REMARKS ON THE VANISHING OBSTACLE LIMIT FOR A 3D VISCOUS INCOMPRESSIBLE FLUID

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Abstract. In [4] the authors considered the bidimensional Navier-Stokes equations in the exterior of an obstacle shrinking to a point and determined the limit velocity. Here we consider the same problem in the three-dimensional case. Assuming that the initial vorticity is smooth, compactly supported and independent of the shrinking obstacle, we prove that the limit velocity is a solution of the Navier-Stokes equations in the full space with the same initial vorticity.

1. Introduction

The investigation of small obstacle limits in an incompressible fluid was initiated in [3]. In that paper, the authors consider the Euler equations in the exterior of a bidimensional obstacle that shrinks homothetically to a point. It is also assumed that the initial vorticity is smooth, compactly supported, independent of the obstacle and that the circulation of the velocity on the boundary of the obstacle is also independent of the size of the obstacle. It is then proved in [3] that the limit velocity is a solution of a PDE that looks like the Euler equation that embeds the Dirac mass of the point the obstacle shrinks to. The initial velocity has a vorticity that also acquires a Dirac mass of this point. The case of several obstacles was treated in [5] and the viscous case was done in [4]. It is proved in [4] that in the case of two-dimensional Navier-Stokes equations, the limit equation is also Navier-Stokes but there is still formation of an additional Dirac mass in the limit vorticity. This is due to the fact that the circulation of the velocity on the boundary of the obstacle is not vanishing. In dimension three, there is no circulation of velocity on the boundary. The aim of this paper is to prove that in the three-dimensional case and for the Navier-Stokes equations, the limit equation is also the Navier-Stokes equation in the full space and that the initial vorticity of the limit velocity is simply the initial vorticity that we prescribe for obstacle-dependent problem. We will also be able to consider more general obstacles as in [4]. Instead of assuming that the obstacle homothetically shrinks to a point, it is sufficient to assume that obstacle is between two balls homothetically shrinking to a point.

More precisely, let $\Pi_\varepsilon = \mathbb{R}^3 \backslash \overline{\Omega}_\varepsilon$ be a smooth, simply connected exterior domain such that there exists the constants $C_1 < C_2$ independent of ε such that $B(0, C_1\varepsilon) \subset \Omega_\varepsilon \subset B(0, C_2\varepsilon)$. We assume that the initial vorticity ω_0 is independent of ε, smooth, divergence free and compactly supported in Π_ε. Let $G_\varepsilon(x, y)$ be the Green function of the domain Π_ε. Since the domain Π_ε is simply connected, we know that there exists a unique velocity u_0^ε associated to the vorticity ω_0, see for example [2]. This velocity is given by the formula

$$u_0^\varepsilon(x) = \int_{\Pi_\varepsilon} \nabla_x G_\varepsilon(x,y) \times \omega_0(y)\, dy. \tag{1}$$

We denote by u_0 the velocity defined on $\mathbb{R}^3$ which is associated to the vorticity ω_0, *i.e.*

$$u_0(x) = -\int_{\Pi_\varepsilon} \frac{x-y}{4\pi|x-y|^3} \times \omega_0(y)\, dy \tag{2}$$

Let u^ε be a weak Leray solution of the Navier-Stokes equations in Π_ε with initial velocity u_0^ε and homogeneous Dirichlet boundary conditions:

$$\begin{cases} \partial_t u^\varepsilon - \nu\triangle u^\varepsilon + u^\varepsilon\cdot\nabla u^\varepsilon = -\nabla p^\varepsilon & \text{in } \Pi_\varepsilon\times(0,\infty),\\ \operatorname{div} u^\varepsilon = 0 & \text{in } \Pi_\varepsilon\times[0,\infty),\\ u^\varepsilon = 0 & \text{on } \partial\Pi_\varepsilon,\\ u^\varepsilon(0,\cdot) = u_0^\varepsilon & \text{in } \Pi_\varepsilon. \end{cases} \tag{3}$$

We also assume that the velocity vanishes at infinity. The aim of this paper is to prove the following theorem.

Theorem 1. *Let ω_0 be a $C^\infty(\mathbb{R}^3)$ divergence free vector field compactly supported in $\mathbb{R}^3\setminus\{0\}$. Let us construct u_0^ε and u_0 as in relations* (1) *and* (2)*. Consider u^ε a weak Leray solution of the Navier-Stokes equations on Π_ε with initial velocity u_0^ε and denote by $\widetilde{u}_\varepsilon$ the extension to $\mathbb{R}^3$ with values 0 on $\overline{\Omega}_\varepsilon$. There exists a sub-sequence of $\widetilde{u}_\varepsilon$ that converges strongly in $L^2_{loc}([0,\infty)\times\mathbb{R}^3)$ to a weak Leray solution of the Navier-Stokes equations in $\mathbb{R}^3$ with initial velocity u_0.*

The proof of this result consists of two parts. We prove first that u_0^ε converges to u_0 strongly in L^2, see Theorem 5 below. We then conclude by showing in Theorem 6 that strong convergence in L^2 for the initial data implies convergence of solutions in the vanishing obstacle limit.

