Isotropic subbundles of $TM \oplus T^*M$

by

Izu Vaisman

ABSTRACT. We define integrable, big-isotropic structures on a manifold M as subbundles $E \subseteq TM \oplus T^*M$ that are isotropic with respect to the natural, neutral metric g of $TM \oplus T^*M$, closed by Courant brackets and such that, if E' is the g-orthogonal bundle, the Courant brackets $[\mathcal{X}, \mathcal{Y}]$, $\mathcal{X} \in \Gamma E, \mathcal{Y} \in \Gamma E'$, belong to $\Gamma E'$. We give the interpretation of such a structure by objects of M, we discuss the local geometry of the structure and we give a reduction theorem.

1 Introduction

All the manifolds and mappings of this paper are assumed of the C^∞ class and the following general notation is used: M is an m-dimensional manifold, $\chi^k(M)$ is the space of k-vector fields, $\Omega^k(M)$ is the space of differential k-forms, Γ indicates the space of global cross sections of a vector bundle, $X, Y, ..$ are either contravariant vectors or vector fields, $\alpha, \beta, ...$ are either covariant vectors or 1-forms, d is the exterior differential and L is the Lie derivative. The Einstein summation convention will be used whenever possible. If required by the context, a linear space V shall be identified with anyone of the spaces $V \oplus 0$, $0 \oplus V$.

The vector bundle $T^{big}M = TM \oplus T^*M$ is called the *big tangent bundle*. It has the natural, non degenerate metric of zero signature (neutral metric)

$$g((X, \alpha), (Y, \beta)) = \frac{1}{2}(\alpha(Y) + \beta(X)), \tag{1.1}$$

*2000 *Mathematics Subject Classification:* 53C99, 53D17.

Key words and phrases: big-isotropic structures, Courant bracket, integrability, reduction.

the non degenerate, skew-symmetric 2-form

$$\omega((X,\alpha),(Y,\beta)) = \frac{1}{2}(\alpha(Y) - \beta(X)) \tag{1.2}$$

and the Courant bracket of cross sections [4]

$$[(X,\alpha),(Y,\beta)] = ([X,Y], L_X\beta - L_Y\alpha + \frac{1}{2}d(\alpha(Y) - \beta(X))). \tag{1.3}$$

A maximal, g-isotropic subbundle $D \subseteq T^{big}M$ is called an almost Dirac structure and, if $\Gamma(D)$ is closed by the Courant bracket, D is an integrable or a Dirac structure. Then, the triple $(D, pr_{TM}, [\,,\,])$ is a Lie algebroid.

The almost Dirac structure D produces the generalized distribution $\mathcal{D} = pr_{TM}D$ endowed with a leaf-wise differentiable 2-form ϖ induced by $\omega|_D$. Conversely, D may be recovered from the pair $(\mathcal{D},\varpi)$ by means of the formula

$$D = \{(X,\alpha)\,/\,X \in \mathcal{D},\, \alpha|_{\mathcal{D}} = i(X)\varpi\}. \tag{1.4}$$

Furthermore, by a technical computation that uses (1.4), it follows that D is a Dirac structure iff $\mathcal{D}$ is integrable and the form ϖ is closed on the leaves of $\mathcal{D}$. Thus, a Dirac structure on M is equivalent with a generalized foliation by presymplectic leaves where the leaf-wise presymplectic form is such that the subbundle (1.4) is differentiable [4].

While the Dirac structures were introduced as a framework for constrained dynamics, it is rather the geometry of these structures and their integrability to a conveniently equipped Lie groupoid that were the object of numerous studies.

The aim of the present paper is to understand the geometry of a g-isotropic subbundle $E \subseteq T^{big}M$, where the maximality requirement is dropped; we call them big-isotropic structures. Then, the subbundle E must be discussed in conjunction with its g-orthogonal bundle $E' \supseteq E$ and the corresponding objects on M will be a pair of generalized distributions $\mathcal{E} = pr_{TM}E \subseteq \mathcal{E}' = pr_{TM}E'$ and bilinear mappings $\varpi_x : \mathcal{E}_x \times \mathcal{E}'_x \rightarrow \mathbb{R}$ $(\forall x \in M)$ with a skew symmetric restriction to $\mathcal{E}_x \times \mathcal{E}_x$. Furthermore, it turns out that the convenient definition of the integrability of E is to ask ΓE to be an algebra and $\Gamma E'$ to be a module over ΓE with respect to the Courant bracket.

This definition and the relationship with the triple $(\mathcal{E},\mathcal{E}',\varpi)$ are made precise in Section 2, where we also give several examples. In particular, $\mathcal{E}$

is a generalized foliation and its leaves, called the characteristic leaves of E, inherit a presymplectic form.

In Section 3 we extend the construction of a local, canonical basis of a Dirac structure given in [6] to (integrable) big-isotropic structures.

In Section 4 we discuss the pullback of a big-isotropic structure by a mapping and use this operation and the canonical bases of Section 3 in order to study the structure induced on a characteristic leaf and that induced on a local transversal submanifold of the leaf. We define a property called (strong) local decomposability and extend the Dufour-Wade proof of the essential uniqueness of the transversal structure of a characteristic leaf of a Dirac structure to strongly, locally decomposable, big-isotropic structures.

Finally, in Section 5 we discuss the push-forward of a big-isotropic structure and conditions that ensure the projectability of a big-isotropic structure to the space of leaves of a foliation. The results are used in order to prove a reduction theorem of an integrable, big-isotropic structure of a manifold M to a structure of a quotient space $N/\mathcal{F}$ of a submanifold $N \subseteq M$ by a foliation $\mathcal{F}$.

2 Definitions, examples, first properties

We generalize the notion of a Dirac structure by giving the following definition.

Definition 2.1. A g-isotropic subbundle $E \subseteq T^{big}M$ of rank k $(0 \leq k \leq m)$ will be called a *big-isotropic structure* on M. A big-isotropic structure E is *integrable* if ΓE is closed by the Courant bracket and $\forall(X,\alpha) \in \Gamma(E)$, $\forall(Y,\beta) \in \Gamma(E')$, where E' is the g-orthogonal subbundle $E^{\perp_g}$ of E, one has

$$[(X,\alpha),(Y,\beta)] \in \Gamma(E').$$

Example 2.1. For $k = m$, the integrable, big-isotropic structures are the Dirac structures.

Example 2.2. Let M be a locally product manifold with the structural foliations $\mathcal{F}_1, \mathcal{F}_2$, i.e., each point $x \in M$ has a neighborhood $U \approx V_1 \times V_2$ where V_a is a neighborhood of x in the leaf of $\mathcal{F}_a$ through x $(a = 1,2)$. Equivalently, M has an atlas of local coordinates of the form (x^h, y^u) such that $\mathcal{F}_1$ has the local equations $dy^u = 0$ and $\mathcal{F}_2$ has the local equations

$dx^h = 0$. Then $T^{big}M$ is the direct sum of the g-orthogonal subbundles $T\mathcal{F}_a \oplus T^*\mathcal{F}_a$ and, if E is a maximal isotropic subbundle of $T\mathcal{F}_1 \oplus T^*\mathcal{F}_1$, E is a big-isotropic structure on M with the orthogonal subbundle $E' = E \oplus (T\mathcal{F}_2 \oplus T^*\mathcal{F}_2)$. Furthermore, assume ΓE has local bases (Z_l, ζ_l) such that

$$Z_l = Z_l^h(x)\frac{\partial}{\partial x^h},\ \zeta_l = \zeta_{lh}(x)dx^h \tag{2.1}$$

and that E is Dirac along the leaves of $\mathcal{F}_1$. Then, using Definition 2.1 and the Courant algebroid properties of the Courant bracket [4, 7], it is easy to check that the big-isotropic structure E is integrable. For instance, if $P \in \chi^2(M), \theta \in \Omega^2(M)$ have local expressions that depend only on (x^h) (which is an invariant property) and if $[P, P] = 0, d\theta = 0$, then $graph(\sharp_P|_{T^*\mathcal{F}_1})$, $graph(\flat_\theta|_{T\mathcal{F}_1})$ are integrable, big-isotropic structures of M of the kind described above.

Example 2.3. For any pair of vector subbundles $F \subseteq F' \subseteq TM$, $E = F \oplus ann\, F'$ is a big-isotropic structure on M with the g-orthogonal bundle $E' = F' \oplus ann\, F$. Furthermore, E is integrable iff F is tangent to a foliation and $\Gamma F'$ is invariant by Lie brackets with cross sections of F; this means that F' is a projectable distribution with respect to the foliation F, i.e., F' is projection-related with distributions of the local spaces of leaves [12].

Example 2.4. Let S be a rank k subbundle of TM and $\theta \in \Omega^2(M)$ a differential 2-form. Then,

$$E_\theta = graph(\flat_\theta|_S) = \{(X, \flat_\theta X = i(X)\theta)\,/\,X \in S\} \subseteq T^{big}M \tag{2.2}$$

is a big-isotropic structure on M with the g-orthogonal bundle

$$E'_\theta = \{(Y, \flat_\theta Y + \gamma)\,/\,Y \in TM,\ \gamma \in ann\, S\}. \tag{2.3}$$

For the integrability conditions, we compute the Courant bracket of a cross section of E_θ and one of E'_θ. With the notation of (2.2), (2.3), we get

$$\begin{aligned}[(X, \flat_\theta X), (Y, \flat_\theta Y + \gamma)] = ([X, Y], (L_X i(Y) - L_Y i(X))\theta \\ +d(\theta(X, Y)) + L_X\gamma) = ([X, Y], i([X, Y])\theta + i(X \wedge Y)d\theta + L_X\gamma).\end{aligned} \tag{2.4}$$

The second argument in the left hand side of (2.4) is in E_θ iff $Y \in S$ and $\gamma = 0$. If this happens, the right hand side of (2.4) is in E_θ iff S is involutive and

$$(2.5) \qquad d\theta(X,Y,Z) = 0, \quad \forall X,Y \in \Gamma S, \forall Z \in \chi^1(M).$$

Now, we consider (2.4) for the case $\gamma = 0$ and an arbitrary vector field Y. Then, similar calculations show that the result stays in $\Gamma E'_\theta$ iff the condition (2.5) is satisfied. Furthermore, if S is involutive, $L_X\gamma \in ann\, S$ holds for all $X \in \Gamma S, \gamma \in ann\, S$ and, again, (2.5) is the necessary and sufficient condition for the bracket (2.4) to be in $\Gamma E'_\theta$. Therefore, E_θ is integrable iff S is a foliation and θ satisfies (2.5) (in particular, if θ is closed).

Example 2.5. Let S^* be a subbubdle of rank k of T^*M and $P \in \chi^2(M)$ a differentiable bivector field on M. Then

$$(2.6) \qquad E_P = graph(\sharp_P|_{S^*}) = \{(\sharp_P\sigma = i(\sigma)P, \sigma) \,/\, \sigma \in S^*\}$$

is a big-isotropic structure on M with the g-orthogonal bundle

$$(2.7) \qquad E'_P = \{(\sharp_P\beta + Y, \beta) \,/\, \beta \in T^*M, Y \in ann\, S^*\}.$$

For the integrability conditions we recall that P defines the bracket of 1-forms:

$$(2.8) \qquad \{\alpha,\beta\}_P = L_{\sharp_P\alpha}\beta - L_{\sharp_P\beta}\alpha - d(P(\alpha,\beta)),$$

which is related to the Schouten-Nijehuis bracket $[P,P]$ by the Gelfand-Dorfman formula [5]

$$(2.9) \qquad P(\{\alpha,\beta\}_P,\gamma) = \gamma([\sharp_P\alpha,\sharp_P\beta]) + \frac{1}{2}[P,P](\alpha,\beta,\gamma).$$

With (2.8) and (2.9) we get the Courant bracket

$$(2.10) \qquad [(\sharp_P\sigma,\sigma),(\sharp_P\beta + Y,\beta)] = (\sharp_P(\{\sigma,\beta\}_P - L_Y\sigma)$$
$$-\sharp_{L_YP}\sigma - \frac{1}{2}i(\sigma\wedge\beta)[P,P], \{\sigma,\beta\}_P - L_Y\sigma),$$

where $\sigma \in S^*, Y \in ann\, S^*$. In (2.10), we have a bracket of two cross sections of E_P iff $\beta \in S^*, Y = 0$ and the result is in ΓE_P iff S^* is closed by the bracket (2.8) and

$$(2.11) \qquad [P,P](\sigma_1,\sigma_2,\beta) = 0, \quad \forall\sigma_1,\sigma_2 \in S^*, \forall\beta \in T^*M.$$

Now, we notice that the closure by brackets of S^* implies $\sharp_{L_Y P}\sigma \in ann\, S^*$ for $\sigma \in S^*, Y \in ann\, S^*$; this follows from the following calculation where $\tau \in S^*$:

$$L_Y P(\sigma,\tau) = Y(P(\sigma,\tau)) + L_Y\sigma(\sharp_P\tau) - L_Y\tau(\sharp_P\sigma)$$

$$= -Y(P(\sigma,\tau)) + \sigma([\sharp_P\tau, Y]) - \tau([\sharp_P\sigma, Y]) = \{\sigma,\tau\}_P(Y) = 0.$$

Accordingly, the necessary and sufficient condition for the general bracket (2.10) to belong to $\Gamma E'_P$ is again (2.11). Therefore, E_P is integrable iff S^* is closed by the P-brackets (2.8) and P satisfies condition (2.11) (in particular, P is a Poisson bivector field).

