ORIENTATION REVERSING INVOLUTIONS ON HYPERBOLIC TORI AND SPHERES

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Abstract. This article studies the relationship between simple closed geodesics and orientation reversing involutions on one-holed hyperbolic tori.

1. Simple closed geodesics on the one holed torus

We consider topological surfaces of signature (g, n) (where g is the underlying genus and n is the number of simple boundary curves) with negative Euler characteristic and endowed with a hyperbolic metric such that the boundary curves are simple closed geodesics or cusps. A surface of signature $(1, 1)$ will be called a one holed torus, and for the remainder of the article T will denote such a surface. The boundary geodesic of T will always be denoted η. If the boundary geodesic is a cusp, then in place of a true boundary geodesic, by considering a small enough horocycle neighborhood around the cusp, one obtains the same properties than in the case where the boundary curve is a simple closed geodesic. Notably, the notion of distance to a cusp can be introduced. Geodesics will generally be considered non-oriented and primitive, and the notation for a path or a geodesic will not be distinguished in order to simplify notation.

Cutting T along an interior simple closed geodesic γ gives a surface of signature $(0, 3)$, generally referred to as a pair of pants or a Y-piece. It is given up to isometry by the lengths of its three boundary geodesics. By cutting along the unique perpendicular geodesic path d_η between the two copies of γ, one now obtains a hyperbolic rectangle with the boundary geodesic η.

In figure 1 four additional geodesic curves have been drawn which decompose the rectangle intro four isometric right angled hyperbolic pentagons. The length of h_η is determined by the lengths of γ and η is through the formula for hyperbolic pentagons

$$\sinh(\frac{d_\eta}{2})\sinh(\frac{\gamma}{2}) = \cosh(\frac{\eta}{4}).$$

For every simple closed geodesic γ, there is a unique simple geodesic path from η to η and perpendicular to η, which does not cross γ. These path is labeled h_γ in

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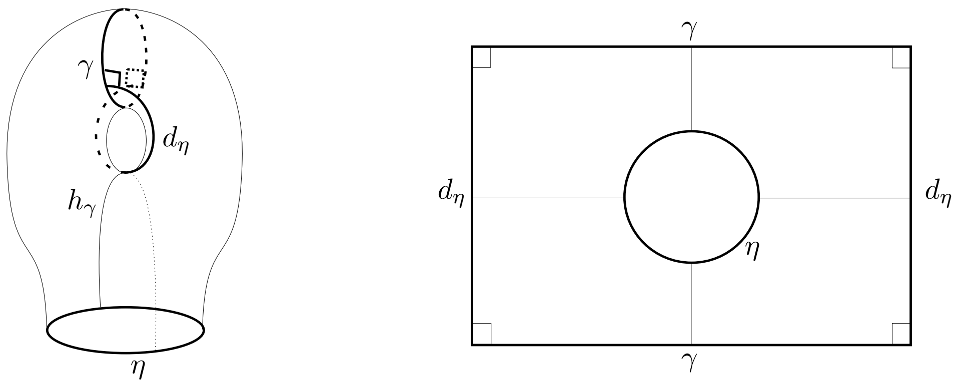

FIGURE 1. The one holed torus

figure 1. Furthermore, if the length of η is known, there is a direct correspondence between the length of γ and the length of h_γ. The formula for the length of γ is

$$\cosh(\frac{\gamma}{2}) = \sinh(\frac{h_\gamma}{2})\sinh(\frac{\eta}{4}).$$

Thus, for two distinct simple closed geodesics α and β on T, if $h_\alpha < h_\beta$ then $\alpha < \beta$ etc.

To describe a one holed torus with a one holed rectangle, you need to explicit how the two copies of γ will be glued together, and this is given by what is known as a twist parameter which we will explain later.

This model is very good for seeing the hyperelliptic involution: it acts as a rotation of angle π around η. This is clearly an isometry of the rectangle, but it also extends to an isometry of order 2 of the torus no matter what the twist parameter is. Notice that it is uniquely defined, leaves all simple closed geodesics invariant, and has three fixed points. One of these fixed points can be located on figure 1 as the midpoint of the two copies of h_η (also the midpoint of h_γ), and the other two are located on γ (on diametrically opposite points) but their location depends on the choice of twist parameter. The three fixed points of the hyperelliptic involution, which shall be denoted τ_h, are the Weierstrass points of T.

From this model it is also easy to construct a covering space which will be useful in the sequel. First take identical copies of the rectangle and paste them along copies of h_η to obtain an infinite strip with holes. Then, depending on a choice of how copies of γ are pasted together, by pasting the strips along the multiple copies of γ, one obtains hyperbolic plane with holes, or a surface of signature $(0,\infty)$, where there is a natural $\mathbb{Z}\times\mathbb{Z}$ action by translation isometries.

