Galloping instability of viscous shock waves

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Abstract

Motivated by physical and numerical observations of time oscillatory "galloping", "spinning", and "cellular" instabilities of detonation waves, we study Poincaré–Hopf bifurcation of traveling-wave solutions of viscous conservation laws. The main difficulty is the absence of a spectral gap between oscillatory modes and essential spectrum, preventing standard reduction to a finite-dimensional center manifold. We overcome this by direct Lyapunov–Schmidt reduction, using detailed pointwise bounds on the linearized solution operator to carry out a nonstandard implicit function construction in the absence of a spectral gap. The key computation is a space-time stability estimate on the transverse linearized solution operator reminiscent of Duhamel estimates carried out on the full solution operator in the study of nonlinear stability of spectrally stable traveling waves.

Contents

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1 Introduction

Motivated by physical and numerical observations of time-oscillatory "galloping" or "pulsating" instabilities of detonation waves [MT, BMR, FW, MT, AT, F1, F2], we study the Poincaré–Hopf bifurcation of viscous shock waves in one spatial dimension. Our main result is to obtain a rigorous (nonlinear) characterization in terms of spectral information; the corresponding spectral problem has been studied already in [LyZ1, LyZ2]. By essentially the same analysis we obtain also a corresponding multi-dimensional result applying to planar viscous shock fronts traveling in a cylinder of finite cross-section, with artificial Neumann or periodic boundary conditions. This gives a simplified mathematical model for time-oscillatory instabilities observed in detonation waves moving within a duct, which, besides the longitudinal galloping instabilities described above, include also transverse "cellular" or "spinning" instabilities. The method of analysis appears to be of general application, in particular, with suitable elaboration, to extend to the originally motivating case of viscous detonation waves of the reactive Navier–Stokes equations with physical viscosity.

From a mathematical standpoint, the main issue in our analysis is that the linearized equations about a standing shock wave have no spectral gap between convective modes corresponding to essential spectra of the linearized operator L about the wave and oscillatory modes corresponding to pure imaginary point spectra. This prohibits the usual PDE analysis by center manifold reduction to a finite-dimensional subspace; likewise, at the linearized level, decay to the center subspace is at time-algebraic rather than time-exponential rate, so that oscillatory and other modes are strongly coupled. We overcome this difficulty by the introduction of a nonstandard Implicit Function Theorem framework suitable for infinite-dimensional Lyapunov–Schmidt reduction in the absence of spectral gap (Section 2.1), together with a finite-dimensional "weak" Implicit Function Theorem suitable for finite-dimensional bifurcation analysis in situations of limited regularity (Section 2.2). The latter, based on the Brouwer Fixed-point Theorem rather than the standard Contraction-mapping construction, seems of interest in its own right.

The study of bifurcation from stability is a natural followup to our previous work on stability of viscous shock and detonation waves; see, e.g., [ZH, MaZ3, MaZ4, HRZ, LyZ1, LyZ2, LRTZ]. Interestingly, the key estimate needed to apply our bifurcation framework turns out to be a space-time stability estimate on the transverse linearized solution operator (meaning the part complementary to oscillatory modes) quite similar to estimates on the full solution operator arising in the stability analysis of viscous traveling waves. To carry out this estimate requires rather detailed pointwise information on the Green function of the linearized operator about the wave; see Sections 1.6.4 and 3. Indeed, we use the full power of the pointwise semigroup techniques developed in [ZH, MaZ1, MaZ3].

1.1 Shocks, detonations, and galloping

The equations of compressible gas dynamics in one spatial dimension, like many equations in continuum mechanics, take the form of hyperbolic conservation laws

$$u_t + f(u)_x = 0 \tag{1.1}$$

(Euler equations) or hyperbolic–parabolic conservation laws

$$u_t + f(u)_x = (B(u)u_x)_x \tag{1.2}$$

(Navier–Stokes equations), depending whether second-order transport effects– in this case, viscosity and heat-conduction– are neglected or included. Here, $x, t \in \mathbb{R}^1$ are spatial location and time, $u \in \mathbb{R}^n$ is a vector of densities of conserved quantities– mass, momentum, and energy in the case of gas dynamics, $f \in \mathbb{R}^n$ is a vector of corresponding fluxes, and $B \in \mathbb{R}^{n\times n}$ is a matrix of transport coefficients.

Such equations are well known to support traveling-wave solutions of form

$$u(x,t) = \bar{u}(x-st) := \begin{cases} u_- & x-st \le 0, \\ u_+ & x-st > 0 \end{cases} \tag{1.3}$$

(discontinuous) and

$$u(x,t) = \bar{u}(x-st), \qquad \lim_{z\to\pm\infty} \bar{u}(z) = u_\pm \tag{1.4}$$

(smooth), respectively, known as ideal and viscous shock waves. These waves may be observed physically; indeed, the corresponding physical objects appear to be quite stable [BE].

Similarly, the equations of reacting, compressible gas dynamics take the form

$$\begin{aligned} u_t + \tilde{f}(u,z)_x - qK\phi(u)z = 0 \\ z_t + (v(u,z)z)_x + K\phi(u)z = 0 \end{aligned} \tag{1.5}$$

(reactive Euler, or Zeldovich–von Neumann–Doering equations (ZND)) or

$$\begin{aligned} u_t + \tilde{f}(u,z)_x - qK\phi(u)z = (\tilde{B}(u,z)u_x)_x, \\ z_t + (v(u,z)z)_x + K\phi(u)z = (\tilde{D}(u,z)z_x)_x \end{aligned} \tag{1.6}$$

(reactive Navier–Stokes equations (rNS)), where $x, t \in \mathbb{R}^1$, $u \in \mathbb{R}^n$ is as before, $z \in \mathbb{R}^r$ is a vector of mass fractions of different reactant species, $v \in \mathbb{R}^1$ is fluid velocity, and $\tilde{f}$, $\tilde{B}$ and $\tilde{D}$ are flux vectors and transport matrices depending (through z) on chemical makeup of the gas, with $\tilde{f}(u,0) = f(u)$, $\tilde{B}(u,0) = B(u)$. The matrix K models reaction dynamics, while $\phi \in \mathbb{R}^1$ is an ignition function serving to "turn on" the reaction: zero for temperatures below a certain critical temperature and positive for temperatures above.

Equations (1.5) and (1.6) support traveling wave solutions

$$u(x,t) = \bar{u}(x-st) \tag{1.7}$$

analogous to (1.3) and (1.4), known as ideal and viscous detonation waves, which may likewise be observed physically. However, in contrast to the shock wave case, detonations appear to be rather unstable. What is typically observed is not the planar, steadily progressing solution (1.7), but rather a nearby solution varying time-periodically about (1.7): that is, an apparent *bifurcation with exchange of stability*. In the typical experimental setting of a detonation moving along a duct, these may be longitudinal "galloping", or "pulsating", instabilities for which the planar structure is maintained, but the form of the profile changes time-periodically, or they may be transverse instabilities in which the front progresses steadily in the longitudinal (i.e., axial) direction, but develops time-oscillatory structure in the transverse (cross-sectional) directions of a form depending on the geometry of the cross-section: "cellular" instabilities for a polygonal (e.g. rectangular) cross-section; "spinning" instabilities for a circular cross-section, with one or more "hot spots", or "combustion heads", moving spirally along the duct.

Stability of shocks and detonations may be studied within a unified mathematical framework; see, e.g., [Er1, Er2, Ko1, Ko2, D, BT, FD, LS, T, K, M1, M2, M3, FM, Me, CJLW] in the inviscid case (1.1), (1.5), and [S, Go1, Go2, KM, KMN, MN, L1, L3, GX, SX, GZ, ZH, Br1, Br2, BrZ, KK, Z1, Z2, Z3, Z4, MaZ2, MaZ3, MaZ4, GMWZ1, GWMZ2, GMWZ3, HZ, BL, LyZ1, LyZ2, JLW, HuZ1, HuZ2, PZ, FS, LRTZ] in the viscous case (1.2), (1.6). In particular, stability in each case has been shown to reduce to spectral considerations accessible by standard normal-modes analysis (see Section 1.2). However, they are studied with different motivations, and yield different results. Historically, it appears that instability motivated the physical study, which has focused on the detonation case and spectral stability criteria [BE]. By contrast, the mathematical study has focused on stability and the somewhat simpler shock wave case, with the main (difficult!) issue being to establish full nonlinear stability assuming that spectral stability holds in the form of a suitable "Lopatinski" or "Evans" condition [M3, Me, Z1, GWMZ2].

Regarding bifurcation, there is strong evidence that the detonation instabilities described above correspond to a *Poincaré–Hopf bifurcation* (indeed, our description above benefits much by hindsight). In particular, Erpenbeck [Er2] and Bourlieux, Majda, and Roytburd [BMR] have carried out formal asymptotics in support of this viewpoint for the one-dimensional (galloping, or pulsating) case, in the context of the ZND equations. More recently, Kasimov and Stewart [KS] have carried out a definitive study in the multi-dimensional case by numerical linearized normal modes analysis, also for the ZND equations, in which they demonstrate that the onset of such instabilities indeed corresponds to crossing of the imaginary axis by a conjugate pair of eigenvalues toward the unstable (positive real part) side, with, moreover, excellent correspondence between observed nonlinear oscillations and the associated normal modes. Depending whether the maximally unstable mode is in the longitudinal or transverse direction, the oscillatory behavior is seen to be of galloping or spinning (cellular) type. However, up to now, there is no rigorous theory for these phenomena.

1.2 Spectral considerations

The main difficulty in the analysis of shock or detonation waves is the lack of a spectral gap between stationary or oscillatory modes and essential spectrum of the linearized operator about the wave. In the inviscid, hyperbolic case (1.1), (1.5), the linearized problem has essential spectrum filling the imaginary axis. In the viscous case (1.2), (1.6), the essential spectrum is, rather, tangent at the origin to the imaginary axis.

For example, without loss of generality taking a standing-wave solution $u = \bar{u}(x)$, $s = 0$ (i.e., working in coordinates moving with the wave), and linearizing (1.2) about $\bar{u}$, we obtain, taking $B \equiv I$ for simplicity,

$$u_t = Lu := u_{xx} - (Au)_x, \qquad A := df(\bar{u}(x)), \tag{1.8}$$

where the asymptotically constant-coefficient operator L is what we have called the linearized operator about the wave. A standard result of Henry ([He], Theorem A.2, chapter 5) on spectrum of operators with asymptotically constant coefficients asserts that the rightmost (i.e. largest real part) envelope of the essential spectrum is the envelope of the union of the rightmost envelopes of the spectra of the limiting, constant-coefficient operators at $\pm\infty$: in this case,

$$L_\pm := \partial_x^2 - A_\pm \partial_x, \qquad A_\pm := df(u_\pm). \tag{1.9}$$

The spectra of $L_\pm$ may be computed by Fourier transform to be the curves traced out by dispersion relations

$$\lambda(\xi) := i a_j^\pm \xi - \xi^2, \qquad \xi \in \mathbb{R}^1, \tag{1.10}$$

where $a_j^\pm$ are the eigenvalues of $A_\pm$. These eigenvalues are *real* and *nonzero* under the physical assumptions that (1.1) be hyperbolic in a neighborhood of $u_\pm$ and the shock be noncharacteristic. Likewise [S], L has always a zero-eigenvalue with eigenfunction $\bar{u}'(x)$, associated with translation of the wave.

In the absence of a spectral gap, *standard stability and bifurcation theorems do not apply*, and so we must carry out a refined analysis.

1.2.1 Spatio-temporal description

The spectral configuration just described translates in the x-t plane, in the stable case that there exist no other eigenvalues λ of L in the nonnegative complex half-plane $\Re\lambda \geq 0$, to the following description developed in [L1, L3, ZH, Z4, MaZ3] of the Green function $G(x,t;y)$ associated with $\partial_t - L$. A point source, or delta-function initial datum, originating at $y > 0$ will propagate initially as an approximate superposition of Gaussians with constant mass (total integral), centered along hyperbolic characteristics $dx/dt = a_j^+$ determined by the asymptotic system at $+\infty$. Those propagating in the positive direction will continue out to $+\infty$; those propagating in the negative direction will continue until they strike the shock layer at approximately $x = 0$, whereupon they will be transmitted and reflected along outgoing characteristic directions $dx/dt = a_k^-$, $a_k^- < 0$ and $dx/dt = a_k^+$, $a_k^+ > 0$. In addition, there will be deposited at the shock layer a certain amount of mass in the stationary eigenmode $\bar{u}'(x)$ (integral $\int \bar{u}(x)dx = u_+ - u_- \neq 0$), corresponding to translation of the background wave.

We refer the reader to [ZH, MaZ3] for further discussion and details. For the present purpose, it suffices to note that the Green function may be modeled qualitatively as the sum of terms

$$K(x,t;y) := ct^{-1/2}e^{-(x-y-at)/4t} \tag{1.11}$$

and

$$J(x,t;y) := c_2\bar{u}'(x)\mathrm{errfn}((-y-at)/2t^{1/2}) \tag{1.12}$$

propagating with noncharacteristic speed $a < 0$, where

$$\mathrm{errfn}(z) := \frac{1}{2\pi}\int_{-\infty}^{z} e^{-\xi^2}d\xi : \tag{1.13}$$

the first modeling moving heat kernels (Gaussian signals), and the second the excitation of the zero-eigenfunction by incoming signals from $y \geq 0$.

From this description, we see explicitly that decaying modes decay time-algebraically and not exponentially, in agreement with the lack of spectral gap. Moreover, the low-frequency/large-time–space behavior is approximately hyperbolic, propagating along characteristics associated with the end-states $u_\pm$.

1.2.2 Evans/Lopatinski determinants and transition to instability

Further information may be obtained at the spectral level by the comparison of Evans and Lopatinski determinants D and Δ for the viscous and inviscid problems (1.2) and (1.1). The Evans function $D(\lambda)$, defined as a Wronskian of functions spanning the decaying manifolds of solutions of the eigenvalue equation

$$(L - \lambda)u = 0 \tag{1.14}$$

associated with L at $x \to +\infty$ and $x \to -\infty$ is an analytic function with domain containing $\{\Re\lambda \geq 0\}$, whose zeroes away from the essential spectrum correspond in location and multiplicity with eigenvalues of L. Its behavior is also closely linked with that of the resolvent kernel of L, i.e., the Laplace transform with respect to time of the Green function G; see [AGJ, GZ, ZH, ZS, Z1, Z2] for history and further details. The corresponding object for the inviscid linearized problem is the Lopatinski determinant $\Delta(\lambda)$ defined in [K, M1, M2, Me], a homogeneous function of degree one: in the one-dimensional case, just linear.

An important relation between these two objects, established in [GZ, ZS, MeZ2], is the low-frequency expansion

$$D(\lambda) = \gamma\Delta(\lambda) + o(|\lambda|) \tag{1.15}$$

quantifying the above observation that low-frequency behavior is essentially hyperbolic, where γ is a constant measuring transversality of the profile $\bar{u}$ as a connecting orbit of equilibria $u_\pm$ in the associated traveling-wave ODE, nonvanishing for transversal connections.

A corresponding expansion

$$D_{rNS}(\lambda) = \gamma\Delta_{CJ}(\lambda) + o(|\lambda|) \tag{1.16}$$

holds for the Evans function D_{rNS} associated with detonation wave solutions of (1.6), where Δ_{CJ} denotes the Lopatinski condition, not for (1.5), but for the still simpler inviscid–instantaneous-reaction-rate *Chapman–Jouguet* model in which detonations are modeled by piecewise constant solutions (1.3) across which combustion proceeds instantaneously; see [Z1, LyZ1, LyZ2, JLW, CJLW] for further details. That is, the low-frequency behavior is not only hyperbolic but "instantaneous", with small-scale details of the profile structure lost. Both (1.15) and (1.16) extend to the corresponding multidimensional problem of a planar shock moving in $\mathbb{R}^d$ (different from the finite-cross-sectional case considered here).

As calculated in [M1] and [LyZ1], respectively, neither Δ nor Δ_{CJ} vanishes for $\lambda \neq 0$ for an ideal-gas equation of state.[1] Likewise, γ does not vanish for any choice of parameters for ideal gas dynamics by a well-known result of Gilbarg [Gi], or, by results of [GS] for reactive gas dynamics with ideal gas equation of state in the ZND-limit $|B| \to 0$. From these observations, combined with (1.15)–(1.16), we may deduce that D and D_{rNS} have for all choices of physical parameters precisely one zero at $\lambda = 0$, corresponding to translation-invariance of the background equations.

In particular, as physical parameters are varied, starting from a stable viscous traveling-wave, transition to instability, signalled by passage from the stable complex half-plane $\{\Re\lambda < 0\}$ to the unstable complex half-plane $\{\Re\lambda > 0\}$ of one or more eigenvalues of the linearized operator about the wave, *cannot occur through passage of a real eigenvalue through the origin, but rather must occur through the passage of one or more nonzero complex conjugate pairs through the imaginary axis:* that is, a Poincaré–Hopf-type configuration. This observation, made in [LyZ1, LyZ2], gives rigorous corroboration at the spectral level of the numerical observations of Kasimov and Stewart [KS]. What remains is to convert this spectral information into a rigorous nonlinear existence result.

1.2.3 Formulation of the problem

From physical/numerical observations, one expects for an ideal gas equation of state that such transition seldom or never occurs for shocks but frequently occurs for detonations. However, from a mathematical point of view, the situation for shocks and detonations appears to be entirely parallel. We may thus phrase a common mathematical problem:

(P) *Let $\bar u^\varepsilon$ denote a family of traveling-wave (either shock or detonation) solutions indexed by bifurcation parameter $\varepsilon \in \mathbb{R}^1$, with associated linearized operators L_ε transitioning from stability to instability at $\varepsilon = 0$. Assuming the "generic" spectral situation that (i) the essential spectrum of each L_ε is contained in $\{\Re\lambda < 0\} \cup \{0\}$, (ii) the translational zero-eigenvalue of each L_ε is simple in the sense that the associated Evans function D_ε vanishes at $\lambda = 0$*

[1] Indeed, this holds for multidimensions as well; see [M1, JLW].

with multiplicity one, and (iii) a single complex conjugate pair of eigenvalues

$$\lambda_\pm(\varepsilon) = \gamma(\varepsilon) \pm i\tau(\varepsilon), \qquad \gamma(0) = 0, \quad \tau(0) \neq 0, \quad d\gamma/d\varepsilon(0) > 0 \tag{1.17}$$

crosses the imaginary axis with positive speed, show that this transition to instability is associated with Poincaré–Hopf bifurcation to nearby time-oscillatory solutions.

We examine this problem in a series of different contexts.

1.3 Model analysis I: the scalar case

Let us first recall the situation considered in [TZ], of a one-parameter family of standing-wave solutions $\bar{u}^\varepsilon(x)$ of a smoothly-varying family of equations

$$u_t = \mathcal{F}(\epsilon, u) := u_{xx} - F(\epsilon, u, u_x) \tag{1.18}$$

(possibly shifts $F(\varepsilon, u, u_x) := f(u, u_x) - s(\varepsilon)u_x$ of a single equation written in coordinates $x \to x - s(\epsilon)t$ moving with traveling-wave solutions of varying speeds $s(\epsilon)$), with linearized operators $L_\epsilon := \partial\mathcal{F}/\partial u|_{u=\bar{u}^\epsilon}$ for which a spectral gap may be recovered in an appropriate exponentially-weighted norm

$$\begin{aligned} \|f\|^2_{H^2_\eta} &:= \sum_{j=0}^{2} \|(d/dx)^j f(x)\|^2_{L^2_\eta}, \\ \|f\|_{L^2_\eta} &:= \|e^{\eta(1+|x|^2)^{1/2}} f(x)\|_{L^2}, \qquad \eta > 0. \end{aligned} \tag{1.19}$$

We call this the *weighted norm condition*.

This approach, introduced by Sattinger [S] applies to the case (see Section 1.2.1) that signals in the far field are convected under the linearized evolution equation $u_t = L_\varepsilon u$ *inward* toward the background profile, hence time-exponentially decaying in the weighted norm $\|\cdot\|_{H^2_\eta}$ penalizing distance from the origin. The method encompasses both (1.2) and (1.6) in the scalar case u, $z \in \mathbb{R}^1$, with artificial viscosity $B = D = I$. It has also interesting applications to certain reaction-diffusion systems and the related Poincaré–Hopf phenomenon of "breathers" [NM, IN, IIM]. However, it does not apply to either shock or detonation waves in the system case $u \in \mathbb{R}^n$, $n \geq 2$.

In this case, we have the following rather complete result, including stability along with bifurcation description.

Proposition 1.1 ([TZ]). *Let $\bar{u}^\varepsilon$, (1.18) be a family of traveling-waves and systems satisfying the weighted norm condition and assumptions (P), with $F \in C^4$. Then, for $a \geq 0$ sufficiently small and $C > 0$ sufficiently large, there is a C^1 function $\epsilon(a)$, $\epsilon(0) = 0$, and a family of solutions*

$$u^a(x,t) = \mathbf{u}^a(x - \sigma^a t, t) \tag{1.20}$$

of (1.18) with $\epsilon = \epsilon(a)$, where $\mathbf{u}^a(\cdot, t)$ is time-periodic and σ^a is a constant drift, with

$$\|\mathbf{u}^a(\cdot,0) - \bar{u}^\varepsilon\|_{H^2_\eta} = a \quad \text{and} \quad \|\mathbf{u}^a(\cdot,t) - \bar{u}^\varepsilon\|_{H^2_\eta} \leq Ca \tag{1.21}$$

for all t. Up to fixed translations in x, t, these are the only nearby solutions of this form, as measured in H^2_η. Moreover, solutions $\mathbf{u}^a$ are time-exponentially phase-asymptotically orbitally stable with respect to H^2_η if $d\epsilon/da > 0$, in the sense that perturbed solutions converge time-exponentially to a specific shift in x and t of the original solution, and unstable if $d\epsilon/da < 0$.[2]

Proposition 1.1 was established in [TZ] by center-manifold reduction, with the main issue being to accomodate the underlying group invariance of translation. The basic idea is to work on the quotient space, reducing the problem from relative to standard Poincaré–Hopf bifurcation, afterward recovering the location $\alpha^a(t)$, driven by the solution on the quotient space, by quadrature. The integral of the periodic driving term yields a periodic part θ^a that may be subsumed in the profile and a drift $\sigma^a t$. See [TZ] for further discussion and details.

Remark 1.2. A consequence of (1.21) (by Sobolev embedding) is

$$|\mathbf{u}^a(x,t) - \bar{u}^\varepsilon| \leq Ce^{-\eta|x|}. \tag{1.22}$$

That is, the existence result in weighted norm space includes quite strong information on the structure of the wave. On the other hand, the stability result is somewhat weakened by the appearance of a spatial weight, being restricted to exponentially decaying perturbations.

[2]In (1.21), we have repaired an obvious error in [TZ], where u^0 appears in place of u^ε, and in (1.20) eliminated a redundant time-periodic translation $\theta^a(t)$.

1.4 Model analysis II: systems of conservation laws

The purpose of the present paper is to extend the analysis initiated in [TZ] for scalar models to the more physically realistic system case. For simplicity, we restrict to the somewhat simpler case of viscous shock solutions of systems with artificial viscosity $B \equiv I$; however, the method of analysis in principle applies also in the general case; see discussion, Section 1.7.

Specifically, consider a one-parameter family of standing viscous shock solutions $\bar{u}^\varepsilon(x)$ of a smoothly-varying family of conservation laws

$$u_t = \mathcal{F}(\epsilon, u) := u_{xx} - F(\epsilon, u)_x, \qquad u \in \mathbb{R}^n \tag{1.23}$$

(typically, shifts $F(\varepsilon, u) := f(u) - s(\varepsilon)u$ of a single equation written in coordinates $x \to x - s(\epsilon)t$ moving with traveling-wave solutions of varying speeds $s(\epsilon)$), with associated linearized operators $L_\epsilon := \partial\mathcal{F}/\partial u|_{u=\bar{u}^\epsilon}$. As discussed further in Section 3, we take $\bar{u}^\varepsilon$ to be of standard *Lax type*, meaning that the hyperbolic convection matrices $A_+^\varepsilon := F_u(u+, \varepsilon)$ and $A_-^\varepsilon := F_u(u_-, \varepsilon)$ at plus and minus spatial infinity have, respectively, $p-1$ negative and $n-p$ positive real eigenvalues for $1 \le p \le n$, where p is the characteristic family associated with the shock: in other words, there are precisely $n-1$ outgoing hyperbolic characteristics in the far field.

This is the only type occurring for gas dynamics with standard (e.g., ideal gas) equation of state; for reacting gas dynamics, the corresponding object is a *strong detonation*, which is the only (nondegenerate) type occurring in the ZND limit $B, D \to 0$ [GS]. The special features of Lax-type shocks (resp. strong detonations), as compared to more general undercompressive shocks (resp. weak detonations) that can occur in other settings, turn out to be important for the analysis (see Section 3), in sharp contrast with the generality of [TZ].

Note, for the system case $n \ge 2$, that there is at least one outgoing characteristic, so that the weighted norm methods of the previous section do not apply. In particular, we see no way to construct a center manifold for this problem, and suspect that one may not exist; the slow (time-algebraic) decay rate of outgoing modes evident in our description of the Green function in Section 1.2.1 suggests an essential obstacle to such construction.

On the other hand, the conservative form of system (1.23) implies that relative mass $\int u(x)dx$ is conserved for all time for perturbations $u = \tilde{u} - \bar{u}^\varepsilon$, $\tilde{u}$ satisfying (1.23). Thus, there is a convenient invariant subspace of (1.23) consisting of perturbations with zero excess mass, on which we could carry out a reduced analysis (though in fact we do not do this) in which the zero eigenvalue associated with translational invariance is removed. For, recall that $\int \bar{u}'(x)dx = u_+ - u_- \neq 0$, so that the associated zero-eigenfunction is not in the subspace. On the other hand, nonzero eigenfunctions always have zero mass [ZH], since $\lambda\phi = L_\varepsilon\phi$ implies $\lambda \int \phi(x)dx = 0$ by divergence form of L_ε, hence the crossing nonzero imaginary eigenvalues $\lambda_\pm(\varepsilon)$ in (P) persist. Thus, the spectral scenario (P) translates in the zero-mass subspace to a standard Poincaré–Hopf scenario, with no additional zero-eigenvalue, for which the translational group-invariance need not be taken into account. Recall, that this was the main issue in the analysis of the scalar case.

