1$.

4. If c has upper level n and lower level m with $m < n$ and c has at least two successors, one of the successors has upper level $n+1$ and lower level 1, another has upper level $m+1$ and lower level 1, all others are in $\mathcal{V}_1$. Moreover by Lemma 4.5, the vertex of upper level $m+1 \leq n$ already exists in the graph.

We deduce that $|\mathcal{V}_0| = 1$, $|\mathcal{V}_1| = k$ and $|\mathcal{V}_n| \leq 2k$ for all $n \geq 2$.

We will always draw the Markov diagram using the following conventions: the vertices are organized such that all vertices in the same column have the same upper level; the columns are sorted such that the upper level increases from the left to the right; horizontally, the graph is presented as a tree, the first branch corresponding to the path labeled by $\underline{u}^0$, the second branch to the path labeled by $\underline{v}^0$, the third to $\underline{u}^1$ and so on. In Figure 4.1, we give an example with $k = 2$, the order $\preceq$ defined by $\delta(0) = +1$ and $\delta(1) = -1$ and the virtual itineraries begin by

$$\underline{u}^0 = 0110\ldots \qquad \underline{v}^0 = 0101\ldots$$
$$\underline{u}^1 = 1101\ldots \qquad \underline{v}^1 = 1011\ldots$$

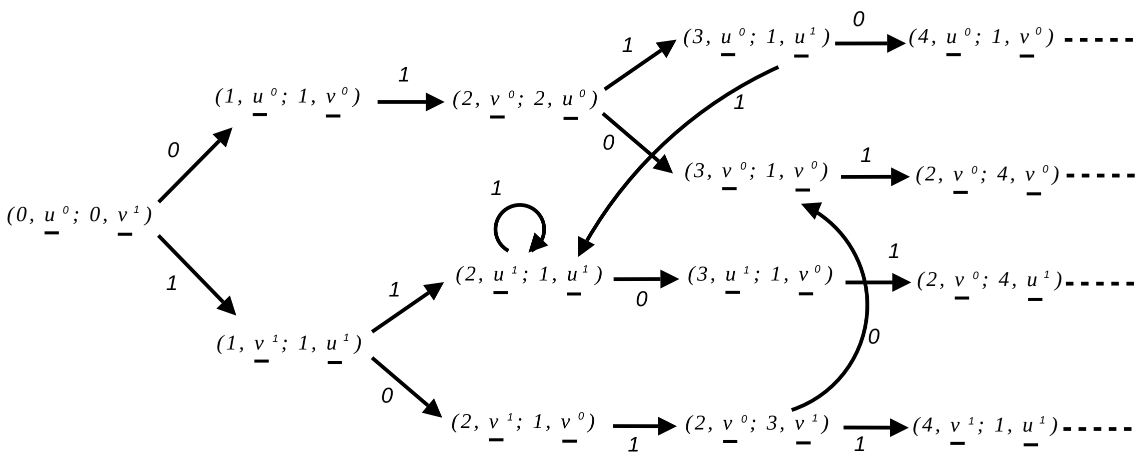

Figure 4.1: An example of a Markov diagram.

Finally we consider the case where the order $\preceq$ is the lexicographic order, ie $s(j) = +1$ for all $j \in \mathtt{A}_k$. In this case, the Markov diagram is simpler, because, if $x_0 = x'_0$, then

$$\underline{x} \preceq \underline{x}' \iff \sigma\underline{x} \preceq \sigma\underline{x}'\,.$$

In particular, if $(p, \underline{a}; q, \underline{b})$ is a vertex of the Markov diagram, then $\underline{a} \in U$ and $\underline{v} \in V$ (see the remark following Lemma 4.3).

Lemma 4.6. *Suppose that $\preceq$ is the lexicographic order and choose $\underline{w} \in \Sigma(\{\underline{u}^j, \underline{v}^j\})$. Suppose that $\underline{w}_{[p,q]} = \underline{u}^i_{[0,q-p]}$ and $\underline{w}_{[q,r]} = \underline{u}^j_{[0,r-q]}$ for some $p < q < r \in \mathbb{Z}_+$ and $i, j \in \mathtt{A}_k$. Then $\underline{w}_{[p,r]} = \underline{u}^i_{[0,r-p]}$. A similar statement is true with $\underline{v}$ instead of $\underline{u}$.*

Proof: We prove the first statement. First notice that $i = w_p$ and $j = w_q = u^i_{q-p}$. Using (4.2) and $\underline{u}^i_{[0,q-p)} = \underline{w}_{[p,q)}$, we have

$$\underline{w} \in \Sigma(\{\underline{u}^j, \underline{v}^j\}) \implies \underline{u}^i \preceq \sigma^p\underline{w} \iff \sigma^{q-p}\underline{u}^i \preceq \sigma^q\underline{w}\,.$$

From (4.1), we deduce that $\underline{u}^j \preceq \sigma^{q-p}\underline{u}^i$, thus

$$\underline{w}_{[q,r]} = \underline{u}^j_{[0,r-q]} \preceq \underline{u}^i_{[q-p,r-p]} \preceq \underline{w}_{[q,r]} \,.$$

We work similarly with $\underline{v}$. $\square$

In other words, if $\underline{w} \in \Sigma(\{\underline{u}^j, \underline{v}^j\})$ contains a prefix of $\underline{u}^i$ (resp. $\underline{v}^i$) and a prefix of $\underline{u}^j$ (resp. $\underline{v}^j$), then these two prefixes are either disjoint or one contains the other. Indeed in Lemma 4.6, we suppose that the tow prefixes are not disjoint (they have a common symbol w_q) and we prove that the first one contains the second one. This property accounts for the following construction: let $\underline{w} \in \mathcal{L}(\Sigma(\{\underline{u}^j, \underline{v}^j\}))$, the $\underline{u}$-**parsing** of $\underline{w}$ is the decomposition of $\underline{w} = \underline{a}^1\underline{a}^2 \dots \underline{a}^n$ in longest prefixes of some $\underline{u}^j$. The word $\underline{a}^1$ is the longest prefix of $\underline{w}$ which is a prefix of $\underline{u}^{w_0}$. If $\underline{w} = \underline{a}^1$, the $\underline{u}$-parsing of $\underline{w}$ is terminated. Otherwise, let $\underline{w}'$ be the word defined by $\underline{w} = \underline{a}^1\underline{w}'$. The word $\underline{a}^2$ is the longest prefix of $\underline{w}'$ which is a prefix of $\underline{u}^{w'_0}$ and so on. Similarly we define the $\underline{v}$-parsing of $\underline{w}$ as the decomposition of $\underline{w}$ in longest prefixes of $\underline{v}^j$.

Lemma 4.7. *Suppose that $\preceq$ is the lexicographic order and choose $\underline{w} \in \mathcal{L}(\Sigma(\{\underline{u}^j, \underline{v}^j\}))$ a non empty word. Define the $\underline{u}$-parsing of $\underline{w} = \underline{a}^1 \dots \underline{a}^n$ and the $\underline{v}$-parsing of $\underline{w} = \underline{b}^1 \dots \underline{b}^m$. Setting $i = a^n_0$, $p = |\underline{a}^n|$, $j = b^m_0$ and $q = |\underline{b}^m|$, then the end vertex of the path starting at the root and labeled by $\underline{w}$ is $(p, i; q, j)$.*

Proof: We establish only the two first elements of the quadruple, the other ones are proved similarly. This is done by induction. If $\underline{w}$ is a word of length 1, then the $\underline{u}$-parsing is trivial: $i = w_0$ and $p = 1$. The edge starting at the root and labeled by w_0 terminates at the vertex $(1, \underline{u}^{w_0}; \cdot, \cdot)$, thus it is true. Now suppose it true for all words of length n and consider the word $\underline{w}j$ of length $n+1$. Let $\underline{w} = \underline{a}^1 \dots \underline{a}^n$ be the $\underline{u}$-parsing of $\underline{w}$ and set $i = a^n_0$ and $p = |\underline{a}^n|$. If $j = u^i_p$, then the $\underline{u}$-parsing of $\underline{w}j$ is $\underline{a}^1 \dots \underline{a}^{n-1}\underline{u}^i_{[0,p]}$. The vertex reached by the path labeled by $\underline{w}j$ is the j-successor of $(p, \underline{u}^i; \cdot, \cdot)$

$$(p, \underline{u}^i; \cdot, \cdot) \xrightarrow{j} (p+1, \underline{u}^i; \cdot, \cdot) \,,$$

by (4.1) and $j = u^i_p$. If $j > u^i_p$, then the $\underline{u}$-parsing of $\underline{w}j$ is $\underline{a}^1 \dots \underline{a}^n u^j_0$. The vertex reached by the path labeled by $\underline{w}j$ is the j-successor of $(p, \underline{u}^i; \cdot, \cdot)$

$$(p, \underline{u}^i; \cdot, \cdot) \xrightarrow{j} (1, \underline{u}^j; \cdot, \cdot) \,,$$

because $j > u^i_p$. $\square$

Corollary 4.8. *Suppose that $\preceq$ is the lexicographic order and choose $\underline{w} \in \mathcal{L}(\Sigma(\{\underline{u}^j, \underline{v}^j\}))$ a non-empty word. Let $\underline{a}$ be the longest suffix of $\underline{w}$ which is a prefix of some $\underline{u}^j$ and $\underline{b}$ be the longest suffix of $\underline{w}$ which is a prefix of some $\underline{v}^j$. Setting $i = a_0$, $p = |\underline{a}|$, $j = b_0$ and $q = |\underline{b}|$, then the path starting at the root and labeled by $\underline{w}$ terminates at the vertex $(p, i; q, j)$.*

Proof: Let $\underline{w} = \underline{a}^1 \dots \underline{a}^n$ the $\underline{u}$-parsing of $\underline{w}$. One has only to show that $\underline{a}^n = \underline{a}$. By definition of $\underline{a}$, the word $\underline{a}^n$ cannot be longer than $\underline{a}$. Suppose it is strictly shorter, then the word $\underline{a}^{n-1}$ terminates inside the word $\underline{a}$. By Lemma 4.6, $\underline{a}^{n-1}$ is contained in $\underline{a}$. But $\underline{a}^{n-1}\underline{a}^n \neq \underline{a}$, since by construction $\underline{a}^{n-1}$ is a longest prefix of some $\underline{u}^j$. Thus $\underline{a}^{n-2}$ terminates inside $\underline{a}$. Carrying on like that, we conclude that $\underline{a}$ contains strictly $\underline{w}$, which is absurd. We work similarly with the $\underline{v}$-parsing. $\square$

For the natural construction of the $\underline{u}$-parsing of $\underline{w}$, we must read $\underline{w}$ from the left to the right and define the words $\underline{a}^j$ by induction. This corollary shows that, if $\preceq$ is the lexicographic order, we can construct the $\underline{u}$-parsing from the right to the left. In particular, when we want to determine the end vertex of a path starting at the root and labeled by $\underline{w}$, we may scan $\underline{w}$ from the end looking for the longest suffix of $\underline{w}$ which is prefix of some $\underline{u}^j$ (or of some $\underline{v}^j$).

4.1.1 Computation of the topological entropy with the Markov diagram

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be an oriented graph. The **adjacency matrix** of $\mathcal{G}$ is the matrix $M = \{M_{i,j}\}_{i,j\in\mathcal{V}}$ where $M_{i,j}$ is defined by

$$M_{i,j} = |\{e \in \mathcal{E} : i(e) = i, t(e) = j\}| \,. \tag{4.3}$$

Suppose that $\mathcal{G}$ is a finite labeled oriented graph which is a presentation of a shift space Σ, then

$$h_{\text{top}}(\Sigma, \sigma) = \log \rho(M)\,, \tag{4.4}$$

where $\rho(M)$ is the spectral radius of M. This is a classical result (see Theorem 4.4.4 in [LM]).

We now report a useful result for the computation of the spectral radius; it is Theorem 1.7 in [BGMY]. We use that theorem in the proof of Proposition 6.15. If $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is a finite oriented graph, a **rome** is a subset $R \subset \mathcal{V}$ such that for all $c \in R$, all walks on $\mathcal{G}$ eventually visit R. Equivalently, R is a rome if the subgraph formed by vertices in $\mathcal{V}\backslash R$ contains no loop. Let i, j be two vertices of a rome R; a simple path from i to j is a path on $\mathcal{G}$ such that all visited vertices between i and j are in $\mathcal{V}\backslash R$. If R is a rome and $r \in \mathbb{R}$, define a rome matrix $A(r) = \{A_{i,j}(r)\}_{i,j\in R}$ by

$$A_{i,j}(r) = \sum_{\pi} r^{1-|\pi|}\,,$$

where the sum runs over all simple paths π from i to j and $|\pi|$ is the length of the path π.

Theorem 4.9. *Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a finite oriented graph, M the adjacency matrix of $\mathcal{G}$ and $R \subset \mathcal{V}$ a rome. The characteristic polynomial of M is*

$$\det(M - \lambda I) = (-\lambda)^m \det(A(\lambda) - \lambda I)\,,$$

where $m = |\mathcal{V}| - |R|$.

Notice that $\det(A(\lambda) - \lambda I)$ is not a characteristic polynomial of the matrix $A(\lambda)$, since $A(\lambda)$ depends on λ.

We recall the formula (3.1), which allows to compute the topological entropy of $\Sigma(\{\underline{u}^j, \underline{v}^j\})$

$$h_{\text{top}}(\Sigma(\{\underline{u}^j, \underline{v}^j\}), \sigma) = \lim_{n\to\infty} \frac{1}{n} \log |\{\underline{w} \in \mathcal{L}(\Sigma(\{\underline{u}^j, \underline{v}^j\})) : |\underline{w}| = n\}\,.$$

By a classical argument of subadditivity (see for example Theorem 4.9 in [W]), we get

$$h_{\text{top}}(\Sigma(\{\underline{u}^j, \underline{v}^j\}), \sigma) = \inf_{n} \frac{1}{n} \log |\{\underline{w} \in \mathcal{L}(\Sigma(\{\underline{u}^j, \underline{v}^j\})) : |\underline{w}| = n\}\,.$$

By construction, there is a 1-to-1 correspondence between the set of words of length n in $\mathcal{L}(\Sigma(\{\underline{u}^j, \underline{v}^j\}))$ and the set of paths on the Markov diagram having length n and starting at the root. This motivates the following definition: let $\mathcal{G}$ be a rooted oriented graph; we define

$$\ell_n(\mathcal{G}) := |\{\text{paths on } \mathcal{G} \text{ having length } n \text{ and starting at the root}\}|\,.$$

Thus we get

$$h_{\mathrm{top}}(\Sigma(\{\underline{u}^j, \underline{v}^j\}), \sigma) = \lim_{n\to\infty} \frac{1}{n} \log \ell_n(\mathcal{G}) = \inf_n \frac{1}{n} \log \ell_n(\mathcal{G})\,. \tag{4.5}$$

The next proposition shows that the topological entropy is upper semi-continuous with respect to the set of virtual itineraries $\{\underline{u}^j, \underline{v}^j\}$.

Proposition 4.10. *Let $\{\underline{u}^j, \underline{v}^j\}$ be a set of virtual itineraries satisfying* (4.1). *For all $\delta > 0$, there exists L_0 such that for all $L \geq L_0$, the following holds: let $\{\tilde{\underline{u}}^j, \tilde{\underline{v}}^j\}$ be a set of virtual itineraries satisfying* (4.1)*; suppose further that $\underline{u}^j$, $\tilde{\underline{u}}^j$ have a common prefix of length L and $\underline{v}^j$, $\tilde{\underline{v}}^j$ have a common prefix of length L, for all $j \in \mathtt{A}_k$. Then*

$$h_{\mathrm{top}}(\Sigma(\{\tilde{\underline{u}}^j, \tilde{\underline{v}}^j\}), \sigma) \leq h_{\mathrm{top}}(\Sigma(\{\underline{u}^j, \underline{v}^j\}), \sigma) + \delta\,.$$

Proof: Given $\Sigma(\{\underline{u}^j, \underline{v}^j\})$ and $\delta > 0$, choose $L(\delta)$ such that

$$\frac{1}{L} \log \ell_L(\mathcal{G}) \leq h_{\mathrm{top}}(\Sigma(\{\underline{u}^j, \underline{v}^j\}), \sigma) + \delta \quad \forall L \geq L(\delta)\,,$$

where $\mathcal{G}$ is the Markov diagram of $\Sigma(\{\underline{u}^j, \underline{v}^j\})$. Choose $L \geq L(\delta)$ and let $\{\tilde{\underline{u}}^j, \tilde{\underline{v}}^j\}$ be a set of virtual itineraries as above. Denote by $\tilde{\mathcal{G}}$ the Markov diagram of $\Sigma(\{\tilde{\underline{u}}^j, \tilde{\underline{v}}^j\})$. Notice that up to vertices of upper level L, the graph $\mathcal{G}$ and $\tilde{\mathcal{G}}$ are the same. Moreover the paths on $\mathcal{G}$ (resp. $\tilde{\mathcal{G}}$) of length L starting at the root visit only vertices of upper level at most L. By (4.5), we get

$$h_{\mathrm{top}}(\Sigma(\{\tilde{\underline{u}}^j, \tilde{\underline{v}}^j\}), \sigma) \leq \frac{1}{L} \log \ell_L(\tilde{\mathcal{G}}) = \frac{1}{L} \log \ell_L(\mathcal{G}) \leq h_{\mathrm{top}}(\Sigma(\{\underline{u}^j, \underline{v}^j\}), \sigma) + \delta\,.$$

$\square$

4.2 Simplification of graphs

As emphasized before, the vertices of the Markov diagram are quadruples, but they represent subintervals of $\Sigma(\{\underline{u}^j, \underline{v}^j\})$. We need this technicality to avoid unexpected identifications and to keep the general properties of the graph independently of the shift $\Sigma(\{\underline{u}^j, \underline{v}^j\})$. But there may be simpler graphs which are presentations of $\Sigma(\{\underline{u}^j, \underline{v}^j\})$. In this section, we study how to define a simpler graph that inherits most of the properties of the Markov diagram.

Definition 4.11. *Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{L})$ be a right-resolving graph. Let $\sim$ be an equivalence relation on $\mathcal{V}$ such that*

$$c_1 \sim c_1' \text{ and } c_1 \xrightarrow{j} c_2 \implies \exists c_2' \text{ such that } c_1' \xrightarrow{j} c_2' \text{ and } c_2 \sim c_2'\,. \tag{4.6}$$

*The graph $\mathcal{G}' = (\mathcal{V}', \mathcal{E}', \mathcal{L}')$ is a **simplification** of $\mathcal{G}$, where $\mathcal{G}'$ is defined by*

1. $\mathcal{V}' := \mathcal{V}/\sim$ *is the quotient set,*

2. $\mathcal{E}'$ *and* $\mathcal{L}'$ *are defined by*

$$[c_1] \xrightarrow{j} [c_2] \iff \forall c_1' \in [c_1], \exists c_2' \in [c_2] \text{ such that } c_1' \xrightarrow{j} c_2' \, .$$

Notice that, in (4.6), there exists a unique vertex c_2' such that $c_1' \xrightarrow{j} c_2'$, since $\mathcal{G}$ is right-resolving. The property (4.6) implies that $\mathcal{E}'$ and $\mathcal{L}'$ are well-defined.

Lemma 4.12. *Let* $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{L})$ *be a right-resolving labeled oriented graph and* $\mathcal{G}' = (\mathcal{V}', \mathcal{E}', \mathcal{L}')$ *the simplification of* $\mathcal{G}$ *generated by the equivalence relation* $\sim$*. For all* $c_1 \in \mathcal{V}$*, the labels of paths on* $\mathcal{G}$ *starting at* c_1 *are identical to the labels of paths on* $\mathcal{G}'$ *starting at* $[c_1]$*. Moreover if a path on* $\mathcal{G}$ *starting at* $c_1 \in \mathcal{V}$ *terminates at the vertex* $c_2 \in \mathcal{V}$*, the path on* $\mathcal{G}'$ *defined by the same labels and starting at* $[c_1]$ *terminates at* $[c_2]$*.*

Proof: The proof is inductive. By Definition 4.11, the two statements are true for the paths of length 1. Suppose they are true for the paths of length n and consider a path labeled by $\underline{w}\, j$ of length $n+1$. After n steps on $\mathcal{G}$, we reach a vertex $c' \in \mathcal{V}$ and after n steps on $\mathcal{G}'$, we reach the vertex $[c']$. Then applying Definition 4.11, $c' \xrightarrow{j} c_2$ implies $[c'] \xrightarrow{j} [c_2]$. □

By this last lemma, we deduce immediately that, if $\mathcal{G}$ is a presentation of Σ and $\mathcal{G}'$ is a simplification of $\mathcal{G}$, then $\mathcal{G}'$ is a presentation of Σ. Notice that a simplification of $\mathcal{G}$ is right-resolving. But the simplification of a simple right-resolving graph is in general no more simple. In the case of the Markov diagram $\mathcal{G}$ (which is simple and right-resolving), we will be interested in simplifications $\mathcal{G}'$ remaining simple. Recall that, for the Markov diagram, all incoming edges at a vertex $c \in \mathcal{V}$ carry the same label. A way to keep $\mathcal{G}'$ simple is to require that, if $c \sim c'$, then all incoming edges at c and c' carry the same label a. Indeed since $\mathcal{G}'$ is right-resolving, there is at most one edge starting at c and labeled by a, thus $\mathcal{G}'$ is simple. Finally we define the root of a simplification: if c is the root $\mathcal{G}$, then the root of $\mathcal{G}'$ is the vertex $[c]$.

In the case of the Markov diagram, we can define an equivalence relation *on quadruples* by

$$(p, \underline{a}; q, \underline{b}) \sim_1 (p', \underline{a}'; q', \underline{b}') \iff [\sigma^p \underline{a}, \sigma^q \underline{b}] = [\sigma^{p'} \underline{a}', \sigma^{q'} \underline{b}'] \, . \tag{4.7}$$

This equivalence relation satisfies (4.6). Indeed let $c = (p, \underline{a}; q, \underline{b})$ and $c' = (p', \underline{a}'; q', \underline{b}')$ be two different vertices in $\mathcal{V}$ such that $c \sim c'$. Suppose that $d \in \mathcal{V}$ and $j \in \mathtt{A}_k$ are such that $c \xrightarrow{j} d$. Then

$$a_p \leq j \leq b_q \iff a'_{p'} \leq j \leq b'_{q'} \iff \text{there exists } d' \text{ such that } c' \xrightarrow{j} d' \, .$$

Moreover the construction of d' from c' depends only on $\sigma^{p'} \underline{a}'$ and $\sigma^{q'} \underline{b}'$, thus $d \sim d'$.

The simplification of the Markov diagram corresponding to this equivalence relation is the follower set graph (see [LM]). In concrete case, $c = (p, \underline{a}; q; \underline{b})$ and $c' = (p', \underline{a}'; q', \underline{b}')$ can be equivalent, only if there are supplementary relations between the virtual itineraries; for example, $\sigma^p \underline{a} = \sigma^{p'} \underline{a}'$ for some $p \neq p'$ or $\underline{a} \neq \underline{a}'$. In such a case, it is possible to construct directly the simplification. As before we begin with the root. Then we add the successors of the root. If some of them are equivalent to the root, we do not draw these vertices, but bring back the corresponding edges at the root. The set $\mathcal{V}_1'$ is formed by the vertices of

$\mathcal{V}_1$ which are not equivalent to the root. Then we construct the successors of all vertices in $\mathcal{V}_1'$ and we draw only the vertices which are not equivalent to the root or to a vertex in $\mathcal{V}_1'$; these vertices belong to $\mathcal{V}_2'$. We continue in this manner. Notice that the location in $\mathcal{G}'$ of the vertex $[c]$ is the location in $\mathcal{G}$ of the vertex $c' \in [c]$ with smallest upper level. We illustrate the construction of simplifications of Markov diagrams by two examples.

Example A (continuation). Suppose that the order $\preceq$ is the lexicographic order and there are two sequences $\underline{u} \equiv \underline{u}^0$ and $\underline{v} \equiv \underline{v}^{k-1}$ such that

$$\underline{u}^j = j\underline{u} \text{ for all } j = 1, \ldots, k-1 \quad \text{and} \quad \underline{v}^j = j\underline{v} \text{ for all } j = 0, \ldots, k-2.$$

Thus, for all $p \geq 1$, we have $\sigma^p \underline{u}^j = \sigma^{p-1}\underline{u}$ for all $j = 1, \ldots, k-1$ and $\sigma^p \underline{v}^j = \sigma^{p-1}\underline{v}$ for $j = 0, \ldots, k-2$. It is easy to check that

$$\Sigma(\{\underline{u}^j, \underline{v}^j\}) = \{\underline{x} \in \Sigma_k^+ : \underline{u} \preceq \sigma^n \underline{x} \preceq \underline{v} \ \forall n \geq 0\} =: \Sigma(\underline{u}, \underline{v}) \,. \tag{4.8}$$

These hypotheses correspond to the case of the shift space $\Sigma_{\alpha,\beta}$ (see (2.14) and (2.15)).

Consider the equivalence relation on quadruples $\sim_1$ defined in (4.7). The simplification of the Markov diagram corresponding to this equivalence relation is denoted by $\mathcal{G}(\underline{u}, \underline{v})$, it will be studied in detail in section 6.1.3. Notice that the root of $\mathcal{G}(\Sigma(\{\underline{u}^j, \underline{v}^j\}))$ is the vertex $(0, \underline{u}; 0, \underline{v})$. Since the order is the lexicographic order, all other vertices of $\mathcal{G}(\Sigma(\{\underline{u}^j, \underline{v}^j\}))$ are of the type $(p, \underline{a}; q, \underline{b})$ with $p, q \geq 1$, $\underline{a} \in U$ and $\underline{v} \in V$. Suppose that $\underline{a} \neq \underline{u}^0$, then $(p, \underline{a}; q, \underline{b}) \sim (p-1, \underline{u}; q, \underline{b})$. Similarly, suppose that $\underline{b} \neq \underline{v}^{k-1}$, then $(p, \underline{a}; q, \underline{b}) \sim (p, \underline{a}; q-1, \underline{v})$. Hence in all equivalence classes, there is a quadruple of the type $(p, \underline{u}; q, \underline{v})$ with $p, q \geq 0$. We choose to represent an equivalence class by a quadruple of the type $(p, \underline{u}; q, \underline{v})$ in this class. In Figure 4.2, we present an example of the graph $\mathcal{G}(\underline{u}, \underline{v})$,

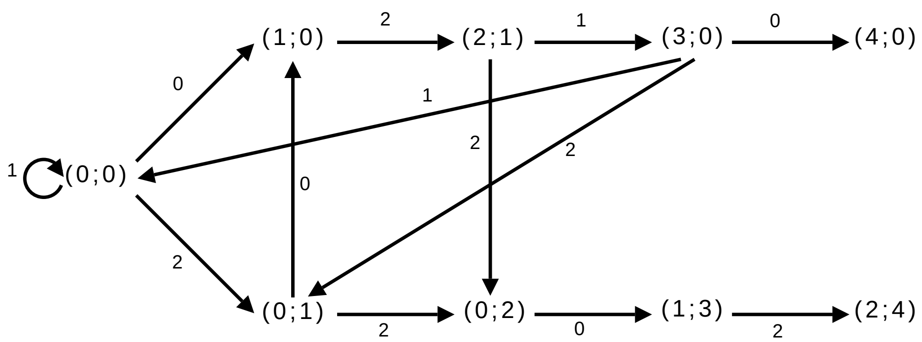

Figure 4.2: The beginning of the graph $\mathcal{G}(\underline{u}, \underline{v})$ for $k = 3$, $\underline{u} = 0210\ldots$ and $\underline{v} = 2202\ldots$. The notation $(p; q)$ stands for the quadruple $(p, \underline{u}; q, \underline{v})$.

when $k = 3$, the beginning of the virtual itineraries are $\underline{u} = 0210\ldots$ and $\underline{v} = 2202\ldots$. The root is the vertex $(0, \underline{u}; 0, \underline{v})$. Consider the path starting at the root and labeled by the word 1. In $\mathcal{G}(\Sigma(\{\underline{u}^j, \underline{v}^j\}))$, it terminates at vertex $(1, \underline{u}^1; 1, \underline{v}^1) \sim_1 (0, \underline{u}; 0, \underline{v})$. Hence in $\mathcal{G}(\underline{u}, \underline{v})$, it terminates at the vertex $(0, \underline{u}; 0, \underline{v})$, ie the root. Similarly consider the path starting at the root and labeled by the word 22. In $\mathcal{G}(\Sigma(\{\underline{u}^j, \underline{v}^j\}))$, it terminates at the vertex $(1, \underline{u}^2; 2, \underline{v}^2) \sim_1 (0, \underline{u}; 2, \underline{v})$. Hence in $\mathcal{G}(\underline{u}, \underline{v})$, it terminates at vertex $(0, \underline{u}; 2, \underline{v})$. We consider in this manner all paths starting at the root to get the graph $\mathcal{G}(\underline{u}, \underline{v})$.

Example B (continuation). Let $k \geq 2$, fix a map $s : \mathtt{A}_k \to \{1, -1\}$. The order on Σ_k^+ is defined by (2.2). Suppose that there is a sequence $\underline{\eta} \equiv \underline{v}^{k-1}$ such that for all $j \in \mathtt{A}_k$

$$\underline{u}^j = \begin{cases} j0^\infty & \text{if } s(j) = +1 \text{ and } s(0) = +1, \\ j0\underline{\eta} & \text{if } s(j) = +1 \text{ and } s(0) = -1, \\ j\underline{\eta} & \text{if } s(j) = -1, \end{cases} \tag{4.9}$$

and for all $j = 0, \ldots, k-2$

$$\underline{v}^j = \begin{cases} j\underline{\eta} & \text{if } s(j) = +1, \\ j0^\infty & \text{if } s(j) = -1 \text{ and } s(0) = +1, \\ j0\underline{\eta} & \text{if } s(j) = -1 \text{ and } s(0) = -1. \end{cases} \tag{4.10}$$

It is easy to check that

$$\Sigma(\{\underline{u}^j, \underline{v}^j\}) = \{\underline{x} \in \Sigma_k^+ : \sigma^n \underline{x} \preceq \underline{\eta} \; \forall n \geq 0\} := \Sigma_{\underline{\eta}} \,.$$

These settings correspond to the case of Example B (see (2.18), (2.19) and (2.20)).

Consider the equivalence relation *on vertices* given by

$$c = (p, \underline{a}; q, \underline{b}) \sim_2 c' = (p', \underline{a}'; q', \underline{b}') \iff (p, \underline{a}; q, \underline{b}) \sim_1 (p', \underline{a}'; q', \underline{b}') \text{ and}$$
$$\text{all incoming edges at } c \text{ and } c' \text{ carry the same label}\,.$$

We emphasize that $\sim_1$ is an equivalence relation on quadruples, we do not need the graph $\Sigma(\{\underline{u}^j, \underline{v}^j\})$ to define $\sim_1$. Whereas $\sim_2$ is an equivalence relation on vertices of $\Sigma(\{\underline{u}^j, \underline{v}^j\})$, because we need the graph to define $\sim_2$. We require the second condition to get a simplification, which is a simple graph. The simplification of the Markov diagram corresponding to $\sim_2$ is denoted by $\mathcal{G}(\underline{\eta})$. For commodity, we denote $\underline{0} \equiv \underline{u}^0$. Notice that $\underline{0} = 0^\infty$ or $\underline{0} = 0\underline{\eta}$, depending on the sign of $s(0)$. An example is presented in Figure 4.2

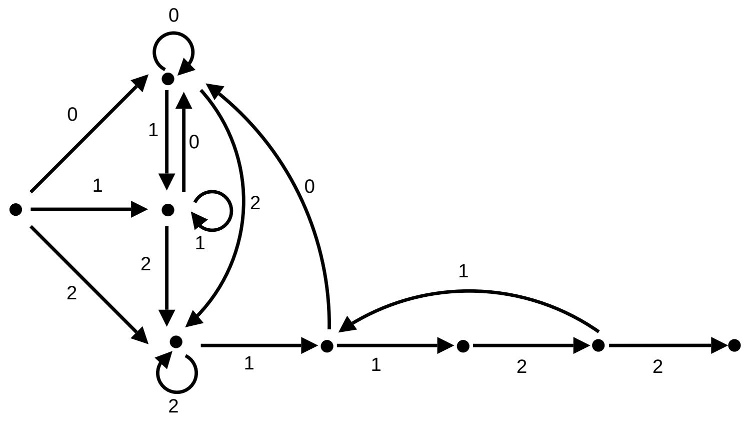

Figure 4.3: The beginning of the graph $\mathcal{G}(\underline{\eta})$ for a generalized β-shift with $k = 3$, $s(j) = (1, -1, -1)$ and $\underline{\eta} = 21122\ldots$.

for $k = 3$, $s(j) = (1, -1, -1)$ and $\underline{\eta} = 21122\ldots$. In this case,

$$\underline{0} \equiv \underline{u}^0 = 0^\infty \,, \quad \underline{u}^1 = 1\underline{\eta} \,, \quad \underline{u}^2 = 2\underline{\eta} \,, \quad \underline{v}^0 = 0\underline{\eta} \,, \quad \underline{v}^1 = 1\underline{0} \,.$$

We work as in Example A. The root is the vertex $(0,\underline{0};0,\underline{\eta})$. Consider the path starting at the root and labeled by the word 1. In $\mathcal{G}(\Sigma(\{\underline{u}^j,\underline{v}^j\}))$, it terminates at the vertex $(1,\underline{v}^1;1,\underline{u}^1)$. Although we have $(1,\underline{v}^1;1,\underline{u}^1)\sim_1(0,\underline{0};0,\underline{\eta})$, this vertex is not equivalent to the root, because of the second condition in the definition of $\sim_2$. Thus we draw a new vertex in the graph $\mathcal{G}(\underline{\eta})$. Similarly, consider the path starting at the root and labeled by the word 2. In $\mathcal{G}(\Sigma(\{\underline{u}^j,\underline{v}^j\}))$, it terminates at vertex $(1,\underline{v}^2;1,\underline{u}^2)$, which is not equivalent to the root; thus we draw a new vertex in $\mathcal{G}(\underline{\eta})$. Consider the path starting at the root and labeled by the word 12. In $\mathcal{G}(\Sigma(\{\underline{u}^j,\underline{v}^j\}))$, it terminates at the vertex $(2,\underline{u}^1;1,\underline{u}^2)\sim_1(1,\underline{v}^2;1,\underline{u}^2)$; moreover the incoming edges at these two vertices carry the label 2. Thus $(2,\underline{u}^1;1,\underline{u}^2)\sim_2(1,\underline{v}^2;1,\underline{u}^2)$ and we draw an edge labeled by 2 from the vertex indexed by 1 to the vertex indexed by 2. We continue in this manner with all paths to get $\mathcal{G}(\underline{\eta})$.

Chapter 5

Maximal measures for piecewise monotone continuous maps

The aim of this chapter is to present the content of Hofbauer's articles [H3] and [H5]. In these papers, the author studied the sets of measures of maximal entropy of piecewise monotone continuous maps. The fundamental idea is the isomorphism modulo small sets. A small set is a negligible set for all measures of maximal entropy. If two dynamical systems are isomorphic after removing small sets, then there is a one-to-one correspondence between their sets of maximal measures. Hofbauer proved that a piecewise monotone continuous map T with positive topological entropy such that the φ-expansion is valid is isomorphic modulo small sets to a countable Markov shift. Thus to describe the set of maximal measures of T, we can look for the maximal measures of a shift space. We begin with the presentation of the isomorphism modulo small sets in a general framework. Then we prove the results of Hofbauer in several steps. Finally we discuss the particular case of piecewise monotone continuous maps such that $|T'(x)|$ is constant; in this case, all measures of maximal entropy are absolutely continuous with respect to Lebesgue measure.

5.1 Isomorphisms modulo small set

Consider a compact measurable dynamical system $(X, \mathcal{B}, T)$. The next notion we want to introduce is the isomorphism modulo small sets. We need a preliminary result.

Lemma 5.1. *Let $(X, \mathcal{B}, T)$ be a measurable dynamical system and $Y \subset X$ such that $T(Y) \subset Y$. Setting $\mathcal{B}_Y := \mathcal{B} \cap Y \equiv \{B \cap Y : B \in \mathcal{B}\}$, then $(Y, \mathcal{B}_Y, T|_Y)$ is a measurable dynamical system.*

Proof: We prove that $(Y, \mathcal{B}_Y)$ is a measurable space: $Y = X \cap Y \in \mathcal{B}_Y$; let $A' \in \mathcal{B}_Y$, ie there exists $A \in \mathcal{B}$ with $A' = A \cap Y$, then

$$Y \backslash A' = Y \backslash (A \cap Y) = Y \backslash A = (X \backslash A) \cap Y \in \mathcal{B}_Y \, ;$$

let $A'_i \in \mathcal{B}_Y$, ie there exists $A_i \in \mathcal{B}$ with $A'_i = A_i \cap Y$, then

$$\bigcap_i A'_i = \bigcap_i (A_i \cap Y) = (\bigcap_i A_i) \cap Y \in \mathcal{B}_Y \, .$$

It remains to prove that $T' := T|_Y$ is measurable: let $A' \in \mathcal{B}_Y$, ie there exists $A \in \mathcal{B}$ with $A' = A \cap Y$, then

$$\begin{aligned}(T')^{-1}(A') &= \{x \in Y : T'(x) \in A'\} = \{x \in Y : T(x) \in A\} \\ &= \{x \in X : T(x) \in A\} \cap Y = T^{-1}(A) \cap Y\,.\end{aligned}$$

The second equality is true, because $T(x) \in Y$ for all $x \in Y$. □

Definition 5.2. *Let $(X, \mathcal{B}, T)$ be a measurable dynamical system with positive topological entropy. A **small set** is a set $N \in \mathcal{B}$ such that $T^{-1}N \subset N$ and for all measures $\mu \in M(X,T)$of maximal entropy, we have $\mu(N) = 0$.*
*Two measurable dynamical systems $(X_1, \mathcal{B}_1, T_1)$ and $(X_2, \mathcal{B}_2, T_2)$ are **isomorphic**, if there exists an invertible map $\psi : X_1 \to X_2$ such that ψ, ψ^{-1} are measurable maps and $T_2 \circ \psi = \psi \circ T_1$.*
*Two measurable dynamical systems having positive topological entropy $(X_1, \mathcal{B}_1, T_1)$ and $(X_2, \mathcal{B}_2, T_2)$ are **isomorphic modulo small sets**, if there exist two small sets $N_1 \subset X_1$, $N_2 \subset X_2$ such that $(X_1', \mathcal{B}_1', T_1')$ and $(X_2', \mathcal{B}_2', T_2')$ are isomorphic, where $X_i' := X_i \backslash N_i$, $\mathcal{B}_i' := \mathcal{B}_i \cap X_i'$ and $T_i' := T_i|_{X_i'}$.*

Notice that $(X_i', \mathcal{B}_i', T_i')$ are measurable dynamical systems. Indeed, $T_i^{-1}(N_i) \subset N_i$ implies

$$T_i(X_i \backslash N_i) \subset T_i(X_i \backslash T_i^{-1}(N_i)) = T_i(T_i^{-1}(X_i) \backslash T_i^{-1}(N_i)) = T_i(T_i^{-1}(X_i \backslash N_i)) \subset X_i \backslash N_i\,.$$

Using Lemma 5.1, we have that $(X_i', \mathcal{B}_i', T_i')$ is a measurable dynamical system. The next lemmas and corollary give sufficient conditions for a set to be small.

Lemma 5.3. *Let $(X, \mathcal{B}, T)$ be a measurable dynamical system with positive topological entropy. If $N \subset X$ is such that $T^{-1}N \subset N$ and for all $\nu \in M(X,T)$*

$$\nu(N) = 1 \implies h_T(\nu) = 0\,,$$

then N is a small set.