Another way of constructing a fundamental domain of the torus is by considering two simple disjoint perpendicular to boundary geodesic paths from η to η. As they both correspond to simple closed geodesics α and β, denote them by h_α and h_β. Cutting along these paths gives an octogon as in figure 3. This octogon can be split into two isometric right angled hexagons by choosing a third boundary to boundary

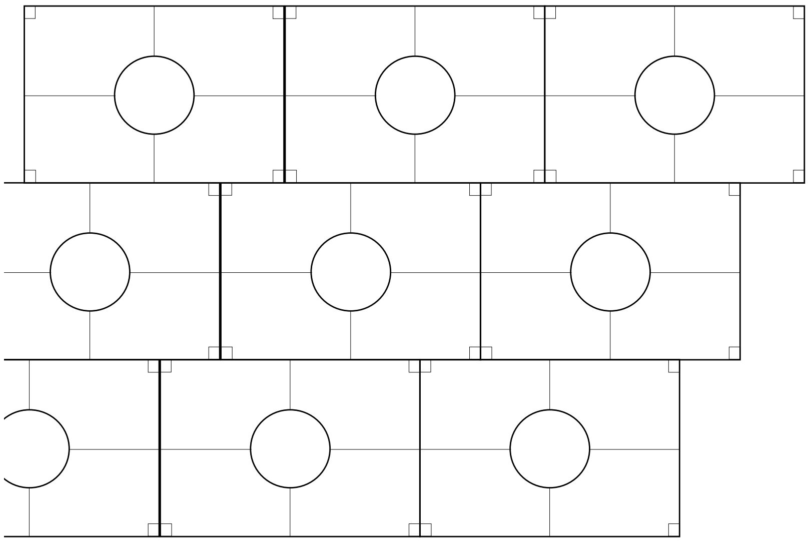

Figure 2. The $\mathbb{Z} \times \mathbb{Z}$ covering of the torus

geodesic on T disjoint from the first two. Notice that there are two possible choices for h_γ.

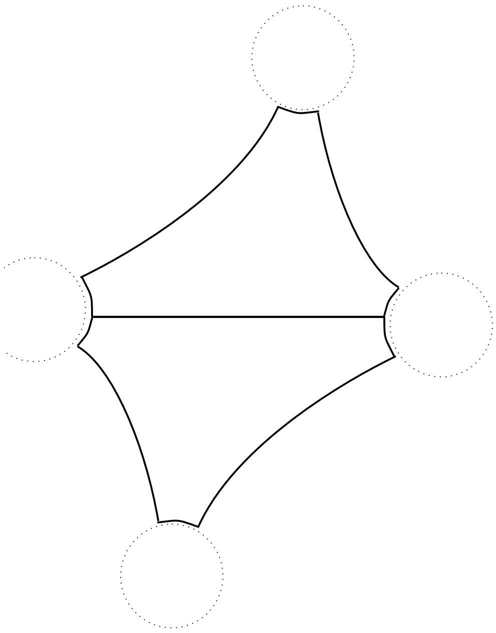

Figure 3. Another fundamental domain

Conversely, for any choice of octogon obtained this way from two copies of a hexagon, a torus is obtained by the regular pasting scheme. The first advantage of this model is that there is no necessity for the introduction of a twist parameter: the three lengths of h_α, h_β and h_γ determine the hexagon, and thus the torus, up to isometry (including the length of η). Also, the three Weierstrass points are now visible: they are the midpoints of h_α, h_β and h_γ. The hyperelliptic involution can now be seen as the rotation of angle π around the midpoint of h_γ. As before, one can consider the $\mathbb{Z} \times \mathbb{Z}$ tiling of a hyperbolic plane with holes obtained by taking copies of this octogon preserving the holes corresponding to copies of η as depicted on figure 4.

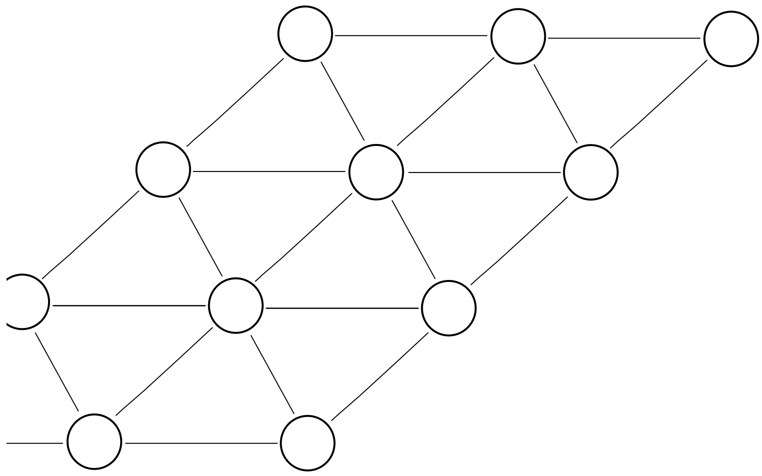

FIGURE 4. Another tiling of the hyperbolic plane with holes

The surface of signature $(0, \infty)$ thus obtained is of course the same one as previously. We will denote this surface $\mathbb{H}_T$. Note that $\mathbb{H}_T = \mathbb{H}_{\tilde{T}}$ if and only if T and $\tilde{T}$ are isometric. For two holes, say η_1 and η_2, denote by $t_{\eta_1\eta_2}$ the translation which brings η_1 to η_2.