In short, the two problems (scalar vs. system) *have essentially complementary mathematical difficulties*, hence little technical contact. Accordingly, the analysis has a quite different flavor in the system case, depending on both the full, pointwise Green function bounds of [MaZ3] and the special, conservative structure of the equations.

Our result in this case, and the main result of the paper, is as follows.

Theorem 1.3. *Let $\bar{u}^\varepsilon$, (1.23) be a family of traveling-waves and systems satisfying assumptions (P), with $F \in C^2$. Then, for $a \geq 0$ sufficiently small and $C > 0$ sufficiently large, there is a function $\epsilon(a)$, continuous at $a = 0$ with $\epsilon(0) = 0$, and a family of time-periodic solutions $u^a(x,t)$ of (1.23) with $\epsilon = \epsilon(a)$, satisfying $\int_{-\infty}^{+\infty} (u^a(x,t) - \bar{u}(x))\, dx \equiv 0$ and the uniform bound*

$$|\mathbf{u}^a(x,t) - \bar{u}^\varepsilon| \leq C(1 + |x|)^{-1}. \tag{1.24}$$

Note that the statement of Theorem 1.3 is considerably weaker than that of Proposition 1.1, asserting no stability information, and only the relatively weak structural information of algebraic decay (1.24) at $\pm\infty$, as compared to the exponential bound (1.22). The method of proof suggests that (1.24) is sharp, or nearly so; see Remark 3.24.

Remark 1.4. With further effort, it appears possible to be recover the information of C^1 regularity of $\varepsilon(a)$ and uniqueness up to translation of solutions; see discussion, Section 5. However, this requires additional properties of the linearized solution operator that may not hold in more general situations.

1.5 Model analysis III: flow in an infinite cylinder

One-dimensional traveling-wave solutions (1.4) of (1.2) with $B \equiv I$ may alternatively be viewed as the restriction to one dimension of a planar viscous shock solution

$$u(x,t) = \bar{u}(x_1 - st) \tag{1.25}$$

of a multidimensional system of viscous conservation laws

$$u_t + \sum f^j(u)_{x_j} = \Delta_x u, \qquad u \in \mathbb{R}^n,\ x \in \mathbb{R}^d,\ t \in \mathbb{R}^+ \tag{1.26}$$

on the whole space. Likewise, traveling-wave solutions (1.25) may be viewed as planar traveling-wave solutions of (1.26) on an infinite cylinder

$$\mathcal{C} := \{x :\ (x_1, \tilde{x}) \in \mathbb{R}^1 \times \Omega\}, \qquad \tilde{x} = (x_2, \dots, x_d) \tag{1.27}$$

with bounded cross-section $\Omega \in \mathbb{R}^{d-1}$, under artificial *Neumann boundary conditions*

$$\partial u/\partial_{\tilde{x}} \cdot \nu_\Omega = 0 \quad \text{for} \quad \tilde{x} \in \partial\Omega, \tag{1.28}$$

or, in the case that Ω is rectangular, *periodic boundary conditions*, $\Omega = T^{d-1}$, T^{d-1} the rectangular torus.

We take this as a simplified mathematical model for flow in a duct, in which we have neglected boundary-layer phenomena along the wall $\partial\Omega$ in order to isolate the oscillatory phenomena of our main interest. Consider a one-parameter family of standing planar viscous shock solutions $\bar{u}^\varepsilon(x_1)$ of a smoothly-varying family of conservation laws

$$u_t = \mathcal{F}(\epsilon, u) := \Delta_x u - \sum_{j=1}^{d} F^j(\epsilon, u)_{x_j}, \qquad u \in \mathbb{R}^n \tag{1.29}$$

in a fixed cylinder $\mathcal{C}$, with Neumann (resp. periodic) boundary conditions– typically, shifts $\sum F^j(\varepsilon, u)_{x_j} := \sum f^j(u)_{x_j} - s(\varepsilon) u_{x_1}$ of a single equation (1.26) written in coordinates $x_1 \to x_1 - s(\epsilon)t$ moving with traveling-wave solutions of varying speeds $s(\epsilon)$– with linearized operators $L_\epsilon := \partial\mathcal{F}/\partial u|_{u=\bar{u}^\epsilon}$. As in the previous subsection, we take $\bar{u}^\varepsilon$ to be of standard Lax type, considered as a shock wave in one dimension. For simplicity, we take $\Omega = [0, 2\pi]^{d-1}$.

Then, we have the following result generalizing Theorem 1.3.

Theorem 1.5. *Let $\bar{u}^\varepsilon$, (1.29) be a family of traveling-waves and systems satisfying assumptions (P), with $F \in C^2$. Then, for $a \geq 0$ sufficiently small and $C > 0$ sufficiently large, there is a function $\epsilon(a)$, continuous at $a = 0$*

with $\epsilon(0) = 0$, and a family of time-periodic solutions $u^a(x,t)$ of (1.29) *with $\epsilon = \epsilon(a)$, satisfying $\int_{\mathcal{C}}(u^a(x,t) - \bar{u}(x_1))\,dx \equiv 0$ and the uniform bound*

$$|\mathbf{u}^a(x,t) - \bar{u}^\varepsilon| \le C(1 + |x_1|)^{-1}. \tag{1.30}$$

Remark 1.6. As in Remark 1.4, C^1 regularity of $\varepsilon(\cdot)$ and uniqueness of nearby solutions may be recoverable with further effort (Section 5). The extension to general cross-sectional geometries is discussed in Section 4.2.

1.6 Idea of the proof

We now briefly discuss the ideas behind the proof of Theorems 1.3 and 1.5. Under the spectral assumptions (P), there exist smooth (in ε) $L(\varepsilon)$-invariant projections onto onto the two-dimensional eigenspace $\Sigma_\pm(\varepsilon)$ of $L(\varepsilon)$ associated with the pair of crossing eigenvalues $\lambda_\pm(\varepsilon) = \gamma(\varepsilon) \pm i\tau(\varepsilon)$ and its complement $\tilde{\Sigma}(\varepsilon)$. Projecting onto these subspaces, and rewriting in polar coordinates the flow on $\Sigma_\pm$, we may thus express (1.23) in standard fashion as

$$\begin{aligned} \dot{r} &= \gamma(\varepsilon) r + N_r(r,\theta,v,\varepsilon),\\ \dot{\theta} &= \tau(\varepsilon) + N_\theta(r,\theta,v,\varepsilon),\\ \dot{v} &= \tilde{L}(\varepsilon) v + N_v(r,\theta,v,\varepsilon), \end{aligned} \tag{1.31}$$

where the "transverse linearized operator" $\tilde{L}$ is the restriction of $L(\varepsilon)$ to $\tilde{\Sigma}(\varepsilon)$, and N_j are higher-order terms coming from the nonlinear part of (1.23): N_r and N_v quadratic order in r, v and their derivatives, and N_θ linear order.

We are precisely interested in the case that $\tilde{L}$ has no spectral gap, i.e., $\Re\sigma(L) \not\le -\theta < 0$ for any $\theta > 0$. One may think for example to the finite-dimensional case that $\tilde{L}(0)$ has additional pure-imaginary spectra besides $\lambda_\pm(0) = \pm i\tau(0)$. Thus, the center manifold may be of higher dimension, and one cannot follow the usual course of reducing to a center manifold involving only r, θ, ε and applying the standard, two-dimensional Poincaré–Hopf Theorem. Instead, we proceed by a direct analysis, combining the two-dimensional Poincaré return map construction with Lyapunov–Schmidt reduction.

1.6.1 Return map construction

Specifically, truncating $|v| \le Cr$, $C >> 1$, in the arguments of N_r, N_θ, we obtain

$$|N_\theta| \le C_2 r, \tag{1.32}$$

and therefore $\dot\theta \ge \tau(0)/2 > 0$ for ε, r sufficiently small. Thus, in seeking periodic solutions

$$(r,\theta,v)(T) = (r,\theta,v)(0), \tag{1.33}$$

we may eliminate θ, solving for $T(a,b,\varepsilon)$ as a function of initial data

$$(a,b) := (r,v)(0) \tag{1.34}$$

and the bifurcation parameter ε, with $T(0,0,0) = 2\pi/\tau(0)$, and seek solutions (1.33) as fixed points

$$(a,b) = (r,v)(T(a,b,\varepsilon)) \tag{1.35}$$

of the Poincaré return map $(r,v)(T(\cdot,\cdot,\varepsilon))$.

Using Duhamel's formula/variation of constants, we may express (1.35) in a standard way (see, e.g., [HK]) as

$$\begin{aligned} 0 = \Pi_1(a,b,\varepsilon) &= \big(e^{\gamma(\varepsilon)T(a,b,\varepsilon)} - 1\big)a + \mathcal{N}_1, \\ 0 = \Pi_2(a,b,\varepsilon) &= \big(e^{\tilde L(\varepsilon)T(a,b,\varepsilon)} - \mathrm{Id}\big)b + \mathcal{N}_2, \end{aligned} \tag{1.36}$$

where

$$\begin{aligned} \mathcal{N}_1 &:= \int_0^{T(a,b,\varepsilon)} e^{\gamma(\varepsilon)(T(a,b,\varepsilon)-s)} N_1(r,\theta,v,\varepsilon)(s)ds, \\ \mathcal{N}_2 &:= \int_0^{T(a,b,\varepsilon)} e^{\tilde L(\varepsilon)(T(a,b,\varepsilon)-s)} N_2(r,\theta,v,\varepsilon)(s)ds \end{aligned} \tag{1.37}$$

are, formally, quadratic order terms in a, b. Note that (r,θ,v) are functions of (a,b,ε) as well as s, through the flow of (1.31), with

$$\|(r,v)\| \le C\|(a,b)\|$$

for any "reasonable" norm in the sense that (1.31) are locally well-posed (in practice, no restriction). By quadratic dependence of $\mathcal{N}_j$, we have, evidently, that $(a,b,\varepsilon) = (0,0,\varepsilon)$ is a solution of (1.36) for all ε.

1.6.2 Standard reduction

Continuing in this standard fashion, we should next like to perform a Lyapunov-Schmidt reduction, using the Implicit Function Theorem to solve the Π_2 equation for b in terms of (a, ε), i.e., to find a function

$$b = \beta(a, \varepsilon), \qquad \beta(0,0) = 0, \tag{1.38}$$

satisfying $\Pi_2(a, \beta(a, \varepsilon), \varepsilon) \equiv 0$: equivalently, to reduce to the nullcline of Π_2. From $|\mathcal{N}_2| \le C|(a,b)|^2$ in (1.36)(ii), we find, further, that $|\beta(a, \varepsilon)| \le C_3|a|^2$, from which straightforward Duhamel/Gronwall estimates on (1.31)(i),(ii) yield

$$|v| \le C_2 r^2 \tag{1.39}$$

for $s \in [0, T]$, justifying a posteriori the truncation $|v| \le Cr$, of N_θ performed in the first step, provided that (a, b, ε) are taken sufficiently small: specifically, small enough that r remains less than or equal to C/C_2 for $s \in [0, T]$.

Substituting into the Π_1 equation, we would then obtain a reduced, scalar bifurcation problem

$$0 = \pi(a, \varepsilon) := \Pi_1(a, \beta(a, \varepsilon), \varepsilon), \qquad \pi(0, \varepsilon) \equiv 0 \tag{1.40}$$

that is solvable in the usual way (i.e., by dividing out a and applying the Implicit Function Theorem a second time, using

$$\partial_\varepsilon\big(a^{-1}\pi\big)(0,0) = \partial_\varepsilon(\gamma/\tau)(0) = \partial_\varepsilon\gamma(0)/\tau(0) \neq 0$$

to solve for ε as a function of a; see, e.g., [HK], or Section 2.2.1).

1.6.3 Regime of validity

Let us ask ourselves now under what circumstances this basic reduction procedure may actually be carried out. Formal differentiation of (1.36) yields

$$\partial_b \Pi_2(0,0,0) = e^{2\pi(\tilde{L}/\tau)(0)} - \mathrm{Id}.$$

In the finite-dimensional ODE case, it is thus necessary and sufficient in order to apply the (standard) Implicit Function Theorem that $(e^{2\pi(\tilde{L}/\tau)(0)} - \mathrm{Id})$ be invertible, i.e., that $\tilde{L}(0)$ have no "resonant" oscillatory modes, i.e., pure imaginary eigenvalues that are integer multiples of the crossing modes

$\lambda_\pm(0) = \pm i\tau(0)$. Note that this includes interesting cases that do not yield to the standard center-manifold reduction, for which $\tilde L$ has no spectral gap. The analogous criterion in the infinite-dimensional (e.g., PDE) setting is that all possible resonant modes $ni\tau(0)$, $n \in \mathbb{Z}$, *lie in the resolvent set of* $(\tilde L)$.

This is encouraging, and shows that the requirements of Lyapunov-Schmidt reduction are much less than those for center-manifold reduction: in particular, only spectral separation (from both $\pm i\tau(0)$ and their aliases $ni\tau(0)$) rather than spectral gap is needed. Unfortunately, in the setting (1.23), (1.29) of our interest we have not even spectral separation, since essential spectra of $\tilde L(\varepsilon)$ accumulates at $\lambda = 0$ for every ε. In this case, or others for which separation fails, we must follow a different approach.

1.6.4 Refined analysis

Alternatively, we may rewrite (1.36)(ii) formally as a fixed-point equation

$$b = \big(\mathrm{Id} - e^{\tilde L(\varepsilon)T(a,b,\varepsilon)}\big)^{-1}\mathcal{N}_2(a,b,\varepsilon), \tag{1.41}$$

then apply the Contraction-mapping Principle to carry out an Implicit Function construction "by hand" (reminiscent of, but not exactly the standard proof of the Implicit Function Theorem). In the standard case that $|e^{\tilde L t}|$ is exponentially decaying ($\sim$ spectral gap), we may rewrite (1.41) using Neumann expansion as

$$b = \sum_{j=0}^\infty e^{j\tilde L(\varepsilon)T(a,b,\varepsilon)}\mathcal{N}_2(a,b,\varepsilon). \tag{1.42}$$

The basis for our analysis is the simple observation that, more generally, provided the series on the righthand side converges (conditionally) for $\|(a,b)\|$ bounded, then (i) (by scaling argument) the righthand side is contractive in b for $\|(a,b,\varepsilon)\|$ sufficiently small, and (ii) (by standard telescoping sum argument, applying $\big(\mathrm{Id} - e^{\tilde L(\varepsilon)T(a,b,\varepsilon)}\big)$ to (1.41)) the resulting solution

$$b = \beta(a,\varepsilon)$$

guaranteed by the Contraction-mapping Principle is in fact a solution of the original equation (1.36)(ii). See Section 2 for further details.

In the context of (1.23), by divergence form of the equation, $N_2 = (n_2)_x$, with $n_2 = O(|(r,v)|^2)$. For simplicity of discussion, model n_2 as a bilinear form in (r, v), so that (by quadratic scaling) $n_2 \in L^1$ for $v \in L^2$, from which we obtain $\mathcal{N}_2 = n_x$, $n \in L^1$ for $b \in L^2$; see Section 3 for further details and the extension to the general case. Thus, replacing the righthand side of (1.42) by its continuous approximant $\mathcal{N}_2$ plus

$$T^{-1} \int_T^\infty e^{\tilde{L}(\varepsilon)t} \mathcal{N}_2 \, dt = T^{-1} \int_T^\infty \big(e^{\tilde{L}(\varepsilon)t} \partial_x\big) n \, dt, \qquad n \in L^1,$$

we see that convergence (as a map from L^2 to L^2) reduces, roughly, to a space-time stability estimate

$$\Big\| \int_T^\infty \int_{-\infty}^{+\infty} \tilde{G}_y(x, t; y) n(y) dy \, dt \Big\|_{L^2(x)} \le C \|n\|_{L^1} \tag{1.43}$$

quite similar to those arising for the full solution operator (again, through Duhamel's principle; see, for example, [Z4, MaZ4]) in the study of nonlinear stability of spectrally stable waves, or, equivalently,

$$\sup_y \Big\| \int_T^\infty \tilde{G}_y(x, t; y) dt \Big\|_{L^2(x)} \le C, \tag{1.44}$$

where $\tilde{G}$ is the Green kernel associated with transverse solution operator $e^{\tilde{L}t}$. (The neglected first term $\mathcal{N}_2$ is C^2 from L^2 to L^2 by the smoothing action of (1.37)(ii); see Section 3.2.)

Note that we have used strongly the specific structure of the nonlinearity, both quadratic dependence and divergence form, in order to have any hope of convergence, since the L^2-operator norm $\big|e^{\tilde{L}t}\big|_{L^2 \to L^2}$ does not decay in t. However, this is still not enough. For, recall the one-dimensional discussion of Section 1.2.1, modeling $\tilde{G}$ qualitatively as the sum of a convected heat kernel

$$K(x, t; y) := ct^{-1/2} e^{-(x-y-at)/4t} \tag{1.45}$$

and an error-function term $J(x, t; y) := c_2 \bar{u}'(x) \mathrm{errfn}((-y - at)/2t^{1/2})$. From the standard bound $\|K_y(\cdot, t; y\|_{L^2} = ct^{-3/4}$, we find that $\|\tilde{G}_y\|_{L^2(x)}$ is not time-integrable, hence $\int_T^\infty \tilde{G}_y(x, t; y) dt$ is not absolutely convergent in $L^2(x)$.

Instead, we must show *conditional convergence*, using detailed knowledge of the propagator $\tilde{G}$ to identify cancellation in $\int_T^\infty \tilde{G}_y(x, t; y) dt$. For example, using $K_y = a^{-1}(K_t - K_{yy})$, we find that

$$
\begin{aligned}
\int_T^t K_y(x,s;y)\,ds &= a^{-1}\big(\int_T^t K_t(x,s;y)\,ds - \int_T^t K_{yy}(x,s;y)\,ds\big) \\
&= a^{-1}\big(K(x,s;y)|_T^t - \int_T^t K_{yy}(x,t;y)\,ds\big)
\end{aligned}
\tag{1.46}
$$

is the sum of $-a^{-1}K(x,T;y) \in L^2(x)$ with terms $a^{-1}K(x,t;y) \sim t^{-1/4}$ and $a^{-1}\int_T^t K_{yy}(x,t;y)\,ds \sim \int_T^t s^{-5/4}ds$ that are respectively decaying and absolutely convergent in $L^2(x)$, hence converges uniformly with respect to y in $L^2(x)$. A similar cancellation argument yields convergence of the error-function term, along with uniform boundedness with respect to y.

Convergence in the latter case is not uniform with respect to y, a detail which necessitates further technicalities: in particular, the weighted estimate

$$|v(x)| \le C(1+|x|)^{-1}$$

leading eventually to (1.24). Likewise, there are further issues associated with limited ε-regularity, addressed by the introduction of a nonstandard, Brouwer-based Implicit Function Theorem; see Section 2.2. However, the main idea is contained in calculation (1.46). The multi-dimensional case goes similarly, with the computation of the critical neutral, zero transverse wave number reducing to the one-dimensional case, and all others to the case of a spectral gap.

Remark 1.7. It is readily checked that the above arguments go through in the general case that n_2 is quadratic order for $|v|_{L^\infty} \le C$ and not bilinear in (v,r), substituting $v \in L^2 \cap L^\infty$ ($\Rightarrow n_2 \in L^1$) for $v \in L^2$, working in the $L^2 \cap L^\infty$ norm in place of L^2, and substituting for (1.44) the estimate

$$\sup_y \Big\| \int_T^\infty \tilde{G}_y(x,t;y)dt \Big\|_{L^2\cap L^\infty(x)} \le C.$$

Similar cancellation estimates are important in the study of asymptotic behavior of stable viscous shock waves [L2, SX, L3, R, HRZ].

Remark 1.8. The family of periodic solutions we obtain (see (1.39)) lies tangent to the (r,θ)-plane corresponding to linearized oscillatory behavior. However, even in the standard case of spectral separation (nonresonance), our argument yields uniqueness of this family only in the wedge $|v| \le Cr$ on which our truncation of the nonlinearity is valid. This is a natural restriction, since the method of proof does not exclude the case that $\tilde{L}$ might have its own crossing conjugate pair with nonresonant frequency, leading to a separate family of periodic solutions tangent to the v-plane. To obtain further information requires more information on $\tilde{L}$ (e.g., a spectral gap).

Remark 1.9. The condition $\int (u^a - \bar{u})dx \equiv 0$ is not imposed a priori, but arises naturally in the course of the calculation, through (1.42) and conservation of mass. A strength of the method is that such structure need not be guessed ahead of time; see Remark 2.6.

1.7 Discussion and open problems

Theorems 1.3 and 1.5 together with the spectral observations of [LyZ1, LyZ2] give rigorous validation in a simplified context of the formal and numerical observations of [BMR, KS]. An interesting problem for future work is to extend these results to the originally-motivating case of detonation waves of the full, reacting compressible Navier–Stokes equations.

We expect that our one-dimensional analysis will extend in straightforward fashion, combining tools developed in [MaZ3, MaZ4, Z1] and [LRTZ] to treat, respectively, nonreacting gas dynamics with physical, partial viscosity and reacting gas dynamics with artificial viscosity. Likewise, we expect that we can readily treat flow in a cylinder for the physical equations with artificial Neumann or periodic boundary conditions. However, the treatment of physical, no-slip boundary conditions (presumably associated with characteristic viscous boundary layers) involves technical and philosophical difficulties beyond the scope of the present analysis, as yet unresolved even for nonreacting, incompressible flows: for example, even the construction of a background traveling profile becomes problematic in this setting. We point out that viscous boundary effects (since also viscosity) are neglected also in the ZND setting of [BMR, KS].

A second natural direction for future investigation is the question of stability of the periodic waves whose existence we have established here. In the absence of a spectral gap, our method of analysis does not directly yield stability as in the case of center manifold reduction, but at best partial information on the location of point spectrum associated with oscillatory modes, with stability presumably corresponding to the standard condition $d\varepsilon/da > 0$. The hope is that we could combine such information with an analysis like that carried out for stationary waves in [ZH, MaZ3], adapted from the autonomous to the time-periodic setting: that is, a generalized Floquet analysis in the PDE setting and in the absence of a spectral gap. We consider this a quite exciting direction for further development of the theory.

Bifurcation in the absence of a spectral gap has been considered by a number of authors in different settings; in particular, it has been studied systematically by Ioss et al [I1, I2, IA, IK, IM] in various contexts using an alternative "spatial dynamics" approach. It would be very interesting to investigate what results could be obtained by this technique in the context of viscous shock waves: more generally, to relate it at a technical level to the one used here. Our approach appears to be somewhat closer to standard center-manifold reduction, in that linearized time-evolutionary estimates appear explicitly in the analysis. On the other hand, the price for this elementary approach is that the needed estimates may in practice (as here) be rather delicate to obtain.

We mention also a recent work of Kunze and Schneider [KuS] in which they analyze pitchfork bifurcation in the absence of a spectral gap using Sattinger's weighted-norm method, as described in Section 1.3, but in a situation where convection is outward, away from the profile layer. This entails the use of "wrong-way" exponentially decaying weights at the linearized level, introducing a spatially-exponentially growing multiplier in quadratic-order source terms, in combination with separate, compensating estimates at the nonlinear level. For similar arguments in the context of nonlinear stability, see, e.g., [PW, Do]. This interesting approach has been used successfully in the shock-wave context to treat the scalar undercompressive case [Do]; however, it does not appear to generalize to the system case.

Remarks 1.10. 1. Birtea et al in [BPRT] successfully carry out a rather complicated bifurcation analysis in the ODE setting without explicit knowledge of the background solution. Similar techniques might perhaps be useful in treating flow in a duct with physcial, nonslip boundary conditions, for which description of the background flow is itself problematic.

2. In carrying out our nonstandard Implicit Function construction for shock waves, we faced the problem of non-uniform convergence of series (1.42) due to lack of spatial localization. We remedied this problem by additional weighted-norm estimates. However, the example of another famous nonstandard Implicit Function construction, namely, Nash–Moser iteration, suggests the alternative approach of introducing a "localizing" step dual to the

smoothing step in the Nash–Moser scheme. It would be interesting to see if this approach could also be carried out, thus avoiding the need for additional analysis associated with pointwise bounds. We note that Nash–Moser iteration combining spatial with the usual frequency cutoffs has been carried out by Klainerman [Kl] in the context of a nonlinear wave equation.

Plan of the paper. In Section 2, we formalize the reduction procedure set out in Section 1.6.4 as an abstract bifurcation framework suitable for application to general discrete dynamical systems in the absence of spectral gap. In Section 3, we recall the pointwise estimates furnished by the methods of [ZH, MaZ3] on the Green function $\tilde{G}$ associated with the transverse linearized solution operator $e^{\tilde{L}t}$ and use these to verify that the shock bifurcation problem indeed fits the hypotheses of our abstract framework, verifying main Theorem 1.3. In Section 4, we describe the extension to multidimensions, verifying Theorem 1.5. Finally, in Section 5, we discuss the finer points of higher-order regularity and uniqueness of solutions.

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2 Abstract bifurcation framework

We begin by formalizing the approach described in Section 1.6.4, providing an abstract framework for the analysis of bifurcations in the absence of a spectral gap for discrete dynamical systems in the general form (1.36): that is, the portion of our analysis occurring *after* the reduction via Poincaré return-map construction to a fixed-point problem. The specifics of the return-map construction are discussed separately in Section 3.2.

Our analysis combines two nonstandard Implicit Function constructions, the first a Lyapunov–Schmidt reduction based on conditional convergence on subspaces, and the second a Brouwer-based "weak" Implicit Function

Theorem requiring minimal regularity: continuity plus differentiability at a point rather than the usual C^1.

