REGULARITY CRITERION FOR SOLUTIONS OF THREE-DIMENSIONAL TURBULENT CHANNEL FLOWS

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Abstract. In this paper we consider the three-dimensional Navier–Stokes equations in infinite channel. We provide a regularity criterion for solutions of the three-dimensional Navier–Stokes equations in terms of the vertical component of the velocity field.

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1. Introduction

Turbulence stands out as a prototype of multi-scale phenomenon that occurs in nature. It involves wide ranges of spatial and temporal scales which makes it very difficult to study analytically and prohibitively expensive to simulate computationally. Turbulent channel flows are considered to be the simplest flows confined within physical boundaries that can be simulated numerically and that demonstrates many of the common features of turbulence. In this paper we consider three-dimensional finite energy turbulent flows of viscous incompressible homogeneous fluids in the infinite channel $\Omega = \mathbb{R}^2 \times [-L, L] \subset \mathbb{R}^3$, subject to the no-slip Dirichlet boundary conditions. These flows are governed by the three-dimensional Navier–stokes system of equations:

$$\frac{\partial u}{\partial t} - \nu \Delta u + (u \cdot \nabla) u + \nabla p = f \qquad \text{in } \Omega \tag{1}$$

$$\nabla \cdot u = 0 \qquad \text{in } \Omega \tag{2}$$

$$u = 0 \qquad \text{on } \partial\Omega \tag{3}$$

$$\lim_{x \to \infty} u(t, x) = 0 \tag{4}$$

$$u(0, x) = u_0(x) \qquad \text{in } \Omega. \tag{5}$$

Here, $u = (u_1, u_2, u_3)$ represents the unknown velocity vector field, and p is the unknown pressure scalar; where $\nu > 0$, the constant kinematic viscosity, f, the body forcing term, and u_0, the initial velocity, are given.

Mathematically, it is well-known that the three-dimensional system (1)–(5) has global (for all time and all initial data) weak solutions (see, e.g., [8], [9], [13], [19], [20], [21] and references therein). The question of well-posedness, in the sense of Hadamard, and in particular the question of uniqueness, of these weak solutions is still an open problem. On the other hand, it is also well-established (see, e.g., [8], [9], [13], [19], [20], [21] and references therein) that the system (1)–(5) possesses a unique strong (regular) solution, which depends continuously on the initial data, for a short interval of time $[0, T_*)$, where T_* depends on the size of initial datum, u_0, on f, ν and L. Moreover, it is also well-known that the existence (for all time) and uniqueness of strong (regular) solutions is guaranteed under suitable additional assumptions

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(see, e.g., [2], [3], [7],[6], [10], [11], [12], [15], [16], [17], [22] and references therein). In particular, some of these recent results involve conditions on only one component of the velocity field of the $3D$ NSE in the whole space $\mathbb{R}^3$ or under periodic boundary conditions (see, e.g., [10], [12], [15], [22]). In this paper, we study this type of sufficient conditions for the global regularity of the $3D$ NSE in the infinite channel Ω, subject to no-slip Dirichlet boundary condition on the physical boundary of the channel. Using the geophysical terminology, our condition is formulated in terms of the third component of the baroclinic mode $\widetilde{u}_3$ (see (16), below, for the definition of the barotropic mode (vertically averaged mode), $\bar{u}$, and the baroclinic mode (the fluctuation about the barotropic mode)). Specifically, our results states that if $\widetilde{u}_3$ satisfies

$$\nabla \widetilde{u}_3 \in L^\infty([0,\infty), L^2(\Omega)), \tag{6}$$

then the strong (regular) unique solution of the 3D Naiver-Stokes equation (1)–(5) exists for all time.

Let us observe that our condition (6) seems to be slightly tighter than the former ones (cf. e.g., [10], [12], [15], [22]). However, unlike the previous works we study here the $3D$ Navier–Stokes in a domain with physical boundaries under the no-slip Dirichlet boundary conditions. Furthermore, we emphasize that the techniques developed here, which are inspired by ideas presented in [5], are totally different than the previous ones.