All simple perpendicular geodesic boundary to boundary paths on T lift to simple perpendicular geodesic boundary to boundary paths on $\mathbb{H}_T$. Conversely, a simple geodesic boundary to boundary path on $\mathbb{H}_T$ projects to such a path on T if and only if it is disjoint from all of its images under the natural $\mathbb{Z} \times \mathbb{Z}$ action on $\mathbb{H}_T$. Such a path will be called *strongly simple*. Furthermore, for two given copies of η in $\mathbb{H}_T$, say η_1 and η_2, exactly one geodesic path between them corresponds to a simple perpendicular geodesic boundary to boundary paths on T. This path may eventually consist of multiple copies of a simple perpendicular geodesic boundary to boundary path on $\mathbb{H}_T$. If it does not consist of multiple copies, it will be called primitive. Using the relationship between simple perpendicular geodesic boundary paths on T and simple closed geodesics, we can characterize simple closed geodesics on T by means of a choice of two disjoint geodesic paths h_α and h_β contained in $\mathbb{H}_T$ verifying the following conditions.

(1) h_α and h_β are disjoint and are disjoint from all copies of both under the $\mathbb{Z} \times \mathbb{Z}$ action.
(2) If h_α joins two holes, say η_0 and η_β, then h_β joins η_0 and η_β.
(3) The parallelogram given by h_α, h_β, $t_{\eta_0\eta_\beta}(h_\alpha)$ and h_β, $t_{\eta_0\eta_\alpha}(h_\beta)$ does not contain any holes in its interior.

A pair of paths that verify these conditions will be called a basis of $\mathbb{H}_T$.

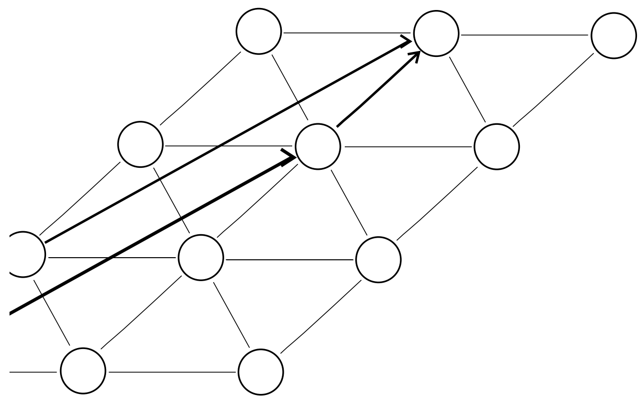

FIGURE 5. A basis for $\mathbb{H}_T$

Using this basis we can now mimick the classical construction of simple closed geodesics on the torus by remarking the following.

Proposition 1.1. *For a choice of basis h_1, h_2 on $\mathbb{H}_T$, the set of simple closed geodesics on T are is in one to one correspondence with $\mathcal{Q} \cup \infty$. More precisely, if γ is a primitive simple closed geodesic on T, then h_γ lifts on $\mathbb{H}_T$ to a unique path given by $a_1h_1 + a_2h_2$ with $a_1 \in \mathbb{Z}$ and $a_2 \in \mathbb{N}$ where a_1 and a_2 are relatively prime.*

Proof. First remark that if a_1 and a_2 are relatively prime, then the path $a_1h_1 + a_2h_2$ is strongly simple (i.e. an admissible path that joins two holes directly), and thus projects to a unique corresponding path on T to which a unique simple closed geodesic is associated.

Conversely, consider γ and h_γ. For a given hole on $\mathbb{H}_T$, say η_0, it is not too difficult to see that there are exactly two lifts of h_γ that touch η_0. Furthermore, they are opposite, i.e. if one corresponds to $a_1h_1 + a_2h_2$, then the other corresponds to $-a_1h_1 - a_2h_2$, and thus by fixing $a_2 \in \mathbb{N}$, the representative is unique. Thus we have $q = \frac{a_1}{a_2}$, and $\infty = \frac{1}{0}$. □

Essentially, this well known association between $\mathcal{Q}$ and simple closed geodesics will not be needed in the sequel, but the association between simple closed geodesics and paths in $\mathbb{H}_T$ will be needed.

2. Isometries and involutions of the torus

The goal of this section is to characterize isometries of the torus by means of simple closed geodesics and their associated heights.

First of all remark that if τ is an isometry of T, then it leaves η invariant. In fact it is determined by its action on η.

Proposition 2.1. *If τ_1 and τ_2 are two isometries of T such that their action on η is the same, then $\tau_1 = \tau_2$.*

Proof. It suffices to prove that if an isometry τ fixes η pointwise, then τ is the identity. If an isometry fixes η pointwise, then it lifts to an isometry of $\mathbb{H}$ that fixes all lifts of η, and is thus the identity on $\mathbb{H}$, and thus τ is also the identity. □

Notice that if a simple closed geodesic γ is left invariant by τ, then h_γ is also invariant. In this case, the two endpoints of h_γ are interchanged, in which case only the midpoint of h_γ is fixed, or the endpoints are fixed in which case all of h_γ is fixed pointwise. This implies that either τ is the identity, or reverses orientation.

