VANISHING VISCOSITY LIMIT FOR INCOMPRESSIBLE FLOW AROUND A SUFFICIENTLY SMALL OBSTACLE

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Abstract. In this article we consider viscous flow in the exterior of an obstacle satisfying the standard no-slip boundary condition at the surface of the obstacle. We look for conditions under which solutions of the Navier-Stokes system in the exterior domain converge to solutions of the Euler system in the full space when both viscosity and the size of the obstacle vanish. We prove that this convergence is true assuming two hypothesis: first, that the initial exterior domain velocity converges strongly (locally) in L^2 to the full-space initial velocity and second, that the diameter of the obstacle is smaller than a suitable constant times viscosity, or, in other words, that the obstacle is sufficiently small. The convergence holds as long as the solution to the limit problem is known to exist and stays sufficiently smooth. To fix the $\mathcal{O}(1)$ spatial scale, we consider flows with an initial vorticity which is compacly supported, vanishes near obstacle, and does not depend on viscosity and the on size of the obstacle. In [3], Iftimie proved that any such vorticity gives rise to a family of exterior flows which converges in L^2 to the corresponding full-space flow. For exterior two dimensional flow, topology implies that the initial velocity is not determined by vorticity alone, but also by its harmonic part. In the case of two dimensional flow, we prove strong convergence of initial data, as required by our main result, if the harmonic part of the family of initial velocities is chosen so that the circulation of the initial flow around the small obstacle vanishes. This work complements the study of incompressible flow around small obstacles, which has been carried out in [3, 4, 5]

1. Introduction

The purpose of the present work is to study the asymptotic behavior of families of solutions of the incompressible Navier-Stokes equations, in two and three space dimensions, in the exterior of a single smooth obstacle, when both viscosity and the size of the obstacle become small. More precisely, let Ω be a smooth, bounded and connected and simply connected domain in $\mathbb{R}^n$,

$n = 2, 3$ and let $\Pi_\varepsilon = \mathbb{R}^n \setminus \varepsilon\Omega$. Let u_0 be a smooth, divergence-free vector field in $\mathbb{R}^n$ which gives rise to a smooth solution u of the Euler equations, defined on an interval $[0, T]$. Let $u^{\nu,\varepsilon} \in L^\infty((0,T); L^2(\Pi_\varepsilon)) \cap C^0_w([0,T); L^2(\Pi_\varepsilon) \cap L^2((0,T); H^1_0(\Pi_\varepsilon))$be a weak Leray solution of the incompressible Navier-Stokes equations, with viscosity ν, in Π_ε, satisfying the no-slip boundary conditions at $\partial\Pi_\varepsilon$. We assume that there exists a constant $C > 0$ such that the following hypothesis holds:

[H] We have that

$$\sup_{\varepsilon \leq C\nu} \|u^{\nu,\varepsilon}(\cdot, 0) - u_0\|_{L^2(\Pi_\varepsilon)} \to 0,$$

as $\nu \to 0$.

We then prove that $\sup_{\varepsilon \leq C\nu} \|u^{\nu,\varepsilon} - u\|_{L^\infty((0,T);L^2(\Pi_\varepsilon)} \to 0$, as $\nu \to 0$.

In addition, we prove two things. First, that if we fix an initial vorticity ω_0 in $\mathbb{R}^3$, smooth, divergence-free and compactly supported in $\mathbb{R}^3 \setminus \{0\}$ and consider $u^{\nu,\varepsilon}(\cdot, 0) = K_{\Pi_\varepsilon}[\omega_0]$, where K_{Π_ε} denotes the Biot-Savart operator in Π_ε, then hypothesis (H) is satisfied. Second, if we fix an initial vorticity ω_0 in $\mathbb{R}^2$, smooth and compactly supported in $\mathbb{R}^2 \setminus \{0\}$ and consider $u^{\nu,\varepsilon}(\cdot, 0) = K_{\Pi_\varepsilon}[\omega_0] + mH_{\Pi_\varepsilon}$, where K_{Π_ε} denotes the Biot-Savart operator in Π_ε, H_{Π_ε} is the normalized generator of the harmonic vector fields in Π_ε and $m = \int \omega_0$, then hypothesis (H) is again satisfied.

A central theme in incompressible hydrodynamics is the vanishing viscosity limit, something naturally associated with the physical phenomena of turbulence and of boundary layers. In particular, a natural question to ask is whether the limiting flow associated with the limit of vanishing viscosity satisfies the incompressible Euler equations. This is known to be true in the absence of material boundaries, see [15, 1] for the two dimensional case and [8, 17] for the three dimensional case. Also, if the boundary conditions are of Navier type, see [2, 14, 7, 21], noncharacteristic, see [18] or for certain symmetric 2D flows, see [16, 13] convergence to an Euler solution remains valid. The most relevant case from the physical point of view corresponds to no slip boundary conditions. In this case, we have results on criteria for convergence to solutions of the Euler system, see [9, 19, 20, 10], but the general problem remains wide open. The present research is motivated by the following question: we have convergence in the absence of boundaries and no information in their presence, but what happens with we only have a little bit of boundary? In other words,

we ask how small a boundary has to be in order not to obstruct convergence in the vanishing viscosity limit.

This problem was one of the main motivations undelying the author's research on incompressible flow around small obstacles. Our previous results include the small obstacle limit for the 2D inviscid equations, see [4, 12] and for the viscous equations, see [5, 3]. The work we present here is a natural outgrowth of this research effort.

The remainder of this article is divided into four sections. In Section 2, we state precisely our main result. In Section 3 we derive our a priori estimates, in Section 4 we put together the estimates of Section 3 and prove our main result and in Section 5 we draw some conclusions.