Example 2.6. The construction indicated in [13] for the lift of a Dirac structure of a manifold M to its tangent manifold TM may also be used for a big-isotropic structure E, E'. More exactly, if we look at the locally free sheaves $\underline{E}, \underline{E}'$ of germs of cross sections of E, E', the formulas

$$\begin{aligned} \underline{tg(E)} &= span\{(X^C,\alpha^C),(X^V,\alpha^V)\,/\,(X,\alpha)\in\underline{E}\},\\ \underline{tg(E)'} &= span\{(X^C,\alpha^C),(X^V,\alpha^V)\,/\,(X,\alpha)\in\underline{E}'\}, \end{aligned} \tag{2.12}$$

where C, V denote the complete and vertical lift, respectively, define locally free sheaves of germs of cross sections of orthogonal subbundles $tg(E), tg(E)' \subseteq T^{big}(TM)$. The formulas for scalar products and Courant brackets of lifts established in [13] show that $tg(E)$ is a big-isotropic structure on TM with the orthogonal bundle $tg(E)'$ and that, if E is integrable, $tg(E)$ is integrable too.

Remark 2.1. 1) The closure of E with respect to the Courant bracket is a more complex notion than the closure of a distribution $\Delta \subseteq TM$ with respect to the Lie bracket. For instance, in the latter case, any vector field $X \in \Delta$ is an infinitesimal automorphism of Δ while, in the former case, if $(X,\alpha) \in \Gamma E$ then X is an infinitesimal automorphism of E (i.e., $(L_X Y, L_X\beta) \in \Gamma E$, $\forall (Y,\beta) \in \Gamma E$) iff $d\alpha(Y,Z) = 0$ for all $Y \in pr_{TM}E, Z \in pr_{TM}E'$. This is an easy consequence of the expression of the Courant bracket and of the isotropy of E. 2) In Examples 2.3 - 2.5 the closure of the g-isotropic subbundle by Courant brackets was enough to ensure integrability; there is no reason for this to happen in the general case.

Now, we shall look for objects of TM that are equivalent with a big-isotropic structure E.

For the algebraic aspects, we refer to a fixed point $x \in M$ and we associate with E the vector spaces

$$\mathcal{E}_x = pr_{T_xM} E_x \subseteq \mathcal{E}'_x = pr_{T_xM} E'_x. \tag{2.13}$$

Then, we define a bilinear mapping $\varpi_x : \mathcal{E}_x \times \mathcal{E}'_x \to \mathbb{R}$ by means of the formula

$$\varpi_x(X, Y) = \omega((X, \alpha), (Y, \beta)) = \alpha(Y) = -\beta(X), \tag{2.14}$$

where $(X, \alpha) \in E_x, (Y, \beta) \in E'_x$. The last equalities hold and the result is independent on the choice of α, β because $(X, \alpha) \perp_g (Y, \beta)$. Of course, $\varpi|_{\mathcal{E}_x \times \mathcal{E}_x}$ is skew-symmetric. Notice that ϖ_x may be identified with a mapping $\flat_{\varpi_x} : \mathcal{E}_x \to \mathcal{E}'^*_x M$, which sends X to $i(X)\varpi$ (with an obvious notation) and it is easy to see that $ker\, \flat_{\varpi_x} = T_xM \cap E_x$. This implies that, if ϖ is non degenerate (i.e., $ker\, \flat_{\varpi_x} = 0$) E_x is the graph of a mapping $pr_{T^*_xM} E_x \to T_xM$. Similarly, if E_x has the property $T^*_xM \cap E_x = 0$ then E_x is the graph of a mapping $\mathcal{E}_x \to T^*_xM$. If we are in one (and the same) of the two cases above $\forall x \in M$, we will say that E is of *the graph type*.

Example 2.7. In Example 2.2 we have $\mathcal{E} = pr_{TM} E, \mathcal{E}' = \mathcal{E} \oplus T\mathcal{F}_2$ and ϖ is the extension of the 2-form of the almost Dirac structure E along $\mathcal{F}_1$ by the value 0 for second arguments in $T\mathcal{F}_2$. In Example 2.3, $\mathcal{E} = F, \mathcal{E}' = F'$ and $\varpi(X, Y) = 0$. In Example 2.4, $\mathcal{E} = S, \mathcal{E}' = TM, \varpi = \theta|_{S \times TM}$ and in Example 2.5,

$$\mathcal{E} = im(\sharp_P|_{S^*}), \mathcal{E}' = \mathcal{E} + ann\, S^*, \varpi(\sharp_P \sigma, \sharp_P \beta + Y) = -P(\sigma, \beta)$$

where $\sigma \in S^*, Y \in ann\, S^*$. In the particular case of Example 2.5 where P is the Lie-Poisson bivector field of the Lie coalgebra $\mathcal{G}^*$ of the connected Lie group G (e.g., see [11]) and $S^* = \mathcal{G}'$ is the Lie subalgebra of the connected subgroup $G' \subseteq G$, $\mathcal{E}$ are the tangent spaces of the orbits of the coadjoint action of G' on $\mathcal{G}^*$.

Proposition 2.1. *For any pair of planes $\mathcal{E}_x \subseteq \mathcal{E}'_x \subseteq T_xM$ and any bilinear mapping $\varpi_x : \mathcal{E}_x \times \mathcal{E}'_x \to \mathbb{R}$ with a skew-symmetric restriction to $\mathcal{E}_x \times \mathcal{E}_x$ there exists a big-isotropic plane $E_x \subseteq T^{big}_x M$ such that* (2.13), (2.14) *are the given planes and mapping.*

Proof. For the given planes and mapping, put

$$E_x = \{(X,\alpha)\,/\,X \in \mathcal{E}_x \,\&\, \forall Y \in \mathcal{E}'_x,\ \alpha(Y) = \varpi_x(X,Y)\},$$
$$E'_x = \{(Y,\beta)\,/\,Y \in \mathcal{E}'_x \,\&\, \forall X \in \mathcal{E}_x,\ \beta(X) = -\varpi_x(X,Y)\}. \tag{2.15}$$

Obviously, covectors α, β as required by (2.15) exist, hence, the projection on the first term of a pair defines epimorphisms $E_x \to \mathcal{E}_x$, $E'_x \to \mathcal{E}'_x$ with the kernels $E_x \cap T^*_xM = ann\,\mathcal{E}'_x$, $E'_x \cap T^*_xM = ann\,\mathcal{E}_x$, respectively. Accordingly, one has the exact sequences

$$0 \to ann\,\mathcal{E}'_x \to E_x \to \mathcal{E}_x \to 0,\ \ 0 \to ann\,\mathcal{E}_x \to E'_x \to \mathcal{E}'_x \to 0, \tag{2.16}$$

and we get

$$dim\,E_x = dim\,\mathcal{E}_x + dim\,ann\,\mathcal{E}'_x,\ dim\,E'_x = dim\,\mathcal{E}'_x + dim\,ann\,\mathcal{E}_x, \tag{2.17}$$

which implies $dim\,E_x + dim\,E'_x = 2m$. Thus, $E'_x, E_x^{\perp_g}$ have the same dimension and, since by (2.15) $E'_x \perp_g E_x$, we get $E'_x = E_x^{\perp_g}$. Furthermore, the skew-symmetry of ϖ on $\mathcal{E}_x \times \mathcal{E}_x$ implies $E_x \subseteq E'_x$, hence, E_x is isotropic. The fact that the given planes and mapping are associated with E_x, E'_x of (2.15) via (2.13), (2.14) is obvious. □

Now, starting with the big-isotropic structure E, let x vary on M. Since E, E' are differentiable vector bundles, the generalized, distributions $\mathcal{E}, \mathcal{E}'$ defined by the spaces (2.13) are differentiable. If $\mathcal{E}$ is a regular distribution (i.e., $dim\,\mathcal{E}_x = const.$, therefore, by (2.17), $dim\,\mathcal{E}'_x = const.$ as well), the structure E will be called *regular*. If E is integrable then $(E, pr_{TM}, [\,,\,])$ is a Lie algebroid and, accordingly, $\mathcal{E}$ is a generalized foliation. Definition 2.1 also implies that if $X \in \Gamma(\mathcal{E}), Y \in \Gamma(\mathcal{E}')$ then $[X,Y] \in \Gamma(\mathcal{E}')$.

It is worth formalizing the status of E' as follows since, presumably, this might be used in a discussion of the integrability of E to a Lie groupoid. Let $B \to M$ be a vector bundle endowed with an anchor (morphism) $\rho : B \to TM$. Assume that there exists a vector subbundle $A \subseteq B$ endowed with a Lie algebroid structure $(A, \rho|_A, [\,,\,]_A)$ and there exists an $\mathbb{R}$-bilinear operation $[\,,\,] : \Gamma A \times \Gamma B \to \Gamma B$ that reduces to $[\,,\,]_A$ for arguments in ΓA. Then, $(B, \rho, [\,,\,])$ will be called a *modular enlargement* of the Lie algebroid A if, $\forall f, h \in C^\infty(M), a \in \Gamma A, b \in \Gamma B$, the following conditions are satisfied:

1) $\rho[a,b] = [\rho a, \rho b]$,

2) $\qquad [fa, hb] = fh[a,b] + f((\rho a)h)b - h((\rho b)f)a,$

3) $\qquad [a_1, [a_2, b]] = [[a_1, a_2], b] + [a_2, [a_1, , b]]$

(the right hand side of 1) is a Lie bracket of vector fields).

With this terminology, if (E, E') is an integrable, big-isotropic structure, E' with $\rho = pr_{TM}$ and the Courant bracket is a modular enlargement of the Lie algebroid E and we shall formulate the following definition.

Definition 2.2. The generalized foliation $\mathcal{E}$ is the *characteristic foliation* of the integrable, big-isotropic structure E and its leaves are the *characteristic leaves*. The generalized distribution $\mathcal{E}'$ is the *characteristic module* of E.

Example 2.8. We can extend the construction of Dirac structures from Lie algebroids [1] as follows. Let $A \to M$ be a Lie algebroid of anchor $\rho_A : A \to TM$ and bracket $[\, ,\,]_A$ and $(B, \rho, [\, ,\,])$ a modular enlargement of A. Assume that one also has a *co-anchor* $\sigma : B \to T^*M$ such that the following properties are satisfied for all $a \in \Gamma A, b \in \Gamma B$:

i) $\qquad < \sigma a, \rho b >= - < \sigma b, \rho a >,$

ii) $\qquad \sigma[a,b] = L_{\rho a}(\sigma b) - L_{\rho b}(\sigma a) + d < \sigma a, \rho b >.$

Furthermore, assume that, $\forall x \in M$, the morphism $(\rho, \sigma) : B \to T^{big}M$ satisfies the condition

$$rank(\rho,\sigma)|_{A_x} + rank(\rho,\sigma)|_{B_x} = 2m.$$

Then, it is easy to check that

$$E_\sigma = \{(\rho a, \sigma a) \,/\, \forall a \in A\}$$

is an integrable, big-isotropic, structure on M with the orthogonal bundle

$$E'_\sigma = \{(\rho b, \sigma b) \,/\, \forall b \in B\}.$$

The characteristic foliation of E_σ is $\rho(A)$ and the characteristic module is $\rho(B)$. In fact, any integrable, big-isotropic, structure E is of this kind, where $A = E$, $B = E'$, the brackets are Courant brackets, the anchor is the projection on TM and the co-anchor is the projection on T^*M.

Concerning the mapping ϖ, we notice that it has the following differentiability property: for any characteristic leaf S of E and for any differentiable vector fields $X, Y \in \chi^1(M)$, such that $X|_S \in \Gamma\mathcal{E}|_S, Y|_S \in \Gamma\mathcal{E}'|_S$, $\varpi(X,Y)$ is a differentiable function on S. Indeed, since $\mathcal{E}|_S$, hence $\mathcal{E}'|_S$ too, has a constant dimension, (2.16) produces exact sequences of differentiable, vector bundles over S. Using differentiable splittings of these sequences, we see that there are 1-forms $\alpha, \beta \in T^*_S M$, which are differentiable along S, such that $(X,\alpha) \in \Gamma E|_S, (Y,\beta) \in \Gamma E'|_S$. Accordingly, (2.14) shows that $\varpi(X,Y)$ is a differentiable function on S. This property will be called *leaf-wise differentiability*. If the structure E is regular, the exact sequences (2.16) have differentiable splittings over M, and the functions $\varpi(X,Y)$ are differentiable on the whole manifold M.

A multilinear mapping

$$\lambda_x : \wedge^{s-1}\mathcal{E}_x \otimes \mathcal{E}'_x \to \mathbb{R}, \tag{2.18}$$

which is defined $\forall x \in M$, has a totally skew-symmetric restriction to $\wedge^s \mathcal{E}_x$ and is leaf-wise differentiable with respect to the characteristic foliation of E, will be called a *truncated s-form* on (M,E). We will denote by $\Omega^s_{tr}(M,E)$ the space of truncated s-forms. Because of the integrability conditions of E, the usual formula for the evaluation of the exterior differential of a differential form makes sense for truncated s-forms on (M,E) and for arguments in $\chi^1(M)$ that belong to $\mathcal{E}$. Moreover, if E is a regular structure, the exterior differential also makes sense if the last argument belongs to $\mathcal{E}'$, while the other arguments belong to $\mathcal{E}$. Whenever it makes sense, we will denote the differential of a truncated form by d_{tr}, . In the regular case, d_{tr} is a coboundary morphism

$$d_{tr} : \Omega^s_{tr}(M,E) \to \Omega^{s+1}_{tr}(M,E), \quad d^2_{tr} = 0 \tag{2.19}$$

and the cohomology spaces $H^s_{tr}(M,E)$ of the cochain complex $(\Omega^s_{tr}(M,E), d_{tr})$ will be the *truncated, de Rham cohomology spaces* of (M,E).