Let us denote by $L^q(\Omega), L^q(\mathbb{R}^2)$, and $H^m(\Omega), H^m(\mathbb{R}^2)$ the usual L^q-Lebesgue and Sobolev spaces, respectively ([1]). We denote by

$$\|\phi\|_q = \begin{cases} \left(\int_\Omega |\phi|^q \, dx_1 dx_2 dx_3\right)^{\frac{1}{q}}, & \text{for every } \phi \in L^q(\Omega) \\ \left(\int_{\mathbb{R}^2} |\phi|^q \, dx_1 dx_2\right)^{\frac{1}{q}}, & \text{for every } \phi \in L^q(\mathbb{R}^2). \end{cases} \tag{7}$$

Let

$$\mathcal{V} \quad = \quad \{v \in C_0^\infty(\Omega) : \nabla \cdot v = 0\}.$$

Since we are interested in flows of finite energy in the infinite channel Ω, we consider the spaces H and V, defined to be the closures of the set $\mathcal{V}$ in $L^2(\Omega)$ under L^2-topology, and in $H^1(\Omega)$ under H^1-topology, respectively. Denote by $P : L^2 \to H$, the orthogonal projection, and let $A = -P\Delta$ be the Stokes operator subject to the homogeneous Dirichlet boundary condition (3). It is well known that the Navier–Stokes equations (NSE) (1)–(5) are equivalent to the functional differential equation (see, e.g., [8], [19], [20],[21])

$$\frac{du}{dt} + \nu A u + B(u,u) = f, \tag{8}$$

$$u(0) = u_0, \tag{9}$$

where $B(u,u) = P((u\cdot\nabla)u)$, the nonlinear (bilinear) term. We say u is a Leray–Hopf weak solution to the system (8)–(9) if u satisfies (see, e.g., [8], [20],[21])

(1) $u \in C([0,T], H\text{-weak}) \cap L^2([0,T], V)$, and $\partial_t u \in L^1([0,T], V')$, where V' is the dual space of V,
(2) the weak formulation:

$$\int_\Omega u(t,x)\cdot\phi(x)\,dx - \int_\Omega u(t_0,x)\cdot\phi(x)\,dx$$

$$= -\int_{t_0}^t \int_\Omega \left(\nu(\nabla u(s,x) : \nabla\phi(x)) + (u(s,x)\cdot\nabla)u(s,x)\cdot\phi(x)\right)\,dx\,ds + \int_{t_0}^t \int_\Omega \left(f(s,x)\cdot\phi(x)\right)\,dx\,ds,$$

for every $\phi \in \mathcal{V}$, and almost every t, $t_0 \in [0,T]$.

(3) the energy inequality:

$$\|u(t)\|_2^2 - \|u(t_0)\|_2^2 + 2\nu \int_{t_0}^t \|\nabla u(s)\|_2^2 \, ds \le 2 \int_{t_0}^t \int_\Omega f(s,x) \cdot u(s,x) \, dx ds, \tag{10}$$

for all $t \in [0,T]$, and for almost every t_0 in the interval $[0,t]$.

Moreover, a weak solution is called strong solution of (8)–(9) on $[0,T]$ if, in addition, it satisfies

$$u \in C([0,T],V) \cap L^2([0,T], H^2(\Omega)).$$

For convenience, we recall the following Gagiliardo-Nirenberg, Ladyzhenskaya, and Sobolev inequalities (cf. e.g., [1], [8], [9] [14], [13] and [19]) in $\mathbb{R}^2$:

$$\|\phi\|_{L^r(\mathbb{R}^2)} \le C_r \|\phi\|_{L^2(\mathbb{R}^2)}^{2/r} \|\phi\|_{H^1(\mathbb{R}^2)}^{\frac{r-2}{r}}, \qquad r < \infty, \tag{11}$$

for every $\phi \in H^1(\mathbb{R}^2)$, and in $\mathbb{R}^3$:

$$\|\psi\|_{L^\alpha(\Omega)} \le C_\alpha \|\psi\|_{L^2(\Omega)}^{\frac{6-\alpha}{2\alpha}} \|\psi\|_{H^1(\Omega)}^{\frac{3(\alpha-2)}{2\alpha}}, \tag{12}$$

for every $u \in H^1(\Omega), 2 \le \alpha \le 6$. Here C_r and C_α are scale invariant constants. We also recall the Pioncaré inequality:

$$\|\nabla v\|_2 \ge \frac{C_0}{L} \|v\|_2 \qquad \forall v \in V \tag{13}$$

$$\|A v\|_2 \ge \frac{C_0}{L} \|\nabla v\|_2 \qquad \forall v \in \mathcal{D}(A), \tag{14}$$

where C_0 is a scale invariant constant. Also, we recall the integral version of Minkowsky inequality for the L^r spaces, $r \ge 1$. Let $\Omega_1 \subset \mathbb{R}^{m_1}$ and $\Omega_2 \subset \mathbb{R}^{m_2}$ be two measurable sets, where m_1 and m_2 are two positive integers. Suppose that $\phi(\xi,\eta)$ is measurable over $\Omega_1 \times \Omega_2$. Then,

$$\left[\int_{\Omega_1} \left(\int_{\Omega_2} |\phi(\xi,\eta)| d\eta \right)^r d\xi \right]^{1/r} \le \int_{\Omega_2} \left(\int_{\Omega_1} |\phi(\xi,\eta)|^r d\xi \right)^{1/r} d\eta. \tag{15}$$

2. Global Existence of the Strong Solution

In this section we will show the global existence of the strong solutions to the three-dimensional Navier–Stokes system (1)–(5) under assumption (6).

We will denote by

$$\overline{\theta}(x_1,x_2) = \frac{1}{2L} \int_{-L}^{L} \theta(x_1,x_2,x_3) \, dx_3 \qquad \text{and} \qquad \widetilde{\theta} = \theta - \overline{\theta}. \tag{16}$$

Following the geophysical fluid dynamics terminology we will call $\overline{\theta}$ the barotropic mode and $\widetilde{\theta}$ the baroclinic mode.

From now on, we will denote by $\nabla_h = (\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2})$ and $\Delta_h = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}$. First, let us prove the following Lemma.

Lemma 1. *Suppose that $\xi(x_1,x_2) \in H^1(\mathbb{R}^2), \phi \in H^1(\Omega)$ and $\psi \in L^2(\Omega)$. Then,*

$$\int_\Omega |\xi| \, |\phi| \, |\psi| \, dx_1 dx_2 dx_3$$
$$\le C \|\xi\|_2^{1/2} \, (\|\xi\|_2 + \|\nabla_h \xi\|_2)^{1/2} \, \|\phi\|_2^{1/2} \, (\|\phi\|_2 + \|\nabla_h \phi\|_2)^{1/2} \, \|\psi\|_2.$$

Proof. Notice that

$$\int_\Omega |\xi|\,|\phi|\,|\psi|\;dx_1dx_2dx_3 = \int_{\mathbb{R}^2}\left[|\xi|\left(\int_{-L}^{L}|\phi|\,|\psi|\;dx_3\right)\right]\;dx_1dx_2.$$

We will estimate the above term by applying the same method used to establish Proposition 2.2 in [4]. First, by Cauchy–Schwarz inequality, we obtain

$$\int_{-L}^{L}|\phi|\,|\psi|\;dx_3 \le \left(\int_{-L}^{L}|\phi|^2\;dx_3\right)^{\frac12}\left(\int_{-L}^{L}|\psi|^2\;dx_3\right)^{\frac12}.$$

Thus, by the above and Hölder inequality, we reach

$$\int_\Omega |\xi|\,|\phi|\,|\psi|\;dx_1dx_2dx_3 \le \int_{\mathbb{R}^2}\left[|\xi|\left(\int_{-L}^{L}|\phi|^2\;dx_3\right)^{\frac12}\left(\int_{-L}^{L}|\psi|^2\;dx_3\right)^{\frac12}\right]dx_1dx_2$$
$$\le \left(\int_{\mathbb{R}^2}|\xi|^4dx_1dx_2\right)^{\frac14}\left[\int_{\mathbb{R}^2}\left(\int_{-L}^{L}|\phi|^2\,dx_3\right)^2dx_1dx_2\right]^{\frac14}\left[\int_\Omega|\psi|^2\,dx\right]^{\frac12}.$$