Proof: Let $\mu \in M(X,T)$ be a measure of maximal entropy. By hypothesis, $h_T(\mu) > 0$ and so $\mu(N) < 1$. Suppose $\alpha := \mu(N) > 0$ and define two measures $\nu_1, \nu_2 \in M(X)$ by

$$\nu_1(A) = \frac{1}{\alpha}\mu(A \cap N) \quad \text{and} \quad \nu_2(A) = \frac{1}{1-\alpha}\mu(A \backslash N) \quad \forall A \in \mathcal{B}\,,$$

such that the support of ν_1 is in N, the support of ν_2 is outside N and $\mu = \alpha\nu_1 + (1-\alpha)\nu_2$. Since $T^{-1}N \subset N$, these two measures are T-invariant. But from the affinity of the entropy map

$$h_T(\mu) = \alpha h_T(\nu_1) + (1-\alpha)h_T(\nu_2) = (1-\alpha)h_T(\nu_2) < h_T(\nu_2)\,,$$

contradicting the hypothesis that μ is a measure of maximal entropy, thus $\mu(N) = 0$. □

Corollary 5.4. *Let (X, d, T) be a compact continuous dynamical system with positive topological entropy. If $N \subset X$ is a closed subset such that $T^{-1}N = N$ and $h_{\text{top}}(N) = 0$, then N is small.*

Proof: N is closed, thus compact and $TN \subset N$. So $(N, d, T|_N)$ is a compact continuous dynamical system. Applying the variational principle, we deduce that $h_T(\mu) = 0$ for all $\mu \in M(X,T)$ such that $\mu(N) = 1$. □

Lemma 5.5. *Let $(X, \mathcal{B}, T)$ be a measurable dynamical system with positive entropy. Let N be a countable set such that $T^{-1}N \subset N$, then N is a small set.*

Proof: Let $\mu \in M(X, T)$ be a measure of maximal entropy. For all $x \in X$, $\mu(\{x\}) = 0$. Indeed, suppose the contrary: there exists $x \in X$ such that $\alpha := \mu(\{x\}) > 0$. Using the T-invariance of μ, we have for all $n \geq 1$

$$\mu(\{T^n x\}) = \mu(T^{-n}\{T^n x\}) \geq \mu(\{x\}) > 0 .$$

Since μ is a probability, there exists $p \geq 1$ such that $T^p x = x$. Define

$$\nu_1 := \frac{1}{p} \sum_{j=0}^{p-1} \delta_{T^j x} \qquad \text{and} \qquad \nu_2 := \frac{1}{1 - p\alpha} (\mu - p\alpha \nu_1)$$

such that $\mu = p\alpha\nu_1 + (1 - p\alpha)\nu_2$ ($p\alpha < 1$, otherwise $h_T(\mu) = 0$). By construction, $\nu_1, \nu_2 \in M(X, T)$ and $h_T(\nu_1) = 0$. Using the affinity of the map $h_T(\cdot)$, we have ($h_T(\nu_1) = 0$)

$$h_T(\mu) = p\alpha h_T(\nu_1) + (1 - p\alpha) h_T(\nu_2) < h_T(\nu_2) ,$$

this contradicts the fact that μ is a measure of maximal entropy. Thus $\mu(\{x\}) = 0$ for all $x \in X$. From the σ-additivity, we conclude that $\mu(N) = 0$. $\square$

If two measurable dynamical systems $(X_1, \mathcal{B}_1, T_1)$ and $(X_2, \mathcal{B}_2, T_2)$ are isomorphic, then there is a bijection between $M(X_1, T_1)$ and $M(X_2, T_2)$. Indeed, let $\psi : X_1 \to X_2$ be an isomorphism and define the map $\psi^* : M(X_1, T_1) \to M(X_2, T_2)$ by

$$\psi^* \mu := \mu \circ \psi^{-1} .$$

This map is well defined, since ψ is measurable. Moreover, it is bijective since the map $(\psi^{-1})^*$ given by $\mu \mapsto (\psi^{-1})^* \mu := \mu \circ \psi$ (also well defined) is the inverse map. We consider now the case of an isomorphism modulo small sets.

Theorem 5.6. *Let $(X_1, \mathcal{B}_1, T_1)$ and $(X_2, \mathcal{B}_2, T_2)$ be two measurable dynamical systems with positive entropy, $N_1 \subset X_1$, $N_2 \subset X_2$ two small sets and $\psi : X_1' := X_1 \backslash N_1 \to X_2' := X_2 \backslash N_2$ an isomorphism. Then the map $\chi : M(X_1, T_1) \to M(X_2, T_2)$ defined by*

$$\big(\chi(\mu)\big)(B) := \mu\big(\psi^{-1}(B \backslash N_2)\big) \qquad \forall B \in \mathcal{B}_2 ,$$

is a bijection between the maximal measures in $M(X_1, T_1)$ and the maximal measures in $M(X_2, T_2)$. Moreover $h_{T_1}(\mu) = h_{T_2}(\chi(\mu))$ for all maximal measures $\mu \in M(X_1, T_1)$.

Proof: Define a map $\omega : M(X_2, T_2) \to M(X_1, T_1)$ by

$$\big(\omega(\mu)\big)(A) = \mu\big(\psi(A \backslash N_1)\big) \qquad \forall A \in \mathcal{B}_1 .$$

If $\mu \in M(X_1, T_1)$ is a maximal measure, then $\mu(N_1) = 0$ and

$$\big(\omega \circ \chi(\mu)\big)(A) = \mu\Big(\psi^{-1}\big(\psi(A \backslash N_1) \backslash N_2\big)\Big) = \mu(A \backslash N_1) = \mu(A) \qquad \forall A \in \mathcal{B}_1 ,$$

Thus ω is an inverse on the left of χ for all maximal measures. Similarly we prove that ω is an inverse on the right for all maximal measure, thus χ is a bijection between the subsets of maximal measures of $M(X_1, T_1)$ and $M(X_2, T_2)$. Finally, consider a maximal measure $\mu \in M(X_1, T_1)$. By definition, $\mu(N_1) = 0$ and by construction of χ, $\chi(\mu)(N_2) = 0$. For a Borel partition α of X_2, define $\alpha' = \psi^{-1}\alpha = \{\psi^{-1}A : A \in \alpha\}$. We have

$$H(\mu, \alpha') = H(\mu, \alpha) .$$

This is true for all Borel partition α, thus the last statement follows. $\square$

5.2 Maximal measures for piecewise monotone continuous maps

To simplify the notation of this section, when we consider a measurable dynamical system $(X, \mathcal{B}, T)$, we will omit the σ-algebra and write (X, T). The interval $[0,1]$ will always be endowed with the Borel σ-algebra; the different shift spaces considered are always endowed with the σ-algebra generated by the cylinders. We consider the measurable dynamical system $([0,1], T)$, where T is a piecewise monotone continuous map of the interval (see Definition 2.1). Suppose that $([0,1], T)$ has positive entropy and the corresponding φ-expansion is valid. We show that the set of maximal measures of $([0,1], T)$ is in bijection with the set of maximal measures for a dynamical system (Σ_M, σ), where Σ_M is a Markov shift on a countable alphabet and σ is the left-shift map. This program was achieved by Hofbauer in [H3]. We split the discussion into three steps. The first one is to show that the coding map $\mathtt{i}$ is an isomorphism modulo small set from $([0,1], T)$ to (Σ_T^+, σ), where $\Sigma_T^+ := \overline{\mathtt{i}(X)}$ and σ is the left shift map. To study the shift space Σ_T^+, we use the Markov diagram: there is a bijection between Σ_T^+ and the infinite paths on the Markov diagram starting at the root. Thus we define Σ_M^+ as the set of infinite sequences of vertices corresponding to a walk on the graph and beginning at the root. Unfortunately the bijection from (Σ_T^+, σ) to (Σ_M^+, σ) is not an isomorphism, since in particular Σ_M^+ is not σ-invariant. Thus we change over from Σ_T^+ to Σ_T, the bi-infinite extension of Σ_T^+ defined by

$$\Sigma_T := \{\underline{w} \in \Sigma_k : \underline{w}_{[n,\infty)} \in \Sigma_T^+ \ \forall n \in \mathbb{Z}\}\,.$$

The second step is to show that there is a bijection κ between the sets of σ-invariant measures on Σ_T^+ and Σ_T and this bijection leaves the measure theoretic entropy invariant. Now we define Σ_M as the set of bi-infinite sequences on the Markov diagram. The third step is to show that there is an isomorphism modulo small sets ψ between (Σ_T, σ) and (Σ_M, σ). For this last step, we consider only the case where all laps f_j are increasing. The general case is treated by Hofbauer in [H5], we briefly sketch the ideas developed in this paper.

5.2.1 The isomorphism $\mathtt{i}$

Let $T : [0,1] \to [0,1]$ be a piecewise monotone continuous map of the interval and define with the same notations as in the previous chapter

$$\Sigma_T^+ := \overline{\mathtt{i}(X)} = \{\underline{x} \in \Sigma_k^+ : \underline{u}^{x_n} \preceq \sigma^n \underline{x} \preceq \underline{v}^{x_n} \quad \forall n \geq 0\}\,.$$

Then (Σ_T^+, σ) is a measurable dynamical system, since σ is continuous and $\sigma(\Sigma_T^+) \subset \Sigma_T^+$.

Proposition 5.7. *Suppose that $([0,1], T)$ has positive entropy and the corresponding φ-expansion is valid. Then $([0,1], T)$ and (Σ_T^+, σ) are isomorphic modulo small sets.*

Proof: From Definition 2.1, the set S is countable and $T^{-1}S \subset S$, hence it is a small set for $([0,1], T)$ by Lemma 5.5. On the other hand,

$$\underline{x} \in \Sigma_T^+ \backslash \mathtt{i}(X) \implies \exists n \geq 0 \text{ and } j \in \mathtt{A}_k \text{ such that } \sigma^n \underline{x} = \underline{u}^j \text{ or } \sigma^n \underline{x} = \underline{v}^j\,,$$

hence $\Sigma_T^+ \backslash \mathtt{i}(X)$ is countable and small for (Σ_T^+, σ). Since the φ-expansion is valid, $\mathtt{i}$ is bijective, continuous and admits a continuous inverse map $\overline{\varphi}_\infty$. Moreover from Theorem 2.15, $\sigma \circ \mathtt{i} = \mathtt{i} \circ T$ on X. $\square$

Notice that (Σ_T^+, σ) is a compact continuous dynamical system. So using the last proposition and the variational principle, we obtain

$$\sup\{h_T(\mu) : \mu \in M([0,1], T)\} = \sup\{h_\sigma(\mu) : \mu \in M(\Sigma_T^+, \sigma)\} = h_{\text{top}}(\Sigma_T^+, \sigma)\,.$$

As emphasized before, the definition of $\overline{h_{\text{top}}}(E, T)$ does not need the continuity of T. In particular, the entropy $\overline{h_{\text{top}}}([0,1], T)$ is well-defined. This quantity is studied in chapter 9 of [BB], in particular it is proved in Theorem 9.4.1 that

$$\overline{h_{\text{top}}}([0,1], T) = h_{\text{top}}(\Sigma_T^+, \sigma)\,.$$

Thus we may formulate a variational principle for the piecewise monotone continuous maps. Let T be a piecewise monotone continuous map such that the φ-expansion is well-defined, then

$$\sup\{h_T(\mu) : \mu \in M([0,1], T)\} = \overline{h_{\text{top}}}([0,1], T)\,.$$

The case $\overline{h_{\text{top}}}([0,1], T) = 0$ is easily treated separately.

5.2.2 Bijection between σ-invariant measures on Σ_T^+ and Σ_T

Given the semi-infinite shift space Σ_T^+, we construct its **natural extension** Σ_T defined by

$$\Sigma_T := \{\underline{w} \in \mathtt{A}_k^{\mathbb{Z}} : \underline{w}_{[n,\infty)} \in \Sigma_T^+ \ \forall n \in \mathbb{Z}\}\,. \tag{5.1}$$

We are interested in maximal measures of Σ_T^+, but we can work with Σ_T, because there is a bijection between the sets of σ-invariant measures on a semi-infinite shift and on its natural extension. Moreover this bijection leaves the measure-theoretical entropy invariant. This result is the content of the next proposition.

Proposition 5.8. *Let $\Sigma^+ \subset \Sigma_k^+$ be a shift space and $\Sigma \subset \Sigma_k$ defined by*

$$\Sigma := \{\underline{w} \in \Sigma_k : \underline{w}_{[n\infty)} \in \Sigma^+ \ \forall n \in \mathbb{Z}\}\,.$$

There exists an invertible map $\kappa : M(\Sigma, \sigma) \to M(\Sigma^+, \sigma)$ such that $h_\sigma(\mu) = h_\sigma\big(\kappa(\mu)\big)$ for all $\mu \in M(\Sigma, \sigma)$.

Proof: The main idea of this proof is that the family of cylinders sets is a semi-algebra generating both σ-algebras on Σ^+ and Σ. Let (notice that $\mathcal{L}(\Sigma_T^+) = \mathcal{L}(\Sigma_T)$)

$$\mathcal{C} := \{{}_n[\underline{w}] : \underline{w} \in \mathcal{L}(\Sigma_T^+), n \in \mathbb{Z}_+\}\,,$$
$$\mathcal{C}' := \{{}_n[\underline{w}] : \underline{w} \in \mathcal{L}(\Sigma_T), n \in \mathbb{Z}\}\,.$$

The family $\mathcal{C}$ is a semi-algebra generating the Borel σ-algebra on Σ^+ and $\mathcal{C}'$ is a semi-algebra generating the Borel σ-algebra on Σ. In this proof, we use the same notation for cylinders in Σ and Σ^+ and it is only the context which indicates in which space we work. Let $\mu \in M(\Sigma, \sigma)$ and define $\tau : \mathcal{C} \to [0,1]$ by

$$\tau({}_n[\underline{w}]) = \mu({}_0[\underline{w}])\,.$$

Then τ is σ-additive, ie for a finite or countable family of disjoint subsets $E_i \in \mathcal{C}$, we have $\tau(\bigcup_i E_i) = \sum_i \tau(E_i)$. Indeed, let $J \subset \mathbb{N}$, $\{\underline{w}^j\}_{j\in J}$ and $\underline{w}$ be words in $\mathcal{L}(\Sigma_T^+)$ and $\{n^j\}_{j\in J} \subset \mathbb{Z}_+$, $n \in \mathbb{Z}_+$ be such that

$$\bigcup_{j\in J} {}_{n^j}[\underline{w}^j] = {}_n[\underline{w}] \qquad \text{and} \qquad {}_{n^i}[\underline{w}^i] \cap_{n^j} [\underline{w}^j] = \emptyset \quad \forall i \neq j\,.$$

Since the cylinders are open and closed subsets of Σ_T, the set ${}_n[\underline{w}]$ is compact and $\{{}_{n^j}[\underline{w}^j] : j \in J\}$ is an open cover of ${}_n[\underline{w}]$. There exists a finite subset $J' \subset J$ such that $\{{}_{n^j}[\underline{w}^j] : j \in J'\}$ is an open cover of ${}_n[\underline{w}]$. But the cylinders ${}_{n^j}[\underline{w}^j]$ are disjoint two by two, thus $J \equiv J'$ and J is finite. Choose $N \in \mathbb{Z}_+$ such that $N + n \geq 0$ and $N + n_j \geq 0$ for all $j \in J$, then, using the σ-invariance of μ several times,

$$\begin{aligned}\tau({}_n[\underline{w}]) = \mu({}_0[\underline{w}]) = \mu(\sigma^{-(N+n)}\,{}_0[\underline{w}]) = \mu(\sigma^{-N}\,{}_n[\underline{w}]) = \mu(\sigma^{-N}\bigcup_{j\in J}{}_{n^j}[\underline{w}^j])\\ = \mu(\bigcup_{j\in J}{}_{n^j}[\underline{w}^j]) = \sum_{j\in J}\mu({}_{n^j}[\underline{w}^j]) = \sum_{j\in J}\mu({}_0[\underline{w}^j]) = \sum_{j\in J}\tau({}_{n^j}[\underline{w}^j])\,.\end{aligned}$$

Since τ is σ-additive on the algebra $\mathcal{C}$, there exists a unique probability measure on the σ-algebra of cylinders extending τ (see [Pa] or Theorem 0.3 in [W]). Define $\kappa(\mu)$ as the unique extension of τ on the σ-algebra of cylinders in Σ^+.
We prove that κ is bijective constructing an inverse map. Let $\nu \in M(\Sigma^+, \sigma)$ and define $\tau' : \mathcal{C}' \to [0, 1]$ by

$$\tau'({}_n[\underline{w}]) = \nu({}_0[\underline{w}])\,.$$

As before, we can prove that τ' is σ-additive. We define a map mapping ν to the unique extension of τ' on the σ-algebra of cylinders in Σ. Since the weight of cylinders is always the same, it is easy to see that this map is the inverse map of κ.
Finally, the invariance of the measure theoretic entropy under κ follows from the Kolmogorov-Sinai Theorem (Theorem 4.17 in [W]). We have $h_\sigma(\mu) = h_\sigma(\mu, \alpha)$ where α is any finite Borel partition of Σ or Σ^+ with cylinder sets, but the corresponding cylinders have the same weight under μ or $\kappa(\mu)$. $\square$

We deduce immediately from this Proposition that κ is a one-to-one between the subsets of maximal measures in $M(\Sigma, \sigma)$ and $M(\Sigma^+, \sigma)$. In particular, we can study the maximal measure on (Σ, σ) instead of (Σ^+, σ) and we will do this with Σ_T.

5.2.3 The isomorphism ψ

The last step is to construct an isomorphism modulo small sets between the shift space of bi-infinite walks on the Markov diagram and the shift space Σ_T. There is a natural way to map a bi-infinite walk on the Markov diagram to an element of Σ_T: one simply has to take the labels of the walk. The inverse map is more complicated: we do not know where to start the walk. This section is devoted to the construction of this map and the study of its properties.

First, we define a map ψ on Σ_T except for an exceptional subset N and show that ψ is an isomorphism. Then we show that N is a small set. The construction of ψ on $\Sigma_T \backslash N$ requires that all laps of T are increasing. We consider only this case, the general case will be treated differently.

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{L})$ be the Markov diagram associated to Σ_T^+ and denote by $\mathcal{O}$ the root of the graph. Since the Markov diagram is simple, we choose to describe paths on $\mathcal{G}$ by sequences of vertices. Denote by $M = \{M_{i,j}\}_{i,j\in\mathcal{V}}$ the adjacency matrix of $\mathcal{G}$ defined by (4.3). Define Σ_M as the set of bi-infinite paths on $\mathcal{G}$

$$\Sigma_M := \{\underline{w} \in \Sigma_{\mathcal{V}} : M_{w_i,w_{i+1}} = 1\,\forall i \in \mathbb{Z}\}\,.$$

One constructs $\psi(\underline{w})$ by the following procedure: fix n and define a walk starting at the root and labeled by $\underline{w}_{[n,\infty)}$, then take the limit $n \to -\infty$; if all coordinates are eventually constant, then $\psi(\underline{w})$ is this limit. Explicitly let $\underline{w} \in \Sigma_T$. For all $n \in \mathbb{Z}$, define a sequence $\underline{z}^n \equiv z^n_n z^n_{n+1} z^n_{n+2} \dots$ with $z^n_j \in \mathcal{V}$ for all $j \geq n$ by $z^n_n = \mathcal{O}$ and $\underline{z}^n$ is the walk labeled by $\underline{w}_{[n,\infty)}$. If for all $j \in \mathbb{Z}$, the limit $z_j := \lim_{n\to-\infty} z^n_j$ exists, then define $\psi(\underline{w}) := \underline{z} \equiv \dots z_{-1}z_0z_1 \dots \in \Sigma_M$. Unfortunately, there is a set where this limit does not exist. For all $\underline{a} \in U \cup V$, we define the sets

$$N_{\underline{a}} := \{\underline{w} \in \Sigma_T : \exists m \in \mathbb{Z} \text{ s.t. } \forall K \leq m, \exists n \leq K \text{ s.t. } \underline{w}_{[n,m)} = \underline{a}_{[0,m-n)}\}\,, \tag{5.2}$$

and

$$N = \bigcup_{\underline{a}\in U\cup V} N_{\underline{a}}\,.$$

Lemma 5.9. *For $\underline{w} \in \Sigma_T \backslash N$, the limit involved in the definition of $\psi(\underline{w})$ exists.*

Proof: By Corollary 4.8, when we want to know at which vertex terminates a path on the Markov diagram starting at the root and labeled by a word $\underline{z} \in \mathcal{L}(\Sigma_T)$, one needs to construct the longest suffix of $\underline{z}$ which is a prefix of some $\underline{u}^j$ and similarly with $\underline{v}^j$. Let $\underline{w} \in \Sigma_T\backslash N$ and $m \in \mathbb{Z}$. Since $\underline{w} \notin N = \bigcup_{\underline{a}\in U\cup V} N_{\underline{a}}$, for each $\underline{a} \in U \cup V$ there exists $K_{\underline{a}}$ such that for each $n \leq K_{\underline{a}}$ we have $\underline{w}_{[n,m)} \neq \underline{a}_{[0,m-n)}$. Let $K := \min_{\underline{a}} K_{\underline{a}}$. For each $l \leq K$, define $r_l > l$ by

$$\underline{w}_{[l,r_l)} = \underline{u}^{w_l}_{[0,r_l-l)} \quad \text{and} \quad w_{r_l} \neq u^{w_l}_{r_l-l}\,.$$

Set $R := \max\{r_l : l \leq K\}$. R exists and $R < m$, since by construction of K, $r_l < m$ for all $l \leq K$. For all $n \leq R$ and all $\underline{a} \in U$, $\underline{w}_{[n,R]} \neq \underline{a}_{[0,R-n]}$. For $n \leq K$, this is true by definition of R; for $n > K$, this is true by Lemma 4.6. Denote by $\{\underline{z}^n\}$ the sequence constructed in the definition of ψ. Then by Lemma 4.7, $z^n_R = (1, \underline{u}^{w_R}; \cdot, \cdot)$ for all $n < R$. Moreover using the construction of successors as in Lemma 4.3, the lower boundary of intervals z^n_t is constant for all $n < R$ and $t \geq R$. In particular, the lower boundary of z^n_m is constant for all $n < R$. Similarly, we construct a R' considering the $\underline{v}$, such that the upper boundary of z^n_m is constant for all $n < R'$. Since $m \in \mathbb{Z}$ is arbitrary, this completes the proof. $\square$

Lemma 5.10. *The map $\psi : \Sigma_T\backslash N \to \Sigma_M$ is a homeomorphism.*

Proof: Define a map $\pi : \Sigma_M \to \Sigma_{\mathcal{V}}$ applying the walk on the Markov diagram $\underline{z} \in \Sigma_M$ to the sequence of labels of $\underline{z}$, ie $\underline{w} = \pi(\underline{z})$ if and only if for all $n \in \mathbb{Z}$, we have $\mathcal{L}(e) = w_n$ with e the unique edge such that $i(e) = z_n$ and $t(e) = z_{n+1}$. We see immediately that $\pi \circ \psi(\underline{w}) = \underline{w}$ for all $\underline{w} \in \Sigma_T$, thus ψ is injective. For the surjectivity, choose $\underline{z} \in \Sigma_M$ and define $\underline{w} = \pi(\underline{z})$. One shows as in Lemma 5.9 that $\psi(\underline{w}) = \underline{z}$. Since $\underline{z} \in \Sigma_M$ is arbitrary, we conclude that $\pi = \psi^{-1}$.
We prove the continuity of ψ and π using the cylinders as a base of neighborhoods. For ψ: let $\underline{w} \in \Sigma_T\backslash N$ and $\underline{z} = \psi(\underline{w})$. Let ${}_m[\underline{x}] \subset \Sigma_M$ be a cylinder containing $\underline{z}$. Since $\underline{w} \notin N$, there exists $K \leq m$ such that for all $j \leq K$, we have

$$\underline{w}_{[j,m)} \neq \underline{a}_{[0,m-j)} \quad \forall \underline{a} \in U \cup V\,.$$

As in Lemma 5.9, we prove that $\psi({}_K[\underline{w}_{[K,m+|\underline{x}|)}]) \subset {}_m[\underline{x}]$. Thus ψ is continuous. For π, the proof is trivial. $\square$

Lemma 5.11. *Suppose that the measurable dynamical system (Σ_T, σ) has positive topological entropy, then the subset $N \subset \Sigma_T$ is a small set.*

Proof: In view of Definition 5.2, any finite or countable union of small sets is still a small set, thus it is enough to prove that $N_{\underline{a}}$ is a small set for all $\underline{a} \in U \cup V$. Fix $\underline{a} \in U \cup V$ and consider $N_{\underline{a}}$. The proof is decomposed in several steps. First we construct a subset $N'_{\underline{a}} \subset N_{\underline{a}}$ such that $\mu(N_{\underline{a}} \backslash N'_{\underline{a}}) = 0$. Then we show that $(N'_{\underline{a}}, \sigma)$ is conjugated to some shift (Σ', σ). Finally we show that $h_{\text{top}}(\Sigma', \sigma)$ is arbitrarily small.

Defining

$$A_m = \{\underline{w} \in \Sigma_T : \forall N \leq m, \exists n \leq N \text{ s.t. } \underline{w}_{[n,m)} = \underline{a}_{[0,m-n)}\},$$

we see directly from formula (5.2) that $N_{\underline{a}} = \bigcup_{m\in\mathbb{Z}} A_m$. From the definition of A_m, we deduce that $A_{m+1} \subset A_m$ and $\sigma^{-1}A_m = A_{m+1}$. Now consider a measure $\mu \in M(\Sigma_T, \sigma)$, then

$$\begin{aligned}\mu(N_{\underline{a}}) &= \mu(\bigcup_{m\in\mathbb{Z}} A_m) = \mu(\bigcup_{m\in\mathbb{Z}} (A_m \backslash A_{m+1}) \cup \bigcap_{m\in\mathbb{Z}} A_m) \\ &= \sum_{m\in\mathbb{Z}} \mu(A_m \backslash A_{m+1}) + \mu(\bigcap_{m\in\mathbb{Z}} A_m) = \mu(\bigcap_{m\in\mathbb{Z}} A_m).\end{aligned}$$

Indeed the σ-invariance of μ implies

$$\mu(A_m \backslash A_{m+1}) = \mu(A_m) - \mu(A_{m+1}) = \mu(A_m) - \mu(\sigma^{-1}A_m) = 0.$$

Thus it remains to prove that $N'_{\underline{a}}$ is a small set, where

$$N'_{\underline{a}} := \bigcap_{m\in\mathbb{Z}} A_m = \{\underline{w} \in \Sigma_T : \forall m \in \mathbb{Z}, \forall N \leq m, \exists n \leq N \text{ s.t. } \underline{z}_{[n,m)} = \underline{a}_{[0,m-n)}\}.$$

Fix $K \in \mathbb{N}$ and take $\underline{w} \in N'_{\underline{a}}$. There exist in $\underline{w}$ prefixes of $\underline{a}$ starting arbitrarily far on the left and ending arbitrarily far on the right. Thus we will construct inductively a sequence of prefixes of $\underline{a}$ found in the sequence $\underline{w}$ such that: the first one has length at least K; given a prefix, the next one is constructed such that it overlaps the previous one by at least K symbols on the left and on the right. Precisely, for $\underline{w} \in \Sigma_T$ and $n \in \mathbb{Z}$, define

$$\ell_n(\underline{w}) := \sup\{t \geq 0 : \underline{w}_{[n,n+t)} = \underline{a}_{[0,t)}\} \in \mathbb{Z}_+ \cup \{\infty\}.$$

Now fix $\underline{w} \in N'_{\underline{a}}$ and define two sequences $\{b_i\}_{i\geq 0} \subset \mathbb{Z}$ and $\{c_i\}_{i\geq 0} \subset \mathbb{Z}$ in the following way

$$b_0 := \max\{n \leq 0 : \ell_n(\underline{w}) \geq K\} \quad \text{and} \quad c_0 := b_0 + \ell_{b_0}(\underline{w}),$$

and inductively for all $i \geq 0$

$$b_{i+1} := \begin{cases} \max\{n \leq b_i - K : n + \ell_n(\underline{w}) \geq c_i + K\} & \text{if } c_i < \infty \\ \max\{n \leq b_i - K : \ell_n(\underline{w}) = \infty\} & \text{if } c_i = \infty \end{cases}$$

and

$$c_{i+1} := \begin{cases} b_{i+1} + \ell_{b_{i+1}}(\underline{w}) & \text{if } c_i < \infty \\ \infty & \text{if } c_i = \infty. \end{cases}$$

By definition, $b_i - b_{i+1} \geq K$ and if $c_i < \infty$, $c_0 - b_0 \geq K$ and $c_{i+1} - c_i \geq K$. The situation is illustrated in Figure 5.2.3. Define a map $\chi : N'_{\underline{a}} \to \Sigma_3$ by $\underline{z} = \chi(\underline{w})$ where

$$z_i = \begin{cases} 1 & \text{if } \exists t \in \mathbb{Z} \text{ such that } b_t = i \\ 2 & \text{if } \exists t \in \mathbb{Z} \text{ such that } c_t = i \\ 0 & \text{otherwise.} \end{cases}$$

This map is well-defined, since the $b_t \neq c_{t'}$ for all $t, t' \in \mathbb{Z}$. Let $\underline{w} \in N'_{\underline{a}}$, then $\underline{z} = \chi(\underline{w})$ satisfies

$$\forall i \in \mathbb{Z}, z_i \neq 0 \implies z_{i+t} = 0 \quad \forall t = 1, \ldots, K-1\,, \tag{5.3}$$
$$\forall i \in \mathbb{Z}, z_i = 2 \implies z_{i+t} \neq 1 \quad \forall t \geq 1\,, \tag{5.4}$$
$$\forall N \in \mathbb{N}, \exists i \leq -N \text{ such that } z_i = 1\,. \tag{5.5}$$

We now check that χ is invertible, continuous and has a continuous inverse. Let

$$\Sigma' := \{\underline{z} \in \Sigma_3 : \underline{z} \text{ satisfy (5.3), (5.4) and (5.5)}\}\,.$$

Define a map $\zeta : \Sigma' \to \Sigma_T$ by $\underline{z} \mapsto \underline{w} := \zeta(\underline{z})$ where $\underline{w}$ is given by the following procedure: for all $i \in \mathbb{Z}$, read $\underline{z}$ backward from i until one finds the first $t \leq i$ such that $z_t \neq 0$ (such a t exists by (5.5)).

1. If $z_t = 1$, then $\underline{w}_{[t,i]} = \underline{a}_{[0,i-t]}$.
2. If $z_t = 2$, then read $\underline{z}$ backward until one finds the first $t' \leq t$ such that $z_{t'} = 1$ and there are on more symbols 1 than symbols 2 in $\underline{z}_{[t',i]}$ (such a t' exists, since by (5.5) there are symbols 1 arbitrarily far on the left and, reading backward, after a symbol 1 there is no symbol 2 by (5.4)). Then $\underline{w}_{[t',i]} = \underline{a}_{[0,i-t']}$.

It is easy to check that $\underline{w} = \zeta \circ \chi(\underline{w})$ for all $\underline{w} \in N'_{\underline{a}}$, thus $\chi : N'_{\underline{a}} \to \chi(N'_{\underline{a}})$ is invertible. The maps χ and ζ are continuous. We use the cylinders as basis of neighborhoods: let $\underline{w} \in N'_{\underline{a}}$ and $\underline{z} = \chi(\underline{w})$. Choose $n \in \mathbb{Z}$ and $\underline{x} \in \mathcal{L}(\Sigma_3)$ such that $\underline{z} \in {}_n[\underline{x}]$. Choose $i \in \mathbb{Z}_+$ such that $b_i \leq n$ and $c_i \geq n + |\underline{x}|$. Then the cylinder ${}_{b_i}[\underline{w}_{[b_i, n+|\underline{x}|)}]$ is mapped by χ inside the cylinder ${}_n[\underline{x}]$, thus χ is continuous. For ζ we proceed similarly.

Finally we calculate the topological entropy of (Σ', σ). Consider the shift space

$$\Sigma'' := \{\underline{z} \in \Sigma_3 : \underline{z} \text{ satisfy (5.3)}\}\,.$$

We compute easily that $h_{\mathrm{top}}(\Sigma'', \sigma) = K^{-1} \log 3$. Since χ is a continuous isomorphism $\chi : N'_{\underline{a}} \to \chi(N'_{\underline{a}}) \subset \Sigma' \subset \Sigma''$, we have

$$h_{\mathrm{top}}(N'_{\underline{a}}, \sigma) = h_{\mathrm{top}}(\chi(N'_{\underline{a}}), \sigma) \leq h_{\mathrm{top}}(\Sigma'', \sigma) = \frac{1}{K} \log 3\,.$$

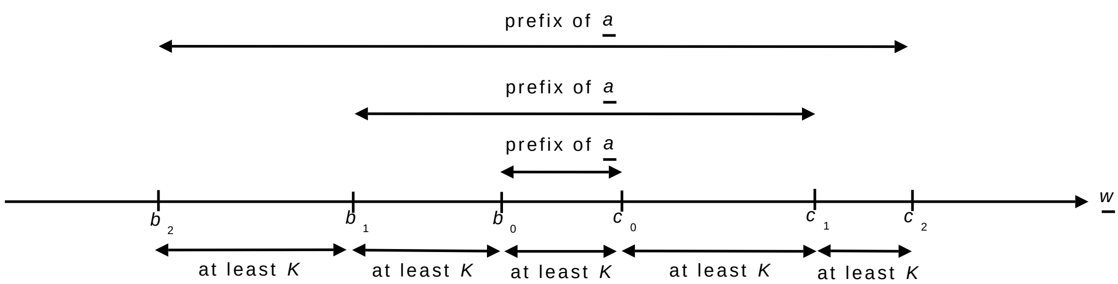

Figure 5.1: Construction of b_i and c_i.

Since K is arbitrary, $h_{\text{top}}(N'_{\underline{a}}) = 0$ and from Lemma 5.3 we conclude that $N'_{\underline{a}}$ is a small set. $\square$

Thus $\psi : \Sigma_T \to \Sigma_M$ is an isomorphism modulo small sets, when all laps of T are increasing. The general case is considered by Hofbauer in [H5]. In this paper, the author proves the following result. Let $p : [0,1] \to [0,1]$ be defined by

$$p(x) = \begin{cases} 2x & \text{if } x \leq 1/2 \\ 2-2x & \text{if } x > 1/2. \end{cases}$$

Given a piecewise monotone continuous map T_1 with k laps, another piecewise monotone continuous map T_2 with $2k$ *increasing* laps is constructed such that $p \circ T_2 = T_1 \circ p$ everywhere T_2 is well-defined. Thus T_2 is a 2-to-1 factor of T_1. Then studying in parallel the Markov diagram of T_1 and T_2, it is proved that there is a 2-to-1 mapping from $N_2 \subset \Sigma_{T_2}$ on $N_1 \subset \Sigma_{T_1}$. Finally it is deduced that N_2 is also a null set.

We conclude Section 5.2.3 by a proposition collecting the results of Lemmas 5.9, 5.10 and 5.11.

Proposition 5.12. *Suppose (Σ_T, σ) has positive topological entropy. Then (Σ_T, σ) and (Σ_M, σ) are isomorphic modulo small sets.*

Finally we discuss what happens when we use a simplification of the Markov diagram. Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{L})$ be the Markov diagram of Σ_T^+ and $\mathcal{G}' = (\mathcal{V}', \mathcal{E}', \mathcal{L}')$ be a simplification of $\mathcal{G}$ associated to the equivalence relation $\sim$. Let $\mathcal{O}'$ be the root of $\mathcal{G}'$. Suppose further that $\mathcal{G}'$ is a simple graph, so we may use sequences of vertices to define paths. We want to define a map $\psi : \Sigma_T \to \Sigma_{\mathcal{V}'}$. The idea to define ψ is exactly the same: we construct a sequence of paths starting at $\mathcal{O}'$ at coordinate n and let n go to $-\infty$. The exceptional set where this process does not converge is a subset of N. Indeed suppose $\underline{w} \in \Sigma_T \backslash N$ and let $\pi : \mathcal{V} \to \mathcal{V}'$ be the projection corresponding to $\sim$, ie $\pi(c) = [c]$. If $\underline{z}^n$ is the sequence appearing in the definition of ψ, then $\pi(z_n^n)\pi(z_{n+1}^n)\dots$ is the sequence appearing in the definition of ψ'. In particular, if for $m \in \mathbb{Z}$ the sequence $\{z_n^m\}_n$ is ultimately constant, then $\{\pi(z_n^m)\}_n$ is also ultimately constant. Thus the map ψ' is well defined on $\Sigma_T \backslash N$. In this thesis, we will often choose to work with a simplification of the Markov diagram and use the result of this section.

5.2.4 Maximal measure for countable Markov shifts

Compiling the results of Propositions 5.7, 5.8 and 5.12, we state the main theorem of this chapter.

Theorem 5.13. *Let T be a piecewise monotone continuous map such that the φ-expansion is valid and the measurable dynamical system $([0,1], T)$ has positive topological entropy. Let Σ_M denote the set of bi-infinite walks on the associated Markov diagram (or a simplification of this graph). Then there is a one-to-one correspondence between the sets of maximal measures for $([0,1], T)$ and for (Σ_M, σ).*

Instead of studying the maximal measures on $([0,1], T)$, we can study the maximal measures on (Σ_M, σ), a Markov shift on a countable alphabet. The fundamental works on this topic are those of Vere-Jones in [VJ1] and [VJ2]. The books of Seneta [Se] and Kitchens [K] are good references.

Consider the adjacency matrix $M = \{M_{i,j}\}_{i,j \in \mathcal{V}}$ of the Markov diagram. The matrix M is **irreducible**, if for all $i, j \in \mathcal{V}$, there exists $n \geq 1$ such that $M_{i,j}^n > 0$. A subset $C \subset \mathcal{V}$

is **strongly connected**, if $M|_{\mathcal{C}} \equiv \{M_{i,j}\}_{i,j\in\mathcal{C}}$ is irreducible. A **communicating class** is a maximal subset of $\mathcal{C} \subset \mathcal{V}$, which is strongly connected. Given a non-irreducible matrix M, we can decompose $\mathcal{V}$ into communicating classes. We get a family $\{\mathcal{C}_i\}_i$ of disjoint subsets of $\mathcal{V}$. In general $\bigcup_i \mathcal{C}_i \neq \mathcal{V}$, because if $M^n_{j,j} = 0$ for all $n \geq 1$, then j does not belong to any communicating class. The decomposition in communicating class is useful, because by Lemma 3.1, any ergodic invariant measure $\mu \in M(\Sigma_M, \sigma)$ is concentrated on a communicating class. Thus the first step is to decompose $\mathcal{V}$ into communicating classes $\{\mathcal{C}_i\}_i$. The number of communicating classes can be finite or countable.

Consider a communicating class $\mathcal{C}_i$ and denote by $M_i := M|_{\mathcal{C}_i}$ the corresponding adjacency matrix. Notice that $\mathcal{C}_i$ can be finite or countable, thus we must use the theory of countable Markov shifts. The following proposition is a compilation of results found in [K], in particular see Proposition 7.2.13.

Proposition 5.14. *Let $M = \{M_{i,j}\}_{i,j\in\mathcal{V}}$ be an irreducible matrix and*

$$\Sigma_M = \{\underline{w} \in \mathcal{V}^{\mathbb{Z}} : M_{w_i,w_{i+1}} = 1 \ \forall i \in \mathbb{Z}\} .$$

Suppose M is a finite matrix, then (Σ_M, σ) has a unique maximal measure.
Suppose M is a countable matrix. If (Σ_M, σ) has a maximal measure, then it is unique.
In both cases, the maximal measure (if it exists) is a Markov measure μ given by

$$p_i = \ell_i r_i \qquad P_{i,j} = M_{i,j}\frac{r_i}{\lambda \ell_j} ,$$
$$\mu(\underline{w}) = p_{w_0} P_{w_0,w_1} P_{w_1,w_2} \dots P_{w_{n-2},w_{n-1}} ,$$

where λ is the spectral radius of M, ℓ and r are the unique left and right eigenvectors for λ, they are normalized such that $\ell \cdot r = 1$. This measure has measure-theoretical entropy $\log \lambda$.