2. Notations and preliminary results

If f is a function defined on Π_ε, we denote by $\widetilde{f}$ the function defined on $\mathbb{R}^3$ which vanishes on $\overline{\Omega}_\varepsilon$ and equals f on Π_ε. If f is regular enough and vanishes on $\partial\Omega_\varepsilon$, then one has that $\nabla\widetilde{f} = \widetilde{\nabla f}$ in $\mathbb{R}^3$. If v is a regular enough vector field defined on Π_ε and tangent to $\partial\Omega_\varepsilon$, then one also has that $\operatorname{div}\widetilde{v} = \widetilde{\operatorname{div} v}$ in $\mathbb{R}^3$. In particular, we have that $\operatorname{div}\widetilde{u}_0^\varepsilon = 0$ in $\mathbb{R}^3$.

Definition 2. *We say that u^ε is a weak Leray solution of* (3) *if*

$$u^\varepsilon \in C^0_w\big([0,\infty);L^2(\Pi_\varepsilon)\big)\cap L^\infty\big([0,\infty);L^2(\Pi_\varepsilon)\big)\cap L^2_{loc}\big([0,\infty);H^1_0(\Pi_\varepsilon)\big)$$

is divergence free, verifies the equation in the sense of distributions, i.e.

$$-\int_0^\infty\int_{\Pi_\varepsilon} u^\varepsilon\cdot\partial_t\varphi + \nu\int_0^\infty\int_{\Pi_\varepsilon}\nabla u^\varepsilon\cdot\nabla\varphi + \int_0^\infty\int_{\Pi_\varepsilon} u^\varepsilon\cdot\nabla u^\varepsilon\cdot\varphi = \int_{\Pi_\varepsilon} u^\varepsilon(0)\cdot\varphi(0) \tag{4}$$

for every divergence free vector field $\varphi \in C_0^\infty([0,\infty)\times\Pi_\varepsilon)$, and moreover u^ε verifies the following energy inequality:

$$\|u^\varepsilon(t)\|^2_{L^2(\Pi_\varepsilon)} + 2\int_0^t \|\nabla u^\varepsilon\|^2_{L^2(\Pi_\varepsilon)} \le \|u^\varepsilon(0)\|^2_{L^2(\Pi_\varepsilon)} \qquad \forall t\ge 0. \tag{5}$$

We will use a similar definition for weak Leray solutions on $\mathbb{R}^3$.

For a divergence free vector field $\varphi\in C_0^\infty(\mathbb{R}^3)$ we define a stream function $\psi = T\varphi$ by

$$\psi(x) = (T\varphi)(x) = -\int_{\mathbb{R}^3} \frac{x-y}{4\pi|x-y|^3}\times\varphi(y)\,dy - \int_{\mathbb{R}^3}\frac{y}{4\pi|y|^3}\times\varphi(y)\,dy.$$

Clearly ψ is divergence free, vanishes in 0 and $\operatorname{curl}\psi=\varphi$. The operator T is bounded from $L^1\cap L^\infty$ to L^∞.

We will use in Section 4 the following approximation of smooth compactly supported divergence free vector fields. Let φ be as above and $\eta\in C^\infty(\mathbb{R}^3)$ be such that $\eta\equiv 0$ on $B(0,C_2)$ and $\eta\equiv 1$ on $\mathbb{R}^3\setminus B(0,2C_2)$. We define $\eta_\varepsilon(x)=\eta(x/\varepsilon)$ and $\varphi_\varepsilon = \operatorname{curl}(\eta_\varepsilon\varphi)$. The vector field φ_ε is smooth, compactly supported, divergence free and vanishes in a neighborhood of the obstacle $\overline{\Omega}_\varepsilon$. We collect in the following lemma several properties relating φ_ε to φ.

Lemma 3. *One has that $\varphi_\varepsilon\to\varphi$ strongly in H^1 and one can decompose $\nabla\varphi_\varepsilon = \xi_\varepsilon + \Xi_\varepsilon$ with $\xi_\varepsilon \rightharpoonup \nabla\varphi$ weak$*$ in L^∞ and $\Xi_\varepsilon\to 0$ strongly in L^2. Moreover, $\operatorname{supp}\xi_\varepsilon\subset\operatorname{supp}\varphi$ for all ε.*

Remark 4. *It will be clear from the proof below that we can allow a time dependence in φ. The results of this lemma will then hold true uniformly with respect to the time variable.*

Proof. We observe first from the explicit expression for η_ε that $\eta_\varepsilon - 1$ and $\nabla\eta_\varepsilon$ converge to 0 in L^2 and $\|\nabla^2\eta_\varepsilon\|_{L^2(\mathbb{R}^3)} = \varepsilon^{-\frac12}\|\nabla^2\eta\|_{L^2(\mathbb{R}^3)}$.