By using Minkowsky inequality (15), we get

$$\left[\int_{\mathbb{R}^2}\left(\int_{-L}^{L}|\phi|^2\,dx_3\right)^2dx_1dx_2\right]^{\frac12} \le \int_{-L}^{L}\left(\int_{\mathbb{R}^2}|\phi|^4\,dx_1dx_2\right)^{\frac12}dx_3.$$

Thanks to (11) with $r=4$, for every fixed x_3 we have

$$\left(\int_{\mathbb{R}^2}|\phi|^4\,dx_1dx_2\right)^{\frac14} \le C\,\|\phi\|_{L^2(\mathbb{R}^2)}^{\frac12}\,\|\phi\|_{H^1(\mathbb{R}^2)}^{\frac12}.$$

As a result of the above and the Cauchy–Schwarz inequality, we obtain

$$\begin{aligned}
&\int_{-L}^{L}\left(\int_{\mathbb{R}^2}|\phi|^4\,dx_1dx_2\right)^{\frac12}dx_3\\
&\le C\int_{-L}^{L}\|\phi\|_{L^2(\mathbb{R}^2)}\,\|\phi\|_{H^1(\mathbb{R}^2)}\,dx_3\\
&\le C\left(\int_{-L}^{L}\|\phi\|_{L^2(\mathbb{R}^2)}^2\;dx_3\right)^{\frac12}\left(\int_{-L}^{L}\|\phi\|_{H^1(\mathbb{R}^2)}^2\;dx_3\right)^{\frac12}\\
&\le C\left(\|\phi\|_2+\|\nabla_h\phi\|_2\right)\;\|\phi\|_2.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\left[\int_{\mathbb{R}^2}\left(\int_{-L}^{L}|\phi|^2\,dx_3\right)^2dx_1dx_2\right]^{\frac14}\\
&\le C\left(\|\phi\|_2+\|\nabla_h\phi\|_2\right)^{1/2}\;\|\phi\|_2^{1/2}.
\end{aligned} \tag{17}$$

By using (11) with $r=4$, we have

$$\left[\int_{\mathbb{R}^2}|\xi|^4\,dx_1dx_2\right]^{\frac14} \le C\|\xi\|_{L^2(\mathbb{R}^2)}^{1/2}\|\xi\|_{H^1(\mathbb{R}^2)}^{1/2}.$$

Thus, by the above and (17), we get

$$\int_\Omega |\xi|\,|\phi|\,|\psi|\;dx_1dx_2dx_3$$
$$\le C\|\xi\|_2^{1/2}\;\left(\|\xi\|_2+\|\nabla_h\xi\|_2\right)^{1/2}\;\|\phi\|_2^{1/2}\;\left(\|\phi\|_2+\|\nabla_h\phi\|_2\right)^{1/2}\;\|\psi\|_2.$$

□

Theorem 2. *Let $f\in L^\infty([0,\infty),L^2(\Omega)), u_0\in V$. Let $u=(u_1,u_2,u_3)$ be a weak solution of the system (8)–(9) in $[0,\infty)$. Suppose that for $T>0$, $\nabla\widetilde{u}_3\in L^\infty([0,T],L^2(\Omega))$; that is, $\widetilde{u}_3$ satisfies*

$$\sup_{0\le t\le T}\|\nabla\widetilde{u}_3(t)\|_2<\infty. \tag{18}$$

Then u is the strong solution of the system (8)–(9) on $[0,T]$.