Theorem 5.15. *Let T be a piecewise monotone continuous map such that the φ-expansion is valid and the measurable dynamical system $([0,1], T)$ has positive topological entropy. Let M denote the adjacency matrix of the corresponding Markov diagram (or a simplification of this graph). Then $([0,1], T)$ has as many maximal measures as M has communicating classes of maximal spectral radius.*

Proof: By Theorem 5.13, we know that there is a one-to-one correspondence between the maximal measures on $([0,1], T)$ and on (Σ_M, σ). The ergodic measures are concentrated on communicating classes. By Proposition 5.14, $([0,1], T)$ has at most as many maximal measures as M has communicating classes of maximal spectral radius.
Take a communicating class $\mathcal{C}$ of maximal spectral radius and define

$$\Sigma_{\mathcal{C}} := \{\underline{w} \in \Sigma_M : w_i \in \mathcal{C} \ \forall i \in \mathbb{Z}\} ,$$

and (recall that $\psi : \Sigma_T \backslash N \to \Sigma_M$ is invertible)

$$\Sigma := \psi^{-1}(\Sigma_{\mathcal{C}}) \subset \Sigma_T .$$

In other words, Σ is the shift space formed by the labels of walks on the Markov diagram remaining in $\mathcal{C}$. This implies

$$\Sigma = \bigcap_{n\geq 0} \left(\bigcup \{ {}_n[\underline{w}] : \underline{w} \in \mathcal{L}(\Sigma), |\underline{w}| = 2n+1 \} \right) .$$

Since the cylinders are closed sets and the union over words of length $2n+1$ is finite, Σ is closed as an intersection of closed sets. Since Σ_T is compact, Σ is compact. Finally (Σ, σ) has at least one maximal measure, since it is an expansive homeomorphism and the entropy map $h_T(\mu)$ is upper semi-continuous (see Theorem 8.2 in [W]). But ψ is an isomorphism modulo small sets between Σ and $\Sigma_{\mathcal{C}}$, thus $\Sigma_{\mathcal{C}}$ has exactly one maximal measure. $\square$

Chapter 6

Two maps of constant slope

Here begins the part of this thesis which contains most of the original results. In this chapter, we study in detail two particular families of piecewise monotone continuous maps. The first one is $T_{\alpha,\beta}(x) := \beta x + \alpha \mod 1$. This map was introduced by Parry in [P2] as a generalization of the β-transformation studied by Rényi in [R] and Parry in [P1]. The shift space $\Sigma_{\alpha,\beta}$ associated to this map is characterized by only two really important virtual itineraries. Thus we introduce the corresponding family of shift spaces

$$\Sigma(\underline{u},\underline{v}) := \{\underline{x} \in \Sigma_k^+ : \underline{u} \preceq \sigma^n \underline{x} \preceq \underline{v} \ \forall n \geq 0\},$$

described by two virtual itineraries $\underline{u}$ and $\underline{v}$. First we introduce an algorithm, based on the φ-expansion, to compute the topological entropy of $\Sigma(\underline{u},\underline{v})$. Then we study an inverse problem: given two sequences $\underline{u},\underline{v} \in \Sigma_k^+$, we want to find α, β such that the map $T_{\alpha,\beta}$ admits $\underline{u}$ and $\underline{v}$ as virtual itineraries. This question was entirely solved in [P1] when $\alpha = 0$. It is much more complicated in the general case $\alpha \in [0,1)$.

The second part of this chapter is devoted to the generalized β-transformations studied by Góra in [G]. These maps are generalization of β-transformations. They also have a unique virtual itinerary, but they involve decreasing laps. Notice that the tent maps belong to the generalized β-transformations. Góra has already constructed the unique invariant measure absolutely continuous with respect to Lebesgue measure. In almost all cases, we prove that this measure is the unique measure of maximal entropy.

6.1 The map $\beta x + \alpha \mod 1$

We consider in detail the map of Example A, ie $T_{\alpha,\beta}(x) = \beta x + \alpha \mod 1$ (see Sections 2.2 and 4.2). We recall some facts we have already presented. Every time we need the validity of the coding, it is mentioned explicitly. Since all laps of $T_{\alpha,\beta}$ are increasing, $\preceq$ is the lexicographic order; this is true in the whole Section 6.1. The number of symbols is $k = \lceil \beta + \alpha \rceil$. Define

$$\gamma := \beta + \alpha - k + 1 \in (0,1].$$

It is convenient to define this parameter, because α and γ play symmetric roles. The set $S_0 = \{a_0, \ldots, a_k\}$ is given by

$$a_0 = 0, \quad a_j = \frac{j-\alpha}{\beta} \quad \text{for } j = 1, \ldots, k-1 \quad \text{and} \quad a_k = 1. \tag{6.1}$$

The coding map corresponding to $T_{\alpha,\beta}$ is denoted by $\mathtt{i}^{\alpha,\beta}$. The representation maps $\varphi^{\alpha,\beta} : [\alpha, \beta+\alpha] \to [0,1]$ and $\overline{\varphi}^{\alpha,\beta} : [0,k] \to [0,1]$ are given by

$$\varphi^{\alpha,\beta}(x) = \frac{x-\alpha}{\beta} \tag{6.2}$$

and

$$\overline{\varphi}^{\alpha,\beta}(x) = \begin{cases} 0 & \text{if } 0 \le x \le \alpha \\ \frac{x-\alpha}{\beta} & \text{if } \alpha \le x \le \beta+\alpha \\ 1 & \text{if } \beta+\alpha \le x \le 1\,. \end{cases} \tag{6.3}$$

There are two important virtual itineraries ($x \in X$)

$$\underline{u}^{\alpha,\beta} \equiv \underline{u}^0 := \lim_{x\downarrow 0} \mathtt{i}^{\alpha,\beta}(x) \quad \text{and} \quad \underline{v}^{\alpha,\beta} \equiv \underline{v}^{k-1} := \lim_{x\uparrow 1} \mathtt{i}^{\alpha,\beta}(x)\,.$$

All virtual itineraries can be expressed with the help of $\underline{u}^{\alpha,\beta}$ and $\underline{v}^{\alpha,\beta}$

$$\underline{u}^j = j\underline{u}^{\alpha,\beta} \text{ for } j = 1,\ldots,k-1 \quad \text{and} \quad \underline{v}^j = j\underline{v}^{\alpha,\beta} \text{ for } j = 0,\ldots,k-2\,.$$

If the φ-expansion is valid, the virtual itineraries satisfy

$$\underline{u}^{\alpha,\beta} \preceq \sigma^n \underline{u}^{\alpha,\beta} \prec \underline{v}^{\alpha,\beta} \quad \text{and} \quad \underline{u}^{\alpha,\beta} \prec \sigma^n \underline{v}^{\alpha,\beta} \preceq \underline{v}^{\alpha,\beta} \quad \forall n \ge 0\,. \tag{6.4}$$

If the φ-expansion is valid, the shift space obtained by the coding is

$$\Sigma_{\alpha,\beta} := \overline{\mathtt{i}^{\alpha,\beta}(X)} = \{\underline{w} \in \Sigma_k^+ : \underline{u}^{\alpha,\beta} \preceq \sigma^n \underline{x} \preceq \sigma^n \underline{v}^{\alpha,\beta}\ \forall n \ge 0\}\,, \tag{6.5}$$

Formula 2.13 applied to $T_{\alpha,\beta}$ gives

$$\sigma^n \underline{u}^{\alpha,\beta} = \lim_{x\downarrow 0} \mathtt{i}^{\alpha,\beta}(T^n_{\alpha,\beta}(x)) \quad \text{and} \quad \sigma^n \underline{v}^{\alpha,\beta} = \lim_{x\uparrow 0} \mathtt{i}^{\alpha,\beta}(T^n_{\alpha,\beta}(x))\,. \tag{6.6}$$

For commodity, we define the orbits of 0 and 1 under $T_{\alpha,\beta}$ as (the limits are taken with $x \in X$)

$$T^n_{\alpha,\beta}(0) := \lim_{x\downarrow 0} T^n_{\alpha,\beta}(x) \quad \text{and} \quad T^n_{\alpha,\beta}(1) := \lim_{x\uparrow 1} T^n_{\alpha,\beta}(x) \quad \forall n \ge 0\,. \tag{6.7}$$

Notice that $T^n_{\alpha,\beta}(0) < 1$ and $T^n_{\alpha,\beta}(1) > 0$ for all $n \ge 0$, since all laps of $T^n_{\alpha,\beta}$ are increasing. From Theorem 2.12, Formula (6.6) and Theorem 2.15, we deduce that ($x \in X$)

$$\overline{\varphi}^{\alpha,\beta}_\infty(\sigma^n \underline{u}^{\alpha,\beta}) = \lim_{x\downarrow 0} \overline{\varphi}^{\alpha,\beta}_\infty(\mathtt{i}^{\alpha,\beta}(T^n_{\alpha,\beta}(x))) = \lim_{x\downarrow 0} T^n_{\alpha,\beta}(x) \equiv T^n_{\alpha,\beta}(0) \quad \forall n \ge 0\,, \tag{6.8}$$

$$\overline{\varphi}^{\alpha,\beta}_\infty(\sigma^n \underline{v}^{\alpha,\beta}) = \lim_{x\uparrow 1} \overline{\varphi}^{\alpha,\beta}_\infty(\mathtt{i}^{\alpha,\beta}(T^n_{\alpha,\beta}(x))) = \lim_{x\uparrow 1} T^n_{\alpha,\beta}(x) \equiv T^n_{\alpha,\beta}(1) \quad \forall n \ge 0\,. \tag{6.9}$$

In particular, these equations applied to $n = 0$ and $n = 1$ give

$$\overline{\varphi}^{\alpha,\beta}_\infty(\underline{u}^{\alpha,\beta}) = 0,\ \overline{\varphi}^{\alpha,\beta}_\infty(\sigma\underline{u}^{\alpha,\beta}) = \alpha \quad \text{and} \quad \overline{\varphi}^{\alpha,\beta}_\infty(\underline{v}^{\alpha,\beta}) = 1,\ \overline{\varphi}^{\alpha,\beta}_\infty(\sigma\underline{v}^{\alpha,\beta}) = \gamma\,. \tag{6.10}$$

These equations point out that $\underline{u}^{\alpha,\beta}$ and $\underline{v}^{\alpha,\beta}$ are $\overline{\varphi}^{\alpha,\beta}$-expansions of 0 and 1. Notice that $\overline{\varphi}^{\alpha,\beta}_\infty(\underline{w}) = 0$ implies that $\overline{\varphi}^{\alpha,\beta}_\infty(\sigma\underline{w}) \le \alpha$. There are many sequences $\underline{w}$ such that $\overline{\varphi}^{\alpha,\beta}_\infty(\underline{w}) = 0$, but only few of them satisfy $\overline{\varphi}^{\alpha,\beta}_\infty(\sigma\underline{w}) = \alpha$. For example, with $\underline{w} = 0^\infty$, we get $\overline{\varphi}_{\infty}^{\alpha,\beta}(\underline{w}) = \overline{\varphi}_{\infty}^{\alpha,\beta}(\sigma\underline{w}) = 0$. As we will see later, $\underline{u}^{\alpha,\beta}$ is the greatest sequence $\underline{w}$ such that $\overline{\varphi}_{\infty}^{\alpha,\beta}(\underline{w}) = 0$. Similar remarks are true for $\underline{v}^{\alpha,\beta}$.

In view of (6.4) and (6.5), we give the following definition. Let $\underline{u}, \underline{v} \in \Sigma_k^+$ be such that $u_0 = 0$, $v_0 = k-1$ and

$$\underline{u} \preceq \sigma^n \underline{u} \preceq \underline{v} \quad \text{and} \quad \underline{u} \preceq \sigma^n \underline{v} \preceq \underline{v} \quad \forall n \geq 0\,. \tag{6.11}$$

Then we set

$$\Sigma(\underline{u}, \underline{v}) := \{\underline{x} \in \Sigma_k^+ : \underline{u} \preceq \sigma^n \underline{x} \preceq \underline{v} \ \forall n \geq 0\}\,. \tag{6.12}$$

Comparing (6.4) and (6.11), notice that we relax the strict inequalities as in Chapter 4. We work in this manner in order to include borderline cases inherited from $T_{\alpha,\beta}$-maps.

6.1.1 The set of $\overline{\varphi}^{\alpha,\beta}$-expansion of 0 and 1

Equations (6.10) show that $\underline{u}^{\alpha,\beta}$ and $\underline{v}^{\alpha,\beta}$ are $\overline{\varphi}^{\alpha,\beta}$-expansions of 0 and 1. Because of the presence of discontinuities in the map $T_{\alpha,\beta}$, there may be other pairs of sequences $\underline{u}, \underline{v}$ verifying (6.10) and (6.11). One of the keys of our work on the map $T_{\alpha,\beta}$ is a good description of these pairs. This is the contents of Propositions 6.2, 6.3 and 6.4. We also take into consideration the borderline cases $\alpha = 1$ and $\gamma = 0$. When $\alpha = 1$ or $\gamma = 0$ the dynamical system $T_{\alpha,\beta}$ is defined using formula (2.10). The orbits of 0 and 1 are defined as before. For example, if $\alpha = 1$ it is the same dynamical system as $T_{0,\beta}$, but with different symbols for the coding of the orbits. The orbit of 0 is coded by $\underline{u}^{1,\beta} = 1^\infty$. Similarly, if $\gamma = 0$ the orbit of 1 is coded by $\underline{v}^{\alpha,\beta} = (k-2)^\infty$. We always assume that $\alpha \in [0,1]$, $\gamma \in [0,1]$ and $\beta \geq 1$.

Lemma 6.1. *Fix $y \in [0,1]\backslash S_0$. The equation*

$$y = \overline{\varphi}^{\alpha,\beta}(j+t), \quad j \in \mathtt{A}_k,\ t \in [0,1]$$

has a unique solution. If $y < y' \in [0,1]\backslash S_0$, then the solutions of the equations

$$y = \overline{\varphi}^{\alpha,\beta}(j+t) \quad \text{and} \quad y' = \overline{\varphi}^{\alpha,\beta}(j'+t')$$

are such that either $j = j'$ and $T_{\alpha,\beta}(y') - T_{\alpha,\beta}(y) = \beta(y'-y)$ or $j < j'$.

Proof: It is sufficient to use (2.10) and $T_{\alpha,\beta}(y) \notin \{0,1\}$:

$$y \notin S_0 \implies y = \varphi^{\alpha,\beta}(j+t) \iff \mathtt{i}_0^{\alpha,\beta}(y) + T_{\alpha,\beta}(y) = j+t\,.$$

For the second statement, we use $\frac{d}{ds}\varphi^{\alpha,\beta}(s) = \frac{1}{\beta}$. □

Proposition 6.2. *Let $0 \leq \alpha < 1$ and assume that the φ-expansion is valid. The following assertions are equivalent.*

1. *There is a unique solution in Σ_k^+ ($\underline{u} = \underline{u}^{\alpha,\beta}$) of the equations*

$$\overline{\varphi}_{\infty}^{\alpha,\beta}(\underline{u}) = 0 \quad \text{and} \quad \overline{\varphi}_{\infty}^{\alpha,\beta}(\sigma\underline{u}) = \alpha\,. \tag{6.13}$$

2. *The orbit of 0 is not periodic or $x = 0$ is a fixed point of $T_{\alpha,\beta}$.*
3. *$\underline{u}^{\alpha,\beta}$ is not periodic or $\underline{u}^{\alpha,\beta} = 0^\infty$.*

Proposition 6.3. *Let $0 < \gamma \leq 1$ and assume that the φ-expansion is valid. The following assertions are equivalent.*

1. *There is a unique solution in Σ_k^+ ($\underline{v} = \underline{v}^{\alpha,\beta}$) of the equations*
$$\overline{\varphi}_\infty^{\alpha,\beta}(\underline{v}) = 1 \quad \textit{and} \quad \overline{\varphi}_\infty^{\alpha,\beta}(\sigma\underline{v}) = \gamma\,. \tag{6.14}$$
2. *The orbit of 1 is not periodic or $x = 1$ is a fixed point of $T_{\alpha,\beta}$.*
3. *$\underline{v}^{\alpha,\beta}$ is not periodic or $\underline{v}^{\alpha,\beta} = (k-1)^\infty$.*

Proof: We prove Proposition 6.2. Assume 1. The validity of the φ-expansion implies that $\underline{u}^{\alpha,\beta}$ is a solution of (6.13). If $\alpha = 0$, then $\underline{u}^{0,\beta} = 0^\infty$ is the only solution of (6.13) since $\underline{x} \neq 0^\infty$ implies $\overline{\varphi}_\infty^{0,\beta}(\underline{x}) > 0$ and $x = 0$ is a fixed point of $T_{0,\beta}$. Let $0 < \alpha < 1$. Using Lemma 6.1 we deduce that $u_0 = 0$ and
$$\alpha = T_{\alpha,\beta}(0) = \overline{\varphi}_\infty^{\alpha,\beta}(u_1 + \overline{\varphi}_\infty^{\alpha,\beta}(\sigma^2\underline{u}))\,.$$
If $\alpha = a_j$, $j = 1, \ldots, k-1$, then (6.13) has at least two solutions, which are $0j(\sigma^2\underline{u}^{\alpha,\beta})$ with $\overline{\varphi}_\infty^{\alpha,\beta}(\sigma^2\underline{u}^{\alpha,\beta}) = T^2(0) = 0$ (see (6.8)), and $0(j-1)\underline{v}^{\alpha,\beta}$ with $\overline{\varphi}_\infty^{\alpha,\beta}(\underline{v}^{\alpha,\beta}) = 1$. Therefore, by our hypothesis we have $\alpha \notin \{a_1, \ldots, a_{k-1}\}$, $u_1 = u_1^{\alpha,\beta}$ and $\overline{\varphi}_\infty^{\alpha,\beta}(\sigma^2\underline{u}^{\alpha,\beta}) = T^2(0) \in (0,1)$. Iterating this argument we conclude that $1 \implies 2$.
Assume 2. If $x = 0$ is a fixed point, then $\alpha = 0$ and $\underline{u}^{0,\beta} = 0^\infty$. If the orbit of 0 is not periodic, (6.6) and the validity of the φ-expansion imply
$$\sigma^n\underline{u}^{\alpha,\beta} = \lim_{x\downarrow 0} \mathtt{i}^{\alpha,\beta}(T_{\alpha,\beta}^n(x)) \succ \lim_{x\downarrow 0} \mathtt{i}^{\alpha,\beta}(x) = \underline{u}^{\alpha,\beta}\,.$$
Assume 3. From (6.8) and the validity of the φ-expansion we get
$$\overline{\varphi}_\infty^{\alpha,\beta}(\sigma^n\underline{u}^{\alpha,\beta}) = T_{\alpha,\beta}^n(0) > \overline{\varphi}_\infty^{\alpha,\beta}(\underline{u}^{\alpha,\beta}) = 0\,,$$
so that the orbit of 0 is not periodic. The orbit of 0 is not periodic if and only if $T_{\alpha,\beta}^n(0) \notin \{a_1, \ldots, a_{k-1}\}$ for all $n \geq 1$. Using Lemma 6.1, we conclude that (6.13) has a unique solution. □

Propositions 6.2 and 6.3 give necessary and sufficient conditions for the existence and uniqueness of the solution of Equations (6.13) and (6.14). In the following discussion we consider the case when there are several solutions. All results are summarized in Proposition 6.4. We assume the validity of the φ-expansion.

Suppose first that the orbit of 1 is not periodic and that the orbit of 0 is periodic, with minimal period $p := \min\{n \geq 1 : T_{\alpha,\beta}^n(0) = 0\} > 1$ (the case $p = 1 \iff \alpha = 0$ is already treated in Proposition 6.2). Hence $0 < \gamma < 1$ and $0 < \alpha < 1$. Let $\underline{u}$ be a solution of Equations (6.13) and suppose furthermore that $\underline{w}$ is a φ-expansion of 1 such that
$$\forall n : \underline{u} \preceq \sigma^n\underline{u} \preceq \underline{w} \quad \text{with} \quad \overline{\varphi}_\infty^{\alpha,\beta}(\underline{w}) = 1\,,\ \overline{\varphi}_\infty^{\alpha,\beta}(\sigma\underline{w}) = \gamma\,.$$
By Lemma 6.1 we conclude that
$$u_j = u_j^{\alpha,\beta} \quad \text{and} \quad T_{\alpha,\beta}^{j+1}(0) = \overline{\varphi}_\infty^{\alpha,\beta}(\sigma^{j+1}\underline{u})\,, \quad j = 1, \ldots, p-2\,.$$
Since $T_{\alpha,\beta}^p(0) = 0$, we have $T_{\alpha,\beta}^{p-1}(0) \in \{a_1, \ldots, a_{k-1}\}$ and the equation
$$T_{\alpha,\beta}^{p-1}(0) = \overline{\varphi}_\infty^{\alpha,\beta}\big(u_{p-1} + \overline{\varphi}_\infty^{\alpha,\beta}(\sigma^p\underline{u})\big)$$

has two solutions. Either $u_{p-1} = u^{\alpha,\beta}_{p-1}$ and $\overline{\varphi}^{\alpha,\beta}_\infty(\sigma^p\underline{u}) = 0$ or $u_{p-1} = u^{\alpha,\beta}_{p-1} - 1$ and $\overline{\varphi}^{\alpha,\beta}_\infty(\sigma^p\underline{u}) = 1$. Let $\underline{a}$ be the prefix of $\underline{u}^{\alpha,\beta}$ of length p and $\underline{a}'$ the word of length p obtained by changing the last letter of $\underline{a}$ into[1] $u^{\alpha,\beta}_{p-1} - 1$. We have $\underline{a}' \prec \underline{a}$. If $u_{p-1} = u^{\alpha,\beta}_{p-1}$, then we can again determine uniquely the next $p-1$ letters u_i. The condition $\underline{u} \preceq \sigma^n\underline{u}$ for $n = p$ implies that we have $u_{2p-1} = u^{\alpha,\beta}_{p-1}$ so that, by iteration, we get the solution $\underline{u} = \underline{u}^{\alpha,\beta}$ for the equations (6.13). If $u_{p-1} = u^{\alpha,\beta}_{p-1} - 1$, then

$$1 = \overline{\varphi}^{\alpha,\beta}_\infty(\sigma^p\underline{u}) = \overline{\varphi}^{\alpha,\beta}_\infty\big(u_p + \overline{\varphi}^{\alpha,\beta}_\infty(\sigma^{p+1}\underline{u})\big)\,.$$

When $\overline{\varphi}^{\alpha,\beta}_\infty(\sigma^p\underline{u}) = 1$, by our hypothesis on $\underline{u}$ we also have $\overline{\varphi}^{\alpha,\beta}_\infty(\sigma^{p+1}\underline{u}) = \gamma$. By Proposition 6.3 the equations

$$\overline{\varphi}^{\alpha,\beta}_\infty(\sigma^p\underline{u}) = 1 \quad \text{and} \quad \overline{\varphi}^{\alpha,\beta}_\infty(\sigma^{p+1}\underline{u}) = \gamma$$

have a unique solution, since we assume that the orbit of 1 is not periodic. The solution is $\sigma^p\underline{u} = \underline{v}^{\alpha,\beta}$, so that $\underline{u} = \underline{a}'\underline{v}^{\alpha,\beta} \prec \underline{u}^{\alpha,\beta}$ is also a solution of (6.13). In that case there is no other solution for (6.13). The borderline case $\alpha = 1$ corresponds to the periodic orbit of the fixed point 0, $\underline{u}^{1,\beta} = 1^\infty$. Notice that $\overline{\varphi}^{1,\beta}_\infty(\sigma\underline{u}^{1,\beta}) \neq 1$. We can also consider $\overline{\varphi}^{1,\beta}_\infty$-expansions of 0 with $u_0 = 0$ and $\overline{\varphi}^{1,\beta}_\infty(\sigma\underline{u}) = 1$. Our hypothesis on $\underline{u}$ implies that $\overline{\varphi}^{1,\beta}_\infty(\sigma^2\underline{u}) = \gamma$. Hence, $\underline{u} = 0\underline{v}^{1,\beta} = \underline{a}'\underline{v}^{\alpha,\beta} \prec \underline{u}^{\alpha,\beta}$ is a solution of (6.13).

We can treat similarly the case when $\underline{u}^{\alpha,\beta}$ is not periodic, but $\underline{v}^{\alpha,\beta}$ is periodic. When both $\underline{u}^{\alpha,\beta}$ and $\underline{v}^{\alpha,\beta}$ are periodic, we have more solutions, but the discussion is similar. Assume that $\underline{u}^{\alpha,\beta}$ has (minimal) period $p > 1$ and $\underline{v}^{\alpha,\beta}$ has (minimal) period $q > 1$. Define $\underline{a}$, $\underline{a}'$ as before, $\underline{b}$ as the prefix of length q of $\underline{v}^{\alpha,\beta}$, and $\underline{b}'$ as the word of length q obtained by changing the last letter of $\underline{b}$ into $v^{\alpha,\beta}_{q-1} + 1$. When $0 < \alpha < 1$ and $0 < \gamma < 1$, one shows as above that the elements $\underline{u} \neq \underline{u}^{\alpha,\beta}$ and $\underline{v} \neq \underline{v}^{\alpha,\beta}$ which are $\overline{\varphi}^{\alpha,\beta}$-expansions of 0 and 1 are of the form

$$\underline{u} = \underline{a}'\underline{b}^{n_1}\underline{b}'\underline{a}^{n_2}\cdots\,,\ n_i \geq 0 \quad \text{and} \quad \underline{v} = \underline{b}'\underline{a}^{m_1}\underline{a}'\underline{b}^{m_2}\cdots\,,\ m_i \geq 0\,.$$

The integers n_i and m_i must be such that (6.11) is verified. The largest solution of (6.13) is $\underline{u}^{\alpha,\beta}$ and the smallest one is $\underline{a}'\underline{v}^{\alpha,\beta}$.

For all $\beta \geq 1$ and $\alpha \in [0,1]$ such that the φ-expansion is valid, define (recall that $k = \lceil \beta + \alpha \rceil$)

$$\mathcal{D}_{\alpha,\beta} := \{(\underline{u},\underline{v}) \in \Sigma^+_k \times \Sigma^+_k : \text{the pair } (\underline{u},\underline{v}) \text{ satisfies (6.11), (6.13) and (6.14)}\}\,.$$

In particular, $(\underline{u}^{\alpha,\beta},\underline{v}^{\alpha,\beta}) \in \mathcal{D}_{\alpha,\beta}$ for all α,β such that the φ-expansion is valid. The next proposition collects all results from Proposition 6.2 to here.

Proposition 6.4. *Let $\beta \geq 1$ and $\alpha \in [0,1]$ be such that the φ-expansion is valid.*

1. *If $\alpha = 0$, then $\underline{u}^{\alpha,\beta} = 0^\infty$. Moreover*
   - (a) *if $\gamma \in (0,1)$ is such that $\underline{v}^{\alpha,\beta}$ is not periodic, then $\mathcal{D}_{\alpha,\beta} = \{(\underline{u}^{\alpha,\beta},\underline{v}^{\alpha,\beta})\}$.*
   - (b) *if $\gamma \in (0,1)$ is such that $\underline{v}^{\alpha,\beta}$ has (minimal) period q, then defining $\underline{b}' = \underline{v}^{\alpha,\beta}_{[0,q-2)}(v^{\alpha,\beta}_{q-1} + 1)$, we have $\mathcal{D}_{\alpha,\beta} = \{(\underline{u}^{\alpha,\beta},\underline{v}^{\alpha,\beta}), (\underline{u}^{\alpha,\beta},\underline{b}'\underline{u}^{\alpha,\beta})\}$.*
   - (c) *if $\gamma = 0$, then $\mathcal{D}_{\alpha,\beta} = \{(\underline{u}^{\alpha,\beta},(k-1)\underline{u}^{\alpha,\beta})\}$.*

[1] $u^{\alpha,\beta}_{p-1} \geq 1$. $u^{\alpha,\beta}_{p-1} = 0$ if and only if $p = 1$ and $\alpha = 0$.

(d) *if* $\gamma = 1$*, then* $\mathcal{D}_{\alpha,\beta} = \{(\underline{u}^{\alpha,\beta}, (k-1)^\infty)\}$*.*

2. *If* $\gamma = 1$*, then* $\underline{v}^{\alpha,\beta} = (k-1)^\infty$*. Moreover*

   (a) *if* $\alpha \in (0,1)$ *is such that* $\underline{u}^{\alpha,\beta}$ *is not periodic, then* $\mathcal{D}_{\alpha,\beta} = \{(\underline{u}^{\alpha,\beta}, \underline{v}^{\alpha,\beta})\}$*.*

   (b) *if* $\alpha \in (0,1)$ *is such that* $\underline{u}^{\alpha,\beta}$ *has (minimal) period* p*, then defining* $\underline{a}' = \underline{u}^{\alpha,\beta}_{[0,p-2)}(u^{\alpha,\beta}_{p-1} - 1)$*, we have* $\mathcal{D}_{\alpha,\beta} = \{(\underline{u}^{\alpha,\beta}, \underline{v}^{\alpha,\beta}, (\underline{a}'\underline{v}^{\alpha,\beta}, \underline{v}^{\alpha,\beta})\}$*.*

   (c) *if* $\alpha = 1$*, then* $\mathcal{D}_{\alpha,\beta} = \{(0\underline{v}^{\alpha,\beta}, \underline{v}^{\alpha,\beta})\}$*.*

3. *If* $\alpha \in (0,1)$ *and* $\gamma \in (0,1)$ *are such that*

   (a) $\underline{u}^{\alpha,\beta}$ *and* $\underline{v}^{\alpha,\beta}$ *are both non periodic, then* $\mathcal{D}_{\alpha,\beta} = \{(\underline{u}^{\alpha,\beta}, \underline{v}^{\alpha,\beta})\}$*.*

   (b) $\underline{u}^{\alpha,\beta}$ *is not periodic and* $\underline{v}^{\alpha,\beta}$ *has (minimal) period* q*, then defining* $\underline{b}'$ *as before, we have* $\mathcal{D}_{\alpha,\beta} = \{(\underline{u}^{\alpha,\beta}, \underline{v}^{\alpha,\beta}), (\underline{u}^{\alpha,\beta}, \underline{b}'\underline{u}^{\alpha,\beta})\}$*.*

   (c) $\underline{u}^{\alpha,\beta}$ *has (minimal) period* p *and* $\underline{v}^{\alpha,\beta}$ *is not periodic, then defining* $\underline{a}'$ *as before, we have* $\mathcal{D}_{\alpha,\beta} = \{(\underline{u}^{\alpha,\beta}, \underline{v}^{\alpha,\beta}), (\underline{a}'\underline{v}^{\alpha,\beta}, \underline{v}^{\alpha,\beta})\}$*.*

   (d) $\underline{u}^{\alpha,\beta}$ *has (minimal) period* p *and* $\underline{v}^{\alpha,\beta}$ *has (minimal) period* q*, then (*$\underline{a} = \underline{u}^{\alpha,\beta}_{[0,p)}$*,* $\underline{b} = \underline{v}^{\alpha,\beta}_{[0,q)}$*,* $\underline{a}'$ *and* $\underline{b}'$ *are defined as before)*

$$\begin{aligned}\underline{u}^{\alpha,\beta} &= \max\{\underline{u} : \exists \underline{v} \ s.t. \ (\underline{u}, \underline{v}) \in \mathcal{D}_{\alpha,\beta}\} = \underline{a}^\infty\,,\\ \underline{\tilde{u}}^{\alpha,\beta} &:= \min\{\underline{u} : \exists \underline{v} \ s.t. \ (\underline{u}, \underline{v}) \in \mathcal{D}_{\alpha,\beta}\} = \underline{a}'(\underline{b})^\infty\,,\end{aligned}$$

and

$$\begin{aligned}\underline{v}^{\alpha,\beta} &= \min\{\underline{v} : \exists \underline{u} \ s.t. \ (\underline{u}, \underline{v}) \in \mathcal{D}_{\alpha,\beta}\} = \underline{b}^\infty\,,\\ \underline{\tilde{v}}^{\alpha,\beta} &:= \max\{\underline{v} : \exists \underline{u} \ s.t. \ (\underline{u}, \underline{v}) \in \mathcal{D}_{\alpha,\beta}\} = \underline{b}'(\underline{a})^\infty\,.\end{aligned}$$

The cases 1a, 1b and 1d were considered by Parry in [P1]. Notice that the cases 2a, 2b and 2c are symmetric formulations of cases 1a, 1b and 1c. Similarly the cases 3b and 3c are symmetric. Notice also that, except for the case 3d, $(\underline{u}^{\alpha,\beta}, \underline{v}^{\alpha,\beta})$ is the unique pair $(\underline{u}, \underline{v}) \in \mathcal{D}_{\alpha,\beta}$ satisfying the strict inequalities (6.4).

6.1.2 An algorithm to find (α, β) with respect to the virtual itineraries

Given a pair $(\underline{u}, \underline{v})$ satisfying $u_0 = 0$, $v_0 = k-1$ and (6.11), we look for (α, β) such that the equations (6.13) and (6.14) are satisfied. In particular, we are interested in the existence and the uniqueness of the solution of this problem. To this end, we describe an algorithm, which assigns to a pair of strings $(\underline{u}, \underline{v})$, satisfying weaker hypotheses than (6.11), a pair of real numbers $(\overline{\alpha}, \overline{\beta}) \in [0,1] \times [1, \infty)$. At the end of the algorithm, there is a condition to check. If this condition is true, then the pair $(\overline{\alpha}, \overline{\beta})$ is the solution we are looking for. The uniqueness needs a result we prove later, but it is stated in Section 6.1.2 for consistency.

We tacitly assume that for all the pairs (α, β) one has $\alpha \in [0,1]$, $\beta \geq 1$ and $\lceil \beta + \alpha \rceil \in [k-1, k]$. In particular, $\beta \geq k-2$ and the map $\overline{\varphi}^{\alpha,\beta}$ verifies

$$0 < \overline{\varphi}^{\alpha,\beta}(t) < 1 \quad \forall t \in (1, k-1)\,.$$

Recall that $\gamma = \alpha + \beta - k + 1$ and notice that our assumptions imply that $0 \leq \gamma \leq 1$.

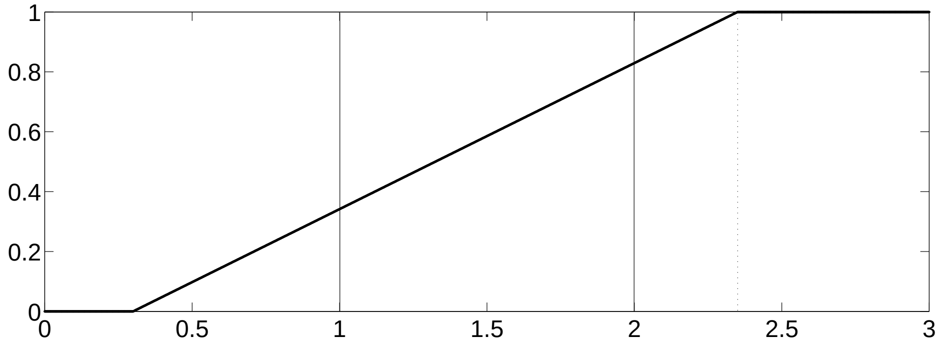

Figure 6.1: The graph of $\overline{\varphi}^{\alpha,\beta}$ with $k = 3$, $\alpha = 0.3$, $\beta = 2.05$ and $\alpha + \beta = 2 + \gamma = 2.35$.

Definition 6.5. *The map $\overline{\varphi}^{\alpha,\beta}$ **dominates** the map $\overline{\varphi}^{\alpha',\beta'}$ if $\overline{\varphi}^{\alpha,\beta}(t) \geq \overline{\varphi}^{\alpha',\beta'}(t)$ for all $t \in [0,k]$ and there exists $s \in [0,k]$ such that $\overline{\varphi}^{\alpha,\beta}(s) > \overline{\varphi}^{\alpha',\beta'}(s)$.*

Lemma 6.6. *If $\overline{\varphi}^{\alpha,\beta}$ dominates $\overline{\varphi}^{\alpha',\beta'}$, then, for all $\underline{x} \in \Sigma_k^+$, $\overline{\varphi}_\infty^{\alpha,\beta}(\underline{x}) \geq \overline{\varphi}_\infty^{\alpha',\beta'}(\underline{x})$. If*

$$0 < \overline{\varphi}_\infty^{\alpha,\beta}(\underline{x}) < 1 \quad or \quad 0 < \overline{\varphi}_\infty^{\alpha',\beta'}(\underline{x}) < 1\,,$$

then the inequality is strict.

Proof: If $\overline{\varphi}^{\alpha,\beta}$ dominates $\overline{\varphi}^{\alpha',\beta'}$, then, by our implicit assumptions on (α,β), we get by inspection of the graphs that

$$\forall t \geq t' \colon\ \overline{\varphi}^{\alpha,\beta}(t) > \overline{\varphi}^{\alpha',\beta'}(t') \quad \text{if} \quad t,t' \in (\alpha, \alpha'+\beta') = (\alpha,\alpha+\beta) \cup (\alpha', \alpha'+\beta')\,,$$

otherwise $\overline{\varphi}^{\alpha,\beta}(t) \geq \overline{\varphi}^{\alpha',\beta'}(t')$. Therefore, for all $n \geq 1$,

$$\overline{\varphi}_n^{\alpha,\beta}(\underline{x}) \geq \overline{\varphi}_n^{\alpha',\beta'}(\underline{x})\,.$$

Suppose that $0 < \overline{\varphi}_\infty^{\alpha,\beta}(\underline{x}) < 1$. Then $x_0 + \overline{\varphi}_\infty^{\alpha,\beta}(\sigma\underline{x}) \in (\alpha,\alpha+\beta)$ and

$$\overline{\varphi}_\infty^{\alpha,\beta}(\underline{x}) = \overline{\varphi}^{\alpha,\beta}(x_0 + \overline{\varphi}_\infty^{\alpha,\beta}(\sigma\underline{x})) > \overline{\varphi}^{\alpha',\beta'}(x_0 + \overline{\varphi}_\infty^{\alpha',\beta'}(\sigma\underline{x})) = \overline{\varphi}_\infty^{\alpha',\beta'}(\underline{x})\,.$$

Similar proof for $0 < \overline{\varphi}_\infty^{\alpha',\beta'}(\underline{x}) < 1$. $\square$

Lemma 6.7. *Let $\alpha = \alpha' \in [0,1]$ and $1 \leq \beta < \beta'$. Then, for $\underline{x} \in \Sigma_k^+$,*

$$0 \leq \overline{\varphi}_\infty^{\alpha,\beta}(\underline{x}) - \overline{\varphi}_\infty^{\alpha,\beta'}(\underline{x}) \leq \frac{|\beta-\beta'|}{\beta'-1}\,.$$

Let $\gamma = \gamma' \in [0,1]$, $0 \leq \alpha' < \alpha \leq 1$ and $\beta' > 1$. Then, for $\underline{x} \in \Sigma_k^+$,

$$0 \leq \overline{\varphi}_\infty^{\alpha',\beta'}(\underline{x}) - \overline{\varphi}_\infty^{\alpha,\beta}(\underline{x}) \leq \frac{|\alpha-\alpha'|}{\beta'-1}\,.$$

The map $\beta \mapsto \overline{\varphi}_\infty^{\alpha,\beta}(\underline{x})$ is continuous at $\beta = 1$.