2.1 Generalized Lyapunov–Schmidt reduction

2.1.1 Setting

Consider a system of difference equations

$$\begin{aligned} \Delta x &= f(x,y,\varepsilon) = (R(x,y,\epsilon) - \mathrm{Id})x + N_1(x,y,\varepsilon), \\ \Delta y &= g(x,y,\varepsilon) = (S(x,y,\varepsilon) - \mathrm{Id})y + N_2(x,y,\varepsilon), \end{aligned} \tag{2.1}$$

associated with a discrete dynamical system

$$\begin{aligned} \hat{x} &= x + f(x,y,\varepsilon) = R(x,y,\epsilon)x + N_1(x,y,\varepsilon), \\ \hat{y} &= y + g(x,y,\varepsilon) = S(x,y,\varepsilon)y + N_2(x,y,\varepsilon), \end{aligned} \tag{2.2}$$

x, $N_1 \in \mathbb{R}^n$, $y \in \mathcal{B}_1$, $N_2 \in \mathcal{B}_2$, where $R(x,y,\varepsilon)$ and $S(x,y,\varepsilon)$ are primary and transverse "linearized" solution operators for one time-step of (2.2),

$$R : \mathbb{R}^n \to \mathbb{R}^n, \qquad S : \mathcal{B}_1 \to \mathcal{B}_1;$$

$\mathcal{B}_2 \subset \mathcal{B}_1$ are Banach spaces with norms $\|\cdot\|_{\mathcal{B}_2} \geq \|\cdot\|_{\mathcal{B}_1}$; $\varepsilon \in \mathbb{R}^m$ is a bifurcation parameter; and N_j are nonlinear terms of quadratic order in (x,y), in the sense that

$$\begin{aligned} |N_1(x,y)| &\leq C(|x| + \|y\|_{\mathcal{B}_1})^2, \\ \|N_2(x,y)\|_{\mathcal{B}_2} &\leq C(|x| + \|y\|_{\mathcal{B}_1})^2, \end{aligned} \tag{2.3}$$

where $|\cdot|$ denotes standard Euclidean norm. In the simplest case, $\mathcal{B}_2$ is a closed subspace of $\mathcal{B}_1$, with common norm $\|\cdot\|_{\mathcal{B}_2} = \|\cdot\|_{\mathcal{B}_1}$.

We assume further that all terms are Lipschitz with respect to their parameters, in the sense that $R(x,y,\varepsilon)$ and $S(x,y,\varepsilon)$ are Lipschitz continuous as functions from $(x,y,\varepsilon) \in \mathbb{R}^n \times \mathcal{B}_1 \times \mathbb{R}^m$ to $C(\mathbb{R}^n \to \mathbb{R}^n)$ and $C(\mathcal{B}_1 \to \mathcal{B}_1)$, respectively, and N_1 and N_2 are Lipschitz continuous as functions from $(x,y,\varepsilon) \in \mathbb{R}^n \times \mathcal{B}_1 \times \mathbb{R}^m$ into $\mathbb{R}^n$ and $\mathcal{B}_2$, satisfying bounds

$$\begin{aligned} |\partial_x N_1(x,y)|_{\mathbb{R}^n \to \mathbb{R}^n}, \quad |\partial_y N_1(x,y)|_{\mathcal{B}_1 \to \mathbb{R}^n} &\leq C(|x| + \|y\|_{\mathcal{B}_1}), \\ |\partial_x N_2(x,y)|_{\mathbb{R}^n \to \mathcal{B}_2}, \quad |\partial_y N_2(x,y)|_{\mathcal{B}_1 \to \mathcal{B}_2} &\leq C(|x| + \|y\|_{\mathcal{B}_1}) \end{aligned} \tag{2.4}$$

consistent with the idea of N_j as bilinear forms.

We have in mind a general discrete dynamical system whose linearized solution operator has a finite-dimensional subspace x that is spectrally separated from its complement y at $\varepsilon = 0$, in which case form (2.2) may always be achieved, with $R = R(\varepsilon)$ and $S = S(\varepsilon)$. We allow dependence of R, S on (x, y) as well as ε in order to admit the slightly more general class of systems (1.36) derived by the return map construction of Section 1.6.1. The case of our particular interest is *bifurcation from a simple eigenvalue*, $n = m = 1$, $R(0, 0, 0) = 1$, of equilibria of (2.2), or, equivalently, zeroes of (2.1).

Remark 2.1. In the return map construction of Section 1.6.1, it is more standard [HK, TZ] to eliminate t instead by using $dt/d\theta = \tau(\varepsilon)^{-1} + O(r)$ to rewrite (1.31) with "$'$" denoting $d/d\theta$ as

$$r' = (\gamma/\tau)(\varepsilon)r + N_1(r, \theta, v, \varepsilon), \\ v' = (\tilde{L}/\tau)(\varepsilon)v + N_2(r, \theta, v, \varepsilon), \tag{2.5}$$

leading to a return map (1.36) of form (2.2) with R, S depending only on ε. However, the nonlinearity

$$N_2 = \big((\tau + N_\theta)^{-1} - \tau\big)\tilde{L}v + (\tau + N_\theta)^{-1}N_v = O(r)\tilde{L}v + O(N_v) \tag{2.6}$$

in (2.5) contains terms $\partial_x(rv)$ that do not take $v \in L^2$ to $\partial_x L^1$, to which the analysis of Section 1.6.4 does not apply. It is for this technical reason that we allow (x, y)-dependence of R, S, gaining needed flexibility at the expense of slight notational and expositional inconvenience.

Example 2.2. In the discussion of Section 1.6.4 for the model problem with $\tilde{G}$ replaced by K, the space $\mathcal{B}_1$ could be taken as $L^2 \cap L^\infty$ and $\mathcal{B}_2$ as $\mathcal{B}_1 \cap \partial_x L^1$, where $\partial_x L^1$ denotes the space of weak derivatives $f = \partial_x F$ of L^1 functions F, with $\|f\|_{L^2\cap L^\infty} := \|f\|_{L^2} + \|f\|_{L^\infty}$, $\|f\|_{\partial_x L^1} := \|F\|_{L^1}$, and $\|f\|_{\mathcal{B}_1\cap\partial_x L^1} := \|f\|_{\mathcal{B}_1} + \|f\|_{\partial_x L^1}$ (see also Remark 1.7). Note that $\mathcal{B}_2$ is in this case not closed with respect to $\|\cdot\|_{\mathcal{B}_1}$; in particular, $\sum_{j=0}^{\infty} S^j y$, $y \in \mathcal{B}_2$ is bounded in $\mathcal{B}_1$, but not (by direct calculation) in $\mathcal{B}_2$, despite that is a convergent sequence with respect to $\|\cdot\|_{\mathcal{B}_1}$ of finite sums $\sum_{j=0}^{N} S^j y$ lying in $\mathcal{B}_2$, since (again, by direct calculation) $S : \mathcal{B}_2 \to \mathcal{B}_2$ is nondecaying, with operator norm $|S|_{\mathcal{B}_2\to\mathcal{B}_2} = 1$.

2.1.2 The equilibrium problem

We seek to solve the equilibrium problem $(f, g) = (0, 0)$ by Lyapunov–Schmidt reduction, that is, to determine a function $y = y(x, \varepsilon)$ satisfying

$$g(x, y(x, \varepsilon), \varepsilon) \equiv 0.$$

Substituting into (2.1), we would then obtain a *reduced*, finite-dimensional equilibrium problem (on nullcline, rather than center manifold)

$$0 = f^*(x, \varepsilon) := f(x, y(x, \varepsilon), \varepsilon), \tag{2.7}$$

presumably amenable to analysis by standard finite-dimensional techniques.

2.1.3 Assumptions

We are interested in the (nonstandard) case that $(S(x, y, \varepsilon) - \mathrm{Id})$ does not possess a bounded inverse from $\mathcal{B}_1 \to \mathcal{B}_1$. Alternative conditions are:

(A) On the subspace $\mathcal{B}_2 \subset \mathcal{B}_1$ containing Range(N_2), $(\mathrm{Id} - S(x, y, \varepsilon))$ has a right inverse $(\mathrm{Id} - S(x, y, \varepsilon))^{-1}$ that is bounded, continuous in (x, y), and Lipschitz in y (with respect to operator norm) as a map from $\mathcal{B}_2 \to \mathcal{B}_1$.

(B) On the subspace $\mathcal{B}_2 \subset \mathcal{B}_1$ containing Range(N_2), $\sum_{j=0}^{\infty} S^j$ is conditionally convergent, uniformly with respect to parameters (x, y, ε), as a map from $\mathcal{B}_2 \to \mathcal{B}_1$, with uniformly Lipschitz limit.

Lemma 2.3. *Condition (B) implies (but is not implied by) (A).*

Proof. By a standard telescoping sums argument, we find that

$$(\mathrm{Id} - S)) \sum_{j=0}^{\infty} S^j = \mathrm{Id}$$

on $\mathcal{B}_2$, hence $(\mathrm{Id} - S)^{-1} := \sum_{j=0}^{\infty} S^j$ is a right inverse of $(\mathrm{Id} - S))$ on the subspace $\mathcal{B}_2$. Moreover, $(\mathrm{Id} - S)^{-1}$ is continuous with respect to (x, y, ε), as the uniform limit of functions $\sum_{j=0}^{N} S^j$ that are continuous in (x, y, ε), and uniformly Lipschitz in y by assumption ☐

Remark 2.4. It is sufficient but not necessary for uniform Lipschitz continuity in y of $\sum_{j=0}^{\infty} S^j$ that approximants $|\sum_{j=0}^{N} \partial_y S^j|_{\mathcal{B}_2\to\mathcal{B}_1}$ be uniformly bounded. For example, in the model computation (1.46) of Section 1.6.4, approximants

$$\sum_{j=0}^{N} \partial_y S^j \sim \int_{T(a,b\varepsilon)}^{NT(a,b\varepsilon)} K_y(x,s;y)\,ds$$

have Lipschitz bound $\partial_{(a,b,\varepsilon)} \sum_{j=0}^{N} \partial_y S^j \sim \Big(K_y(x,NT;y) - K_y(x,T;y)\Big)\partial_{(a,b,\varepsilon)}T$ going to infinity in $L^2(x)$ as $N^{1/4}$ as $N\to\infty$, by $\|K_y(\cdot,t;y)\|_{L^2} \sim t^{-3/4}$, while limit $\int_{T(a,b\varepsilon)}^{+\infty)} K_y(x,s;y)\,ds$ has finite Lipschitz bound $\|K_y(x,T;y)\Big)\partial_{(a,b,\varepsilon)}T\|_{L^2(x)}$.

The condition is trivially satisfied of course, if S depends only on ε.

2.1.4 Basic reduction theorem

Proposition 2.5. *Under the assumptions of Section 2.1.1, together with condition (A) (resp. (B)): (i) $g(x,y,\varepsilon)=0$ is equivalent to*

$$y = \mathcal{T}(x,y,\varepsilon,\omega) := (\mathrm{Id} - S(x,y,\varepsilon))^{-1} N_2(x,y,\varepsilon) + \omega \tag{2.8}$$

for some $\omega \in \mathrm{Ker}(\mathrm{Id} - S(x,y,\varepsilon))$. (ii) For each fixed $|(x,\varepsilon)|$, $\|\omega\|_{\mathcal{B}_1}$ sufficiently small, (2.8) has a unique solution $y = Y(x,\varepsilon,\omega)\in\mathcal{B}_1$, $Y(\cdot)$ continuous in (x,ε,ω), with

$$\|Y(x,\varepsilon,\omega)\|_{\mathcal{B}_1} \le C(\|\omega\|_{\mathcal{B}_1}^2 + |x|^2). \tag{2.9}$$

Proof. Rewriting $g(x,y,\varepsilon)=0$ as $(\mathrm{Id} - S(x,y,\varepsilon))y = N_2(x,y,\varepsilon)\in\mathcal{B}_2$ and applying $(\mathrm{Id} - S(x,y,\varepsilon))^{-1}$ to both sides, we obtain

$$\tilde{y} = (\mathrm{Id} - S(x,y,\varepsilon))^{-1} N_2(x,y,\varepsilon), \tag{2.10}$$

with $\tilde{y} := (\mathrm{Id} - S(x,y,\varepsilon))^{-1}(\mathrm{Id} - S(x,y,\varepsilon)y$. Observing that $(\mathrm{Id} - S(x,y,\varepsilon))\tilde{y} = (\mathrm{Id} - S(x,y,\varepsilon))y$ by property (A), we obtain $\tilde{y} = y - \omega$ for some $\omega \in \mathrm{Ker}(\mathrm{Id} - S(x,y,\varepsilon))$, and thereby (i) as claimed.

By the Lipschitz bounds (2.4)(ii), combined with the assumed boundedness of $(\mathrm{Id} - S(x,y,\varepsilon))^{-1}$ as an operator from $\mathcal{B}_2\to\mathcal{B}_1$, the righthand side $\mathcal{T}$ of (2.8) is contractive in y with respect to norm $\|\cdot\|_{\mathcal{B}_1}$ for $|(x,\varepsilon)|$ sufficiently small, whence we obtain by the Contraction-mapping Principle existence and uniqueness of the solution $y = Y(x, \varepsilon, \omega) \in \mathcal{B}_1$ asserted in (ii), as the iterative limit $\lim_{j\to\infty} \mathcal{T}^j(0)$. Moreover, $Y(\cdot)$ is continuous in its arguments, as the uniform limit of continuous functions. Finally, using (A) and (2.3)(ii) to bound the righthand side of (2.8), we obtain

$$\|y\|_{\mathcal{B}_1} \leq \|\omega\|_{\mathcal{B}_1} + C(|x|^2 + \|y\|_{\mathcal{B}_1}^2),$$

which for $\|y\|_{\mathcal{B}_1}$ sufficiently small gives (2.9), completing the proof. $\square$

Noting that $0 \in \operatorname{Ker}(\mathrm{Id} - S(x, y, \varepsilon))$ for any (x, y, ε), we may choose $\omega = 0$,

$$y(x, \varepsilon) := Y(x, \varepsilon, 0), \tag{2.11}$$

we obtain the desired reduction (2.7). (In general, this is the only reasonable choice, since the rest of the kernel may depend on the solution y.)

Remark 2.6. If $(\mathrm{Id} - S(x, y, \varepsilon))$ has a kernel, as holds always in the traveling-wave context by translational group invariance, then, in order for (A) to hold, $\mathrm{Range}(N_2)$ must lie in the complementary subspace, and so by the choice $\omega = 0$ we automatically project out the kernel and work on the (nonlinearly) invariant complementary subspace. In the shock wave context, $\mathrm{Range}(N_2)$ is zero-mass, preserved by S, and so we find ourselves restricted to perturbations of zero mass. Note, however, that we do not a priori (i.e., by force) restrict to invariant subspaces, but find this in a natural way through the analysis.

A related observation is that, different from the situation in the general reaction–diffusion–convection setting considered in [TZ], the periodic waves we construct vary periodically in time while keeping the speed of the original wave. That is, there is no mean drift, as in the general case. (Proof: conservation of mass together with decay of the perturbation at spatial infinity determines the speed by the same Rankine–Hugoniot conditions as in the shock case.)

Remark 2.7. Assumption (B) is implied by exponential decay of $|S^n|$ as $n \to \infty$, which yields absolute convergence by the ratio test. This in turn is implied by stability plus spectral gap of the transverse operator S, i.e., $|\sigma(S)| < 1$, the standard case. In our case of interest, there is no spectral gap and $|S|^n$ is not exponentially, but algebraically decaying. Thus, the convergence will be conditional, and also depends strongly on the specific structure of N_2.

Remark 2.8. In the case that S is not stable, it may be split into stable/unstable parts and solved by a combined forward/backward scheme. In the parabolic PDE context, there can exist only finitely many unstable modes and so this also fits our framework. For reversible systems treated by spatial dynamics approach in [I1, I2], etc., the unstable subspace could be infinite-dimensional, and our framework based on decay/cancellation seems not likely to work. So, the approach has possible restrictions, but has the advantage of yielding a particularly direct, simple, and natural analysis closely connected to the stability theory.

2.1.5 Refined reduction theorem

Extremely useful in applications is the standard extension of the Contraction-mapping Principle to the case of a map $\mathcal{T}$ that is bounded in a norm $\|\cdot\|_2$ and contractive in a weaker norm $\|\cdot\|_1$ for bounded $\|\cdot\|_2$, so long as the unit ball in norm $\|\cdot\|_2$ is closed with respect to $\|\cdot\|_1$. We present now a similar extension of our reduction procedure that is needed for the application to viscous shock waves in Section 3.

Consider again the situation described in Section 2.1.1, but introducing additional Banach spaces $X_2 \subset X_1$, $X_1 \subset \mathcal{B}_1$ and $X_2 \subset \mathcal{B}_2$, with norms $\|\cdot\|_{X_1} \geq \|\cdot\|_{\mathcal{B}_1}$ and $\|\cdot\|_{X_2} \geq \|\cdot\|_{\mathcal{B}_2}$ stronger than the norms for $\mathcal{B}_1$ and $\mathcal{B}_2$, for which

(C) The unit ball in X_1 is closed in $\mathcal{B}_1$.

In place of (2.3), suppose that $N_2 : X_1 \to X_2$ with

$$
\begin{aligned}
|N_1(x,y)| &\leq C(|x| + \|y\|_{X_1})^2,\\
\|N_2(x,y)\|_{X_2} &\leq C(|x| + \|y\|_{X_1})^2.
\end{aligned}
\tag{2.12}
$$

In place of (2.4), substitute the weaker bounds

$$
\begin{aligned}
|\partial_x N_1(x,y)|_{\mathbb{R}^n\to\mathbb{R}^n}, \quad |\partial_y N_1(x,y)|_{\mathcal{B}_1\to\mathbb{R}^n} &\leq C(|x| + \|y\|_{X_1}),\\
|\partial_x N_2(x,y)|_{\mathbb{R}^n\to\mathcal{B}_2}, \quad |\partial_y N_2(x,y)|_{\mathcal{B}_1\to\mathcal{B}_2} &\leq C(|x| + \|y\|_{X_1}),
\end{aligned}
\tag{2.13}
$$

assuming Lipschitz continuity only for $\|y\|_{X_1}$ uniformly bounded.

In place of (A), (B), substitute:

(A') On subspace $X_2 \subset \mathcal{B}_1$ containing $\mathrm{Range}(N_2|_{X_1})$, $(\mathrm{Id} - S(x,y,\varepsilon))$ has a right inverse $(\mathrm{Id} - S(x,y,\varepsilon))^{-1}$ that is bounded as a map from $X_2 \to X_1$ and, for $\|y\|_{X_2}$ uniformly bounded, is continuous in (x,ε) and Lipschitz in y from $\mathcal{B}_2 \to \mathcal{B}_1$, uniformly with respect to (x,y,ε).

(B') On subspace $X_2 \subset \mathcal{B}_1$ containing $\mathrm{Range}(N_2|_{X_1})$, $\sum_{j=0}^{\infty} S(x,y,\varepsilon)^j$ is conditionally convergent, uniformly with respect to parameters (x,y,ε), as a map from $X_2 \to \mathcal{B}_1$, uniformly bounded as a map from $X_2 \to X_1$, and uniformly Lipschitz from $\mathcal{B}_2 \to \mathcal{B}_1$ for $\|y\|_{X_2}$ uniformly bounded.

Lemma 2.9. *Condition (B') implies (but is not implied by) (A').*

Proof. Essentially a repetition of the proof of Lemma 2.3, with property (C) giving $X_2 \to X_1$. □

Proposition 2.10. *Under the assumptions above, together with condition (A') (resp. (B')): (i) $g(x,y,\varepsilon) = 0$ for $(x,y,\varepsilon) \in \mathbb{R}^n \times X_1 \times \mathbb{R}^m$ is equivalent to*

$$y = \mathcal{T}(x,y,\varepsilon,\omega) := (\mathrm{Id} - S(x,y,\varepsilon))^{-1} N_2(x,y,\varepsilon) + \omega \tag{2.14}$$

for some $\omega \in \mathrm{Ker}(\mathrm{Id} - S(x,y,\varepsilon)) \cap X_1$. (ii) (ii) For each fixed $|(x,\varepsilon)|$, $\|\omega\|_{X_1}$ sufficiently small, (2.14) *has a unique solution $y = Y(x,\varepsilon,\omega) \in X_1$, $Y(\cdot)$ continuous in (x,ε,ω) with respect to the weaker norm $\|\cdot\|_{\mathcal{B}_1}$, with*

$$\|Y(x,\varepsilon,\omega)\|_{X_1} \le C(\|\omega\|_{X_1}^2 + |x|^2). \tag{2.15}$$

Proof. Following the same steps as for the proof of Proposition 2.5, we obtain, first, (i), and, second, that $\mathcal{T}(x,\cdot,\varepsilon,\omega)$ for $|(x,\varepsilon)|$, $\|\omega\|_{X_1}$ sufficiently small is bounded from the unit ball in X_1 to itself and contractive with respect to the weaker norm $\|\cdot\|_{\mathcal{B}_1}$. By the extended Contraction-mapping Principle described above (just the standard Contraction-mapping Principle considering the unit ball in X_1 as a closed set in $\mathcal{B}_1$), we may conclude the existence of a unique solution $y = Y(x,\varepsilon,\omega) \in X_1$, expressible as the iterative limit $\lim_{j\to\infty} \mathcal{T}^j(0)$ in $\mathcal{B}_1$. Moreover, as the uniform limit of continuous functions, $Y(\cdot)$ is continuous in its arguments with respect to norm $\|\cdot\|_{\mathcal{B}_1}$. Finally, using (A') and (2.12)(ii) to bound the righthand side of (2.8), we obtain

$$\|y\|_{\mathcal{B}_1} \le \|\omega\|_{\mathcal{B}_1} + C(|x|^2 + \|y\|_{\mathcal{B}_1}^2),$$

which for $\|y\|_{\mathcal{B}_1}$ sufficiently small gives (2.9), completing the proof. □

A sufficient condition for (C) is that X_1 be compactly embedded in $\mathcal{B}_1$. However, this is far from necessary.

Example 2.11. In the application to viscous shock waves in Section 3, we will take $\mathcal{B}_1 = L^2$, $\mathcal{B}_2 = \partial_x L^1 \cap L^2$, $X_1 = \{f : |f(x)| \le C(1+|x|)^{-1}\}$, and $X_2 = \{F = \partial_x f : |f(x)| \le C(1+|x|)^{-2}\} \cap X_1$, with $\|f\|_{X_1} := \|f(x)(1+|x|)\|_{L^\infty}$ and $\|\partial_x f\|_{X_2} := \|f(x)(1+|x|)^2\|_{L^\infty} + \|\partial_x f\|_{X_1}$. Compare Example 2.2.

Remark 2.12. Proposition 2.10 admits also the further refinement that X_j could be taken as subsets rather than subspaces of $\mathcal{B}_j$ (in the above argument, for example, we effectively work with balls in X_j). However, this elaboration is not needed in the contexts treated here, and for simplicity we omit it.

2.2 Brouwer-based Implicit Function Theorem

We augment the above reduction procedure with a nonstandard Implicit Function Theorem based on the Brouwer Fixed-point Theorem rather than the usual Contraction-mapping Principle, which features minimal regularity requirements. Designed for use together with our reduction procedure, it seems also of interest in its own right.

Lemma 2.13 (Brouwer-based IFT). *Let $f(x,\varepsilon)$ be continuous in a neighborhood of the origin, $(x,\varepsilon) \in \mathbb{R}^N$, $f(0,0)=0$, with $f(0,\varepsilon)$ differentiable at $\varepsilon = 0$ (only!), and $\partial_\epsilon f(0,0)$ invertible. Then, for $|x|$ sufficiently small, there exists a solution $\varepsilon = \varepsilon(x)$ of $f(x,\varepsilon(x)) = 0$, $\varepsilon(0) = 0$, with $\varepsilon(\cdot)$ continuous at $x = 0$ (only).*

Remark 2.14. Notice that no regularity is asserted for $\varepsilon(x)$ except at the origin. Likewise, differentiability of f is assumed only at the origin, and only in ε.

Proof. Expand

$$f_\varepsilon(0,0)^{-1} f(x,\varepsilon) = \varepsilon + \delta(\varepsilon) + N(x,\varepsilon), \tag{2.16}$$

where $\delta := f_\varepsilon(0,0)^{-1}(f(0,\varepsilon) - \varepsilon$, and $N(x,\varepsilon) := f_\varepsilon(0,0)^{-1}(f(x,\varepsilon) - f(0,\varepsilon))$. By differentiability, $\delta = o(|\varepsilon|)$, while by uniform continuity $N \to 0$ as $|x| \to 0$. Thus, for $|x|$ sufficiently small, $S(x,\varepsilon) := -\delta(\varepsilon) - N(x,\varepsilon)$ takes $B(0,C)$ to itself, whence a solution exists by continuity with respect to ε and the Brouwer fixed-point Theorem. □

Remark 2.15. The proof could be described as contractive–small decomposition.

Assuming further regularity in x, we may obtain further information using the following refinement.

Lemma 2.16. *Let $f(x,\varepsilon)$ be continuous in (x,ε), $f(0,0)=0$, $f(0,\cdot)$ differentiable at the origin, $f_\varepsilon(0,0)$ invertible, and $f(\cdot,\varepsilon)$ Hölder continuous in a neighborhood of the origin for fixed ε sufficiently small, with Hölder exponent $0<\alpha\le 1$. Then, for $|x|$ sufficiently small, there exists a solution $\varepsilon=\varepsilon(x)$ of $f(x,\varepsilon(x))\equiv 0$, with $|\varepsilon(x)|\le C|x|^\alpha$ (i.e., $\varepsilon(0)=0$ and ε Hölder continuous at the origin).*

Proof. An exact copy of the previous proof, but incorporating the new bound $|N(x,\varepsilon)|\le C|x|^\alpha$ coming from Hölder continuity to work on ball $B(0,2C|x|^\alpha)$. □

Remark 2.17. Of course, under suitable compactness hypotheses, Lemma 2.13 has a straightforward extension via the Schauder fixed-point theorem to the infinite-dimensional case.