Proposition 2.2. *For any integrable, big-isotropic structure E one has*

$$d_{tr}\varpi(\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3) = 0, \quad \forall \mathcal{X}_a \in \mathcal{E},\ a = 1,2,3. \tag{2.20}$$

If either E is an almost Dirac structure or E is a regular big-isotropic structure, E is integrable iff $\mathcal{E}$ is involutive, the corresponding distribution $\mathcal{E}'$ is invariant by Lie brackets with vector fields of $\mathcal{E}$ and (2.20) *with $\mathcal{X}_3$ replaced by $\mathcal{Y} \in \mathcal{E}'$ holds.*

Proof. For almost Dirac structures the result is known [4]. We prove the result for regular, big-isotropic structures and we will get the first assertion on our way. It was already shown that $\mathcal{E}$ and $\mathcal{E}'$ satisfy the required conditions for any integrable, big-isotropic structure. Now, let $X_1, X_2 \in \mathcal{E}, Y \in \mathcal{E}'$ be differentiable vector fields on M and $\alpha_1, \alpha_2, \beta$ differentiable 1-forms such that $(X_1, \alpha_1), (X_2, \alpha_2) \in E, (Y, \beta) \in E'$. (The existence of α, β is ensured by the regularity hypothesis.) Then, keeping in mind the g-orthogonality relations among these pairs and using (2.14), we get

$$(d_{tr}\varpi)(X_1, X_2, Y) = (d_C\omega)((X_1, \alpha_1), (X_2, \alpha_2), (Y, \beta))$$

$$= X_1(\alpha_2(Y)) - X_2(\alpha_1(Y)) + Y(\alpha_1(X_2)) + \alpha_1([X_2, Y]) - \alpha_2([X_1, Y])$$

$$- < L_{X_1}\alpha_2 - L_{X_2}\alpha_1 - d(\alpha_1(X_2)), Y >= 0,$$

where d_C is the operator defined by the usual expression of the exterior differential of a Lie algebroid with Courant brackets instead of Lie brackets.

The previous calculation makes sense and remains true in the non regular case if we also assume $Y \in \mathcal{E}$. This proves the first assertion of the proposition.

Back to the regular case, if we start with a big-isotropic structure E which satisfies the required hypotheses then, for arguments as above, we get

$$d_{tr}\varpi(X_1, X_2, Y) = 2g([(X_1, \alpha_1), (X_2, \alpha_2)], (Y, \beta))$$

$$= 2g([(X_2, \alpha_2), (Y, \beta)], (X_1, \alpha_1)).$$

Thus, if ϖ is d_{tr}-closed, E is closed by Courant brackets and E' is closed by Courant brackets with cross sections of E.

For the non regular case, we do not get the converse result; from condition (2.20) it only follows that $[\Gamma E, \Gamma E] \in \Gamma E'$. □

Corollary 2.1. *For any big-isotropic structure E and for each point $x \in M$, there exists a canonical extension of E_x to an almost Dirac space $D_x(E) \subseteq T^{big}_x M$. If differentiable in x, these spaces define a canonical almost Dirac extension $D(E)$ of E and if E is integrable so is $D(E)$.*

Proof. The restriction of ϖ to $\mathcal{E} \times \mathcal{E}$ is a leaf-wise differentiable 2-form and we may use (1.4) and define

$$D_x(E) = \{(X, \alpha) \,/\, X \in \mathcal{E}_x \,\&\, \forall Y \in \mathcal{E}_x,\ \alpha(Y) = \varpi_x(X, Y)\}. \tag{2.21}$$

From (2.15), it follows that $E_x \subseteq D_x(E)$ and that we have

(2.22) $D_x(E) = \{(X,\alpha) + (0,\gamma) \,/\, (X,\alpha) \in E_x,\, \gamma \in ann\,\mathcal{E}_x\} = E_x + ann\,\mathcal{E}_x.$

Since $E_x \cap ann\,\mathcal{E}_x = ann\,\mathcal{E}'_x$, (2.17) and (2.22) show that $dim\,D_x(E) = m$. Notice that $D_x(E) \subseteq E'_x$. If $D(E)$ is differentiable and E is integrable, $\mathcal{E}$ is a generalized foliation and $D(E)$ is integrable because of (2.20). □

For instance, in the case of Example 2.3, $D(E) = F \oplus ann\,F$ and, in the case of Example 2.2, we get $D(E) = E \oplus T^*\mathcal{F}_2$.

Example 2.9. Let M be a manifold endowed with a regular, involutive, k-dimensional subbundle $\mathcal{E} \subseteq TM$ and with a d_{tr}-closed, truncated 2-form ϖ of the pair $(\mathcal{E}, \mathcal{E}' = TM)$. Then, the hypotheses of the regular case of Proposition 2.2 are satisfied and via (2.15), we get the corresponding integrable, big-isotropic structure $E_\varpi = graph\,\flat_\varpi \subseteq T^{big}M$ of rank k. Let $\tilde{\mathcal{E}}$ be a complementary subbundle of $\mathcal{E}$ $(TM = \mathcal{E} \oplus \tilde{\mathcal{E}})$ and denote by a prime and a double prime the projections of a vector on $\mathcal{E}, \tilde{\mathcal{E}}$, respectively. Then, we can extend ϖ to a 2-form $\theta \in \Omega^2(M)$ by the formula

$$\theta(Y_1, Y_2) = \varpi(Y'_1, Y'_2) + \varpi(Y'_1, Y''_2) - \varpi(Y'_2, Y''_1)$$

and we get $E_\varpi = E_\theta$, where E_θ was defined in Example 2.4. We would also like to comment on the following particular case. If $\varpi|_{\mathcal{E}\times\mathcal{E}}$ is non degenerate on each leaf of $\mathcal{E}$ we will say that the structure $graph\,\flat_\varpi$ is non degenerate and there exists a regular Poisson bivector field $\Pi \in \chi^2(M)$ (Schouten-Nijenhuis bracket $[\Pi,\Pi] = 0$) with the symplectic foliation $\mathcal{E}$. Furthermore, since the non-degeneracy of $\varpi|_{\mathcal{E}\times\mathcal{E}}$ implies that $ker\,\flat_\varpi = 0$ and $im\,\flat_\varpi \cap ann\,\mathcal{E} = 0$ where $\flat_\varpi : \mathcal{E} \to \mathcal{E}'^* = T^*M$, we deduce that $T^*M = im\,\flat_\varpi \oplus ann\,\mathcal{E}$, therefore, ϖ also defines a normal bundle Q of the foliation $\mathcal{E}$ which may be seen as $Q = \mathcal{E}^{\perp_\varpi}$. Conversely, if a regular Poisson structure Π with symplectic foliation $\mathcal{E}$ and a normal bundle Q of $\mathcal{E}$ are given, we may extend the leaf-wise symplectic form of Π by the value zero on Q to a closed, truncated form ϖ and we will have the corresponding, integrable, big-isotropic structure $graph\,\flat_\varpi$. Thus, a non degenerate structure $graph\,\flat_\varpi$ is equivalent with a regular Poisson structure together with a normal bundle of its symplectic foliation. More exactly, if the Poisson bivector field is P and the normal bundle is Q, one has $E = \{(\sharp_P\lambda, \lambda) \,/\, \lambda \in ann\,Q\}$ and $E' = E \oplus (Q \oplus Q^*$.

We finish this section by indicating that, like a Dirac structure [4], an integrable, big-isotropic structure (E, E') on M allows for a partial Hamiltonian formalism as follows. A function $f \in C^\infty(M)$ will be called a *strong-Hamiltonian function* if there exists a vector field $X_f \in \chi^1(M)$ such that $(X_f, df) \in \Gamma E$. Similarly, if $(X_f, df) \in \Gamma E'$ f is a *Hamiltonian function*. We will denote by $C^\infty_{sHam}(M)$ the set of Hamiltonian functions and by $C^\infty_{Ham}(M)$ the set of weak Hamiltonian functions. The vector field X_f is a (*strong*) *Hamiltonian vector field* of f and any $Z \in \chi^1(M)$ which is (strong) Hamiltonian for some f is a (*strong*) *Hamiltonian vector field*. The fields X^1_f, X^2_f are Hamiltonian vector fields of the same function f iff $X^2_f - X^1_f \in ann\, pr_{T^*M}E$ and are strong-Hamiltonian vector fields of f iff $X^2_f - X^1_f \in ann\, pr_{T^*M}E'$. Similarly, Z is strong-Hamiltonian (Hamiltonian) for two functions f_1, f_2 iff $df_2 - df_1 \in ann\, \mathcal{E}'$ (respectively, $df_2 - df_1 \in ann\, \mathcal{E}$). We will denote by $\chi_{Ham}(M), \chi_{sHam}$, respectively, the set of Hamiltonian and strong Hamiltonian vector fields.

Furthermore, if $f \in C^\infty_{sHam}(M)$ and $h \in C^\infty_{Ham}(M)$ the following bracket is well defined

$$\{f, h\} = X_f h = \varpi(X_f, X_h) = -X_h f \tag{2.23}$$

and does not depend on the choice of the strong-Hamiltonian vector fields of the functions f, h. The bracket (2.23) will be called the *Poisson bracket* of the two functions. Formula (1.3) shows that $\{f, h\} \in C^\infty_{Ham}(M)$ and one of its Hamiltonian vector fields is $[X_f, X_h]$. Moreover, if both $f, h \in C^\infty_{sHam}(M)$, their Poisson bracket is skew symmetric and belongs to $C^\infty_{sHam}(M)$. Now, if we notice that $d_{tr}\varpi(X_f, X_h, X_l)$ makes sense $\forall f, h \in C^\infty_{sHam}(M), \forall l \in C^\infty_{Ham}(M)$ (since the functions $\varpi(X_f, X_l)$, etc. are differentiable), the computation done during the proof of Proposition 2.2 now yields $d_{tr}\varpi(X_f, X_h, X_l) = 0$. This is easily seen to be equivalent with the *Leibniz property*

$$\{\{f, h\}, l\} = \{f, \{h, l\}\} - \{h, \{f, l\}\}, \tag{2.24}$$

which restricts to the Jacobi identity on $C^\infty_{sHam}(M)$. Therefore, $C^\infty_{sHam}(M)$ with the Poisson bracket is a Lie algebra and $C^\infty_{Ham}(M)$ is a module over this Lie algebra. Using the Poisson bracket (2.23) it also follows easily that $\chi_{sHam}(M)$ is a Lie subalgebra of $\chi^1(M)$ and $\chi_{Ham}(M)$ is a module over the former.

3 Canonical local bases

We will discuss local properties of a k-dimensional, big-isotropic structure E by constructing canonical, local bases in the neighborhood of a fixed point $x_0 \in M$, as constructed by Dufour and Wade in the Dirac case [6]. In what follows the notation is the same as in Section 2.

We begin with vectors $X^0_a, Y^0_h \in T_{x_0}M$ where X^0_a, $a = 1, ..., dim\,\mathcal{E}_{x_0}$, is a basis of $\mathcal{E}_{x_0}$ and Y^0_h, $h = 1, ..., dim\,\mathcal{E}'_{x_0} - dim\,\mathcal{E}_{x_0} \overset{(2.17)}{=} m - k$, is a basis of a complement of $\mathcal{E}_{x_0}$ in $\mathcal{E}'_{x_0}$. Then, there exist covariant vectors $\xi^a_0, \eta^h_0 \in T^*_{x_0}M$ such that (X^0_a, ξ^a_0) are linearly independent elements of E_{x_0} and $(X^0_a, \xi^a_0), (Y^0_h, \eta^h_0)$ is a basis of a complement of $ker(E'_{x_0} \to \mathcal{E}'_{x_0})$. Since $ker(E'_{x_0} \to \mathcal{E}'_{x_0}) = ann\,\mathcal{E}_{x_0}$, if we add a basis $(0, \zeta^s_0)$, $(s = 1, ..., dim\,ann\,\mathcal{E}_{x_0})$ of $ann\,\mathcal{E}_{x_0}$ we get a basis of E'_{x_0}. Moreover, since $ann\,\mathcal{E}_{x_0} \supseteq ann\,\mathcal{E}'_{x_0}$, we may ask the basis $(0, \zeta^s_0)$ to consist of elements $(0, \kappa^u_0), (0, \nu^q_0)$, $u = 1, ..., dim\,ann\,\mathcal{E}'_{x_0}$, $q = 1, ..., dim\,ann\,\mathcal{E}_{x_0} - dim\,ann\,\mathcal{E}'_{x_0} \overset{(2.17)}{=} m - k$, where $(0, \kappa^u_0)$ is a basis of $ann\,\mathcal{E}'_{x_0}$. Then, since $ker(E_{x_0} \to \mathcal{E}_{x_0}) = ann\,\mathcal{E}'_{x_0}$, $(X^0_a, \xi^a_0), (0, \kappa^u_0)$ is a basis of E_{x_0}. Furthermore, we shall need vectors $Z^0_\sigma \in T_{x_0}M$, $\sigma = 1, ..., dim\,M - dim\,\mathcal{E}'_{x_0} = dim\,ann\,\mathcal{E}'_{x_0}$, which are a basis of a complement of $\mathcal{E}'_{x_0}$ in $T_{x_0}M$. Notice the important fact that the indices h, q, on one side, and u, σ, on the other side, have the same range.