Since $\varphi_\varepsilon = \eta_\varepsilon\varphi + \nabla\eta_\varepsilon\times\psi$, we have that

$$\begin{aligned}\|\varphi_\varepsilon-\varphi\|_{L^2(\mathbb{R}^3)} &\le \|(\eta_\varepsilon-1)\varphi\|_{L^2(\mathbb{R}^3)} + \|\nabla\eta_\varepsilon\times\psi\|_{L^2(\mathbb{R}^3)}\\ &\le \|(\eta_\varepsilon-1)\|_{L^2(\mathbb{R}^3)}\|\varphi\|_{L^\infty(\mathbb{R}^3)} + \|\nabla\eta_\varepsilon\|_{L^2(\mathbb{R}^3)}\|\psi\|_{L^\infty(\mathbb{R}^3)} \xrightarrow{\varepsilon\to 0} 0,\end{aligned}$$

and similarly

$$\begin{aligned}\|\nabla(\varphi_\varepsilon-\varphi)\|_{L^2(\mathbb{R}^3)} \le \|(\eta_\varepsilon-1)\|_{L^2(\mathbb{R}^3)}\|\varphi\|_{L^\infty(\mathbb{R}^3)} &+ C\|\nabla\eta_\varepsilon\|_{L^2(\mathbb{R}^3)}(\|\varphi\|_{L^\infty(\mathbb{R}^3)}+\|\nabla\psi\|_{L^\infty(\mathbb{R}^3)})\\ &+ C\|\nabla^2\eta_\varepsilon\|_{L^2(\mathbb{R}^3)}\|\psi\|_{L^\infty(B(0,2\varepsilon C_2))}\xrightarrow{\varepsilon\to 0} 0,\end{aligned}$$

where we used that $\psi(0)=0$ to deduce that $\|\psi\|_{L^\infty(B(0,2\varepsilon C_2))} = O(\varepsilon)$.

Next, we set $\xi_\varepsilon = \eta_\varepsilon\nabla\varphi$ and $\Xi_\varepsilon = \nabla\varphi_\varepsilon - \eta_\varepsilon\nabla\varphi$ so that $\operatorname{supp}\xi_\varepsilon\subset\operatorname{supp}\varphi$. The term Ξ_ε was already estimated above and proved to be convergent to 0 in L^2 as $\varepsilon\to 0$. The sequence ξ_ε is bounded in L^∞ and converges to $\nabla\varphi$ in L^2. By uniqueness of limits in the sense of distributions, the limit of every sub-sequence of ξ_ε weak$*$ convergent in L^∞ must necessarily be $\nabla\varphi$. Since all weak limits are $\nabla\varphi$, we deduce that the whole sequence must converge, *i.e.* $\xi_\varepsilon\rightharpoonup\nabla\varphi$ weak$*$ in L^∞. This completes the proof of the lemma. □

3. An estimate for the initial velocity

We prove in this section the following convergence result for the initial velocities.

Theorem 5. *One has that $\widetilde{u}_0^\varepsilon, u_0 \in L^2(\mathbb{R}^3)$ and $\widetilde{u}_0^\varepsilon \to u_0$ strongly in $L^2(\mathbb{R}^3)$ as $\varepsilon \to 0$. More precisely, there exists a constant C independent of ε such that $\|\widetilde{u}_0^\varepsilon - u_0\|_{L^2(\mathbb{R}^3)} \leq C\varepsilon^{\frac{1}{2}}$.*

We start by writing the Green function G_ε under the form $G_\varepsilon(x,y) = -\frac{1}{4\pi|x-y|} + \gamma_\varepsilon(x,y)$. Since $-\frac{1}{4\pi|x-y|}$ is the fundamental solution of the laplacian in $\mathbb{R}^3$, one has that the function $\gamma_\varepsilon(x,y)$ verifies the following properties:

$$\gamma_\varepsilon(x,y) = \gamma_\varepsilon(y,x) \quad \forall x,y \in \Pi_\varepsilon,$$

$$\triangle_x \gamma_\varepsilon(x,y) = \triangle_y \gamma_\varepsilon(x,y) = 0 \quad \forall x,y \in \Pi_\varepsilon,$$

$$\gamma_\varepsilon(x,y) = \frac{1}{4\pi|x-y|} \quad \text{if } x \in \Pi_\varepsilon \text{ and } y \in \partial\Pi_\varepsilon \text{ or } x \in \partial\Pi_\varepsilon \text{ and } y \in \Pi_\varepsilon,$$

$$\gamma_\varepsilon(x,y) \to 0 \text{ as } |x| \to \infty \text{ or } |y| \to \infty.$$

The initial velocity can be decomposed as follows:

$$\begin{aligned} u_0^\varepsilon(x) &= \int_{\Pi_\varepsilon} \nabla_x G_\varepsilon(x,y) \times \omega_0(y)\, dy \\ &= -\int_{\Pi_\varepsilon} \frac{x-y}{4\pi|x-y|^3} \times \omega_0(y)\, dy + \int_{\Pi_\varepsilon} \nabla_x \gamma_\varepsilon(x,y) \times \omega_0(y)\, dy \\ &\equiv u_0(x) + v_\varepsilon(x). \end{aligned} \tag{6}$$

It is obvious that u_0 is smooth and can be bounded by $O(1/|x|^2)$ as $|x| \to \infty$, so $u_0 \in L^2(\mathbb{R}^3)$. It is therefore sufficient to prove that $\|u_0^\varepsilon - u_0\|_{L^2(\mathbb{R}^3)} \leq C\varepsilon^{\frac{1}{2}}$ for some constant C. Let $K = \operatorname{supp}\omega_0$. We write

$$\|u_0^\varepsilon - u_0\|_{L^2(\mathbb{R}^3)} \leq \|u_0\|_{L^2(\Omega_\varepsilon)} + \|v_\varepsilon\|_{L^2(\Pi_\varepsilon)} \leq C\varepsilon^{\frac{3}{2}} + \int_K \|\nabla_x \gamma_\varepsilon(x,y)\|_{L^2_x(\Pi_\varepsilon)} |\omega_0(y)|\, dy.$$

The rest of this proof consists in proving that the last term above is $O(\varepsilon^{\frac{1}{2}})$. In order to do that, we need some bounds on the function $\gamma_\varepsilon(x,y)$.