Proposition 2.2. *The only non-trivial orientation preserving involution of T is the hyperelliptic involution τ_h.*

Proof. If τ is an orientation preserving involution, then τ acts on η as either the identity, or τ^2 does.The latter is the only non trivial possibility and in this case, because τ is orientation preserving, it acts as a rotation of angle π on η. The hyperelliptic involution has the same action on η and this concludes the argument. □

Proposition 2.3. *If τ is an orientation reversing isometry of T then it is an involution. Furthermore, there is a unique simple closed geodesic γ such that τ fixes h_γ pointwise. Either* $\mathrm{Fix}(\tau) = \gamma \cup h_\gamma$ *or* $\mathrm{Fix}(\tau) = h_\gamma$ *and τ acts as a rotation of angle π on γ.*

Proof. Consider the action of τ on η. Notice that τ necessarily reverses the orientation of η. It follows that τ fixes exactly two diametrically opposite points on η, say p_1 and p_2. Thus τ^2 fixes η pointwise and is thus the identity.

For the second part, consider the complete geodesic path c leaving p_1 at a perpendicular angle to η. Clearly, $c \subset \mathrm{Fix}(\tau)$ and c cannot intersect η in different points from p_1 and p_2. Also, c is necessarily simple. It follows that c is a simple geodesic path from p_1 to p_2, and is perpendicular to η. This path is the height for some simple closed geodesic γ, and thus we denote it by h_γ. Any other fixed point path from η to η would have to be geodesic, perpendicular to η and intersect η in p_1 or p_2, and by unicity of geodesics, unicity of h_γ follows.

Finally, it is clear that $\tau(\gamma) = \gamma$. Cutting T along γ one obtains a pair of pants Y with boundary geodesics η, and the two copies of γ, say γ_1 and γ_2. On Y, τ acts as the symmetry along h_γ, and all fixed points of τ contained in ∂Y lie on h_γ. The only issue remaining is the action of τ on γ. As τ is an involution, then if τ does not fix γ pointwise, then τ acts as a rotation of angle π on γ or has exactly two fixed points. In the latter case, this would imply fixed points for τ in ∂Y not lying on h_γ, and this concludes the argument. □

From this the following result follows.

Corollary 2.4. *A torus T admits a orientation reversing involution as an involution if and only if it admits a simple closed geodesic such that the twist parameter is* 0 *or* $\frac{1}{2}$.

Proof. From the preceding proposition, an orientation reversing involution τ either verifies $\mathrm{Fix}(\tau) = \gamma \cup h_\gamma$ or $\mathrm{Fix}(\tau) = h_\gamma$ for some simple closed geodesic γ. It is easy to verify that in the first case $tw_\gamma = 0$ and in the second $tw_\gamma = \frac{1}{2}$. Conversely, denote by γ the simple closed geodesic with twist parameter 0 or $\frac{1}{2}$. On Y constructed as in the previous proposition, the involution given by the symmetry along h_γ naturally extends to an involution on T. □

Note that in the case of closed surfaces, there is a much more general result of this type found in [2].

Proposition 2.5. *For a non-trivial, τ, if there is a simple closed geodesic γ on T such that $\tau(\gamma) = \gamma$, then τ is an involution.*

Proof. If $\tau(\gamma) = \gamma$ then $\tau(h_\gamma) = h_\gamma$. If τ fixes h_γ pointwise, then, by what precedes, τ is an orientation reversing involution. If not, then τ^2 does fix h_γ pointwise, and is not the identity then it is an orientation reversing involution. But this is not possible, because if τ is orientation reversing, then τ^2 is not, and if τ is orientation preserving, then τ^2 is as well. □

This proposition deals with systoles.

Proposition 2.6. *If σ is a systole of T and τ an isometry, then either $\tau(\sigma) = \sigma$ or $i(\sigma, \tau(\sigma)) = 1$.*

Proof. If $\tau(\sigma) \neq \sigma$ then $i(\sigma, \tau(\sigma)) \geq 1$. If $i(\sigma, \tau(\sigma)) > 1$, then $i(h_\sigma, h_{\tau(\sigma)}) \geq 1$. In that case it is easy to see that there is a non-homotopically trivial boundary to boundary path shorter than h_σ which in turn produces an interior simple closed geodesic shorter than σ, a contradiction. □

The fact that a systole cannot intersect its image more than once is in fact true on any surface that is not of signature $(0, 4)$. From these different facts stated above, we get a geometric interpretation of the well-known result concerning the order of an isometry of the torus.

Corollary 2.7. *An isometry τ of T has order $n \leq 6$ where n divides* 12.

Proof. Let σ be a systole of T. If τ has order > 2 then by the previous propositions, $i(\sigma, \tau(\sigma)) = 1$ and thus $h_\sigma \cap h_{\tau(\sigma)} = \emptyset$. Now either
1. $h_{\tau^2(\sigma)} = h_\sigma$ or
2. $h_{\tau^2(\sigma)} \cap h_{\tau(\sigma)} \cap h_\sigma = \emptyset$.

In case 1, τ^2 fixes σ and thus $\tau^4 = id$. In case 2, because there cannot be more than three disjoint heights on T, it follows that $h_{\tau^3(\sigma)} = h_\sigma$. Thus τ^3 fixes σ and $\tau^6 = id$. □

Notice that if T has an isometry τ of order 3, then $\tau \circ \tau_h$ is an isometry of order 6. In the case where τ is of order 4, then $\tau \circ \tau_h$ is another isometry of order 4. For a given length of η, both these situations are attained by unique tori (up to isometry).