2. Statement of the main result

Let us first consider our problem in dimension two. We consider ω_0 smooth, compactly supported in the plane minus the origin and let $\omega = \omega(x,t)$ the solution of the Euler equations with initial data ω_0. Set $u = K * \omega$, where K is the kernel of the Biot-Savart law in the plane. The velocity u satisfies the incompressible 2D Euler equations with pressure p. Set $m = \int \omega_0 dx$. Let Γ be a smooth Jordan curve, dividing the plane into two open connected components: a bounded one denoted Ω, and an unbounded one, denoted Π. We use the following notation: $\Omega_\varepsilon \equiv \varepsilon\Omega$, $\Pi_\varepsilon = \varepsilon\Pi$ and $\Gamma_\varepsilon = \partial\Omega_\varepsilon$. Let $K^\varepsilon = \nabla^\perp \Delta_\varepsilon^{-1}$, where Δ_ε is the Dirichlet Laplacian in Π_ε. Let H^ε be the generator of the harmonic vector fields in Π_ε, normalized so that its circulation around Γ_ε is one. We consider $u^{\nu,\varepsilon} = u^{\nu,\varepsilon}(x,t)$ to be the unique solution of the initial-boundary value problem:

$$
(1) \qquad \begin{cases} \partial_t u^{\nu,\varepsilon} + u^{\nu,\varepsilon} \cdot \nabla u^{\nu,\varepsilon} = -\nabla p^{\nu,\varepsilon} + \nu \Delta u^{\nu,\varepsilon}, & \text{in } \Pi_\varepsilon \times (0,\infty) \\ \text{div } u^{\nu,\varepsilon} = 0 & \text{in } \Pi_\varepsilon \times [0,\infty) \\ u^{\nu,\varepsilon}(x,t) = 0 & \text{for } x \in \Gamma_\varepsilon, t > 0 \\ u^{\nu,\varepsilon}(t=0) = K^\varepsilon[\omega_0] + mH^\varepsilon \end{cases}
$$

In dimension three, Π_ε is simply connected, so that if ω_0 is smooth, divergence-free, vector field compactly supported in Π_ε then we take $u^{\nu,\varepsilon}(t=0)$ to be the unique divergence free vector field, tangent to $\partial\Pi_\varepsilon$ whose curl is ω_0. We again consider equation (1) supplemented by this initial data instead of $u^{\nu,\varepsilon}(t=0) = K^\varepsilon[\omega_0] + mH^\varepsilon$.

Our objective is to prove the following result

Theorem 1. *Fix $T > 0$ such that the strong solution u of the incompressible Euler equation in $\mathbb{R}^n$ ($n = 2, 3$) exists up to time T, i.e. $u \in C^0\big([0,T]; Lip(\mathbb{R}^n)\big) \cap C^1\big([0,T]; L^\infty(\mathbb{R}^n)\big)$. There exist constants $C_1 = C_1(\Omega, \omega_0) > 0$ and $C_2 = C(T, \omega_0) > 0$, such that*

$$\sup_{0<\varepsilon<C_1\nu} \|u^{\nu,\varepsilon}(\cdot,t) - u(\cdot,t)\|_{L^\infty((0,T);L^2(\Pi_\varepsilon))} \leq C_2\nu \qquad \forall t \in [0,T].$$

3. Convergence of the initial velocity and preliminary results

We will assume throughout this paper that the initial vorticity is smooth, compactly supported with support disjoint of the origin. In dimension two, ω_0 is scalar while in the three-dimensional case ω_0 is assumed to be a divergence free vector field. In both cases, we denote by u_0 the velocity associated to ω_0 in $\mathbb{R}^n$, $n = 2, 3$.

Let $m = \int \omega_0$. We set

$$u_0^\varepsilon = K^\varepsilon[\omega_0] + mH^\varepsilon$$

in the case of the dimension two. The additional term mH^ε is due to the fact that the exterior domain Π_ε is not simply connected.

In dimension three the domain Π_ε is simply connected, so there is a unique velocity field which is divergence free, tangent to the boundary with curl ω_0. We denote this unique vector field by u_0^ε.

In dimension two, we know that there is a unique global solution of the incompressible Navier-Stokes equations in the exterior domain Π_ε with initial velocity u_0^ε, see [11]. We denote it by $u^{\nu,\varepsilon}$. In dimension three, it is proved in [3] that the initial velocity u_0^ε is square-integrable. Accordingly, there exists a global weak Leray solution $u^{\nu,\varepsilon}$, *i.e.*

$$u^{\nu,\varepsilon} \in L^\infty\big([0,\infty); L^2(\Pi_\varepsilon\big) \cap C_w^0\big([0,\infty); L^2(\Pi_\varepsilon\big) \cap L^2_{loc}\big([0,\infty); H_0^1(\Pi_\varepsilon\big),$$

$u^{\nu,\varepsilon}$ verifies the equation in the sense of distributions and the following energy inequality holds true:

$$\|u^{\nu,\varepsilon}(t)\|_{L^2(\Pi_\varepsilon)} + 2\nu \int_0^t \|\nabla u^{\nu,\varepsilon}(s)\|_{L^2(\Pi_\varepsilon)}\, ds \leq \|u_0^\varepsilon\|_{L^2(\Pi_\varepsilon)} \qquad \forall t \geq 0. \tag{2}$$

We prove now three lemmas. The first one concerns L^2 convergence of the initial data.

Lemma 2. *Fix ε_0 such that the support of ω_0 does not intersect Ω_ε for any $\varepsilon < \varepsilon_0$. There exists a constant $C > 0$, depending on Ω and ω_0 such that*

$$\|u_0^\varepsilon - u_0\|_{L^2(\Pi_\varepsilon)} \leq C\varepsilon.$$

Proof. The proof in dimension three can be found in [3]. We assume now that the space dimension is two.

We begin with a construction whose details can be found in [4]. In Section 2 of [4], an explicit formula for both K^ε and H^ε can be found in terms of a conformal map T, which takes Π into the exterior of the unit disk centered at zero. The construction of T and its behavior near infinity are contained in Lemma 2.1 of [4]. Using identities (3.5) and (3.6) in [4], we have that the vector field H^ε can be written explictly as

$$H^\varepsilon = H^\varepsilon(x) = \frac{1}{2\pi\varepsilon} DT^t(x/\varepsilon)\, \frac{(T(x/\varepsilon))^\perp}{|T(x/\varepsilon)|^2}.$$

and the operator K^ε can be written as an integral operator with kernel $\mathcal{K}^\varepsilon$, given by

$$\mathcal{K}^\varepsilon = \frac{1}{2\pi\varepsilon} DT^t(x/\varepsilon)\Big(\frac{(T(x/\varepsilon) - T(y/\varepsilon))^\perp}{|T(x/\varepsilon) - T(y/\varepsilon)|^2} - \frac{(T(x/\varepsilon) - (T(y/\varepsilon))^*)^\perp}{|T(x/\varepsilon) - (T(y/\varepsilon))^*|^2}\Big),$$

where $x^* = x/|x|^2$ denotes the inversion with respect to the unit circle. Furthermore, we recall Theorem 4.1 of [4], which proves that

$$\|u_0^\varepsilon\|_{L^\infty(\Pi_\varepsilon)} \leq C\|\omega_0\|_{L^\infty}^{1/2}\|\omega_0\|_{L^1}^{1/2}. \tag{3}$$