For fixed y, the function $x \mapsto \gamma_\varepsilon(x,y)$ is harmonic on the exterior domain Π_ε, vanishes at infinity and the trace on $\partial\Pi_\varepsilon$ is known. To bound this function we can use the Kelvin transform which is a well-known tool to treat exterior domain problems for harmonic functions. More precisely, it can be proved that the function $z \mapsto \frac{1}{|z|}\gamma_\varepsilon(\frac{\varepsilon^2 z}{|z|^2}, y)$ is harmonic on its domain of definition and can be extended smoothly up to $z = 0$, see for example [1, Chapter 4]. Let Ω_ε^* be the image of Π_ε by the mapping $x \mapsto \frac{\varepsilon^2 x}{|x|^2}$ to which we add the point $\{0\}$. Then Ω_ε^* is a smooth open set such that $B(0, \varepsilon/C_2) \subset \Omega_\varepsilon^* \subset B(0, \varepsilon/C_1)$. For $z \in \Omega_\varepsilon^* \setminus \{0\}$, we denote $h_y^\varepsilon(z) = \frac{1}{|z|}\gamma_\varepsilon(\frac{\varepsilon^2 z}{|z|^2}, y)$ and we extend it smoothly to $z = 0$. Then

$$\begin{cases} \triangle_z h_y^\varepsilon = 0 & \text{in } \Omega_\varepsilon^*, \\ h_y^\varepsilon(z) = \frac{1}{4\pi|z|\left|\frac{\varepsilon^2 z}{|z|^2} - y\right|} & \text{for } z \in \partial\Omega_\varepsilon^*. \end{cases} \tag{7}$$

Clearly, for $z \in \partial\Omega_\varepsilon^*$ and $y \in K$ one has that $\frac{\varepsilon^2 z}{|z|^2} \in \partial\Omega_\varepsilon$ so $|\frac{\varepsilon^2 z}{|z|^2} - y| \geq d(y, \partial\Omega_\varepsilon) \geq C(K)$ for ε small enough (here $C(K)$ is a constant depending solely on the compact K). Since we also have that $\partial\Omega_\varepsilon^* \subset B(0, \varepsilon/C_1) \setminus B(0, \varepsilon/C_2)$, we deduce that the boundary data in (7) is bounded by C/ε with C a constant independent of ε (in fact, it is exactly of order $O(1/\varepsilon)$). By the maximum principle, we infer that

$$|h_y^\varepsilon(z)| \leq \frac{C}{\varepsilon} \qquad \forall z \in \Omega_\varepsilon^*, y \in K. \tag{8}$$

We observe next that $\gamma_\varepsilon(x,y) = \frac{\varepsilon^2}{|x|} h_y^\varepsilon(\frac{\varepsilon^2 x}{|x|^2})$. An easy computation shows that

$$|\nabla_x \gamma_\varepsilon(x,y)| \leq \frac{\varepsilon^2}{|x|^2} \Big|h_y^\varepsilon\Big(\frac{\varepsilon^2 x}{|x|^2}\Big)\Big| + C\frac{\varepsilon^4}{|x|^3}\Big|\nabla_z h_y^\varepsilon\Big(\frac{\varepsilon^2 x}{|x|^2}\Big)\Big| \leq C\frac{\varepsilon}{|x|^2} + C\frac{\varepsilon^4}{|x|^3}\Big|\nabla_z h_y^\varepsilon\Big(\frac{\varepsilon^2 x}{|x|^2}\Big)\Big|, \tag{9}$$

for all $z \in \Omega_\varepsilon^*$ and $y \in K$. We used above the bound (8). The first term on the right-hand side is bounded in $L^2(\Pi_\varepsilon)$ by $O(\varepsilon^{\frac{1}{2}})$. Indeed, one can write

$$\Big\|\frac{\varepsilon}{|x|^2}\Big\|_{L^2(\Pi_\varepsilon)} \leq \varepsilon\Big(\int_{B(0,C_1\varepsilon)^c} \frac{1}{|x|^4}\,dx\Big)^{\frac{1}{2}} = C\varepsilon\Big(\int_{C_1\varepsilon}^\infty \frac{1}{r^2}\,dr\Big)^{\frac{1}{2}} = C'\varepsilon^{\frac{1}{2}}. \tag{10}$$

It remains to bound the $L^2(\Pi_\varepsilon)$ norm of the last term in (9). Since the Jacobian of the application $z \mapsto \frac{\varepsilon^2 z}{|z|^2}$ is bounded by $C\frac{\varepsilon^6}{|z|^6}$, we obtain from (9) and (10) after making the change of variables $x = \frac{\varepsilon^2 z}{|z|^2}$ that

$$\|\nabla_x \gamma_\varepsilon(x,y)\|_{L^2(\Pi_\varepsilon)} \leq C\varepsilon^{\frac{1}{2}} + C\varepsilon\|\nabla_z h_y^\varepsilon\|_{L^2(\Omega_\varepsilon^*)} \tag{11}$$