Now we shall extend the basis $\mathcal{B}_0 = \{(X^0_a, \xi^a_0), (0, \kappa^u_0), (Y^0_h, \eta^h_0), (0, \nu^q_0)\}$ to a basis of cross sections of E, E' over a neighborhood U of x_0 in M; we will allow U to undergo as many restrictions as needed for the correctness of the various constructions below without changing its name.

Clearly, we may assume that there exists a basis of T_UM that consists of local vector fields (X_a, Y_h, Z_σ) with the values $(X^0_a, Y^0_h, Z^0_\sigma)$ at x_0 and we denote by $(\theta^a, \phi^h, \psi^\sigma)$ the corresponding, dual, local basis of T^*_UM, i.e.,

$$\theta^a(X_b) = \delta^a_b, \theta^a(Y_h) = 0, \theta^a(Z_\sigma) = 0,$$

$$\phi^h(X_b) = 0, \phi^h(Y_l) = \delta^h_l, \phi^h(Z_\sigma) = 0,$$

$$\psi^\sigma(X_b) = 0, \psi^\sigma(Y_k) = 0, \psi^\sigma(Z_\tau) = \delta^\sigma_\tau.$$

Accordingly, an extension of the basis $\mathcal{B}_0$ to a basis of (E, E') over U has an expression of the form

$$
\begin{aligned}
&\mathcal{X}_a = (A^b_a X_b + A'^h_a Y_h + A''^\sigma_a Z_\sigma, \alpha^a_b\theta^b + \alpha'^a_h\phi^h + \alpha''^a_\sigma\psi^\sigma),\\
&\Xi_u = (B^a_u X_a + B'^h_u Y_h + B''^\sigma_u Z_\sigma, \beta^u_a\theta^a + \beta'^u_h\phi^h + \beta''^u_\sigma\psi^\sigma),\\
&\mathcal{Y}_h = (C^a_h X_a + C'^l_h Y_l + C''^\sigma_h Z_\sigma, \gamma^h_a\theta^a + \gamma'^h_l\phi^l + \gamma''^h_\sigma\psi^\sigma),\\
&\Theta_q = (L^a_q X_a + L'^h_q Y_h + L''^\sigma_q Z_\sigma, \lambda^q_a\theta^a + \lambda'^q_h\phi^h + \lambda''^q_\sigma\psi^\sigma),
\end{aligned} \tag{3.1}
$$

where we use the Einstein summation convention, $\mathcal{X}_a, \Xi_u$ is a basis of $E|_U$, $\mathcal{Y}_h, \Theta_q$ completes the former to a basis of $E'|_U$ and

$$
\begin{aligned}
&A^b_a(x_0) = \delta^b_a, A'^h_a(x_0) = 0, A''^\sigma_a(x_0) = 0,\\
&B^a_u(x_0) = 0, B'^h_u(x_0) = 0, B''^\sigma_u(x_0) = 0,\\
&C^a_h(x_0) = 0, C'^l_h(x_0) = \delta^l_h, C''^\sigma_h(x_0) = 0,\\
&L^a_q(x_0) = 0, L'^h_q(x_0) = 0, L''^\sigma_q(x_0) = 0.
\end{aligned} \tag{3.2}
$$

We shall change this basis in order to simplify the expressions (3.1). However, we keep denoting the elements of the new bases by the same letters as in (3.1). Firstly, in view of (3.2), we may assume that the matrix (A^b_a) is non degenerate on U and change the vectors $\mathcal{X}_a$ by the matrix $(A^b_a)^{-1}$. As a result we get a basis (3.1) where $A^b_a = \delta^b_a$. Similarly, we may get $C'^l_h = \delta^l_h$. Then, the new basis may be changed by $\Xi_u \mapsto \Xi_u - B^a_u\mathcal{X}_a$ and get a new basis (3.1) where $B^a_u = 0$. Similarly, we may get $C^a_h = 0, L^a_q = 0$, then, with the change $\Theta_q \mapsto \Theta_q - L'^h_q\mathcal{Y}_h$, also get $L'^h_q = 0$.

Thus, any big-isotropic structure (E, E') has local bases of the form

$$
\begin{aligned}
&\mathcal{X}_a = (X_a + A'^h_a Y_h + A''^\sigma_a Z_\sigma, \alpha^a_b\theta^b + \alpha'^a_h\phi^h + \alpha''^a_\sigma\psi^\sigma),\\
&\Xi_u = (B'^h_u Y_h + B''^\sigma_u Z_\sigma, \beta^u_a\theta^a + \beta'^u_h\phi^h + \beta''^u_\sigma\psi^\sigma),\\
&\mathcal{Y}_h = (Y_h + C''^\sigma_h Z_\sigma, \gamma^h_a\theta^a + \gamma'^h_l\phi^l + \gamma''^h_\sigma\psi^\sigma),\\
&\Theta_q = (L''^\sigma_q Z_\sigma, \lambda^q_a\theta^a + \lambda'^q_h\phi^h + \lambda''^q_\sigma\psi^\sigma),
\end{aligned} \tag{3.3}
$$

where $(\mathcal{X}_a, \Xi_u)$ is a basis of E and (3.2) holds.

Furthermore, the following g-orthogonality conditions must be satisfied:

$$
\mathcal{X}_a \perp_g \mathcal{X}_b,\ \mathcal{X}_a \perp_g \Xi_u,\ \Xi_u \perp_g \Xi_v,\ \mathcal{X}_a \perp_g \mathcal{Y}_h, \tag{3.4}
$$

$$
\Xi_u \perp_g \mathcal{Y}_h,\ \mathcal{X}_a \perp_g \Theta_q,\ \Xi_u \perp_g \Theta_q.
$$

In particular, the second, fifth and sixth conditions (3.4), taken at x_0 give

$$\beta^u_a(x_0) = 0,\ \beta'^u_h(x_0) = 0,\ \lambda^q_a(x_0) = 0.$$

This implies that the 1-forms $\beta''^u_\sigma\psi^\sigma$, on one hand, and $\lambda'^q_h\phi^h + \lambda''^q_\sigma\psi^\sigma$, on the other hand are linearly independent at x_0 and on a neighborhood U of x_0, which may be used for further simplifications of the basis: i) we may linearly change Ξ_u by the inverse of the matrix (β''^u_σ) and get a new basis (3.3) where $\beta''^u_\sigma = \delta^u_\sigma$, ii) after change i), subtract $\sum_\sigma \alpha''^a_\sigma\Xi_\sigma$, $\sum_\sigma \gamma''^h_\sigma\Xi_\sigma$, $\sum_\sigma \lambda''^q_\sigma\Xi_\sigma$ from $\mathcal{X}_a, \mathcal{Y}_h, \Theta_q$, respectively, and get rid of the terms with ψ^σ in $\mathcal{X}_a, \mathcal{Y}_h, \Theta_q$, iii) change ii) reaches a situation where the forms $\lambda'^q_h\phi^h$ are independent and we will change the Θ_q by the inverse of the matrix (λ'^q_h) and obtain $\lambda'^q_h = \delta^q_h$ for the new basis, iv) subtract $\sum_l \gamma'^h_l\Theta_l$ from $\mathcal{Y}_h$ and get $\gamma'^h_l = 0$ in the new basis, v) since change ii) alters the coefficients of Y_h in $\mathcal{Y}_h$ and adds terms in Y_h to Θ_q, we correct that by changing the new $\mathcal{Y}_h$ with the corresponding coefficient matrix (which is non degenerate on U) and by subtracting the necessary linear combination of $\mathcal{Y}_h$ from Θ_q. The result is a local basis of (E, E') that looks as follows

$$\begin{aligned} \mathcal{X}_a &= (X_a + A'^h_a Y_h + A''^\sigma_a Z_\sigma, \alpha^a_b\theta^b + \alpha'^a_h\phi^h), \\ \Xi_u &= (B'^h_u Y_h + B''^\sigma_u Z_\sigma, \beta^u_a\theta^a + \beta'^u_h\phi^h + \psi^u), \\ \mathcal{Y}_h &= (Y_h + C''^\sigma_h Z_\sigma, \gamma^h_a\theta^a), \\ \Theta_q &= (L''^\sigma_q Z_\sigma, \lambda^q_a\theta^a + \phi^q), \end{aligned} \tag{3.5}$$

where (3.2) are still valid.

It is easy to see that, if the vector fields X_a, Y_h, Z_σ are fixed, there is only one basis of (E, E') which is of the form (3.5). For this reason we will say that the basis (3.5) is a *canonical, local basis* of the big-isotropic structure E.

Now, we consider the integrable case. Let $\mathcal{U}$ be a neighborhood of the point x_0 on the characteristic leaf $\mathcal{S}$ through x_0. Then $dim\,\mathcal{E}|_\mathcal{U} = const.$ and, in view of (2.17), $dim\,\mathcal{E}'|_\mathcal{U} = const.$ too. Furthermore, on the neighborhood U of x_0 in M there are coordinates (x^a, y^h, z^σ) such that x_0 has the coordinates $(0,0,0)$, the equations of $\mathcal{U}$ are $y^h = 0, z^\sigma = 0$, $\mathcal{E}|_\mathcal{U} = span\{(\partial/\partial x^a)|_\mathcal{U}\}$ and $\mathcal{E}'|_\mathcal{U} = span\{(\partial/\partial x^a)|_\mathcal{U}, (\partial/\partial y^h)|_\mathcal{U}\}$. Indeed, we may assume that U is a tubular neighborhood of $\mathcal{U}$ where the tangent space of the tubular fibers at the points of $\mathcal{U}$ is a direct sum of a complementary space of $\mathcal{E}$ in $\mathcal{E}'$ and a complementary space of $\mathcal{E}'$ in TM; then, take x^a coordinates on $\mathcal{U}$ and y^h, z^σ coordinates along the tubular fibers such that $(\partial/\partial y^h)|_{\mathcal{U}}$ span the chosen complement of $\mathcal{E}$ in $\mathcal{E}'$ and $(\partial/\partial z^\sigma)|_{\mathcal{U}}$ span the chosen further complement in $T_{\mathcal{U}}M$. The tubular structure of the neighborhood U will be important and we will denote by $\mathcal{F}$ the foliation of U by the tubular fibers, for later use. It is easy to see that we can construct bases (3.1) where

$$X_a = \frac{\partial}{\partial x^a},\, Y_h = \frac{\partial}{\partial y^h} + \chi_h^\sigma \frac{\partial}{\partial z^\sigma},\, Z_\sigma = \frac{\partial}{\partial z^\sigma} \;\; (\chi_h^\sigma|_{\mathcal{U}} = 0). \tag{3.6}$$

If these values are inserted in (3.5), the result takes the following form (with new coefficients):

$$\begin{aligned}
\mathcal{X}_a &= (\tfrac{\partial}{\partial x^a} + A'^h_a \tfrac{\partial}{\partial y^h} + A''^\sigma_a \tfrac{\partial}{\partial z^\sigma}, \alpha^a_b dx^b + \alpha'^a_h dy^h),\\
\Xi_u &= (B'^h_u \tfrac{\partial}{\partial y^h} + B''^\sigma_u \tfrac{\partial}{\partial z^\sigma}, \beta^u_a dx^a + \beta'^u_h dy^h + dz^u),\\
\mathcal{Y}_h &= (\tfrac{\partial}{\partial y^h} + C''^\sigma_h \tfrac{\partial}{\partial z^\sigma}, \gamma^h_a dx^a),\\
\Theta_q &= (L''^\sigma_q \tfrac{\partial}{\partial z^\sigma}, \lambda^q_a dx^a + dy^q),
\end{aligned} \tag{3.7}$$

where (3.2) holds $\forall x \in \mathcal{U}$ and (3.4) holds on U, which means that we have

$$\begin{aligned}
&\alpha'^a_h + \gamma^h_a = 0,\ \lambda^q_a + A'^q_a = 0,\ \beta'^u_h + C''^u_h = 0,\ L''^u_q + B'^q_u = 0,\\
&\beta^u_a + A''^u_a + \alpha'^a_h B'^h_u + \beta'^u_h A'^h_a = 0,\ B''^u_v + B''^v_u + \beta'^u_h B'^h_v + \beta'^v_h B'^h_u = 0,\\
&\alpha^a_b + \alpha^b_a + \alpha'^a_h A'^h_b + \alpha'^b_h A'^h_a = 0.
\end{aligned} \tag{3.8}$$

We also notice that, if we are interested in bases of E', without requiring them to be a prolongation of a basis of E, we may repeat the subtraction trick between $\mathcal{X}_a, \Xi_u$ and Θ_q and between $\mathcal{X}_a, \Xi_u$ and $\mathcal{Y}_h$ as well and get a basis of the form (3.7) with the supplementary conditions

$$\alpha'^a_h = 0,\ \beta'^u_h = 0,\ A'^h_a = 0,\ B'^h_u = 0; \tag{3.9}$$

the new pairs $\mathcal{X}_a, \Xi_u$ may not belong to E any more and (3.8) does not hold.