Proof. Let $u_0\in V$. Following, for instance, the Galerkin method one can show that there exists a unique strong solution u for the system (8)–(9), with the initial datum u_0, for a short interval of time (see, e.g., [8], [13], [19],[20] and [21]. Suppose that $[0,T_*)$ is the maximal interval of existence of this strong solution u. It is also well known (see, e.g., the above references) that there exists a Leray-Hopf weak solution for the system (8)–(9), with the same initial datum u_0, which exists globally in time, i.e. for all time $t\ge 0$. Most importantly, following the work of J. Sather and J. Serrin in [18] on can show that all the Leray-Hopf weak solutions coincide with the unique strong solution, u, on the interval $[0,T_*)$.

To conclude our proof we need to show that $T<T_*$. Suppose, arguing by contradiction, that $T_*\le T$, and that (18) holds. If we show that the $\limsup_{t\to T_*^-}\|u(t)\|_{H^1}<\infty$ then $[0,T_*)$ is not a maximal interval of existence, which leads to a contradiction.

For the rest of this proof we consider the strong solution, u, in the the interval $[0,T_*)$. From the energy inequality of (10), which is satisfied by all the Leray-Hopf weak solutions (see, for example, [8], [13], [19], [20] or [21] for details), we have

$$\|u(t)\|_2^2\le C\frac{L^4\,F^2}{\nu^2}+e^{-\frac{\nu\,t}{L^2}}\|u_0\|_2^2, \tag{19}$$

$$\nu\int_0^t\|\nabla u(s)\|_2^2\,ds\le C\frac{L^2\,F^2\,t}{\nu}+\|u_0\|_2^2, \tag{20}$$

for all $t\in[0,T_*)$, where

$$F=\|f\|_{L^\infty([0,\infty),L^2(\Omega))}. \tag{21}$$

In particular, since $t<T_*\le T$ we have

$$\|u(t)\|_2^2+\nu\int_0^t\|\nabla u(s)\|_2^2\,ds\le K_1, \tag{22}$$

where

$$K_1=C\frac{F^2\,(L^4+\nu\,T)}{\nu^2}+2\|u_0\|_2^2. \tag{23}$$

Taking the inner product of the equation (8) with $-\Delta_h u$ in H, and notice that $P\partial_{x_i}=\partial_{x_i}P$ for $i=1,2$, we get

$$\frac{1}{2}\frac{d\|\nabla_h u\|_2^2}{dt}+\nu\,\|\Delta_h u\|_2^2+\nu\,\|\nabla_h u_z\|_2^2=-\int_\Omega(f-B(u,u))\cdot\Delta_h u\;dx_1dx_2dx_3.$$