To understand the behavior for ε small in the expressions above, we need to understand the behavior of $T(x)$ for large x. We use Lemma 1 in [6], which is a more detailed version of Lemma 2.1 in [4], to find that there exists a constant $\beta > 0$ such that

$$T(x/\varepsilon) = \beta x \varepsilon^{-1} + h(x/\varepsilon), \tag{4}$$

with $h = h(x)$ a bounded, holomorphic function on Π_1 satisfying $|Dh(x)| \leq C/|x|^2$. Therefore,

$$|DT(x/\varepsilon) - \beta\mathbb{I}| \leq C\frac{\varepsilon^2}{|x|^2}. \tag{5}$$

We will need a further estimate on the bounded holomorphic function $h = h(z) = T(z) - \beta z$, namely that

$$|h(z_1) - h(z_2)| \leq C\frac{|z_1 - z_2|}{|z_1||z_2|},$$

for some constant $C > 0$ independent of z_1, z_2. This estimate holds since, by construction (see Lemma 2.1 in [4]), we have that $h(z) = g(1/z)$ with g a holomorphic function on $(\Pi_1)^*$, whose derivatives are bounded in the closure of $(\Pi_1)^*$. Here, $(\Pi_1)^*$ denotes the image of Π_1 through the the mapping $x \mapsto x^* = x/|x|^2$ to which we add $\{0\}$. (Clarify what is needed on the domain in order to guarantee that a global C^1 bound implies a global Lipschitz bound. Clearly convexity is enough, as is star-shapedness, but the question is whether a C^1 bound on the boundary is enough.) Therefore we have

$$|h(z_1) - h(z_2)| = \Big|g\Big(\frac{1}{z_1}\Big) - g\Big(\frac{1}{z_2}\Big)\Big| \leq C\Big|\frac{1}{z_1} - \frac{1}{z_2}\Big| = C\frac{|z_1 - z_2|}{|z_1||z_2|}. \tag{6}$$

In order to estimate $\|u_0^\varepsilon - u_0\|_{L^2(\Pi_\varepsilon)}$ we use the fact that the support of ω_0 is contained in Π_ε for ε sufficiently small ε to write

$$2\pi[u_0^\varepsilon(x) - u_0(x)] =$$

$$\int_{\Pi_\varepsilon} \Big(\frac{1}{\varepsilon} DT^t(x/\varepsilon)\, \frac{(T(x/\varepsilon) - T(y/\varepsilon))^\perp}{|T(x/\varepsilon) - T(y/\varepsilon)|^2} - \frac{(x-y)^\perp}{|x-y|^2}\Big)\omega_0(y)\, dy$$

$$+ \int_{\Pi_\varepsilon} \frac{1}{\varepsilon} DT^t(x/\varepsilon)\Big(\frac{(T(x/\varepsilon))^\perp}{|T(x/\varepsilon)|^2} - \frac{(T(x/\varepsilon) - (T(y/\varepsilon))^*)^\perp}{|T(x/\varepsilon) - (T(y/\varepsilon))^*|^2}\Big)\omega_0(y)\, dy$$

$$\equiv \mathfrak{A}^\varepsilon + \mathfrak{B}^\varepsilon.$$

Let us begin by estimating $\mathfrak{B}^\varepsilon$. We make the change of variables $\eta = \varepsilon T(y/\varepsilon)$, whose Jacobian is $J = |\det(DT^{-1})(y/\varepsilon)|$, a bounded function. Additionally, we set $z = \varepsilon T(x/\varepsilon)$. With this we find:

$$\mathfrak{B}^\varepsilon = DT^t(x/\varepsilon) \int_{\{|\eta|>\varepsilon\}} \Big(\frac{z^\perp}{|z|^2} - \frac{(z - \varepsilon^2\eta^*)^\perp}{|z - \varepsilon^2\eta^*|^2}\Big)\omega_0(\varepsilon T^{-1}(\eta/\varepsilon))\, J d\eta.$$

We observe now that there exists ρ independently of ε such that the support of $\omega_0(\varepsilon T^{-1}(\eta/\varepsilon))$ is contained in the set $\{|\eta| > \rho\}$. Therefore we can write

$$|\mathfrak{B}^\varepsilon| \leq C \int_{\{|\eta|>\rho\}} \frac{\varepsilon^2|\eta^*|}{|z||z - \varepsilon^2\eta^*|} |\omega_0(\varepsilon T^{-1}(\eta/\varepsilon))|\, J d\eta \leq C\frac{\varepsilon^2}{|z|^2},$$

where C depends on the support of ω_0, on the L^1-norm of ω_0 and on the domain Ω through the bounds on the conformal map T and its derivatives. Finally, we use this estimate in the integral of the square of $\mathfrak{B}^\varepsilon$:

$$\int_{\Pi_\varepsilon} |\mathfrak{B}^\varepsilon|^2\, dx \leq C\varepsilon^4 \int_{\{|z|>\varepsilon\}} \frac{1}{|z|^4}\, dz \leq C\varepsilon^2,$$

as desired.

Next we treat $\mathfrak{A}^\varepsilon$. First we re-write $\mathfrak{A}^\varepsilon$ in a more convenient form:

$$\mathfrak{A}^\varepsilon = \int_{\Pi_\varepsilon} \frac{1}{\beta} DT^t(x/\varepsilon)\Big(\frac{\beta}{\varepsilon}\frac{(T(x/\varepsilon)-T(y/\varepsilon))^\perp}{|T(x/\varepsilon)-T(y/\varepsilon)|^2} - \frac{(x-y)^\perp}{|x-y|^2}\Big)\omega_0(y)\,dy$$

$$+ \int_{\Pi_\varepsilon} \Big(\frac{1}{\beta}DT^t(x/\varepsilon) - \mathbb{I}\Big)\frac{(x-y)^\perp}{|x-y|^2}\,\omega_0(y)\,dy$$

$$\equiv \mathfrak{A}^\varepsilon{}_1 + \mathfrak{A}^\varepsilon{}_2.$$

By (5), the term $\mathfrak{A}^\varepsilon{}_2$ can be easily estimated:

$$|\mathfrak{A}^\varepsilon{}_2| \le \frac{\varepsilon^2}{|x|^2}\int_{\Pi_\varepsilon} \frac{1}{|x-y|}\,|\omega_0(y)|\,dy \le C\frac{\varepsilon^2}{|x|^2},$$

so this reduces to an estimate similar to the one we found for $\mathfrak{B}^\varepsilon$.