2.2.1 Application to bifurcation

Happily, Lemma 2.13 is precisely what is needed to apply the Lyapunov–Schmidt method in finite-parameter bifurcation analyses. For example, consider a one-parameter bifurcation problem

$$\begin{aligned} 0 = f(x,y,\varepsilon) &:= \gamma(x,y,\varepsilon)x + N_1(x,y,\varepsilon),\\ 0 &= g(x,y,\varepsilon), \end{aligned} \tag{2.17}$$

$x\in\mathbb{R}^1$ scalar, $(f,g)(0,0,\varepsilon)\equiv 0$, where $\gamma(0,0,0)=0$, $\partial_\varepsilon\gamma(0,0,0)$ exists and is nonzero, and

$$|N_1|\le C(|(x,y)|^2).$$

Assume, moreover, that f is continuous in all variables.

Now, suppose that by some technique we may carry out the Lyapunov–Schmidt reduction, obtaining a solution $y=y(x,\varepsilon)$, $|y|\le C|x|^2$ of

$$g(x,y(x,\varepsilon),\varepsilon)\equiv 0,$$

that is merely *continuous* in (x,ε). Substituting into (2.17)(i), we obtain the reduced problem

$$\begin{aligned} 0 = \tilde f(x,\varepsilon) &:= f(x,y(x,y(x,\varepsilon),0),\\ &= \tilde\gamma(\varepsilon)x + \tilde N(x,\varepsilon), \end{aligned} \tag{2.18}$$

$\tilde\gamma(\varepsilon) := \gamma(0,0,\varepsilon)$, $\tilde N(x,\varepsilon) := N_1(x, y(x,\varepsilon),\varepsilon) + \big(\gamma(x, y(x,\varepsilon)),\varepsilon) - \gamma(0,0,\varepsilon)\big)x$, where

$$|\tilde N| \le C|x|^2 \tag{2.19}$$

uniformly in ε, and therefore $N_* := \tilde N(x,\varepsilon)/x$ is continuous in (x,ε) by (2.19) together with the assumed continuity of f and y.

Dividing (2.18) by x, therefore, we obtain, finally,

$$0 = f_*(x,\varepsilon) := \tilde\gamma(\varepsilon) + N_*(x,\varepsilon), \tag{2.20}$$

where f_* is continuous and $f_*(0,\varepsilon) \equiv \tilde\gamma(\varepsilon)$, satisfying the hypotheses of Lemma 2.13. Note that the variable y may be infinite-dimensional, and that we have made no assumptions at all about regularity of g. In particular, returning to the original setting of (2.1)–(2.2), we obtain the following general result on bifurcation from a simple eigenvalue.

Corollary 2.18. *Let* (2.1) *satisfy the assumptions described in Section 2.1.1 together with condition (A) (resp. (B)), or, alternatively, the assumptions of Section 2.1.5 together with condition (A') (resp. (B')). Further, suppose that* $x \in \mathbb{R}^1$, *with* $(f,g)(0,0,\varepsilon) \equiv 0)$, $R(0,0,0) = 1$ *and* $\partial_\varepsilon R(0,0,0)$ *exists and is nonzero. Then, for* $a \in \mathbb{R}^1$ *sufficiently small, there exists a function* $\varepsilon = \varepsilon(a)$, *continuous at* $a = 0$, *with* $\varepsilon(0) = 0$, *and a continuous function* $y = y(x,\varepsilon) \in \mathcal{B}_1$, *such that*

$$(f,g)(a, y(a,\varepsilon(a),\varepsilon(a)) \equiv (0,0): \tag{2.21}$$

that is, there occurs a bifurcation of equilibria at $\varepsilon = 0$ *from the trivial solution* $(x,y) \equiv (0,0)$.

Example 2.19. The main applications we have in mind are: (i) finite-dimensional Poincaré–Hopf bifurcation *with* resonant transverse modes, but in a direction orthogonal to the range of nonlinear term N_2. (ii) infinite-dimensional Poincaré–Hopf bifurcation with essential spectrum approaching resonant modes. The first is rather special, implying existence of an (non-linearly) invariant subspace complementary to $\mathrm{Ker}(\mathrm{Id} - S(\varepsilon))$ (see Remark 2.6); the second appears to be rather general. In each case, $(\mathrm{Id} - S(\varepsilon))$ has a bounded right inverse only, and only on the range of N_2.

2.3 Uniqueness and higher regularity

We now address the related issues of uniqueness and regularity with respect to a of the bifurcation function $(b, \varepsilon)(\cdot) = (Y(\cdot, \varepsilon(\cdot), \varepsilon(\cdot))$ described in Corollary 2.18 and higher regularity with respect to ε of the reduction function $Y(\cdot)$ described in Proposition 2.1. Consider the strengthened versions of (A), (B):

(A") On the subspace $\mathcal{B}_2 \subset \mathcal{B}_1$ containing Range(N_2), $(\mathrm{Id} - S(x, y, \varepsilon))$ has a right inverse $(\mathrm{Id} - S(x, y, \varepsilon))^{-1}$ that is both bounded and C^k, $k \geq 1$, in (x, y, ε) as a map from $\mathcal{B}_2 \to \mathcal{B}_1$.

(B") On subspace $\mathcal{B}_2 \subset \mathcal{B}_1$ containing Range(N_2), $\sum_{j=0}^{\infty} S(x, y, \varepsilon)^j$ is conditionally convergent, uniformly with respect to (x, y, ε), and $C^k(x, y, \varepsilon)$ as a map from $\mathcal{B}_2 \to \mathcal{B}_1$.

Lemma 2.20. *Condition (B") implies (A").*

Proof. Identical with that of Lemma 2.3. □

2.3.1 Basic regularity theorem

Proposition 2.21. *Under the assumptions of Section 2.1.1, together with condition (A") (resp. (B")), with $R, S, N_1, N_2 \in C^k$, $k \geq 1$ (Frechet sense) in all arguments: (i) $g(x, y, \varepsilon) = 0$ is equivalent to 2.8. (ii)* (2.8) *has a unique solution $y = Y(x, \varepsilon, \omega)$, C^k in (x, ε, ω), for $|(x, \varepsilon)|$, $\|\omega\|_{\mathcal{B}_1}$ sufficiently small.*

Proof. Identical with that of Proposition 2.5. □

Now consider the simplest situation that $\ell := \dim \mathrm{Ker}(\mathrm{Id} - S(x, y, \varepsilon))$ is independent of (x, y, ε) and $\mathrm{Ker}(\mathrm{Id} - S(x, y, \varepsilon))$ is independent of y, with $\mathrm{Ker}(\mathrm{Id} - S(x, y, \varepsilon))$ smoothly (C^k) parametrized by $\omega \in \mathbb{R}^\ell$, with $\omega = 0$ mapping to the origin. This is always the case, for example, in the situation (as for viscous shock waves) that $\mathrm{Ker}(\mathrm{Id} - S(x, y, \varepsilon))$ corresponds to one or more group-invariances of the underlying equations.

Corollary 2.22. *Let* (2.1) *satisfy the assumptions of Proposition 2.21. Further, suppose that $x \in \mathbb{R}^1$, with $(f, g)(0, 0, \varepsilon) \equiv 0$, $R(0, 0, 0) = 1$ and $\partial_\varepsilon R(0, 0, 0) \neq 0$, and that* $\mathrm{Ker}(\mathrm{Id} - S(x, y, \varepsilon))$ *is both independent of y and parametrized by $\omega \in \mathbb{R}^\ell$ as described just above. Then, for $a \in \mathbb{R}^1$, $\omega \in \mathbb{R}^\ell$*

sufficiently small, there exist C^k functions $\varepsilon = \varepsilon(a,\omega)$, $\varepsilon(0,0) = 0$, and $y = y(x,\varepsilon,\omega) \in \mathcal{B}_1$, $y(0,0,0) = 0$, $k \geq 1$, such that

$$(f,g)(a, y(a, \varepsilon(a,\omega), \omega), \varepsilon(a,\omega)) \equiv (0,0) : \tag{2.22}$$

that is, there occurs an ℓ-fold C^k bifurcation of equilibria at $\varepsilon = 0$ from the trivial solution $(x,y) \equiv (0,0)$. Moreover, these are the unique nontrivial equilibrium solutions nearby $(a,b,\varepsilon) \equiv (0,0,\varepsilon)$, as measured in norm $\|\cdot\|_{\mathcal{B}_1}$.

Proof. The proof is essentially identical with that of Corollary 2.18, but incorporating ω in 2.8, and relying in place of the Brouwer-based Implicit Function Theorem of Lemma 2.2 on the standard Implicit Function Theorem based on the Contraction-mapping Principle, which yields uniqueness along with existence. $\square$

Remark 2.23. In the situation that $\mathrm{Ker}(\mathrm{Id} - S(x,y,\varepsilon))$ corresponds to group-invariance(s) of the underlying equations, we may conclude that the solutions $y(a) = y(a, \varepsilon(a,0), \omega), \varepsilon(a,0))$ constructed in Corollary 2.18 are unique up to group invariance. When $\dim \mathrm{Ker}(\mathrm{Id} - S(x,y,\varepsilon)) \neq$ constant, on the other hand, it appears difficult to say anything about existence (or nonexistence) of solutions other than these. When $\mathrm{Ker}(\mathrm{Id} - S(x,y,\varepsilon))$ depends on y, it is difficult to say anything about any solution other than the one corresponding to $\omega = 0$.

Remark 2.24. In the application to Poincaré–Hopf bifurcation, there is a further issue of truncation carried out in the return map construction, limiting our uniqueness conclusions to a wedge $|v| \leq Cr$; see Remark 1.8.

2.3.2 Lipschitz version

To obtain existence and uniqueness as in Corollary 2.22, together with Lipschitz continuity of $\varepsilon = \varepsilon(a,\omega)$ and $y = y(x,\varepsilon,\omega)$, it is enough to take R, S, N_1, N_2 Lipschitz, with

$$|D_{(x,y)}N_j| \leq C|(x,y)|, \qquad |D_\varepsilon N_j| \leq C|(x,y)|^2, \tag{2.23}$$

$$|R(x,y,\varepsilon) - R(0,0,\varepsilon)|,\ |D_\epsilon\big(R(x,y,\varepsilon) - R(0,0,\varepsilon)\big)| \leq C|x,y,\varepsilon| \tag{2.24}$$

from $\mathcal{B}_2 \to \mathcal{B}_1$, for $\|y\|_{X_2}$ uniformly bounded, and (A”), (B”) replaced by refined, Lipschitz versions ((B”’)⇒(A”’)):

(A"') On subspace $X_2 \subset \mathcal{B}_1$ containing $\mathrm{Range}(N_2|_{X_1})$, $(\mathrm{Id}-S(x,y,\varepsilon))$ has a right inverse $(\mathrm{Id}-S(x,y,\varepsilon))^{-1}$ that is bounded as a map from $X_2 \to X_1$ and Lipschitz in (x,y,ε) from $\mathcal{B}_2 \to \mathcal{B}_1$ for $\|y\|_{X_2}$ uniformly bounded.

(B') On subspace $X_2 \subset \mathcal{B}_1$ containing $\mathrm{Range}(N_2|_{X_1})$, $\sum_{j=0}^{\infty} S(x,y,\varepsilon)^j$ is conditionally convergent, uniformly with respect to parameters (x,y,ε), as a map from $X_2 \to \mathcal{B}_1$, uniformly bounded as a map from $X_2 \to X_1$, and uniformly Lipschitz in (x,y,ε) from $\mathcal{B}_2 \to \mathcal{B}_1$ for $\|y\|_{X_2}$ uniformly bounded.

Proposition 2.25. *Let* (2.1) *satisfy* (2.23), (2.24), *and (A"') (resp. (B"')). Further, suppose that* $x \in \mathbb{R}^1$, *with* $(f,g)(0,0,\varepsilon) \equiv 0$, $R(0,0,0) = 1$, $R(0,0,\cdot)$ *differentiable,* $\partial_\varepsilon R(0,0,0) \neq 0$, *and that* $\mathrm{Ker}(\mathrm{Id} - S(x,y,\varepsilon))$ *is both independent of* y *and parametrized by* $\omega \in \mathbb{R}^\ell$ *as described above. Then, for* $a \in \mathbb{R}^1$, $\omega \in \mathbb{R}^\ell$ *sufficiently small, there exist Lipschitz functions* $\varepsilon = \varepsilon(a,\omega)$, $\varepsilon(0,0) = 0$, *and* $y = y(x,\varepsilon,\omega) \in \mathcal{B}_1$, $y(0,0,0) = 0$, *such that*

$$(f,g)(a, y(a,\varepsilon(a,\omega),\omega), \varepsilon(a,\omega)) \equiv (0,0) : \tag{2.25}$$

that is, there occurs an ℓ*-fold Lipschitz bifurcation of equilibria at* $\varepsilon = 0$ *from the trivial solution* $(x,y) \equiv (0,0)$. *Moreover, these are the unique nontrivial equilibrium solutions nearby* $(a,b,\varepsilon) \equiv (0,0,\varepsilon)$, *as measured in norm* $\|\cdot\|_{\mathcal{B}_1}$.

Proof. Carrying out the same reduction procedure as in Proposition 2.5, we obtain a Lipschitz function $y = y(x,\varepsilon,\omega)$ satisfying the $g(x, y(x,\varepsilon,\omega),\varepsilon) \equiv 0$, with $|y|, |D_\varepsilon y| = O(|x|^2 + |\omega|)$. Substituting this function into equation $f = 0$, and reducing as in (2.20) to an equation of the form treated in Lemma 2.13, we find that $\delta + N$ in (2.16) is contractive with respect to ε for $|(x,\varepsilon,\omega)|$ sufficiently small: the first term because it is C^1 in ε with zero-derivative at $\varepsilon = 0$, the second by Lipschitz bound $|D_\varepsilon N| \leq C|x,\varepsilon,\omega|$), which follows from (2.23), (2.24) applied to

$$\begin{aligned} N(x,\varepsilon,\omega) = x^{-1}(N_1(x, y(x,\varepsilon)),\varepsilon) - N_2(0,0,\varepsilon)) \\ + (R(x, y(x,\varepsilon,\omega),\varepsilon),\varepsilon) - R(0,0,\varepsilon). \end{aligned} \tag{2.26}$$

$\square$

Remark 2.26. In the Poincaré–Hopf context, $R(x,y,\varepsilon) = e^{\gamma(\varepsilon)T(x,y,\varepsilon)}$ with $\gamma \in C^k$, T Lipschitz, and $\gamma(0) = 0$; see Section 3.2. Thus, (2.24) follows by

$$|R(x,y,\varepsilon) - R(0,0,\varepsilon)| \leq C|\gamma(\varepsilon)||T(x,y,\varepsilon) - T(0,0,\varepsilon)| \leq C_2|\varepsilon|\,|(x,y)| \tag{2.27}$$

and

$$
\begin{aligned}
|D_\varepsilon\big(R(x,y,\varepsilon)-R(0,0,\varepsilon)\big)| &\le C|D_\varepsilon\gamma(\varepsilon)||T(x,y,\varepsilon)-T(0,0,\varepsilon)| \\
&\quad + C|\gamma(\varepsilon)||D_\varepsilon(T(x,y,\varepsilon)-T(0,0,\varepsilon))| \\
&\le C_2(|(x,y)|+|\varepsilon|),
\end{aligned}
\tag{2.28}
$$

so that the argument of Proposition 2.25 applies provided we can obtain by some method a Lipschitz reduction $y=y(x,\varepsilon,\omega)$.

3 Application to viscous shock waves

We now apply the framework of Section 2 to the problem of bifurcation of viscous shock waves. As described in Section 1.4, consider a one-parameter family of standing viscous shock solutions

$$
u(x,t)=\bar u^\varepsilon(x), \qquad \lim_{z\to\pm\infty}\bar u^\varepsilon(z)=u_\pm^\varepsilon \quad (\text{constant for fixed } \varepsilon),
\tag{3.1}
$$

of a smoothly-varying family of conservation laws

$$
u_t=\mathcal{F}(\epsilon,u):=u_{xx}-F(\epsilon,u)_x, \qquad u\in\mathbb{R}^n,
\tag{3.2}
$$

with associated linearized operators

$$
L_\epsilon:=\partial\mathcal{F}/\partial u|_{u=\bar u^\epsilon}=-\partial_x A^\varepsilon(x)+\partial_x^2,
\tag{3.3}
$$

$A^\varepsilon(x):=F_u(\bar u^\varepsilon(x),\varepsilon)$, denoting $A^\varepsilon_\pm:=\lim_{z\to\pm\infty}A^\varepsilon(z)=F_u(u_\pm,\varepsilon)$. Profiles $\bar u^\varepsilon$ satisfy the standing-wave ODE

$$
u'=F(u,\varepsilon)-F(u_-,\varepsilon).
\tag{3.4}
$$

We recall the standard assumptions of [ZH, Z4, Z1]:

(H0) $F\in C^k$, $k\ge 2$.

(H1) $\sigma(A^\varepsilon_\pm)$ real, distinct, and nonzero.

(H2) Considered as connecting orbits of (3.4), $\bar u^\varepsilon$ are transverse and unique up to translation, with dimensions of the stable subspace $S(A^\varepsilon_+)$ and the unstable subspace $U(A^\varepsilon_-)$ summing for each ε to $n+1$.

Remark 3.1. Condition (H2) implies in part that $\bar{u}^\varepsilon$ is of standard *Lax type*, i.e., the hyperbolic convection matrices $A^\varepsilon_\pm := F_u(u_\pm, \varepsilon)$ at $\pm\infty$ have, respectively, $p-1$ negative and $n-p$ positive eigenvalues for $1 \le p \le n$, where p is the characteristic family associated with the shock. For further discussion, see [ZH, Z1, HZ].

To (H0)–(H2) we adjoin the generalized spectral (i.e., Evans function) condition:

($\mathcal{D}_\varepsilon$) On a neighborhood of $\{\Re\lambda \ge 0\} \setminus \{0\}$, the only zeroes of D are (i) a zero of multiplicity one at $\lambda = 0$, and (ii) a crossing conjugate pair of zeroes $\lambda_\pm(\varepsilon) = \gamma(\varepsilon) + i\tau(\varepsilon)$ with $\gamma(0) = 0$, $\partial_\varepsilon \gamma(0) > 0$, and $\tau(0) \ne 0$.

Lemma 3.2. *Conditions (H0)–(H2) and ($\mathcal{D}_\varepsilon$) are equivalent to conditions (P)(i)–(iii) of the introduction together with $F \in C^k$, $k \ge 2$, simplicity and nonvanishing of $\sigma(A^\varepsilon_\pm)$, and the Lax condition*

$$\dim S(A^\varepsilon_+) + \dim U(A^\varepsilon_-) = n + 1, \tag{3.5}$$

with $\bar{u}^\varepsilon$ (linearly and nonlinearly) stable for $\varepsilon < 0$ and unstable for $\varepsilon \ge 0$.

Proof. By (H0), (H2) and standard ODE theory, solution $\bar{u}^\varepsilon$ of (3.4) is C^3 in x, C^2 in (x, ε), and decays at exponential rate in first two derivatives to endstates $u^\varepsilon_\pm$ as $x \to \pm\infty$. Thus, L_ε is asymptotically constant-coefficient, and its essential spectrum may be computed as in (1.9)–(1.10), Section 1.2, to lie entirely within a parabolic region:

$$\sigma_{\mathrm{ess}}(L_\varepsilon) \subset \{\Re\lambda \le -\theta|\Im\lambda|^2\}, \qquad \theta > 0, \tag{3.6}$$

verifying (P)(i). On the other hand, computation (1.10) shows that (P)(i) implies (H2) in the generic situation that $\sigma(A^\varepsilon_\pm)$ is simple and nonvanishing. Likewise, recalling [AGJ, GZ] that to the right of the essential spectrum boundary, zeroes of D_ε correspond to eigenvalues of L_ε, we find that ($\mathcal{D}_\varepsilon$)(i) and (ii) are exactly (P)(ii) and (iii). Finally, recalling the *stability criterion* (see, e.g., [ZH, Z4, MaZ3, MaZ4]) that, under (H0)–(H2), linear and nonlinear stability of $\bar{u}^\varepsilon$ is equivalent to

($\mathcal{D}$) On $\{\Re\lambda \ge 0\}$, D_ε has a single zero at $\lambda = 0$,

we find that $\bar{u}^\varepsilon$ is stable precisely for $\varepsilon < 0$. □

Remark 3.3. The undercompressive shocks studied in [HZ] give an example of shocks satisfying (P) but not (3.5). Their weaker decay properties are insufficient for the convergence argument of Section 1.6.4; see Remark 3.9.

Finally, we introduce the Banach spaces $\mathcal{B}_1 = L^2$, $\mathcal{B}_2 = \partial_x L^1 \cap L^2$,

$$X_1 = \{f : |f(x)| \le C(1+|x|)^{-1}\},$$

and

$$X_2 = \partial_x \{f : |f(x)| \le C(1+|x|)^{-2}\} \cap X_1,$$

equipped with norms $\|f\|_{\mathcal{B}_1} = \|f\|_{L^2}$, $\|\partial_x f\|_{\mathcal{B}_2} = \|f\|_{L^1} + \|\partial_x f\|_{L^2}$,

$$\|f\|_{X_1} = \|(1+|x|)f\|_{L^\infty}, \quad \text{and} \quad \|\partial_x f\|_{X_2} = \|(1+|x|)^2 f\|_{L^\infty} + \|\partial_x f\|_{X_1},$$

where ∂_x is taken in the sense of distributions. By inspection, we have that $\mathcal{B}_2 \subset \mathcal{B}_1$, $X_2 \subset X_1$, $X_1 \subset \mathcal{B}_1$, $X_2 \subset \mathcal{B}_2$, and the closed unit ball in X_1 is closed as a subset of $\mathcal{B}_1$.

3.1 Linearized bounds

Lemma 3.4. *Associated with eigenvalues $\lambda_\pm(\varepsilon)$ of L_ε are right and left eigenfunctions $\phi^\varepsilon_\pm$ and $\tilde\phi^\varepsilon_\pm \in C^k(x,\varepsilon)$, $k \ge 2$, exponentially decaying in up to q derivatives as $x \to \pm\infty$, and L_ε-invariant projection*

$$\Pi f := \sum_{j=\pm} \phi^\varepsilon_j(x)\langle \tilde\phi^\varepsilon_j, f\rangle \tag{3.7}$$

onto the total (oscillatory) eigenspace $\Sigma^\varepsilon := \mathrm{Span}\{\phi^\varepsilon_\pm\}$, bounded from L^q or $\mathcal{B}_2$ to $W^{2,p} \cap X_2$ for any $1 \le q, p \le \infty$. Moreover,

$$\phi^\varepsilon_\pm = \partial_x \Phi^\varepsilon_\pm, \tag{3.8}$$

with $\Phi^\varepsilon \in C^{k+1}$ exponentially decaying in up to $k+1$ derivatives as $x \to \pm\infty$.

Proof. From simplicity of $\lambda_\pm$, we obtain either by standard spectral perturbation theory [Kat] or by direct Evans-function calculations [GJ1, GJ2, ZH] that there exist $\lambda_\pm(\cdot)$, $\phi^\varepsilon_\pm(\cdot) \in L^2$ with the same smoothness $C^2(\varepsilon)$ assumed on F. The exponential decay properties in x then follow by standard asymptotic ODE theory; see, e.g., [GZ, Z1]. Finally, recall the observation of [ZH] that, by divergence form of L_ε, we may integrate $L_\varepsilon \phi = \lambda\phi$ from $x = -\infty$

to $x = +\infty$ to obtain $\lambda \int_{-\infty}^{+\infty} \phi(x)dx = 0$, and thereby (since $\lambda_\pm \neq 0$ by assumption)

$$\int_{-\infty}^{+\infty} \phi_\pm(x)dx = 0, \tag{3.9}$$

from which we obtain by integration (3.8) with the stated properties of $\Phi_\pm$. From (3.8) and representation (3.7), we obtain by Hölder's inequality the stated bounds on projection Π. $\square$

Defining $\tilde\Pi^\varepsilon := \mathrm{Id} - \Pi^\varepsilon$, $\tilde\Sigma^\varepsilon := \mathrm{Range}\tilde\Pi^\varepsilon$, and $\tilde L_\varepsilon := L_\varepsilon \tilde\Pi^\varepsilon$, denote by

$$G(x,t;y) := e^{L_\varepsilon t}\delta_y(x) \tag{3.10}$$

the Green kernel associated with the linearized solution operator e^{Lt} of the linearized evolution equations $u_t = L_\varepsilon u$, and

$$\tilde G(x,t;y) := e^{\tilde L_\varepsilon t}\tilde\Pi\delta_y(x) \tag{3.11}$$

the Green kernel associated with the transverse linearized solution operator $e^{\tilde L_\varepsilon t}\tilde\Pi$. By direct computation, $G = \mathcal{O} + \tilde G$, where

$$\mathcal{O}(x,t;y) := e^{(\gamma(\varepsilon)+i\tau(\varepsilon))t}\phi_+(x)\tilde\phi_+^t(y) + e^{(\gamma(\varepsilon)-i\tau(\varepsilon))t}\phi_-(x)\tilde\phi_-^t(y). \tag{3.12}$$

Supressing the parameter ε, denote by $a_j^\pm$, $l_j^\pm$, and $r_j^\pm$ the eigenvalues and left and right eigenvectors of $A_\pm^\varepsilon$.