Proof: Let $\alpha = \alpha' \in [0,1]$ and $1 \leq \beta < \beta'$. For $t, t' \in [0,k]$,

$$|\overline{\varphi}^{\alpha,\beta'}(t') - \overline{\varphi}^{\alpha,\beta}(t)| \leq |\overline{\varphi}^{\alpha,\beta'}(t') - \overline{\varphi}^{\alpha,\beta'}(t)| + |\overline{\varphi}^{\alpha,\beta'}(t) - \overline{\varphi}^{\alpha,\beta}(t)| \leq \frac{|t-t'|}{\beta'} + \frac{|\beta-\beta'|}{\beta'}\,,$$

because the maximum of $|\overline{\varphi}^{\alpha,\beta'}(t) - \overline{\varphi}^{\alpha,\beta}(t)|$ is taken at $\alpha+\beta$. By induction

$$|\overline{\varphi}_n^{\alpha,\beta'}(x_0,\ldots,x_{n-1}) - \overline{\varphi}_n^{\alpha,\beta}(x_0,\ldots,x_{n-1})| \leq |\beta-\beta'| \sum_{j=1}^{n} (\beta')^{-j}\,.$$

Since $\beta' > 1$ the sum is convergent. This proves the first statement. The second statement is proved similarly using

$$|\overline{\varphi}^{\alpha',\beta'}(t') - \overline{\varphi}^{\alpha,\beta}(t)| \leq |\overline{\varphi}^{\alpha',\beta'}(t') - \overline{\varphi}^{\alpha',\beta'}(t)| + |\overline{\varphi}^{\alpha',\beta'}(t) - \overline{\varphi}^{\alpha,\beta}(t)| \leq \frac{|t-t'|}{\beta'} + \frac{|\alpha-\alpha'|}{\beta'}$$

which is valid for $\gamma = \gamma' \in [0,1]$ and $0 \leq \alpha' < \alpha \leq 1$. We prove the last statement. Given $\varepsilon > 0$ there exists n^*

$$\overline{\varphi}_{n^*}^{\alpha,1}(\underline{x}) \geq \overline{\varphi}_{\infty}^{\alpha,1}(\underline{x}) - \varepsilon\,.$$

Since $\beta \mapsto \overline{\varphi}_{n^*}^{\alpha,\beta}(\underline{x})$ is continuous, there exists β' so that for $1 \leq \beta \leq \beta'$,

$$\overline{\varphi}_{n}^{\alpha,\beta}(\underline{x}) \geq \overline{\varphi}_{n^*}^{\alpha,\beta'}(\underline{x}) \geq \overline{\varphi}_{n^*}^{\alpha,1}(\underline{x}) - \varepsilon \quad \forall n \geq n^*\,.$$

Hence

$$\overline{\varphi}_{\infty}^{\alpha,1}(\underline{x}) - 2\varepsilon \leq \overline{\varphi}_{\infty}^{\alpha,\beta}(\underline{x}) \leq \overline{\varphi}_{\infty}^{\alpha,1}(\underline{x})\,.$$

□

Corollary 6.8. *Given* $\underline{x}$ *and* $0 \leq \alpha^* \leq 1$, *let*

$$g_{\alpha^*}(\gamma) := \overline{\varphi}_{\infty}^{\alpha^*,\beta(\gamma)}(\underline{x}) \quad \textit{with} \quad \beta(\gamma) := \gamma - \alpha^* + k - 1\,.$$

For $k \geq 3$ *the map* g_{α^*} *is continuous and non-increasing on* $[0,1]$. *If* $0 < g_{\alpha^*}(\gamma_0) < 1$, *then the map is strictly decreasing in a neighborhood of* γ_0. *If* $k = 2$ *then the same statements hold on* $[\alpha^*, 1]$.

Corollary 6.9. *Given* $\underline{x}$ *and* $0 < \gamma^* \leq 1$, *let*

$$h_{\gamma^*}(\alpha) := \overline{\varphi}_{\infty}^{\alpha,\beta(\alpha)}(\underline{x}) \quad \textit{with} \quad \beta(\alpha) := \gamma^* - \alpha + k - 1\,.$$

For $k \geq 3$ *the map* h_{γ^*} *is continuous and non-increasing on* $[0,1]$. *If* $0 < h_{\gamma^*}(\alpha_0) < 1$, *then the map is strictly decreasing in a neighborhood of* α_0. *If* $k = 2$ *then the same statements hold on* $[0, \gamma^*)$.

Proposition 6.10. *Let* $k \geq 2$, $\underline{u}, \underline{v} \in \Sigma_k^+$ *verifying* $u_0 = 0$ *and* $v_0 = k-1$ *and*

$$\sigma\underline{u} \preceq \underline{v} \quad \textit{and} \quad \underline{u} \preceq \sigma\underline{v}\,.$$

If $k = 2$ *we also assume that* $\sigma\underline{u} \preceq \sigma\underline{v}$. *Then there exist* $\overline{\alpha} \in [0,1]$ *and* $\overline{\beta} \in [1,\infty)$ *so that* $\overline{\gamma} \in [0,1]$. *If* $\overline{\beta} > 1$, *then*

$$\overline{\varphi}_{\infty}^{\overline{\alpha},\overline{\beta}}(\sigma\underline{u}) = \overline{\alpha} \quad \textit{and} \quad \overline{\varphi}_{\infty}^{\overline{\alpha},\overline{\beta}}(\sigma\underline{v}) = \overline{\gamma}\,.$$

Proof: We consider separately the cases $\sigma\underline{v} = 0^\infty$ and $\sigma\underline{u} = (k-1)^\infty$ (i.e. $u_j = k-1$ for all $j \geq 1$). If $\sigma\underline{v} = 0^\infty$, then $\underline{u} = 0^\infty$ and $\underline{v} = (k-1)0^\infty$; we set $\overline{\alpha} := 0$ and $\overline{\beta} := k-1$ ($\overline{\gamma} = 0$). If $\sigma\underline{u} = (k-1)^\infty$, then $\underline{v} = (k-1)^\infty$ and $\underline{u} = 0(k-1)^\infty$; we set $\overline{\alpha} := 1$ and $\overline{\beta} := k$.

From now on we assume that $0^\infty \prec \sigma\underline{v}$ and $\sigma\underline{u} \prec (k-1)^\infty$. Set $\alpha_0 := 0$ and $\beta_0 := k$. We consider in detail the case $k = 2$, so that we also assume that $\sigma\underline{u} \preceq \sigma\underline{v}$.

Step 1. Set $\alpha_1 := \alpha_0$ and solve the equation

$$\overline{\varphi}_\infty^{\alpha_1,\beta}(\sigma\underline{v}) = \beta + \alpha_1 - k + 1\,.$$

There exists a unique solution, β_1, such that $k-1 < \beta_1 \leq k$. Indeed, the map

$$G_{\alpha_1}(\gamma) := g_{\alpha_1}(\gamma) - \gamma \quad \text{with} \quad g_{\alpha_1}(\gamma) := \overline{\varphi}_\infty^{\alpha_1,\beta(\gamma)}(\sigma\underline{v}) \text{ and } \beta(\gamma) := \gamma - \alpha_1 + k - 1$$

is continuous and strictly decreasing on $[\alpha_1, 1]$ (see Corollary 6.8). If $\sigma\underline{v} = (k-1)^\infty$, then $G_{\alpha_1}(1) = 0$ and we set $\beta_1 := k$ and we have $\gamma_1 = 1$. If $\sigma\underline{v} \neq (k-1)^\infty$, then there exists a smallest $j \geq 1$ so that $v_j \leq (k-2)$. Therefore $\overline{\varphi}_\infty^{\alpha_1,k}(\sigma^j\underline{v}) < 1$ and

$$\overline{\varphi}_\infty^{\alpha_1,k}(\sigma\underline{v}) = \overline{\varphi}_{j-1}^{\alpha_1,k}\big(v_1,\ldots,v_{j-1} + \overline{\varphi}_\infty^{\alpha_1,k}(\sigma^j\underline{v})\big) < 1\,,$$

so that $G_{\alpha_1}(1) < 0$. On the other hand, since $\sigma\underline{v} \neq 0^\infty$, $\overline{\varphi}_\infty^{\alpha_1,k-1}(\sigma\underline{v}) > 0$, so that $G_{\alpha_1}(0) > 0$. There exists a unique $\gamma_1 \in (0,1)$ with $G_{\alpha_1}(\gamma_1) = 0$. Define $\beta_1 := \beta(\gamma_1) = \gamma_1 - \alpha_1 + k - 1$.

Step 2. Solve in $[0, \gamma_1)$ the equation

$$\overline{\varphi}_\infty^{\alpha,\beta(\alpha)}(\sigma\underline{u}) = \alpha \quad \text{with} \quad \beta(\alpha) := \gamma_1 - \alpha + k - 1 = \beta_1 + \alpha_1 - \alpha\,.$$

If $\sigma\underline{u} = 0$, then set $\overline{\alpha} := 0$ and $\overline{\beta} := \beta_1$. Let $\sigma\underline{u} \neq 0$. There exists a smallest $j \geq 1$ such that $u_j \geq 1$. This implies that $\overline{\varphi}_\infty^{\alpha_1,\beta_1}(\sigma^j\underline{u}) > 0$ and consequently

$$\overline{\varphi}_\infty^{\alpha_1,\beta(\alpha_1)}(\sigma\underline{u}) = \overline{\varphi}_{j-1}^{\alpha_1,\beta_1}\big(u_1,\ldots,u_{j-1} + \overline{\varphi}_\infty^{\alpha_1,\beta_1}(\sigma^j\underline{u})\big) > 0\,.$$

Since $\sigma\underline{u} \preceq \sigma\underline{v}$,

$$0 < \overline{\varphi}_\infty^{\alpha_1,\beta_1}(\sigma\underline{u}) \leq \overline{\varphi}_\infty^{\alpha_1,\beta_1}(\sigma\underline{v}) = \gamma_1\,.$$

We have $\gamma_1 = 1$ only in the case $\sigma\underline{v} = (k-1)^\infty$; in that case we also have $\overline{\varphi}_\infty^{\alpha_1,\beta_1}(\sigma\underline{u}) < 1$. By Corollary 6.9, for any $\alpha > \alpha_1$ we have $\overline{\varphi}_\infty^{\alpha_1,\beta_1}(\sigma\underline{u}) > \overline{\varphi}_\infty^{\alpha,\beta(\alpha)}(\sigma\underline{u})$. Therefore, the map

$$H_{\gamma_1}(\alpha) := h_{\gamma_1}(\alpha) - \alpha \quad \text{with} \quad h_{\gamma_1}(\alpha) := \overline{\varphi}_\infty^{\alpha,\beta(\alpha)}(\sigma\underline{u})$$

is continuous and strictly decreasing on $[0, \gamma_1)$, $H_{\gamma_1}(\alpha_1) > 0$ and $\lim_{\alpha\uparrow\gamma_1} H_{\gamma_1}(\alpha) < 0$. There exists a unique $\alpha_2 \in (\alpha_1, \gamma_1)$ such that $H_{\gamma_1}(\alpha_2) = 0$. Set $\beta_2 := \gamma_1 - \alpha_2 + k - 1 = \alpha_1 + \beta_1 - \alpha_2$ and $\gamma_2 := \alpha_2 + \beta_2 - k + 1 = \gamma_1$. Since $\alpha_2 \in [0, \gamma_1)$, we have $\beta_2 > 1$. Hence

$$\alpha_1 < \alpha_2 < \gamma_1 \quad \text{and} \quad 1 < \beta_2 < \beta_1 \quad \text{and} \quad \gamma_2 = \gamma_1\,. \tag{6.15}$$

If $\sigma\underline{v} = (k-1)^\infty$, $\gamma_2 = 1$ and we set $\overline{\alpha} := \alpha_2$ and $\overline{\beta} := \beta_2$.

Step 3. From now on $\sigma\underline{u} \neq 0^\infty$ and $\sigma\underline{v} \neq (k-1)^\infty$. Set $\alpha_3 := \alpha_2$ and solve in $[\alpha_3, 1]$ the equation

$$\overline{\varphi}_\infty^{\alpha_3,\beta(\gamma)}(\sigma\underline{v}) = \gamma \quad \text{with} \quad \beta(\gamma) := \gamma - \alpha_3 + k - 1\,.$$

By Lemma 6.6 ($k = 2$),

$$\overline{\varphi}_\infty^{\alpha_3,\beta(\alpha_3)}(\sigma\underline{v}) = \overline{\varphi}_\infty^{\alpha_2,1}(\sigma\underline{v}) \geq \overline{\varphi}_\infty^{\alpha_2,1}(\sigma\underline{u}) > \overline{\varphi}_\infty^{\alpha_2,\beta_2}(\sigma\underline{u}) = \alpha_2\,,$$

since $0 < \alpha_2 < 1$. On the other hand by Corollary 6.9,

$$\overline{\varphi}_{\infty}^{\alpha_3,\beta(\gamma_1)}(\sigma\underline{v}) = \overline{\varphi}_{\infty}^{\alpha_3,1+\gamma_1-\alpha_3}(\sigma\underline{v}) < \overline{\varphi}_{\infty}^{\alpha_1,1+\gamma_1-\alpha_1}(\sigma\underline{v}) = \overline{\varphi}_{\infty}^{\alpha_1,\beta_1}(\sigma\underline{v}) = \gamma_1\,, \tag{6.16}$$

since $0 < \gamma_1 < 1$. Therefore, the map G_{α_3} is continuous and strictly decreasing on $[\alpha_3, 1]$, $G_{\alpha_3}(\alpha_3) > 0$ and $G_{\alpha_3}(\gamma_1) < 0$. There exists a unique $\gamma_3 \in (\alpha_3, \gamma_1)$ such that $G_{\alpha_3}(\gamma_3) = 0$. Set $\beta_3 := \gamma_3 - \alpha_3 + k - 1$, so that $\beta_3 < \gamma_1 - \alpha_2 + k - 1 = \beta_2$. Hence

$$\alpha_3 = \alpha_2 \quad \text{and} \quad 1 < \beta_3 < \beta_2 \quad \text{and} \quad 0 < \gamma_3 < \gamma_2 < 1\,. \tag{6.17}$$

Step 4. Solve in $[0, \gamma_3)$ the equation

$$\overline{\varphi}_{\infty}^{\alpha,\beta(\alpha)}(\sigma\underline{u}) = \alpha \quad \text{with} \quad \beta(\alpha) := \gamma_3 - \alpha + k - 1 = \beta_3 + \alpha_3 - \alpha\,.$$

By Lemma 6.6

$$\overline{\varphi}_{\infty}^{\alpha_3,\beta(\alpha_3)}(\sigma\underline{u}) = \overline{\varphi}_{\infty}^{\alpha_3,\beta_3}(\sigma\underline{u}) > \overline{\varphi}_{\infty}^{\alpha_3,\beta_2}(\sigma\underline{u}) = \overline{\varphi}_{\infty}^{\alpha_2,\beta_2}(\sigma\underline{u}) = \alpha_2\,, \tag{6.18}$$

since $0 < \alpha_2 < 1$. On the other hand,

$$0 < \overline{\varphi}_{\infty}^{\alpha_3,\beta(\alpha_3)}(\sigma\underline{u}) = \overline{\varphi}_{\infty}^{\alpha_3,\beta_3}(\sigma\underline{u}) \leq \overline{\varphi}_{\infty}^{\alpha_3,\beta_3}(\sigma\underline{v}) = \gamma_3 < 1\,.$$

By Corollary 6.9

$$\overline{\varphi}_{\infty}^{\alpha,\beta(\alpha)}(\sigma\underline{u}) < \overline{\varphi}_{\infty}^{\alpha_3,\beta(\alpha_3)}(\sigma\underline{u}) \quad \forall \alpha \in (\alpha_3, \gamma_3)\,.$$

Therefore, the map

$$H_{\gamma_3}(\alpha) := h_{\gamma_3}(\alpha) - \alpha \quad \text{with} \quad h_{\gamma_3}(\alpha) := \overline{\varphi}_{\infty}^{\alpha,\beta(\alpha)}(\sigma\underline{u})$$

is continuous and strictly decreasing on $[\alpha_3, \gamma_3)$, $H_{\gamma_3}(\alpha_3) > 0$ and $\lim_{\alpha\uparrow\gamma_3} H_{\gamma_3}(\alpha) < 0$. There exists a unique $\alpha_4 \in (\alpha_3, \gamma_3)$. Set $\beta_4 := \gamma_3 - \alpha_4 + k - 1 = \alpha_3 + \beta_3 - \alpha_4$ and $\gamma_4 := \alpha_4 + \beta_4 - k + 1 = \gamma_3$. Hence

$$\alpha_3 < \alpha_4 < \gamma_3 \quad \text{and} \quad 1 < \beta_4 < \beta_3 \quad \text{and} \quad \gamma_4 = \gamma_3\,. \tag{6.19}$$

Repeating steps 3 and 4 we get two monotone sequences $\{\alpha_n\}$ and $\{\beta_n\}$. We set $\overline{\alpha} := \lim_{n\to\infty} \alpha_n$ and $\overline{\beta} := \lim_{n\to\infty} \beta_n$.

We consider briefly the changes which occur when $k \geq 3$. Step 1 remains the same. In step 2 we solve the equation $H_{\gamma_1}(\alpha) = 0$ on $[0, 1)$ instead of $[0, \gamma_1)$. The proof that $H_{\gamma_1}(\alpha_1) > 0$ remains the same. We prove that $\lim_{\alpha\uparrow 1} H_{\gamma_1}(\alpha) < 0$. Corollary 6.9 implies that

$$\gamma_1 = \overline{\varphi}_{\infty}^{\alpha_1,\beta_1}(\sigma\underline{v}) = \overline{\varphi}_{\infty}^{\alpha_1,\beta(\alpha_1)}(\sigma\underline{v}) > \overline{\varphi}_{\infty}^{\alpha,\beta(\alpha)}(\sigma\underline{v}) \quad \forall \alpha > \alpha_1\,.$$

Since $\sigma\underline{u} \preceq \underline{v}$ and $\beta(\alpha_1) = \beta_1$,

$$\overline{\varphi}_{\infty}^{\alpha,\beta(\alpha)}(\sigma\underline{u}) \leq \overline{\varphi}^{\alpha,\beta(\alpha)}\big(v_0 + \overline{\varphi}_{\infty}^{\alpha,\beta(\alpha)}(\sigma\underline{v})\big) \leq \overline{\varphi}^{\alpha_1,\beta(\alpha_1)}\big(v_0 + \overline{\varphi}_{\infty}^{\alpha,\beta(\alpha)}(\sigma\underline{v})\big) < 1\,.$$

Instead of (6.15) we have

$$\alpha_1 < \alpha_2 < 1 \quad \text{and} \quad 1 < \beta_2 < \beta_1 \quad \text{and} \quad \gamma_2 = \gamma_1\,.$$

Estimate (6.16) is still valid in step 3 with $k \geq 3$. Hence $G_{\alpha_3}(\gamma_1) < 0$. We solve the equation $G_{\alpha_3}(\gamma) = 0$ on $[0, \gamma_1]$. We have

$$\overline{\varphi}_{\infty}^{\alpha_3,\beta(\gamma_1)}(\sigma\underline{u}) = \overline{\varphi}_{\infty}^{\alpha_2,\beta_2}(\sigma\underline{u}) = \alpha_2\,.$$

By Corollary 6.8 we get

$$\overline{\varphi}_\infty^{\alpha_3,\beta(\gamma)}(\sigma\underline{u}) > \overline{\varphi}_\infty^{\alpha_2,\beta_2}(\sigma\underline{u}) = \alpha_2 \quad \forall\gamma < \gamma_1\,.$$

Since $\underline{u} \preceq \sigma\underline{v}$,

$$\overline{\varphi}_\infty^{\alpha_3,\beta(0)}(\sigma\underline{v}) \geq \overline{\varphi}^{\alpha_3,\beta(0)}\big(u_0 + \overline{\varphi}_\infty^{\alpha_3,\beta(0)}(\sigma\underline{u})\big) \geq \overline{\varphi}^{\alpha_2,\beta(\gamma_1)}\big(u_0 + \overline{\varphi}_\infty^{\alpha_3,\beta(0)}(\sigma\underline{u})\big) > 0\,.$$

Estimate (6.18) is still valid in step 4 so that $H_{\gamma_3}(\alpha_3) > 0$. Corollary 6.9 implies that

$$\gamma_3 = \overline{\varphi}_\infty^{\alpha_3,\beta_3}(\sigma\underline{v}) = \overline{\varphi}_\infty^{\alpha_3,\beta(\alpha_3)}(\sigma\underline{v}) > \overline{\varphi}_\infty^{\alpha,\beta(\alpha)}(\sigma\underline{v}) \quad \forall\alpha > \alpha_3\,.$$

Therefore

$$\overline{\varphi}_\infty^{\alpha_3,\beta(\alpha_3)}(\sigma\underline{u}) \leq \overline{\varphi}^{\alpha,\beta(\alpha)}\big(v_0 + \overline{\varphi}_\infty^{\alpha,\beta(\alpha)}(\sigma\underline{v})\big) \leq \overline{\varphi}^{\alpha_3,\beta(\alpha_3)}\big(v_0 + \overline{\varphi}_\infty^{\alpha,\beta(\alpha)}(\sigma\underline{v})\big) < 1\,.$$

Instead of (6.19) we have

$$\alpha_3 < \alpha_4 < 1 \quad \text{and} \quad 1 < \beta_4 < \beta_3 \quad \text{and} \quad \gamma_4 = \gamma_3\,.$$

Assume that $\overline{\beta} > 1$. Then $1 < \overline{\beta} \leq \beta_n$ for all n. We have

$$\overline{\varphi}_\infty^{\alpha_n,\beta_n}(\sigma\underline{v}) = \gamma_n\,, \quad n \text{ odd}$$

and

$$\overline{\varphi}_\infty^{\alpha_n,\beta_n}(\sigma\underline{u}) = \alpha_n\,, \quad n \text{ even}\,.$$

Let $\overline{\gamma} = \overline{\alpha} + \overline{\beta} - k + 1$. For n odd, let $\beta_n^* := \overline{\gamma} - \alpha_n + k - 1$; using Lemma 6.7 we get

$$\begin{aligned}|\overline{\varphi}_\infty^{\overline{\alpha},\overline{\beta}}(\sigma\underline{v}) - \overline{\gamma}| &\leq |\overline{\varphi}_\infty^{\overline{\alpha},\overline{\beta}}(\sigma\underline{v}) - \overline{\varphi}_\infty^{\alpha_n,\beta_n^*}(\sigma\underline{v})| + |\overline{\varphi}_\infty^{\alpha_n,\beta_n^*}(\sigma\underline{v}) - \overline{\varphi}_\infty^{\alpha_n,\beta_n}(\sigma\underline{v})| + |\gamma_n - \gamma| \\ &\leq \frac{1}{\overline{\beta} - 1}(2|\overline{\alpha} - \alpha_n| + |\overline{\beta} - \beta_n|) + |\gamma_n - \gamma|\,,\end{aligned}$$

since $\beta_n^* = \overline{\beta} + \overline{\alpha} - \alpha_n$. Letting n go to infinity, we get $\overline{\varphi}_\infty^{\overline{\alpha},\overline{\beta}}(\sigma\underline{v}) = \overline{\gamma}$. Similarly we prove $\overline{\varphi}_\infty^{\overline{\alpha},\overline{\beta}}(\sigma\underline{v}) = \overline{\alpha}$. □

Corollary 6.11. *Suppose that $(\underline{u},\underline{v})$, respectively $(\underline{u}',\underline{v}')$, verify the hypotheses of Proposition 6.10 with $k \geq 2$, respectively with $k' \geq 2$. If $k \geq k'$, $\underline{u} \preceq \underline{u}'$ and $\underline{v}' \preceq \underline{v}$, then $\overline{\beta}' \leq \overline{\beta}$ and $\overline{\alpha}' \geq \overline{\alpha}$.*

Proof: We consider the case $k = k'$, whence $\sigma\underline{v}' \preceq \sigma\underline{v}$. From the proof of Proposition 6.10 we get $\gamma_1' \leq \gamma_1$ and $\alpha_1' \geq \alpha_1$. Suppose that $\gamma_j' \leq \gamma_j$ and $\alpha_j' \geq \alpha_j$ for $j = 1,\ldots,n$. If n is even, then $\alpha_{n+1}' = \alpha_n'$ and $\alpha_{n+1} = \alpha_n$. We prove that $\gamma_{n+1}' \leq \gamma_{n+1}$. We have

$$\gamma_{n+1}' = \overline{\varphi}_\infty^{\alpha_{n+1}',\beta(\gamma_{n+1}')}(\sigma\underline{v}') \leq \overline{\varphi}_\infty^{\alpha_{n+1}',\beta(\gamma_{n+1}')}(\sigma\underline{v}) \leq \overline{\varphi}_\infty^{\alpha_{n+1},\beta(\gamma_{n+1}')}(\sigma\underline{v}) \implies \gamma_{n+1} \geq \gamma_{n+1}'\,.$$

If n is odd, then $\gamma_{n+1}' = \gamma_n'$ and $\gamma_{n+1} = \gamma_n$. We prove that $\alpha_{n+1}' \geq \alpha_{n+1}$. We have

$$\begin{aligned}\alpha_{n+1} = \overline{\varphi}_\infty^{\alpha_{n+1},\beta(\alpha_{n+1})}(\sigma\underline{u}) &\leq \overline{\varphi}_\infty^{\alpha_{n+1},\beta(\alpha_{n+1})}(\sigma\underline{u}') = \overline{\varphi}_\infty^{\alpha_{n+1},\gamma_{n+1}-\alpha_{n+1}+k-1)}(\sigma\underline{u}') \\ &\leq \overline{\varphi}_\infty^{\alpha_{n+1},\gamma_{n+1}'-\alpha_{n+1}+k-1)}(\sigma\underline{u}') \implies \alpha_{n+1}' \geq \alpha_{n+1}\,.\end{aligned}$$

□

We state a uniqueness result. The proof uses Theorem 6.16.

Proposition 6.12. *Let $k \geq 2$, $\underline{u}, \underline{v} \in \Sigma_k^+$, $u_0 = 0$ and $v_0 = k-1$, and assume that* (6.11) *holds. Then there is at most one solution $(\alpha, \beta) \in [0,1] \times [1, \infty)$ for the equations*

$$\overline{\varphi}_\infty^{\alpha,\beta}(\sigma\underline{u}) = \alpha \quad \textit{and} \quad \overline{\varphi}_\infty^{\alpha,\beta}(\sigma\underline{v}) = \gamma\,.$$

Proof: Assume that there are two solutions (α_1, β_1) and (α_2, β_2) with $\beta_1 \leq \beta_2$. If $\alpha_2 > \alpha_1$, then

$$\alpha_2 - \alpha_1 = \overline{\varphi}_\infty^{\alpha_2,\beta_2}(\sigma\underline{u}) - \overline{\varphi}_\infty^{\alpha_1,\beta_1}(\sigma\underline{u}) \leq 0\,,$$

which is impossible. Therefore $\alpha_2 \leq \alpha_1$. If $\beta_1 = \beta_2$, then

$$0 \geq \alpha_2 - \alpha_1 = \overline{\varphi}_\infty^{\alpha_2,\beta_2}(\sigma\underline{u}) - \overline{\varphi}_\infty^{\alpha_1,\beta_1}(\sigma\underline{u}) \geq 0\,,$$

which implies $\alpha_2 = \alpha_1$. Therefore we assume that $\alpha_2 \leq \alpha_1$ and $\beta_1 < \beta_2$. However, Theorem 6.16 implies that

$$\log_2 \beta_1 = h_{\mathrm{top}}(\Sigma(\underline{u}, \underline{v}), \sigma) = \log_2 \beta_2\,,$$

which is impossible. □

6.1.3 The topological entropy of $\Sigma(u, v)$

The key of this section is a good description of the graph $\mathcal{G}(\underline{u}, \underline{v})$ defined in Example A at the end of Chapter 4. We recall in details the construction of $\mathcal{G}(\underline{u}, \underline{v})$. Set $\underline{u}^0 \equiv \underline{u}$, $\underline{v}^{k-1} \equiv \underline{v}$ and

$$\underline{u}^j = j\underline{u} \ \forall j = 1, \ldots, k-1 \quad \text{and} \quad \underline{v}^j = j\underline{v} \ \forall j = 0, \ldots, k-2\,. \tag{6.20}$$

Then $\Sigma(\{\underline{u}^j, \underline{v}^j\}) = \Sigma(\underline{u}, \underline{v})$. Let $\mathcal{G}' = (\mathcal{V}', \mathcal{E}', \mathcal{L}')$ denote the Markov diagram of $\Sigma(\{\underline{u}^j, \underline{v}^j\})$. Using the relations (6.20), we define an equivalence relation on quadruples by

$$(p, \underline{a}; q, \underline{b}) \sim_1 (p', \underline{a}'; q', \underline{b}') \iff [\sigma^p \underline{a}, \sigma^q \underline{b}] = [\sigma^{p'} \underline{a}', \sigma^{q'} \underline{b}']\,.$$

The graph $\mathcal{G}(\underline{u}, \underline{v}) = (\mathcal{V}, \mathcal{E}, \mathcal{L})$ is the simplification of $\mathcal{G}'$ corresponding to this equivalence relation. The next proposition summarizes the structure of the graph $\mathcal{G}(\underline{u}, \underline{v})$. Given a word $\underline{w} \in \mathcal{L}(\Sigma(\underline{u}, \underline{v}))$, let $u(\underline{w})$ denote the longest suffix of $\underline{w}$ which is a prefix of $\underline{u}$ and $v(\underline{w})$ denote the longest suffix of $\underline{w}$ which is a prefix of $\underline{v}$.

Proposition 6.13. *The vertices of the graph $\mathcal{G}(\underline{u}, \underline{v})$ are quadruples $(p, \underline{u}; q, \underline{v})$ with $p, q \geq 0$. The root is the vertex $(0, \underline{u}; 0, \underline{v})$. There are $k-2$ edges labeled by j for $j = 1, \ldots, k-2$ going from the root to the root. The edge starting at the root and labeled by 0 terminates at the vertex $(1, \underline{u}; 0, \underline{v})$; the edge starting at the root and labeled by $k-1$ terminate at the vertex $(0, \underline{u}; 1, \underline{v})$. Let $c = (p, \underline{u}; q, \underline{v})$ be a vertex of $\mathcal{G}(\underline{u}, \underline{v})$ with $p \geq 1$ or $q \geq 1$, ie c is not the root. If c has only one successor c', ie $u_p = v_q$, then $c \xrightarrow{u_p} c' = (p+1, \underline{u}; q+1, \underline{v})$. If c has at least two successors, ie $u_p < v_q$, then*

$$\begin{aligned}
&c \xrightarrow{u_p} (p+1, \underline{u}; 0, \underline{v})\,,\\
&c \xrightarrow{v_q} (0, \underline{u}; q+1, \underline{v})\,,\\
&c \xrightarrow{j} (0, \underline{u}; 0, \underline{v}) \quad \forall u_p < j < v_q\,.
\end{aligned}$$

Moreover, if $p > q$ and c has at least two successors, then the path on $\mathcal{G}(\underline{u},\underline{v})$, starting at the root and labeled by $\underline{v}_{[0,q)}$ terminates at the vertex $(t,\underline{u};q,\underline{v})$ for some $t < q$; this last vertex has at least two successors, one of them is $(0,\underline{u};q+1;\underline{v})$. If $p < q$ and c has at least two successors, then the path on $\mathcal{G}(\underline{u},\underline{v})$, starting at the root and labeled by $\underline{u}_{[0,p)}$ terminates at the vertex $(p,\underline{u};s,\underline{v})$ for some $s < p$; this last vertex has at least two successors, one of them is $(p+1,\underline{u};0,\underline{v})$.
Let $\underline{w} \in \mathcal{L}(\Sigma(\underline{u},\underline{v}))$ and set $p = |u(\underline{w})|$ and $q = |v(\underline{w})|$. Then the path on $\mathcal{G}(\underline{u},\underline{v})$, starting at the root and labeled by $\underline{w}$, terminates at the vertex $(p,\underline{u};q,\underline{v})$.

Proof: The proof is immediate. It is a compilation of Lemmas 4.3, 4.5, 4.7 and Corollary 4.8. Each time we get a vertex $(p,\underline{u}^j;\cdot,\cdot)$ in the Markov diagram of $\Sigma(\{\underline{u}^j,\underline{v}^j\})$ with $p \geq 1$ and $j \neq 0$, we replace this vertex by $(p-1,\underline{u};\cdot,\cdot)$ in $\mathcal{G}(\underline{u},\underline{v})$; each time we get a vertex $(\cdot,\cdot;q,\underline{v}^j)$ in the Markov diagram of $\Sigma(\{\underline{u}^j,\underline{v}^j\})$ with $q \geq 1$ and $j \neq k-1$, we replace this vertex by $(\cdot,\cdot;q-1,\underline{v})$ in $\mathcal{G}(\underline{u},\underline{v})$. □

The upper level of $c = (p,\underline{u};q,\underline{v})$ is $\max\{p,q\}$ and the lower level of c is $\min\{p,q\}$. It is still true that the upper level of a vertex is the length of the shortest path from the root to this vertex. Notice also that there are only two vertices in $\mathcal{V}_n$ for all $n \geq 1$. This remark allows us to index the vertices of $\mathcal{G}(\underline{u},\underline{v})$ by the set of words (recall that the index of a vertex c is the labels of the shortest path from the root to c)

$$\{\varepsilon\} \cup \bigcup_{n\geq 1}\{\underline{u}_{[0,n)}\} \cup \bigcup_{n\geq 1}\{\underline{v}_{[0,n)}\}\,.$$

The vertices labeled by the prefixes of $\underline{u}$ form the upper branch of the graph, the vertices labeled by the prefixes of $\underline{v}$ form the lower branch. We also recall a fundamental property of the Markov diagram. By Lemma 4.3, if a path on $\mathcal{G}(\underline{u},\underline{v})$ starts at the root and terminates at the vertex $(p,\underline{u};q,\underline{v})$ with $p,q \geq 1$, then the last but one visited vertex is $(p-1,\underline{u};q-1,\underline{v})$. Set $r = \min\{p,q\}$; repeating this step r times, we conclude that, r steps before the end of the path, the path visits the vertex $(p-r,\underline{u};q-r,\underline{v})$.

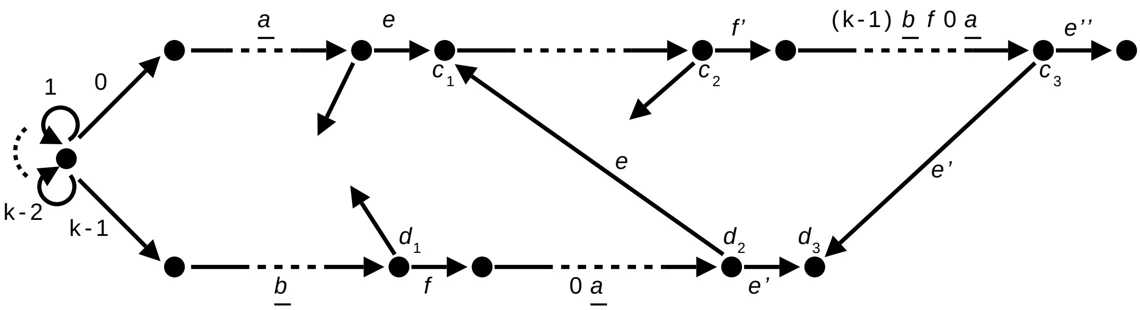

Figure 6.2: The beginning of the graph $\mathcal{G}(\underline{u},\underline{v})$.

A typical situation is sketched in Figure 6.2. All the edges starting at the root, except for the edges labeled by 0 and $k-1$, terminate at the root. Let d_2 be the vertex indexed by $\underline{w} = (k-1)\underline{b}f0\underline{a}$. Define d_1 as the last vertex before d_2, where there are at least two outgoing edges; it is indexed by $(k-1)\underline{b}$. Since d_1 has at least two successors, $f > 0$. Notice that the f-successor of d_1 is a vertex $d' = (0,\underline{u};\cdot,\cdot)$ and d' has only one successor by hypothesis. Since $u_0 = 0$, the outgoing edge at d' is labeled by 0. Set $p = |\underline{a}|+1$ and $q = |\underline{a}|+|\underline{b}|+3$. By Corollary 4.8, $u(\underline{w}) = 0\underline{a}$ and $v(\underline{w}) = \underline{w}$. Thus the vertex d_2 is

$$d_2 = (p,\underline{u};q,\underline{v})\,.$$

d_2 has at least two successors, thus $e < e'$. Moreover the e-successor of d_2 is the vertex $c_1 = (p+1, \underline{u}; \cdot, \cdot)$. Let c_3 be a vertex of the upper line such that $c_3 \xrightarrow{e'} d_3$ and let c_2 the last vertex before c_3, where there is at least two outgoing edges. The shortest path from c_2 to c_3 must be indexed by $f'(k-1)\underline{b}f0\underline{a}$ and $e'' < e'$. Moreover $e'' \geq e$, since by inequalities (6.11), $0\underline{a}e \ldots = \underline{u} \preceq \sigma^n \underline{u} = 0\underline{a}e'' \ldots$ for n well chosen.

We give now the setting of the next lemma and the next proposition, which are an adaptation of Lemmas 2 and 3 in [H4]. They form the main result of that paper, where Hofbauer gives a proof of the uniqueness of the measure of maximal entropy for the map $T_{\alpha,\beta}$. Let $k \geq 2$ and $\underline{a}, \underline{b}$ be two non periodic words, ie there is no word $\underline{x}$ such that $\underline{a} = \underline{x}^n$ for some $n \geq 2$ and similarly with $\underline{b}$. Set $p = |\underline{a}|$ and $q = |\underline{b}|$. Suppose that either $p \geq 2$ and $a_0 = 0$ or $\underline{a} = 1$ is a word of length 1; similarly either $q \geq 2$ and $b_0 = k-1$ or $\underline{b} = k-2$ is a word of length 1. Define $\underline{a}' = \underline{a}_{[0,p-1)}(a_{p-1}+1)$ and $\underline{b}' = \underline{b}_{[0,q-1)}(b_{q-1}-1)$. Then we set

$$\underline{u} = \underline{a}^\infty\,, \quad \underline{\tilde{u}} = \underline{a}'\underline{b}^\infty\,, \quad \underline{v} = \underline{b}^\infty \quad \text{and} \quad \underline{\tilde{v}} = \underline{b}'\underline{a}^\infty\,.$$

Suppose further that the pairs $(\underline{u}, \underline{v})$ and $(\underline{\tilde{u}}, \underline{\tilde{v}})$ satisfy (6.11). These hypotheses correspond to the settings of Proposition 6.4, in particular cases 1c, 2c and 3d.

Lemma 6.14. *In the above settings, the vertices of the graph $\mathcal{G}(\underline{u}, \underline{v})$ are indexed by the words*

$$\{\varepsilon\} \cup \bigcup_{n=1}^{p-1} \{\underline{a}_{[0,n)}\} \cup \bigcup_{n=1}^{p-1} \{\underline{b}_{[0,n)}\}\,.$$

Let $r = |v(\underline{a}_{[0,p-1)})|$ and c_1 denote the vertex indexed by $\underline{a}_{[0,p-1)}$. If $u_{p-1} < v_r$, then the edge, starting at c_1 and labeled by u_{p-1}, terminates at the root. If $u_{p-1} = v_r$, then the edge, starting at c_1 and labeled by u_{p-1}, terminates at the vertex indexed by $\underline{v}_{[0,r]}$. If $k = 2$, then $u_{p-1} = v_r$.
Let $s = |u(\underline{b}_{[0,q-1)})|$ and d_1 denote the vertex indexed by $\underline{b}_{[0,q-1)}$. If $u_s < v_{q-1}$, then the edge, starting at d_1 and labeled by v_{q-1}, terminates at the root. If $u_s = v_{q-1}$, then the edge, starting at d_1 and labeled by v_{q-1}, terminates at the vertex indexed by $\underline{u}_{[0,s]}$. If $k = 2$, then $u_s = v_{q-1}$.