Next we examine $\mathfrak{A}^\varepsilon{}_1$. We use the expression for T given in (4) to write

$$\mathfrak{A}^\varepsilon{}_1 = \frac{DT^t(\frac{x}{\varepsilon})}{\beta}\int_{\Pi_\varepsilon}\Big(\frac{(x-y+(\frac{\varepsilon}{\beta})[h(\frac{x}{\varepsilon})-h(\frac{y}{\varepsilon})])^\perp}{|x-y+(\frac{\varepsilon}{\beta})[h(\frac{x}{\varepsilon})-h(\frac{y}{\varepsilon})]|^2} - \frac{(x-y)^\perp}{|x-y|^2}\Big)\omega_0(y)\,dy.$$

With this we have:

$$|\mathfrak{A}^\varepsilon{}_1| \le C\int_{\Pi_\varepsilon} \frac{|\frac{\varepsilon}{\beta}|[h(\frac{x}{\varepsilon})-h(\frac{y}{\varepsilon})]|}{|x-y+(\frac{\varepsilon}{\beta})[h(\frac{x}{\varepsilon})-h(\frac{y}{\varepsilon})]||x-y|}\,|\omega_0(y)|\,dy.$$

We will make use several times of the estimate we obtained for h given in (6). First

$$|\frac{\varepsilon}{\beta}|[h(\frac{x}{\varepsilon})-h(\frac{y}{\varepsilon})]| \le C\frac{\varepsilon^2|x-y|}{|\beta||x||y|}. \tag{7}$$

Using (7) gives

$$|\mathfrak{A}^\varepsilon{}_1| \le C\int_{\Pi_\varepsilon} \frac{\varepsilon^2}{|x-y+(\frac{\varepsilon}{\beta})[h(\frac{x}{\varepsilon})-h(\frac{y}{\varepsilon})]||\beta||x||y|}\,|\omega_0(y)|\,dy.$$

Let R, $r>0$ be such that the support of ω_0 is contained in the disk of radius R and outside the disk of radius r. We will estimate $\mathfrak{A}^\varepsilon{}_1$ in two regions: $|x|\ge 2R$ and $|x|<2R$. Also, recall that the estimate of the L^2-norm of $\mathfrak{A}^\varepsilon{}_1$ is to be performed in Π_ε so we may assume throughout that $|x|\ge C\varepsilon$. Suppose first that $|x|\ge 2R$. Then we find:

$$|\mathfrak{A}^\varepsilon{}_1| \le C\frac{\varepsilon^2}{|x|^2}.$$

Above we used that $r < |y| \leq |x|/2$ and hence $|x-y+(\frac{\varepsilon}{\beta})[h(\frac{x}{\varepsilon})-h(\frac{y}{\varepsilon})]| \geq C|x|$ if ε is sufficiently small, since h is bounded. Finally, in the region $C\varepsilon \leq |x| < 2R$ we have:

$$|\mathfrak{A}^{\varepsilon}{}_1| \leq C \int_{\Pi_\varepsilon} \frac{\varepsilon^2}{||x-y| - \varepsilon^2(|x-y|/|x||y|)|\,|\beta|\,|x|\,|y|} |\omega_0(y)|\, dy.$$

Above $|y|$ is of order 1 and $|x| > C\varepsilon$, so

$$||x-y| - \varepsilon^2(|x-y|/|x||y|)| \geq \frac{|x-y|}{2}$$

for ε small enough. Therefore

$$|\mathfrak{A}^{\varepsilon}{}_1| \leq C\frac{\varepsilon^2}{|x|} \int_{\Pi_\varepsilon} \frac{|\omega_0(y)|}{|x-y|}\, dy \leq C\frac{\varepsilon^2}{|x|}.$$

Clearly this last portion has L^2-norm in the region $|x| < 2R$ bounded by $C\varepsilon$.

□

Remark: Note that if we were willing to confine our analysis to the exterior of a small disk, the proof above would be much simpler. Indeed, let $\Omega_\varepsilon = \{|x| > \varepsilon\}$. Then the conformal map T is the identity, so $\mathfrak{A}^\varepsilon \equiv 0$ and all that is needed is the easier estimate for $\mathfrak{B}^\varepsilon$. (Comment for ourselves: we may wish to prove that the estimate in the lemma is sharp, which can probably be done explicitly for the exterior of the disk, with ω_0 a dirac at, say, $(1,0)$.)

To state the second lemma, we first require some notation. In dimension two, we denote by $\psi = \psi(x,t)$ the stream function for the velocity field u, chosen so that $\psi(0,t) = 0$. In dimension three, ψ denotes the unique divergence free vector field vanishing at infinity whose curl is u. In other words, we set

$$\psi(x,t) = \int_{\mathbb{R}^2} \frac{(x-y)^\perp \cdot u(y,t)}{2\pi|x-y|^2} dy + \int_{\mathbb{R}^2} \frac{y^\perp \cdot u(y,t)}{2\pi|y|^2} dy$$

in dimension two so that $u = \nabla^\perp \psi$ and

$$\psi(x,t) = -\int_{\mathbb{R}^3} \frac{x-y}{4\pi|x-y|^3} \times u(y,t) dy - \int_{\mathbb{R}^3} \frac{y}{4\pi|y|^3} \times u(y,t) dy$$

in dimension three so that $u = \operatorname{curl} \psi$. In both dimensions one has that ψ and $\nabla\psi$ are uniformly bounded on the time interval $[0,T]$.