3.1.1 Short time estimates

Lemma 3.5. *For $0 \le t \le T$, any fixed $T > 0$, and some $C = C(T)$,*

$$\|e^{L_\varepsilon t}\partial_x f\|_{\mathcal{B}_2},\ \|e^{\tilde L_\varepsilon t}\tilde\Pi\partial_x f\|_{\mathcal{B}_2} \le Ct^{-1/2}\|f\|_{L^1\cap L^2}. \tag{3.13}$$

$$\|e^{L_\varepsilon t}\partial_x f\|_{X_2},\ \|e^{\tilde L_\varepsilon t}\tilde\Pi\partial_x f\|_{X_2} \le Ct^{-1/2}\|(1+|x|)^2 f\|_{L^\infty}, \tag{3.14}$$

Proof. From standard parabolic semigroup bounds

$$|e^{L_\varepsilon t}|_{L^2\to L^2} \le Ct, \qquad |e^{L_\varepsilon t}\partial_x|_{L^2\to L^2} \le Ct^{-1/2}, \tag{3.15}$$

and properties $e^{L_\varepsilon t} = e^{L_\varepsilon t}\Pi + e^{\tilde L_\varepsilon t}\tilde\Pi$ and $\|\Pi\partial_x f\|_{L^2} \le |f|_{L^2}$, we obtain

$$\|e^{L_\varepsilon t}\partial_x f\|_{L^2},\ \|e^{\tilde L_\varepsilon t}\tilde\Pi\partial_x f\|_{L^2} \le Ct^{-1/2}\|f\|_{L^2}. \tag{3.16}$$

Likewise, we may obtain integrated bounds

$$\Big\| \int e^{L_\varepsilon t}\partial_x f\Big\|_{L^1}, \ \Big\| \int e^{\tilde L_\varepsilon t}\tilde \Pi \partial_x f\Big\|_{L^1} \le C\|f\|_{L^1} \tag{3.17}$$

using the divergence form of L_ε, by integrating the linearized equations with respect to x to obtain linearized equations $U_t = \mathcal{L}_\varepsilon U$ for integrated variable

$$U(x,t,\varepsilon) := \int_{-\infty}^x u(z,t,\varepsilon)dz, \qquad u(\cdot,t) := e^{L_\varepsilon t}\partial_x f,$$

with linearized operator $\mathcal{L}_\varepsilon := -A^\varepsilon(x)\partial_x + \partial_x^2$ of the same parabolic form as L_ε, then applying standard parabolic semigroup estimates (alternatively, short-time bounds as in Lemma 3.12) to bound

$$\|U(\cdot,t,\varepsilon)\|_{L^1} = \|e^{\mathcal{L}_\varepsilon t} f\|_{L^1} \le C\|f\|_{L^1},$$

the $e^{\tilde L t}\tilde \Pi$ bound then following by relation $e^{L_\varepsilon t} = e^{L_\varepsilon t}\Pi + e^{\tilde L_\varepsilon t}\tilde\Pi$ together with $\| \int \Pi \partial_x f\|_{L^1} \le |f|_{L^1}$. Combining (3.16) and (3.17), we obtain (3.13). Bounds (3.14) follow by an identical argument, substituting for L^2 and L^1 the weighted spaces $\|(1+|x|)\cdot\|_{L^\infty}$ and $\|(1+|x|)^2\cdot\|_{L^\infty}$. $\square$

3.1.2 Pointwise Green function bounds

Proposition 3.6 ([ZH, Z4, MaZ3]). *Under assumptions (H0)–(H2), (P),*

$$\tilde G = \mathcal{E} + \mathcal{S} + \mathcal{R}, \tag{3.18}$$

where, for $y \le 0$ and all $t \ge 0$:

$$\mathcal{E}(x,t;y) := \sum_{a_k^->0} [c^0_{k,-}]\bar U'(x) l_k^{-t} \left(\mathrm{errfn}\left(\frac{y+a_k^- t}{\sqrt{4t}}\right) - \mathrm{errfn}\left(\frac{y-a_k^- t}{\sqrt{4t}}\right)\right) \tag{3.19}$$

and

$$
\begin{aligned}
&\mathcal{S}(x,t;y)\\
&:= \chi_{\{t\ge 1\}} \sum_{a_k^-<0} r_k^- l_k^{-t} (4\pi t)^{-1/2} e^{-(x-y-a_k^- t)^2/4t}\\
&+ \chi_{\{t\ge 1\}} \sum_{a_k^->0} r_k^- l_k^{-t} (4\pi t)^{-1/2} e^{-(x-y-a_k^- t)^2/4t} \left(\frac{e^{-x}}{e^x+e^{-x}}\right)\\
&+ \chi_{\{t\ge 1\}} \sum_{a_k^->0,\, a_j^-<0} [c_{k,-}^{j,-}] r_j^- l_k^{-t} (4\pi \bar\beta_{jk}^- t)^{-1/2} e^{-(x-z_{jk}^-)^2/4\bar\beta_{jk}^- t} \left(\frac{e^{-x}}{e^x+e^{-x}}\right)\\
&+ \chi_{\{t\ge 1\}} \sum_{a_k^->0,\, a_j^+>0} [c_{k,-}^{j,+}] r_j^+ l_k^{-t} (4\pi \bar\beta_{jk}^+ t)^{-1/2} e^{-(x-z_{jk}^+)^2/4\bar\beta_{jk}^+ t} \left(\frac{e^{x}}{e^x+e^{-x}}\right)
\end{aligned}
\tag{3.20}
$$

denote excited and scattering terms and

$$
\begin{aligned}
\mathcal{R}(x,t;y) &= O(e^{-\eta(|x-y|+t)})\\
&+ \sum_{k=1}^n O\left((t+1)^{-1/2} e^{-\eta x^+} + e^{-\eta|x|}\right) t^{-1/2} e^{-(x-y-a_k^- t)^2/Mt}\\
&+ \sum_{a_k^->0,\, a_j^-<0} \chi_{\{|a_k^- t|\ge|y|\}} O((t+1)^{-1/2} t^{-1/2}) e^{-(x-z_{jk}^-)^2/Mt} e^{-\eta x^+},\\
&+ \sum_{a_k^->0,\, a_j^+>0} \chi_{\{|a_k^- t|\ge|y|\}} O((t+1)^{-1/2} t^{-1/2}) e^{-(x-z_{jk}^+)^2/Mt} e^{-\eta x^-},
\end{aligned}
\tag{3.21}
$$

$$
\begin{aligned}
&\mathcal{R}_t(x,t;y)\\
&= O(e^{-\eta(|x-y|+t)})\\
&+ \sum_{k=1}^n O\left((t+1)^{-1/2} e^{-\eta x^+} + e^{-\eta|x|}\right) (1+t)^{1/2} t^{-3/2} e^{-(x-y-a_k^- t)^2/Mt}\\
&+ \sum_{a_k^->0,\, a_j^-<0} \chi_{\{|a_k^- t|\ge|y|\}} O(t^{-3/2}) e^{-(x-z_{jk}^-)^2/Mt} e^{-\eta x^+},\\
&+ \sum_{a_k^->0,\, a_j^+>0} \chi_{\{|a_k^- t|\ge|y|\}} O(t^{-3/2}) e^{-(x-z_{jk}^+)^2/Mt} e^{-\eta x^-},
\end{aligned}
\tag{3.22}
$$

$$
\begin{aligned}
\mathcal{R}_y(x,t;y) &= \partial_t r(x,t;y) + O(e^{-\eta(|x-y|+t)}) \\
&\quad + \sum_{k=1}^{n} O\Big((t+1)^{-1/2}e^{-\eta x^+} + e^{-\eta|x|}\Big)\, t^{-1}e^{-(x-y-a_k^- t)^2/Mt} \\
&\quad + \sum_{a_k^->0,\, a_j^-<0} \chi_{\{|a_k^- t|\ge |y|\}} O((t+1)^{-1/2}t^{-1}) e^{-(x-z_{jk}^-)^2/Mt} e^{-\eta x^+} \\
&\quad + \sum_{a_k^->0,\, a_j^+>0} \chi_{\{|a_k^- t|\ge |y|\}} O((t+1)^{-1/2}t^{-1}) e^{-(x-z_{jk}^+)^2/Mt} e^{-\eta x^-},
\end{aligned}
\tag{3.23}
$$

$$
\begin{aligned}
r(x,t;y) &= O\big(e^{-\eta|y|}(1+t)^{-1/2}\big) \\
&\qquad \times \Big(\sum_{a_k^-<0} e^{-(x-y-a_k^- t)^2/Mt} + \sum_{a_k^+>0} e^{-(x-y-a_k^+ t)^2/Mt} \Big),
\end{aligned}
\tag{3.24}
$$

$$
\begin{aligned}
&\partial_t^k \mathcal{R}_y(x,t;y) \\
&\quad = O(e^{-\eta(|x-y|+t)}) \\
&\quad + \sum_{k=1}^{n} O\Big((t+1)^{-1/2}e^{-\eta x^+} + e^{-\eta|x|}\Big)(1+t)^{1/2} t^{-(k+3)/2} e^{-(x-y-a_k^- t)^2/Mt} \\
&\quad + \sum_{a_k^->0,\, a_j^-<0} \chi_{\{|a_k^- t|\ge |y|\}} O(t^{-(k+3)/2}) e^{-(x-z_{jk}^-)^2/Mt} e^{-\eta x^+}, \\
&\quad + \sum_{a_k^->0,\, a_j^+>0} \chi_{\{|a_k^- t|\ge |y|\}} O(t^{-(k+3)/2}) e^{-(x-z_{jk}^+)^2/Mt} e^{-\eta x^-} \\
&\quad + O\big(e^{-\eta|y|}(1+t)^{-(k+2)/2}\big)\Big(\sum_{a_k^-<0} e^{-(x-y-a_k^- t)^2/Mt} + \sum_{a_k^+>0} e^{-(x-y-a_k^+ t)^2/Mt} \Big),
\end{aligned}
\tag{3.25}
$$

$k \ge 1$, a faster decaying residual, where $a_j^\pm$, l_j, r_j as defined above are the eigenvalues and left and right eigenvectors of $A_\pm^\varepsilon$, $\eta, M > 0$ are positive constants,

$$
z_{jk}^\pm(y,t) := a_j^\pm \left(t - \frac{|y|}{|a_k^-|} \right)
\tag{3.26}
$$

and

$$\bar\beta^{\pm}_{jk}(x,t;y) := \frac{|x^{\pm}|}{|a^{\pm}_j t|} + \frac{|y|}{|a^{-}_k t|}\left(\frac{a^{\pm}_j}{a^{-}_k}\right)^2 \tag{3.27}$$

are approximate scattered characteristic paths and effective diffusion rates along them, scattering coefficients $[c^{j,i}_{k,-}]$*,* $i=-,0,+$ *are constant,* $x^{\pm}$ *denotes the positive/negative part of* x*,* $\chi_{\{t\ge 1\}}$ *is a smooth cutoff function in* t*, identically one for* $t\ge 1$ *and identically zero for* $t\le 1/2$*, and indicator function* $\chi_{\{|a^-_k t|\ge |y|\}}$ *is one for* $|a^-_k t|\ge |y|$ *and zero otherwise.*

Moreover, for $y\le 0$*,* $k\ge 1$*, some* $M>0$*,*

(3.28)

$$\begin{aligned}
&\partial_t^k \tilde G_y(x,t;y)\\
&\quad=\sum_{k=1}^n O\left(t^{-1/2}e^{-\eta x^+}+e^{-\eta|x|}\right)(1+t)^{k/2}t^{-(k+1/2)}e^{-(x-y-a_k^- t)^2/Mt}\\
&\quad+\sum_{a_k^->0,\,a_j^-<0}\chi_{\{|a_k^- t|\ge|y|\}}O\big((1+t)^{-(k+2)/2}\big)e^{-(x-z_{jk}^-)^2/Mt}e^{-\eta x^+},\\
&\quad+\sum_{a_k^->0,\,a_j^+>0}\chi_{\{|a_k^- t|\ge|y|\}}O\big((1+t)^{-(k+2)/2}\big)e^{-(x-z_{jk}^+)^2/Mt}e^{-\eta x^-}\\
&\quad+O\big(e^{-\eta|y|}(1+t)^{-(k+2)/2}\big)\Big(\sum_{a_k^-<0}e^{-(x-y-a_k^- t)^2/Mt}+\sum_{a_k^+>0}e^{-(x-y-a_k^+ t)^2/Mt}\Big).
\end{aligned}$$

Symmetric bounds hold for $y\ge 0$*.*

Proof. Evidently, it is equivalent to establish decomposition

$$G=\mathcal{O}+\mathcal{E}+\mathcal{S}+\mathcal{R}$$

of the full Green function. This problem has been treated in [ZH, MaZ3], starting with Inverse Laplace Transform representation

$$G(x,t;y)=e^{Lt}\delta_y(x)=\oint_\Gamma e^{\lambda t}(\lambda-L_\varepsilon)^{-1}\delta_y(x)d\lambda\,, \tag{3.29}$$

where

$$\Gamma := \partial\{\lambda : \Re\lambda \le \eta_1-\eta_2|\Im\lambda|\}$$

is an appropriate sectorial contour, $\eta_1,\eta_2>0$; estimating the resolvent kernel $G^\varepsilon_\lambda(x,y) := (\lambda-L_\varepsilon)^{-1}\delta_y(x)$ using Taylor expansion in λ, asymptotic ODE techniques in x, y, and judicious decomposition into various scattering, excited, and residual modes; then, finally, estimating the contribution of various modes to (3.29) by Riemann saddlepoint (Stationary Phase) method, moving contour Γ to a optimal, “minimax” positions for each mode, depending on the values of (x, y, t).

In the present case, we may first move Γ to a contour Γ' enclosing (to the left) all spectra of L_ε except for the crossing pair $\lambda_\pm(\varepsilon)$, to obtain

$$G(x,t;y) = \oint_{\Gamma'} e^{\lambda t}(\lambda - L_\varepsilon)^{-1} d\lambda + \sum_{j=\pm} \text{Residue}_{\lambda_j(\varepsilon)}\big(e^{\lambda t}(\lambda - L_\varepsilon)^{-1}\delta_y(x)\big),$$

where $\text{Residue}_{\lambda_j(\varepsilon)}\big(e^{\lambda t}(\lambda - L_\varepsilon)^{-1}\delta_y(x)\big) = \mathcal{O}(x,t;y)$, then estimate the remaining term $\oint_{\Gamma'} e^{\lambda t}(\lambda - L_\varepsilon)^{-1} d\lambda$ on minimax contours as just described. See the proof of Proposition 7.1, [MaZ3], for a detailed discussion of minimax estimates $\mathcal{E} + \mathcal{S} + \mathcal{R}$ and of Proposition 7.7, [MaZ3], for a complementary discussion of residues incurred at eigenvalues in $\{\Re\lambda \geq 0\} \setminus \{0\}$.

We have repaired in (3.21), (3.23) two minor omissions in the bounds of [MaZ3], pointed out already in [R, HR, HRZ]. Specifically, (i) in the first term on the righthand side of (3.21), (3.23), we have replaced the term $O(e^{-\eta t}e^{-|x-y|^2/Mt})$ appearing in [MaZ3, Z1], with a corrected version $O(e^{-\eta(|x-y|+t)})$, and (ii) we have added a missing term

(3.30)
$$\partial_t r = \partial_t O\big(e^{-\eta|y|}(1+t)^{-1/2}\big)\Big(\sum_{a_k^-<0} e^{-(x-y-a_k^- t)^2/Mt} + \sum_{a_k^+>0} e^{-(x-y-a_k^+ t)^2/Mt}\Big),$$

which may alternatively be estimated as in [R, HR, HRZ] as

$$\partial_t r = O\big(e^{-\eta|y|}(1+t)^{-1}\big)\Big(\sum_{a_k^-<0} e^{-(x-y-a_k^- t)^2/Mt} + \sum_{a_k^+>0} e^{-(x-y-a_k^+ t)^2/Mt}\Big), \tag{3.31}$$

to the righthand side of (3.23).

As discussed in [R], the first correction concerns only bookkeeping, and comes from the fact that approximate Gaussians appearing in $\mathcal{S}$ decay only as $e^{-\eta x}$ in the far field and not as does the total solution as $O(e^{-\eta t}e^{-|x-y|^2/Mt})$. This is only an artifact of our way of displaying the solution. The second is more fundamental, and deserves further comment. Namely, different from the constant-coefficient case, the (approximate) Gaussians appearing in the Green function for the variable coefficient case involve convection, diffusion, and also directional parameters that depend on x and y through $\bar{u}^\varepsilon(x)$. Thus, x- and y-derivatives do not always bring down additional additional factors $t^{-1/2}$ of temporal decay as for constant-coefficient Gaussians, but may instead, through derivatives falling on spatially-dependent parameters, bring down factors $e^{-\eta|x|}$ or $e^{-\eta|y|}$ of spatial decay; see [ZH, Z4] for further discussion. The neglected term (3.31), which is harmless for the stability analyses of [R, MaZ2, MaZ4, Z1], may be recognized as a term of this form.

Unfortunately, term (3.31), though harmless in the stability analysis, is unacceptable for our bifurcation analysis (see Remark 3.9), and thus must be treated in a different way. Reviewing the analysis of [MaZ3], Proposition 7.7, we see that term (3.31) results from a term in (3.29) of form

$$\partial_y \oint_{\Gamma_1} e^{\lambda t} \lambda \phi(x) \tilde{\psi}(y)\, d\lambda,$$

where Γ_1 is a fixed finite arc of a larger contour Γ, $\tilde{\psi}(y)$, $\partial_y \tilde{\psi}(y) \sim e^{-\eta|y|}$ is a fast-decaying mode and $\phi(x) \sim e^{c_1 \Re\lambda + c_2 \Re(\lambda^2)}$ is a slow-decaying mode; the factor λ reflects the fact that this is a Taylor remainder term in the low-frequency expansion about $\lambda = 0$. The "old" estimate (3.31) makes use of the fact that each factor of λ introduces (by a simple scaling argument) an additional factor $(1+t)^{-1/2}$ of temporal decay in the Riemann saddle-point/Stationary Phase estimate from which the final bounds are obtained.

The "new" estimate (3.30) follows by a still simpler argument, making use of the fact that a factor of λ corresponds identically to time-differentiation:

$$\partial_y \oint_{\Gamma_1} e^{\lambda t} \lambda \phi(x) \tilde{\psi}(y)\, d\lambda = \partial_t \oint_{\Gamma_1} e^{\lambda t} \partial_y \phi(x) \tilde{\psi}(y)\, d\lambda,$$

and estimating $\oint_{\Gamma_1} e^{\lambda t} \partial_y \phi(x) \tilde{\psi}(y)\, d\lambda$ as before.

Conversely, a time-derivative on the Green function can always be traded for a factor of λ in the integrand of (3.29), yielding faster temporal decay by factor $(1+t)^{1/2} t^{-1} \sim t^{-1/2}$. We use this observation to generate estimates (3.22), (3.25), and (3.28) not stated in [MaZ3, Z1]. Note that conglomerate high-derivative bound (3.28) does not contain the "bookkeeping" error term $O(e^{-\eta(|x-y|+t)})$ deriving from our way of decomposing the solution, and is in fact much easier to get than the more detailed individual bounds, requiring only modulus-based stationary-phase estimates like those used to bound residual $\mathcal{R}$. $\square$

Remark 3.7. The effect of t-derivatives for variable-coefficient equations in bringing down additional factors of temporal decay is in contrast to that of x- and y-derivatives, which eventually bring down also $e^{-\eta|x|}$ and $e^{-\eta|y|}$ factors not adding extra decay [ZH, HZ].

Remark 3.8. Integrating the pointwise bounds of Proposition 3.6, we obtain

$$\|(\tilde{G}-\mathcal{E})_y\|_{L^2(x)} \le Ct^{-3/4}, \qquad \|(\tilde{G}-\mathcal{E})_{yt}\|_{L^2(x)} \le Ct^{-5/4}, \tag{3.32}$$

similarly as for the convected heat kernel K in the analysis of Section 1.6.4.

Remark 3.9. Note that $\tilde{G}$ bounds indeed agree with the model given in Section 1.2.1. Specifically, $\mathcal{E}$ terms are exactly superpositions of terms of form J. Likewise, $\mathcal{S}$ terms near their centers are well-approximated by moving Gaussians K, so satisfy similar bounds; see [R]. Thus, the model analysis well-represents the primary terms $\mathcal{E}$ and $\mathcal{S}$ in the transverse Green function expansion for Lax-type viscous shocks.

Undercompressive shocks give a different model

$$G \sim K(x,t;y)p(y) + J(x,t;y)q(y)$$

with $|p_y|, |q_y| \sim e^{-\eta|y|}$, and thus

$$(Kp)_y \sim Ke^{-\eta|y|}, \tag{3.33}$$

insufficient for the argument. For further discussion, see [Z4, HZ]. A related, more subtle point arising also in the Lax shock case is that terms $\sim e^{-\eta|y|}|K_y|$ appearing in R_y bound (3.31) are also not acceptable, since the $e^{-\eta|y|}$ term is no help in convolution against an L^1 source, and $|K_y|$ does not have the sign cancellation of K_y. Our replacement with bound (3.30) amounts to showing that, even though the y-derivative no longer introduces cancellation for these terms, they already possess an additional t-derivative that leads to cancellation by another route.

3.2 Return map construction

Given a family of solutions $\{\tilde{u}^\varepsilon\}$ of (3.2), define the perturbation variable

$$u(x,t,\varepsilon) := \tilde{u}^\varepsilon(x,t) - \bar{u}^\varepsilon(x), \tag{3.34}$$

satisfying nonlinear perturbation equations

$$u_t - L_\varepsilon u = Q(u,\varepsilon)_x, \qquad u(x,0,\varepsilon) = u_0(x,\varepsilon), \tag{3.35}$$

where

$$Q(u,\varepsilon) := -F(\tilde u^\varepsilon,\varepsilon) + F(\bar u^\varepsilon,\varepsilon) + F_u(\bar u^\varepsilon,\varepsilon) = \mathcal{O}(|u|^2) \tag{3.36}$$

so long as u satisfies a uniform L^∞ bound $|u| \le C$. Decomposing

$$u = r\cos\theta\phi_+ + r\sin\theta\phi_- + v, \tag{3.37}$$

where $r\cos\theta\phi_+ + r\sin\theta\phi_- \in \Sigma$, $v := \tilde\Pi u \in \tilde\Sigma$, and coordinatizing as (r,θ,v), we obtain after a brief calculation

$$\begin{aligned} \dot r &= \gamma(\varepsilon) r + N_r(r,\theta,v,\varepsilon),\\ \dot\theta &= \tau(\varepsilon) + N_\theta(r,\theta,v,\varepsilon),\\ \dot v &= \tilde L_\varepsilon v + N_v(r,\theta,v,\varepsilon). \end{aligned} \tag{3.38}$$

Lemma 3.10. *Away from $r=0$, N_r and N_θ are C^k, $k\ge 2$, from $(r,\theta,v)\in \mathbb{R}^1\times[0,2\pi]\times L^q$ to $\mathbb{R}^1$, for any $0\le q\le\infty$, with*

$$|N_r| \le C(|r|^2 + \|v\|_{L^q}^2), \quad \text{and} \quad |N_\theta| \le C(|r| + \|v\|_{L^q}). \tag{3.39}$$

Likewise (for all r), N_v is C^k, $k\ge 2$, from $(r,\theta,v)\in\mathbb{R}^1\times[0,2\pi]\times X_1$ to X_2' defined as $\partial_x\{f : \|(1+|x|^2)f\|_{L^\infty} < +\infty\}$ and, for $(|r| + \|v\|_{X_1}) \le C$, Lipschitz from $(r,\theta,v)\in\mathbb{R}^1\times[0,2\pi]\times\mathcal{B}_1$ to $\partial_x(L^1\cap L^2)$, with

$$\|N_v\|_{X_2'} \le C(|r|^2 + \|v\|_{X_1}^2) \quad \text{and} \quad \|DN_v\|_{\partial_x(L^1\cap L^2)} \le C(|r| + \|v\|_{X_1}). \tag{3.40}$$

Proof. Direct calculation, using (3.36) and the Π-bounds of Lemma 3.4. $\square$

Now, truncate N_r and N_θ, replacing them with

$$\begin{aligned} \hat N_r(r,\theta,v) &:= N_r(r,\theta,\hat v(r,v)),\\ \hat N_\theta(r,\theta,v) &:= N_\theta(r,\theta,\hat v(r,v)), \end{aligned} \tag{3.41}$$

$$\hat v := \psi(C|r|/|v|)v, \tag{3.42}$$

where $\psi \in \mathbb{R}^1$ is a C^∞ "truncation" function with

$$\psi(z) = \begin{cases} 1 & z \geq 1, \\ z & z \leq 1/2, \end{cases} \qquad \psi'(z) = \begin{cases} 0 & z \geq 1, \\ 1 & z \leq 1/2, \end{cases} \tag{3.43}$$

to obtain $|\hat{v}| \leq C|r|$, and thus, by (3.39) with $q = \infty$,

$$|\hat{N}_r| \leq Cr^2, \quad |\hat{N}_\theta| \leq Cr. \tag{3.44}$$

By direct calculation, using $|\psi|$, $|\psi'| \leq 1$ and $\psi' \equiv 0$ for $v = 0$, $r \neq 0$,

$$\hat{v}_r = C\mathrm{sgn}(r)(v/|v|)\psi'(C|r|/|v|) \tag{3.45}$$

and

$$\hat{v}_v = \psi(C|r|/|v|)\mathrm{Id} - (|r|/|v|)\psi'(C|r|/|v|)C(vv^T/|v|^2) \tag{3.46}$$

are bounded for bounded v, r and C^∞ for $r \neq 0$, with $|D^2_{(r,v)}\hat{v}| \leq Cr^{-1}$.