Let $g \in C_0^\infty(\mathbb{R}^3)$ be such that $g \equiv 1$ on $\Omega_1^* \setminus B(0, \frac{1}{C_2})$ and $g \equiv 0$ on $B(0, \frac{1}{C_1})^c \cup B(0, \frac{1}{2C_2})$. Let us introduce the function

$$H_\varepsilon(z) = \frac{g(\frac{z}{\varepsilon})}{4\pi|z|\Big|\frac{\varepsilon^2 z}{|z|^2} - y\Big|}, \qquad z \in \mathbb{R}^3.$$

We observe that $H_\varepsilon \in C_0^\infty(\mathbb{R}^3)$, that $\operatorname{supp} H_\varepsilon \subset \{\frac{\varepsilon}{2C_2} \leq |z| \leq \frac{\varepsilon}{C_1}\}$ and $H_\varepsilon = h_y^\varepsilon$ on $\partial\Omega_\varepsilon^*$. Therefore, for $z \in \operatorname{supp} H_\varepsilon$ and $y \in K$, one has that $C_1\varepsilon \leq \Big|\frac{\varepsilon^2 z}{|z|^2}\Big| \leq 2C_2\varepsilon$ so that $\Big|\frac{\varepsilon^2 z}{|z|^2} - y\Big| \geq d(y, B(0, 2C_2\varepsilon)) \geq C(K)$. Since $|z| \simeq \varepsilon$ for $z \in \operatorname{supp} H_\varepsilon$, we obtain after a few calculations that there exists a constant C independent of ε (but depending on K) such that

$$\|H_\varepsilon\|_{L^\infty(\mathbb{R}^3)} \leq \frac{C}{\varepsilon}, \qquad \|\nabla H_\varepsilon\|_{L^\infty(\mathbb{R}^3)} \leq \frac{C}{\varepsilon^2}, \qquad \|\triangle H_\varepsilon\|_{L^\infty(\mathbb{R}^3)} \leq \frac{C}{\varepsilon^3}. \tag{12}$$

Since $h_y^\varepsilon - H_\varepsilon$ vanishes on $\partial\Omega_\varepsilon^*$, one can integrate by parts and use (8), (12) to write

$$\begin{aligned}\|\nabla(h_y^\varepsilon - H_\varepsilon)\|_{L^2(\Omega_\varepsilon^*)}^2 &= -\int_{\Omega_\varepsilon^*} (h_y^\varepsilon - H_\varepsilon)\triangle(h_y^\varepsilon - H_\varepsilon) = \int_{\Omega_\varepsilon^*} (h_y^\varepsilon - H_\varepsilon)\triangle H_\varepsilon \\ &\leq (\sup_{\Omega_\varepsilon^*} |h_y^\varepsilon| + \sup_{\Omega_\varepsilon^*} |H_\varepsilon|) \sup_{\Omega_\varepsilon^*} |\triangle H_\varepsilon| \operatorname{mes}(\Omega_\varepsilon^*) \leq \frac{C}{\varepsilon}.\end{aligned}$$

We use again (12) to infer that

$$\|\nabla h_y^\varepsilon\|_{L^2(\Omega_\varepsilon^*)} \leq \|\nabla(h_y^\varepsilon - H_\varepsilon)\|_{L^2(\Omega_\varepsilon^*)} + \|\nabla H_\varepsilon\|_{L^2(\Omega_\varepsilon^*)} \leq C\varepsilon^{-\frac{1}{2}}.$$

Using this in (11) shows that $\|\nabla_x \gamma_\varepsilon(x,y)\|_{L^2(\Pi_\varepsilon)}$ is uniformly bounded by $C\varepsilon^{\frac{1}{2}}$ for $y \in K = \operatorname{supp}\omega_0$. The proof of Theorem 5 is completed.

4. Convergence of solutions

The aim of this section is to prove a general convergence result: strong convergence in L^2 for the initial data implies convergence of weak Leray solutions in the vanishing obstacle limit. Throughout this section we drop the previous assumptions on the initial vorticity and the special forms of the initial velocities u_0^ε and u_0. We will prove the following result.

Theorem 6. *Suppose that u_0^ε is divergence free, tangent to the boundary, vanishes at infinity and belongs to $L^2(\Pi_\varepsilon)$. Let $u_0 \in L^2(\mathbb{R}^3)$ be a divergence free vector field such that $\widetilde{u}_0^\varepsilon \to u_0$ strongly in $L^2(\mathbb{R}^3)$. Let u^ε be a weak Leray solution of the Navier-Stokes equations on Π_ε with initial velocity u_0^ε. There exists a sub-sequence of $\widetilde{u}_\varepsilon$ that converges strongly in $L^2_{loc}([0,\infty)\times\mathbb{R}^3)$ to a weak Leray solution of the Navier-Stokes equations in $\mathbb{R}^3$ with initial velocity u_0.*

We proceed now with the proof of this theorem. Since u^ε is a weak Leray solution and u_0^ε is bounded in $L^2(\Pi_\varepsilon)$, the energy inequality (5) implies that $\widetilde{u}_\varepsilon$ is bounded in $L^\infty(\mathbb{R}_+; L^2(\Pi_\varepsilon)) \cap L^2_{loc}([0,\infty); H^1(\Pi_\varepsilon))$. We require now some temporal estimates for u^ε.