Let $R > 0$ be such that the ball of radius R, centered at the origin, contains Ω. Let $\varphi = \varphi(r)$ be a smooth function on $\mathbb{R}_+$ such that $\varphi(r) \equiv 0$ if $0 \leq r \leq$

$R+1$, $\varphi \geq 0$ and $\varphi(r) \equiv 1$ if $r \geq R+2$. Set $\varphi^\varepsilon = \varphi^\varepsilon(x) = \varphi(|x|/\varepsilon)$ and

$$u^\varepsilon = \nabla^\perp(\varphi^\varepsilon \psi)$$

in dimension two and

$$u^\varepsilon = \operatorname{curl}(\varphi^\varepsilon \psi)$$

in dimension three. In both dimension two and three, the vector field is divergence free and vanishes in the neighborhood of the boundary.

We also choose the pressure $p = p(x,t)$ so that $p(0,t) = 0$.

Lemma 3. *Fix $T > 0$. There exist constants K_i, $i = 1, \ldots, 5$ such that, for any $0 < \varepsilon < \varepsilon_0$ and any $0 \leq t < T$ we have:*

(1) $\|\nabla u^\varepsilon\|_{L^2}^2 \leq K_1$,
(2) $\|u^\varepsilon\|_{L^\infty} \leq K_2$,
(3) $\|u^\varepsilon - u\|_{L^2} + \|u^\varepsilon - \varphi^\varepsilon u\|_{L^2} \leq K_3 \varepsilon$,
(4) $\|\nabla\psi\nabla\varphi^\varepsilon\|_{L^\infty} + \|\psi\nabla^2\varphi^\varepsilon\|_{L^\infty} \leq K_4/\varepsilon$,
(5) $\|p\nabla\varphi^\varepsilon\|_{L^2} + \|\partial_t\psi\nabla\varphi^\varepsilon\|_{L^2} \leq K_5\varepsilon$.

Above, we used the notation $\|\nabla\psi\nabla\varphi^\varepsilon\|_{L^\infty} = \sum_{i,j} \|\partial_i\psi\partial_j\varphi^\varepsilon\|_{L^\infty}$ in dimension two and $\|\nabla\psi\nabla\varphi^\varepsilon\|_{L^\infty} = \sum_{i,j,k} \|\partial_i\psi_k\partial_j\varphi^\varepsilon\|_{L^\infty}$ in dimension three. Similar notations were used for the other terms.

Proof. Some of the inequalities above can be improved in dimension three. However, it turns out that these improvements have no effect on the the statement of Theorem 1. Therefore, to avoid giving separate proofs in dimension three we chosed to state these weaker estimates.

Recall that both u and ∇u are uniformly bounded. First we write

$$\partial_i u^\varepsilon = \partial_i \nabla^\perp(\varphi^\varepsilon\psi) = u\partial_i\varphi^\varepsilon + \psi\partial_i\nabla^\perp\varphi^\varepsilon + \varphi^\varepsilon\partial_i u$$

in dimension two and

$$\partial_i u^\varepsilon = \partial_i \operatorname{curl}(\varphi^\varepsilon\psi) = u\partial_i\varphi^\varepsilon + \partial_i\nabla\varphi^\varepsilon \times \psi + \varphi^\varepsilon\partial_i u$$

in dimension three. The support of the first two terms of the right-hand sides of the relations above are contained in the annulus $\varepsilon(R+1) < |x| < \varepsilon(R+2)$, whose Lebesgue measure is $\mathcal{O}(\varepsilon^2)$. Furthermore, $|\nabla\varphi^\varepsilon| = \mathcal{O}(1/\varepsilon)$, $|\nabla^2\varphi^\varepsilon| = \mathcal{O}(1/\varepsilon^2)$ and $|\psi(x,t)| = \mathcal{O}(\varepsilon)$ for $|x| < \varepsilon(R+2)$, since $\psi(0,t) = 0$. Taking L^2 norms in the expressions above gives the first estimate.

Next we observe that $u^\varepsilon = \varphi^\varepsilon u + \psi \nabla^\perp \varphi^\varepsilon$ or $u^\varepsilon = \varphi^\varepsilon u + \nabla \varphi^\varepsilon \times \psi$. Clearly $\varphi^\varepsilon u$ is bounded and to bound the second term, we use again that $\psi(0,t) = 0$, which proves the second estimate. For the third estimate, observe that $u^\varepsilon - u$ and $u^\varepsilon - \varphi^\varepsilon u$ are bounded, as we have just proved, and have support in the ball $|x| < \varepsilon(R+2)$. For the fourth estimate, we use again that $\psi(0,t) = 0$. The last estimate follows from two facts: that the functions whose L^2-norm we are estimating have support on the ball $|x| < \varepsilon(R+2)$ and that they are both bounded, since $p(0,t) = 0$ and $\psi_t(0,t) = 0$. □

Finally, we require a modified Poincaré inequality, stated below. We include a sketch of the proof for completeness.

Lemma 4. *Let Ω be the obstacle under consideration and let R be such that $\Omega \subset B_R$. Consider the scaled obstacles Ω_ε and the exterior domains Π_ε. Then, if $W \in H_0^1(\Pi_\varepsilon)$ we have*

$$\|W\|_{L^2(\Pi_\varepsilon \cap B_{(R+2)\varepsilon})} \leq K_6\, \varepsilon\, \|\nabla W\|_{L^2(\Pi_\varepsilon \cap B_{(R+2)\varepsilon})}.$$

Proof. The proof proceeds in two steps. First we establish the result in the case $\varepsilon = 1$. Suppose, by contradiction, that there exists a sequence $\{W^k\} \subset H_0^1(\Pi_1)$ such that $\|W^k\|_{L^2(\Pi_1 \cap B_{R+2})} > k \|\nabla W^k\|_{L^2(\Pi_1 \cap B_{R+2})}$. Set $V^k = W^k / \|W^k\|_{L^2(\Pi_1 \cap B_{R+2})}$. Then $V^k \in H_0^1(\Pi_1)$, with unit L^2 norm in $\Pi_1 \cap B_{R+2}$, while the L^2-norm of its gradient vanishes as $k \to \infty$. Thus, passing to a subsequence if necessary, $V^k \to V$, weakly in H^1 and strongly in L^2 (on $\Pi_1 \cap B_{R+2}$) so that $\|V\|_{L^2(\Pi_1 \cap B_{R+2})} = 1$ and $\nabla V = 0$. Since this set is connected, it follows that V is constant in $\Pi_1 \cap B_{R+2}$. By continuity of the trace, the trace of V on $\Gamma = \partial \Pi_1$ must vanish, which shows that $V \equiv 0$ in $\Pi_1 \cap B_{R+2}$, a contradiction.