By integration by parts we get

$$
\begin{aligned}
&-\int_\Omega B(u,u)\cdot\Delta_h u\,dx_1dx_2dx_3 = \int_\Omega \sum_{l=1}^2 \frac{\partial u_k}{\partial x_j}\frac{\partial u_j}{\partial x_l}\frac{\partial u_k}{\partial x_l}\,dx_1dx_2dx_3\\
&= \int_\Omega \Bigg\{ \left(\frac{\partial u_1}{\partial x_1}\right)^3 + \left(\frac{\partial u_2}{\partial x_2}\right)^3 + \left(\frac{\partial u_1}{\partial x_1}+\frac{\partial u_2}{\partial x_2}\right)\left[\left(\frac{\partial u_1}{\partial x_2}\right)^2 + \left(\frac{\partial u_2}{\partial x_1}\right)^2 + \frac{\partial u_1}{\partial x_2}\,\frac{\partial u_2}{\partial x_1}\right]\\
&\quad + \sum_{k,l=1}^2 \frac{\partial u_k}{\partial x_3}\frac{\partial u_3}{\partial x_l}\frac{\partial u_k}{\partial x_l} + \sum_{j,l=1}^2 \frac{\partial u_3}{\partial x_j}\frac{\partial u_j}{\partial x_l}\frac{\partial u_3}{\partial x_l} + \sum_{l=1}^2 \frac{\partial u_3}{\partial x_3}\frac{\partial u_3}{\partial x_l}\frac{\partial u_3}{\partial x_l}\Bigg\}\,dx_1dx_2dx_3\\
&= \int_\Omega \Bigg\{ -\frac{\partial u_3}{\partial x_3}\left[\left(\frac{\partial u_1}{\partial x_1}\right)^2 + \left(\frac{\partial u_2}{\partial x_2}\right)^2 - \frac{\partial u_1}{\partial x_1}\frac{\partial u_2}{\partial x_2} + \left(\frac{\partial u_1}{\partial x_2}\right)^2 + \left(\frac{\partial u_2}{\partial x_1}\right)^2 + \frac{\partial u_1}{\partial x_2}\,\frac{\partial u_2}{\partial x_1}\right]\\
&\quad + \sum_{k,l=1}^2 \left(\frac{\partial u_k}{\partial x_l}\frac{\partial u_3}{\partial x_3}\frac{\partial u_k}{\partial x_l} + u_k\frac{\partial u_3}{\partial x_3}\frac{\partial^2 u_k}{\partial x_l\partial x_l} - u_k\frac{\partial u_3}{\partial x_l}\frac{\partial^2 u_k}{\partial x_l\partial x_3}\right)\\
&\quad + \sum_{j,l=1}^2 \frac{\partial u_3}{\partial x_j}\frac{\partial u_j}{\partial x_l}\frac{\partial u_3}{\partial x_l} - \sum_{l=1}^2\left(\frac{\partial u_1}{\partial x_1}+\frac{\partial u_2}{\partial x_2}\right)\frac{\partial u_3}{\partial x_l}\frac{\partial u_3}{\partial x_l}\Bigg\}\,dx_1dx_2dx_3\\
&= \int_\Omega \Bigg\{ -\frac{\partial \widetilde{u}_3}{\partial x_3}\left[\left(\frac{\partial u_1}{\partial x_1}\right)^2 + \left(\frac{\partial u_2}{\partial x_2}\right)^2 - \frac{\partial u_1}{\partial x_1}\frac{\partial u_2}{\partial x_2} + \left(\frac{\partial u_1}{\partial x_2}\right)^2 + \left(\frac{\partial u_2}{\partial x_1}\right)^2 + \frac{\partial u_1}{\partial x_2}\,\frac{\partial u_2}{\partial x_1}\right]\\
&\quad + \sum_{k,l=1}^2 \left(\frac{\partial u_k}{\partial x_l}\frac{\partial \widetilde{u}_3}{\partial x_3}\frac{\partial u_k}{\partial x_l} + u_k\frac{\partial \widetilde{u}_3}{\partial x_3}\frac{\partial^2 u_k}{\partial x_l\partial x_3} - u_k\frac{\partial \widetilde{u}_3}{\partial x_l}\frac{\partial^2 u_k}{\partial x_l\partial x_3}\right)\\
&\quad -\overline{u}_3\left[\sum_{j,l=1}^2\left[\frac{\partial}{\partial x_j}\left(\frac{\partial u_j}{\partial x_l}\frac{\partial u_3}{\partial x_l}\right) + \frac{\partial}{\partial x_l}\left(\frac{\partial u_j}{\partial x_3}\frac{\partial u_j}{\partial x_l}\right)\right] - \sum_{l=1}^2 \frac{\partial}{\partial x_l}\left[\left(\frac{\partial u_1}{\partial x_1}+\frac{\partial u_2}{\partial x_2}\right)\frac{\partial u_3}{\partial x_l}\right]\right]\\
&\quad + \sum_{j,l=1}^2 \frac{\partial \widetilde{u}_3}{\partial x_j}\frac{\partial u_j}{\partial x_l}\frac{\partial u_3}{\partial x_l} - \sum_{l=1}^2\left(\frac{\partial u_1}{\partial x_1}+\frac{\partial u_2}{\partial x_2}\right)\frac{\partial \widetilde{u}_3}{\partial x_l}\frac{\partial u_3}{\partial x_l}\Bigg\}\,dx_1dx_2dx_3.
\end{aligned}
$$