Lemma 3.11. *As functions from $(r, \theta, v) \in \mathbb{R}^1 \times [0, 2\pi] \times L^q$ to $\mathbb{R}^1$, for any $0 \leq q \leq \infty$, $\hat{N}_r$ and $\hat{N}_\theta$ are C^k, $k \geq 2$, for r away from 0. At $r = 0$, $\hat{N}_r$ is C^1 with Lipschitz first derivative, and $\hat{N}_\theta$ is Lipschitz. Moreover,*

$$|D\hat{N}_r| \leq Cr, \quad |D\hat{N}_\theta|, |D^2\hat{N}_r| \leq C \quad \text{for } |r| \leq C. \tag{3.47}$$

Proof. Immediate, using the chain rule, $|D_{r,v}\hat{v}| \leq C$, $|D^2_{(r,v)}\hat{v}| \leq Cr^{-1}$, and

$$|D_{(r,\theta,\hat{v})}\hat{N}_r| \leq C(\|\hat{v}\|_\infty + |r|), \quad |D_{(r,\theta,\hat{v})}\hat{N}_\theta|, |D^2_{(r,\theta,\hat{v})}\hat{N}_r| \leq C. \tag{3.48}$$

□

Consider the truncated system

$$\begin{aligned} \dot{r} &= \gamma(\varepsilon)r + \hat{N}_r(r, \theta, v, \varepsilon), \\ \dot{\theta} &= \tau(\varepsilon) + \hat{N}_\theta(r, \theta, v, \varepsilon), \\ \dot{v} &= \tilde{L}_\varepsilon v + N_v(r, \theta, v, \varepsilon). \end{aligned} \tag{3.49}$$

Proposition 3.12. *For $0 \leq t \leq T$, any fixed $C_1, T > 0$, some $C > 0$, and $|a|$, $\|b\|_{X_1}$, $|\varepsilon|$ sufficiently small, system (3.49) with initial data $(r_0, \theta_0, v_0) = (a, 0, b)$ possesses a solution $(r, \theta, v)(a, b, \varepsilon, t) \in \mathbb{R}^1 \times \mathbb{R}^1 \times X_1$ that for $a \neq 0$*

is C^{k+1} in t and C^k in (a,b,ε), $k\geq 2$, with respect to the weaker norm $\mathcal{B}_1$, and for $a=0$ is C^1 in t and Lipschitz in (a,b,ε) with respect to $\mathcal{B}_1$, with

$$\begin{aligned} C^{-1}|a| \leq |r(t)| \leq C|a|,\\ \|v(t)\|_{X_1} \leq C(\|b\|_{X_1} + |a|^2), \end{aligned} \tag{3.50}$$

and

$$|D_{(a,b)}(r,\theta,v)(t)|_{\mathbb{R}^1\times[0,2\pi]\times\mathcal{B}_1\to\mathcal{B}_1} \leq C. \tag{3.51}$$

In particular, for $\|b\|_{X_1}\leq C_1|a|$, all $0\leq t\leq T$,

$$\|v(t)\|_{X_1} \leq C|r(t)|. \tag{3.52}$$

Proof. Existence and uniquess follow by a standard Contraction–mapping argument, using Duhamel's Principle to express (3.49) as

$$\begin{aligned} r(t) &= e^{\gamma(\varepsilon)t}a + \int_0^t e^{\gamma(\varepsilon)(t-s)}\hat{N}_r(r,\theta,v,\varepsilon)(s)ds,\\ \theta(t) &= \tau(\varepsilon)t + \int_0^t e^{\gamma(\varepsilon)(t-s)}\hat{N}_\theta(r,\theta,v,\varepsilon)(s)ds,\\ v(t) &= e^{\tilde{L}_\varepsilon t}b + \int_0^t e^{\tilde{L}_\varepsilon(t-s)}N_v(r,\theta,v,\varepsilon)(s)ds. \end{aligned} \tag{3.53}$$

Specifically, estimates (3.39)–(3.40) together with short-time bounds (3.14)–(3.13) give, for $E:=\sup_{t\in[0,T]}(|r|+\|v\|_{X_1})$ sufficiently small, uniform boundedness in X_1 of the righthand side of (3.53) by $C(|a|+|b|_{X_1})$, and contractivity with respect to $\mathcal{B}_1$, whence we obtain existence and uniqueness by the bounded in strong norm/contractive in weak norm principle described in Section 2. The resulting solution is Lipschitz in (a,b,ε) and C^1 in time with Lipschitz first derivative, by the corresponding properties of the righthand side of (3.53), inherited through uniform convergence of the iterative contraction-mapping construction.

Bound (3.50)(i) follows readily from (3.53) by Gronwall's inequality, using the "decoupled" estimate $|\hat{N}_r|\leq C|r|^2$ for $\hat{N}_r$. Bound (3.50)(ii) is a standard Gronwall bound that requires no comment. Combining (3.50)(i)–(ii), we obtain evidently (3.52) for $|a|$ sufficiently small. Bound (3.51) follows by the Lipschitz bound (with respect to (r,v)) CE on the righthand side of (3.53) used to obtain existence, and the fact that $E:=\sup_{t\in[0,T]}(|r|+\|v\|_{X_1})\leq$

$C(|a|+\|b\|_{X_1})$ for $(|a|+\|b\|_{X_1})$ sufficiently small, by (3.50). Finally, (3.50)(i) shows for $a \neq 0$ that $r(t)$ remains bounded from zero, whence the solution is C^k, $k \geq 2$ in (a, b, ε) and C^{k+1} in t, by the corresponding properties of the righthand side of (3.53) away from $r = 0$. □

Remark 3.13. Since $\|v\|_{X_1}$ controls $\|v\|_{L^\infty}$, the bound (3.52) allows us to justify a posteriori the truncation step provided that we can later find a solution with $\|b\|_{X_1} \leq C|a|^2$, and $|a|$, $\|b\|_{X_1}$ sufficiently small. That is, such a solution satisfies not only the truncated system (3.49) but also the original equations (3.38).

Lemma 3.14. *For $|a|$, $\|b\|_{X_1}$, $|\varepsilon|$ sufficiently small, there exists a function $T = T(a, b, \varepsilon)$,*

$$T(0, 0, \varepsilon) \equiv 2\pi/\tau(\varepsilon), \tag{3.54}$$

Lipschitz in (a, b, ε) with respect to $\mathcal{B}_1$, such that there exists a unique solution $(r, \theta, v)(a, b, \varepsilon, t)$ of (3.49) on $0 \leq t \leq T(a, b, \varepsilon)$, with

$$\theta(a, b, \varepsilon, T(a, b, \varepsilon)) = 2\pi. \tag{3.55}$$

Proof. Existence and uniqueness on $[0, T_*]$, $T_* > 2\pi/\tau(0)$, follow by Proposition 3.12. Likewise, (3.50) shows for $(a, b) = (0, 0)$ that $(r, v) \equiv (0, 0)$ for all t, hence $\theta(t) = \tau(\varepsilon)t$, giving (3.54). More generally, existence of $T(\cdot)$ satisfying (3.55) with the stated regularity follows by the Implicit Function Theorem applied around $(a, b, \varepsilon, T) = (0, 0, 0, 2\pi/\tau(0))$, using the fact, by (3.49)(ii), that $\partial\theta/\partial t = \tau(\varepsilon) + O(|r|) \neq 0$ for all $0 \leq t \leq T_*$. □

Substituting $t = T(a, b, \varepsilon)$ in 3.53, we may express the Poincaré return map $(a, b, \varepsilon) \to (\hat{a}, \hat{b}) := (r, v)(a, b, \varepsilon, T(a, b, \varepsilon))$ as a discrete dynamical system

$$\begin{aligned} \hat{a} &= R(a, b, \epsilon)a + N_1(a, b, \varepsilon), \\ \hat{b} &= S(a, b, \varepsilon)b + N_2(a, b, \varepsilon) \end{aligned} \tag{3.56}$$

of the form (2.2) studied in Section 2, with a, ε, $N_1 \in \mathbb{R}^1$ and $b \in \mathcal{B}_1$, $N_2 \in \mathcal{B}_2$, where

$$\begin{aligned} R(a, b, \varepsilon) &:= e^{\gamma(\varepsilon)T(a,b,\varepsilon)}, \\ S(a, b, \varepsilon) &:= e^{\tilde{L}_\varepsilon T(a,b,\varepsilon)} \end{aligned} \tag{3.57}$$

are primary and transverse linearized solution operators for one time-step $T(a,b,\varepsilon)$ of continuous system (3.49), and

$$
\begin{aligned}
N_1(a,b,\varepsilon) &:= \int_0^{T(a,b,\varepsilon)} e^{\gamma(\varepsilon)(T(a,b,\varepsilon)-s)} \hat N_r(r,\theta,v,\varepsilon)(s)ds,\\
N_2(a,b,\varepsilon) &:= \int_0^{T(a,b,\varepsilon)} e^{\tilde L_\varepsilon(T(a,b,\varepsilon)-s)} N_v(r,\theta,v,\varepsilon)(s)ds.
\end{aligned}
\tag{3.58}
$$

Evidently, small-amplitude periodic solutions of (3.49) with period T close to $T(0,0,0)=2\pi/\tau(0)$ are equivalent to fixed points of the Poincaré return map (equilibria of (3.56)). Moreover, on the wedge $\{\|b\|_{X_1}\le C_1|a|\}$ (see (3.52)), these are equivalent to small-amplitude periodic solutions of the original (untruncated) system (3.2), by Remark 3.13.

Lemma 3.15. *Under assumptions (H0)–(H2), (P), $R(0,0,\cdot)$ is differentiable for all ε sufficiently small, with $\partial_\varepsilon R(0,0,0)=\partial_\varepsilon\gamma(0)2\pi/\tau(0)\neq 0$ and $R(0,0,0)=1$; N_1 is quadratic order and Lipschitz from $\mathbb{R}^1\times L^q\times\mathbb{R}^1\to\mathbb{R}^1$ for any $1\le q\le\infty$, with*

$$
|N_1|\le C|a|^2, \quad |D_{(a,b,\varepsilon)}N_1|\le C|a|
\tag{3.59}
$$

(note: stronger than conditions (2.12)*(i),* (2.13)*(i)); and N_2 is quadratic order from $\mathbb{R}^1\times X_1\times\mathbb{R}^1\to X_2$ and Lipschitz from $\mathbb{R}^1\times\mathcal{B}_1\times\mathbb{R}^1\to\mathcal{B}_2$ for $\|b\|_{X_1}$ uniformly bounded, satisfying* (2.12)*(ii),* (2.13)*(ii).*

Proof. From (3.54), we find that $R(0,0,\varepsilon)=e^{2\pi\gamma(\varepsilon)/\tau(\varepsilon)}$ is C^k, $k\ge 2$ for ε sufficiently small, with $R(0,0,0)=0$ and $\partial_\varepsilon R(0,0,0)=\partial_\varepsilon\gamma(0)2\pi/\tau(0)\neq 0$ as claimed. Likewise, the bounds on N_1 follow easily from Duhamel representation (3.58)(i), bounds (3.50)(i) and (3.47), and ODE bound

$$
|e^{\gamma(\varepsilon)(T(a,b,\varepsilon)-s)}|\le C
$$

on the primary linearized solution operator. Finally, the crucial bounds on N_2 follow from Duhamel representation (3.58)(ii), bounds (3.50)(ii), (3.51), and (3.40), and semigroup bounds (3.14), (3.13) on the transverse linearized solution operator. □

Remark 3.16. Note that we have made use of parabolic smoothing, reflected by integrability of singularity $t^{-1/2}$ in the bound on $e^{\tilde L_\varepsilon t}\tilde\Pi\partial_x$. Compare the analogous computations in the proof of Theorem 1.3, [TZ] on construction of a center manifold in the context of model problem I.

At this point, we have reformulated model problem II in the abstract framework of Sections 2.1.5 and 2.2.1, using only the conservative structure of the equations, the elementary semigroup estimates (3.13), (3.14), and direct computation. It remains to verify the refined right invertibility condition (B') $\Rightarrow$ (A')– essentially a linearized stability estimate– for which we shall require the detailed pointwise bounds (3.18)–(3.27) of Section 3.1.2.

3.3 Model problem revisited

To motivate the computations to follow, we return first in more detail to the model analysis of Section 1.6.4, taking

$$\tilde{G}(x,t;y) = K(x,t;y) + J(x,t;y), \tag{3.60}$$

where $K(x,t;y) := ct^{-1/2}e^{-(x-y-at)/4t}$ models the scattering term $\mathcal{S}$ and $J(x,t;y) := c_2\bar{u}'(x)\mathrm{errfn}((-y-at)/2t^{1/2})$ the excited term $\mathcal{E}$ in decomposition (3.18), so that the iterated transverse solution kernel takes form

$$S^j(a,b,\varepsilon)\delta_y(x) = K(x, jT(a,b,\varepsilon); y) + J(x, jT(a,b,\varepsilon); y), \tag{3.61}$$

T as described in Lemma 3.14, and thus

$$\sum_{j=1}^{\infty} S^j(a,b,\varepsilon)\delta_y(x) = \sum_{j=1}^{+\infty} K(x, jT(a,b,\varepsilon); y) + \sum_{j=1}^{+\infty} J(x, jT(a,b,\varepsilon); y). \tag{3.62}$$

We assume noncharacteristicity, $a < 0$. Define associated operators

$$\mathcal{K}_N f(x) := \int_{-\infty}^{+\infty} \sum_{j=1}^{N+1} K(x, jT(a,b,\varepsilon); y) f(y)\, dy \tag{3.63}$$

and

$$\mathcal{J}_N f(x) := \int_{-\infty}^{+\infty} \sum_{j=1}^{N+1} J(x, jT(a,b,\varepsilon); y) f(y)\, dy. \tag{3.64}$$

3.3.1 Scattering estimate (K term)

Approximating the kernel $K_N := \sum_{j=1}^{N+1} K(x, jT(a,b,\varepsilon); y)$ by integral

$$\hat{K}_N(x,y,a,b,\varepsilon) := T(a,b,\varepsilon)^{-1} \int_{T(a,b,\varepsilon)}^{NT(a,b,\varepsilon)} K(x,t;y)\, dt,$$

define the "continuization error" kernel (suppressing (a,b,ε) dependence)
(3.65)

$$\theta_N(x,y) := \sum_{j=1}^{N+1} K(x, jT(a,b,\varepsilon); y) - T(a,b,\varepsilon)^{-1} \int_{T(a,b,\varepsilon)}^{NT(a,b,\varepsilon)} K_y(x,t;y)\, dt$$

and associated continuization error operator

$$\Theta f(x) := \int_{-\infty}^{+\infty} \theta(x,t,\varepsilon) f(y)\, dy. \tag{3.66}$$

Lemma 3.17. *$\int_T^{+\infty} K_y(x,t;y)dt$ is convergent for each x, y, with*

$$\Big|\int_T^{+\infty} K_y(x,t;y)dt\Big|, \int_T^{+\infty} |K_{yy}(x,t;y)|dt, \int_T^{+\infty} |K_{yt}(x,t;y)|dt \le C(1+|x-y|)^{-1} \tag{3.67}$$

for $C>0$ independent of x, y, $T \ge T_0 > 0$.

Proof. From $K_y = a^{-1}(K_t - K_{yy})$, we have

$$\begin{aligned}\int_T^t K_y(x,s;y)\, ds &= a^{-1}\big(\int_T^t K_t(x,s;y)\, ds - \int_T^t K_{yy}(x,s;y)\, ds\big)\\ &= a^{-1}\big(K(x,s;y)|_T^t - \int_T^t K_{yy}(x,s;y)\, ds\big)\\ &= -a^{-1}K(x,T;y) - a^{-1}\int_T^t K_{yy}(x,s;y)\, ds.\end{aligned} \tag{3.68}$$

Since $|K_{yy}| \le Ct^{-3/2}$ is absolutely integrable, this gives convergence for each fixed x, y. Moreover, the first term clearly satisfies (3.67)(i). Thus, to establish (3.67)(i), it is sufficient to establish (3.67)(ii), or, by $|K_{yy}| \le Ct^{-3/2}e^{-(x-y-at)^2/Ct}$, the uniform bound

$$\int_T^{+\infty} |x-y| t^{-3/2} e^{-(x-y-at)^2/Ct}\, dt \le C.$$

Observing that

$$\begin{aligned}|x-y|t^{-3/2}e^{-(x-y-at)^2/Ct} &\le (|x-y-at|/t^{1/2})t^{-1}e^{-(x-y-at)^2/Ct}\\ &\quad + |a|t^{-1/2}e^{-(x-y-at)^2/Ct}\\ &\le Ct^{-1/2}e^{-(x-y-at)^2/C_2 t},\end{aligned} \tag{3.69}$$

we find in turn that it is sufficient to show

$$\int_T^{+\infty} t^{-1/2} e^{-(x-y-at)^2/Ct}\, dt \le C, \tag{3.70}$$

which follows by

$$\begin{aligned}
&\int_T^{+\infty} t^{-1/2} e^{-(x-y-at)^2/Ct}\, dt \\
&= \Big(\int_{|x-y-at|\le t/C} + \int_{|x-y-at|\ge t/C} \Big) t^{-1/2} e^{-(x-y-at)^2/Ct}\, dt \\
&\le C \int_{-\infty}^{+\infty} |x-y|^{-1/2} e^{-(x-y-at)^2/C|x-y|}\, dt + \int_0^{+\infty} t^{-1/2} e^{-t/C^3} dt.
\end{aligned} \tag{3.71}$$

Bound (3.67)(iii) follows similarly, using $|K_{yt}| \le Ct^{-3/2} e^{-(x-y-at)^2/Ct}$. □

Lemma 3.18. *For some $C > 0$, all $x \in \mathbb{R}^1$,*

$$\int_{-\infty}^{+\infty} (1+|x-y|)^{-1}(1+|y|)^{-2} dy \le C(1+|x|)^{-1}. \tag{3.72}$$

Proof. Dividing into commensurate and incommensurate parts, we have

$$\begin{aligned}
&\int_{-\infty}^{+\infty} (1+|x-y|)^{-1}(1+|y|)^{-2} dy \\
&\qquad = \Big(\int_{|x-y|\le |x|/C} + \int_{|x-y|\ge |x|/C}\Big)(1+|x-y|)^{-1}(1+|y|)^{-2} dy \\
&\qquad \le C_2 \int_{x(1-1/C)}^{x(1+1/C)} (1+|x|)^{-2} dy + C_2 \int_{-\infty}^{+\infty} (1+|x|)^{-1}(1+|y|)^{-2} dy \\
&\qquad \le C_3 (1+|x|)^{-1}.
\end{aligned} \tag{3.73}$$

□

Corollary 3.19. *$\hat{\mathcal{K}}_N$, Θ_N, and $\mathcal{K}_N$ are uniformly convergent with respect to (a,b,ε) as $N \to \infty$ as operators from $\mathcal{B}_2 \to \mathcal{B}_1$, with limits $\hat{\mathcal{K}}_\infty$, Θ_∞, and $\mathcal{K}_\infty$ uniformly bounded from $X_2 \to X_1$ and Lipschitz in (a,b) from $\mathcal{B}_2 \to \mathcal{B}_1$.*

Proof. By the same calculation as in (3.68),

$$\begin{aligned}
T\partial_y \hat{K}_N(x,y) &= a^{-1}(K(x,(N+1)T;y) - a^{-1}K(x,T;y) \\
&\qquad - a^{-1} \int_T^{(N+1)t} K_{yy}(x,s;y)\, ds,
\end{aligned} \tag{3.74}$$

hence, by the triangle inequality,

$$\begin{aligned}
\|(\hat{\mathcal{K}}_N - \hat{\mathcal{K}}_\infty)\partial_x f\|_{L^2} &= \Big\| \int_{-\infty}^{+\infty} \partial_y (\hat{K}_N(x,y) - \hat{K}_\infty) f(y) dy \Big\|_{L^2(x)} \\
&\le C \sup_y \Big(\|K(x,NT;y)\|_{L^2(x)} \\
&\qquad + \int_{NT}^{+\infty} \|K_{yy}(x,t;y)\|_{L^2(x)}\, ds, \Big) \|f\|_{L^1} \\
&= C_2 \Big((NT)^{-1/4} + \int_{NT}^{+\infty} t^{-5/4} \Big) \|f\|_{L^1} \\
&= 2C_2 (NT)^{-1/4} \|f\|_{L^1}
\end{aligned} \tag{3.75}$$

goes to zero as $N \to +\infty$, and so $\hat{\mathcal{K}}_N$ is uniformly convergent from $\mathcal{B}_2 \to \mathcal{B}_1$.

By the Fundamental Theorem of Calculus,

$$|K(x,t;y) - K(x,NT;y)| = |\int_{NT}^{t} K_t(x,s;y) ds| \le T \int_{NT}^{(N+1)T} |K_t(x,t;y)| dt,$$

hence $|\theta_n(x,y)| \le \int_T^{(N+1)T} |K_t(x,y)| dt$ and, likewise,

$$|\partial_y \theta_n(x,y)| \le \int_T^{(N+1)T} |K_{yt}(x,y)| dt. \tag{3.76}$$

Thus,

$$\begin{aligned}
\|(\Theta_N - \Theta_\infty)\partial_x f\|_{L^2} &= \Big\| \int_{-\infty}^{+\infty} \partial_y (\theta_N(x,y) - \theta_\infty) f(y) dy \Big\|_{L^2(x)} \\
&\le \Big\| \int_{-\infty}^{+\infty} \int_{NT}^{+\infty} |K_{yt}(x,y;t)| |f(y)|\, dy\, dt \Big\|_{L^2(x)} \\
&\le C \big(\sup_y \int_{NT}^{+\infty} \|K_{yt}(x,t;y)\|_{L^2(x)}\, ds \big) \|f\|_{L^1} \\
&= 2C_2 (NT)^{-1/4} \|f\|_{L^1}
\end{aligned} \tag{3.77}$$

goes to zero as $N \to +\infty$, and so Θ_N is uniformly convergent from $\mathcal{B}_2 \to \mathcal{B}_1$. Combining, we have $\mathcal{K}_N = \hat{\mathcal{K}}_N + \Theta_N$ uniformly convergent from $\mathcal{B}_2 \to \mathcal{B}_1$.

Further, using

$$T \partial_y \hat{K}_\infty(x,y) = -a^{-1} K(x,T;y) - a^{-1} \int_T^{+\infty} K_{yy}(x,t;y)\, ds, \tag{3.78}$$

$|K(x,T;y| \leq Ce^{|x-y|^2/CT} \leq C_2(1+|x-y|)^{-1}$, and Lemma 3.17, we obtain

$$|\partial_y \hat{K}_\infty(x,y)|, \quad |\partial_y \theta_\infty(x,y)| \leq C(1+|x-y|)^{-1}, \tag{3.79}$$

and thus

$$|\partial_y K_\infty(x,y)| \leq C(1+|x-y|)^{-1}. \tag{3.80}$$

By Lemma 3.18, we thus have

$$\begin{aligned} \|\hat{\mathcal{K}}_\infty(\partial_x f)\|_{X_1} &= \sup_x (1+|x|)\Big|\int_{-\infty}^{+\infty} \partial_y \hat{K}_\infty(x,y) f(y) dy\Big| \\ &\leq \sup_x (1+|x|)\int_{-\infty}^{+\infty} C(1+|x-y|)^{-1}(1+|y|)^{-1}\|\partial f\|_{X_2} dy \\ &\leq C\|\partial_x f\|_{X_2} \end{aligned} \tag{3.81}$$

as claimed, and similarly for Θ_∞. Thus, $\mathcal{K}_\infty = \hat{\mathcal{K}}_\infty + \Theta_\infty$ is bounded from $X_2 \to X_1$ as well.

Finally, we establish uniform Lipschitz continuity in (a,b) of $\hat{\mathcal{K}}_\infty$, Θ_∞, and thereby $\mathcal{K}_\infty = \hat{\mathcal{K}}_\infty + \Theta_\infty$. Differentiating $\partial_y \hat{K}_\infty(x,y) := \int_T^{+\infty} K_y(x,t;y)dt$ with respect to (a,b), we easily obtain

$$\begin{aligned} \|\partial_{(a,b)} \partial_y \hat{K}_\infty(x,y)\|_{L^2(x)} &= \| - K_y(x,T;y)\partial_{(a,b)} T\|_{L^2(x)} \\ &\leq \|K_y(x,T;y)\|_{L^2(x)} |\partial_{(a,b)} T|, \end{aligned} \tag{3.82}$$

and thus Lipschitz continuity of $\hat{\mathcal{K}}_\infty$ from $\mathcal{B}_2 \to \mathcal{B}_1$.

Lipschitz continuity of Θ_∞ requires a bit more care, and an additional observation of general use. Namely, for a function like K_y for which higher t-derivatives decay successively faster, the total truncation error Θ_∞ can be partially evaluated as a time-independent function plus an arbitrarily rapidly converging integral in time, by using successively higher order numerical quadrature formulae. In particular, the bound

$$|\partial_y \theta_\infty(x,y)| \leq \int_T^{+\infty} |K_{yt}(x,y)| dt$$

of (3.76) coming from first-order quadrature, can be improved to

$$|\partial_y \theta_\infty(x,y)| \leq |(1/2)K_y(x,y)| + C\int_T^{+\infty} |K_{ytt}(x,y)| dt \tag{3.83}$$

by using the second-order Trapezoid rule

(3.84)

$$
\begin{aligned}
&(f(1)/2 + f(2) + \cdots + f(N-1) + f(N)/2) - \int_1^N f(t)dt \\
&\quad = \sum_{j=1}^{N-1} \Big(\int_0^1 \int_0^x (-xz) f''(j+z) dz dx + \int_0^1 \int_x^1 (xz + x - z) f''(j+z) dz dx \Big)
\end{aligned}
$$

and $(1/2)K_y(x, NT; y) \to 0$ in $L^2(x)$ to rewrite

$$
\partial_y \theta_\infty(x,y) = (1/2)K_y(x,T;y) + O\Big(\int_T^{+\infty} |K_{ytt}(x,y)| dt \Big), \tag{3.85}
$$

obtaining convergence at rate $\int \|K_{ytt}(x,y)\|_{L^2(x)} dt \sim \int t^{-7/4} \sim t^{-3/4}$ instead of $t^{-1/4}$ as for (3.76).