We conclude the proof with a scaling argument. Let $W \in H_0^1(\Pi_\varepsilon)$ and set $Y = Y(x) = W(\varepsilon x)$. Then $Y \in H_0^1(\Pi_1)$. Using the first step we deduce that there exists a constant K_6 such that $\|Y\|_{L^2(\Pi_1 \cap B_{R+2})} \leq K_6 \|\nabla Y\|_{L^2(\Pi_1 \cap B_{R+2})}$. Undoing the scaling we find:

$$\|Y\|^2_{L^2(\Pi_1 \cap B_{R+2})} = \int_{\Pi_1 \cap B_{R+2}} |W(\varepsilon x)|^2\, dx = \frac{\|W\|^2_{L^2(\Pi_\varepsilon \cap B_{(R+2)\varepsilon})}}{\varepsilon^2};$$

$$\|\nabla Y\|^2_{L^2(\Pi_1 \cap B_{R+2})} = \int_{\Pi_1 \cap B_{R+2}} \varepsilon^2 |\nabla W\,(\varepsilon x)|^2\, dx = \|\nabla W\|^2_{L^2(\Pi_\varepsilon \cap B_{(R+2)\varepsilon})}.$$

The desired result follows immediately.

□

4. Proof of the main result

We are now ready to prove our result, Theorem 1.

4.1. **Case $n = 2$.** We start with the planar case. The velocity field u^ε is divergence free and satisfies the equation

$$u_t^\varepsilon = -\varphi^\varepsilon u \cdot \nabla u - \varphi^\varepsilon \nabla p + \partial_t \psi \nabla^\perp \varphi^\varepsilon.$$

We set $W^{\nu,\varepsilon} \equiv u^{\nu,\varepsilon} - u^\varepsilon$. The vector field $W^{\nu,\varepsilon}$ is divergence free, vanishes on the boundary and satisfies:

$$\partial_t W^{\nu,\varepsilon} - \nu \Delta W^{\nu,\varepsilon} = -u^{\nu,\varepsilon} \cdot \nabla u^{\nu,\varepsilon} - \nabla p^{\nu,\varepsilon} + \nu \Delta u^\varepsilon + \varphi^\varepsilon u \cdot \nabla u + \varphi^\varepsilon \nabla p - \partial_t \psi \nabla^\perp \varphi^\varepsilon.$$

We perform an energy estimate, multiplying this equation by $W^{\nu,\varepsilon}$ and integrating over Π_ε. We obtain

$$\frac{1}{2}\frac{d}{dt}\|W^{\nu,\varepsilon}\|_{L^2}^2 + \nu\|\nabla W^{\nu,\varepsilon}\|_{L^2}^2 = -\nu \int_{\Pi_\varepsilon} \nabla W^{\nu,\varepsilon} \cdot \nabla u^\varepsilon \, dx \tag{8}$$

$$- \int_{\Pi_\varepsilon} W^{\nu,\varepsilon} \cdot [(u^{\nu,\varepsilon} \cdot \nabla) u^{\nu,\varepsilon}] \, dx + \int_{\Pi_\varepsilon} W^{\nu,\varepsilon} \cdot [(\varphi^\varepsilon u \cdot \nabla) u] \, dx$$

$$+ \int_{\Pi_\varepsilon} W^{\nu,\varepsilon} \cdot \varphi^\varepsilon \nabla p \, dx - \int_{\Pi_\varepsilon} W^{\nu,\varepsilon} \cdot \partial_t \psi \nabla^\perp \varphi^\varepsilon \, dx.$$

We will examine each one of the five terms in the right-hand-side of identity (8).

We look at the first term. We use Cauchy-Schwarz and Young's inequalities to obtain

$$\left| \nu \int_{\Pi_\varepsilon} \nabla W^{\nu,\varepsilon} \cdot \nabla u^\varepsilon \, dx \right| \leq \frac{\nu}{2}\Big(\|\nabla W^{\nu,\varepsilon}\|_{L^2}^2 + \|\nabla u^\varepsilon\|_{L^2}^2 \Big). \tag{9}$$

Next we look at the second and third terms together. We write

$$
\begin{aligned}
|\mathcal{I}| \equiv & \Big| -\int_{\Pi_\varepsilon} W^{\nu,\varepsilon}\cdot[(u^{\nu,\varepsilon}\cdot\nabla)u^{\nu,\varepsilon}]\,dx + \int_{\Pi_\varepsilon} W^{\nu,\varepsilon}\cdot[(\varphi^\varepsilon u\cdot\nabla)u]\,dx\Big| \\
= & \Big|\int_{\Pi_\varepsilon}\big\{W^{\nu,\varepsilon}\cdot[u^\varepsilon\cdot\nabla(u-u^\varepsilon)] - W^{\nu,\varepsilon}\cdot(W^{\nu,\varepsilon}\cdot\nabla u^\varepsilon) \\
& \qquad\qquad + W^{\nu,\varepsilon}\cdot[((\varphi^\varepsilon u-u^\varepsilon)\cdot\nabla)u]\big\}\,dx\Big| \\
\le & \Big|\int_{\Pi_\varepsilon} W^{\nu,\varepsilon}\cdot[u^\varepsilon\cdot\nabla(u-u^\varepsilon)]\,dx\Big| + \Big|\int_{\Pi_\varepsilon}(W^{\nu,\varepsilon}\cdot\nabla u^\varepsilon)\cdot W^{\nu,\varepsilon}\,dx\Big| \\
& \qquad\qquad + \Big|\int_{\Pi_\varepsilon} W^{\nu,\varepsilon}\cdot[((\varphi^\varepsilon u-u^\varepsilon)\cdot\nabla)u]\,dx\Big| \\
= & \Big|\int_{\Pi_\varepsilon}(u-u^\varepsilon)\cdot[(u^\varepsilon\cdot\nabla)W^{\nu,\varepsilon}]\,dx\Big| + \Big|\int_{\Pi_\varepsilon}(W^{\nu,\varepsilon}\cdot\nabla u^\varepsilon)\cdot W^{\nu,\varepsilon}\,dx\Big| \\
& \qquad\qquad + \Big|\int_{\Pi_\varepsilon} W^{\nu,\varepsilon}\cdot[((\varphi^\varepsilon u-u^\varepsilon)\cdot\nabla)u]\,dx\Big| \\
\le & \|u^\varepsilon\|_{L^\infty}\|u-u^\varepsilon\|_{L^2}\|\nabla W^{\nu,\varepsilon}\|_{L^2} + \|W^{\nu,\varepsilon}\|_{L^2}\|\varphi^\varepsilon u-u^\varepsilon\|_{L^2}\|\nabla u\|_{L^\infty} \\
& \qquad\qquad + \Big|\int_{\Pi_\varepsilon}(W^{\nu,\varepsilon}\cdot\nabla u^\varepsilon)\cdot W^{\nu,\varepsilon}\,dx\Big|.
\end{aligned}
$$