Then, from the above and the Cauchy–Schwarz inequality we obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d\|\nabla_h u\|^2}{dt} + \nu\|\Delta_h u\|_2^2 + \nu\|\nabla_h u_z\|_2^2 \le F\,\|\Delta_h u\|_2 + C\int_\Omega |\overline{u}_3|\,|\nabla u|\,|\nabla_h\nabla u|\,dx_1dx_2dx_3\\
&\quad +C\int_\Omega |\nabla\widetilde{u}_3|\,|\nabla_h u|^2\,dx_1dx_2dx_3 + C\int_\Omega |u|\,|\nabla\widetilde{u}_3|\,|\nabla_h\nabla u|\,dx_1dx_2dx_3.
\end{aligned}
$$

where F is given in (21). By applying Lemma 1 we obtain

$$
\begin{aligned}
&\int_\Omega |\overline{u}_3|\,|\nabla u|\,|\nabla_h\nabla u|\,dx_1dx_2dx_3\\
&\le C\|\overline{u}_3\|_2^{1/2}\,\left(\|\overline{u}_3\|_2 + \|\nabla_h\overline{u}_3\|_2\right)^{1/2}\,\|\nabla u\|_2^{1/2}\left(\|\nabla u\|_2 + \|\nabla_h\nabla u\|_2\right)^{1/2}\|\nabla_h\nabla u\|_2\\
&\le C\|\overline{u}_3\|_2\,\|\nabla u\|_2\,\|\nabla_h\nabla u\|_2 + C\|\overline{u}_3\|_2^{1/2}\,\|\nabla_h\overline{u}_3\|_2^{1/2}\,\|\nabla u\|_2^{1/2}\|\nabla_h\nabla u\|_2^{1/2}\|\nabla_h\nabla u\|_2\\
&\le C\|u\|_2\,\|\nabla u\|_2\,\|\nabla_h\nabla u\|_2 + C\|u\|_2^{1/2}\,\|\nabla_h u\|_2^{1/2}\,\|\nabla u\|_2^{1/2}\|\nabla_h\nabla u\|_2^{3/2}. \qquad (24)
\end{aligned}
$$

By Hölder inequality, we reach

$$\int_\Omega |\nabla \widetilde{u}_3| \, |\nabla_h u|^2 \, dx_1 dx_2 dx_3$$
$$\leq C\|\nabla \widetilde{u}_3\|_2 \, \|\nabla_h u\|_4^2 \leq C\|\nabla \widetilde{u}_3\|_2 \, \|\nabla_h u\|_2^{1/2} \left(\|\nabla_h u\|_2 + \|\nabla_h \nabla u\|_2\right)^{3/2}. \tag{25}$$

And also by Hölder inequality, we obtain

$$\int_\Omega |u| \, |\nabla \widetilde{u}_3| \, |\nabla_h \nabla u| \, dx_1 dx_2 dx_3$$
$$\leq C \int_{\mathbb{R}^2} \left\{\|u(x_1, x_2, \cdot)\|_\infty \|\nabla \widetilde{u}_3(x_1, x_2, \cdot)\|_2 \, \|\nabla_h \nabla u(x_1, x_2, \cdot)\|_2\right\} \, dx_1 dx_2$$
$$\leq C \int_{\mathbb{R}^2} \left\{\|u(x_1, x_2, \cdot)\|_2^{1/2} \|\frac{\partial u}{\partial x_3}(x_1, x_2, \cdot)\|_2^{1/2} \|\nabla \widetilde{u}_3(x_1, x_2, \cdot)\|_2 \, \|\nabla_h \nabla u(x_1, x_2, \cdot)\|_2\right\} \, dx_1 dx_2$$
$$\leq C \left\{\int_{\mathbb{R}^2} \|u(x_1, x_2, \cdot)\|_2^4 \, dx_1 dx_2\right\}^{1/8} \left\{\int_{\mathbb{R}^2} \|\frac{\partial u}{\partial x_3}(x_1, x_2, \cdot)\|_2^4 \, dx_1 dx_2\right\}^{1/8}$$
$$\times \left\{\int_{\mathbb{R}^2} \|\nabla \widetilde{u}_3(x_1, x_2, \cdot)\|_2^4 \, dx_1 dx_2\right\}^{1/4} \|\nabla_h \nabla u\|_2$$
$$\leq C\|u\|_2^{1/4} \|\nabla_h u\|_2^{1/4} \left\|\frac{\partial u}{\partial x_3}\right\|_2^{1/4} \left\|\frac{\partial \nabla_h u}{\partial x_3}\right\|_2^{1/4} \|\nabla \widetilde{u}_3\|_2^{1/2} \, \|\nabla_h \nabla \widetilde{u}_3\|_2^{1/2} \, \|\nabla_h \nabla u\|_2. \tag{26}$$