Using Trapezoid formula (3.84) to express

$$
\begin{aligned}
\partial_y \theta_\infty(x,y) &= (1/2)K_y(x,T;y) \\
&\quad + \sum_{j=1}^{N-1} \Big(\int_0^1 \int_0^s (-s\tau) K_{ytt}(x, \big((j+\tau)T\big); y) d\tau ds \\
&\quad + \int_0^1 \int_s^1 \big((s\tau + s - \tau)T\big) K_{ytt}(x, \big((j+\tau)T\big); y) d\tau ds \Big),
\end{aligned} \tag{3.86}
$$

we find that

(3.87)

$$
\begin{aligned}
&\partial_{(a,b)} \partial_y \theta_\infty(x,y) \\
&\quad = (1/2)K_{yt}(x,T;y)\partial_{(a,b)}T \\
&\qquad + \sum_{j=1}^{N-1} \Big(\int_0^1 \int_0^s (-s\tau)(j+\tau) K_{yttt}(x, \big((j+\tau)T\big); y) d\tau ds \\
&\qquad + \int_0^1 \int_s^1 \big((s\tau + s - \tau)T\big)(j+\tau) K_{yttt}(x, \big((j+\tau)T\big); y) d\tau ds \Big) \partial_{(a,b)}T \\
&\quad = O\big(T^{-1/4} + T^{-1} \int_T^{+\infty} t^{-5/4} dt\big) = O\big(T^{-1/4} + t^{-1/4}T^{-1}\big)
\end{aligned}
$$

is uniformly bounded in $L^2(x)$, from which we may conclude that Θ_∞ is uniformly Lipschitz from $\mathcal{B}_2 \to \mathcal{B}_1$. $\square$

Remark 3.20. Similarly, we could use Simpson's rule to express

$$\partial_y \theta_\infty(x,y) = (1/6)K_y(x,T;y) + (5/6)K_y(x,2T;y) + O\Big(\int_T^{+\infty} |K_{ytttt}(x,y)|dt\Big),$$

obtaining convergence at rate $\int \|K_{ytttt}(x,y)\|_{L^2(x)}dt \sim \int t^{-11/4} \sim t^{-7/4}$, and so on. *Together with the favorable t-derivative bounds noted in* (3.28)*, this allows us to neglect continuization error for any practical purpose* of determining convergence or boundedness.

Remark 3.21. In our calculations of Lipschitz continuity of both $\hat{\mathcal{K}}_\infty$ and Θ_∞, it is critical that we choose an alternative, more rapidly-converging representation of the limit in order to detect regularity. Recall the quite relevant Remark 2.4.

3.3.2 Excited estimate (J term)

Approximating the kernel $J_N := \sum_{j=1}^{N+1} J(x, jT(a,b,\varepsilon); y)$ by integral

$$\hat{J}_N(x,y,a,b,\varepsilon) := T(a,b,\varepsilon)^{-1} \int_{T(a,b,\varepsilon)}^{NT(a,b,\varepsilon)} J(x,t;y)\,dt,$$

define the continuization error kernel (suppressing (a,b,ε) dependence)
(3.88)

$$\psi_N(x,y) := \sum_{j=1}^{N+1} J(x, jT(a,b,\varepsilon); y) - T(a,b,\varepsilon)^{-1} \int_{T(a,b,\varepsilon)}^{NT(a,b,\varepsilon)} J_y(x,t;y)\,dt$$

and associated continuization error operator

$$\Psi f(x) := \int_{-\infty}^{+\infty} \psi(x,t,\varepsilon) f(y)\,dy. \tag{3.89}$$

Corollary 3.22. *$\hat{\mathcal{J}}_N$, Ψ_N, and $\mathcal{J}_N$ are bounded and convergent, uniformly in (a,b,ε), as $N \to \infty$ as operators from $X_2 \to X_1$, with limits $\hat{\mathcal{J}}_\infty$, Ψ_∞, and $\mathcal{J}_\infty$ uniformly Lipschitz in (a,b) from $\mathcal{B}_2 \to \mathcal{B}_1$.*

Proof. Observing that $|J_y| \le e^{-\eta|x|}|K|$, we may factor out $e^{-\eta|x|/2}$ to bound

$$
\begin{aligned}
\Big\| \int J_\infty(x,y)\partial_y f(y)dy \Big\|_{X_1} &= \Big\| \int \int_T^{+\infty} J_y(x,t;y) f(y) dt dy \Big\|_{X_1} \\
&\le \int_T^{+\infty} \Big\| \int |J_y(x,t;y)|\,|f(y)| dy \Big\|_{X_1} dt \\
&\le C \Big\| \int_T e^{-\eta|x|/2} |K(x,t;y)| dt \Big\|_{L^\infty(x,y)} |f|_{L^1} \\
&\le C \Big\| \int_T e^{-\eta|x|/2} |K(x,t;y)| dt \Big\|_{L^\infty(x,y)} |\partial_x f|_{X_2}.
\end{aligned}
\tag{3.90}
$$

Recalling, by (3.70), that $\int_T^{+\infty} |K(x,t;y)| dt \le C$, we thus have $\hat{\mathcal{J}}_\infty$ uniformly bounded from $X_2 \to X_1$.

Recalling that $|f(y)| \le \|\partial_x f\|_{X_2}(1+|y|)^{-2}$, we may substitute in the third line of (3.90) the more careful estimate

$$
C \Big\| \int_T^{+\infty} e^{-\eta|x|/2} |K(x,t;y)|(1+|y|)^{-1/2} dt \Big\|_{L^\infty(x,y)} |\partial_x f|_{X_2}
$$

to obtain

$$
\begin{aligned}
&\Big\| \int \int_{NT}^{+\infty} J_y(x,t;y) f(y) dt dy \Big\|_{X_1} \\
&\qquad \le C \Big\| \int_{NT}^{+\infty} e^{-\eta|x|/2} |K(x,t;y)|(1+|y|)^{-1/2} dt \Big\|_{L^\infty(x,y)} \\
&\qquad\quad \times |\partial_x f|_{X_2} \\
&\qquad \le C_2 (1+NT)^{-1/2} |\partial_x f|_{X_2}
\end{aligned}
\tag{3.91}
$$

and thus uniform convergence of $\hat{\mathcal{J}}_N$ from $X_2 \to X_1$ as well.

Finally, from

$$
\partial_y \hat{J}_\infty(x,y,a,b,\varepsilon) := T(a,b,\varepsilon)^{-1} \int_{T(a,b,\varepsilon)}^{+\infty} J_y(x,t;y)\, dt,
\tag{3.92}
$$

we obtain

$$
\begin{aligned}
\partial_{(a,b)} \partial_y \hat{J}_\infty(x,y,a,b,\varepsilon) &= -(\partial_{(a,b)} T / T^2) \int_{T(a,b,\varepsilon)}^{+\infty} J_y(x,t;y)\, dt \\
&\quad - T^{-1} J_y(x, T(a,b,\varepsilon); y)
\end{aligned}
\tag{3.93}
$$

and thus, by Lipschitz continuity of T, and $T(0,0,0)=2\pi/\tau(0)\neq 0$,

$$\left\|\partial_{(a,b)}\partial_y \hat{J}_\infty(x,y,a,b,\varepsilon)\right\|_{L^2(x)} \le C\Big\|\int_{T(a,b,\varepsilon)}^{+\infty} J_y(x,t;y)\,dt\Big\|_{L^2(x)} + C\|J_y(x,T(a,b,\varepsilon);y)\|_{L^2(x)} \tag{3.94}$$

is uniformly bounded, from which we obtain (see discussion, Section 1.6.4) that $\mathcal{J}_\infty$ is uniformly Lipschitz from $\mathcal{B}_2\to\mathcal{B}_1$.

The treatment of continuization errors Ψ_N and Ψ_∞ is straightforward; see Remark 3.20. The results for $\mathcal{J}_N$ and $\mathcal{J}_\infty$ then follow by $\mathcal{J}_N=\hat{\mathcal{J}}_N+\Psi_N$. □

Combining Corollary 3.19 and Corollary 3.22, we obtain immediately:

Proposition 3.23. *Model operator $S(a,b,\varepsilon)$ defined in* (3.60), (3.61) *satisfies refined condition (B'), i.e., $\sum_{j=1}^\infty S^j=\lim_{N\to\infty}(\mathcal{K}_N+\mathcal{J}_N)$ is uniformly bounded as an operator from $X_2\to X_1$ and, for $\|\cdot\|_{X_2}$ uniformly bounded, is uniformly convergent from $\mathcal{B}_2\to\mathcal{B}_1$, with limit uniformly Lipschitz with respect to (a,b).*

3.4 Proof of the Main Theorem

The one-dimensional case is now straightforward.

Proof of Theorem 1.3. By Lemma 3.15 together with Corollary 2.18, it is sufficient to establish that $S(a,b,\varepsilon):=e^{\tilde{L}_\varepsilon T(a,b,\varepsilon)}\tilde{\Pi}$ satisfies condition (B') of Section 1.6.4, i.e., $\sum_{j=0}^N S^j$ is uniformly bounded as an operator from $X_2\to X_1$ and, for $\|\cdot\|_{X_2}$ uniformly bounded, is uniformly convergent from $\mathcal{B}_2\to\mathcal{B}_1$, with limit uniformly Lipschitz with respect to (a,b).

Following the model analysis of Section 3.3, approximate $\Sigma_N:=\sum_{j=1}^{N+1}S^j$ by its continuization $\hat{\Sigma}_N$, with associated kernel

$$\hat{\sigma}_N(x,y):=\hat{\Sigma}_N\delta_y(x)=T(a,b,\varepsilon)^{-1}\int_{T(a,b,\varepsilon)}^{NT(a,b,\varepsilon)}\tilde{G}(x,t;y)dt. \tag{3.95}$$

By Remark 3.20, and the Green function bounds (3.28), the continuization error $\Theta_N:=\Sigma_N-\hat{\Sigma}_N$ is uniformly bounded as an operator from $X_2\to X_1$ and, for $\|\cdot\|_{X_2}$ uniformly bounded, is uniformly convergent from $\mathcal{B}_2\to\mathcal{B}_1$, with limit uniformly Lipschitz with respect to (a,b). Thus, we may discard Θ_N in what follows, and work directly with continuization $\hat{\Sigma}_N$.

Following (3.18), decompose

$$\hat{\Sigma}_N = \hat{\mathcal{E}}_N + \hat{\mathcal{S}}_N + \hat{\mathcal{R}}_N, \tag{3.96}$$

with kernels $\hat{E}_N := \hat{\mathcal{E}}_N \delta_y(x)$, $\hat{S}_N := \hat{\mathcal{S}}_N \delta_y(x)$, $\hat{R}_N := \hat{\mathcal{R}}_N \delta_y(x)$ defined by

$$\begin{aligned}
\hat{E}_N &:= T(a,b,\varepsilon)^{-1} \int_{T(a,b,\varepsilon)}^{NT(a,b,\varepsilon)} \mathcal{E}^\varepsilon(x,t;y)dt, \\
\hat{S}_N &:= T(a,b,\varepsilon)^{-1} \int_{T(a,b,\varepsilon)}^{NT(a,b,\varepsilon)} \mathcal{S}^\varepsilon(x,t;y)dt, \\
\hat{R}_N &:= T(a,b,\varepsilon)^{-1} \int_{T(a,b,\varepsilon)}^{NT(a,b,\varepsilon)} \mathcal{R}^\varepsilon(x,t;y)dt,
\end{aligned} \tag{3.97}$$

where $\mathcal{E}^\varepsilon$, $\mathcal{S}^\varepsilon$, $\mathcal{R}^\varepsilon$ are as in (3.18), but with ε-dependence indicated. Hereafter, we suppress their ε-dependence as in previous sections, writing $\mathcal{E}$, $\mathcal{S}$, $\mathcal{R}$.

The operators $\hat{\mathcal{E}}_N$ and $\hat{\mathcal{S}}_N$ may be estimated as were $\hat{\mathcal{K}}_N$ and $\hat{\mathcal{J}}_N$ in Sections 3.3.1 and 3.3.2, respectively. Indeed, $\hat{\mathcal{E}}_N$ is exactly a superposition of terms of form $\hat{J}_N$, while $\hat{\mathcal{S}}_N$ is a superposition of approximate Gaussian terms obeying the same estimates used to bound $\hat{K}_N$, so can be handled in the same way; see [R, HR, HRZ] for similar calculations. Thus, the residual term $\hat{\mathcal{R}}_N$ is the only one requiring comment.

By bounds (3.23) and (3.24), $\mathcal{R}_y$ may be decomposed into terms

$$\sum_{k=1}^n O(e^{-\eta|x|}) t^{-1} e^{-(x-y-a_k^- t)^2/Mt}$$

of order $|J_y|$, J as defined in Section 3.3.2, terms

$$\begin{aligned}
&\sum_{k=1}^n O((t+1)^{-1/2} e^{-\eta x^+} t^{-1}) e^{-(x-y-a_k^- t)^2/Mt} \\
&+ \sum_{a_k^->0,\, a_j^-<0} \chi_{\{|a_k^- t|\ge|y|\}} O((t+1)^{-1/2} t^{-1}) e^{-(x-z_{jk}^-)^2/Mt} e^{-\eta x^+} \\
&+ \sum_{a_k^->0,\, a_j^+>0} \chi_{\{|a_k^- t|\ge|y|\}} O((t+1)^{-1/2} t^{-1}) e^{-(x-z_{jk}^+)^2/Mt} e^{-\eta x^-},
\end{aligned} \tag{3.98}$$

of order $|K_{yy}|$, a neglible term of order $e^{-\eta(|x-y|+t)}$, and the time-derivative

$\partial_t r(x,t;y)$ of terms

$$r(x,t;y) = O\big(e^{-\eta|y|}(1+t)^{-1/2}\big) \times \Big(\sum_{a_k^- <0} e^{-(x-y-a_k^- t)^2/Mt} + \sum_{a_k^+ >0} e^{-(x-y-a_k^+ t)^2/Mt} \Big) \tag{3.99}$$

of order $|K|$.

Terms of order $|J_y|$ may be estimated exactly as was J_y in the proof of Corollary 3.22, since those arguments depended only on modulus bounds. Likwise, terms of order $|K_{yy}|$ may be handled by the modulus-bound arguments used in the proofs of Lemma 3.17 and Corollary 3.19 to bound terms of the same order. Terms of order $e^{-\eta(|x-y|+t)}$ may be handled by a similar argument, using $|\int_{NT}^{+\infty} e^{-\eta(|x-y|+t)}dt| \le Ce^{-\eta NT}e^{-\eta|x-y|}$. Finally, derivative terms may be estimated by a cancellation argument like that used to bound K_y, integrating in time to obtain $\int_{NT}^{MT} \partial_t r(x,t;y)dt = r(x,t;y)|_{NT}^{MT}$, hence

$$\Big\| T^{-1} \int_{NT}^{MT} \partial_t r(x,t;y)dt \Big\|_{L^2(x)} \le C\big((NT)^{-1/4} + (MT)^{-1/4}\big), \tag{3.100}$$

giving uniform convergence from $\mathcal{B}_2 \to \mathcal{B}_1$. Likewise, the limiting kernel

$$T^{-1} \int_T^{+\infty} \partial_t r(x,t;y)dt = -T^{-1} r(x,T;y) = O\big(e^{-\eta|x-y|^2}\big)$$

is clearly bounded by $C(1+|x-y|)^{-1}$, hence the limiting operator is bounded from $X_2 \to X_1$ by the argument used in Corollary 3.19 to treat K_y. Lipschitz continuity of all operators is straightforward, similarly as in the proof of Corollary 3.19, since (a,b) derivatives of limiting operators $T^{-1}\int_T^{+\infty}$ pass onto the factor T^{-1} and the lower limit of integration only. $\square$

Remark 3.24. The method of proof, especially the $(1+|x-y|)^{-1}$ decay rate on the kernel $\hat{K}_\infty(x,y)$ of $\sum S^j$ associated with scattering terms (Lemma 3.17), suggests strongly that $(1+|x|)^{-1}$ decay bound of (1.24) on the perturbation $\mathbf{u}^a(x,t) - \bar{u}^\varepsilon$ is in fact sharp. Recall that substantial cancellation was already taken into account in bounding the kernel.

4 The multidimensional case

We now briefly describe the extension to the multidimensional case, model problem III, Section 1.5. Consider a one-parameter family of standing planar viscous shock solutions $\bar u^\varepsilon(x_1)$ of a smoothly-varying family of conservation laws

$$u_t = \mathcal{F}(\epsilon, u) := \Delta_x u - \sum_{j=1}^d F^j(\epsilon, u)_{x_j}, \qquad u \in \mathbb{R}^n \tag{4.1}$$

in an infinite cylinder

$$\mathcal{C} := \{x:\ (x_1, \tilde x) \in \mathbb{R}^1 \times \Omega\}, \qquad \tilde x = (x_2, \dots, x_d) \tag{4.2}$$

$\Omega \in \mathbb{R}^{d-1}$ bounded, with Neumann boundary conditions

$$\partial u/\partial_{\tilde x} \cdot \nu_\Omega = 0 \quad \text{for} \quad \tilde x \in \partial\Omega, \tag{4.3}$$

(or, in the case that Ω is rectangular, periodic boundary conditions), with associated linearized operators

$$L(\epsilon) := \partial \mathcal{F}/\partial u|_{u=\bar u^\epsilon} = -\sum_{j=1}^d \partial_{x_j} A^j(x_1, \varepsilon) + \Delta_x, \tag{4.4}$$

$A^j(x, \varepsilon) := F^j_u(\bar u^\varepsilon(x), \varepsilon)$, denoting $A^j_\pm(\varepsilon) := \lim_{z\to\pm\infty} A^j(z, \varepsilon) = F_u(u_\pm, \varepsilon)$. Profiles $\bar u^\varepsilon$ satisfy the standing-wave ODE

$$u' = F^1(u, \varepsilon) - F^1(u_-, \varepsilon). \tag{4.5}$$

Following [Z4, Z1], assume:

(H0) $F^j \in C^k$, $k \geq 2$.

(H1) $\sigma(A^1_\pm(\varepsilon))$ real, distinct, and nonzero, and $\sigma(\sum \xi_j A^j_\pm(\varepsilon))$ real and semisimple for $\xi \in \mathbb{R}^d$.

(H2) Considered as connecting orbits of (4.5), $\bar u^\varepsilon$ are transverse and unique up to translation, with dimensions of the stable subpace $S(A^1_+)$ and the unstable subspace $U(A^1_-)$ summing for each ε to $n+1$.

As in Remark 3.1, (H2) implies that $\bar u^\varepsilon$ is of standard Lax type.

4.1 Rectangular geometry

For clarity of exposition, we specialize now to the case of a rectangular cross-section Ω with periodic boundary conditions, without loss of generality

$$\Omega = [0, 2\pi]^d. \tag{4.6}$$

This case closely resembles that of the whole space, making the analysis particularly transparent. In particular, we may take the discrete Fourier transform in transverse directions $\tilde{x} = (x_2, \ldots, x_d)$ to obtain a family of linearized ordinary differential operators in x_1,

$$L_{\tilde{\xi}}(\varepsilon) := L_0(\varepsilon) - \sum_{j=2}^{d} i\xi_j A^j(x_1, \varepsilon) - |\tilde{\xi}|^2, \tag{4.7}$$

indexed by $\tilde{\xi} \in \mathbb{Z}^{d-1}$, where $L_0(\varepsilon) := \partial_{x_1}^2 - \partial_{x_1} A^1(x_1, \varepsilon)$ is the one-dimensional linearized operator of (3.3), and $\tilde{\xi} = (\xi_2, \ldots, \xi_d)$ are the frequency variables associated with coordinates $\tilde{x} = (x_2, \ldots, x_d)$.

Associated to each $L_\xi(\varepsilon)$, we may define an Evans function

$$D^\varepsilon(\tilde{\xi}, \lambda) \tag{4.8}$$

as in the one-dimensional case, whose zeroes correspond in location and multiplicity with eigenvalues of $L_{\tilde{\xi}}$. Moreover, the eigenvalues of L consist of the union of eigenvalues of $L_{\tilde{\xi}}$ for all $\tilde{\xi} \in \mathbb{Z}^{d-1}$, with associated eigenfunctions

$$W(x) = e^{i\tilde{\xi}\cdot\tilde{x}} w(x_1), \tag{4.9}$$

where w is the eigenfunction of $L_{\tilde{\xi}}$.

To (H0)–(H2) we adjoin the Evans function condition:

($\mathcal{D}_\varepsilon$) On a neighborhood of $\mathbb{Z}^{d-1} \times \{\Re\lambda \geq 0\} \setminus \{0, 0\}$, the only zeroes of $D(\tilde{\xi}, \cdot)$ are (i) a zero of multiplicity one at $(\tilde{\xi}, \lambda) = (0, 0)$, and (ii) a crossing conjugate pair of zeroes $\lambda_\pm(\varepsilon) = \gamma(\varepsilon) + i\tau(\varepsilon)$ of some $L_{\tilde{\xi}_*}$, with $\gamma(0) = 0$, $\partial_\varepsilon \gamma(0) > 0$, and $\tau(0) \neq 0$.

Lemma 4.1. *Conditions (H0)–(H2) and ($\mathcal{D}_\varepsilon$) are equivalent to conditions (P)(i)–(iii) of the introduction together with $F \in C^k$, $k \geq 2$, simplicity and nonvanishing of $\sigma(A^1_\pm(\varepsilon))$, semisimplicity of $\sigma(\sum \xi_j A^j_\pm(\varepsilon))$ and the Lax condition*

$$\dim S(A^1_+(\varepsilon)) + \dim U(A^1_-(\varepsilon)) = n + 1, \tag{4.10}$$

with $\bar{u}^\varepsilon$ (linearly) stable for $\varepsilon < 0$ and unstable for $\varepsilon \geq 0$.

Proof. Essentially the same as that of Lemma 3.2, but in the final assertion (here, just a comment) substituting for the one-dimensional linearized stability analysis of [ZH, MaZ3] a "one-and-one-half dimensional" stability analysis like that used below to verify condition (B'); see Remark 4.7. □

Remark 4.2. Eigenvalues crossing at transverse wave-number $\tilde{\xi}_* = 0$ correspond to the one-dimensional case considered in Section 3, hence the multidimensional subsumes the one-dimensional analysis. Such crossings correspond to longitudinal "galloping" or "pulsating" instabilities. Eigenvalues crossing at nonzero wave number correspond in the rectangular geometry to transverse "cellular" instabilities as discussed in [KS].

Introduce Banach spaces $\mathcal{B}_1 = L^2$, $\mathcal{B}_2 = \partial_x L^1 \cap L^2$,

$$X_1 = \{f : |f(x)| \le C(1+|x_1|)^{-1}\},$$

and

$$X_2 = \partial_x\{f : |f(x)| \le C(1+|x_1|)^{-2}\} \cap X_1,$$

equipped with norms $\|f\|_{\mathcal{B}_1} = \|f\|_{L^2}$, $\|\partial_x f\|_{\mathcal{B}_2} = \|f\|_{L^1} + \|\partial_x f\|_{L^2}$,

$$\|f\|_{X_1} = \|(1+|x_1|)f\|_{L^\infty}, \quad \text{and} \quad \|\partial_x f\|_{X_2} = \|(1+|x_1|)^2 f\|_{L^\infty} + \|\partial_x f\|_{X_1},$$

where ∂_x is taken in the sense of distributions. By inspection, we have that $\mathcal{B}_2 \subset \mathcal{B}_1$, $X_2 \subset X_1$, $X_1 \subset \mathcal{B}_1$, $X_2 \subset \mathcal{B}_2$, and the closed unit ball in X_1 is closed as a subset of $\mathcal{B}_1$.

4.1.1 Linearized estimates

Lemma 4.3. *Associated with eigenvalues $\lambda_\pm(\varepsilon)$ of L_ε are right and left eigenfunctions $\phi^\varepsilon_\pm = e^{i\tilde{\xi}_*\cdot\tilde{\xi}} w_\pm(x_1)$ and $\tilde{\phi}^\varepsilon_\pm = e^{i\tilde{\xi}_*\cdot\tilde{\xi}}\tilde{w}_\pm(x_1) \in C^k(x,\varepsilon)$, $k \ge 2$, exponentially decaying in up to q derivatives as $x_1 \to \pm\infty$, and L_ε-invariant projection*

$$\Pi f := \sum_{j=\pm} \phi^\varepsilon_j(x)\langle\tilde{\phi}^\varepsilon_j, f\rangle \tag{4.11}$$

onto the total (oscillatory) eigenspace $\Sigma^\varepsilon := \mathrm{Span}\{\phi^\varepsilon_\pm\}$, bounded from L^q or $\mathcal{B}_2$ to $W^{2,p} \cap X_2$ for any $1 \le q, p \le \infty$. Moreover,

$$\phi^\varepsilon_\pm = \partial_x \Phi^\varepsilon_\pm, \tag{4.12}$$

with $\Phi^\varepsilon \in C^{k+1}$ exponentially decaying in up to $k+1$ derivatives as $x \to \pm\infty$.

Proof. Essentially identical to that of Lemma 3.4. $\square$

Defining $\tilde\Pi^\varepsilon := \mathrm{Id} - \Pi^\varepsilon$, $\tilde\Sigma^\varepsilon := \mathrm{Range}\tilde\Pi^\varepsilon$, and $\tilde L(\varepsilon) := L(\varepsilon)\tilde\Pi^\varepsilon$, denote by

$$G(x,t;y) := e^{L(\varepsilon)t}\delta_y(x) \tag{4.13}$$

the Green kernel associated with the linearized solution operator e^{Lt} of the linearized evolution equations $u_t = L(\varepsilon)u$, and

$$\tilde G(x,t;y) := e^{\tilde L(\varepsilon)t}\tilde\Pi\delta_y(x) = e^{L(\varepsilon)t}\tilde\Pi\delta_y(x) \tag{4.14}$$

the Green kernel associated with the transverse linearized solution operator $e^{\tilde L(\varepsilon)t}\tilde\Pi$. Evidently, $G = \mathcal{O} + \tilde G$, where

$$\mathcal{O}(x,t;y) := e^{(\gamma(\varepsilon)+i\tau(\varepsilon))t}\phi_+(x)\tilde\phi_+^t(y) + e^{(\gamma(\varepsilon)-i\tau(\varepsilon))t}\phi_-(x)\tilde\phi_-^t(y). \tag{4.15}$$

Lemma 4.4 (Short-time estimates). *For $0 \le t \le T$, any fixed $T > 0$, and some $C = C(T)$,*

$$\|e^{L(\varepsilon)t}\partial_x f\|_{\mathcal{B}_2},\ \|e^{\tilde L(\varepsilon)t}\tilde\Pi\partial_x f\|_{\mathcal{B}_2} \le Ct^{-1/2}\|f\|_{L^1\cap L^2}. \tag{4.16}$$

$$\|e^{L(\varepsilon)t}\partial_x f\|_{X_2},\ \|e^{\tilde L(\varepsilon)t}\tilde\Pi\partial_x f\|_{X_2} \le Ct^{-1/2}\|(1+|x_1|)^2 f\|_{L^\infty}, \tag{4.17}$$

Proof. Essentially identical to that of Lemma 3.5. $\square$

Note that the Fourier transform completely decouples the linearized problem in the sense that the Fourier-transformed operators

$$\tilde G_{\tilde\xi}(x_1,t;y_1) = \mathcal{F}G\mathcal{F}^{-1}$$

acting in frequency space, $\mathcal{F}$ denoting Fourier transform, have simple form

$$\tilde G_{\tilde\xi}(x,t;y) = \begin{cases} e^{L_{\tilde\xi}(\varepsilon)t}\delta_{y_1}(x_1) & \tilde\xi \ne \tilde\xi_*, \\ e^{L_{\tilde\xi}(\varepsilon)t}\tilde\Pi_{\tilde\xi_*}\delta_{y_1}(x_1) & \tilde\xi = \tilde\xi_*, \end{cases} \tag{4.18}$$

$\tilde\Pi_{\tilde\xi} := \mathrm{Id} - \Pi_{\tilde\xi}$, where

$$\Pi_{\tilde\xi} f := \sum_{j=\pm} w_j^\varepsilon(x)\langle \tilde w_j^\varepsilon, f\rangle$$

is the one-dimensional projection onto the oscillatory subspace of $L_{\tilde\xi_*}$.