Fix $i=1,2$ and note that

$$
\partial_i u^\varepsilon = \partial_i\psi\nabla^\perp\varphi^\varepsilon + \psi\partial_i\nabla^\perp\varphi^\varepsilon + \partial_i\varphi^\varepsilon u + \varphi^\varepsilon\partial_i u. \tag{10}
$$

Therefore,

$$
\begin{aligned}
|\mathcal{I}| \le & \|u^\varepsilon\|_{L^\infty}\|u-u^\varepsilon\|_{L^2}\|\nabla W^{\nu,\varepsilon}\|_{L^2} + \|W^{\nu,\varepsilon}\|_{L^2}\|\varphi^\varepsilon u-u^\varepsilon\|_{L^2}\|\nabla u\|_{L^\infty} \\
& + (\|\nabla\psi\nabla\varphi^\varepsilon\|_{L^\infty} + \|\psi\nabla^2\varphi^\varepsilon\|_{L^\infty})\|W^{\nu,\varepsilon}\|^2_{L^2(A_\varepsilon)} + \|\varphi^\varepsilon\nabla u\|_{L^\infty}\|W^{\nu,\varepsilon}\|^2_{L^2},
\end{aligned} \tag{11}
$$

where A_ε is the set $\Pi_\varepsilon\cap B_{(R+2)\varepsilon}$, which contains the support of $\nabla\varphi^\varepsilon$.

Next we look at the fourth and fifth terms. Recall that we choose the pressure p in such a way that $p(0,t)=0$. We find

$$
\begin{aligned}
|\mathcal{J}| \equiv & \Big|\int_{\Pi_\varepsilon} W^{\nu,\varepsilon}\cdot\varphi^\varepsilon\nabla p\,dx - \int_{\Pi_\varepsilon} W^{\nu,\varepsilon}\cdot\partial_t\psi\nabla^\perp\varphi^\varepsilon\,dx\Big| \\
\le & \Big|\int_{\Pi_\varepsilon} W^{\nu,\varepsilon}\cdot p\nabla\varphi^\varepsilon\,dx\Big| + \Big|\int_{\Pi_\varepsilon} W^{\nu,\varepsilon}\cdot\partial_t\psi\nabla^\perp\varphi^\varepsilon\,dx\Big|.
\end{aligned}
$$

We estimate each term above to obtain

$$
|\mathcal{J}| \le (\|p\nabla\varphi^\varepsilon\|_{L^2} + \|\partial_t\psi\nabla^\perp\varphi^\varepsilon\|_{L^2})\|W^{\nu,\varepsilon}\|_{L^2}. \tag{12}
$$

We use estimates (9), (11) and (12) in the energy identity (8), together with the estimates obtained in Lemmas 3 and 4 to deduce that

$$\frac{1}{2}\frac{d}{dt}\|W^{\nu,\varepsilon}\|_{L^2}^2+\nu\|\nabla W^{\nu,\varepsilon}\|_{L^2}^2\leq\frac{\nu}{2}\|\nabla W^{\nu,\varepsilon}\|_{L^2}^2+\frac{\nu}{2}K_1+K_2K_3\varepsilon\|\nabla W^{\nu,\varepsilon}\|_{L^2}$$

$$+\frac{K_4}{\varepsilon}K_6^2\varepsilon^2\|\nabla W^{\nu,\varepsilon}\|_{L^2}^2+\|\varphi^\varepsilon\nabla u\|_{L^\infty}\|W^{\nu,\varepsilon}\|_{L^2}^2+\varepsilon(K_5+K_3\|\nabla u\|_{L^\infty})\|W^{\nu,\varepsilon}\|_{L^2}$$

$$\leq\frac{\nu}{2}\|\nabla W^{\nu,\varepsilon}\|_{L^2}^2+\frac{\nu}{2}K_1+\frac{\nu}{4}\|\nabla W^{\nu,\varepsilon}\|_{L^2}^2+K_2^2K_3^2\frac{\varepsilon^2}{\nu}+K_4K_6^2\varepsilon\|\nabla W^{\nu,\varepsilon}\|_{L^2}^2$$

$$+K_0\|W^{\nu,\varepsilon}\|_{L^2}^2+\frac{\|W^{\nu,\varepsilon}\|_{L^2}^2}{2}+\frac{\widetilde{K}_5^2\varepsilon^2}{2}.$$

Above we have used the notation $K_0=\sup_\varepsilon\|\varphi^\varepsilon\nabla u\|_{L^\infty}$ and $\widetilde{K}_5^2=(K_5+K_3\|\nabla u\|_{L^\infty})^2$.

At this point we choose ε so that

$$0<\varepsilon<\min\left\{\varepsilon_0\,,\,\frac{\nu}{8K_4\,K_6^2}\right\}.$$

With this choice, letting $y=y(t)=\|W^{\nu,\varepsilon}\|_{L^2}^2$, we obtain

$$\frac{dy}{dt}\leq\nu K_1+2K_2^2K_3^2\frac{\varepsilon^2}{\nu}+\widetilde{K}_5^2\varepsilon^2+(2K_0+1)y\leq C_1'\nu+C_2'y. \tag{13}$$

The stated result follows from this estimate through Gronwall's inequality in standard fashion together with the estimate on the initial data $W^{\nu,\varepsilon}(x,0)$ from Lemma 2 and item (3) of Lemma 3. This concludes the proof in the bidimensional case.

4.2. **Case $n=3$.** The proof in dimension three is similar to the previous one. There are two differences: notation and the justification that we can multiply the equation of $W^{\nu,\varepsilon}$ by $W^{\nu,\varepsilon}$.