It follows easily that, if the local coordinates (x^a, y^h, z^σ) such that

$$y^h|_{\mathcal{U}} = 0,\ z^\sigma|_{\mathcal{U}} = 0,\ \left.\frac{\partial}{\partial y^h}\right|_{\mathcal{U}} \in (\mathcal{E}'\backslash\mathcal{E})|_{\mathcal{U}},\ \left.\frac{\partial}{\partial z^\sigma}\right|_{\mathcal{U}} \in (TM\backslash\mathcal{E}')|_{\mathcal{U}} \tag{3.10}$$

are chosen, the basis of the form (3.7) where $\mathcal{X}_a, \Xi_u \in E$ and $\mathcal{Y}_h, \Theta_q \in E'\backslash E$ is unique. A similar fact holds for the basis of E' which satisfies the conditions

(3.9). Accordingly, the basis (3.7) will be called a *canonical, local basis* of the integrable, big-isotropic structure (E, E') and the basis where (3.9) also holds is a *canonical basis* of E'.

Remark 3.1. In the Dirac case $k = m$, the basis (3.7) reduces to

$$\mathcal{X}_a = (\frac{\partial}{\partial x^a} + A''^{\sigma}_a \frac{\partial}{\partial z^\sigma}, \alpha^a_b dx^b),\ \Xi_u = (B''^{\sigma}_u \frac{\partial}{\partial z^\sigma}, \beta^u_a dx^a + dz^u), \tag{3.11}$$

which are the formulas given in [6]. In the almost Dirac case, we have the corresponding formulas deduced from (3.5)

$$\mathcal{X}_a = (X_a + A''^{\sigma}_a Z_\sigma, \alpha^a_b \theta^b),\ \Xi_u = (B''^{\sigma}_u Z_\sigma, \beta^u_a \theta^a + \psi^u). \tag{3.12}$$

For (3.11) and (3.12) the conditions (3.8) reduce to

$$\alpha^b_a + \alpha^a_b = 0,\ B''^{\sigma}_u + B''^{u}_\sigma = 0,\ \beta^u_a + A''^{u}_a = 0. \tag{3.13}$$

As an example, we use these formulas for a straightforward proof of the fact that, in the two-dimensional case $m = 2$, any almost Dirac structure D is Dirac. Indeed, the points of M may be classified into three classes $(0, 1, 2)$ where $dim(pr_{TM}D) = 0, 1, 2$, respectively. Obviously, any point of class 2 has a neighborhood U of class 2 and D is integrable on U. Formulas (3.12) and (3.13) show that any point of class 1 is regular, hence, it also has a neighborhood where D is integrable. If a point of class 0 has a neighborhood U of points of class 0, D is integrable on U. Finally, if a point x_0 of class 0 has no such neighborhood, every neighborhood of x_0 has points that are of class 2 (necessarily), hence, x_0 is the limit of points that have neighborhoods where D is closed by Courant brackets and, by continuity, x_0 also has a neighborhood where D is integrable. A similar analysis, where it is simpler to use Proposition 2.2 instead of the local, canonical bases shows that a big-isotropic structure E with $k = 1, m = 2$ must be integrable.

4 Local geometry of a big-isotropic structure

We begin with a preparatory discussion of the operation of pullback of a big-isotropic structure by a mapping (see [3, 4] for the Dirac case), which is of a more general interest. Let $f : N^n \to M^m$ be a mapping of manifolds, $x \in N, y = f(x)$ a pair of corresponding points, and E an arbitrary vector subbundle of $T^{big}M$. We denote by f_* the differential of f and by f^* the transposed mapping of f_*. Then

$$f^*(E_y) = \{(X, f^*\alpha) \,/\, X \in T_xN,\, \alpha \in T^*_yM,\\ (f_*X, \alpha) \in E_y\} \tag{4.1}$$

is the pullback of E at x.

Proposition 4.1. *Let E be a rank k, big-isotropic subbundle of $T^{big}M$. Then $f^*(E_y)$ is isotropic in T^{big}_xN and its g_N-orthogonal space is $(f^*(E_y))' = f^*(E'_y)$. Furthermore, if f is an embedding of N in M, if E is an integrable, big-isotropic structure on M and if $f^*E = \cup_{y\in M} f^*(E_y)$ is a differentiable subbundle of $T^{big}N$ then f^*E is an integrable, big-isotropic structure on N. Finally, the condition*

$$E_y \cap ker\, f^*_y = E'_y \cap ker\, f^*_y \tag{4.2}$$

characterizes the situation where the dimension of $f^*(E_y)$ is $n - m + k$.

Proof. It is easy to check that $f^*(E_y)$ is an isotropic subspace of (T^{big}_xN, g_N) and we will compute its dimension. Obviously, we have

$$dim\, ker\, f_{*x} = n - rank_x f,\; dim\, ker\, f^*_y = m - rank_x f. \tag{4.3}$$

Then, define the space

$$S_y = \{(f_*X, \alpha) \in E_y \,/\, X \in T_xN,\, \alpha \in T^*_yM\} = E_y \cap (im\, f_* \oplus T^*_yM), \tag{4.4}$$

and notice that the correspondence $(f_*X, \alpha) \mapsto (X, f^*\alpha)$ produces an isomorphism

$$S_y/(S_y \cap ker\, f^*_y) \approx f^*(E_y)/ker\, f_{*x}. \tag{4.5}$$

Therefore,

$$\begin{aligned} dim\, f^*(E_y) &= dim\, ker\, f_{*x} + dim\, S_y - dim(S_y \cap ker\, f^*_y) \\ &= dim\, ker\, f_{*x} + dim\, S_y - dim(E_y \cap ker\, f^*_y) \end{aligned} \tag{4.6}$$

(the equality $S_y \cap ker\, f^*_y = E_y \cap ker\, f^*_y$ follows from the definition of S_y since $ker\, f^*_y \subseteq im\, f_{*x} \oplus T^*_yM$).

Furthermore, we notice the equality

$$im\, f_{*x} \oplus T^*_y M = (ker\, f^*_y)^{\perp_{g_M}}, \tag{4.7}$$

which follows since the left hand side is included in the right hand side and the two spaces have the same dimension by (4.3). Accordingly, we have

$$S_y = (E'_y)^{\perp_{g_M}} \cap (ker\, f^*_y)^{\perp_{g_M}} = (E'_y + ker\, f^*_y)^{\perp_{g_M}}, \tag{4.8}$$

whence, using the classical relation between the dimensions of the sum and intersection of two linear subspaces, we get

$$\begin{aligned} dim\, S_y &= 2m - dim(E'_y + ker\, f^*_y) \\ &= dim\, E_y - dim\, ker\, f^*_y + dim(E'_y \cap ker\, f^*_y). \end{aligned} \tag{4.9}$$

If we combine (4.6), (4.9) and (4.3) we get

$$dim\, f^*(E_y) = n - m + k + dim(E' \cap ker\, f^*_y) - dim(E \cap ker\, f^*_y). \tag{4.10}$$

The same procedure with the roles of E and E' interchanged uses the space $S'_y = E'_y \cap (im\, f_* \oplus T^*_y M)$, which (like in (4.9)) has the dimension

$$dim\, S'_y = dim\, E'_y - dim\, ker\, f^*_y + dim(E_y \cap ker\, f^*_y), \tag{4.11}$$

and leads to

$$dim\, f^*(E'_y) = n + m - k - dim(E' \cap ker\, f^*_y) + dim(E \cap ker\, f^*_y). \tag{4.12}$$

Hence, $dim\, f^*(E_y) + dim\, f^*(E'_y) = 2n$ and, since it is easy to check the g_N-orthogonality of these two spaces, we have $f^*(E'_y) = (f^*(E_y))^{\perp_{g_N}}$.

For the second assertion of the proposition, it suffices to notice that $X \in T_x N$ belongs to $pr_{T_xN} f^*(E_y)$ iff $f_* X \in pr_{T_yM} E_y$. If f is an embedding and if f^*E is a differentiable subbundle (in particular, $dim\, f^*(E_y) = const.$), a field $X \in \Gamma(pr_{T_xN}(f^*E))$ has an f-related field $f_*X \in \Gamma(pr_{T_yM} E)$. Moreover, since $dim\, ker\, f^*_y = m - n = const.$, a cross section of f^*E must be of the form $(X, f^*\alpha)$ where (f_*X, α) is a differentiable cross section of E, and the same holds for f^*E' and E'. If these facts are taken into consideration, a straightforward examination of the Courant brackets shows that the integrability conditions for E imply the integrability conditions for f^*E.

Finally, hypothesis (4.2) is equivalent with $dim(E' \cap ker\, f^*_y) = dim(E \cap ker\, f^*_y)$ and, by (4.10), we have the required dimension for $f^*(E_y)$ iff (4.2) holds. □

Remark 4.1. 1) By (4.10), $dim\, f^*(E_y) = const.$ iff

$$dim(E'_y \cap ker\, f^*_y) - dim(E_y \cap ker\, f^*_y) = const.$$

By (4.9) and (4.11), this condition is equivalent with

$$dim\, S'_y - dim\, S_y = 2(m-k) - (dim(E' \cap ker\, f^*_y) + dim(E \cap ker\, f^*_y)) = const.$$

2) The algebraic part of Proposition 4.1 holds for any linear mapping $l : V \to W$ between linear spaces and any isotropic subspace $E \subseteq W \oplus W^*$. The significance of the equality $dim\, f^*(E_y) = n - m + k$ is the codimensional invariance property $n - dim\, f^*(E_y) = m - dim\, E_y$. Hypothesis (4.2) holds in the Dirac case ($E = E'$) and in the case where f is a submersion ($ker\, f^*_y = 0$).

Corollary 4.1. *With the notation of Proposition* 4.1 *and of its proof, if* f *is an embedding and if* $dim\, S_y = const.$, $dim\, S'_y = const.$ *the pullbacks* f^*E, f^*E' *are differentiable.*

Proof. By the definition of S_y, S'_y (see (4.4)), if the dimensions of these spaces do not depend on x, S_y, S'_y are differentiable with respect to x; then, by Remark 4.1 1) $dim\, f^*(E_y) = const.$ and $dim\, f^*(E'_y) = const.$ On the other hand, since f is an embedding, $ker\, f_{*x} = 0$ and formula (4.5) yields

$$S_y/(E_y \cap ker\, f^*_y) \approx f^*(E_y), S'_y/(E'_y \cap ker\, f^*_y) \approx f^*(E'_y). \tag{4.13}$$

Formula (4.13) and the constant dimensions of the spaces therein imply the differentiability of f^*E, f^*E' and Proposition 4.1 allows us to conclude. □

Now, we shall discuss some local properties of an integrable, big-isotropic structure E.

Proposition 4.2. *The pullback of an integrable, big-isotropic structure* E *to a characteristic leaf* $\mathcal{S}$ *is the same as the pullback of its Dirac extension* $D(E)$ *and it is a presymplectic structure of* $\mathcal{S}$*.*

Proof. Consider the neighborhoods $U, \mathcal{U}$ where one has the canonical basis (3.7) and $\mathcal{S} \cap U$ has the equations $y^h = 0, z^\sigma = 0$. Then, at the points of $\mathcal{U}$, $\gamma \in \Omega^1(M)$ belongs to $ann\mathcal{E}|_{\mathcal{U}}$ iff $\gamma|_{\mathcal{U}} = \gamma_h dy^h + \gamma_\sigma dz^\sigma$. This implies that $i^*\gamma = 0$ ($i : \mathcal{U} \to M$), whence, by (2.22), $i^*D(E) = i^*E$. Now, (3.2) and (3.8), which hold for the coordinate values $(x^a, 0, 0)$, give $\beta^u_a(x^c, 0, 0) = 0, \alpha^b_a(x^c, 0, 0) = -\alpha^a_b(x^c, 0, 0)$ and, accordingly, (4.1) shows that

$$i^*E = \{(f_a \frac{\partial}{\partial x^a}, f_a\alpha^a_b(x^c, 0, 0)dx^b) \,/\, f_a = f_a(x^c)\}.$$

Therefore, i^*E is the presymplectic structure defined by the 2-form of components $\alpha^a_b(x^c, 0, 0)$, which is just $\varpi|_{\mathcal{E}\times\mathcal{E}}$. $\square$

Remark 4.2. The canonical basis (3.7) also provides a local expression of the Dirac extension $D(E)$ over the neighborhood U. Indeed, (3.7) shows that the annihilator of $span\{pr_{TM}\mathcal{X}_a\}$ is spanned by the 1-forms $dy^h - A'^h_a dx^a, dz^\sigma - A''^\sigma_a dx^a$ and $ann\,\mathcal{E}$ consists of the forms

$$\gamma = \varphi_h(dy^h - A'^h_a dx^a) + \psi_\sigma(dz^\sigma - A''^\sigma_a dx^a)$$

where

$$\varphi_h B'^h_u + \psi_\sigma B''^\sigma_u = 0. \tag{4.14}$$

Therefore,

$$D_U(E) = \{f^a\mathcal{X}_a + s^u\Xi_u + (0, \varphi_h(dy^h - A'^h_a dx^a) + \psi_\sigma(dz^\sigma - A''^\sigma_a dx^a))\} \tag{4.15}$$

where $f^a, s^u, \varphi_h, \psi_\sigma \in C^\infty(U)$ and (4.14) holds. In particular, if the structure E is regular then $B'^h_u \equiv 0, B''^\sigma_u \equiv 0$ and $D(E)$ is a differentiable, Dirac structure.