Proof: Up to the vertex indexed by prefixes of $\underline{u}$ of length less than p, the structure of the graph is typical and described in Proposition 6.13. Indeed p is the minimal period of $\underline{u}$, thus there cannot be supplementary identifications. Now consider the vertex g_1 indexed by $\underline{u}_{[0,p-1)}$; it is the quadruple $(p-1, \underline{u}; m-1, \underline{v})$ with $m < p$. The situation is illustrated in Figure 6.3. If $u_{p-1} < v_{m-1}$, the successors of g_1 are given by

$$g_1 = (p-1, \underline{u}; m-1, \underline{v}) \xrightarrow{j} \begin{cases} (p, \underline{u}; 0, \underline{v}) \sim_1 (0, \underline{u}; 0, \underline{v}) & \text{if } j = u_{p-1}, \\ (0, \underline{u}; 0, \underline{v}) & \text{if } u_{p-1} < j < v_{m-1}, \\ (0, \underline{u}; m, \underline{v}) & \text{if } j = v_{m-1}. \end{cases}$$

All these successors already exist in $\mathcal{G}(\underline{u}, \underline{v})$. If $u_{p-1} = v_{m-1}$, the only successor of g_1 is given by

$$g_1 = (p-1, \underline{u}; m-1, \underline{v}) \overset{v_{m-1}}{\longrightarrow} = (p, \underline{u}; m, \underline{v}) \sim_1 (0, \underline{u}; m, \underline{v})\,.$$

In this later case, we prove that the vertex g' indexed by $\underline{v}_{[0,m)}$ is the quadruple $(0, \underline{u}; m, \underline{v})$. By the fundamental property of the Markov diagram, $\underline{u}_{[p-m,p-1)} = \underline{v}_{[0,m-1)}$. By hypothesis, $u_{p-1} = v_{m-1}$. By Proposition 6.13, the vertex g' is $(t, \underline{u}; m, \underline{v})$ for some $0 \leq t < m$. Suppose that $t > 0$; it means that $\underline{u}_{[0,t)} = \underline{v}_{[m-t,m)}$. Thus $\underline{u}_{[p-t,p)} = \underline{u}_{[0,t)}$. Since $\sigma^p \underline{u} = \underline{u}$, we conclude that $\sigma^{p-t}\underline{u} = \underline{u}$ and this contradicts the hypothesis that p is the minimal period of $\underline{u}$. Thus the only successor of g_1 is identical to g'. Finally notice that $u_{p-1} > 0$. Indeed, if $p = 1$, then $u_0 = 1$ by hypothesis. Suppose $p \geq 2$ and $u_{p-1} = 0$, then by inequalities (6.11),

$$\underline{u} \preceq \sigma^{p-1}\underline{u} = 0\underline{u} \preceq 0(\sigma\underline{u}) = \underline{u}\,.$$

This implies $\sigma^{p-1}\underline{u} = \underline{u}$, but this contradicts the minimality of p. If $k = 2$, $0 < u_{p-1} \leq v_{m-1} \leq v_0 = 1$, thus $u_{p-1} = v_{m-1}$. Similarly, we consider the vertices labeled by prefixes of $\underline{v}$. □

Proposition 6.15. *In the above setting, if $h_{\text{top}}(\Sigma(\underline{u},\underline{v}),\sigma) > 0$, then $h_{\text{top}}(\Sigma(\underline{u},\underline{v}),\sigma) = h_{\text{top}}(\Sigma(\underline{\tilde{u}},\underline{\tilde{v}}),\sigma)$.*

Proof: Let M (resp. $\tilde{M}$) denote the adjacency matrix of the graph $\mathcal{G}(\underline{u},\underline{v})$ (resp. $\mathcal{G}(\underline{\tilde{u}},\underline{\tilde{v}})$). By Formula 4.4, we must compute the spectral radius $\rho(M)$ and $\rho(\tilde{M})$, thus we compare $\mathcal{G}(\underline{u},\underline{v})$ and $\mathcal{G}(\underline{\tilde{u}},\underline{\tilde{v}})$. The graph $\mathcal{G}(\underline{u},\underline{v})$ is described in Lemma 6.14. We consider now the graph $\mathcal{G}(\underline{\tilde{u}},\underline{\tilde{v}})$. Up to the vertices of upper level $p-1$, the graph is the same as $\mathcal{G}(\underline{u},\underline{v})$ since the virtual itineraries are the same. Consider the vertex g_1 indexed by $\underline{\tilde{u}}_{[0,p-1)} = \underline{u}_{[0,p-1)}$; as before, it is the quadruple $(p-1,\underline{\tilde{u}};m-1,\underline{\tilde{v}})$ with $m < p$. Since $\tilde{u}_{p-1} + 1 = u_{p-1}$, the successors of g_1 are

$$g_1 = (p-1,\underline{\tilde{u}};m-1;\underline{\tilde{v}}) \xrightarrow{j} \begin{cases} (p,\underline{\tilde{u}};0,\underline{\tilde{v}}) \sim_1 (0,\underline{b}^\infty;0,\underline{b}'\underline{a}^\infty) & \text{if } j = \tilde{u}_{p-1}, \\ (0,\underline{\tilde{u}};0,\underline{\tilde{v}}) & \text{if } \tilde{u}_{p-1} < j < \tilde{v}_{m-1}, \\ (0,\underline{\tilde{u}};m,\underline{\tilde{v}}) & \text{if } j = \tilde{v}_{m-1}. \end{cases}$$

Comparing the outgoing edges at g_1 in the two graphs, we conclude that all edges which are present in $\mathcal{G}(\underline{u},\underline{v})$ are also present in $\mathcal{G}(\underline{\tilde{u}},\underline{\tilde{v}})$. There is only one additional edge leaving g_1 in $\mathcal{G}(\underline{\tilde{u}},\underline{\tilde{v}})$. It is labeled by $\tilde{u}_{p-1}$ and it terminates at the vertex $(p,\underline{\tilde{u}};0,\underline{\tilde{v}}) \sim_1 (0,\underline{b}^\infty;0,\underline{b}'\underline{a}^\infty)$. Similarly we consider the vertex h_1 indexed by $\underline{v}_{[0,q-1)} = \underline{\tilde{v}}_{[0,q-1)}$ in the two graphs. There is only one additional edge starting at h_1 in $\mathcal{G}(\underline{\tilde{u}},\underline{\tilde{v}})$. It is labeled by $\tilde{v}_{q-1}$ and terminates at the vertex $(0,\underline{\tilde{u}};0,\sigma^q\underline{\tilde{v}}) \sim_1 (0,\underline{a}'\underline{b}^\infty;0,\underline{a}^\infty)$.

Consider the vertex $g_2 = (p,\underline{\tilde{u}};0,\underline{\tilde{v}}) \sim_1 (0,\underline{b}^\infty;0,\underline{b}'\underline{a}^\infty)$ indexed by $\underline{\tilde{u}}_{[0,p)}$ in $\mathcal{G}(\underline{\tilde{u}},\underline{\tilde{v}})$. Since $\underline{b}_{[0,q-1)} = \underline{b}'_{[0,q-1)}$, there is only one path of length $q-1$ starting at g_2. It is labeled by $\underline{b}_{[0,q-1)}$ and terminates at the vertex $(p+q-1,\underline{\tilde{u}};q-1,\underline{\tilde{v}}) \sim_1 (0,b_q\underline{b}^\infty;0,(b_q+1)\underline{a}^\infty)$. This last vertex has two outgoing edges

$$\begin{aligned} &(p+q-1,\underline{\tilde{u}};q-1,\underline{\tilde{v}}) \xrightarrow{b_q} (p+q,\underline{\tilde{u}};0,\underline{\tilde{v}}) \sim_1 (p,\underline{\tilde{u}};0,\underline{\tilde{v}})\,, \\ &(p+q-1,\underline{\tilde{u}};q-1,\underline{\tilde{v}}) \xrightarrow{b_q+1} (0,\underline{\tilde{u}};q,\underline{\tilde{v}})\,. \end{aligned}$$

The first one is the vertex g_2 indexed by $\underline{\tilde{u}}_{[0,p)}$, the second one is the vertex h_2 indexed by $\underline{\tilde{v}}_{[0,q)}$. Similarly, we consider all paths starting at h_2. The graph $\mathcal{G}(\underline{\tilde{u}},\underline{\tilde{v}})$ is also finite. The graph $\mathcal{G}(\underline{u},\underline{v})$ is a subgraph of $\mathcal{G}(\underline{\tilde{u}},\underline{\tilde{v}})$. The supplementary part of $\mathcal{G}(\underline{\tilde{u}},\underline{\tilde{v}})$ is a communicating class; let $\tilde{\mathcal{G}}$ denote the subgraph $\mathcal{G}(\underline{\tilde{u}},\underline{\tilde{v}})\backslash\mathcal{G}(\underline{u},\underline{v})$. The situation is illustrated in Figure 6.3. The spectral spectral radius of $\tilde{M}$ is the maximum between the spectral radius of M and the spectral radius of the adjacency matrix of $\tilde{\mathcal{G}}$ (see Lemma 4.4.3 in [LM]). Using Theorem 4.9, we compute easily the characteristic polynomial of the adjacency matrix of $\tilde{\mathcal{G}}$. The rome is $\{g_2,h_2\}$ and the characteristic polynomial is

$$\lambda^{p+q} - \lambda^p - \lambda^q = \lambda^q(\lambda^p - \lambda^{p-q} - 1)\,.$$

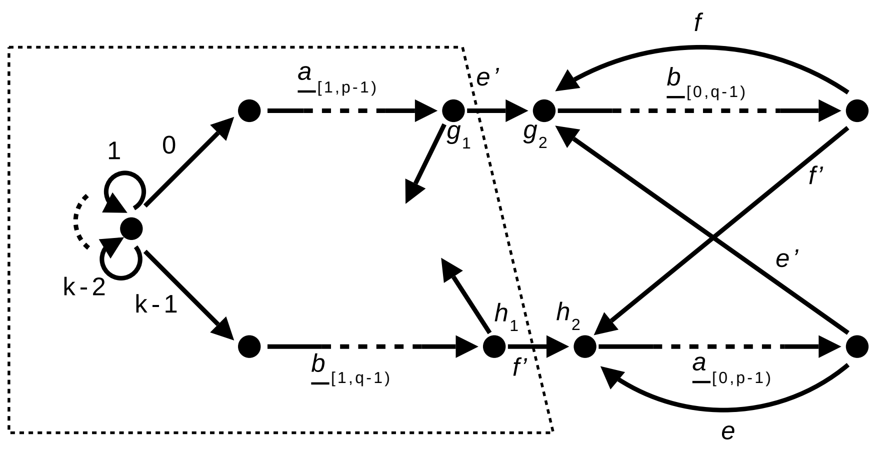

Figure 6.3: The graph $\mathcal{G}(\underline{\tilde{u}}, \underline{\tilde{v}})$. The box indicates the subgraph $\mathcal{G}(\underline{u}, \underline{v})$. The labels are $e = u_{p-1}$, $e' = e + 1$, $f = b_{q-1}$ and $f' = f - 1$.

Let λ^* denote the largest root of this polynomial; notice that $\lambda^* \leq 2$. To prove the Proposition, it is sufficient to show that $\rho(M)$ is larger than or equal to λ^*.

If $k \geq 4$, there are $k-2$ loops of length 1 at the root, thus $\rho(M) \geq k - 2 \geq 2 \geq \lambda^*$. If the path starting at the root and labeled by $\underline{a}$ ends at the root, which could happen only for $k \geq 3$ (see Lemma 6.14), then there is a subgraph of $\mathcal{G}$ consisting of two cycles passing trough the root, one of length p or of length q and another one of length 1. This also implies that $\rho(M) \geq \lambda^*$. Similarly, if the path starting at the root and labeled by $\underline{b}$ ends at the root, then $\rho(M) \geq \lambda^*$. It remains the case, $k \leq 3$, the path starting at the root and labeled by $\underline{a}$ does not end at the root and the path starting at root an labeled by $\underline{b}$ does not end at the root.

Let $\mathcal{H}$ be a communicating class of $\mathcal{G}(\underline{u}, \underline{v})$ whose adjacency matrix has spectral radius strictly greater than 1 ($\mathcal{H}$ exists by hypothesis). If the root is a vertex of $\mathcal{H}$, which happens only if $k = 3$, then we conclude as above that $\rho(M) \geq \lambda^*$. Hence, we assume that the root is not a vertex of $\mathcal{H}$. The vertices of $\mathcal{H}$ are indexed by prefixes of $\underline{a}_{[0,p-1)}$ and $\underline{b}_{[0,q-1)}$. From here, we use recursively the most important properties of the graph $\mathcal{G}(\underline{u}, \underline{v})$, which are emphasized in Figure 6.2 and its comments. Let $c_1 \in \mathcal{H}$ be the first (ie most on the left) vertex of the upper branch and $d_1 \in \mathcal{H}$ the first vertex of the lower branch. Since $\mathcal{H}$ is a communicating class, there are edges starting in $\mathcal{H}$ and terminating at c_1 and d_1. Let $c_2 \in \mathcal{H}$ be the first vertex in the upper branch, which has at least two outgoing edges (in fact, it has two edges, because the root is not in $\mathcal{H}$). Similarly, let $d_2 \in \mathcal{H}$ be the first vertex of the lower branch, which has two outgoing edges. We claim that there is an edge from c_2 to d_1 and from d_2 to c_1.

Suppose it is not the case: for example, there is no edge from c_2 to d_1. Let $c_3 \in \mathcal{H}$ be the first vertex in the upper branch such that there is an edge from c_3 to d_1. We define by induction two finite sequences of vertices $\{e_i\}$ and $\{f_i\}$. The vertex e_i is the last vertex before c_3 in the upper branch, having two outgoing edges. Then denote by f_1' the vertex of the lower branch where terminates an edge starting at e_1. The vertex f_1 is the vertex immediately before f_1' in the lower branch; it has two outgoing edges. Then denote by e_2' the vertex of the upper branch where terminates an edge starting at f_1. The vertex e_2 is the vertex immediately before e_2' in the upper branch; it has two outgoing edges. Continue in this manner until $f_i = d_2$ or $e_i' = c_1$. We prove that the sequences are finite. $\mathcal{H}$ is a communicating class, thus e_i and f_i are in $\mathcal{H}$. Moreover the upper level (ie the length of the shortest path from the root to the vertex) of f_i is strictly smaller than the upper level of e_i; similarly the upper level of e_{i+1} is strictly smaller than the upper level of f_i. Finally, $f_i' = d_1$ is impossible, because this contradicts the definition of c_3. Suppose the sequences end under condition $f_n = d_2$. The situation is illustrated in Figure 6.4.

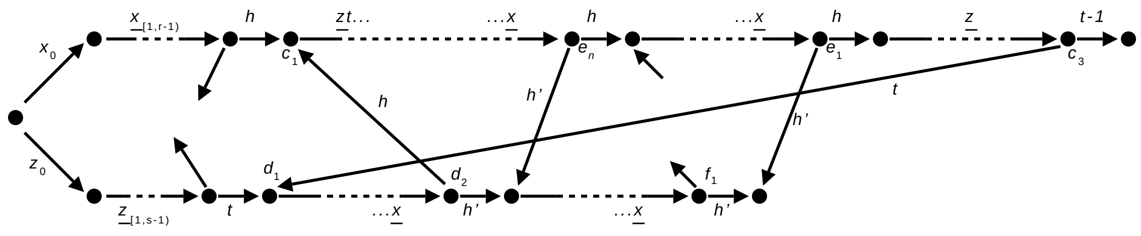

Figure 6.4: The beginning of the graph $\mathcal{G}(\underline{u}, \underline{v})$.

Let $\underline{x}h$ be the word which indexes the vertex c_1 and $\underline{z}t$ be the word which indexes d_1. Set $r = |\underline{x}h|$ and $s = |\underline{z}t|$. Since d_1 is the first vertex in $\mathcal{H}$ of the lower branch, $\underline{u}_{[r,r+s)} = \underline{z}t$. Since there is an edge from d_2 to c_1, $\underline{x}$ is a suffix of the word which indexes d_2 and $h' = h + 1$ (otherwise there is a third edge from d_2 to the root). This implies that $\underline{x}$ is a suffix of the word which indexes the vertex e_n and the outgoing edges at e_n are labeled by h and h'. By induction, $\underline{x}$ is a suffix of the word which indexes the vertex e_1 and the outgoing edges at e_1 are labeled by h and h'. For the same reasons, $\underline{z}$ is a suffix of the word which indexes the vertex c_3 and the outgoing edges at c_3 are labeled by t and $t-1$. Thus there exists m such that $\sigma^m \underline{u} = \underline{x}h\underline{z}(t-1)\ldots \prec \underline{x}h\underline{z}t\ldots = \underline{u}$; this contradicts the inequalities (2.16). Similarly, we consider the case where the sequences $\{e_i\}$ and $\{f_i\}$ end under condition $e_i' = c_1$; this situation is also absurd. Thus there is an edge from c_2 to d_1 and an edge from d_2 to c_1.

Since the spectral radius of the communicating class $\mathcal{H}$ is strictly greater than 1, there exists at least one edge from some other vertex of $\mathcal{H}$ to c_1 or d_1, say c_1. Let d_3 be the vertex where starts this additional edge terminating at c_1. Then there are two cycles in $\mathcal{H}$ rooted at c_1: one follows the upper branch from c_1 to c_2, then it follows the lower branch from d_1 to d_2, then it goes back to c_1; the other one follows the upper branch from c_1 to c_2, then the lower branch from d_1 to d_3, then it goes back to c_1. By Proposition 6.13, the first cycle has length $r + s$, the second cycle has length $r + s + w$ where w is the length of the path following the lower branch from d_2 to d_2. Moreover $r + s \leq p$ and $r + s + w \leq q$. The situation, in particular the length of the different paths, is illustrated in Figure 6.5. Using Theorem 4.9 with $\{c_1, d_1\}$ as a rome, we show that $\rho(M) \geq \lambda^*$. $\square$

Theorem 6.16. *Let $k \geq 2$ and let $\underline{u} \in \Sigma_k^+$ and $\underline{v} \in \Sigma_k^+$, such that $u_0 = 0$, $v_0 = k - 1$ and*

$$\underline{u} \preceq \sigma^n \underline{u} \preceq \underline{v} \quad \forall n \geq 0 \quad \textit{and} \quad \underline{u} \preceq \sigma^n \underline{v} \preceq \underline{v} \quad \forall n \geq 0\,.$$

If $k = 2$ we also assume that $\sigma\underline{u} \preceq \sigma\underline{v}$. Let $\overline{\alpha}$ and $\overline{\beta}$ be the two real numbers defined by the algorithm of Proposition 6.10. Then

$$h_{\mathrm{top}}(\Sigma(\underline{u}, \underline{v}), \sigma) = \log \overline{\beta}\,.$$

If $k = 2$ and $\sigma\underline{v} \prec \sigma\underline{u}$, then $h_{\mathrm{top}}(\Sigma(\underline{u}, \underline{v}), \sigma) = 0$.

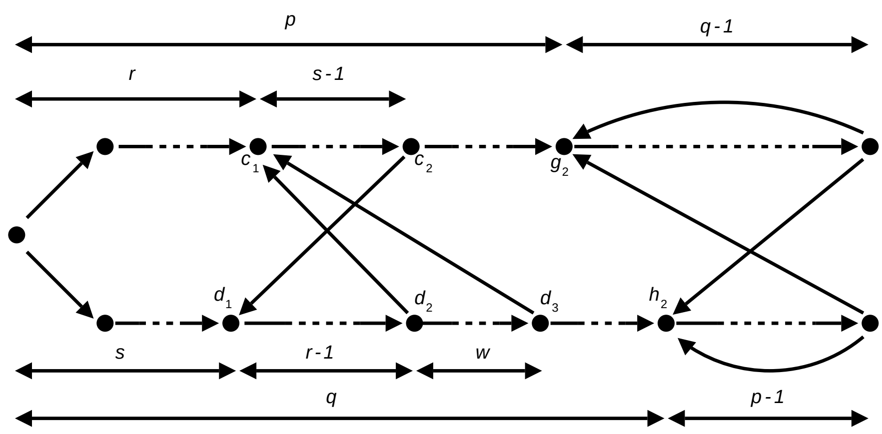

Figure 6.5: The graph $\mathcal{G}(\underline{\tilde{u}}, \underline{\tilde{v}})$ with the length of the paths.

Proof: Let $\overline{\beta} > 1$. By Propositions 6.4 and 6.10 we have

$$\Sigma(\underline{u}^{\overline{\alpha},\overline{\beta}}, \underline{v}^{\overline{\alpha},\overline{\beta}}) \subset \Sigma(\underline{u}, \underline{v}) \subset \Sigma(\underline{\tilde{u}}^{\overline{\alpha},\overline{\beta}}, \underline{\tilde{v}}^{\overline{\alpha},\overline{\beta}}) .$$

From Proposition 6.15 we get

$$h_{\rm top}(\Sigma(\underline{u}^{\overline{\alpha},\overline{\beta}}, \underline{v}^{\overline{\alpha},\overline{\beta}}), \sigma) = h_{\rm top}(\Sigma(\underline{\tilde{u}}^{\overline{\alpha},\overline{\beta}}, \underline{\tilde{v}}^{\overline{\alpha},\overline{\beta}}), \sigma) = \log \overline{\beta} .$$

Let $\lim_n \alpha_n = \overline{\alpha}$ and $\lim_n \beta_n = \overline{\beta} = 1$. We have $\alpha_n < 1$ and $\beta_n > 1$ (see proof of Proposition 6.10). Let

$$\underline{u}^n := \underline{\tilde{u}}^{\alpha_n,\beta_n} \quad \text{and} \quad \underline{v}^n := \underline{\tilde{v}}^{\alpha_n,\beta_n} .$$

By Proposition 6.4,

$$\underline{v}^{\alpha_1,\beta_1} \preceq \underline{v} \preceq \underline{v}^1 .$$

By monotonicity,

$$\overline{\varphi}_\infty^{\alpha_2,\beta_2}(\sigma\underline{v}^1) \leq \overline{\varphi}_\infty^{\alpha_1,\beta_1}(\sigma\underline{v}^1) = \gamma_1 = \gamma_2 = \overline{\varphi}_\infty^{\alpha_2,\beta_2}(\sigma\underline{v}^2) .$$

Therefore $\underline{v}^1 \preceq \underline{v}^2$ ($v_0^1 = v_0^2$) and by Proposition 6.4,

$$\underline{u}^2 \preceq \underline{u} \preceq \underline{u}^{\alpha_2,\beta_2} \quad \text{and} \quad \underline{v} \preceq \underline{v}^2 .$$

By monotonicity,

$$\overline{\varphi}_\infty^{\alpha_3,\beta_3}(\sigma\underline{u}^3) = \alpha_3 = \alpha_2 = \overline{\varphi}_\infty^{\alpha_2,\beta_2}(\sigma\underline{u}^2) \leq \overline{\varphi}_\infty^{\alpha_3,\beta_3}(\sigma\underline{u}^2) .$$

Therefore $\underline{u}^3 \preceq \underline{u}^2$ and

$$\underline{u}^3 \preceq \underline{u} \quad \text{and} \quad \underline{v}^{\alpha_3,\beta_3} \preceq \underline{v} \preceq \underline{v}^3 .$$

Iterating this argument we conclude that

$$\underline{u}^n \preceq \underline{u} \quad \text{and} \quad \underline{v} \preceq \underline{v}^n .$$

These inequalities imply

$$h_{\rm top}(\Sigma(\underline{u}, \underline{v}), \sigma) \leq h_{\rm top}(\Sigma(\underline{u}^n, \underline{v}^n), \sigma) = \log \beta_n \to 0 \quad \text{for } n \to \infty .$$

Finally let $k=2$ and $\sigma\underline{v}\prec\sigma\underline{u}$. If $\sigma\underline{u}=(1)^\infty$, then $\underline{v}_j=0$ for a single value of j, so that $h_{\rm top}(\Sigma(\underline{u},\underline{v}),\sigma)=0$. Suppose that $\sigma\underline{u}\neq(1)^\infty$ and fix any $\beta>1$. The function $\alpha\mapsto\overline{\varphi}_\infty^{\alpha,\beta}(\sigma\underline{u})$ is continuous and decreasing since $\overline{\varphi}^{\alpha,\beta}$ dominates $\overline{\varphi}^{\alpha',\beta}$ if $\alpha<\alpha'$. There exists $\alpha\in(0,1)$ such that $\overline{\varphi}_\infty^{\alpha,\beta}(\sigma\underline{u})=\alpha$. If $\underline{v}_0<\underline{v}_0^{\alpha,\beta}$, then $\underline{v}\prec\underline{v}^{\alpha,\beta}$ and $\Sigma(\underline{u},\underline{v})\subset\Sigma(\underline{u},\underline{v}^{\alpha,\beta})$, whence $h_{\rm top}(\Sigma(\underline{u},\underline{v}),\sigma)\leq\log\beta$. If $\underline{v}_0=\underline{v}_0^{\alpha,\beta}=1$, then

$$\overline{\varphi}_\infty^{\alpha,\beta}(\sigma\underline{v})\leq\overline{\varphi}_\infty^{\alpha,\beta}(\sigma\underline{u})=\alpha<\gamma=\overline{\varphi}_\infty^{\alpha,\beta}(\sigma\underline{v}^{\alpha,\beta})\,.$$

The map $\overline{\varphi}_\infty^{\alpha,\beta}$ is continuous and non-decreasing on Σ_k^+ so that $\sigma\underline{v}\prec\sigma\underline{v}^{\alpha,\beta}$, whence $\underline{v}\prec\underline{v}^{\alpha,\beta}$ and $h_{\rm top}(\Sigma(\underline{u},\underline{v}),\sigma)\leq\log\beta$. Since $\beta>1$ is arbitrary, $h_{\rm top}(\Sigma(\underline{u},\underline{v}),\sigma)=0$. □

6.1.4 The inverse problem

In this section we solve the inverse problem for $\beta x+\alpha \mod 1$, namely the question we address is the following: given two sequences $\underline{u}$ and $\underline{v}$ verifying

$$\underline{u}\preceq\sigma^n\underline{u}\prec\underline{v}\quad\text{and}\quad\underline{u}\prec\sigma^n\underline{v}\preceq\underline{v}\quad\forall n\geq 0\,,\tag{6.21}$$

can we find $\alpha\in[0,1)$ and $\beta\in(1,\infty)$ so that $\underline{u}=\underline{u}^{\alpha,\beta}$ and $\underline{v}=\underline{v}^{\alpha,\beta}$?

Lemma 6.17. *Let $\alpha\in[0,1)$ and $\beta\geq1$ be such that the $\overline{\varphi}^{\alpha,\beta}$-expansion is valid. Suppose that $\overline{\varphi}_\infty^{\alpha,\beta}(\underline{u})=0$, $\overline{\varphi}_\infty^{\alpha,\beta}(\sigma\underline{u})=\alpha$ and $\underline{u}\preceq\sigma^n\underline{u}$ for all $n\geq0$. Then*

$$\underline{u}=\underline{u}^{\alpha,\beta}\iff\overline{\varphi}_\infty^{\alpha,\beta}(\sigma^n\underline{u})<1\quad\forall n\geq0\,.$$

Similarly, suppose that $\overline{\varphi}_\infty^{\alpha,\beta}(\underline{v})=1$, $\overline{\varphi}_\infty^{\alpha,\beta}(\sigma\underline{v})=\gamma$ and $\sigma^n\underline{v}\preceq\underline{v}$ for all $n\geq0$. Then

$$\underline{v}=\underline{v}^{\alpha,\beta}\iff\overline{\varphi}_\infty^{\alpha,\beta}(\sigma^n\underline{v})>0\quad\forall n\geq0\,.$$

Proof: We prove only the first statement. Suppose that $\underline{u}=\underline{u}^{\alpha,\beta}$. Then by (6.8), $\overline{\varphi}_\infty^{\alpha,\beta}(\sigma^n\underline{u})=T_{\alpha,\beta}^n(0)<1$ for all $n\geq0$. On the other hand, suppose that $\overline{\varphi}_\infty^{\alpha,\beta}(\sigma^n\underline{u})<1$ for all $n\geq0$. If $\underline{u}^{\alpha,\beta}$ is not periodic, then by Proposition 6.2, $\underline{u}^{\alpha,\beta}$ is the unique sequence satisfying $\overline{\varphi}_\infty^{\alpha,\beta}(\underline{x})=0$ and $\overline{\varphi}_\infty^{\alpha,\beta}(\sigma\underline{x})=\alpha$, thus $\underline{u}=\underline{u}^{\alpha,\beta}$. If $\underline{u}^{\alpha,\beta}$ has minimal period p, then by Lemma 6.1, $\underline{u}_{[0,p-1)}=\underline{u}_{[0,p-1)}^{\alpha,\beta}$. We have two choices for u_{p-1}. Either $u_{p-1}=u_{p-1}^{\alpha,\beta}-1$ and $\overline{\varphi}_\infty^{\alpha,\beta}(\sigma^p\underline{u})=1$ or $u_{p-1}=u_{p-1}^{\alpha,\beta}$ and $\overline{\varphi}_\infty^{\alpha,\beta}(\sigma^p\underline{u})=0$. The first choice is impossible by hypothesis. Thus $\underline{u}_{[0,p)}=\underline{u}_{[0,p)}^{\alpha,\beta}$. Repeating the argument, we conclude that $\underline{u}=\underline{u}^{\alpha,\beta}$. □

Proposition 6.18. *Suppose that the φ-expansion is valid. Let $\underline{u}$ be a solution of (6.13) and $\underline{v}$ a solution of (6.14). If (6.21) holds, then*

$$\underline{u}^{\alpha,\beta}=\underline{u}\iff\forall n\geq0:\overline{\varphi}_\infty^{\alpha,\beta}(\sigma^n\underline{u})<1\iff\forall n\geq0:\overline{\varphi}_\infty^{\alpha,\beta}(\sigma^n\underline{v})>0\iff\underline{v}^{\alpha,\beta}=\underline{v}\,.$$

Proof: The φ-expansion is valid, so that (6.8) is true,

$$\forall n\geq0:\overline{\varphi}_\infty^{\alpha,\beta}(\sigma^n\underline{u}^{\alpha,\beta})=T_{\alpha,\beta}^n(0)<1\,.$$

Lemma 6.17 implies

$$\underline{u}=\underline{u}^{\alpha,\beta}\iff\forall n\geq0:\overline{\varphi}_\infty^{\alpha,\beta}(\sigma^n\underline{u})<1\,.$$

Similarly

$$\underline{v}=\underline{v}^{\alpha,\beta}\iff\forall n\geq0:\overline{\varphi}_\infty^{\alpha,\beta}(\sigma^n\underline{v})>0\,.$$

Let $\underline{x} \prec \underline{x}'$, $\underline{x}, \underline{x}' \in \Sigma(\underline{u}, \underline{v})$. Let $\ell := \min\{m \geq 0 : x_m \neq x'_m\}$. Then

$$\overline{\varphi}_\infty^{\alpha,\beta}(\underline{x}) = \overline{\varphi}_\infty^{\alpha,\beta}(\underline{x}') \implies \overline{\varphi}_\infty^{\alpha,\beta}(\sigma^{\ell+1}\underline{x}) = 1 \quad \text{and} \quad \overline{\varphi}_\infty^{\alpha,\beta}(\sigma^{\ell+1}\underline{x}') = 0\,.$$

Indeed,

$$\overline{\varphi}_{\ell+1}^{\alpha,\beta}\big(x_0, \ldots, x_{\ell-1}, x_\ell + \overline{\varphi}_\infty^{\alpha,\beta}(\sigma^{\ell+1}\underline{x})\big) = \overline{\varphi}_{\ell+1}^{\alpha,\beta}\big(x_0, \ldots, x_{\ell-1}, x'_\ell + \overline{\varphi}_\infty^{\alpha,\beta}(\sigma^{\ell+1}\underline{x}')\big)$$

Therefore $x'_\ell = x_\ell + 1$, $\overline{\varphi}_\infty^{\alpha,\beta}(\sigma^{\ell+1}\underline{x}) = 1$ and $\overline{\varphi}_\infty^{\alpha,\beta}(\sigma^{\ell+1}\underline{x}') = 0$. Suppose that $\overline{\varphi}_\infty^{\alpha,\beta}(\sigma^n\underline{u}) = 1$, and apply the above result to $\sigma^n\underline{u}$ and $\underline{v}$ to get the existence of m with $\overline{\varphi}_\infty^{\alpha,\beta}(\sigma^m\underline{v}) = 0$. □

Definition 6.19. *Let $\underline{u} \in \Sigma_k^+$ with $u_0 = 0$ and $\underline{u} \preceq \sigma^n\underline{u}$ for all $n \geq 0$. Define the sequence $\widehat{\underline{u}} \in \Sigma_k^+$ as*

$$\widehat{\underline{u}} := \sup\{\sigma^n\underline{u} : n \geq 0\}\,.$$

For all $\underline{u} \in \Sigma_k^+$ with $u_0 = 0$ and $\underline{u} \preceq \sigma^n\underline{u}$ for all $n \geq 0$, we have

$$\sigma^n\widehat{\underline{u}} \preceq \widehat{\underline{u}} \quad \forall n \geq 0\,.$$

Indeed there exists a sequence $\{n_j\}_j$ so that $\widehat{\underline{u}} = \lim_j \sigma^{n_j}\underline{u}$. By continuity

$$\sigma^n\widehat{\underline{u}} = \lim_{j\to\infty} \sigma^{n+n_j}\underline{u} \preceq \widehat{\underline{u}}\,.$$

We explain the ideas developed in Theorems 6.20 and 6.21. Fix $\underline{u} \in \Sigma_k^+$ with $u_0 = 0$ and $\underline{u} \preceq \sigma^n\underline{u}$. Choose $\underline{v} \in \Sigma_k^+$ such that $v_0 = k-1$ and (6.21) holds. By equations (6.10), the only possible pair (α, β) such that $\underline{u} = \underline{u}^{\alpha,\beta}$ and $\underline{v} = \underline{v}^{\alpha,\beta}$ is the pair $(\overline{\alpha}, \overline{\beta})$ constructed by the algorithm of Proposition 6.10. By Proposition 6.18, a necessary and sufficient condition for $\underline{u} = \underline{u}^{\overline{\alpha},\overline{\beta}}$ and $\underline{v} = \underline{b}^{\overline{\alpha},\overline{\beta}}$ is

$$\overline{\varphi}_\infty^{\overline{\alpha},\overline{\beta}}(\sigma^n\underline{u}) < 1 \quad \forall n \geq 0\,.$$

For our fixed $\underline{u}$, we construct in Theorem 6.20 a critical $\widehat{\beta}$ such that this equation is true for all $\overline{\beta} > \widehat{\beta}$ and false for all $\overline{\beta} < \widehat{\beta}$. This critical $\widehat{\beta}$ is constructed using $\widehat{\underline{u}}$. In Theorem 6.21, we find a critical sequence $\underline{u}_*$ such that the algorithm applied to $\underline{v} \succ \underline{u}_*$ gives $\overline{\beta} > \widehat{\beta}$.
Example. We consider the strings $\underline{u}' = (01)^\infty$ and $\underline{v}' = (110)^\infty$. One can prove that $\underline{u}' = \underline{u}^{\alpha,\beta}$ and $\underline{v}' = \underline{v}^{\alpha,\beta}$ where β is the largest root of

$$x^3 - x - 1 = 0$$

and $\alpha = (1+\beta)^{-1}$. With the notations of Proposition 6.4 we have

$$\underline{a} = 01 \quad \underline{a}' = 00 \quad \underline{b} = 110 \quad \underline{b}' = 111\,.$$

Let

$$\underline{u} := (00110111)^\infty = (\underline{a}'\underline{b}\underline{b}')^\infty\,.$$

We have

$$\widehat{\underline{u}} = (11100110)^\infty = (\underline{b}'\underline{a}'\underline{b})^\infty\,.$$

By definition $\overline{\varphi}_\infty^{\alpha,\beta}(\sigma\underline{u}) = \alpha$. We have

$$(\underline{b})^\infty \preceq \widehat{\underline{u}} \preceq \underline{b}'(\underline{a})^\infty\,.$$

From Proposition 6.4 and Proposition 6.15, we conclude that $\log\beta = h_{\rm top}(\Sigma(\underline{u}, \widehat{\underline{u}}), \sigma)$.