Proposition 2.8. *For given boundary geodesic length, there is a unique torus with an isometry of order* 4 *(resp. order* 6*). These tori will be denoted* T_4 *(resp.* T_6*).*

Proof. Consider T with an isometry τ of order 4, and σ a systole of T. As seen in the proof of the previous corollary, $\tau(h_\sigma) \cap h_\sigma$. The endpoints of h_σ and $\tau(h_\sigma)$ divide η into four arcs of equal length. This implies that $T \setminus h_\sigma \setminus \tau(h_\sigma)$ is a hyperbolic right-angled octogon, as in the following figure.

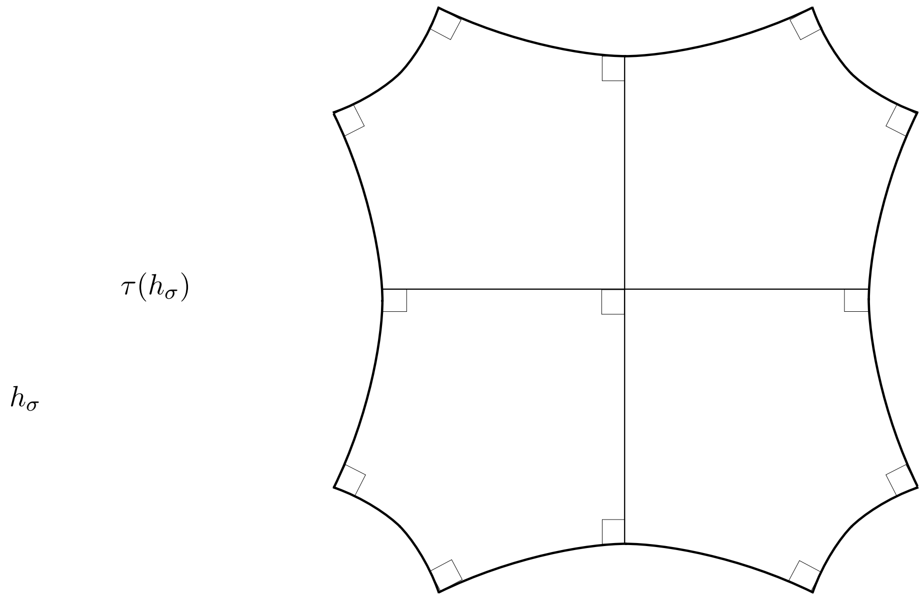

Figure 6. The torus T_4

If we denote x the length of h_σ, then the octogon has four non-adjacent sides of length x and the remaining sides of length $\frac{\eta}{4}$. It can thus be decomposed into five right angled pentagons as in figure 6. The two transversal geodesic lines correspond to the two systoles of the surface. The three sides of a pentagon on the octogon are of length $\frac{x}{2}, \frac{\eta}{4}$ and $\frac{x}{2}$. By hyperbolic plane geometry, the value of x is uniquely determined by the length of η. By the standard formula for right-angled pentagons, the length of the systoles is given by the formula

$$cosh^2(\frac{\sigma}{2}) - 1 = cosh(\frac{\eta}{4})$$

and thus $cosh(\frac{\sigma}{2}) = \sqrt{2}\cosh(\frac{\eta}{4})$. Notice that the full isometry group of this torus is generated by two orientation reversing involutions (for example the two involutions obtained by taking the symmetry along the two heights of the two systoles) and the hyperelliptic involution. In all, the surface admits 5 different involutions.

In the case where there is an isometry τ of order 6, then by the equivalent reasoning, there are three disjoint equal length heights. The endpoints of the three heights divide η into six arcs of equal length. By cutting along the three heights, one obtains two isometric right-angled hexagons. Each hexagon has three non adjacent edges of length the length of h_σ, and the three remaining edges are of length $\frac{\eta}{6}$. For a fixed length of η, this hexagon is uniquely determined, and thus T_6 is determined uniquely. Notice that the systole length of T_6 is given by the formula (i.e. [5], [4])

$$\cosh(\frac{\sigma}{2}) = \cosh(\frac{\eta}{6}) + \frac{1}{2}.$$

Consider the three heights of the three systoles of T_6 and the three orientation reversing involutions obtained by taking the symmetry along them. It is not to difficult to see that the three involutions plus the hyperelliptic involution generate the full isometry group of T_6. □

The result for T_6 is well known, and this torus has maximum size systole among all tori of same boundary length, see for instance [5].