By (24)–(26) and Young's inequality we get

$$\frac{d\|\nabla_h u\|_2^2}{dt} + \nu\|\nabla_h \nabla u\|_2^2 \leq CF^2 + C\|u\|_2^2 \|\nabla u\|_2^2$$
$$+C\left(\|u\|_2^{1/2} \, \|\nabla u\|_2^2 + \|\nabla \widetilde{u}_3\|_2^4 + \|u\|_2^2 \left\|\frac{\partial u}{\partial x_3}\right\|_2^2 \|\nabla \widetilde{u}_3\|_2^4\right) \|\nabla_h u\|_2^2,$$

where F is given in (21). Thanks to Gronwall inequality, we obtain, for all $t \in [0, T_*)$,

$$\|\nabla_h u(t)\|_2^2 + \nu \int_0^t \|\nabla_h \nabla u(s)\|_2^2 \, ds \leq K_2,$$

where

$$K_2 = e^{CK_1^2 + C\left(T + K_1^2\right) \max_{0 \leq s \leq T} \|\nabla \widetilde{u}_3(s)\|_2^4} \left[\|u_0\|_{H^1(\Omega)}^2 + F^2 + CK_1^2\right]. \tag{27}$$

Recall that $\|u\|_V^2 = \int u \cdot Au \, dx_1 dx_2 dx_3$. It is well know that $\|u\|_V^2$ is equivalent to $\|\nabla u\|_2^2$ (see, e.g., [8]). Taking the inner product of the equation (8) with Au in H, we get

$$\frac{1}{2} \frac{d\|u\|_V^2}{dt} + \nu \, \|Au\|_2^2 = \int_\Omega (f - B(u, u)) \cdot Au \, dx_1 dx_2 dx_3$$
$$\leq F\|Au\|_2 + C\|u\|_6 \, \|\nabla u\|_3 \, \|Au\|$$
$$\leq F\|Au\|_2 + C\|\nabla_h u\|_2^{2/3} \, \|\frac{\partial u}{\partial x_3}\|_2^{1/3} \, \|\nabla u\|_2^{1/2} \, \|Au\|^{3/2}.$$

Here, we used (cf., e.g., [9], p. 33)

$$\|u\|_6 \leq C \left\|\frac{\partial u}{\partial x_1}\right\|_2^{1/3} \left\|\frac{\partial u}{\partial x_2}\right\|_2^{1/3} \left\|\frac{\partial u}{\partial x_3}\right\|_2^{1/3}.$$

By Young's inequality we obtain

$$\frac{d\|u\|_V^2}{dt} + \nu\,\|Au\|_2^2 \le CF^2 + C\|\nabla_h u\|_2^{8/3}\,\|\frac{\partial u}{\partial x_3}\|_2^{4/3}\,\|u\|_V^2.$$

By Gronwall inequality and (27), we obtain, for all $t \in [0, T_*)$,

$$\|u(t)\|_V^2 + \nu \int_0^t \|Au(s)\|_2^2\, ds \le K,$$

where

$$K = e^{CK_2^{4/3} K_1^{2/3}} \left[\|u_0\|_{H^1(\Omega)}^2 + F^2\right]. \tag{28}$$

Therefore,

$$\limsup_{t\to T_*^-} \|u(t)\|_V^2 \le K,$$

which leads to a contradiction that $[0, T_*)$ is the maximal interval of existence, and this completes the proof.

□

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