Theorem 6.20. *Let $k \geq 2$ and let $\underline{u} \in \Sigma_k^+$ and $\underline{v} \in \Sigma_k^+$, such that $u_0 = 0$, $v_0 = k-1$ and (6.21) holds. If $k = 2$ we also assume that $\sigma\underline{u} \preceq \sigma\underline{v}$. Set $\log\widehat{\beta} := h_{\rm top}(\Sigma(\underline{u}, \widehat{\underline{u}}), \sigma)$. Let $\overline{\alpha}$ and $\overline{\beta}$ be defined by the algorithm of Proposition 6.10. Then*

1. *If $\widehat{\beta} < \overline{\beta}$, then $\underline{u} = \underline{u}^{\overline{\alpha},\overline{\beta}}$ and $\underline{v} = \underline{v}^{\overline{\alpha},\overline{\beta}}$.*
2. *If $\widehat{\beta} = \overline{\beta} > 1$ and $\underline{u}^{\overline{\alpha},\overline{\beta}}$ and $\underline{v}^{\overline{\alpha},\overline{\beta}}$ are not both periodic, then $\underline{u} = \underline{u}^{\overline{\alpha},\overline{\beta}}$ and $\underline{v} = \underline{v}^{\overline{\alpha},\overline{\beta}}$.*
3. *If $\widehat{\beta} = \overline{\beta} > 1$ and $\underline{u}^{\overline{\alpha},\overline{\beta}}$ and $\underline{v}^{\overline{\alpha},\overline{\beta}}$ are both periodic, then $\underline{u} \neq \underline{u}^{\overline{\alpha},\overline{\beta}}$ and $\underline{v} \neq \underline{v}^{\overline{\alpha},\overline{\beta}}$.*

Proof: Let $\widehat{\beta} < \overline{\beta}$. Suppose that $\underline{u} \neq \underline{u}^{\overline{\alpha},\overline{\beta}}$ or $\underline{v} \neq \underline{v}^{\overline{\alpha},\overline{\beta}}$. By Proposition 6.18 $\underline{u} \neq \underline{u}^{\overline{\alpha},\overline{\beta}}$ and $\underline{v} \neq \underline{v}^{\overline{\alpha},\overline{\beta}}$, and there exists n such that $\overline{\varphi}_\infty^{\overline{\alpha},\overline{\beta}}(\sigma^n\underline{u}) = 1$. Hence $\overline{\varphi}_\infty^{\overline{\alpha},\overline{\beta}}(\widehat{\underline{u}}) = 1$. If $\overline{\gamma} > 0$, then $\widehat{\underline{u}}_0 = v_0 = k-1$ whence $\sigma\widehat{\underline{u}} \preceq \sigma\underline{v}$, so that $\overline{\varphi}_\infty^{\overline{\alpha},\overline{\beta}}(\sigma\widehat{\underline{u}}) = \overline{\gamma}$. By Propositions 6.4 and 6.15 we deduce that

$$\log\widehat{\beta} = h_{\rm top}(\Sigma(\underline{u}, \widehat{\underline{u}}), \sigma) = h_{\rm top}(\Sigma(\underline{u}, \underline{v}), \sigma) = \log\overline{\beta}\,,$$

a contradiction. If $\overline{\gamma} = 0$, either $\widehat{\underline{u}}_0 = k-1$ and $\overline{\varphi}_\infty^{\overline{\alpha},\overline{\beta}}(\sigma\widehat{\underline{u}}) = \overline{\gamma}$, and we get a contradiction as above, or $\widehat{\underline{u}}_0 = k-2$ and $\overline{\varphi}_\infty^{\overline{\alpha},\overline{\beta}}(\sigma\widehat{\underline{u}}) = 1$. In the latter case, since $\sigma\widehat{\underline{u}} \preceq \widehat{\underline{u}}$, we conclude that $\widehat{\underline{u}}_1 = k-2$ and $\overline{\varphi}_\infty^{\overline{\alpha},\overline{\beta}}(\sigma^2\widehat{\underline{u}}) = 1$. Using $\sigma^n\widehat{\underline{u}} \preceq \widehat{\underline{u}}$ we get $\widehat{\underline{u}} = (k-2)^\infty = \underline{v}^{\overline{\alpha},\overline{\beta}}$, so that $h_{\rm top}(\Sigma(\underline{u}, \widehat{\underline{u}}), \sigma) = h_{\rm top}(\Sigma(\underline{u}, \underline{v}), \sigma)$, a contradiction.
We prove 2. Suppose for example that $\underline{u}^{\overline{\alpha},\overline{\beta}}$ is not periodic. This implies that $\overline{\alpha} < 1$, so that Proposition 6.2 implies that $\underline{u} = \underline{u}^{\overline{\alpha},\overline{\beta}}$. We conclude using Proposition 6.18. Similar proof if $\underline{v}^{\overline{\alpha},\overline{\beta}}$ is not periodic.
We prove 3. By Proposition 6.18, $\underline{u} = \underline{u}^{\overline{\alpha},\overline{\beta}}$ or $\underline{v} = \underline{v}^{\overline{\alpha},\overline{\beta}}$ if and only if $\underline{u} = \underline{u}^{\overline{\alpha},\overline{\beta}}$ and $\underline{v} = \underline{v}^{\overline{\alpha},\overline{\beta}}$. Suppose $\underline{u} = \underline{u}^{\overline{\alpha},\overline{\beta}}$, then $\underline{u}$ is periodic so that $\widehat{u} = \sigma^p\underline{u}$ for some p. This implies that

$$\overline{\varphi}_\infty^{\overline{\alpha},\overline{\beta}}(\sigma\widehat{\underline{u}}) \leq \overline{\varphi}_\infty^{\overline{\alpha},\overline{\beta}}(\widehat{\underline{u}}) = \overline{\varphi}_\infty^{\overline{\alpha},\overline{\beta}}(\sigma^p\underline{u}) < 1\,.$$

by Proposition 6.18. Let $\widehat{\underline{u}}_0 \equiv \widehat{k} - 1$. We can apply the algorithm of Proposition 6.10 to the pair $(\underline{u}, \widehat{\underline{u}})$ and get two real numbers $\widetilde{\alpha}$ and $\widetilde{\beta}$ (if $\widehat{k} = 2$, using $\widehat{\beta} > 1$ and Theorem 6.16, we have $\sigma\underline{u} \preceq \sigma\widehat{\underline{u}}$). Theorem 6.16 implies $\widehat{\beta} = \widetilde{\beta}$, whence $\widetilde{\beta} = \overline{\beta}$. The map $\alpha \mapsto \overline{\varphi}_\infty^{\alpha,\overline{\beta}}(\sigma\underline{u})$ is continuous and decreasing, so that $\alpha \mapsto \overline{\varphi}_\infty^{\alpha,\overline{\beta}}(\sigma\underline{u}) - \alpha$ is strictly decreasing, whence there exists a unique solution to the equation $\overline{\varphi}_\infty^{\alpha,\overline{\beta}}(\sigma\underline{u}) - \alpha = 0$, which is $\overline{\alpha} = \widetilde{\alpha}$. Therefore $\overline{\varphi}_\infty^{\overline{\alpha},\overline{\beta}}(\sigma\widehat{\underline{u}}) < 1$ and we must have $\widehat{k} = k$, whence

$$\overline{\varphi}_\infty^{\overline{\alpha},\overline{\beta}}(\sigma\widehat{\underline{u}}) = \overline{\alpha} + \overline{\beta} - k + 1 = \overline{\varphi}_\infty^{\overline{\alpha},\overline{\beta}}(\sigma\underline{v})\,.$$

But this implies $\overline{\varphi}_\infty^{\overline{\alpha},\overline{\beta}}(\widehat{\underline{u}}) = 1$, a contradiction. $\square$

Theorem 6.21. *Let $k \geq 2$ and let $\underline{u} \in \Sigma_k^+$ and $\underline{v} \in \Sigma_k^+$, such that $u_0 = 0$, $v_0 = k-1$ and (6.21) holds. If $k = 2$ we also assume that $\sigma\underline{u} \preceq \sigma\underline{v}$. Let $\overline{\alpha}$ and $\overline{\beta}$ be defined by the algorithm of Proposition 6.10. If $h_{\rm top}(\Sigma(\underline{u}, \widehat{\underline{u}}), \sigma) > 1$, then there exists $\underline{u}_* \succeq \widehat{\underline{u}}$ such that*

$$\underline{u}_* \prec \underline{v} \implies \underline{u} = \underline{u}^{\overline{\alpha},\overline{\beta}} \text{ and } \underline{v} = \underline{v}^{\overline{\alpha},\overline{\beta}}$$
$$\underline{u}_* \succ \underline{v} \implies \underline{u} \neq \underline{u}^{\overline{\alpha},\overline{\beta}} \text{ and } \underline{v} \neq \underline{v}^{\overline{\alpha},\overline{\beta}}$$

Proof: As in the proof of Theorem 6.20 we define $\widetilde{k}$ and, by the algorithm of Proposition 6.10 applied to the pair $(\underline{u}, \widehat{\underline{u}})$, two real numbers $\widetilde{\alpha}$ and $\widetilde{\beta}$. By Theorem 6.16, $\log \widetilde{\beta} = h_{\mathrm{top}}(\Sigma(\underline{u}, \widehat{\underline{u}}), \sigma)$. We set

$$\underline{u}_* := \begin{cases} \underline{v}_*^{\widetilde{\alpha},\widetilde{\beta}} & \text{if } \underline{v}^{\widetilde{\alpha},\widetilde{\beta}} \text{ is periodic} \\ \underline{v}^{\widetilde{\alpha},\widetilde{\beta}} & \text{if } \underline{v}^{\widetilde{\alpha},\widetilde{\beta}} \text{ is not periodic}. \end{cases}$$

It is sufficient to show that $\underline{u}_* \prec \underline{v}$ implies $\overline{\beta} > \widetilde{\beta}$ (see Theorem 6.20 point 2). Suppose the contrary, $\overline{\beta} = \widetilde{\beta}$. Then

$$1 = \overline{\varphi}_\infty^{\widetilde{\alpha},\overline{\beta}}(\widehat{\underline{u}}) \leq \overline{\varphi}_\infty^{\widetilde{\alpha},\overline{\beta}}(\underline{v}).$$

We have $\overline{\varphi}_\infty^{\overline{\alpha},\overline{\beta}}(\underline{v}) = 1$ and for $\alpha > \overline{\alpha}$, $\overline{\varphi}_\infty^{\alpha,\overline{\beta}}(\underline{v}) < 1$ (see Lemma 6.6). Therefore $\widetilde{\alpha} \leq \overline{\alpha}$. On the other hand, applying Corollary 6.11 we get $\widetilde{\alpha} \geq \overline{\alpha}$ so that $\widetilde{\alpha} = \overline{\alpha}$ and $\widetilde{k} = k$. From Propositions 6.3 or 6.4, we get $\underline{v} \preceq \underline{u}_*$, a contradiction.

Suppose that $\underline{u}_* \succ \underline{v}$. We have $\widehat{u} \preceq \underline{v} \prec \underline{u}_*$, whence $h_{\mathrm{top}}(\Sigma(\underline{u}, \widehat{\underline{u}}), \sigma) = h_{\mathrm{top}}(\Sigma(\underline{u}, \underline{u}_*), \sigma)$ and therefore $\overline{\beta} = \widetilde{\beta}$. As above we show that $\overline{\alpha} = \widetilde{\alpha}$. Notice that if $\underline{u}^{\widetilde{\alpha},\widetilde{\beta}}$ is not periodic, then by Proposition 6.2 $\underline{u}^{\widetilde{\alpha},\widetilde{\beta}} = \underline{u}$. If $\underline{v}^{\widetilde{\alpha},\widetilde{\beta}}$ is not periodic, then by Proposition 6.3 $\underline{v}^{\widetilde{\alpha},\widetilde{\beta}} = \underline{v}$. If $\underline{v}^{\widetilde{\alpha},\widetilde{\beta}}$ is periodic, then inequalities (6.21) imply that we must have $\underline{v}_*^{\widetilde{\alpha},\widetilde{\beta}} \prec \underline{v}$. Therefore we may have $\underline{u}_* \succ \underline{v}$ and inequalities (6.21) only if $\underline{u}^{\widetilde{\alpha},\widetilde{\beta}}$ and $\underline{v}^{\widetilde{\alpha},\widetilde{\beta}}$ are periodic. Suppose that it is the case. If $\underline{u}$ is not periodic, then using Proposition 6.18 the second statement is true. If $\underline{u}$ is periodic, then $\widehat{\underline{u}} = \sigma^p \underline{u}$ for some p, whence $\overline{\varphi}_\infty^{\widetilde{\alpha},\widetilde{\beta}}(\underline{u}) = 1$; by Proposition 6.18, $\underline{u} \neq \underline{u}^{\widetilde{\alpha},\widetilde{\beta}}$. □

6.2 Generalized β-transformations

We consider the map of Example B. We recall briefly some facts we have already presented. Fix $k \geq 2$, $\beta \in (k-1, k]$ and a map $s : \mathtt{A}_k \to \{1, -1\}$. We often write the map $s(j)$ as a vector with k coordinates. The set S_0 is given by

$$a_j = \frac{j}{\beta} \text{ for } j \in \mathtt{A}_k, \quad a_k = 1.$$

The generalized β-transformation T_β is defined by

$$T_\beta(x) = \begin{cases} \beta x - j & \text{if } x \in I_j \text{ and } s(j) = +1, \\ 1 - (\beta x - j) & \text{if } x \in I_j \text{ and } s(j) = -1. \end{cases}$$

Notice that T_β depends on the map s, however we do not write this dependence explicitly in T_β as well as in all the notations which follow. The corresponding coding map is $\mathtt{i}^\beta$ and the representation function is $\overline{\varphi}^\beta$. There is only one important virtual itinerary

$$\underline{\eta}^\beta := \lim_{x \uparrow 1} \mathtt{i}^\beta(x).$$

All other virtual itineraries can be expressed with the help of $\underline{\eta}^\beta$ (see (2.18) and (2.19)). The shift space obtained by the coding is

$$\Sigma_\beta := \overline{\mathtt{i}^\beta(X)} = \{\underline{x} \in \Sigma_k^+ : \sigma^n \underline{x} \preceq \underline{\eta}^\beta \ \forall n \geq 0\}. \tag{6.22}$$

Moreover the virtual itinerary $\underline{\eta}^\beta$ satisfies

$$\sigma^n \underline{\eta}^\beta \preceq \underline{\eta}^\beta \quad \forall n \geq 0\,. \tag{6.23}$$

Notice that $\beta \mapsto \underline{\eta}^\beta$ is a strictly increasing map. Indeed, consider $\beta < \beta' \in (k-1, k]$, then

$$h_{\text{top}}(\Sigma_\beta, \sigma) = \log \beta < \log \beta' = h_{\text{top}}(\Sigma_{\beta'}, \sigma)\,.$$

Moreover by (6.22), we have

$$\underline{\eta} \preceq \underline{\eta}' \iff \Sigma_{\underline{\eta}} \subset \Sigma_{\underline{\eta}'}\,,$$

thus $\underline{\eta}^\beta \prec \eta^{\beta'}$. Finally notice that, if $s = (1, -1)$ or $s = (-1, -1)$, then $\underline{\eta}^\beta = 10\ldots$. Indeed, in both cases $T_\beta(1) = 2 - \beta$ and $a_1 = \frac{1}{\beta}$. Moreover for $\beta \in (1, 2]$, we have

$$2 - \beta < \frac{1}{\beta} \iff \beta^2 - 2\beta + 1 = (\beta - 1)^2 > 0\,.$$

This remark must be compared to Lemma II.2.1 in [CE], where the situation is considered in a combinatorial point of view.

Before studying in detail some properties of the generalized β-transformations, we must introduce the tent map, which is an extensively studied particular case of the generalized β-transformations (see for example [DGP], [CE] or [MS]). Let $c = 1/2$, $\beta \in (1, 2]$ and define the tent map $g_\beta : [0, 1] \to [0, 1]$ by

$$g_\beta(x) = \begin{cases} \beta x & \text{if } x \leq c, \\ \beta(1 - x) & \text{if } x \geq c. \end{cases}$$

The map g_β is continuous and it is a member of the family of unimodal maps. Let $c_n = c_n(\beta) = g_\beta^n(c)$ for all $n \geq 1$. The non-wandering set $\Omega(g_\beta)$ of g_β is included in

$$\Omega(g_\beta) \subset \{0\} \cup [c_2, c_1]\,.$$

If $\beta \in (\sqrt{2}, 2]$, then $\Omega(g_\beta)$ is exactly this set. The interval $[c_2, c_1]$ is called the core of the tent map g_β. The non trivial dynamic is concentrated on the core. Finally we recall the concept of **renormalization** in the special case of the tent map (see [MS]). Suppose that $\beta \in (2^{(2^{-m})}, 2^{(2^{-m+1})}]$ for some $m \geq 1$. Then, there are m intervals $J_1, \ldots, J_m$ with disjoint interior such that:

1. $g_\beta(J_i) = J_{i+1}$ for all $i = 1, \ldots, m-1$ and $g_\beta(J_m) = J_1$,
2. for all $i = 1, \ldots, m$, the map $g_\beta^m|_{J_i}$ is conjugated to the map g_{β^m} by an affine map from J_i onto $[0, 1]$.

In other words, all asymptotic properties of the tent maps can be established for $\beta \in (\sqrt{2}, 2]$, then extended on $\beta \in (1, 2]$ using the renormalization. This concludes our brief reminder about the tent map.

As we will see in this section, the different results about generalized β-transformations are easier established when $\beta > 2$. Thus we will often consider separately the cases $k \geq 3$ (ie $\beta > 2$) and $k = 2$. This latter case splits in four cases:

1. If $s = (+1, +1)$: this case, corresponding to the β-transformations $x \mapsto \beta x \mod 1$, is the easiest.

2. If $s = (+1, -1)$: this case corresponds to a tent map of slope β. Indeed the map T_β is conjugated (by an affine map preserving the orientation) to $g_\beta|_{[0,c_1]}$. All results we prove are known for the tent maps. The structure of the proofs is often the same: we prove a result for all $\beta \in (\sqrt{2}, 2]$ and then extend this result on $\beta \in (1, 2]$ thanks to the renormalization.

3. If $s = (-1, +1)$: this case also corresponds to a tent map of slope β. Indeed the map T_β is conjugated (by an affine map changing the orientation) to $g_\beta|_{[c_2,c_1]}$. We never consider this case, because it is conjugated to the case $s = (+1, -1)$ restricted to a well-chosen interval.

4. If $s = (-1, -1)$: this case is the most difficult. The techniques used for the cases 2 and 3 often work for all $\beta \in (\beta_0, 2]$ with $\beta_0 > 1$; but contrary to the tent map case, we cannot use the renormalization to extend the proofs on $\beta \in (1, 2]$.

In a recent paper [G], Góra studied the generalized β-transformations; his approach is very similar to the one of Parry's papers [P1] and [P2]. By the general theorem of Lasota and Yorke, we know that all generalized β-transformations T_β admit a T_β-invariant probability measure which is absolutely continuous with respect to Lebesgue measure. For all generalized β-transformations T_β, Góra constructed the density (with respect to Lebesgue measure) of a T_β-invariant probability measure μ_β and he proved that μ_β is the unique measure absolutely continuous with respect to Lebesgue measure.

6.2.1 Uniqueness of the maximal measure

We prove that the map T_β has a unique measure of maximal entropy in almost all cases. The only gap is $k = 2$ and $s(j) = (-1, -1)$, where the uniqueness of the maximal measure is proved only for $\beta \in (\sqrt[3]{2}, 2]$. We use the method presented in chapter 5 and in particular Theorem 5.15. Thus the order $\preceq$ being fixed, we choose $\underline{\eta} \in \Sigma_k^+$ such that

$$\sigma^n \underline{\eta} \preceq \underline{\eta} \quad \forall n \geq 0\,,$$

and we study the graph $\mathcal{G}(\underline{\eta})$ defined in Example B at the end of chapter 4. Recall the relations (4.9) and (4.10), expressing all virtual itineraries $\underline{u}^j$ and $\underline{v}^j$ with the help of $\underline{\eta}$. With these notations

$$\Sigma(\{\underline{u}^j, \underline{v}^j\}) := \{\underline{x} \in \Sigma_k^+ : \underline{u}^{x_n} \preceq \sigma^n \underline{x} \preceq \underline{v}^{x_n} \ \forall n \geq 0\} = \{\underline{x} \in \Sigma_k^+ : \sigma^n \underline{x} \preceq \underline{\eta} \ \forall n \geq 0\} =: \Sigma_{\underline{\eta}}\,.$$

We denote by $\mathcal{G}(\Sigma(\{\underline{u}^j, \underline{v}^j\}))$ the Markov diagram of $\Sigma(\{\underline{u}^j, \underline{v}^j\})$. Recall also that $\sim_1$ is an equivalence relation on the quadruples defined by

$$(p, \underline{a}; q, \underline{b}) \sim_1 (p', \underline{a}'; q', \underline{b}') \iff [\sigma^p \underline{a}, \sigma^q \underline{b}] = [\sigma^{p'} \underline{a}', \sigma^{q'} \underline{b}']\,.$$

We define a second equivalence relation on the vertices of $\mathcal{G}(\Sigma(\{\underline{u}^j, \underline{v}^j\}))$ by

$$c \equiv (p, \underline{a}; q, \underline{b}) \sim_2 c' \equiv (p', \underline{a}'; q', \underline{b}') \iff (p, \underline{a}; q, \underline{b}) \sim_1 (p', \underline{a}'; q', \underline{b}') \text{ and all incoming edges at } c \text{ and } c' \text{ carry the same label}\,.$$

The graph $\mathcal{G}(\underline{\eta})$ is the simplification of the Markov diagram according to the equivalence relation $\sim_2$. Finally, recall that we write

$$\underline{0} \equiv \underline{u}^0 := \lim_{x \downarrow 0} \mathtt{i}^\beta(x) \quad \text{and} \quad \underline{\eta} \equiv \underline{v}^{k-1} := \lim_{x \uparrow 1} \mathtt{i}^\beta(x)\,.$$

If $s(0) = 1$, then $\underline{0} = 0^\infty$; if $s(0) = -1$, then $\underline{0} = 0\underline{\eta}$.

For Section 6.2.1, we modify slightly the notation so that it is better suited for the study of $\mathcal{G}(\underline{\eta})$. Notice that the root of $\mathcal{G}(\Sigma(\{\underline{u}^j, \underline{v}^j\}))$ is the quadruple $(0, \underline{0}; 0, \underline{\eta})$. The other vertices of $\mathcal{G}(\Sigma(\{\underline{u}^j, \underline{v}^j\}))$ are of the type $(p, \underline{a}; q, \underline{b})$ with $p, q \geq 1$ and $\underline{a}, \underline{b} \in U \cup V$. We consider three cases:

1. If $\underline{a} \notin \{\underline{0}, \underline{\eta}\}$ and $\underline{0} = 0^\infty$, then $(p, \underline{a}; q, \underline{b}) \sim_1 (p-1, \underline{a}'; q, \underline{b})$ with $\underline{a}' \in \{\underline{0}, \underline{\eta}\}$. But $\underline{\eta}$ being the maximal element of $\Sigma(\{\underline{u}^j, \underline{v}^j\})$, we have $\underline{a}' = \underline{0}$; thus $(p, \underline{a}; q, \underline{b}) \sim_1 (0, \underline{0}; q, \underline{b})$.

2. If $\underline{a} \notin \{\underline{0}, \underline{\eta}\}$ and $\underline{0} = 0\underline{\eta}$, then $(p, \underline{a}; q, \underline{b}) \sim_1 (p-1, \underline{a}'; q, \underline{b})$ with $\underline{a}' \in \{\underline{0}, \underline{\eta}\}$. But $\underline{\eta}$ being the maximal element of $\Sigma(\{\underline{u}^j, \underline{v}^j\})$, we have $\underline{a}' = \underline{0}$; thus $(p, \underline{a}; q, \underline{b}) \sim_1 (p-1, \underline{0}; q, \underline{b})$. If $p \geq 2$, then $(p, \underline{a}; q, \underline{b}) \sim_1 (p-2, \underline{\eta}; q, \underline{b})$.

3. If $\underline{b} \notin \{\underline{u}^0, \underline{v}^{k-1}\}$, then $(p, \underline{a}; q, \underline{b}) \sim_1 (p, \underline{a}; q-1, \underline{b}')$ with $\underline{b}' \in \{\underline{0}, \underline{\eta}\}$. But $\underline{0}$ being the minimal element of $\Sigma(\{\underline{u}^j, \underline{v}^j\})$, we have $\underline{b}' = \underline{\eta}$; thus $(p, \underline{a}; q, \underline{b}) \sim_1 (p, \underline{a}; q-1, \underline{\eta})$.

Looking at the three cases, we see that for all vertices $c = (p, \underline{a}; q, \underline{b})$ of $\mathcal{G}(\Sigma(\{\underline{u}^j, \underline{v}^j\}))$, there is a quadruple $(p', \underline{a}'; q', \underline{\eta}) \sim_1 (p, \underline{a}; q, \underline{b})$ with $p', q' \geq 0$, $\underline{a}' \in \{\underline{0}, \underline{\eta}\}$ and, if $\underline{a}' = \underline{0}$, then $p' = 0$. We are now able to define our new notation. This notation has two main characteristics. First it contains a supplementary information, the label of all edges incoming at a vertex; this is useful to deal with the equivalence relation $\sim_2$. Secondly there is a precise rule to choose between all equivalent (for $\sim_1$) notations of a quadruple. Henceforth, the root of $\mathcal{G}(\Sigma(\{\underline{u}^j, \underline{v}^j\}))$ is denoted by $(\mathcal{R}; 0, \underline{0}, 0, \underline{\eta})$ (recall that there is no incoming edge at the root); the symbol $\mathcal{R}$ is an exceptional symbol identifying the root. A vertex $c = (p, \underline{a}; q, \underline{b})$ of $\mathcal{G}(\Sigma(\{\underline{u}^j, \underline{v}^j\}))$ (except for the root) is denoted by $(a_{p-1}; p', \underline{a}'; q', \underline{\eta})$ with $p', \underline{a}', q'$ defined by:

1. $(p', \underline{a}'; q', \underline{\eta}) \sim_1 (p, \underline{a}; q, \underline{b})$,

2. $p', q' \geq 0$, p', q' are minimal, $\underline{a}' \in \{\underline{0}, \underline{\eta}\}$ and $p' = 0$ if $\underline{a}' = \underline{0}$.

Notice that this notation is well-defined: since c is not the root, $p \geq 1$. Moreover, there exist unique $p', q', \underline{a}'$ satisfying the requests. We get $\mathcal{G}(\underline{\eta})$ by identifying the vertices of $\mathcal{G}(\Sigma(\{\underline{u}^j, \underline{v}^j\}))$ having the same (new) notation. As before, we use an unordered notation

$$\langle j; p, \underline{a}; q, \underline{b} \rangle = \begin{cases} (j; p, \underline{a}; q, \underline{b}) & \text{if } \sigma^p \underline{a} \preceq \sigma^q \underline{b}\,, \\ (p; q, \underline{b}; p, \underline{a}) & \text{if } \sigma^p \underline{a} \succ \sigma^q \underline{b} \end{cases}.$$

The next proposition gives the main properties of the graph $\mathcal{G}(\underline{\eta})$.

Proposition 6.22. *The root of $\mathcal{G}(\underline{\eta})$ is the vertex $(\mathcal{R}; 0, \underline{0}; 0, \underline{\eta})$. All other vertices of $\mathcal{G}(\underline{\eta})$ are of the type $(j; p, \underline{a}; q, \underline{\eta})$ with $j \in \mathtt{A}_k$, $p, q \geq 0$, $\underline{a} \in \{\underline{0}, \underline{\eta}\}$; moreover, $p = 0$ if $\underline{a} = \underline{0}$. The root has k successors. For all $0 \leq j \leq k-2$, the j-successor of the root is the vertex $(j; 0, \underline{0}; 0, \underline{\eta})$. The $k-1$-successor of the root is the vertex $(k-1; 0, \underline{0}; 1, \underline{\eta})$ if $s(k-1) = 1$, it is the vertex $(k-1; 1, \underline{\eta}; 0, \underline{\eta})$ if $s(k-1) = -1$. Let $c = (j; p, \underline{a}; q, \underline{\eta})$ be a vertex of $\mathcal{G}(\underline{\eta})$, except for the root. If c has only one successor c', ie $a_p = \eta_q$, then*

$$c \xrightarrow{a_p} c' = \begin{cases} \langle a_p; p+1, \underline{a}; q+1, \underline{\eta} \rangle & \text{if } \underline{a} = \underline{\eta}, \\ (a_p; 0, \underline{0}; q+1, \underline{\eta}) & \text{if } \underline{a} = \underline{0} \text{ and } s(0) = +1, \\ (a_p; q+1, \underline{\eta}; 0, \underline{\eta}) & \text{if } \underline{a} = \underline{0} \text{ and } s(0) = -1. \end{cases}$$

If c has at least two successors, ie $a_p < \eta_q$, then

$$c \overset{\eta_q}{\to} \begin{cases} (\eta_q; 0, \underline{0}; q+1, \underline{\eta}) & \textit{if } s(\eta_q) = +1, \\ (\eta_q; q+1, \underline{\eta}; 0, \underline{\eta}) & \textit{if } s(\eta_q) = -1, \end{cases}$$

$$c \overset{j}{\to} (j; 0, \underline{0}; 0, \underline{\eta}) \quad \forall a_p \leq j < \eta_q\,.$$

Proof: We apply Lemmas 4.3, 4.5, the equivalence relation $\sim_2$ and our convention of notation. □

If $s = (1, -1)$ or $s = (-1, -1)$, we already noticed that $\underline{\eta}^\beta = 10\ldots$. In Figure 6.6, we draw the beginning of the graphs $\mathcal{G}(\underline{\eta})$ in both cases.

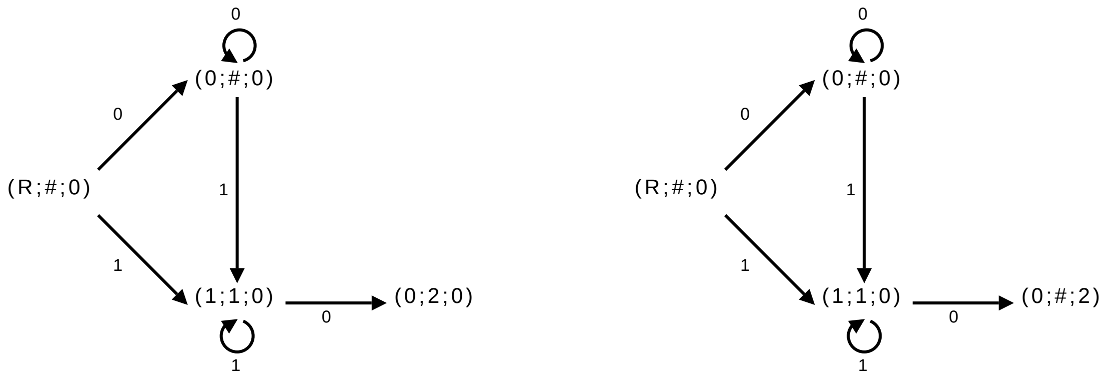

Figure 6.6: The beginning of $\mathcal{G}(\underline{\eta})$. On the left, the case $s = (1, -1)$ and $\underline{\eta} = 10\ldots$; on the right, the case $s = (-1, -1)$ and $\underline{\eta} = 10\ldots$. The notation $(j; p; q)$ stands for the vertex $(j; p, \underline{\eta}; q, \underline{\eta})$ and the notation $(j; \#; q)$ for the vertex $(j; 0, \underline{0}; q, \underline{\eta})$.

Theorem 6.23. *Let $k \geq 2$, $s : \mathtt{A}_k \to \{1, -1\}$ and $\beta \in (k-1, k]$ and consider the corresponding generalized β-transformation T_β. If $s \neq (-1, -1)$, then T_β has a unique measure of maximal entropy. If $s = (-1, -1)$ and $\beta > \sqrt[3]{2}$, then T_β has a unique measure of maximal entropy.*

Proof: Let $\underline{\eta} = \underline{\eta}^\beta$. By Theorem 5.15, we must show that $\mathcal{G}(\underline{\eta})$ has a unique communicating class of maximal spectral radius. Let $c = (j; p, \underline{a}; q, \underline{\eta})$ be a vertex of $\mathcal{G}(\underline{\eta})$. Let $|c|$ denote the length of the corresponding interval in $[0, 1]$, ie

$$|c| = |\overline{\varphi}^\beta_\infty(\sigma^q \underline{\eta}) - \overline{\varphi}^\beta_\infty(\sigma^p \underline{a})|\,.$$

By the monotonicity of the map $x \mapsto \mathtt{i}^\beta(x)$, we have that $x \in X \cap [\overline{\varphi}^\beta_\infty(\sigma^p \underline{a}), \overline{\varphi}^\beta_\infty(\sigma^q \underline{b})]$ implies $\mathtt{i}^\beta(x) \in [\sigma^p \underline{a}, \sigma^q \underline{b}]$. Now we distinguish the cases.

1. If $\beta > 2$, we prove that all vertices of $\mathcal{G}(\underline{\eta})$ without the root are a communicating class. Let V denote the set of vertices indexed by j for $j < k-1$. From any vertex of V, we can reach in one step all successors of the root. If we prove that there is a path from each vertex c_0 to a vertex of V, we are done. Let c_0 be a vertex of $\mathcal{G}(\underline{\eta})$. If c_0 has $n \geq 3$ successors, then by Lemma 6.22, $n-2$ of them are in V. Otherwise if c_0 has one successor, define c_1 as this successor; if c_0 has two successors, define c_1 as the one such that $|c_1|$ is maximal. Since the slope of T_β is $\pm\beta$, we have $|c_1| \geq \frac{\beta}{2}|c_0|$.

If c_1 has at least three successors, then at least one of them is in V. Otherwise define c_2 as the successor of c_1 such that $|c_2|$ is maximal. Continuing in this manner, we find a vertex c_n such that

$$|c_n| \geq \left(\frac{\beta}{2}\right)^n |c_0|\,.$$

Since $\beta/2 > 1$ and $|c_n| \leq 1$, this sequence is finite and there is a path from c_0 to V.

2. If $s = (1, 1)$, this is the β-transformation and it is well-know that it has a unique measure of maximal entropy (see [T] and [H1]). By the way, it is easy to see that all vertices of $\mathcal{G}(\underline{\eta})$ except for the root form a communicating class.

3. If $s = (1, -1)$. The beginning of the graph is drawn in Figure 6.6 Suppose first that $\beta \in (\sqrt{2}, 2]$. Let V be the set of all vertices of $\mathcal{G}(\underline{\eta})$ except for the root and the vertex indexed by the word 0. We prove that V is strongly connected. Let $v \in V$ be the vertex indexed by the word 1; we have $v = (1; 1, \underline{\eta}; 0, \underline{\eta})$. There is a path from v to c for all $c \in V$. Consider a vertex $c_0 \in V$. Define a sequence $\{c_i\}_i$ by induction: c_i is the successor of c_{i-1} such that $|c_i|$ is maximal. We claim that there exists n such that c_n and c_{n+1} have both two successors. If not, then for all $m \geq 1$,

$$|c_{2m}| \geq \frac{\beta^2}{2}|c_{2m-2}| \geq \cdots \geq \left(\frac{\beta^2}{2}\right)^m |c_0|\,.$$

But $\beta^2/2 > 1$ and $|c_{2m}| < 1$, thus we have a contradiction. Now suppose that $c_n = (j; p, \underline{\eta}; q, \underline{\eta})$ (notice that the case $(j; 0, \underline{0}; q, \underline{\eta})$ is impossible, because $\underline{v}^0 = 0\underline{\eta}$ and $\underline{u}^1 = 1\underline{\eta}$). We consider separately the cases $p < q$ and $p > q$. In both cases, there is a path of length 2 from c_n to the vertex $(1; 1, \underline{\eta}; 0, \underline{\eta})$ (see Figure 6.7). Thus V is strongly connected. Either V is a communicating class or we must add the vertex indexed by the word 0 to obtain a communicating class. In both cases, there is a unique communicating class of maximal spectral radius, thus by Theorem 5.15 a unique measure of maximal entropy. Using the renormalization, we extend this result to all $\beta \in (1, 2]$.

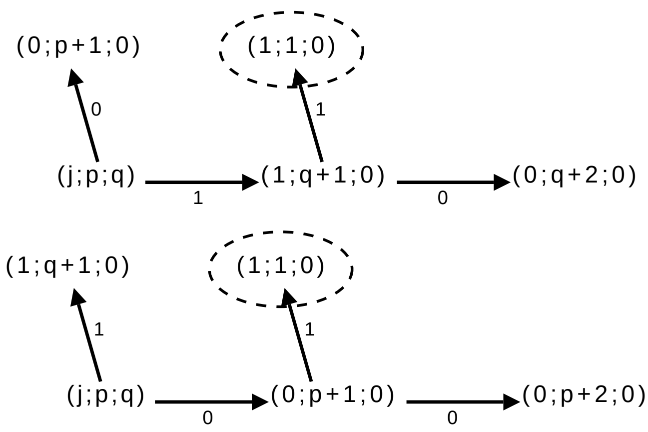

Figure 6.7: On the top, the case $p < q$; on the bottom, the case $p > q$. The notation $(j; p; q)$ stands for the vertex $(j; p, \underline{\eta}; q, \underline{\eta})$.

4. If $s = (-1, 1)$. This case is conjugated to the case $s = (1, -1)$ restricted to an appropriate interval, thus there is also a unique measure of maximal entropy.

5. If $s = (-1, -1)$, the proof is very similar to the case $s = (1, -1)$. The beginning of the graph is drawn in Figure 6.6. Suppose $\beta \in (\sqrt{2}, 2]$. Let V be the set all vertices of $\mathcal{G}(\underline{\eta})$ except for the root and the vertex indexed by the word 0. We prove that V is strongly connected. Let $v \in V$ be the vertex indexed by the word 1; we have $v = (1; 1, \underline{\eta}; 0, \underline{\eta})$. There is a path from v to c for all $c \in V$. Consider a vertex $c_0 \in V$. As in case $s = (1, -1)$, there is a path from c_0 to a vertex c_n such that c_n has two successors and one of them, called c_{n+1}, has also two successors. There are three cases to consider: $c_n = (j; 0, \underline{0}; q, \underline{\eta})$, $c_n = (j; p, \underline{\eta}; q, \underline{\eta})$ with $p > q$ and $c_n = (j; p, \underline{\eta}; q, \underline{\eta})$ with $p < q$. The first one is trivial. We illustrate the other cases in Figure 6.8. Since there is a path from the vertex $(0; 0, \underline{0}; 0, \underline{\eta})$ to $(1; 1, \underline{\eta}; 0, \underline{\eta})$, V is strongly connected. Either V is a communicating class or we must add the vertex labeled by the word 0 to have a communicating class. In both cases, there is a unique communicating class with maximal spectral radius.

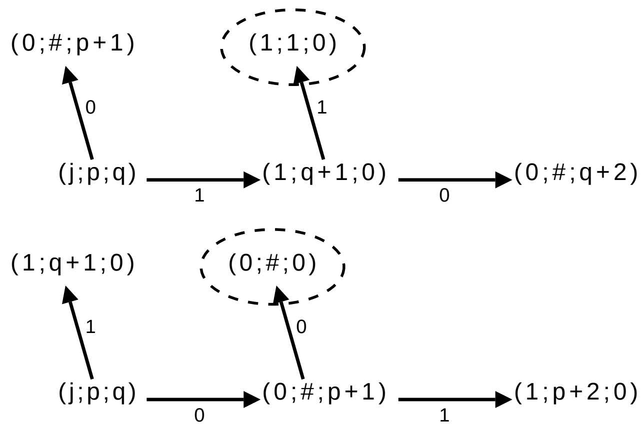

Figure 6.8: If $\beta \in (\sqrt{2}, 2]$. On the top, the case $p < q$; on the bottom, the case $p > q$. The notation $(j; p; q)$ stands for the vertex $(j; p, \underline{\eta}; q, \underline{\eta})$ and the notation $(j; \#; q)$ for the vertex $(j; 0, \underline{0}; q, \underline{\eta})$.

If $\beta \in (\sqrt[3]{2}, \sqrt{2}]$, let V' be the set V without the vertex indexed by the word 1. Let $v' \in V'$ be the vertex indexed by 10; we have $v' = (0; 0, \underline{0}; 2, \underline{\eta})$. Arguing as before, we prove that V' is strongly connected. There is a path from v' to all vertices in V'. Let $c_0 \in V'$, there exists a path from c_0 to c_n and two vertices c_{n+1}, c_{n+2} such that $c_n \to c_{n+1} \to c_{n+2}$. Moreover two vertices between c_n, c_{n+1}, c_{n+2} have two successors. If c_{n+1} is one of them, we recover the case treated in Figure 6.8. Otherwise we consider the three cases $c_n = (j; 0, \underline{0}; q, \underline{\eta})$, $c_n = (j; p, \underline{\eta}; q, \underline{\eta})$ with $p > q$ and $c_n = (j; p, \underline{\eta}; q, \underline{\eta})$ with $p < q$. The first one is trivial. We illustrate the other cases in Figure 6.9. Since there is a path from the vertex $(1; 1, \underline{\eta}; 0, \underline{\eta})$ to $(0; 0, \underline{0}; 2, \underline{\eta})$, V' is strongly connected. As before, there is a unique communicating class with maximal spectral radius.

$\square$

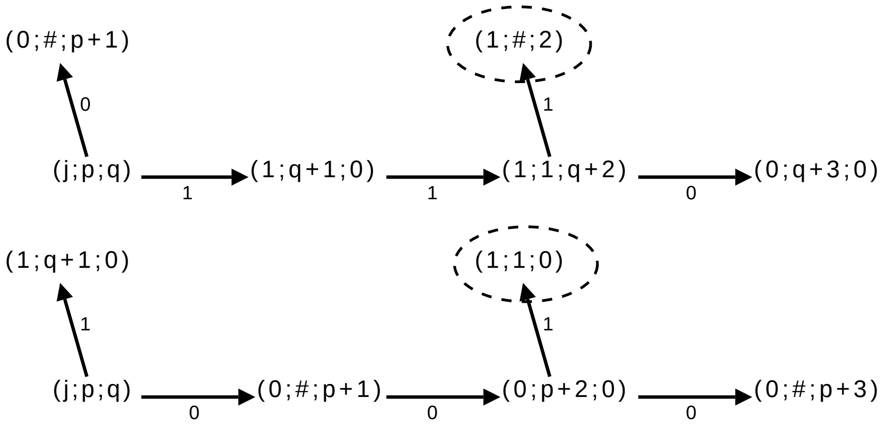

Figure 6.9: If $\beta \in (\sqrt[3]{2}, \sqrt{2}]$. On the top, the case $p < q$; on the bottom, the case $p > q$. The notation $(j;p;q)$ stands for the vertex $(j;p,\underline{\eta};q,\underline{\eta})$ and the notation $(j;\#;q)$ for the vertex $(j;0,\underline{0};q,\underline{\eta})$.

Chapter 7

Normality

This chapter is devoted to the study of the normality for the maps $T_{\alpha,\beta}$ and the generalized β-transformations. The word "normal" has several meanings. We use this word in the sense which was used by Borel when he spoke of normal number. Let (X, T) be a measurable dynamical system and consider an ergodic measure $\mu \in M(X, T)$. Roughly speaking, a point $x \in X$ is μ-normal, if the frequency of times when the orbit of x visits any set A tends to $\mu(A)$. We will make precise this definition using the weak*-topology. By the Birkhoff Ergodic Theorem, we know that μ-almost all points $x \in X$ are μ-normal. We only consider piecewise monotone continuous maps with $X = [0, 1]$. In particular, if μ is equivalent to Lebesgue measure, then Lebesgue almost all points are μ-normal. However given a point $x \in [0, 1]$, it is often very hard to answer the question: is the point x μ-normal or not? Our approach consists to fix $x \in [0, 1]$ and a family of expansions described by a parameter $\kappa \in K$. Then we estimate the size of the set of parameters $\kappa \in K$ for which x is μ_κ-normal, where μ_κ is the unique measure of maximal entropy.

First we consider the family of maps $T_{\alpha,\beta}$. Let $\mu_{\alpha,\beta} \in M([0, 1], T_{\alpha,\beta})$ denote the measure of maximal entropy. In Theorem 7.8, we prove that, for any $x \in [0, 1]$, the point x is $\mu_{\alpha,\beta}$-normal for Lebesgue almost all parameters (α, β). To this end, we prove in Theorem 7.6 an intermediate result: for all $x \in [0, 1]$ and all $\alpha \in [0, 1)$ (except for $x = \alpha = 0$), the point x is $\mu_{\alpha,\beta}$-normal for Lebesgue almost all β. In Theorem 7.10, we prove a result that seems to be paradoxical: the plane of parameters (α, β) is filled by disjoint analytic curves along which the orbit of $x = 0$ is at most at one point $\mu_{\alpha,\beta}$-normal. Finally we consider the case of generalized β-transformations T_β. Let $\mu_\beta \in M([0, 1], T_\beta)$ denote the measure of maximal entropy. In Theorem 7.13, we prove that the point $x = 1$ is μ_β-normal for almost all β. As in Theorem 6.23, there is a gap for $s = (-1, -1)$ and β too small. The proofs of Theorems 7.6 and 7.13 are very similar. They are inspired by a paper of Schmeling [S]. In that paper, Schmeling considered the β-transformations βx mod 1 and he proved that the point $x = 1$ is μ_β-normal for almost all β (μ_β is the measure of maximal entropy). Finally, notice that we recover a result of Bruin in [Br], where he proved that the turning point of a tent map g_β is μ_β-normal for almost all β (as usual, μ_β is the measure of maximal entropy). Indeed, the tent map is a particular generalized β-transformation and Theorem 7.13 is a generalization of Bruin's results.