Corollary 2.9. *All isometries of the torus different from the hyperelliptic involution are the product of at most two orientation reversing involutions.*

Proof. If τ is an orientation preserving isometry, then it is of order $n \leq 6$ which divides 12. If $n = 2$ then τ is the hyperelliptic involution. If $n = 3$ (resp. 4, 6) in the preceding proposition we saw that the surface with τ as an isometry is unique. In all cases, all isometries are a product of two orientation reversing involutions. Furthermore, if τ is an orientation reversing isometry of a torus then it is an involution, and this concludes the argument. □

3. Geodesics of equal length and isometries of the torus

If τ is an isometry of T, different from the hyperelliptic involution, then there are an infinity of simple closed geodesics of T that are not globally fixed by τ. This implies the existence of distinct equal length simple closed geodesics γ_1 and γ_2 such that $\tau(\gamma_1) = \gamma_2$. A natural question is to ask under what conditions does the existence of two distinct equal length geodesics imply the existence of an isometry between them. For the specific case of T_6 with η a cusp, there is a conjecture that states that between simple closed geodesics of equal length there is always an isometry of T_6 from one to the other. This conjecture is equivalent to a old number theory conjecture concerning Markov triples (i.e. [3]) and is still open, despite strong partial results (i.e. [6], [1]). There is another conjecture stated in [7], that says that there are at most 6 simple geodesics of equal length on a one holed torus. This is a stronger conjecture that implies that between simple closed geodesics of equal length on any T_6 there is an isometry. It also implies that any T_4 would have the same property.

Here we show, for any torus, necessary and sufficient conditions for the existence of an orientation reversing involution between two simple closed geodesics of equal length. In general, we have the following proposition.

Proposition 3.1. *Let T be a hyperbolic torus and let γ and $\tilde{\gamma}$ be simple closed geodesics of equal length on T. The two following statements are equivalent:*

1. There exists an isometry φ of T such that $\varphi(\gamma) = \tilde{\gamma}$.

2. The twist parameter $t_\gamma \in]-\frac{1}{2}, \frac{1}{2}]$ for γ and the twist parameter $t_{\tilde{\gamma}} \in]-\frac{1}{2}, \frac{1}{2}]$ for $\tilde{\gamma}$ verify $t_\gamma = \pm t_{\tilde{\gamma}}$.

Proof. □

For a torus with two simple closed geodesics of equal length, say γ_1 and γ_2, consider the rectangle with geodesic boundary (as in section 1) for one of the two geodesics.

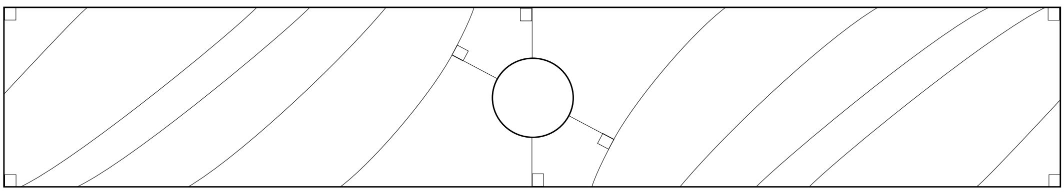

FIGURE 7. Two geodesics with 9 intersection points

Notice that because the two geodesics are of equal length, the distance between them and η is equal. This creates two pairs of isometric trirectangles as in the figure. Furthermore, if one cuts T along both γ_1 and γ_2, then one obtains a collection of $n-1$ hyperbolic quadrilaterals where n is the number of intersection points between the two geodesics, and one quadrilateral with boundary geodesic η. Denote by D_1 and D_2 the two rectangles for the two geodesics.

NEEDS CORRECTION

Consider the following subspace of $\mathbb{H}_T$ which is an n-sheeted covering of T, which we shall call checkerboard for γ_1 and γ_2, and which we shall denote $C_{\gamma_1\gamma_2}$.

Number the boxes a_{ij} of the checkerboard like the elements of a matrix $n \times n$. Then denote by k_1 the element a_{2k_1} corresponding to where the copy of η is situated, and by k_2 the element a_{k_12} corresponding to where the copy of η is situated. We now have the following proposition.

Proposition 3.2. *There is an orientation reversing involution between γ_1 and γ_2 if and only if $\ell(\gamma_1) = \ell(\gamma_2)$ and $k_1 = k_2$.*

Proof. Suppose there is an involution τ between γ_1 and γ_2 that reverses orientation. Thus $\tau(C_{\gamma_1\gamma_2}) = C_{\gamma_1\gamma_2}$, and the two geodesics are interchanged by τ. Necessarily, $k_1 = k_2$.