Now, at x_0 , we consider the local transversal submanifold $\mathcal{Q}_0$ of the characteristic leaf $\mathcal{S}$ given by the equations $x^a = 0$ and denote by $\iota : \mathcal{Q}_0 \to M$ the corresponding embedding. For this embedding, (3.7) shows that the spaces S, S' of Corollary 4.1 are given by

$$S = span\{\Xi_u|_{\mathcal{Q}_0}\}, \;\; S' = span\{\Xi_u|_{\mathcal{Q}_0}, \mathcal{Y}_h|_{\mathcal{Q}_0}, \Theta_q|_{\mathcal{Q}_0}\},$$

hence, S, S' have a constant dimension. Then, Corollary 4.1 tells us that $\mathcal{Q}_0$ has an induced, integrable, big-isotropic structure $E^{tr}_{x_0} = \iota^*E$, which will be called the *transversal structure* of E at x_0.

Furthermore, with (4.4) and (3.7), it follows that $E \cap ker\,\iota^* = S \cap ker\,\iota^* = 0$ and we see that E^{tr}_x is isomorphic with the bundle $S|_{\mathcal{Q}_0}$. In particular, Ξ_u

$(\mathrm{mod}\, x^a = 0)$ is a basis of the transversal structure $E^{tr}_{x_0}$ along $\mathcal{Q}_0$. We also notice that $E^{tr} \cap (T\mathcal{F} \oplus ann\, T\mathcal{F}) = 0$ and, in particular, $E^{tr}_{x_0}$ is of the graph type (it is the graph of a mapping $\Lambda \to T\mathcal{Q}_0$, where Λ is a field of subspaces of $T^*\mathcal{Q}_0$).

Any local transversal submanifold $\mathcal{T}_0$ of $\mathcal{U}$ at x_0 inherits a transversal structure since there exists a tubular neighborhood such that $\mathcal{Q}_0 = \mathcal{T}_0$. Hence, we may speak of the transversal structure in a generic way, which, however, does not mean that the transversal structures defined on different submanifolds $\mathcal{T}_0$ are equivalent. We shall address this question but we need more preparations first. The following considerations are inspired by the case of foliation-coupling Dirac structures [15] and may be used in a discussion of the coupling between a big-isotropic structure and a foliation, which we do not intend to develop.

Recall that U has the tubular foliation $\mathcal{F}$ of equations $x^a = const.$. Define the field of subspaces

$$H(E,\mathcal{F}) = \{Z \in TM \,/\, \exists \alpha \in ann\, T\mathcal{F},\ (Z,\alpha) \in E\}. \tag{4.16}$$

Using the basis (3.7 we get

$$H(E,\mathcal{F}) = \{f^a pr_{TM}\mathcal{X}_a \,/\, \sum_a f^a \alpha'^a_u = 0,\ f^a \in C^\infty(U)\}, \tag{4.17}$$

therefore, $H(E,\mathcal{F}) \cap T\mathcal{F} = 0$. Generally, $H(E,\mathcal{F})$ may not have a constant dimension; we will say that $H(E,\mathcal{F})$ is the *pseudo-normal* bundle of $\mathcal{F}$. $H(E,\mathcal{F})$ is a true normal bundle, i.e.,

$$T_U M = H(E,\mathcal{F}) \oplus T\mathcal{F}, \tag{4.18}$$

iff

$$\alpha'^a_h = 0 \quad (\overset{(3.8)}{\Leftrightarrow}\ \gamma^h_a = 0). \tag{4.19}$$

We would like to notice that condition (4.19) has other interesting interpretations too. One of them is obtained if we define the field of subspaces

$$\mathcal{H}(E',\mathcal{F}) = \{\theta \in T^*M \,/\, \exists Z \in T\mathcal{F}, (Z,\theta) \in E'\}, \tag{4.20}$$

which we call the *pseudo-conormal bundle of $\mathcal{F}$ modulo E*. Using (3.7), it follows that $\theta \in \mathcal{H}(E',\mathcal{F})$ iff

$$\theta = \varphi_u(\beta^u_a dx^a + \beta'^u_h dy^h + dz^u) + \psi_h \gamma^h_a dx^a + \mu_q(\lambda^q_a dx^a + dy^q).$$

Accordingly, we see that

$$(4.21)\quad T^*_U M = \mathcal{H}(E', \mathcal{F}) + ann\, T\mathcal{F},\ \mathcal{H}(E', \mathcal{F}) \cap ann\, T\mathcal{F} = span\{\gamma^h_a dx^a\}.$$

Therefore, we have

$$(4.22)\qquad T^*_U M = \mathcal{H}(E', \mathcal{F}) \oplus ann\, T\mathcal{F}$$

iff (4.19) holds.

For another interpretation of condition (4.19) let us define the field of subspaces

$$(4.23)\qquad H(E', \mathcal{F}) = \{Z \in TM \,/\, \exists \alpha \in ann\, T\mathcal{F},\, (Z, \alpha) \in E'\}.$$

Notice that $H(E, \mathcal{F}) \subseteq H(E', \mathcal{F})$. From (3.7), we get

$$(4.24)\qquad H(E', \mathcal{F}) = span\{pr_{TM}\mathcal{X}_a - \sum_{q,\sigma} \alpha'^a_q L''^\sigma_q \frac{\partial}{\partial z^\sigma}, pr_{TM}\mathcal{Y}_h\},$$

Therefore, $\mathcal{E}'|_U = H(E', \mathcal{F}) + T\mathcal{F}$ and $H(E', \mathcal{F}) \cap T\mathcal{F}$ is the (trivial) $(m-k)$-dimensional bundle with the basis $\{pr_{TM}\mathcal{Y}_h\}$. Together with (4.24), this implies

$$(4.25)\qquad (ann\, \mathcal{F})^* \approx H(E', \mathcal{F})/(H(E', \mathcal{F}) \cap T\mathcal{F}).$$

On the other hand, let us recall the truncated 2-form $\varpi : \mathcal{E} \times \mathcal{E}' \to \mathbb{R}$ and consider its *second flat morphism* $\flat'_\varpi : \mathcal{E}' \to \mathcal{E}^*$ given by $(\flat'_\varpi Y)(X) = \varpi(X, Y)$. It follows that condition (4.19) holds iff

$$(4.26)\qquad H(E', \mathcal{F}) \cap T\mathcal{F} \subseteq ker\, \flat'_\varpi.$$

This again is an interpretation of (4.19). Notice also that if (4.19) holds then $\forall X \in H(E', \mathcal{F})$ the corresponding 1-form α such that $(X, \alpha) \in E'$ is uniquely defined.

The interesting fact that follows from (4.18) is that $E|_U$ decomposes along $\mathcal{F}$ and $H(E, \mathcal{F})$. Indeed, using the canonical basis (3.7) we see that (4.18) implies

$$T^*\mathcal{F} \approx ann\, H(E, \mathcal{F}) = span\{dy^h - A'^h_a dx^a,\, dz^\sigma - A''^\sigma_a dx^a\}$$

and

$$E \cap (T\mathcal{F} \oplus T^*\mathcal{F}) = span\{\Xi_u\}.$$

Thus, this intersection produces the transversal structures on the fibers of $\mathcal{F}$. We also have $H^*(E,\mathcal{F}) \approx ann\, T\mathcal{F}$ and

$$E \cap (H(E,\mathcal{F}) \oplus H^*(E,\mathcal{F})) = span\{\mathcal{X}_a\}.$$

Therefore, the following decomposition holds

$$E|_U = [E \cap (T\mathcal{F} \oplus T^*\mathcal{F})] \oplus [E \cap (H(E,\mathcal{F}) \oplus H^*(E,\mathcal{F}))]. \tag{4.27}$$

This property justifies the following definition.

Definition 4.1. A big-isotropic structure E on M is called *locally decomposable* if each point $x \in M$ has a tubular neighborhood U of the characteristic slice (a neighborhood of the characteristic leaf) $\mathcal{U}$ through x, with the tubular foliation $\mathcal{F}$, where the decomposition (4.18) holds. If (4.18) holds for any tubular neighborhood, E will be called *strongly, locally decomposable*.

All the Dirac structures are strongly, locally decomposable [6]; a non-Dirac example follows.

Example 4.1. Take $M = \mathbb{R}^3$ with coordinates (x, y, z) and

$$E = span\{(\frac{\partial}{\partial x}, 0), (0, dz)\}.$$

E is a regular, integrable, big-isotropic structure of dimension 2 with

$$E' = E \oplus span\{(\frac{\partial}{\partial y}, 0), (0, dy)\}.$$

The given basis is canonical and local decomposability holds. New tubular coordinates are defined by

$$\tilde{x} = \tilde{x}(x, y, z),\ \tilde{y} = \tilde{y}(x, y, z),\ \tilde{z} = \tilde{z}(x, y, z),$$

where (see (3.10)

$$\tilde{y}(x, 0, 0) = 0,\ \tilde{z}(x, 0, 0) = 0,\ \left.\frac{\partial \tilde{y}}{\partial x}\right|_{(x,0,0)} = 0,\ \left.\frac{\partial \tilde{z}}{\partial x}\right|_{(x,0,0)} = 0,$$

$$\left.\frac{\partial \tilde{z}}{\partial y}\right|_{(x,0,0)} = 0,\ \frac{\partial \tilde{x}}{\partial x} \neq 0,\ \frac{\partial \tilde{y}}{\partial y} \neq 0,\ \frac{\partial \tilde{z}}{\partial z} \neq 0.$$

If we make this change in the generators of E and produce a canonical basis out of the result we see that (4.19), hence local decomposability, also holds for the new tubular neighborhood.

The following example shows that local decomposability of an (integrable), big-isotropic structure does not imply strong, local decomposability.

Example 4.2. Take $M = \mathbb{R}^5$ with the canonical coordinates (x^1, x^2, y^1, y^2, z). Define

$$E = span\{\mathcal{X}_1 = (\frac{\partial}{\partial x^1}, dx^2 + dy^1), \tag{4.28}$$
$$\mathcal{X}_2 = (\frac{\partial}{\partial x^2}, -dx^1 + dy^2),\ \Xi_1 = (0, dz)\}.$$

This is a regular, 3-dimensional, integrable, big-isotropic structure and the orthogonal bundle E' has the supplementary generators

$$\mathcal{Y}_1 = (\frac{\partial}{\partial y^1}, -dx^1),\ \mathcal{Y}_2 = (\frac{\partial}{\partial y^2}, -dx^2),\ \Theta_1 = (0, dy^1),\ \Theta_2 = (0, dy^2). \tag{4.29}$$

The characteristic leaf through the origin is the (x^1, x^2)-plane, $\mathbb{R}^5$ may be seen as a tubular neighborhood of this leaf and the basis given by (4.28), (4.29) is canonical. Accordingly, the local decomposability property does not hold for the tubular foliation that consists of the family of 3-dimensional planes with coordinates (y^1, y^2, z). Now, consider the following coordinate transformation

$$\tilde{x}^1 = x^1 - y^2,\ \tilde{x}^2 = x^2 + y^1,\ \tilde{y}^1 = y^1,\ \tilde{y}^2 = y^2,\ \tilde{z} = z.$$

If we express the pairs (4.28), (4.29) and then produce a canonical basis of (E, E') with respect to the new coordinates we get

$$\tilde{\mathcal{X}}_1 = (\frac{\partial}{\partial \tilde{x}^1}, d\tilde{x}^2),\ \tilde{\mathcal{X}}_2 = (\frac{\partial}{\partial \tilde{x}^2}, -d\tilde{x}^1),\ \tilde{\Xi} = (0, d\tilde{z}),$$
$$\tilde{\mathcal{Y}}_1 = (\frac{\partial}{\partial \tilde{y}^1}, 0),\ \tilde{\mathcal{Y}}_2 = (\frac{\partial}{\partial \tilde{y}^2}, 0),\ \tilde{\Theta}_1 = (0, d\tilde{y}^1),\ \tilde{\Theta}_2 = (0, d\tilde{y}^2).$$

Thus, with respect to the tubular fibers defined by the new coordinates the local decomposability property holds.

It is known that the transversal structure of a Dirac structure is well defined up to a natural equivalence [6]. We will show that the same holds for the big-isotropic structures that are strongly, locally decomposable. The basis (3.7) also yields a transversal structure on each submanifold $\mathcal{Q}_x$ defined by $x^a = x^a(x)$ $(x \in \mathcal{U})$. We will denote by E^{tr} the family of transversal structures E^{tr}_x, $x \in \mathcal{U}$ and prove the following lemma.