4.1. **Temporal estimates.** Let $\varphi \in C_0^\infty(\mathbb{R}^3)$ be a divergence free test vector field. We construct φ_ε as in Section 2. Taking the product of the equation of u^ε with φ_ε and integrating in space and time from s to t yields

$$\begin{aligned}
\big|\langle u^\varepsilon(t),\varphi_\varepsilon\rangle - \langle u^\varepsilon(s),\varphi_\varepsilon\rangle\big| &= \Big|\nu\int_s^t\int_{\Pi_\varepsilon}\nabla u^\varepsilon\cdot\nabla\varphi_\varepsilon - \int_s^t\int_{\Pi_\varepsilon} u^\varepsilon\cdot\nabla u^\varepsilon\cdot\varphi_\varepsilon\Big| \\
&\leq \nu\int_s^t \|\nabla u^\varepsilon\|_{L^2(\Pi_\varepsilon)}\|\nabla\varphi_\varepsilon\|_{L^2(\mathbb{R}^3)} + \int_s^t \|u^\varepsilon\|_{L^2(\Pi_\varepsilon)}\|\nabla u^\varepsilon\|_{L^2(\Pi_\varepsilon)}\|\nabla\varphi_\varepsilon\|_{L^2(\mathbb{R}^3)} \\
&\leq C(t-s)^{\frac{1}{2}}\|\nabla\varphi_\varepsilon\|_{L^2(\mathbb{R}^3)}\|\nabla u^\varepsilon\|_{L^2(\mathbb{R}_+\times\Pi_\varepsilon)}\big(1+\|u^\varepsilon\|_{L^\infty(\mathbb{R}_+;L^2(\Pi_\varepsilon))}\big) \\
&\leq C(t-s)^{\frac{1}{2}}\|\varphi\|_{H^2(\mathbb{R}^3)}\|u_0^\varepsilon\|_{L^2(\Pi_\varepsilon)}(1+\|u_0^\varepsilon\|_{L^2(\Pi_\varepsilon)})
\end{aligned} \tag{13}$$

where we used (5) and the constant C is independent of ε, s and t. Let us define F_ε on $\mathbb{R}_+\times\mathbb{R}^3$ by means of

$$\langle F_\varepsilon(t),\varphi\rangle = \langle u^\varepsilon(t), \nabla\eta_\varepsilon\times\psi\rangle$$

Clearly F_ε is bounded in $L^\infty(\mathbb{R}_+; H^{-2}(\mathbb{R}^3))$ by $C\varepsilon^{\frac{1}{2}}$ and from (13) one has that

$$|\langle \overline{u}_\varepsilon(t) + F_\varepsilon(t) - \overline{u}_\varepsilon(s) - F_\varepsilon(s),\varphi\rangle| \leq C(t-s)^{\frac{1}{2}}\|\varphi\|_{H^2(\mathbb{R}^3)}$$

so that

$$\|\mathbb{P}[\eta_\varepsilon\widetilde{u}_\varepsilon(t) + F_\varepsilon(t) - \eta_\varepsilon\widetilde{u}_\varepsilon(s) - F_\varepsilon(s)]\|_{H^{-2}(\mathbb{R}^3)} \leq C(t-s)^{\frac{1}{2}},$$

where $\mathbb{P}$ denotes the usual Leray projector in $\mathbb{R}^3$, *i.e.* the L^2 orthogonal projection on the subspace of divergence free vector fields. We conclude that the functions $\mathbb{P}(\eta_\varepsilon\widetilde{u}_\varepsilon + F_\varepsilon)$ are equicontinuous (in time) in $C^0([0,\infty); H^{-2}(\mathbb{R}^3))$.

4.2. **Passing to the limit.** Given the bounds (5) and by the Ascoli theorem, we can extract a sub-sequence again denoted by $\widetilde{u}_\varepsilon$ such that

$$\widetilde{u}_\varepsilon \rightharpoonup u \quad \text{in } L^\infty(\mathbb{R}_+; L^2(\mathbb{R}^3)) \text{ weak}* \tag{14}$$

$$\widetilde{u}_\varepsilon \rightharpoonup u \quad \text{in } L^2_{loc}([0,\infty); H^1(\mathbb{R}^3)) \text{ weakly} \tag{15}$$

$$\mathbb{P}(\eta_\varepsilon\widetilde{u}_\varepsilon + F_\varepsilon) \to v \quad \text{in } C^0([0,\infty); H^{-3}_{loc}(\mathbb{R}^3)) \text{ strongly} \tag{16}$$

for some limit vector fields u and v

$$u \in L^\infty(\mathbb{R}_+; L^2(\mathbb{R}^3)) \cap L^2_{loc}([0,\infty); H^1(\mathbb{R}^3)), \quad v \in C^0([0,\infty); H^{-3}_{loc}(\mathbb{R}^3)).$$

We observe now that $\operatorname{div}\widetilde{u}_\varepsilon = 0$ so necessarily $\operatorname{div} u = 0$ and $\mathbb{P}(\widetilde{u}_\varepsilon) = \widetilde{u}_\varepsilon$. Therefore $\mathbb{P}(\eta_\varepsilon\widetilde{u}_\varepsilon + F_\varepsilon) = \widetilde{u}_\varepsilon + \mathbb{P}[(\eta_\varepsilon - 1)\widetilde{u}_\varepsilon + F_\varepsilon]$. Since $F_\varepsilon \to 0$ strongly in $L^\infty(\mathbb{R}_+; H^{-2}(\mathbb{R}^3))$, we infer from (16) that

$$\widetilde{u}_\varepsilon + \mathbb{P}[(\eta_\varepsilon - 1)\widetilde{u}_\varepsilon] \to v \quad \text{in } C^0([0,\infty); H^{-3}_{loc}(\mathbb{R}^3)) \text{ strongly.} \tag{17}$$