7.1 Definitions

Let (X, d, T) be a compact measurable dynamical system (for the definitions, see Section 3.1.1). Recall that the set of Borel probability measures $M(X)$ is endowed with the weak*-topology; in particular, $M(X)$ and $M(X,T)$ are compact. For all $x \in X$ and all $n \geq 1$, the **empirical measure** of order n at x is

$$\mathcal{E}_n(x) := \frac{1}{n}\sum_{i=0}^{n-1} \delta_x \circ T^{-i} \in M(X)\,,$$

where δ_x is the Dirac mass at x. Let $V_T(x) \subset M(X,T)$ denote the set of all cluster points of $\{\mathcal{E}_n(x)\}_{n\geq 1}$ in the weak*-topology.

Definition 7.1. *Let $\mu \in M(X,T)$ be an ergodic measure and $x \in X$. The orbit of x under T is μ-**normal**, if $V_T(x) = \{\mu\}$, ie for all continuous $f \in C(X)$, we have*

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} f(T^i x) = \int f d\mu.$$

To estimate the size of sets, we use the Hausdorff dimension and the topological entropy. We recall the definition of the Hausdorff dimension $\dim_H E$; one has to compare this definition to the one of the topological entropy $h_{\mathrm{top}}(E,T)$ (see Definition 3.4). The similarity of these two definitions is the key of Lemma 7.2, which links $h_{\mathrm{top}}(E,T)$ and $\dim_H E$ for the shift spaces. Let (X,d) be a metric space and $E \subset X$. Let $\mathcal{D}_\varepsilon(E)$ be the set of all finite or countable covers of E with sets of diameter smaller than ε. For all $s \geq 0$, define

$$H_\varepsilon(E,s) := \inf\{\sum_{B\in\mathcal{C}} (\operatorname{diam} B)^s : \mathcal{C} \in \mathcal{D}_\varepsilon(E)\}\,.$$

The s-**Hausdorff outer measure** of E is $H(E,s) := \lim_{\varepsilon\to 0} H_\varepsilon(E,s)$. The **Hausdorff dimension** of E is

$$\dim_H E := \inf\{s \geq 0 : H(E,s) = 0\}\,.$$

The next lemma can be found in [Fu].

Lemma 7.2. *For $\beta > 1$, consider the dynamical system $(\Sigma_k^+, d_\beta, \sigma)$ (the metric d_β is defined by (2.1)). Let $E \subset \Sigma_k^+$ be such that $\sigma(E) \subset E$, then*

$$\dim_H E \leq \frac{h_{\mathrm{top}}(E,\sigma)}{\log\beta}\,.$$

Proof: Let $\varepsilon \in (0,1), s \geq 0, n \geq 0$ and $\mathcal{C} \in \mathcal{G}_n(E,\sigma,\varepsilon)$. Since $\operatorname{diam} B_m(x,\varepsilon) \leq \varepsilon\beta^{-m+1} \leq \varepsilon\beta^{-n+1}$ for all $B_m(x,\varepsilon) \in \mathcal{C}$, $\mathcal{C}$ is a cover of E with sets of diameter smaller than $\varepsilon\beta^{-n+1}$. Moreover

$$\sum_{B_m(x,\varepsilon)\in\mathcal{C}} \operatorname{diam}\,(B_m(x,\varepsilon))^{\frac{s}{\log\beta}} \leq (\varepsilon\beta)^{\frac{s}{\log\beta}} \sum_{B_m(x,\varepsilon)\in\mathcal{C}} e^{-ms}\,.$$

Thus $H_\delta(E, \frac{s}{\log\beta}) \leq (\varepsilon\beta)^{\frac{s}{\log\beta}} C_n(E,\sigma,\varepsilon,s)$ with $\delta = \varepsilon\beta^{-n+1}$. Taking the limit $n \to \infty$, we obtain

$$H(E, \frac{s}{\log\beta}) \leq (\varepsilon\beta)^{\frac{s}{\log\beta}} C(E,\sigma,\varepsilon,s)\,.$$

If $s > h_{\rm top}(E,\sigma,\varepsilon)$, then $H(E,\frac{s}{\log\beta}) = 0$ and $\frac{s}{\log\beta} \geq \dim_H E$. This is true for all $s > h_{\rm top}(E,\sigma,\varepsilon)$, thus

$$\dim_H E \leq \frac{h_{\rm top}(E,\sigma,\varepsilon)}{\log\beta} \leq \frac{h_{\rm top}(E,\sigma)}{\log\beta}\,.$$

□

The next lemma is a classical result about the Hausdorff dimension, it is Proposition 2.3 in [F]. The proof is very similar to the one of the previous lemma.

Lemma 7.3. *Let $(X,d),(X',d')$ be two metric spaces and $\rho : X \to X'$ be an α-Hölder continuous map with $\alpha \in (0,1]$. Let $E \in X$, then*

$$\dim_H \rho(E) \leq \frac{\dim_H E}{\alpha}\,.$$

Finally we report Theorem 4.1 from [PS]. This theorem is used to estimate the topological entropy of sets we are interested in.

Theorem 7.4. *Let (X,d,T) be a compact continuous dynamical system and $F \subset M(X,T)$ be a closed subset. Define*

$$E := \{x \in X : V_T(x) \cap F \neq \emptyset\}\,.$$

Then

$$h_{\rm top}(E,T) \leq \sup_{\nu\in F} h_T(\nu)\,.$$

7.2 Normality for the map $\beta x + \alpha \mod 1$

In this section, we study the normality for the map $T_{\alpha,\beta} = \beta x + \alpha \mod 1$ with $\beta > 1$ and $\alpha \in [0,1)$. Notice that the set S, where $T_{\alpha,\beta}$ is not well-defined, depends on the parameters α,β. We want to work with $x \in [0,1]$ fixed and α,β varying. This could be a problem, because $T_{\alpha,\beta}(x)$ may be not defined for a big subset of the set of parameters. Since all laps of $T_{\alpha,\beta}$ are increasing, there is a convenient way to modify the definitions of the map $T_{\alpha,\beta}$ and the coding $\mathtt{i}^{\alpha,\beta}$, in such a way that $T_{\alpha,\beta}$ is well defined for all $x \in [0,1)$. Until here, the map $T_{\alpha,\beta}$ was defined on $X_1 \equiv [0,1]\backslash S_0$ and $T^n_{\alpha,\beta}$ was defined on $X \equiv [0,1]\backslash S$ for all $n \geq 0$. Henceforth, we extend the definition of $T_{\alpha,\beta}$ on $[0,1)$ by right-continuity. We summarize the modifications. The set S_0 remains unchanged, the intervals I_j are now defined as $I_j := [a_j, a_{j+1})$ for all $j \in \mathtt{A}_k$. The maps $f_j : I_j \to [0,1)$ are always given by

$$f_j(x) := \beta x + \alpha - j\,,$$

and the map $T_{\alpha,\beta} : [0,1) \to [0,1)$ is defined by $T_{\alpha,\beta}|_{I_j} = f_j$. Defined in this manner, the map $T_{\alpha,\beta}$ is right-continuous. The coding map $\mathtt{i}^{\alpha,\beta} : [0,1) \to \Sigma_k^+$ is defined using the intervals I_j

$$\mathtt{i}^{\alpha,\beta}(x) := \mathtt{i}_0^{\alpha,\beta}(x)\mathtt{i}_1^{\alpha,\beta}(x)\ldots \quad \text{with } \mathtt{i}_n^{\alpha,\beta}(x) = j \iff T^n_{\alpha,\beta}(x) \in I_j.$$

As in Lemma 2.4, we can prove that $\mathtt{i}^{\alpha,\beta}$ is right-continuous on $[0,1)$. The map $\overline{\varphi}^{\alpha,\beta} : [0,k] \to [0,1]$ remains unchanged; in particular, the map $\overline{\varphi}^{\alpha,\beta}_\infty : \Sigma_k^+ \to [0,1]$ is continuous by Theorem 2.12. Using the right-continuity of $\mathtt{i}^{\alpha,\beta}$ and the continuity of $\overline{\varphi}_{\infty}^{\alpha,\beta}$, we check that the point 2 of Theorem 2.15 becomes

$$T^n_{\alpha,\beta}(x) = \overline{\varphi}_{\infty}^{\alpha,\beta} \circ \sigma^n \circ \mathtt{i}^{\alpha,\beta}(x) \quad \forall x \in [0,1)\,. \tag{7.1}$$

The definition of the virtual itineraries $\underline{u}^{\alpha,\beta}$ and $\underline{v}^{\alpha,\beta}$ remains unchanged. It is easy to check that $\underline{u}^{\alpha,\beta} = \mathtt{i}^{\alpha,\beta}(0)$, $\overline{\mathtt{i}^{\alpha,\beta}([0,1))} = \Sigma_{\alpha,\beta}$ and the inequalities (6.4) remain true. Finally, notice that it is also possible to extend the definitions of $T_{\alpha,\beta}$ and $\mathtt{i}^{\alpha,\beta}$ on $(0,1]$ by left-continuity. In this case, we define the intervals $I_j := (a_j, a_{j+1}]$ and (7.1) is true for all $x \in (0,1]$.

In [P2], Parry constructed a measure $\mu_{\alpha,\beta} \in M([0,1], T_{\alpha,\beta})$, which is absolutely continuous with respect to Lebesgue measure. Its density is (λ is the Lebesgue measure on $[0,1]$)

$$h_{\alpha,\beta}(x) := \frac{d\mu_{\alpha,\beta}}{d\lambda}(x) = \frac{1}{N_{\alpha,\beta}} \sum_{n\geq 0} \frac{1_{x<T^n_{\alpha,\beta}(1)} - 1_{x<T^n_{\alpha,\beta}(0)}}{\beta^{n+1}}\,, \tag{7.2}$$

with $N_{\alpha,\beta}$ the normalization factor. In [Ha], Halfin proved that $h_{\alpha,\beta}(x)$ is nonnegative for all $x \in [0,1]$. By Formula (3.6), the topological entropy of the dynamical system $(\Sigma_{\alpha,\beta}, \sigma)$ is $\log\beta$. Hofbauer showed in [H4] that it has a unique measure of maximal entropy $\hat{\mu}_{\alpha,\beta} \in M(\Sigma_{\alpha,\beta}, \sigma)$ (Proposition 6.15 is a big part of this proof). By Proposition 5.7, the dynamical system $([0,1], T_{\alpha,\beta})$ has a unique measure of maximal entropy given by $\hat{\mu}_{\alpha,\beta} \circ (\overline{\varphi}_{\infty}^{\alpha,\beta})^{-1}$. Finally, Hofbauer proved in [H2] that $\hat{\mu}_{\alpha,\beta} \circ (\overline{\varphi}_{\infty}^{\alpha,\beta})^{-1}$ is absolutely continuous with respect to Lebesgue measure. Since the measures $\mu_{\alpha,\beta}$ and $\hat{\mu}_{\alpha,\beta} \circ (\overline{\varphi}_{\infty}^{\alpha,\beta})^{-1}$ have both strictly positive densities at 0 and 1, they are equal

$$\mu_{\alpha,\beta} = \hat{\mu}_{\alpha,\beta} \circ (\overline{\varphi}_{\infty}^{\alpha,\beta})^{-1}\,.$$

7.2.1 Normality in the whole plane (α, β)

We give a lemma which shows that for given x and α, there is exponential separation between the orbits of x under the two different dynamical systems T_{α,β_1} and T_{α,β_2}.

Lemma 7.5. *Let $x \in [0,1)$, $\alpha \in [0,1)$ and $1 < \beta_1 \leq \beta_2$. Define $l = \min\{n \geq 0 : \mathtt{i}^1_n(x) \neq \mathtt{i}^2_n(x)\}$ with $\mathtt{i}^j(x) = \mathtt{i}^{\alpha,\beta_j}$ for $j = 1,2$. If $x \neq 0$, then*

$$\beta_2 - \beta_1 \leq \frac{\beta_2}{x}\beta_2^{-l}\,.$$

If $x = 0$ and $\alpha \neq 0$, then

$$\beta_2 - \beta_1 \leq \frac{\beta_2^2}{\alpha}\beta_2^{-l}\,.$$

Proof: Let $\delta := \beta_2 - \beta_1 \geq 0$. We prove by induction that for all $m \geq 1$, $\mathtt{i}^1_{[0,m)}(x) = \mathtt{i}^2_{[0,m)}(x)$ implies

$$T_2^m(x) - T_1^m(x) \geq \beta_2^{m-1}\delta x\,,$$

where $T_i = T_{\alpha,\beta_i}$. For $m = 1$,

$$T_2(x) - T_1(x) = \beta_2 x + \alpha - \mathtt{i}^2_0(x) - (\beta_1 x + \alpha - \mathtt{i}^1_0(x)) = \delta x\,.$$

Suppose that this is true for m, then $\mathtt{i}^1_{[0,m+1)} = \mathtt{i}^2_{[0,m+1)}$ implies

$$\begin{aligned} T_2^{m+1}(x) - T_1^{m+1}(x) &= \beta_2 T_2^m(x) + \alpha - \mathtt{i}^2_m(x) - (\beta_1 T_1^m(x) + \alpha - \mathtt{i}^1_m(x)) \\ &= \beta_2(T_2^m(x) - T_1^m(x)) + \delta T_1^m(x) \geq \beta_2^m \delta x \,. \end{aligned}$$

On the other hand, $1 \geq T_2^m(x) - T_1^m(x) \geq \beta_2^{m-1}\delta x$. Thus $\delta \leq \frac{\beta_2^{-m+1}}{x}$ for all m such that $\mathtt{i}^1_{[0,m)} = \mathtt{i}^2_{[0,m)}$. If $x = 0$, then $T_1(x) = T_2(x) = \alpha$ and we can apply the first statement to $y = \alpha > 0$. □

Now we can state our first theorem and its corollary about the normality of orbits under $T_{\alpha,\beta}$. The proof of the theorem is inspired by the proof of Theorem C in [S], where the case $x = 1$ and $\alpha = 0$ is considered. The 1-dimensional Lebesgue measure is denoted by λ.

Theorem 7.6. *Let $x \in [0,1)$ and $\alpha \in [0,1)$ except for $(x,\alpha) = (0,0)$. Then the set*

$$\{\beta > 1 : \textit{the orbit of } \mathtt{i}^{\alpha,\beta}(x) \textit{ under } \sigma \textit{ is } \hat{\mu}_{\alpha,\beta}\textit{-normal}\}$$

has full Lebesgue measure.

Corollary 7.7. *Let $x \in [0,1)$ and $\alpha \in [0,1)$ except for $(x,\alpha) = (0,0)$. Then the set*

$$\{\beta > 1 : \textit{the orbit of } x \textit{ under } T_{\alpha,\beta} \textit{ is } \mu_{\alpha,\beta}\textit{-normal}\}$$

has full Lebesgue measure.
If $\alpha = 0$, then the orbit of $x = 0$ under $T_{\alpha,\beta}$ is never $\mu_{\alpha,\beta}$-normal.

Notice that the theorem and its corollary may also be formulated for $x \in (0,1]$ using a left-continuous extensions of $T_{\alpha,\beta}$ and $\mathtt{i}^{\alpha,\beta}$ on $(0,1]$.
Proof of the theorem: We briefly sketch the proof. It is sufficient to consider a finite interval $[\underline{\beta},\overline{\beta}]$, since there is a countable cover by such intervals. We use the uniqueness of the measure of maximal entropy $\hat{\mu}_{\alpha,\beta}$: for $\underline{x} \in \Sigma_{\alpha,\beta}$ not $\hat{\mu}_{\alpha,\beta}$-normal, there exists $\nu \in V_\sigma(\underline{x})$ such that $h_\sigma(\nu) < h_\sigma(\hat{\mu}_{\alpha,\beta}) = \log\beta$. Therefore we cover the set of abnormal β in $[\underline{\beta},\overline{\beta}]$ by sets Ω_N, $N \in \mathbb{N}$,

$$\Omega_N := \{\beta \in [\underline{\beta},\overline{\beta}] : \{\mathcal{E}_n(\mathtt{i}^{\alpha,\beta}(x))\}_n \text{ clusters on } \nu \text{ with } h_\sigma(\nu) < (1 - 1/N)\log\beta\}\,.$$

We consider each Ω_N separately and cover them by appropriate intervals, which we generically denote by $[\beta_1,\beta_2]$. The main idea is to imbed $\{\mathtt{i}^{\alpha,\beta}(x) : \beta \in [\beta_1,\beta_2]\}$ in a shift space $\Sigma^* := \Sigma(\underline{u}^*,\underline{v}^*)$ with $\underline{u}^*$ and $\underline{v}^*$ well chosen. Writing $D^* \subset \Sigma^*$ for the range of the imbedding, we estimate the Hausdorff dimension of the subset of D^* corresponding to points $\mathtt{i}^{\alpha,\beta}(x)$ which are not $\hat{\mu}_{\alpha,\beta}$-normal. Then we estimate the coefficient of Hölder continuity of the map ρ_* defined as the inverse of the imbedding. This gives us an estimate of the Hausdorff dimension of the non $\hat{\mu}_{\alpha,\beta}$-normal points in the interval $[\beta_1,\beta_2]$.

To obtain uniform estimates, we restrict our proof to the interval $[\underline{\beta},\overline{\beta}]$ with $1 < \underline{\beta} < \overline{\beta} < \infty$. All shift spaces below are subshifts of Σ_k with $k = \lceil \alpha + \overline{\beta} \rceil$. Let $\Omega := \{\beta \in [\underline{\beta},\overline{\beta}] : \mathtt{i}^{\alpha,\beta}(x) \text{ is not } \hat{\mu}_{\alpha,\beta}\text{-normal}\}$. For $\beta \in \Omega$, we have $V_\sigma(\mathtt{i}^{\alpha,\beta}(x)) \neq \{\hat{\mu}_{\alpha,\beta}\}$. Since $\hat{\mu}_{\alpha,\beta}$ is the unique $T_{\alpha,\beta}$-invariant measure of maximal entropy $\log\beta$, there exist $N \in \mathbb{N}$ and $\nu \in V_\sigma(\mathtt{i}^{\alpha,\beta}(x))$ such that $h_\sigma(\nu) < (1 - 1/N)\log\beta$. Setting

$$\Omega_N := \{\beta \in [\underline{\beta},\overline{\beta}] : \exists \nu \in V_\sigma(\mathtt{i}^{\alpha,\beta}(x)) \text{ s.t. } h_\sigma(\nu) < (1 - 1/N)\log\beta\}\,,$$

we have $\Omega = \bigcup_{N\geq 1}\Omega_N$. We will prove that $\dim_H \Omega_N < 1$, so that $\lambda(\Omega_N) = 0$ for all $N \geq 1$.

For $N \in \mathbb{N}$ fixed, define $\varepsilon := \frac{\beta\log\beta}{2N-1} > 0$ and $\delta := \log\bigl(1+\varepsilon/\overline{\beta}\bigr)$. Let $\beta \in [\underline{\beta},\overline{\beta}]$. Following Proposition 4.10, choose L_β such that

$$h_{\mathrm{top}}(\Sigma_{\underline{u}',\underline{v}'},\sigma) \leq h_{\mathrm{top}}(\Sigma_{\alpha,\beta},\sigma) + \delta/2\,,$$

for all pairs $(\underline{u}',\underline{v}')$ satisfying (6.11), $\underline{u}'_{[0,L_\beta)} = \underline{u}^{\alpha,\beta}_{[0,L_\beta)}$ and $\underline{v}'_{[0,L_\beta)} = \underline{v}^{\alpha,\beta}_{[0,L_\beta)}$. Choose $q_\beta \in \mathbb{Q}$ such that $\log\beta - \delta/2 \leq \log q_\beta \leq \log\beta$. Let

$$J(\beta,L_\beta,q_\beta) := \{\beta' \in [q_\beta,\overline{\beta}] : \underline{u}^{\alpha,\beta'}_{[0,L_\beta)} = \underline{u}^{\alpha,\beta}_{[0,L_\beta)}, \underline{v}^{\alpha,\beta'}_{[0,L_\beta)} = \underline{v}^{\alpha,\beta}_{[0,L_\beta)}\}\,.$$

This set is an interval, since the maps $\beta' \mapsto \underline{u}^{\alpha,\beta'}$ and $\beta' \mapsto \underline{v}^{\alpha,\beta'}$ are both monotone increasing. Moreover $\beta \in J(\beta,L_\beta,q_\beta)$. Notice also that the family $\{J(\beta,L_\beta,q_\beta) : \beta \in [\underline{\beta},\overline{\beta}]\}$ is countable. Indeed the interval $J(\beta,L_\beta,q_\beta)$ is entirely characterized by $\underline{u}^{\alpha,\beta}_{[0,L_\beta)}$, $\underline{v}^{\alpha,\beta}_{[0,L_\beta)}$ and q_β. But there are only countably many triples in $\mathtt{A}_k^*\times\mathtt{A}_k^*\times\mathbb{Q}$. Thus $\{J(\beta,L_\beta,q_\beta) : \beta \in [\underline{\beta},\overline{\beta}]\}$ is a countable cover of $[\underline{\beta},\overline{\beta}]$. To prove that $\lambda(\Omega_N) = 0$, it is sufficient to prove that $\lambda(\Omega_N \cap J(\beta,L_\beta,q_\beta)) = 0$ for all $\beta \in [\underline{\beta},\overline{\beta}]$. The interval $J(\beta,L_\beta,q_\beta)$ may be open, closed or neither open nor closed. We need to work on a closed interval, thus we prove an equivalent result: for any closed interval $[\beta_1,\beta_2] \subset J(\beta,L_\beta,q_\beta)$, we have $\lambda(\Omega_N \cap [\beta_1,\beta_2]) = 0$.

Let $\underline{u}^j = \underline{u}^{\alpha,\beta_j}$ and $\underline{v}^j = \underline{v}^{\alpha,\beta_j}$. Using (6.4) and the monotonicity of $\beta \mapsto \underline{u}^{\alpha,\beta}$ and $\beta \mapsto \underline{v}^{\alpha,\beta}$, we have

$$\begin{aligned} &\underline{u}^1 \preceq \sigma^n\underline{u}^1 \preceq \underline{v}^1 \preceq \underline{v}^2 &&\forall n\geq 0\,,\\ &\underline{u}^1 \preceq \underline{u}^2 \preceq \sigma^n\underline{v}^2 \preceq \underline{v}^2 &&\forall n \geq 0\,. \end{aligned}$$

Hence the couple $(\underline{u}^1,\underline{v}^2)$ satisfies (6.11) and we set $\Sigma^* := \Sigma(\underline{u}^1,\underline{v}^2)$ and

$$D^* := \{\underline{z} \in \Sigma^* : \exists\beta \in [\beta_1,\beta_2] \text{ s.t. } \underline{z} = \mathtt{i}^{\alpha,\beta}(x)\}\,.$$

We define a map $\rho_* : D^* \to [\beta_1,\beta_2]$ by $\rho_*(\underline{z}) = \beta \iff \mathtt{i}^{\alpha,\beta}(x) = \underline{z}$. This map is well defined: by definition of D^*, for all $\underline{z} \in D^*$ there exists a $\beta \in [\beta_1,\beta_2]$ such that $\underline{z} = \mathtt{i}^{\alpha,\beta}(x)$; moreover this β is unique, since by Lemma 7.5, $\beta \mapsto \mathtt{i}^{\alpha,\beta}(x)$ is strictly increasing. On the other hand, for all $\beta \in [\beta_1,\beta_2]$, we have from (6.5)

$$\underline{u}^1 \preceq \underline{u}^{\alpha,\beta} \preceq \sigma^n\mathtt{i}^{\alpha,\beta}(x) \preceq \underline{v}^{\alpha,\beta} \preceq \underline{v}^2 \quad \forall n \geq 0\,,$$

whence $\mathtt{i}^{\alpha,\beta}(x) \in \Sigma^*$ and $\rho_* : D^* \to [\beta_1,\beta_2]$ is surjective. Let $\log\beta_* := h_{\mathrm{top}}(\Sigma^*,\sigma)$; then by Proposition 4.10

$$\log\beta_* = h_{\mathrm{top}}(\Sigma^*,\sigma) \leq h_{\mathrm{top}}(\Sigma_{\alpha,\beta},\sigma) + \delta/2 = \log\beta + \delta/2\,.$$

By definition of q_β, we have $\log\beta - \delta/2 \leq \log q_\beta \leq \log\beta_1$, thus $\log\beta^* \leq \log\beta_1 + \delta$ and

$$\beta_* - \beta_1 \leq \beta_1\bigl(\mathrm{e}^\delta - 1\bigr) \leq \varepsilon\,. \tag{7.3}$$

Let us compute the coefficient of Hölder continuity of $\rho_* : (D^*, d_{\beta_*}) \to [\beta_1,\beta_2]$. Let $\underline{z} \neq \underline{z}' \in D^*$ and $n = \min\{l \geq 0 : z_l \neq z'_l\}$, then $d_{\beta_*}(\underline{z},\underline{z}') = \beta_*^{-n}$. By Lemma 7.5,

$$|\rho_*(\underline{z}) - \rho_*(\underline{z}')| \leq C\rho_*(\underline{z})^{-n} \leq C\beta_1^{-n} = C(d_{\beta_*}(\underline{z},\underline{z}'))^{\frac{\log\beta_1}{\log\beta_*}}\,,$$

where

$$C = \begin{cases} \frac{\overline{\beta}}{x} & \text{if } x \neq 0 \\ \frac{\overline{\beta}^2}{\alpha} & \text{if } x = 0\,. \end{cases}$$

By equation (7.3) and the choice of ε, we have

$$\begin{aligned} \beta_* - \beta_1 \leq \frac{\underline{\beta}\log\underline{\beta}}{2N-1} &\implies \beta_* - \beta_1 \leq \frac{\beta_1 \log\beta_1}{2N-1} \\ &\iff 1 + \frac{\beta_* - \beta_1}{\beta_1\log\beta_1} \leq 1 + \frac{1}{2N-1} \\ &\iff \frac{\log\beta_1 + \frac{\beta_*-\beta_1}{\beta_1}}{\log\beta_1} \leq \frac{2N}{2N-1} \\ &\implies \frac{\log\beta_1}{\log\beta_*} \geq \frac{\log\beta_1}{\log\beta_1 + \frac{\beta_*-\beta_1}{\beta_1}} \geq 1 - \frac{1}{2N}\,. \end{aligned}$$

In last line, we use the concavity of the logarithm, so the first order Taylor development is an upper estimate. Thus ρ_* has Hölder-exponent $1 - \frac{1}{2N}$.

Define

$$G_N^* := \{\underline{z} \in \Sigma^* : \exists \nu \in V_\sigma(\underline{z}) \text{ s.t. } h_\sigma(\nu) < (1-1/N)\log\beta_*\}\,.$$

Let $\beta \in \Omega_N \cap [\beta_1, \beta_2]$. Then there exists $\nu \in V_\sigma(\mathtt{i}^{\alpha,\beta}(x))$ such that

$$h_\sigma(\nu) < (1-1/N)\log\beta \leq (1-1/N)\log\beta_*\,.$$

Since $\mathtt{i}^{\alpha,\beta}(x) \in D^* \subset \Sigma^*$, we have $\mathtt{i}^{\alpha,\beta}(x) \in G_N^*$. Using the surjectivity of ρ_*, we obtain $\Omega_N \cap [\beta_1,\beta_2] \subset \rho_*(G_N^* \cap D^*)$. We claim that $h_{\text{top}}(G_N^*, \sigma) \leq (1-1/N)\log\beta_*$. This implies, using Lemmas 7.3 and 7.2,

$$\begin{aligned} \dim_H(\Omega_N \cap [\beta_1,\beta_2]) &\leq \dim_H \rho_*(G_N^* \cap D^*) \\ &\leq \frac{\dim_H G_N^*}{1-\frac{1}{2N}} \leq \frac{h_{\text{top}}(G_N^*,\sigma)}{(1-\frac{1}{2N})\log\beta_*} \leq \frac{1-\frac{1}{N}}{1-\frac{1}{2N}} < 1\,. \end{aligned}$$

Thus $\lambda(\Omega_N \cap [\beta_1,\beta_2]) = 0$.

It remains to prove $h_{\text{top}}(G_N^*,\sigma) \leq (1-1/N)\log\beta_*$. Recall that (see Definition 3.2)

$$h_\sigma(\nu) = \sup_{\mathcal{A}} \lim_n \frac{1}{n} H(\nu, \bigvee_{j=0}^{n-1} T^{-j}\mathcal{A})\,,$$

where the supremum runs over all finite Borel partitions $\mathcal{A}$. By Theorem 3.3, the partition $\mathcal{A}_1 := \{[j] : j \in \mathtt{A}_k\}$ is such that $h_\sigma(\nu) = h_\sigma(\nu, \mathcal{A}_1)$. Thus

$$h_\sigma(\nu) = \lim_n \frac{1}{n} H(\nu, \mathcal{A}_n)\,,$$

where $\mathcal{A}_n := \{[\underline{w}] : \underline{w} \in \mathcal{L}(\Sigma^*), |\underline{w}| = n\}$. Since the cylinders are both open and closed, for all $\underline{w} \in \mathcal{L}(\Sigma^*)$, the map $\nu \mapsto \nu([\underline{w}])$ is continuous in the weak*-topology (see Theorem 30.10 in [Ba]). Thus

$$\nu \mapsto H(\nu, \mathcal{A}_n) = -\sum_{[\underline{w}] \in \mathtt{A}_n} \nu([\underline{w}])\log\nu([\underline{w}])$$

is continuous in the weak*-topology, as a finite sum of continuous functions. Moreover $\frac{1}{n}H(\nu, \mathcal{A}_n)$ is decreasing in n. For all $m \geq 1$, we set

$$F_N^*(m) := \{\nu \in M(\Sigma^*, \sigma) : \frac{1}{m}H(\nu, \mathcal{A}_m) \leq (1 - 1/N)\log\beta_*\}\,,$$
$$G_N^*(m) := \{\underline{z} \in \Sigma^* : V_\sigma(\underline{z}) \cap F_N^*(m) \neq \emptyset\}\,.$$

Let $\underline{z} \in G_N^*$, then there exists $\nu \in V_\sigma(\underline{z})$ such that $h_\sigma(\nu) < (1 - \frac{1}{N})\log\beta_*$. Since $\frac{1}{m}H(\nu, \mathcal{A}_m) \downarrow h_\sigma(\nu)$, there exists $m \geq 1$ such that $\frac{1}{m}H(\nu, \mathcal{A}_m) \leq (1-1/N)\log\beta_*$, whence $\nu \in F_n^*(m)$ and $\underline{z} \in G_N^*(m)$. This implies $G_N^* \subset \bigcup_{m\geq 1} G_N^*(m)$. Since $\nu \mapsto H(\nu, \mathcal{A}_n)$ is continuous, $F_N^*(m)$ is closed for all $m \geq 1$. Finally we obtain using Theorem 7.4

$$h_{\mathrm{top}}(G_N^*, \sigma) = \sup_m h_{\mathrm{top}}(G_N^*(m), \sigma) \leq \sup_m \sup_{\nu \in F_N^*(m)} h_\sigma(\nu)$$
$$\leq \sup_m \sup_{\nu \in F_N^*(m)} \frac{1}{m}H(\nu, \mathcal{A}_m) \leq (1 - 1/N)\log\beta_*\,.$$

□

Proof of the Corollary: Let $\beta > 1$ be such that the orbit of $\mathtt{i}^{\alpha,\beta}(x)$ under σ is $\hat{\mu}_{\alpha,\beta}$-normal. Let $f \in C([0,1])$, then $\hat{f} : \Sigma_{\alpha,\beta} \to \mathbb{R}$ defined by $\hat{f} := f \circ \overline{\varphi}_\infty^{\alpha,\beta}$ is continuous, since $\overline{\varphi}_\infty^{\alpha,\beta}$ is continuous. Using $\mu_{\alpha,\beta} := \hat{\mu}_{\alpha,\beta} \circ (\overline{\varphi}_\infty^{\alpha,\beta})^{-1}$, we have

$$\int_{[0,1]} f d\mu_{\alpha,\beta} = \int_{\Sigma_{\alpha,\beta}} \hat{f} d\hat{\mu}_{\alpha,\beta} = \lim_{n\to\infty} \sum_{i=0}^{n-1} \hat{f}(\sigma^i \mathtt{i}^{\alpha,\beta}(x))$$
$$= \lim_{n\to\infty} \sum_{i=0}^{n-1} f(\overline{\varphi}^{\alpha,\beta}(\sigma^i \mathtt{i}^{\alpha,\beta}(x))) = \lim_{n\to\infty} \sum_{i=0}^{n-1} f(T_{\alpha,\beta}^i(x))\,.$$

The second equality comes from the $\hat{\mu}_{\alpha,\beta}$-normality of the orbit of $\mathtt{i}^{\alpha,\beta}(x)$ under σ, the last one is (7.1) which is true for all $x \in [0,1)$ with our convention for the extension of $T_{\alpha,\beta}$ and $\mathtt{i}^{\alpha,\beta}$ on $[0,1)$.
The last statement is trivial: if $\alpha = 0$, then $T_{\alpha,\beta}^n(0) = 0$ for all $n \geq 0$. □

The next step is to consider the question of $\mu_{\alpha,\beta}$-normality in the whole plane (α, β) instead of working with α fixed. Define $\mathcal{R} := [0,1) \times (1, \infty)$.

Theorem 7.8. *For all $x \in [0,1)$, the set*

$$\mathcal{N}(x) := \{(\alpha, \beta) \in \mathcal{R} : \textit{the orbit of } x \textit{ under } T_{\alpha,\beta} \textit{ is } \mu_{\alpha,\beta}\textit{-normal}\}$$

has full 2-dimensional Lebesgue measure.

Proof: We only have to prove that $\mathcal{N}(x)$ is measurable and to apply Fubini's Theorem and Corollary 7.7. The first step is to prove that for all $x \in [0,1)$ and all $n \geq 0$, the maps $(\alpha, \beta) \mapsto \mathtt{i}^{\alpha,\beta}(x)$ and $(\alpha, \beta) \mapsto T_{\alpha,\beta}^n(x)$ are measurable. First notice that for all $n \geq 1$

$$T_{\alpha,\beta}^n(x) = \beta^n x + \alpha\frac{\beta^n - 1}{\beta - 1} - \sum_{j=0}^{n-1} \mathtt{i}_j^{\alpha,\beta}(x)\ \beta^{n-j-1}\,. \tag{7.4}$$

The proof by induction is immediate. To prove that $(\alpha, \beta) \mapsto \mathtt{i}^{\alpha,\beta}(x)$ is measurable, it is enough to prove that for all $n \geq 0$ and for all words $\underline{w} \in \mathtt{A}_k^*$ of length n

$$\{(\alpha, \beta) \in \mathcal{R} : \mathtt{i}_{[0,n)}^{\alpha,\beta}(x) = \underline{w}\}$$

is measurable, since the σ-algebra on Σ_k^+ is generated by the cylinders. This set is the subset of $\mathbb{R}^2$ such that

$$\begin{cases} \beta > 1\,, \\ 0 \le \alpha < 1\,, \\ w_j < \beta T^j_{\alpha,\beta}(x) + \alpha \le w_j + 1 \quad \forall 0 \le j < n\,. \end{cases}$$

Using (7.4), this system of inequalities can be rewritten

$$\begin{cases} \beta > 1\,, \\ 0 \le \alpha < 1\,, \\ \alpha > \frac{\beta-1}{\beta^{j+1}-1}\left(\sum_{i=0}^{j} w_i\beta^{j-i} - \beta^{j+1}x\right) & \forall 0 \le j < n\,, \\ \alpha \le \frac{\beta-1}{\beta^{j+1}-1}\left(1 + \sum_{i=0}^{j} w_i\beta^{j-i} - \beta^{j+1}x\right) & \forall 0 \le j < n\,. \end{cases}$$

From this, the measurability of $\mathtt{i}^{\alpha,\beta}$ follows. If $(\alpha,\beta) \mapsto \mathtt{i}^{\alpha,\beta}(x)$ is measurable, then by formula (7.4), $(\alpha,\beta) \mapsto T^n_{\alpha,\beta}(x)$ is clearly measurable for all $n \ge 0$. Then for all $f \in C([0,1])$ and all $n \ge 1$, the map $(\alpha,\beta) \mapsto S_n(f) := \frac{1}{n}\sum_{i=0}^{n-1} f(T^i_{\alpha,\beta}(x))$ is measurable and consequently

$$\{(\alpha,\beta) : \lim_{n\to\infty} S_n(f) \text{ exists}\}$$

is a measurable set.

On the other hand, if $f \in C([0,1])$, then $(\alpha,\beta) \mapsto \int f d\mu_{\alpha,\beta}$ is measurable. Indeed

$$\int f d\mu_{\alpha,\beta} = \int f h_{\alpha,\beta} d\lambda$$

and in view of equation (7.2) and the measurability of $(\alpha,\beta) \mapsto T_{\alpha,\beta}(x)$, the map $(\alpha,\beta) \mapsto h_{\alpha,\beta}$ is clearly measurable. Therefore

$$\{(\alpha,\beta) : \lim_{n\to\infty} S_n(f) = \int f d\mu_{\alpha,\beta}\}$$

is measurable for all $f \in C([0,1])$. Since $[0,1]$ endowed with the euclidian metric is a complete and separable metric space, there exists a countable subset $\{f_m\}_{m\in\mathbb{N}} \subset C([0,1])$ which is dense with respect to the uniform convergence (see Lemma 31.4 in [Ba]). Then setting

$$D_m := \{(\alpha,\beta) \in \mathcal{R} : \lim_{n\to\infty} S_n(f_m) = \int f_m d\mu_{\alpha,\beta}\}\,,$$

we have $\mathcal{N}(x) = \bigcap_{m\in\mathbb{N}} D_m$, whence it is a measurable set. □

7.2.2 Normality along particular curves

We have shown that for a given $x \in [0,1)$, the orbit of x under $T_{\alpha,\beta}$ is $\mu_{\alpha,\beta}$-normal for almost all (α,β). The orbits of 0 and 1 are of particular interest (see equations (7.2)). Now we show that through any point (α_0,β_0), there passes a curve defined by $\alpha = \alpha(\beta)$ such that the orbit of 0 under $T_{\alpha(\beta),\beta}$ is $\mu_{\alpha(\beta),\beta}$-normal for at most one β. A trivial example of such a curve is $\alpha = 0$, since $x = 0$ is a fixed point. The idea is to consider curves along which the coding of 0 is constant, ie to define $\alpha(\beta)$ such that $\underline{u}^{\alpha(\beta),\beta}$ is constant.

Define

$$\mathcal{U} := \{\underline{u} \in \Sigma_k^+ : \exists\ (\alpha, \beta) \in \mathcal{R} \text{ s.t. } \underline{u} = \underline{u}^{\alpha,\beta}\}\,.$$

We define an equivalence relation in $\mathcal{R}$ by

$$(\alpha, \beta) \sim (\alpha', \beta') \iff \underline{u}^{\alpha,\beta} = \underline{u}^{\alpha',\beta'}\,.$$

An equivalence class is denoted by $[\underline{u}] := \{(\alpha, \beta) \in \mathcal{R} : \underline{u}^{\alpha,\beta} = \underline{u}\}$. The next lemma describes $[\underline{u}]$.

Lemma 7.9. *Let $\underline{u} \in \mathcal{U}$ and set*

$$\alpha(\beta) = (\beta - 1) \sum_{j\geq 0} \frac{u_j}{\beta^{j+1}}\,.$$

Then there exists $\beta_{\underline{u}} \geq 1$ such that

$$[\underline{u}] = \{(\alpha(\beta), \beta) : \beta \in J_{\underline{u}}\}$$

with $J_{\underline{u}} = (\beta_{\underline{u}}, \infty)$ or $J_{\underline{u}} = [\beta_{\underline{u}}, \infty)$.