First, about notation. One has to replace everywhere the term $\partial_t\psi\nabla^\perp\varphi^\varepsilon$ by $\nabla\varphi^\varepsilon\times\partial_t\psi$ and also relation (10) becomes

$$\partial_i u^\varepsilon=\nabla\varphi^\varepsilon\times\partial_i\psi+\partial_i\nabla\varphi^\varepsilon\times\psi+\partial_i\varphi^\varepsilon u+\varphi^\varepsilon\partial_i u.$$

These two modifications are just changes of notations. These new terms are of the same type as the old ones, so the estimates that follow are not affected.

Second, we multiplied the equation of $W^{\nu,\varepsilon}$ by $W^{\nu,\varepsilon}$. The solution $u^{\nu,\varepsilon}$, and therefore $W^{\nu,\varepsilon}$ too, is not better than $L^\infty(0,T;L^2(\Pi_\varepsilon))\cap L^2(0,T;H^1(\Pi_\varepsilon))$. But it is well-known that some of the trilinear terms that appear when multiplying the equation of $W^{\nu,\varepsilon}$ by $W^{\nu,\varepsilon}$ are not well defined in dimension three with this regularity only. In other words, one cannot multiply directly the equation

of $W^{\nu,\varepsilon}$ by $W^{\nu,\varepsilon}$. Nevertheless, there is a classical trick that allow to do this multiplication if the the weak solution $u^{\nu,\varepsilon}$ verifies the energy inequality. What we would like to do, is to subtract the equation of u^ε from the equation of $u^{\nu,\varepsilon}$ and to multiply the result by $u^{\nu,\varepsilon} - u^\varepsilon$. This is the same as multiplying the equation of $u^{\nu,\varepsilon}$ by $u^{\nu,\varepsilon}$, adding the equation of u^ε times u^ε and subtracting the equation of $u^{\nu,\varepsilon}$ times u^ε and the equation of u^ε times $u^{\nu,\varepsilon}$. Since u^ε is smooth, all these operations are legitimate except for the multiplication of the equation of $u^{\nu,\varepsilon}$ by $u^{\nu,\varepsilon}$. Formally, multiplying the equation of $u^{\nu,\varepsilon}$ by $u^{\nu,\varepsilon}$ and integrating in space and time from 0 to t yields the energy equality, *i.e.* relation (2) where the sign $\leq$ is replaced by $=$. Since we assumed that the energy inequality holds true, the above operations are justified provided that the relation we get at the end is an inequality instead of an equality. But an inequality is of course sufficient for our purposes. Finally, to be completely rigorous, one has to integrate in time from the begining. That is, we would obtain at the end relation (13) integrated in time. Clearly, the result of the application of the Gronwall lemma in (13) is the same as in (13) integrated in time. This completes the proof in dimension three.

5. Comments and conclusions

Let us assume that $\varepsilon < C_1\nu$, so that we are in the context of Theorem 1. According to the physical understanding of incompressible flow past a bluff body at high Reynolds number, the flow $u^{\nu,\varepsilon}$ is a turbulent perturbation of the smooth background flow u and the difference $u^{\nu,\varepsilon} - u$ is called the *wake* of the obstacle. The turbulence is caused by vorticity shed by the obstacle through boundary layer separation. The main difficulty in studying the vanishing viscosity limit in the presence of boundaries is the fact that, although the Navier-Stokes equations do have a vorticity form, valid in the bulk of the fluid, the vorticity equation does not satisfy a useful boundary condition, so that we cannot control the amount of vorticity added to the flow by the boundary layer. In the proof of Theorem 1, we found a way of controlling the kinetic energy of the wake without making explicit reference to the vorticity. At this point, it is reasonable to ask whether we can control the vorticity content of the wake as well. To answer that, we introduce the *enstrophy* $\Omega^{\nu,\varepsilon}(t)$ of the

flow:

$$\Omega^{\nu,\varepsilon}(t) \equiv \frac{1}{2}\int_{\Pi_\varepsilon} |\operatorname{curl} u^{\nu,\varepsilon}|^2\, dx.$$

Of course, enstrophy measures how much vorticity is in the flow, but its behavior as $\nu \to 0$ is also important to understand the statistical structure of the turbulent wake.

Corollary 5. *For any $T > 0$ there exists a constant $C > 0$, independent of ν such that*

$$\int_0^T \Omega^{\nu,\varepsilon}(t)\, dt \le C.$$

Proof. We go back to relation (13) and include the viscosity term that was ignored there:

$$\frac{dy}{dt} + \frac{\nu}{4}\|\nabla W^{\nu,\varepsilon}\|_{L^2}^2 \le C_1'\nu + C_2' y.$$

We next integrate in time to obtain

$$\|W^{\nu,\varepsilon}(\cdot,T)\|_{L^2}^2 - \|W^{\nu,\varepsilon}(\cdot,0)\|_{L^2}^2 + \frac{\nu}{4}\int_0^T \|\nabla W^{\nu,\varepsilon}(\cdot,t)\|_{L^2}^2\, dt$$

$$\le C_1' T\nu + C_2' \int_0^T \|W^{\nu,\varepsilon}(\cdot,t)\|_{L^2}^2\, dt.$$

Now we use Theorem 1 and ignore a term with good sign to obtain

$$\frac{\nu}{4}\int_0^T \|\nabla W^{\nu,\varepsilon}(\cdot,t)\|_{L^2}^2\, dt \le CT\nu + \|W^{\nu,\varepsilon}(\cdot,0)\|_{L^2}^2 \le C'T\nu,$$

where we used Lemma 2 together with item (3) from Lemma 3 to estimate the initial data term. From this we conclude that

$$\int_0^T \|\nabla W^{\nu,\varepsilon}(\cdot,t)\|_{L^2}^2\, dt \le C.$$

Finally, we observe that

$$\Omega^{\nu,\varepsilon} \le C\|\nabla W^{\nu,\varepsilon}\|_{L^2}^2 + C\|\nabla u^\varepsilon\|_{L^2}^2 \le C\|\nabla W^{\nu,\varepsilon}\|_{L^2}^2 + CK_1,$$

by item (1) in Lemma 3. This concludes the proof.

□

Acknowledgments: This work was done during the Special Semester in Fluid Mechanics at the Centre Interfacultaire Bernoulli, EPFL; the authors wish to express their gratitude for the hospitality received.

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