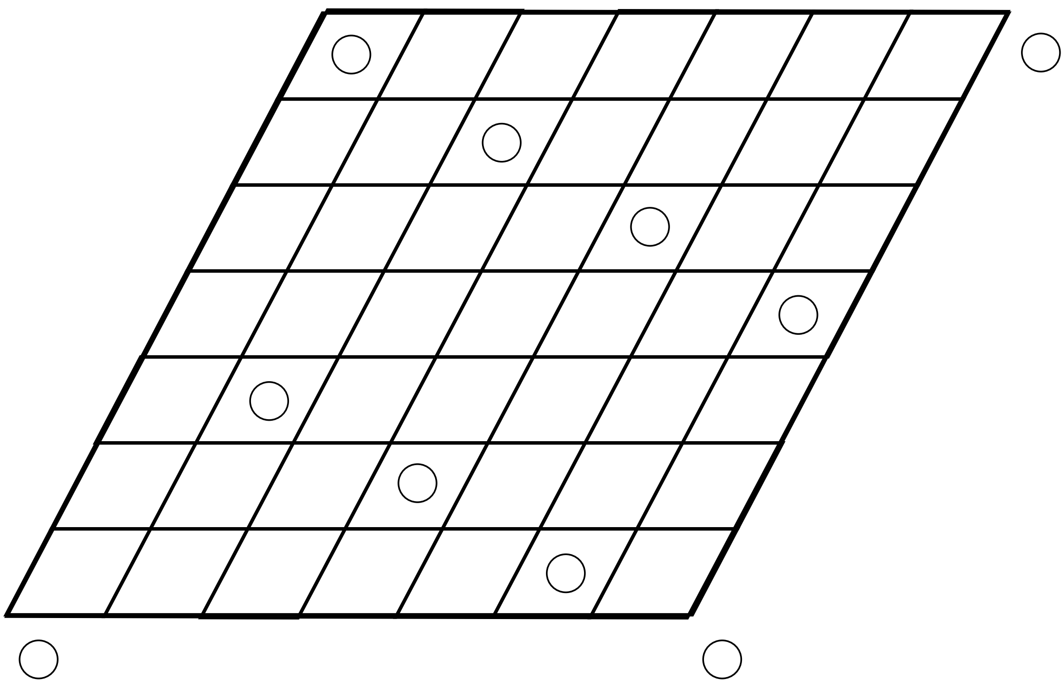

FIGURE 8. The checkerboard for two geodesics with 7 intersection points

Now suppose $k_1 = k_2$. Fix γ_1 and consider what happens to γ_2 under twist deformations along γ_1. By the convexity of geodesic length functions under twist deformations, it follows that there are at most two twist parameters for which the length of γ_2 is equal to that of γ_1. The same reasoning holds for when γ_2 is fixed.

Now consider the oriented angle θ as in the following figure. As the geodesics are in symmetric situations, it suffices to show that their twist parameter must be the same (or opposite) in a situation where they are of equal length, and have same value of θ.

Fix $\theta(0)$ as the value when the two geodesics are of equal length and let $\theta(t)$ denote the angle under twisting $t \in \mathbb{R}$. Under twisting, $\theta(t)$ changes, and it is not too difficult to see that $\theta(t)$ is monotonous (non-constant). So for a fixed value of $\theta(0)$, there is at most one twist parameter where the geodesics are of equal length and where they intersect at angle $\theta(0)$ as in the figure. This concludes the argument. □

If we denote by n the number of intersection points between two intersecting geodesics γ_1 and γ_2, then $(k_1, n) = (k_2, n) = 1$, otherwise the geodesics would not be primitive. Furthermore, for a given k_1, the value for k_2 is given by Bezout, i.e. k_2 is the smallest integer such that

$$k_1 \, k_2 = 1 \mod n.$$

Lemma 3.3. *Consider the set of inversable elements of $\mathbb{Z}/n\mathbb{Z}$, denoted $(\mathbb{Z}/n\mathbb{Z})^\times$. Then all elements $k \in (\mathbb{Z}/n\mathbb{Z})^\times$ verify $k^2 = 1$ if and only if n divides 24.*

Proof. Consider the decomposition of n into prime factors $n = p_1^{\alpha_1} \dots p_r^{\alpha_r}$. Then it is well known that

$$(\mathbb{Z}/n\mathbb{Z})^\times \simeq \Pi_{i=1}^r (\mathbb{Z}/p_i^{\alpha_i}\mathbb{Z})^\times.$$

From this we can calculate $(\mathbb{Z}/n\mathbb{Z})^\times$ by

$$(\mathbb{Z}/p^\alpha\mathbb{Z})^\times \simeq \mathbb{Z}/\varphi(p^\alpha)\mathbb{Z} = \mathbb{Z}/p^{\alpha-1}(p-1)\mathbb{Z}.$$

It follows that $(\mathbb{Z}/p^\alpha\mathbb{Z})^\times$ contains an element of order $p-1$. As we want all elements of $(\mathbb{Z}/n\mathbb{Z})^\times$ to be of order at most 2, it follows that all prime factors of n are either 2 or 3. Furthermore, if $\alpha > 1$, then $2 \in \mathbb{Z}/\varphi(3^\alpha)\mathbb{Z}$ is of order greater than 2. Finally, if $\alpha > 3$, then $3 \in \mathbb{Z}/\varphi(2^\alpha)\mathbb{Z}$ has order greater than 2. From this we deduce that $n = 2^{\alpha_1}3^{\alpha_2}$ with $\alpha_1 \leq 3$ and $\alpha_2 \leq 1$. Finally, for all such n, we have that $(\mathbb{Z}/n\mathbb{Z})^\times$ is a finite product of $\mathbb{Z}/2\mathbb{Z}$s, and this concludes the proof. □

Theorem 3.4. *There is an orientation reversing involution between γ_1 and γ_2 verifying $\ell(\gamma_1) = \ell(\gamma_2)$ if $i(\gamma_1, \gamma_2) = n$ and n divides 24. Conversely, for any length of η, and for all n that does not divide 24, then there is a torus T with boundary geodesic η and two simple closed geodesics γ_1 and γ_2 verifying $\ell(\gamma_1) = \ell(\gamma_2)$ such that there is no orientation reversing involution between them.*

Proof. Suppose that γ_1 and γ_2 verify $\ell(\gamma_1) = \ell(\gamma_2)$, $i(\gamma_1, \gamma_2) = n$ and n divides 24. From the previous lemma, $k_1 = k_2$. From proposition 3.2, there is an orientation reversing involution between them.