Proof: If $\underline{u} = 0^\infty$, then the statement is trivially true with $\alpha(\beta) \equiv 0$ and $\beta_{\underline{u}} = 1$. Suppose $\underline{u} \neq 0^\infty$. First we prove that

$$(\alpha, \beta) \sim (\alpha', \beta) \implies \alpha = \alpha'\,,$$

then

$$(\alpha, \beta) \in [\underline{u}] \implies (\alpha(\beta'), \beta') \in [\underline{u}] \quad \forall \beta' \geq \beta\,.$$

Let $(\alpha, \beta) \in [\underline{u}]$. Using (7.1), we have $\overline{\varphi}_\infty^{\alpha,\beta}(\sigma\underline{u}) = T_{\alpha,\beta}(0) = \alpha$. Since the map $\alpha \mapsto \overline{\varphi}_\infty^{\alpha,\beta}(\sigma\underline{u}) - \alpha$ is continuous (Theorem 2.12) and strictly decreasing (Lemma 6.1), the first statement is true. Let $\beta' > \beta$. By Corollary 6.8, we have that $\overline{\varphi}^{\alpha,\beta}(\sigma\underline{u}) > \varphi^{\alpha,\beta'}(\sigma\underline{u})$. Therefore there exists a unique $\alpha' < \alpha$ such that $\varphi^{\alpha',\beta'}(\sigma\underline{u}) = \alpha'$. We prove that $\underline{u}^{\alpha',\beta'} = \underline{u}$. By Proposition 6.4, we have $\underline{u} \preceq \underline{u}^{\alpha',\beta'}$. By Proposition 6.15, we have

$$h_{\rm top}(\Sigma_{\underline{u},\underline{v}^{\alpha',\beta'}}, \sigma) = h_{\rm top}(\Sigma_{\alpha',\beta'}, \sigma) = \log \beta'\,.$$

Since $\Sigma_{\alpha,\beta} = \Sigma_{\underline{u},\underline{v}^{\alpha,\beta}}$ and $\beta' > \beta$, we must have $\underline{v}^{\alpha,\beta} \prec \underline{v}^{\alpha',\beta'}$. Therefore

$$\begin{aligned}
&\underline{u} \preceq \sigma^n\underline{u} \prec \underline{v}^{\alpha,\beta} \prec \underline{v}^{\alpha',\beta'} \quad \forall n \geq 0\\
&\underline{u} \preceq \underline{u}^{\alpha',\beta'} \prec \sigma^n\underline{v}^{\alpha',\beta'} \preceq \underline{v}^{\alpha',\beta'} \quad \forall n \geq 0\,,
\end{aligned}$$

are the inequalities (6.21) for the pair $(\underline{u}, \underline{v}^{\alpha',\beta'})$. We can apply Proposition 6.10 and Theorem 6.20 to this pair and get $\underline{u} = \underline{u}^{\alpha',\beta'}$. It remains to show that $\alpha' = \alpha(\beta')$. Applying the formula (2.17) to $\underline{u}^{\alpha,\beta}$, we have for all $(\alpha, \beta) \in \mathcal{R}$

$$\alpha = (\beta - 1) \sum_{j\geq 0} \frac{u_j^{\alpha,\beta}}{\beta^{j+1}}\,.$$

Since for all $\beta > \beta_{\underline{u}}$, we have $\underline{u} \in \Sigma_{\alpha,\beta}$, this completes the proof. □

For each $\underline{u} \in \mathcal{U}$, the equivalence class $[\underline{u}]$ defines an curve in $\mathcal{R}$, which is strictly monotone decreasing (except for $\underline{u} = 0^\infty$),

$$[\underline{u}] = \{(\alpha, \beta) : \alpha = (\beta - 1) \sum_{j \geq 0} \frac{u_j}{\beta^{j+1}},\ \beta \in I_{\underline{u}}\}\,.$$

Since $\beta > 1$ and $u_j \leq k - 1$, the sum is uniformly bounded on any interval $[\beta_0, \infty)$ with $\beta_0 > 1$. Thus each curve is analytic. Moreover these curves are disjoint two by two and their union is $\mathcal{R}$.

Theorem 7.10. *Let $(\alpha, \beta) \in \mathcal{R}$, $\underline{u} = \underline{u}^{\alpha,\beta}$ and define $\alpha(\beta)$ and $\beta_{\underline{u}}$ as in Lemma 7.9. Then for all $\beta > \beta_{\underline{u}}$, the orbit of $x = 0$ under $T_{\alpha(\beta),\beta}$ is not $\mu_{\alpha(\beta),\beta}$-normal.*

Proof: Let $\hat{\nu} \in M(\Sigma_k^+, \sigma)$ (with k large enough) be a cluster point of $\{\mathcal{E}_n(\underline{u})\}_{n \geq 1}$. By Lemma 7.9, $\underline{u}^{\alpha(\beta),\beta} = \underline{u}$ for any $\beta > \beta_{\underline{u}}$. Therefore

$$h_\sigma(\hat{\nu}) \leq h_{\text{top}}(\Sigma_{\alpha(\beta),\beta}, \sigma) = \log \beta \quad \forall \beta > \beta_{\underline{u}}$$

and $\hat{\nu}$ is not a measure of maximal entropy. Moreover $\mu_{\alpha(\beta),\beta} = \hat{\mu}_{\alpha(\beta),\beta} \circ (\overline{\varphi}_\infty^{\alpha(\beta),\beta})^{-1}$ is the unique measure of maximal entropy for $T_{\alpha(\beta),\beta}$. Thus $\nu_\beta := \hat{\nu} \circ (\overline{\varphi}_\infty^{\alpha(\beta),\beta})^{-1} \in M([0,1], T_{\alpha(\beta),\beta})$ is not a measure of maximal entropy for all $\beta > \beta_{\underline{u}}$. □

Recall that

$$\mathcal{N}(0) = \{(\alpha, \beta) \in \mathcal{R} : \text{the orbit of } 0 \text{ under } T_{\alpha,\beta} \text{ is } \mu_{\alpha,\beta}\text{-normal}\}\,.$$

By Theorem 7.8, $\mathcal{N}(0)$ has full Lebesgue measure. On the other hand, by Theorem 7.10, we can decompose $\mathcal{R}$ into a family of disjoint analytic curves such that each curve meets $\mathcal{N}(0)$ in at most one point. This situation seems to be paradoxical, but it is very similar to the one presented in [Mi] by Milnor following an idea of Katok.

Finally notice that, in Lemma 7.9, we construct curves along which the coding of $x = 0$ is constant. What happens when we consider $x \in [0, 1]$? From (2.17), it is possible to show that for any $x \in [0, 1]$ and any $\underline{x}$, we have

$$\{(\alpha, \beta) \in \mathcal{R} : \mathtt{i}^{\alpha,\beta}(x) = \underline{x}\} \subset \{(\alpha, \beta) : \alpha = (\beta - 1)(-x + \sum_{j \geq 0} \frac{x_j}{\beta^{j+1}})\}\,.$$

We can define $J(x, \underline{x}) \in (1, \infty)$ as the unique set such that

$$\{(\alpha, \beta) \in \mathcal{R} : \mathtt{i}^{\alpha,\beta}(x) = \underline{x}\} = \{(\alpha, \beta) : \beta \in J(x, \underline{x}) \text{ and } \alpha = (\beta - 1)(-x + \sum_{j \geq 0} \frac{x_j}{\beta^{j+1}})\}\,.$$

Unfortunately, we have no proof that $J(x, \underline{x})$ is an interval except if $x = 0$ or $x = 1$.

7.3 Normality in generalized β-transformations

In this section, we consider the question of the normality in the generalized β-transformations T_β. We work as before: $x = 1$ is fixed and we estimate the size of the subset of the parameters for which x is μ_β-normal. The structure of the proof is very similar. Since the tent maps are a particular example of generalized β-transformations (they correspond to $k = 2$ and $s = (1, -1)$), we recover a result of Bruin in [Br]. Notice that the proofs are very different.

Contrarily to the case of $T_{\alpha,\beta}$, we consider only $x = 1$, because we do not have a proof of the exponential separation of orbits for all $x \in [0,1]$, but only for $x = 1$. However, the orbit of 1 is the most important orbit in this dynamical system. For example, it appears explicitly in the density of the invariant measure μ_β (see [G]). The orbit of 1 is defined by $T_\beta^0(1) := 1$, $T_\beta(1) \equiv \gamma := \lim_{x\uparrow 1} T_\beta(x)$ and for all $n \geq 2$,

$$T_\beta^n(1) = T_\beta^{n-1}(\gamma) \quad \text{if } \gamma \in X.$$

If $\gamma \notin X$, then the orbit of 1 is not well-defined. Remember that the set $S = S(\beta)$ depends on the parameter β. For the map $T_{\alpha,\beta}$, we solved this problem by an extension of the definition of the orbit on $[0,1)$; this was possible, because all laps of $T_{\alpha,\beta}$ are increasing. In this case, we will simply show that the set of β such that $\gamma \notin X$ is countable, hence negligible.

Lemma 7.11. *For any family of generalized β-transformations defined by $(s_n)_{0\leq n<k}$, the set $\{\beta \in (k-1,k] : \gamma \in S(\beta)\}$ is countable.*

Proof: For a fixed $n \geq 1$, we study the map $\beta \mapsto T_\beta^n(\gamma)$. This map is well defined everywhere in $(k-1,k]$ except for finitely many points and it is continuous on each interval where it is well defined. Indeed this is true for $n = 1$. Suppose it is true for n, then $T_\beta^{n+1}(1)$ is well defined and continuous wherever $T_\beta^n(1)$ is well defined and continuous, except for $T_\beta^n(1) \in S_0(\beta)$. By the induction hypothesis, there exists a finite family of disjoint open intervals J_i and continuous functions $g_i : J_i \to [0,1]$ such that $(k-1,k]\backslash(\bigcup_i J_i)$ is finite and

$$T_\beta^n(x) = g_i(\beta) \quad \text{if } \beta \in J_i\,.$$

Then

$$\{\beta \in (k-1,k] : T_\beta^n(1) \text{ is well defined and } T_\beta^n(1) \in S_0(\beta)\} = \bigcup_{i,j}\{\beta \in J_i : g_i(\beta) = \frac{j}{\beta}\}\,.$$

We claim that $\{\beta \in J_i : g_i(\beta) = \frac{j}{\beta}\}$ has finitely many points. From the form of the map T_β, it follows immediately that each $g_i(\beta)$ is a polynomial of degree n. Since $\beta > 1$,

$$g_i(\beta) = \frac{j}{\beta} \quad \Longleftrightarrow \quad \beta g_i(\beta) - j = 0\,.$$

This polynomial equation has at most $n+1$ roots. In fact, using the monotonicity of the map $\beta \mapsto \underline{\eta}^\beta$, we can prove that this set has at most one point. The lemma follows, since $S(\beta) = \bigcup_{n\geq 0} S_n(\beta)$. □

Notice that, using the strict monotonicity of $\beta \mapsto \underline{\eta}^\beta$, we could prove that, for all i and j, the set $\{\beta \in J_i : g_i(\beta) = \frac{j}{\beta}\}$ has at most one point. But the proof we gave here is easily adapted to prove that, for all $x \in (0,1)$, the set $\{\beta \in (k-1,k] : x \in S(\beta)\}$ is countable. The next lemma is an equivalent of Lemma 7.5 in the case of the generalized β-transformations. It states the exponential separation of the orbits of $x = 1$ under two maps T_{β_1} and T_{β_2}. Notice that we consider only $x = 1$, contrary to Lemma 7.5.

Lemma 7.12. *Consider a family $\{T_\beta\}_{\beta>1}$ of generalized β-transformations defined by a sequence $s = (s_n)_{0\leq n<k}$. Let $k-1 < \beta_1 \leq \beta_2 \leq k$ and $\underline{\eta}^j := \underline{\eta}^{\beta_j}$ for $j = 1,2$; define $l := \min\{n \geq 0 : \underline{\eta}_n^1 \neq \underline{\eta}_n^2\}$.*
If $k \geq 3$, for all $\beta_0 > 2$, there exists K such that $\beta_1 \geq \beta_0$ implies

$$\beta_2 - \beta_1 \leq K\beta_2^{-l}\,.$$

If $s=(+1,+1)$, then
$$\beta_2-\beta_1\le k\beta_2^{-l}\,.$$
If $s=(+1,-1)$ or $(-1,+1)$, then for all $\beta_0>1$, there exists K such that $\beta_1\ge\beta_0$ implies
$$\beta_2-\beta_1\le K\beta_2^{-l}\,.$$
If $s=(-1,-1)$, then there exists $\beta_0>1$ and K such that $\beta_1\ge\beta_0$ implies
$$\beta_2-\beta_1\le K\beta_2^{-l}\,.$$

The proof is very similar to the proof of Brucks and Misiurewicz for Proposition 1 of [BM], see also Lemma 23 of Sands in [Sa].

Proof: Let $\delta:=\beta_2-\beta_1\ge 0$ and denote $T_j=T_{\beta_j}$ and $\mathtt{i}^j=\mathtt{i}^{\beta_j}$ for $j=1,2$. Let $b_1,b_2\in[0,1]$ such that $r:=\mathtt{i}_0^1(b_1)=\mathtt{i}_0^2(b_2)$. We consider four cases according to the signs of b_2-b_1 and s_r. If $b_2-b_1\ge 0$ and $s_r=1$, then
$$T_2(b_2)-T_1(b_1)=\beta_2b_2-r-(\beta_1b_1-r)=\beta_2b_2-(\beta_2-\delta)b_1\ge\beta_2(b_2-b_1)\,.$$
If $b_2-b_1\le 0$ and $s_r=1$, then
$$T_1(b_1)-T_2(b_2)=\beta_1b_1-r-(\beta_2b_2-r)=(\beta_2-\delta)b_1-\beta_2b_2\ge\beta_2(b_1-b_2)-\delta\,.$$
If $b_2-b_1\ge 0$ and $s_r=-1$, then
$$T_1(b_1)-T_2(b_2)=1-(\beta_1b_1-r)-[1-(\beta_2b_2-r)]=\beta_2b_2-(\beta_2-\delta)b_1\ge\beta_2(b_2-b_1)\,.$$
If $b_2-b_1\le 0$ and $s_r=-1$, then
$$T_2(b_2)-T_1(b_1)=1-(\beta_2b_2-r)-[1-(\beta_1b_1-r)]=(\beta_2-\delta)b_1-\beta_2b_2\ge\beta_2(b_1-b_2)-\delta\,.$$
In four cases, we have
$$|T_2(b_2)-T_1(b_1)|\ge\beta_2|b_2-b_1|-\delta\,.$$
Applying this formula n times, we find that $\mathtt{i}^1_{[0,n)}(b_1)=\mathtt{i}^2_{[0,n)}(b_2)$ implies
$$|T_2^n(b_2)-T_1^n(b_1)|\ge\beta_2^n\left(|b_2-b_1|-\frac{\delta}{\beta_2-1}\right)\,.$$

Consider the case $k\ge 3$. Set $b_i=T_i(1)$ for $i=1,2$; we have
$$|b_2-b_1|=\delta>\frac{\delta}{\beta_0-1}\ge\frac{\delta}{\beta_2-1}\,.$$
Using $|T_2^n(b_2)-T_1^n(b_1)|\le 1$, we conclude that for all $\beta_0\le\beta_1\le\beta_2$, if $\underline{\eta}^1_{[0,n)}=\underline{\eta}^2_{[0,n)}$ then
$$\delta\le\frac{\beta_0-1}{\beta_0-2}\,\beta_2^{-n+1}\,.$$
For the case $s=(+1,+1)$, we can apply Lemma 7.5 with $\alpha=0$ and $x=1$.
The case $s=(+1,-1)$ and $(-1,+1)$ are considered in Lemma 23 of [Sa].

For the case $s = (-1,-1)$: for a fixed n, we want to find β_0 such that for all $\beta_0 \le \beta_1 \le \beta_2$ such that β_1, β_2 belongs to the same interval of continuity of $\beta \mapsto T_\beta^n(1)$, we have

$$|T_2^n(1) - T_1^n(1)| > \frac{\delta}{\beta_2 - 1}\,. \tag{7.5}$$

Then setting $b_i = T_i^n(1)$, we conclude as in the case $k \ge 3$. Formula (7.5) is true, if $|\frac{d}{d\beta}T_\beta^n(1)| > \frac{1}{\beta-1}$ for all $\beta \ge \beta_0$. We have $T_\beta(1) = 2-\beta$, $T_\beta^2(1) = 1-\beta(2-\beta) = (\beta-1)^2$ and

$$T_\beta^3(1) = \begin{cases} 1-\beta(\beta-1)^2 & \text{if } (\beta-1)^2 < 1/\beta, \\ 2-\beta(\beta-1)^2 & \text{if } (\beta-1)^2 > 1/\beta. \end{cases}$$

Thus $\frac{d}{d\beta}T_\beta^3(1) = -3\beta^2 + 4\beta - 1$ and $|\frac{d}{d\beta}T_\beta^3(1)| = 3\beta^2 - 4\beta + 1$ for all $\beta \in (1,2]$. Then

$$|\frac{d}{d\beta}T_\beta^3(1)| > \frac{1}{\beta-1} \iff 3\beta^3 - 7\beta^2 + 5\beta - 2 > 0 \iff \beta > 1.52818\ldots\,.$$

Thus the last claim is proved for $\beta_0 = 1.52818$. Considering greater n, it is possible to obtain smaller β_0. This is illustrated in Figure 7.1, where we plot the graphs of $T_\beta^n(1)$ and $|\frac{d}{d\beta}T_\beta^n(1)|$ for $n = 1,2,3,4,5$. With $n = 4$, we get $\beta_0 = 1.5$; with $n = 5$, we get $\beta_0 = 1.40796\ldots$. □

In the tent map case, the separation of orbits is proved for $\beta \in (\sqrt{2}, 2]$ and then extended arbitrarily near $\beta_0 = 1$ using the renormalization. In the case $s = (-1,-1)$, there is no such argument and we are forced to increase n to obtain a lower bound β_0.

Now we turn to the question of normality for generalized β-transformations. The structure of the proof is very similar to the proof of Theorem 7.6 and Corollary 7.7.

Theorem 7.13. *Consider a family $\{T_\beta\}_{k-1<\beta\le k}$ of generalized β-transformations defined by a sequence $s = (s_n)_{0\le n<k}$. Let β_0 be defined as in Lemma 7.12. Then the set*

$$\{\beta > \beta_0 : \textit{the orbit of } \underline{\eta}^\beta \textit{ under } \sigma \textit{ is } \hat{\mu}_\beta\textit{-normal}\}$$

has full λ-measure.

Corollary 7.14. *Consider a family $\{T_\beta\}_{\beta>1}$ of generalized β-transformations defined by a sequence $s = (s_n)_{n\ge 0}$. Let β_0 be defined as in Lemma 7.12. Then the set*

$$\{\beta > \beta_0 : \textit{the orbit of } 1 \textit{ under } T_\beta \textit{ is } \mu_\beta\textit{-normal}\}$$

has full λ-measure.

Proof of Theorem: Let

$$B_0 := \{\beta \in (\beta_0, \infty) : 1 \notin S(\beta)\}\,.$$

From Lemma 7.11, this subset has full Lebesgue measure. To obtain uniform estimates, we restrict our proof to the interval $[\underline{\beta}, \overline{\beta}]$ with $\beta_0 < \underline{\beta} < \overline{\beta} < \infty$. Let $k := \lceil \overline{\beta} \rceil$ and $\Omega := \{\beta \in [\underline{\beta}, \overline{\beta}] \cap B_0 : \underline{\eta}^\beta$ is not $\hat{\mu}_\beta$-normal$\}$. As before, setting

$$\Omega_N := \{\beta \in [\underline{\beta}, \overline{\beta}] \cap B_0 : \exists \nu \in V_\sigma(\underline{\eta}^\beta) \text{ s.t. } h_\sigma(\nu) < (1-1/N)\log\beta\}\,,$$

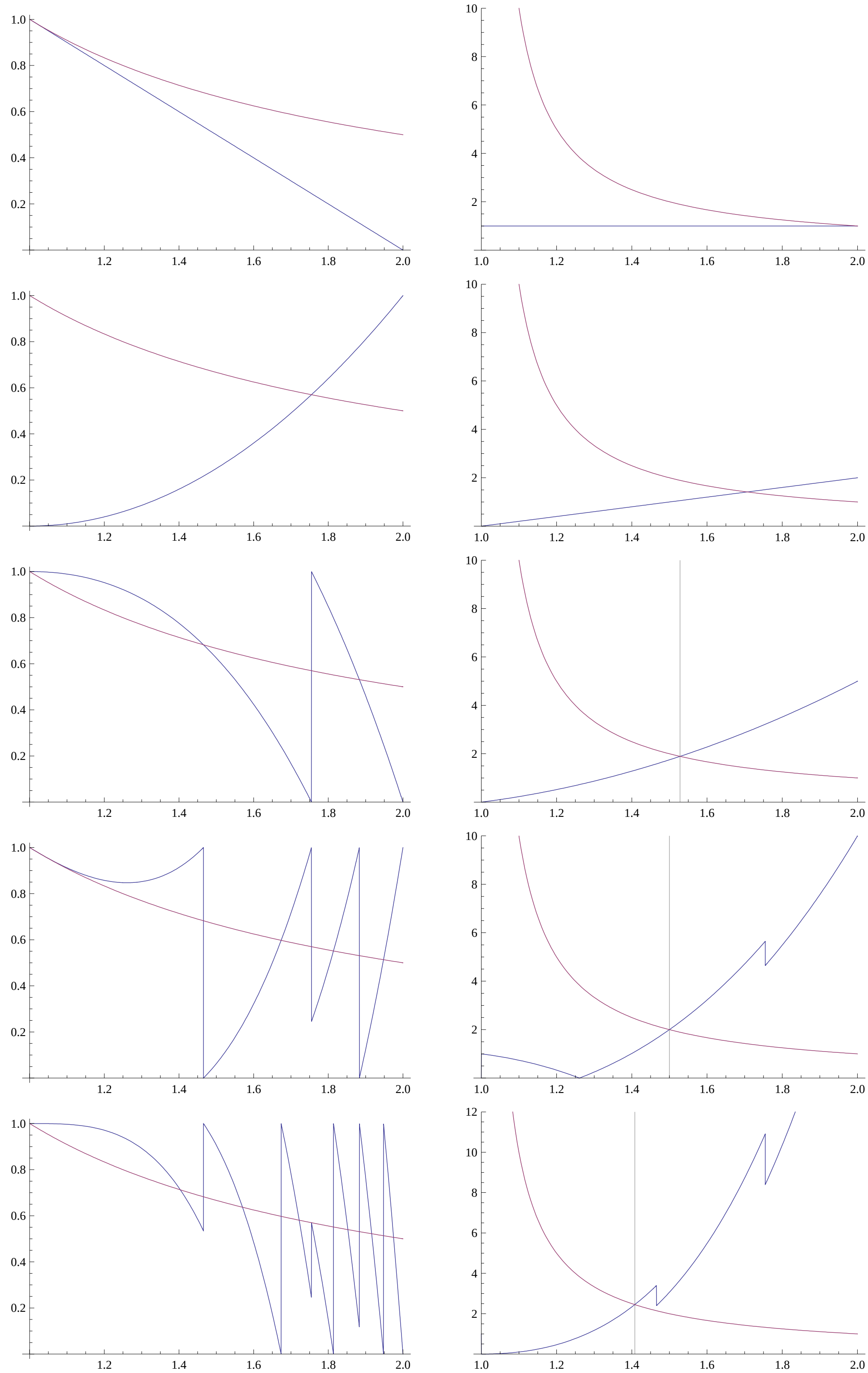

Figure 7.1: In the left column, we plot in red the graphs of $T_\beta^n(1)$ for $n = 1, \ldots, 5$ and in blue the critical point $a_1 = \frac{1}{\beta}$. In the right column, we plot in red the graph of $|\frac{d}{d\beta}T_\beta^n(1)|$ for $n = 1, \ldots, 5$ and in blue the graph of $(\beta - 1)^{-1}$; for $n = 3, 4, 5$, we add the critical value β_0.

we have $\Omega = \bigcup_{N\geq 1}\Omega_N$. We prove that $\dim_H \Omega_N < 1$. For $N \in \mathbb{N}$ fixed, define $\varepsilon := \frac{\beta\log\beta}{2N-1} > 0$ and L such that $\underline{\eta}^{\beta}_{[0,L)} = \underline{\eta}^{\beta'}_{[0,L)}$ implies $|\beta - \beta'| \leq \varepsilon$ (see Lemma 7.12). Consider the family of subsets of $[\underline{\beta}, \overline{\beta}]$ of the following type

$$J(\underline{w}) = \{\beta \in [\underline{\beta}, \overline{\beta}] : \underline{\eta}^{\beta}_{[0,L)} = \underline{w}\}\,,$$

where $\underline{w}$ is a word of length L. $J(\underline{w})$ is either empty or it is an interval. We cover the non-closed $J(\underline{w})$ with countably many closed intervals if necessary. We prove that $\lambda(\Omega_N \cap [\beta_1, \beta_2]) = 0$ where $\beta_1 < \beta_2$ are such that $\underline{\eta}^{\beta_1}_{[0,L)} = \underline{\eta}^{\beta_2}_{[0,L)}$.

Let $\underline{\eta}^j = \underline{\eta}^{\beta_j}$. Let

$$D^* := \{\underline{z} \in \Sigma_{\underline{\eta}^2} : \exists \beta \in [\beta_1, \beta_2] \cap B_0 \text{ s.t. } \underline{z} = \underline{\eta}^{\beta}\}\,.$$

Define $\rho_* : D^* \to [\beta_1, \beta_2] \cap B_0$ by $\rho_*(\underline{z}) = \beta \Leftrightarrow \underline{\eta}^{\beta} = \underline{z}$. As before, from formula (6.22) and strict monotonicity of $\beta \mapsto \underline{\eta}^{\beta}$, we deduce that ρ_* is well defined and surjective. We compute the coefficient of Hölder continuity of $\rho_* : (D^*, d_{\beta_*}) \to [\beta_1, \beta_2]$. Let $\underline{z} \neq \underline{z}' \in D^*$ and $n = \min\{l \geq 0 : z_l \neq z'_l\}$, then $d_{\beta_*}(\underline{z}, \underline{z}') = \beta_*^{-n}$. By Lemma 7.12, there exists C such that

$$|\rho_*(\underline{z}) - \rho_*(\underline{z}')| \leq C\rho_*(\underline{z})^{-n} \leq C\beta_1^{-n} = C(d_{\beta_*}(\underline{z}, \underline{z}'))^{\frac{\log\beta_1}{\log\beta_*}}\,.$$

By the choice of L and ε, we have

$$\frac{\log\beta_1}{\log\beta_*} \geq 1 - \frac{1}{2N}\,,$$

thus ρ_* has Hölder-exponent of continuity $1 - \frac{1}{2N}$. Define

$$G^*_N := \{\underline{z} \in \Sigma^* : \exists \nu \in V_\sigma(\underline{z}) \text{ s.t. } h_\sigma(\nu) < (1 - 1/N)\log\beta_*\}\,.$$

As before, we have $\Omega_N \cap [\beta_1, \beta_2] \subset \rho_*(G^*_N \cap D^*)$ and $h_{\text{top}}(G^*_N, \sigma) \leq (1 - 1/N)\log\beta_*$. Finally $\dim_H(\Omega_N \cap [\beta_1, \beta_2]) < 1$ and $\lambda(\Omega_N \cap [\beta_1, \beta_2]) = 0$. $\square$

Proof of the Corollary: The proof is similar to the proof of Corollary 7.7. Equation (7.1) holds, since we work on B_0. $\square$

7.4 Concluding remarks

We recall Theorem C of Schmeling's paper [S]. Schmeling considers the family of β-transformations $T_\beta := \beta x \mod 1$; it corresponds to generalized β-transformations with $s(j) = +1$ for all $j \in \mathtt{A}_k$. Schmeling's Theorem asserts that for Lebesgue almost all $\beta > 1$, the virtual itinerary $\underline{\eta}^{\beta}$ is $\hat{\mu}_\beta$-normal. This theorem is a particular case of Theorem 7.6 (take $\alpha = 0$ and $x = 1$) and Theorem 7.13 (take $s(j) = +1$ for all $j \in \mathtt{A}_k$).

The global structure of the proofs of Theorems 7.6 and 7.13 is similar. The method is inspired from the proof of Schmeling. In particular, the imbedding of the orbits $\mathtt{i}^{\kappa}(x)$ in a well chosen shift space is an idea of Schmeling (κ is a generic notation for the parameter: $\kappa = (\alpha, \beta)$ in Theorem 7.6 and $\kappa = \beta$ in Theorem 7.13). However, these theorems are non trivial generalizations of Schmeling's Theorem.

For the β-transformations as well as for the generalized β-transformations, there is only one important virtual itinerary. Recall that $\Sigma_\beta \equiv \Sigma_{\underline{\eta}^{\beta}}$, $h_{\text{top}}(\Sigma_\beta, \sigma) = \log\beta$ and

$$\underline{\eta} \preceq \underline{\eta}' \iff \Sigma_{\underline{\eta}} \subset \Sigma_{\underline{\eta}'}\,.$$

Thus the map $\beta \mapsto \underline{\eta}^\beta$ is strictly monotone increasing and $\beta \le \beta'$ implies that $\Sigma_\beta \subset \Sigma_{\beta'}$. Using this remark, the choice of the shift space where the orbits $\mathtt{i}^\beta(x)$ are imbedded is easy. We have

$$\mathtt{i}^\beta(x) \in \Sigma_{\beta_2} \quad \forall \beta \le \beta_2 \,,$$

and we choose to imbed the set $\{\mathtt{i}^\beta(x) : x \in [\beta_1, \beta_2]\}$ in Σ_{β_2}. This choice is easy, but above all the shift space Σ_{β_2} is itself a generalized β-shift.

For the maps $T_{\alpha,\beta}$, the situation is slightly more complicated. By Lemma 7.5, we know that the maps $\beta \mapsto \underline{u}^{\alpha,\beta}$ and $\beta \mapsto \underline{v}^{\alpha,\beta}$ are strictly monotone increasing (except for $\beta \mapsto \underline{u}^{0,\beta}$ which is constant). Thus, for all $\alpha > 0$, it is false that

$$\beta_1 \le \beta_2 \implies \Sigma_{\alpha,\beta_1} \subset \Sigma_{\alpha,\beta_2} \,,$$

and we cannot imbed the set $\{\mathtt{i}^{\alpha,\beta}(x) : \beta \in [\beta_1, \beta_2]\}$ in the shift space Σ_{α,β_2}. Fortunately, there exists a natural choice of this shift space. It is $\Sigma^* := \Sigma(\underline{u}^1, \underline{v}^2)$, because

$$\{\mathtt{i}^{\alpha,\beta}(x) : \beta \in [\beta_1, \beta_2]\} \subset \Sigma^* \,.$$

Nevertheless this choice creates troubles. Indeed $\Sigma^* \neq \Sigma_{\alpha,\beta}$ for some $\beta \in [\beta_1, \beta_2]$. When we must compute the exponent of Hölder-continuity, it is more complicated. In the previous case, we have only to control the length of the interval $[\beta_1, \beta_2]$; this is done by Lemma 7.12. In the case of $T_{\alpha,\beta}$, we control the length of the interval $[\beta_1, \beta_2]$ by Lemma 7.5, but we must also estimate the distance between β_1 and β_*, where β_* is defined by $\log \beta_* = h_{\mathrm{top}}(\Sigma^*, \sigma)$; this is done by Proposition 4.10.

It is interesting to notice that Σ^* is defined as $\Sigma(\underline{u}^1, \underline{v}^2)$. In particular, we do not prove that there exists a pair (α_*, β_*) such that $\Sigma_* = \Sigma_{\alpha_*,\beta_*}$. In fact, this is a corollary of Theorem 6.21, but we do not need this fact. Notice that, in Theorem 7.6, we work with α fixed and $\alpha_* < \alpha$ (if $\alpha > 0$). Thus we imbed the orbits in a really different dynamical system. It could be a little bit surprising that we do not use the existence of (α_*, β_*) such that $\Sigma_* = \Sigma_{\alpha_*,\beta_*}$. Indeed, the shift spaces of the type $\Sigma_{\alpha,\beta}$ are a subfamily of the shift spaces of the type $\Sigma(\underline{u}, \underline{v})$, because

$$\Sigma_{\alpha,\beta} = \Sigma(\underline{u}^{\alpha,\beta}, \underline{v}^{\alpha,\beta}) \,.$$

But some ergodic properties are different: for example, Hofbauer proved in [H4] that a shift space of the type $\Sigma_{\alpha,\beta}$ has a unique measure of maximal entropy, whereas he showed that there exist shift spaces of the type $\Sigma(\underline{u}, \underline{v})$ having two measures of maximal entropy.

Another non trivial adaptation of the proof of Schmeling is the estimate of $h_{\mathrm{top}}(G_N^*, \sigma)$. In his estimate, Schmeling uses the fact that the measure of maximal entropy is equivalent to Lebesgue measure and its density is uniformly bounded below and above. This is true for the β-transformations. For the maps $T_{\alpha,\beta}$, it is not true in general; the condition $\beta > 2$ is sufficient (for more details see [H6]). For the generalized β-transformations, it is not true in general, but $k \ge 3$ is a sufficient condition. Our proof uses Theorem 7.4 to estimate the topological entropy of the set G_N^*; this adaption is necessary to cover all maps $T_{\alpha,\beta}$.

Bibliography

[AKM] R. Adler, A. Konheim, and M. McAndrew, *Topological entropy*, Trans. Amer. Math. Soc., **114** (1965), 309-319.

[Ba] H. Bauer, *Measure and integration theory*, vol. 26 of de Gruyter studies in mathematics, de Gruyter, Berlin, 2001.

[BGMY] L. Block, J. Guckenheimer, M. Misiurewicz, and L.-S. Young, *Periodic Points and topological entropy of one-dimensional maps*, in Global theory of dynamical systems, no. 819 in Lecture notes in mathematics, Springer-Verlag, 1980.

[B1] R. Bowen, *Entropy for group endomorphisms and homogeneous spaces*, Trans. Amer. Math. Soc., **153** (1971), 401-414.

[B2] R. Bowen, *Topological entropy for noncompact sets*, Trans. Amer. Math. Soc., **184** (1973), 125-136.

[B3] R. Bowen, *On axiom A diffeomorphisms*, vol. 35 of Regional conference series in mathematics, Amer. Math. Soc, Providence, 1978.

[BB] K. Brucks and H. Bruin, *Topics from one-dimensional dynamics*, vol. 470 of London Mathematical Society student texts, Cambridge University Press, Cambridge, 2004.

[BM] K. Brucks and M. Misiurewicz, *The trajectory of the turning point is dense for almost all tent maps*, Ergod. Th. and Dynam. Sys., **16** (1996), 1173-1183.

[Br] H. Bruin, *For almost every tent map, the turning point is typical*, Fund. Math., **155** (1998), 215-235.

[CE] P. Collet and J.-P. Eckmann, *Iterated maps on the interval as dynamical systems*, Birkhäuser, Basel, 1980.

[MS] W. de Melo and S. van Strien, *One-dimensional dynamics*, Springer-Verlag, Berlin, 1993.

[DGP] B. Derrida, A. Gervois, and Y. Pomeau, *Iteration of endomorphisms on the real axis and representation of numbers*, Ann. Inst. Henri Poincaré, **29** (1978), 305-356.

[F] K. Falconer, *Fractal geometry*, Wiley, Chichester, 2003.

[Fu] H. Furstenberg, *Disjointness in ergodic theory, minimal sets, and a problem in Diophantine approximation*, Math. Sys. Theo., **1** (1967), 1-49.

[G] P. Góra, *Invariant densities for generalized β-maps*, Ergod. Th. and Dynam. Sys., **27** (2007), 1-16.

[Ha] S. Halfin, *Explicit construction of invariant measures for a class of continuous state Markov processes*, The Annals of Probability, **3** (1975), 859-864.

[H1] F. Hofbauer, *β-shifts have unique maximal measure*, Mh. Math., **85** (1978), 189-198.

[H2] F. Hofbauer, *Maximal measures for piecewise monotonically increasing transformations on* $[0,1]$, in Ergodic Theory, vol. 729 of Lecture Notes in Mathematics, 1979, 66-77.

[H3] F. Hofbauer, *On intrinsic ergodicity of piecewise monotonic transformations with positive entropy*, Israel J. Math., **34** (1979), 213-237.

[H4] F. Hofbauer, *Maximal measures for simple piecewise monotonic transformations*, Z. Wahrschein. Gebiete, **52** (1980), 289-300.

[H5] F. Hofbauer, *On intrinsic ergodicity of piecewise monotonic transformations with positive entropy II*, Israel J. Math., **38** (1981), 107-115.

[H6] F. Hofbauer, *The maximal measure for linear mod one transformations*, J. London Math. Soc., **23** (1981), 92-112.

[K] B. Kitchens, *Symbolic dynamics : one-sided, two-sided and countable state Markov shifts*, Springer, 1998.

[LM] D. Lind and B. Marcus, *An introduction to symbolic dynamics and coding*, Cambridge University Press, Cambridge, 1995.

[Ma] R. Mané, *Ergodic theory and differentiable dynamics*, Springer-Verlag, Heidelberg, 1987.

[MSS] N. Metropolis, M. Stein, and P. Stein, *On finite limit sets for transformations of the unit interval*, J. of Combin. Theory, **15** (1973), 25-44.

[Mi] J. Milnor, *Fubini foiled: Katok's paradoxical example in measure theory*, The Math. Intelligencer, **19** (1997), 30-32.

[MS] M. Misiurewicz and W. Szlenk, *Entropy of piecewise monotone mappings*, Studia Math., **67** (1980), 45-63.

[P1] W. Parry, *On the β-expansions of real numbers*, Acta Math. Acad. Sci. Hung., **11** (1960), 401-416.

[P2] W. Parry, *Representations for real numbers*, Acta Math. Acad. Sci. Hung., **15** (1964), 95-105.

[Pa] K. Parthasarathy, *Introduction to probability and measure*, Hindustan Book Agency, 2005.

[Pe] Y.B. Pesin, *Dimension theory in dynamical systems: contemporary views and applications*, The University of Chicago Press, Chigaco, 1997.

[PS] C.-E. Pfister and W. Sullivan, *On the topological entropy of satured sets*, Ergod. Th. and Dynam. Sys., **27** (2007), 926-956.

[R] A. Rényi, *Representations for real numbers and their ergodic properties*, Acta Math. Acad. Sci. Hung., **8** (1957), 477-493.

[Sa] D. Sands, *Topological conditions for positive Lyapunov exponent in unimodal maps*, PhD thesis, St John's College, 1993.

[S] J. Schmeling, *Symbolic dynamics for β-shifts and self-normal numbers*, Ergod. Th. and Dynam. Sys., **17** (1997), 675-694.

[Se] E. Seneta, *Non-negative matrices and Markov chains*, Springer, New York, 2006.

[T] Y. Takahashi, *Isomorphisms of β-automorphisms to Markov automorphisms*, Osaka J. Math., **10** (1973), 175-184.

[VJ1] D. Vere-Jones, *Geometric ergodicity in denumerable Markov Chains*, Quarterly J. of Math., **13** (1962), 7-28.

[VJ2] D. Vere-Jones, *Ergodic properties of nonnegative matrices*, Pacific J. of Math., **22** (1967), 361-386.

[W] P. Walters, *An introduction to ergodic theory*, vol. 79 of Graduate texts in mathematics, Springer-Verlag, New-York, 1982.

Curriculum Vitae

Bastien Faller

Institut d'analyse et de calcul scientifique (IACS)
Ecole polytechnique fédérale de Lausanne (EPFL)
CH-1015 Lausanne

Born on November 29th 1980 in Basel, Switzerland

Education and degrees

1996 - 1999	Maturité fédérale latin-grec, Lausanne.
1999 - 2004	Diploma studies in physics at the EPFL, Lausanne.
April 2004	Physics diploma thesis on *Le gaz de Bose au-delà du champ moyen* under the supervision of Pr. Philippe Martin.
2004 - 2008	Ph. D. studies in mathematics under the supervision of Pr. Charles-Edouard Pfister at the IACS(EPFL), Lausanne.

Professional background

2004	Teaching at the Collège d'Oron, Switzerland.
2004 - 2008	Research and teaching assistant at the IACS(EPFL), Lausanne.