But we know that $\mathbb{P}$ is bounded in any H^s, $s \in \mathbb{R}$, so

$$\begin{aligned}\|\mathbb{P}[(\eta_\varepsilon - 1)\widetilde{u}_\varepsilon]\|_{H^{-3}} &\leq C\|\mathbb{P}[(\eta_\varepsilon - 1)\widetilde{u}_\varepsilon]\|_{L^1(\mathbb{R}^3)} \leq C\|(\eta_\varepsilon - 1)\widetilde{u}_\varepsilon\|_{L^2(\mathbb{R}^3)} \\ &\leq C\|\eta_\varepsilon - 1\|_{L^2(\mathbb{R}^3)}\|\widetilde{u}_\varepsilon\|_{L^2(\mathbb{R}^3)} \leq C\|\eta_\varepsilon - 1\|_{L^2(\mathbb{R}^3)}\|u_0^\varepsilon\|_{L^2(\mathbb{R}^3)} \longrightarrow 0,\end{aligned}$$

uniformly with respect to t. We infer from (17) that

$$\widetilde{u}_\varepsilon \to v \quad \text{in } L^\infty_{loc}([0,\infty); H^{-3}_{loc}(\mathbb{R}^3)) \text{ strongly.}$$

Next, using that $\widetilde{u}_\varepsilon$ is bounded in $L^2_{loc}([0,\infty); H^1(\mathbb{R}^3))$ and the interpolation inequality $\|\widetilde{u}_\varepsilon\|_{L^2(W)} \leq \|\widetilde{u}_\varepsilon\|^{\frac{1}{4}}_{H^{-3}(W)}\|\widetilde{u}_\varepsilon\|^{\frac{3}{4}}_{H^1(W)}$ that holds true for every bounded open set W, we conclude that $\widetilde{u}_\varepsilon \to v$ strongly in $L^2_{loc}(\mathbb{R}_+ \times \mathbb{R}^3)$. By uniqueness of limits in the sense of distributions, we infer that $u = v$ and therefore

$$\widetilde{u}_\varepsilon \to u \quad \text{in } L^2_{loc}(\mathbb{R}_+ \times \mathbb{R}^3) \text{ strongly.} \tag{18}$$

With these informations, it is easy to pass to the limit in the equation of u^ε and obtain that u is a weak solution of the Navier-Stokes equations in $\mathbb{R}^3$. Indeed, let $\varphi \in C_0^\infty([0,\infty) \times \mathbb{R}^3)$ be a divergence free test vector field and define φ_ε as in Section 2. Relation 4 with φ_ε instead of φ gives

$$-\int_0^\infty \int_{\mathbb{R}^3} \widetilde{u}_\varepsilon \cdot \partial_t\varphi_\varepsilon + \nu \int_0^\infty \int_{\mathbb{R}^3} \nabla\widetilde{u}_\varepsilon \cdot \nabla\varphi_\varepsilon - \int_0^\infty \int_{\mathbb{R}^3} (\widetilde{u}_\varepsilon \otimes \widetilde{u}_\varepsilon) \cdot \nabla\varphi_\varepsilon = \int_{\mathbb{R}^3} \widetilde{u}_\varepsilon(0) \cdot \varphi_\varepsilon(0). \tag{19}$$

From Lemma 3 we know that $\partial_t\varphi_\varepsilon \to \partial_t\varphi$ strongly in $L^1(\mathbb{R}_+; L^2(\mathbb{R}^3))$, that $\varphi_\varepsilon(0) \to \varphi(0)$ strongly in $L^1(\mathbb{R}_+; L^2(\mathbb{R}^3))$ and that $\nabla\varphi_\varepsilon \to \nabla\varphi$ strongly in $L^2(\mathbb{R}_+ \times \mathbb{R}^3)$. Given (14), (15) and the convergence of , we deduce that the right-hand side and the first two terms on the left-hand side of (19) converge to the expected limit. Using the decomposition $\nabla\varphi_\varepsilon = \xi_\varepsilon + \Xi_\varepsilon$ given in Lemma 3, we write

$$\int_0^\infty \int_{\mathbb{R}^3} (\widetilde{u}_\varepsilon \otimes \widetilde{u}_\varepsilon) \cdot \nabla\varphi_\varepsilon = \int_0^\infty \int_{\mathbb{R}^3} (\widetilde{u}_\varepsilon \otimes \widetilde{u}_\varepsilon) \cdot \xi_\varepsilon + \int_0^\infty \int_{\mathbb{R}^3} (\widetilde{u}_\varepsilon \otimes \widetilde{u}_\varepsilon) \cdot \Xi_\varepsilon.$$

Given (18) and that $\xi_\varepsilon \rightharpoonup \nabla\varphi$ weak$*$ in $L^\infty(\mathbb{R}_+ \times \mathbb{R}^3)$ with supports included into a compact independent of ε, one has that

$$\int_0^\infty \int_{\mathbb{R}^3} (\widetilde{u}_\varepsilon \otimes \widetilde{u}_\varepsilon) \cdot \xi_\varepsilon \overset{\varepsilon\to 0}{\longrightarrow} \int_0^\infty \int_{\mathbb{R}^3} (u \otimes u) \cdot \nabla\varphi.$$