Proposition 4.5 (Global bounds). *Under assumptions (H0)–(H2), (P): (i) $\tilde{G}_0$ satisfies the pointwise bounds stated in Proposition 3.6. (ii) For $\xi \neq 0$, the Fourier transformed solution operators $e^{L_{\tilde{\xi}}(\varepsilon)t}$ for $\tilde{\xi} \neq \tilde{\xi}_*$, $e^{L_{\tilde{\xi}}(\varepsilon)t}\tilde{\Pi}_{\tilde{\xi}_*}$ for $\tilde{\xi} = \tilde{\xi}_*$ satisfy for some C, $\eta > 0$, the exponential bounds*

$$
\begin{aligned}
\|(1+|x_1|)e^{L_{\tilde{\xi}}(\varepsilon)t} f\|_{L^\infty(x_1)} &\leq Ce^{-\eta t}\|(1+|x_1|)f\|_{L^\infty(x_1)}, \\
\|(1+|x_1|)e^{L_{\tilde{\xi}}(\varepsilon)t}\tilde{\Pi}_{\tilde{\xi}_*} f\|_{L^\infty(x_1)} &\leq Ce^{-\eta t}\|(1+|x_1|)f\|_{L^\infty(x_1)}.
\end{aligned}
\tag{4.19}
$$

Proof. Assertion (i) follows immediately from the observation that L_0 is exactly the one-dimensional linearized operator studied in Section 3.1, together with the fact that the bounds on the restriction to $\tilde{\Sigma}$ in case $\tilde{\xi}_* = 0$ are the same as the bounds on the full solution operator in the case $\tilde{\xi} \neq 0$ of linearized one-dimensional stability.

Assertion (ii) follows for $|\tilde{\xi}|$ large by standard semigroup/asymptotic ODE estimates. For $|\tilde{\xi}|$ bounded but nonzero, it follows by standard semigroup estimates, together with the assumption that there are no eigenvalues of $L_{\tilde{\xi}}$ other than the crossing pair at $\tilde{\xi} = \tilde{\xi}_*$ and the computation as in (1.10) for the one-dimensional case that $\sigma_{\text{ess}}(L_{\tilde{\xi}}) \subset \{\lambda : \Re\lambda \leq -2\eta|\tilde{\xi}|^2\}$ for some $\eta > 0$. □

Remark 4.6. The argument for (ii) of course fails at $\tilde{\xi} = 0$, due to the lack of spectral gap.

4.1.2 Proof of the main theorem

Proof of Theorem 1.5. We restrict for definiteness to the case of periodic boundary conditions. The proof in the Neumann case is essentially identical.

Using the bounds of Lemmas 4.3 and 4.4, we may carry out the polar coordinate and Poincaré return map construction of Section 3.2 essentially unchanged to obtain a multidimensional version of Lemma 3.15 (statement unchanged), reducing the problem to the abstract form studied in Section 2. Thus, it is sufficient to establish the linearized estimate (B"). But, this follows easily using the one-dimensional bounds of Proposition 4.5(i) and the calculations of the one-dimensional case together with the $\xi \neq 0$ bounds of Proposition 4.5(ii).

For example, uniform boundedness of $\int_T^{+\infty} e^{\tilde{L}(\varepsilon)t}dt$ from $X_2 \to X_1$ follows by

$$
\begin{aligned}
\Big\| \int_T^{+\infty} e^{\tilde L(\varepsilon)t} f\,dt\Big\|_{X_1} &:= \Big\|(1+|x_1|)\int_T^{+\infty} e^{\tilde L(\varepsilon)t} f\,dt\Big\|_{L^\infty(x_1,\tilde x)} \\
&\le \sum_{\tilde\xi} \Big\|(1+|x_1|)\int_T^{+\infty} e^{\tilde L_{\tilde\xi}(\varepsilon)t}\hat f\,dt\Big\|_{L^\infty(x_1)}
\end{aligned}
\tag{4.20}
$$

and the observation, by Proposition 4.5(ii), Hausdorff–Young's inequality, and the fact that L^∞ controls L^1 on bounded domains, that

$$
\begin{aligned}
\sum_{\tilde\xi\neq 0} \Big\|(1+|x_1|)\int_T^{+\infty} e^{\tilde L_{\tilde\xi}(\varepsilon)t}\hat f\,dt\Big\|_{L^\infty(x_1)} &\le \sum_{\tilde\xi\neq 0}\Big(\int_T^{+\infty} Ce^{-\eta|\tilde\xi|^2 t}\,dt\Big) \\
&\qquad \times \|(1+|x_1|)\hat f(\cdot,\tilde\xi)\|_{L^\infty(x_1)} \\
&\le C_2\sup_{\tilde\xi}\|(1+|x_1|)\hat f(\cdot,\tilde\xi)\|_{L^\infty(x_1)} \\
&\le C_2\|(1+|x_1|)f\|_{L^1(\tilde x;L^\infty(x_1))} \\
&\le C_3\|(1+|x_1|)f\|_{L^\infty(x)} \\
&= C_3\|f\|_{X_1},
\end{aligned}
\tag{4.21}
$$

together with the computation, using the one-dimensional estimates carried out in Section 3, and denoting $f=\partial_{x_1}F_1+\partial_{\tilde x}\tilde F$, so that $\hat f=\partial_{x_1}\hat F_1$, of

$$
\begin{aligned}
\Big\|(1+|x_1|)\int_T^{+\infty} e^{\tilde L_0(\varepsilon)t}\hat f\,dt\Big\|_{L^\infty(x_1)} &= \Big\|(1+|x_1|)\int_T^{+\infty} e^{\tilde L_0(\varepsilon)t}\partial_{x_1}\hat F_1\,dt\Big\|_{L^\infty(x_1)} \\
&\le C\|(1+|x_1|)^2\hat F(\cdot,0)\|_{L^\infty(x_1)} \\
&\le C\sup_{\tilde\xi}\|(1+|x_1|)^2\hat F(\cdot,\tilde\xi)\|_{L^\infty(x_1)} \\
&\le C\|(1+|x_1|)^2 F\|_{L^1(\tilde x;L^\infty(x_1))} \\
&\le C_2\|(1+|x_1|)^2 F\|_{L^\infty(x)} \\
&:= C_2\|f\|_{X_2}.
\end{aligned}
\tag{4.22}
$$

Other computations follow similarly. □

Remark 4.7. We point out that we have in passing set up a framework suitable for the linearixe stability analysis of flow in a duct, a problem of interest in its own right and somewhat different from either the one-dimensional case considered in [ZH, MaZ3] or the multi-dimensional case considered in [ZS, Z1]. It would be very interesting to try to carry out a full nonlinear stability analysis by this technique.

4.2 General cross-sectional geometry

We may treat general cross-sections Ω by separation of variables, decomposing

$$L(\varepsilon, x_1, \partial_x) = L_0(\varepsilon, x_1, \partial_{x_1}) + M(\varepsilon, x_1, \partial_{\tilde{x}}),$$

where

$$M := -\sum_{j=2}^{d} A^j(x_1, \varepsilon)\partial_{x_j} + \Delta_{\tilde{x}},$$

and expanding the perturbation u in eigenfunctions $w_j(\varepsilon, x_1)$ of $M(\varepsilon, x_1, \partial_{\tilde{x}})$ on domain Ω as

$$u(x,t) = \sum_{k=0}^{+\infty} \alpha_k(x_1) w_k(\varepsilon, x_1)(\tilde{x}),$$

$\alpha_k =: \hat{u}(k)$, to recover a decoupled system

$$\begin{aligned} \partial_t \alpha_k &= L_k \alpha_k \\ &:= (L_0 + \nu_k)\alpha_k + O(|\partial_{x_1} w_k| + |\partial_{x_1}^2 w_k|)\alpha_k + O(|\partial_{x_1} w_k|)\partial_{\tilde{x}}\alpha_k, \end{aligned} \tag{4.23}$$

where $\nu_k = \nu_k(x_1)$ are the eigenvalues of M associated with w_k, with both ν_k and w_k converging exponentially in x_1 to limits as $x_1 \to \pm\infty$, that is, for which L_k is of the same basic form as $L_{\tilde{\xi}}$ in Section 4.1.2. Thus, we can carry out the entire Evans function construction of Section 4.1, with k replacing $\tilde{\xi}$.

More, since $w_0 \equiv 1$ for Neumann boundary conditions, we have again that $L_k = L_0$ for $k = 0$, justifying our notation. Likewise, augmenting P(i) with the assumption that $\nu_k \leq -\nu_* < 0$ for all $k \geq 1$, we recover a spectral gap for all L_k, $k \geq 1$. From these two observations, we obtain Proposition 4.5 exactly as before. Lemma 4.4 holds also, indeed was independent of Ω. Thus, we may carry out the entire argument of Theorem 1.5 essentially unchanged, provided that we can establish the generalized Hausdorff–Young inequalities

$$\begin{aligned} \|\hat{u}\|_{\ell^\infty(k,x_1)} &\leq C\|u\|_{L^\infty(x_1, L^1(\tilde{x}))} \\ \|u\|_{L^\infty(x)} &\leq C\|\hat{u}\|_{L^\infty(x_1, \ell^1(k,x_1))}, \end{aligned} \tag{4.24}$$

which follow, for example, if one can establish uniform supremum bounds on the right and left eigenfunctions of M by asymptotic eigenvalue–eigenfunction estimates as $k \to \infty$.

Alternatively, we could perform a more general analysis allowing viscosities with cross-derivatives, and also avoiding the need for asymptotic spectral analysis, by expanding u instead in eigenfunctions of $\Delta_{\tilde{x}}$ (or, more generally, the second-order elliptic operator in derivatives of $\tilde{x}$ appearing in operator L), and noting that for Neumann boundary conditions the first eigenfunction w_0 is still $\equiv 1$. Thus, though different wave numbers no longer completely decouple, we still obtain a cascaded system in which the zero wave-number equation is again just $\partial_t w_0 = L_0 w_0$ as in the one-dimensional case, and we can perform an analysis as before, but dividing only into the two blocks $k = 0$ and $k \geq 1$, treating coupling terms in the $k \geq 1$ part as source terms driving the equation.

Estimating the one-dimensional part as before in (4.22) and for the $k \geq 1$ part substituting for (4.20)–(4.21) the Sobolev/Parseval estimates

(4.25)
$$\begin{aligned}
&\Big\| \int_T^{+\infty} e^{\tilde{L}(\varepsilon)t} \Pi_{k\geq 1} f \, dt \Big\|_{X_1} := \Big\| (1+|x_1|) \int_T^{+\infty} e^{\tilde{L}(\varepsilon)t} f \, dt \Big\|_{L^\infty(x_1,\tilde{x})} \\
&\quad \leq \Big\| (1+|x_1|) \int_T^{+\infty} e^{\tilde{L}(\varepsilon)t} \Pi_{k\geq 1} f \, dt \Big\|_{L^\infty(x_1, H^{d-1}(\tilde{x}))} \\
&\quad \leq \sqrt{ \sum_{k\geq 1} \Big\| \int_T^{+\infty} (1+|x_1|) \| (1+|\nu_k|)^{(d-1)/2} e^{\tilde{L}_k(\varepsilon)t} \hat{f} \|_{\ell^2(k)} \, dt \Big\|^2_{L^\infty(x_1)} } \\
&\quad \leq \| (1+|x_1|) \hat{f}(\cdot, \tilde{\xi}) \|_{L^\infty(x_1), \ell^2(k)} \sqrt{ \sum_{k\geq 1} \Big(\int_T^{+\infty} (1 + t^{-(d-1)/2}) C e^{-\eta|\nu_k| t} \, dt \Big)^2 } \\
&\quad \leq C_2 \| (1+|x_1|) \hat{f}(\cdot, \tilde{\xi}) \|_{L^\infty(x_1), \ell^2(k)} \\
&\quad \leq C_2 \| (1+|x_1|) f \|_{L^2(\tilde{x}; L^\infty(x_1))} \\
&\quad \leq C_3 \| (1+|x_1|) f \|_{L^\infty(x)} = C_3 \| f \|_{X_1},
\end{aligned}$$

we would then obtain the result without recourse to asymptotic eigenvalue estimates. We shall not pursue these issues further here.

5 Higher regularity

Finally, we outline a strategy for obtaining Lipschitz regularity of $\varepsilon(\cdot)$ and uniqueness up to translation of solutions, using not (B"') but an alternative argument special to the problem at hand. Specifically, we show that the part of $\sum_{j=0}^{\infty} S^j$ associated with primary terms $\mathcal{E} + \mathcal{S}$ indeed satisfies $\mathcal{B}_1$ (that is, L^2) Lipschitz continuity with respect to ε, condition (B"'), but the part associated with residual $\mathcal{R}$ apparently does not, or at least this seems difficult to show. We give an additional, more specialized argument to recover Lipschitz continuity of the reduction function $b = b(a, \varepsilon)$ from $\mathbb{R}^{n+1}$ to L^p, $2 < p < +\infty$, and thereby (exploiting a useful asymmetry between nonlinear parts N_1 and N_2) Lipschitz continuity of the reduced, scalar bifurcation problem.

The last part of the argument is speculative. Rigorous proof would require detailed estimation of (y, t, ε)-derivatives of the residual $\mathcal{R}$, a straightforward but tedious step that we leave for the future. Finally, assuming Lipschitz continuity may be achieved, we indicate how this may be improved to C^1.

5.1 Lipschitz regularity

Consider again the model problem,

$$\tilde{G}^\varepsilon(x, t; y) = K^\varepsilon(x, t; y) + J^\varepsilon(x, t; y), \tag{5.1}$$

now including ε-dependence, where

$$K(x, t; y) := ct^{-1/2} e^{-(x-y-a(\varepsilon)t)/4t} \tag{5.2}$$

and

$$J(x, t; y) := c_2(\bar{u}^\varepsilon)'(x)\mathrm{errfn}((-y - a(\varepsilon)t)/2t^{1/2}), \tag{5.3}$$

$a(\varepsilon) < a_0 < 0$ and C^1 in ε. By inspection, this is faithful to the approximation $\tilde{G} \sim \mathcal{E}^\varepsilon + \mathcal{S}^\varepsilon$ obtained by neglecting residual $\mathcal{R}^\varepsilon$ in (3.18).

5.1.1 Model computation

We first show how to verify (B"') for approximation (5.1). By Remark 3.20, it is sufficient to treat the continuous approximants $\hat{K}_\infty$, $\hat{J}_\infty$. The J-term may be shown to be Lipschitz in ε by a modulus-bound computation

$$
\begin{aligned}
&\|\partial_\varepsilon T \hat{J}^\varepsilon_\infty \partial_x f\|_{X_1} \\
&\qquad = \Big\| \int J^\varepsilon_y(x,T;y) f(y) dy + \int \int_{T(a,b,\varepsilon)}^{+\infty} J^\varepsilon_{yy}(x,t;y) \partial_\varepsilon a t f(y)\, dt dy \Big\|_{X_1} \\
&\qquad \le C\|\partial_x f\|_{X_2} + \Big\| \int_{T(a,b,\varepsilon)}^{+\infty} e^{-\eta|x|}(1+|y|)^{\alpha-1} K^\varepsilon_y(x,t;y) t\, dt \Big\|_{L^\infty} \|\partial_x f\|_{X_2} \\
&\qquad \le C(\alpha)\|\partial_x f\|_{X_2}
\end{aligned}
\tag{5.4}
$$

similar to (3.91), for any $0<\alpha\le 1/2$.

The K-term by (3.78) may be expressed as

$$
\partial_y \hat{K}^\varepsilon_\infty = -(aT)^{-1}\Big(K(x,T;y) - \int_T^{+\infty} K_{yy}(x,t;y)\, ds \Big), \tag{5.5}
$$

yielding

$$
\begin{aligned}
\|\partial_\varepsilon a T \partial_y \hat{K}^\varepsilon_\infty\|_{L^2(x)} \le \Big\| K^\varepsilon_y(x,T;y)(\partial_\varepsilon a) T \Big\|_{L^2(x)} + \Big\| K^\varepsilon_{yy}(x,T;y)(\partial_\varepsilon T) \Big\|_{L^2(x)} \\
+ \Big\| \int_{T(a,b,\varepsilon)}^{+\infty} K^\varepsilon_{yyy}(x,t;y)(\partial_\varepsilon a) t\, dt \Big\|_{L^2(x)}.
\end{aligned}
\tag{5.6}
$$

The first two terms on the righthand side of (5.6) are clearly bounded. The third term may be bounded by a cancellation estimate like that used to bound $\hat{K}_\infty$ in the first place, using $K_{yyy} = -a^{-1}K_{yyt} + a^{-1}K_{yyyy}$ and integration by parts to obtain

$$
\begin{aligned}
a\int_{T(a,b,\varepsilon)}^{+\infty} K^\varepsilon_{yyy}(x,t;y) t\, dt = K^\varepsilon_{yy}(x,T;y) T + \int_{T(a,b,\varepsilon)}^{+\infty} K_{yy}(x,t;y)\, dt \\
+ \int_{T(a,b,\varepsilon)}^{+\infty} K^\varepsilon_{yyyy}(x,t;y) t\, dt,
\end{aligned}
\tag{5.7}
$$

where the last two terms by $|K_{yyyy}|t \sim |K_{yy}|$ are similar order and uniformly absolutely convergent in $L^2(x)$.

5.1.2 Residual estimate

One may guess that $|\partial_\varepsilon \mathcal{R}^\varepsilon_y(x,t;y)| \sim |\mathcal{R}^\varepsilon_{yy}(x,t;y)t|$ similarly as for primary terms, which would lead to problems with terms involving $e^{-\eta|y|}$ factors, for which successive y-derivatives do not improve decay (see Remark 3.7). However, the reality seems to be more favorable. Estimating at the resolvent kernel level, we find that ε-derivatives falling on such exponential terms bring neither faster decay *nor* an additional factor of t, and so such terms may be estimated by the same modulus bounds as used in verifying the simpler (B'). This gives modulus bound

$$\begin{aligned} |\partial_\varepsilon(\mathcal{R}^\varepsilon_y - \partial_t r^\varepsilon)| &\sim |K_{yy}| + |K_{yyy}t| + |J_{yy}| + |J_{yyy}t| \\ &\sim |K_y| + |J_y|, \end{aligned} \tag{5.8}$$

r as in (3.23). We emphasize that this is speculative, and not a rigorous bound, however.

Strategically expressing

$$T\partial_y \hat{R}^\varepsilon_\infty = r^\varepsilon(x,T;y)f(y) + \int_T^{+\infty} (R^\varepsilon_y - r^\varepsilon_t)(x,t;y)f(y)\,dt \tag{5.9}$$

and differentiating with respect to ε, we thus find, modulo terms that are clearly bounded,

$$\begin{aligned} \|\partial_\varepsilon\partial_y \hat{R}^\varepsilon_\infty\|_{L^2} &\sim \int_T^{+\infty} (R^\varepsilon_y - r^\varepsilon_t)(x,t;y)f(y)\,dt \\ &\sim \Big\| \int_{T(a,b,\varepsilon)}^{+\infty} (|K_y| + |J_y|)\,dt \Big\|_{L^2}, \end{aligned} \tag{5.10}$$

where $\int |J_y| dt$ by the computations of Section 3.3 is bounded (indeed, convergent) in L^2, but $\int |K_y| dt$ is divergent. Thus, it is not clear to us that condition (B"') should be satisfied for the full problem. At the least, such a result would involve a delicate higher-derivative cancellation estimate like those used for the principal terms, requiring considerable care in the (y,t,ε)-derivative estimates for $\mathcal{R}$.

Instead, we suggest a different, more special strategy based on modulus bounds alone. Note, by an estimate like that used to show (3.67)(ii)–(iii), that

$$\int_{T(a,b,\varepsilon)}^{+\infty} |K_y|\,dt \le C(1+|x-y|)^{-1/2}. \tag{5.11}$$

Thus term $\int |K_y| dt$ just misses boundedness in L^2, being uniformly bounded in any L^p, $p > 2$. Likewise, for $\|b\|_{X_1} \le \varepsilon$ sufficiently small, N_2 is Lipschitz, contractive from any L^p, $p < \infty$ to $\mathcal{B}_2$, since $(1+|x-y|)^{-1}$ is in L^q for any $q > 1$, whence $\mathcal{T} = (I-S)^{-1}N_2$ is contractive in b from $L^p \to \mathcal{B}_1$, for any $p < \infty$, with $\|\partial_b \mathcal{T}\|_{L^p \to \mathcal{B}_1} < 1/2$. Thus, in the fixed point iteration $b_{n+1}(a,\varepsilon) := T(a, b_n(a,\varepsilon), \varepsilon)$, we can bound

$$\begin{aligned} \|\partial_\varepsilon b_{n+1}(a,\varepsilon)\|_{L^p} &\le \|\partial_\varepsilon \mathcal{T}(a, b_n(a,\varepsilon), \varepsilon)\|_{L^p} \\ &\qquad + \|\partial_b \mathcal{T}(a, b_n(a,\varepsilon), \varepsilon)\|_{L^p \to \mathcal{B}_1} \|\partial_\varepsilon b_n\|_{L^p} \\ &\le \sup_{\|b\|_{X_1} \le \varepsilon} \|\partial_\varepsilon \mathcal{T}(a,b,\varepsilon)\|_{L^p} + (1/2)\|\partial_\varepsilon b_n(a,\varepsilon)\|_{L^p} \end{aligned} \tag{5.12}$$

to obtain by iteration a uniform bound $\|\partial_\varepsilon b_n(a,\varepsilon)\|_{L^p} \le 2\sup_b \|\partial_\varepsilon \mathcal{T}(a,b,\varepsilon)\|_{L^p}$ showing that $b(x,\varepsilon)$ is uniformly Lipschitz in ε with respect to L^p, any $2 < p < \infty$, even if it is not Lipschitz with respect to $\mathcal{B}_1 = L^2$.

Recalling the useful asymmetry that N_1 is Lipschitz from any L^p to $\mathbb{R}^n$, we find that the Lipshitz function $b(x,\varepsilon) : \mathbb{R}^{n+m} \to L^p$ yields a Lipschitz reduced function $\tilde{N}(a,\varepsilon) := N_1(a, b(a,\varepsilon), \varepsilon)$ from $\mathbb{R}^{n+m} \to \mathbb{R}^n$ in (2.18), from which we recover Lipschitz regularity of the bifurcation and uniqueness up to translation of solutions, as described in Section 2.3.2.

Remark 5.1. By the same technique, we may streamline the treatment of the principal terms, estimating the K-term by modulus bounds in (5.6) as $\int_T^{+\infty} |K_{yyy}t|dt \sim \int_T^{+\infty} |K_y|dt$, thus avoiding the higher-derivative cancellation estimate (5.7).

5.2 C^1 regularity

We conclude by describing briefly how, given a unique Lipschitz reduction map $b = b(a,\varepsilon)$ established by any means, to recover a C^1 bifurcation as in the standard (positive spectral gap) case described in Proposition 1.1. Recall, first, that N_j in the original system were in fact C^1 in all parameters away from $a = 0$, so that this property is inherited (as uniform limit of continuous functions $\partial_\varepsilon b_n$; see estimate (5.12)) by the reduction function $b(a,\varepsilon)$. Thus, we need only consider the vicinity of $a = 0$.

Next, recalling Remark 2.1, observe that we could perform the return map construction in θ- rather than t-coordinates to obtain a nicer, C^k, $k \ge 2$, first equilibrium equation $\hat{f}(a,b) = 0$, but a degraded second equilibrium equation to which our framework (B') does not apply. Since the second equation is only used to determine the reduction function $b(\cdot)$, this is unimportant, and we may work in the more favorable θ-variables. Combining these observations, and recalling derivative estimate $|DN_1| \leq C(|a|+|b|)$ and Lipschitz estimate $|Db| \leq C|a|$, we find that $a^{-1}N_1(a, b(a,\varepsilon), \varepsilon)$ is in fact C^1 at $a = 0$, with $D_{a,\varepsilon}N_1(a, b(a,\varepsilon), \varepsilon) = \partial_a N_1(a, 0, \varepsilon)$. Thus, the reduced scalar equation (2.18) is C^1 and we obtain a C^1 bifurcation.

Remark 5.2. The change to θ-coordinates is indeed necessary in order remove Lipschitz dependence of the intervening return time $T(x, y, \varepsilon)$ in the r-equation. For, otherwise, there appears a linear term

$$\big(e^{\gamma(\varepsilon)T(x,y,\varepsilon)} - 1\big)r$$

that is differentiable only at $\varepsilon = 0$, where $\gamma(0) = 0$. Essentially, by changing to θ-variables, we take advantage of partial cancellation between derivatives of this linear term and the nonlinear term N_1 that are difficult detect by other means.

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