Conversely, if n does not divide 24, then by the previous lemma there exists a $\tilde{k}$ such that $(\tilde{k}, n) = 1$, and $\tilde{k}^2 \neq 1 \mod n$. Now consider a geodesic on a torus, say γ_1, and consider a geodesic that intersects it n times, so as to obtain a checkerboard with $k_1 = \tilde{k}$, and thus $k_2 \neq k_1$. Using the convexity of geodesic length functions under twisting, one can twist along the longest of the two geodesics until the geodesics are of equal length. As before, notice this does not change the configuration of the checkerboard, and by proposition 3.2, this implies that there is no orientation reversing involution between the two geodesics, now of equal length, and with n intersection points. □

Although the result is complete for orientation reversing involutions, there may still be an isometry between geodesics of equal length, but of course it would have to be a product of two such involutions, and it could only occur on the surfaces T_4 or T_6.

4. Simple closed geodesics on a four holed sphere

This section is devoted to the relationship between simple closed geodesics on a torus and those of a sphere with four boundary geodesics where all boundary geodesics are of equal length. Although it seems to be common knowledge that the simple closed geodesics of both types of surfaces are closely related, the exact length relationship between the geodesics of corresponding surfaces seems not to have been studied before.

To see this relationship, first consider a surface X of signature $(0, 4)$ with boundary geodesics, say $\eta_1, \dots, \eta_4$, of equal length ℓ. A simple closed geodesic γ on X separates X into two pairs of pants. Let us denote these pairs of pants $P_1 = (\eta_1, \eta_3, \gamma)$ and (η_2, η_4, γ). The length of γ, ℓ and a twist parameter determine the surface up to isometry. Consider the unique perpendicular geodesic simple path c_1 between η_1 and η_3 (resp. c_2 between η_2 and η_4). Consider the midpoints p_1 and p_2 of the paths c_1 and c_2. The rotation of angle π around p_1 (resp. p_2) is an isometry of P_1 (resp. P_2). It is not too difficult to see that the two isometries induce a global isometry σ_1 of X with the two fixed points p_1 and p_2. This isometry is a hyperelliptic involution (i.e. orientation preserving of order 2 and the quotient is topologically a sphere). The isometry σ_1 sends η_1 to η_3. Of course the isometry depends on our original choice of simple closed geodesic γ, but in fact there are exactly three distinct isometries constructed this way, say σ_1, σ_2 and σ_3, which we shall call the three hyperelliptic involutions of X. Each is determined by its action on the set of boundary geodesics. (By construction, a boundary geodesic is never invariant under such an isometry and there are thus only three possibilities for the image of η_1 for instance.) The set of all the fixed points of the three hyperelliptic involutions is thus a set of six points, say $\{p_1, \dots, p_6\}$.

Proposition 4.1. *Every simple closed geodesic γ on X passes through exactly four of the points in $\{p_1, \dots, p_6\}$.*

Proof. □

From this, the following corollary is immediate.

Corollary 4.2. *All simple closed geodesics of X are globally invariant by any one of its hyperelliptic involutions.*

We can now discuss the relationship between simple closed geodesics of a one-holed torus T and the simple closed geodesics of a four holed sphere.

Theorem 4.3. *Let T be a one holed torus with boundary geodesic η. There is a one to one correspondence φ between the simple closed geodesics of T and the simple closed geodesics of a uniquely defined (up to isometry) four holed sphere X with boundary geodesics all of length $\ell(\eta)/2$ such that $\ell(\varphi(\gamma)) = 2\ell(\gamma)$. Furthermore, if $\mathrm{int}(\alpha, \beta) = k$, then $\mathrm{int}(\varphi(\alpha), \varphi(\beta)) = 2k$. The reciprocal holds as well (i.e. a one to one correspondence $\tilde{\varphi}$ between the simple closed geodesics of a four holed sphere X and the simple closed geodesics of a one holed torus with opposite properties).*

Note that, for given boundary length ℓ, $\varphi = \tilde{\varphi}^{-1}$ can be seen as a one to one application between the set of all tori with boundary length ℓ and the set of four holed spheres with boundary lengths all equal to 2ℓ.

Proof. □

The idea is now to use this result to see how the results of the previous section translate into the case of the four holed sphere. For the remainder of the article, X will be a four holed sphere with four boundary geodesics $\eta_1, \dots, \eta_4$ of equal length. The first remark to be made is that, up to conjugation with their hyperelliptic involutions, the set of isometries of X is isomorphic to the set of isometries of its corresponding $\tilde{\varphi}(X)$.

Corollary 4.4. *There is an orientation reversing involution between γ_1 and γ_2 verifying $\ell(\gamma_1) = \ell(\gamma_2)$ if $i(\gamma_1, \gamma_2) = n$ and n divides 24. Conversely, for any length of η, and for all n that does not divide 24, then there is an X and two simple closed geodesics on X γ_1 and γ_2 verifying $\ell(\gamma_1) = \ell(\gamma_2)$ such that there is no orientation reversing involution between them.*

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