Next, we use the Sobolev embedding $H^1 \subset L^6$ and a Hölder inequality to write

$$\begin{aligned}\Big|\int_0^\infty \int_{\mathbb{R}^3} (\widetilde{u}_\varepsilon \otimes \widetilde{u}_\varepsilon) \cdot \Xi_\varepsilon\Big| &\le C \int_0^M \|\widetilde{u}_\varepsilon\|^2_{L^6(\mathbb{R}^3)} \|\Xi_\varepsilon\|_{L^{\frac{3}{2}}(\mathbb{R}^3)} \\ &\le C\|\widetilde{u}_\varepsilon\|^2_{L^2((0,M);H^1)} \|\Xi_\varepsilon\|^2_{L^\infty((0,M);L^{\frac{3}{2}})} \overset{\varepsilon\to 0}{\longrightarrow} 0,\end{aligned}$$

where M is such that $\operatorname{supp}\varphi \subset [0, M] \times \mathbb{R}^3$.

We conclude that sending $\varepsilon \to 0$ in (19) results in

$$-\int_0^\infty \int_{\mathbb{R}^3} u \cdot \partial_t\varphi + \nu \int_0^\infty \int_{\mathbb{R}^3} \nabla u \cdot \nabla\varphi - \int_0^\infty \int_{\mathbb{R}^3} (u \otimes u) \cdot \nabla\varphi = \int_{\mathbb{R}^3} u(0) \cdot \varphi(0),$$

which is the weak formulation of the Navier-Stokes equations in $\mathbb{R}^3$. To finish the proof of Theorem 6, it remains to prove that the solution u verifies the energy inequality. This is done using the following classical liminf argument. We apply the $\liminf_{\varepsilon\to 0}$ to (5) to obtain

$$\liminf_{\varepsilon\to 0} \|\widetilde{u}_\varepsilon(t)\|^2_{L^2(\mathbb{R}^3)} + 2\nu \liminf_{\varepsilon\to 0} \int_0^t \|\nabla\widetilde{u}_\varepsilon\|^2_{L^2(\mathbb{R}^3)} \le \|u_0\|^2_{L^2(\mathbb{R}^3)} \qquad \forall t \ge 0. \tag{20}$$

Let us fix the time t. From (5) we know that the sequence $\widetilde{u}_\varepsilon(t)$ is bounded in L^2. We also have that $\widetilde{u}_\varepsilon(t)$ converges in H^{-3} to $u(t)$, so the limit of every sub-sequence weakly convergent in L^2 must necessarily be $u(t)$. Since all L^2 weak limits are the same, we conclude that the whole sequence must converge to $u(t)$ in L^2 weakly, so

$$\|u(t)\|_{L^2(\mathbb{R}^3)} \le \liminf_{\varepsilon\to 0} \|\widetilde{u}_\varepsilon(t)\|_{L^2(\mathbb{R}^3)}. \tag{21}$$

One can prove in a similar manner that $\nabla\widetilde{u}_\varepsilon \rightharpoonup \nabla u$ weakly in $L^2\big((0,t) \times \mathbb{R}^3\big)$, so

$$\|\nabla u(t)\|_{L^2((0,t)\times\mathbb{R}^3)} \le \liminf_{\varepsilon\to 0} \|\nabla\widetilde{u}_\varepsilon(t)\|_{L^2((0,t)\times\mathbb{R}^3)}. \tag{22}$$

The energy inequality for u now follows from relations (20), (21) and (22).

We omitted to prove the weak continuity in time with values in L^2 of u. In fact, using the "uniqueness of limit" argument as above, it immediately follows that a function belonging to the space $L^\infty\big([0,\infty); L^2\big) \cap C^0([0,\infty); H^{-3}_{loc})$ automatically verifies this time continuity property. The proof of Theorem 6 is completed.

Remark 7. *It is clear from the proof that if we assume that the initial velocities u_0^ε converge only weakly to u_0, then we can still prove convergence of u^ε to some u solution of the Navier-Stokes equation in the sense of Definition 2 but without the energy inequality. The strong convergence of u_0^ε to u_0 is required only to prove the energy inequality.*

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References

1. S. Axler, P. Bourdon, and W. Ramey, *Harmonic function theory*, second ed., Graduate Texts in Mathematics, vol. 137, Springer-Verlag, New York, 2001.
2. W. Borchers and H. Sohr, *On the equations* $\operatorname{rot}\mathbf{v} = \mathbf{g}$ *and* $\operatorname{div}\mathbf{u} = f$ *with zero boundary conditions*, Hokkaido Math. J. **19** (1990), no. 1, 67–87.
3. D. Iftimie, M. C. Lopes Filho, and H. J. Nussenzveig Lopes, *Two dimensional incompressible ideal flow around a small obstacle*, Comm. Partial Differential Equations **28** (2003), no. 1-2, 349–379.
4. ______, *Two-dimensional incompressible viscous flow around a small obstacle*, Math. Ann. **336** (2006), no. 2, 449–489.
5. M. C. Lopes Filho, *Vortex dynamics in a two dimensional domain with holes and the small obstacle limit*, To appear in SIAM J. Math. Anal.

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