Lemma 4.1. *Let E be a locally decomposable, integrable, big-isotropic structure on M, x_0 a point of M and U a tubular neighborhood of x_0 with the tubular foliation $\mathcal{F}$. Then, any $\mathcal{F}$-projectable vector field $Z \in H(E,\mathcal{F})$ is an infinitesimal automorphism of the transversal structure E^{tr}.*

Proof. Since Z is $\mathcal{F}$-projectable, the flow of Z sends leaves of $\mathcal{F}$ to leaves of $\mathcal{F}$. As explained earlier, the transversal structure E^{tr} on the leaves of $\mathcal{F}$ is induced by the vector bundle $S = span\{\Xi_u\}$ (see (3.7)). Equivalently, E^{tr} is induced also by the bundle $\tilde{E}^{tr} = S \oplus ann\, T\mathcal{F}$ (obviously, the intersection of the terms is zero and $\tilde{E}^{tr}$ is an integrable, big-isotropic structure on U of the same dimension k like E). Thus, the conclusion will be obtained if we show that Z is an infinitesimal automorphism of $\tilde{E}^{tr}$. If we denote

$$\mathcal{X}_a = (V_a, \nu^a),\ \Xi_u = (W_u, \xi^u), \tag{4.30}$$

the projectability of Z is equivalent with asking $Z = \zeta^a(x^b)V_a$ and Z is an infinitesimal automorphism of $\tilde{E}^{tr}$ iff

$$(L_{V_a}W_u, L_{V_a}\xi^u) \in \Gamma\tilde{E}^{tr}. \tag{4.31}$$

The integrability of E implies the existence of local functions f^a, φ^u such that

$$\begin{aligned}[\mathcal{X}_a, \Xi_u] &= ([V_a, W_u], L_{V_a}\xi^u - L_{W_u}\nu^a + d(\nu^a(W_u))) \\ &= ([V_a, W_u], L_{V_a}\xi^u - i(W_u)d\nu^a) = (L_{V_a}W_u, L_{V_a}\xi^u) \\ &\qquad -(0, (W_u\alpha^a_b)dx^b) = f^a\mathcal{X}_a + \varphi^u\Xi_u,\end{aligned}$$

where we have used the local decomposability condition $\alpha'^a_h = 0$. Since $[V_a, W_u]$ does not contain $\partial/\partial x^a$, we must have $f^a = 0$ and (4.31) follows. □

Now, we can prove

Proposition 4.3. *Let E be a strongly, locally decomposable, integrable, big-isotropic structure on a manifold M. Then, the transversal structure $E^{tr}_{x_0}$ is well defined up to a structure preserving diffeomorphism.*

Proof. The proof given for the Dirac structures in [6] also holds here. First we look at local transversal submanifolds $\mathcal{T}_0, \mathcal{T}_1$ of the characteristic slice $\mathcal{U}$ at $x_0 \neq x_1 \in \mathcal{U}$. Then, there exist a tubular neighborhood such that $\mathcal{T}_0, \mathcal{T}_1$ are fibers of the tubular foliation $\mathcal{F}$ and a diffeomorphism Φ, which is a composition of transformations of flows of $\mathcal{F}$-projectable vector fields

$Z \in H(E, \mathcal{F})$, that sends $\mathcal{T}_0$ onto $\mathcal{T}_1$. By Lemma 4.1, Φ sends the transversal structure on $\mathcal{T}_0$ onto the transversal structure on $\mathcal{T}_1$. Now, if we have two different local transversal submanifolds $\mathcal{T}_0, \mathcal{T}'_0$ at the same point x_0 of $\mathcal{U}$, we take a loop of $\mathcal{U}$ at x_0, break it into a finite number of pieces, and go from $\mathcal{T}_0$ to $\mathcal{T}'_0$ through intermediate transversal submanifolds defined at the breaking points. The composition of the diffemorphisms Φ defined as above between the intermediate transversal manifolds gives us the required equivalence of the transversal structures on $\mathcal{T}_0$ and $\mathcal{T}'_0$. □

5 Reduction of big-isotropic structures

Recently, the generalization of symplectic reduction to Dirac structures was discussed by several authors, in particular [2, 10]. In this section we discuss a reduction scheme for big-isotropic structures.

We begin by defining a push forward procedure. The notation will be similar to that of Proposition 4.1; in particular, we consider the mapping $f : N \to M$ and the corresponding points $y = f(x)$. But, we start with a rank k subbundle $E \subseteq T^{big}N$. Then, we define the push forward of E by

$$f_*(E_x) = \{(f_*X, \alpha) \,/\, X \in T_xN,\ \alpha \in T^*_yM,\ (X, f^*_y\alpha) \in E_x\}. \tag{5.1}$$

Proposition 5.1. *If the bundle E is a big-isotropic structure then $f_*(E_x)$ is an isotropic subspace of (T^{big}_yM, g_M) and its orthogonal space is $f_*(E'_x)$. Furthermore, iff*

$$E_x \cap ker\, f_{*x} = E'_x \cap ker\, f_{*x} \tag{5.2}$$

the dimension of $f_*(E_x)$ is $m - n + k$.

Proof. The isotropy of $f_*(E_x)$ is obvious. Then, let us define the space

$$\Sigma = \{(X, f^*\alpha) \in E_x \,/\, X \in T_xN,\ \alpha \in T^*_yM\} = E_x \cap (T_xN \oplus im\, f^*_y) \tag{5.3}$$

$$= (E'_x + ker\, f_{*x})^{\perp_{g_N}}.$$

The last equality (5.3) follows from

$$(ker\, f_{*x})^{\perp_{g_N}} = T_xN \oplus im\, f^*_y, \tag{5.4}$$

which holds because the right hand side is included in the left hand side and the former has the dimension required for the orthogonal space of the latter. The correspondence $(X, f^*\alpha) \mapsto (f_*X, \alpha)$ produces an isomorphism

$$\Sigma/(E_x \cap ker\, f_{*x}) \approx f_*(E_x)/ker\, f_y^*, \tag{5.5}$$

whence

$$\begin{aligned}
dim\, f_*(E_x) &= dim\, ker\, f_y^* + dim\, \Sigma - dim(E_x \cap ker\, f_{*x}) \\
&\overset{(4.3)}{=} m - rank_x f + dim\, \Sigma - 2n + dim(E_x \cap ker\, f_{*x})^{\perp_{g_N}} \\
&\overset{(5.4)}{=} m - rank_x f + dim\, \Sigma - 2n + dim((T_xN \oplus im\, f_y^*) + E'_x) \\
&= m - rank_x f + dim\, \Sigma - 2n + dim(T_xN \oplus im\, f_y^*) \\
&\quad + dim\, E'_x - dim((T_xN \oplus im\, f_y^*) \cap E'_x) \\
&= m + n - k + (dim\, \Sigma - dim\, \Sigma'),
\end{aligned}$$

where Σ' is the space (5.3) for E'.

The same calculations with the roles of E and E' interchanged give

$$dim\, f_*(E'_x) = m - n + k - (dim\, \Sigma - dim\, \Sigma'),$$

hence, $dim\, f_*(E_x) + dim\, f_*(E'_x) = 2m$. Since it is trivial to check that the two spaces are g_M-orthogonal, we get the required relation $f_*(E'_x) \perp_{g_M} f_*(E_x)$.

Furthermore, from the last expression of Σ in (5.3) and the similar expression of Σ' we get

$$\begin{aligned}
dim\, \Sigma - dim\, \Sigma' &= dim\, \Sigma'^{\perp_{g_N}} - dim\, \Sigma^{\perp_{g_N}} \\
&= dim\, E_x + dim\, ker\, f_{*x} - dim(E_x \cap ker\, f_{*x}) \\
&\quad - dim\, E'_x - dim\, ker\, f_{*x} + dim(E'_x \cap ker\, f_{*x}) \\
&= 2k - 2n + (dim(E'_x \cap ker\, f_{*x}) - dim(E_x \cap ker\, f_{*x})).
\end{aligned}$$

Accordingly, we obtain

$$dim\, f_*(E_x) = m - n + k + (dim(E'_x \cap ker\, f_{*x}) - dim(E_x \cap ker\, f_{*x})), \tag{5.6}$$

which justifies the last assertion of the proposition. □

Remark 5.1. Proposition 5.1 holds for any linear mapping $l : V \to W$ between linear spaces and any isotropic subspace $E \subseteq V \oplus V^*$. The significance of the equality $dim\, f^*(E_y) = m - n + k$ is the codimensional invariance property $n - dim\, E_x = m - dim\, f_*(E_x)$. Hypothesis (5.2) holds in the Dirac case ($E = E'$) and in the case where f is an immersion ($kef\, f_* = 0$). If f is a submersion and the difference between the dimensions of Σ, Σ' is constant then $dim\, f_*(E_x)$ is the same $\forall x \in N$. However, there may not be a well defined subbundle f_*E because $f^-1(y)$ may have more than one point.

Proposition 5.2. *If f is a submersion then $f_*f^*(E_y) = E_y$. If f is an immersion then $f^*f_*(E_x) = E_x$.*

Proof. By looking at the formulas (4.1), (5.1) we see that $(Z, \lambda) \in f_*f^*(E_y)$ iff $(Z, \lambda) = (f_*X, \alpha + \beta)$ where $(f_*X, \alpha) \in E_y$ and $\beta \in ker\, f^*_y$. Since, if f is a submersion $ker\, f^*_y = 0$, we get $(Z, \lambda) \in E_y$, hence, $f_*f^*(E_y) \subseteq E_y$. On the other hand, again since f is a submersion, any pair of E_y is of the form (f_*X, α), hence, of the form $(Z, \lambda) \in f_*f^*(E_y)$. Thus, $E_y \subseteq f_*f^*(E_y)$ and the first assertion is proven. The proof of the second assertion is similar. $\square$

Another kind of preparation that we need concerns the notion of projectability. Let M be an m-dimensional, differentiable manifold and $\mathcal{F}$ a foliation of M by p-dimensional leaves. In what follows, the terms projectable and foliated are synonymous and describe objects related with corresponding objects of the space of leaves via the natural projection. In particular, a vector field is foliated if it can be projected onto the space of leaves and a differential form is projectable if it is the pullback of a form on the space of leaves (such forms are called basic forms by most authors).

Definition 5.1. An arbitrary subbundle $E \subseteq T^{big}M$ of rank k is called *foliated* or *projectable* if each point $x \in M$ has an open neighborhood U such that the quotient manifold $Q_U = U/(\mathcal{F}|_U)$ is endowed with a subbundle $\Delta_U \subseteq T^{big}Q_U$ that satisfies the condition $E|_U = \pi^*(\Delta_U)$ ($\pi : U \to U/(\mathcal{F}|_U)$).

In Definition 5.1, π is the natural projection and $\pi^*(\Delta_U)$ is obtained by the pullback of Δ_U, i.e.,

$$E_x = \{(X, \pi^*\alpha) \,/\, X \in T_xM,\, (\pi_*X, \alpha) \in \Delta_{\pi(x)}\} \;\; (x \in U). \tag{5.7}$$

Proposition 5.3. *The subbundle E is foliated iff it satisfies the following two conditions: a) $E \supseteq T\mathcal{F}$, b) each point $x \in M$ has an open neighborhood U*

such that $\Gamma E|_U$ has a basis (X_i, ξ^i) $(i = 1, ..., k)$ with projectable vector fields X_i and projectable 1-forms ξ^i. Furthermore, condition b) may be replaced by b') every $Y \in \Gamma T\mathcal{F}$ is an infinitesimal automorphism of E, i.e., $\forall (X, \xi) \in \Gamma E$ one has $(L_Y X, L_Y \xi) \in \Gamma E$.

Proof. If E is projectable, since $(0, 0) \in \Delta_U$, formula (5.7) shows that a) holds. Then, assume that (V_u, λ^u) $(u = 1, ..., q = m - p)$ is a local basis of the cross sections of Δ_U and put $V_u = [X_u]_{T\mathcal{F}}, \xi_u = \pi^* \lambda^u$ $(X \in \chi^1(U))$, where the brackets denote equivalence classes modulo $T\mathcal{F}$. It follows easily that $\Gamma E|_U$ has a local basis that consists of (X_u, ξ^u) and of a basis of $\Gamma T\mathcal{F}$. Hence b) also holds. Conversely, if we have the properties a), b), we can change the basis provided by b) to a basis of the form $((X_u, \xi^u), (Y_a, 0))$ where Y_a is a local basis of $T\mathcal{F}$ and $([X_u]_{T\mathcal{F}}, \lambda^u)$, $\xi^u = \pi^* \lambda^u$, is a basis for the local structure Δ_U required by Definition 5.1. For the last assertion, if b) holds and if we put $(X, \xi) = \sum_{i=1}^k f^i (X_i, \xi^i)$ then

$$(L_Y X, L_Y \xi) = \sum_{i=1}^k (Y f^i)(X_i, \xi^i) \in \Gamma E.$$

Conversely, assume that a), b') hold. For any $x \in M$, we can take a cubical, open neighborhood with $\mathcal{F}$-adapted local coordinates (z^a, y^u) (i.e., $\mathcal{F}$ has the local equations $dy^u = 0$ and $z^a(x) = 0, y^u(x) = 0$). Then, we can take an arbitrary basis of E of the form $(\partial/\partial z^a, 0), (V_u, \lambda^u))$ along the transversal slice $z^a = 0$ and move (V_u, λ^u) along the slice $y^u = 0$ by the linear holonomy of the foliation $\mathcal{F}$ (e.g., [9]). By hypothesis b') the flows of the tangent vectors of the leaves preserve E, hence, the result of the previous procedure consists of cross sections of E and we get a projectable basis as required by b). □

Corollary 5.1. *If E is a big-isotropic structure on M and ΓE is closed by Courant brackets, E is foliated iff $E \supseteq T\mathcal{F}$.*

Proof. . For a foliated subbundle E the indicated condition is condition a) of Proposition 5.3. Conversely, since ΓE is closed by Courant brackets, with the notation of b'), Proposition 5.3, we have

$$[(Y, 0), (X, \xi)] = (L_Y X, L_Y \xi) \in \Gamma E.$$

Hence, conditions a) and b') hold and we are done. □

Proposition 5.4. *For a big-isotropic, projectable subbundle $E \subseteq T^{big}M$, ΓE is closed by Courant brackets iff the spaces $\Gamma\Delta_U$ of the local projected subbundles Δ_U are closed by Courant brackets.*

Proof. For two differentiable, local, cross sections of E of the form prescribed by (5.7) the Courant bracket has the following expression

$$[(X,\pi^*\alpha),(Y,\pi^*\beta)] = ([X,Y],\pi^*(L_{\pi_*X}\beta - L_{\pi_*Y}\alpha) + \frac{1}{2}d(\alpha(\pi_*Y)-\beta(\pi_*X))) \tag{5.8}$$

and it is related by (5.7) with the Courant bracket $[(\pi_*X,\alpha),(\pi_*Y,\beta)]$ on Q_U. Thus, if $\Gamma\Delta$ is closed by Courant brackets, the left hand side of (5.8) is in ΓE. Furthermore, the Courant bracket satisfies the property that, $\forall f \in C^\infty(M)$, one has

$$[(X,\alpha),f(Y,\beta)] = f[(X,\alpha),(Y,\beta)] + (Xf)(Y,\beta) - g((X,\alpha),f(Y,\beta))(0,df) \tag{5.9}$$

and the isotropy of E yields

$$[f(X,\pi^*\alpha),h(Y,\pi^*\beta)] = fh[(X,\pi^*\alpha),(Y,\pi^*\beta)] + f(Xh)(X,\pi^*\alpha) - h(Yf)(Y,\pi^*\beta),$$

where $f,h \in C^\infty(M)$ may not be projectable. Accordingly, the existence of projectable bases of E shows that the bracket-closure of $\Gamma\Delta_U$ implies the bracket-closure of ΓE. The converse result is a straightforward consequence of the existence of the projectable bases of E. □

Remark 5.2. From (5.9) it follows that if $E \subseteq T^{big}M$ is closed by Courant brackets then either E is isotropic or $E \supseteq T^*M$.

We may apply the previous general results to speak of projectable, big-isotropic structures, integrable or not, and of projectable (almost) Dirac structures.

Proposition 5.5. *If E is a projectable, big-isotropic structure on the foliated manifold $(M,\mathcal{F})$ its orthogonal bundle E' is projectable as well. Moreover, the corresponding, local, projected bundles Δ_U are big-isotropic on the local transversal manifolds $\mathcal{Q}_U$, the orthogonal bundles Δ'_U are the local, projected bundles of E', and E is integrable iff the structures Δ_U are integrable.*

Proof. It is easy to check conditions a) and b') of Proposition 5.3 for E'. (In particular, for b') take $(X,\alpha) \in \Gamma E, (Y,\beta) \in \Gamma E', Z \in T\mathcal{F}$, express $Z(g((X,\alpha),(Y,\beta))) = 0$ and use b') for E.) For the other assertions, use Proposition 5.4 and its proof. □

Example 5.1. Let $P \in \chi^2(M)$ be a bivector field on $(M,\mathcal{F})$ and define

$$D_P = T\mathcal{F} \oplus \{(\sharp_P\alpha, \alpha) \,/\, \alpha \in ann(T\mathcal{F})\}. \tag{5.10}$$

This is an almost Dirac structure, which does not depend on the $T\mathcal{F}$-component of P in the following sense. If $\nu\mathcal{F}$ is a normal bundle of $\mathcal{F}$, i.e., $TM = \nu\mathcal{F} \oplus T\mathcal{F}$, and if we put

$$P = \frac{1}{2}P^{ab}Z_a \wedge Z_b + P^{au}Z_a \wedge Y_u + \frac{1}{2}P^{uv}Y_u \wedge Y_v, \tag{5.11}$$

where (z^a, y^u) are the local coordinates used in the proof of Proposition 5.3 and $Z_a = \partial/\partial z^a, Y_u = \partial/\partial y^u - t^a_u Z_a$ (for some coefficients t^a_u), D_P is spanned by $(Z_a, 0)$ and $(P^{uv}Y_v, dy^u)$. Conditions a), b') of Proposition 5.3 show that the projectability of D_P is equivalent with the projectability of the bivector field P, i.e., $\partial P^{uv}/\partial z^a = 0$. Furthermore, using Proposition 5.5, we see that the integrability of D_P holds iff $[P,P]|_{ann(T\mathcal{F})} = 0$ (Schouten-Nijenhuis bracket), hence D_P is equivalent with the *transversely Hamiltonian structure* defined by P on $(M,\mathcal{F})$ [14].

Example 5.2. In a similar way, let ω be a foliated 2-form on $(M,\mathcal{F})$ and $\nu\mathcal{F}$ be a chosen normal bundle. Define

$$D_\omega = T\mathcal{F} \oplus \{(X, \flat_\omega X) \,/\, X \in \nu\mathcal{F}\}. \tag{5.12}$$

Then D_ω is an almost Dirac structure. If we put

$$\omega = \frac{1}{2}\omega_{uv}(y)dy^u \wedge dy^v, \tag{5.13}$$

we see that D_ω is spanned by $(Z_a, 0), (Y_u, \omega_{uv}Y_v)$, therefore, D_ω is foliated and independent on the choice of $\nu\mathcal{F}$. Furthermore, D_ω is integrable iff ω is closed, i.e., D_ω is equivalent with a foliated presymplectic form.

Now, we will discuss the concept of reduction. The general geometric framework may be described as follows. Let M be a manifold, E a subbundle of $T^{big}M$ and $\iota : N \hookrightarrow M$ an embedded submanifold. Assume that N is E-*proper*, meaning that the pullback ι^*E is differentiable. Then, assume that N is endowed with a foliation $\mathcal{F}$ such that $T\mathcal{F} \subseteq \iota^*E$, a condition that is equivalent with

$$\forall Z \in T\mathcal{F},\ \exists \alpha \in ann\, TN \text{ such that } (Z, \alpha) \in E|_N \tag{r}$$

and will be called the *reducibility condition*. Then, ι^*E may be foliated. If it is so and if the quotient space $N/\mathcal{F}$ is a paracompact, Hausdorff manifold Q, ι^*E projects to a subbundle $E^{red}_N \subseteq T^{big}Q$ that has the restrictions Δ_U of Definition 5.1 over the projections $\pi(U)$. Accordingly, $\pi^*E^{red}_N = \iota^*E$ and Proposition 5.2 shows that we may write

$$E^{red}_N = \pi_*(\iota^*E). \tag{5.14}$$

The bundle E^{red}_N will be called the *reduction of E via $(N, \mathcal{F})$*. This is a generalization of the framework described in the paper of Stiénon-Xu [10].

Now, the following result is immediate.

Proposition 5.6. *Let E be an integrable, big-isotropic structure on the manifold M. Let $\iota : N \to M$ be an embedded submanifold such that the fields of subspaces $E \cap (TN \oplus T^*_N M), E' \cap (TN \oplus T^*_N M)$ are of a constant dimension. Let $\mathcal{F}$ be a foliation of N by the fibers of a submersion $\pi : N \to Q$ such that the natural projection*

$$pr_{T\mathcal{F}} : E \cap (T\mathcal{F} \oplus ann\, TN) \to T\mathcal{F} \tag{5.15}$$

is surjective. Then, there exists a well defined, integrable, reduced, big-isotropic structure E^{red} on Q that satisfies the condition (5.14).

Proof. By Corollary 4.1 ι^*E is an integrable, big-isotropic structure of N and condition (5.15) is equivalent with the reducibility condition (r). Thus, the reduced structure E^{red} may exist and Corollary 5.1 tells us that E^{red} exists indeed. Then, Proposition 5.5 shows that E^{red} is an integrable, big-isotropic structure on Q. ☐

Corollary 5.2. *Let E be an integrable, big-isotropic structure on the manifold M. Assume that the connected, Lie group G acts on M and the action preserves E and fixes an embedded submanifold $\iota : N \to M$. Assume that the restriction of the action of G to N is proper and free and denote by $\mathcal{F}$ the foliation of N by the orbits of G. Finally, assume that the reducibility condition* (r) *holds for the infinitesimal transformations Z of G on N. Then, there* *exists a Hausdorff manifold $Q = N/G$ endowed with a reduced, integrable, big-isotropic structure E^{red}.*

Proof. Under the hypotheses, condition (r) holds for any $Z \in \chi^1(N)$ and the fields of subspaces $E \cap (TN \oplus T^*_N M)$, $E' \cap (TN \oplus T^*_N M)$ have a constant dimension. □

Example 5.3. We shall apply Proposition 5.6 to a Poisson structure $E_P = E'_P = graph\,\sharp_P$, where $P \in \chi^2(M)$ is a Poisson bivector field on M. Assume that the submanifold $\iota^* : N \hookrightarrow M$ is such that $dim(E_P \cap (TN \oplus T^*_N M)) = const.$ and that N has a foliation $\mathcal{F}$ with the quotient manifold $Q = M/\mathcal{F}$. Then, N has the Dirac structure

$$\iota^* E_P = \{(\sharp_P\alpha, \iota^*\alpha) \,/\, \sharp_P\alpha \in TN, \alpha \in T^*_N M\}.$$

Furthermore, the reducibility condition (r) is equivalent to

$$T\mathcal{F} \subseteq \sharp_P(ann\,TN). \tag{5.16}$$

Therefore, if we assume that (5.16) holds, Q has the reduced Dirac structure $E^{red} = \pi_*\iota^*(E_P)$, which, pointwisely, turns out to be

$$E^{red} = \{(\pi_*\sharp_P\tilde{\lambda}, \lambda) \,/\, \lambda \in T^*Q,\; [\tilde{\lambda}]_{ann\,TN} = \pi^*\lambda,\; \sharp_P\tilde{\lambda} \in TN\}. \tag{5.17}$$

The reduced structure E^{red} is Dirac. If we want a Poisson reduced structure we have to add the condition $E^{red} \cap TQ = 0$, which, by (5.17), is equivalent to

$$\sharp_P(ann\,TN) \cap TN \subseteq T\mathcal{F}.$$

A more general Poisson reduction scheme, where $T\mathcal{F} = V \cap TN$ for a vector bundle V over N that satisfies adequate conditions, was given by Marsden and Ratiu [8]. The present example corresponds to the case $V = \sharp_P(ann\,TN)$, while, however, we do not ask $\sharp_P(ann\,TN)$ to be a regular vector bundle.

Acknowledgement. Part of the work on this paper was done during the author's visit to the Bernoulli Center of the École Polytechnique Fédérale de Lausanne, Switzerland, in June-August 2006, and the author wishes to express his gratitude to the Center and to professor Tudor Raţiu, the director of the Bernoulli Center, in particular, for the invitation and support.

References

[1] H. Bursztyn, M. Crainic, A. Weinstein, C. Zhu, Integration of twisted Dirac structures, Duke Math. J., 123 (2004), 549-607

[2] H. Bursztyn, G. R. Cavalcanti and M. Gualtieri, Reduction of Courant algebroids and generalized complex structures, arXiv:math.DG/0509640.

[3] H. Bursztyn and O. Radko, Gauge equivalence of Dirac structures and symplectic groupoids, Ann. Inst. Fourier (Grenoble), 53 (2003), 309-337.

[4] T. J. Courant, Dirac Manifolds, Transactions Amer. Math. Soc., 319 (1990), 631-661.

[5] I. M. Gelfand and I. Ya. Dorfman, The Schouten bracket and Hamiltonian operators. Funkt. Anal. Prilozhen. 14(3) (1980), 71-74.

[6] J.-P. Dufour and A. Wade, On the local structure of a Dirac manifold, arXiv:math.SG/0405257.

[7] Z.-J. Liu, A. Weinstein and P. Xu, Manin triples for Lie bialgebroids, J. Diff. Geom., 45 (1997), 547-574.

[8] J. E. Marsden and T. Ratiu, Reduction of Poisson manifolds, Lett. Math. Phys., 11 (1986), 161-169.

[9] P. Molino, Riemannian foliations, Progress in Math., vol. 73, Birkhäuser, Boston, 1988.

[10] M. Stiénon and P. Xu, Reduction of Generalized Complex Structures, arXiv:DG/0509393.

[11] I. Vaisman, Lectures on the geometry of Poisson manifolds, Progress in Math., vol. 118 Birkhäuser Verlag, Boston, 1994.

[12] I. Vaisman, Transversal Twistor Spaces of Foliations, Annal. Global Analysis and Geom., 19 (2001), 209-234.

[13] I. Vaisman, Tangent Dirac Structures and Submanifolds, Intern. J. of Geom. Methods in Modern Physics, 2 (2005), 759-775.

[14] I. Vaisman, Poisson structures on foliated manifolds, Travaux Math. - Publ. de l'Univ. de Luxembourg (4^{th} Conference on Poisson Geometry; C. Molitor-Braun, N. Poncin and M. Schlichenmaier eds.), 16 (2005), 139-161.

[15] I. Vaisman, Foliation-coupling Dirac structures, J.Geom. Phys., 56 (2006), 917-938.

Department of Mathematics
University of Haifa, Israel
E-mail: vaisman@math.haifa.ac.il