SLOWLY DECAYING SOLUTIONS TO INCOMPRESSIBLE NAVIER-STOKES SYSTEM

To the memory of Tetsuro Miyakawa, our teacher and friend

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Abstract. We study the large time asymptotics of solutions to the Cauchy problem in $\mathbb{R}^n$ for the incompressible Navier-Stokes system. Imposed assumptions on initial data imply that, at the first approximation, solutions look as solutions to the linear heat equation. The main goal of this work is to derive second terms of the asymptotic expansions of solutions and to extend, in this way, the results by Fujigaki & Miyakawa (SIAM J. Math. Anal. **33** (2001), 523–544) and Gallay & Wayne (R. Soc. London Philos. Trans. Ser. A Math. Phys. Eng. Sci. **360** (2002), 2155–2188).

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1. Introduction. We consider the Cauchy problem for the Navier-Stokes system in $\mathbb{R}^n$ with $n \geq 2$

$$\begin{aligned} u_t - \Delta u + (u \cdot \nabla) u &= -\nabla p \quad \text{for} \quad (x,t) \in \mathbb{R}^n \times (0,\infty), \\ \nabla \cdot u &= 0, \\ u(x,0) &= u_0(x). \end{aligned} \tag{1}$$

Here $x \in \mathbb{R}^n$ denotes the space variable, $t \geq 0$ is the elapsed time, $u = u(x,t) = (u_1(x,t), ..., u_n(x,t))$ and $p = p(x,t)$ correspond to the unknown velocity vector and the scalar pressure, respectively. Moreover, $u_0(x) = (u_0^1(x), \ldots, u_0^n(x))$ is a given initial velocity. In system (1), all physical constants are normalized to 1.

The purpose of this paper is to derive asymptotic profiles, as $t \to \infty$, of solutions to system (1) which decay in time rather slowly (cf. our standing assumptions (4)-(6), below). First, however, we recall that there is a large literature discussing the asymptotic properties of weak and strong solutions to (1). Indeed, J. Leray in his pioneer work [18] raised the question on the decay properties of weak solutions which provoked several mathematicians to study the large time behavior of their L^2-norm because it correspond to the energy, *cf. e.g.* [22, 23, 24, 14, 3, 12] and reference therein. These results are often completed by space-time estimates of solution to (1), cf. [1, 2, 4, 5, 6, 13, 19, 20] (both lists are by no mean complete).

The present paper is motivated by the higher order asymptotic expansion of weak and strong solutions to (1) obtained by Carpio [8] and improved by Fujigaki & Miyakawa [9] and Gallay & Wayne [10, 11] under the condition $(1+|x|)u_0 \in L^1(\mathbb{R}^n)$. Note that such an initial datum decays at infinity faster than a general function from $L^p(\mathbb{R}^n)$ with $p \geq 1$. We emphasize this fact because, on the other hand, for initial conditions decaying at infinity as homogeneous functions of degree -1, the large time asymptotics of corresponding solutions is described by self-similar solutions to the Navier-Stokes system, cf. [21, 7], for more details.

The goal of this work is, roughly speaking, to show the correlation between the decay rate of the initial datum as $|x| \to \infty$ and the asymptotic expansion of the corresponding solution as $t \to \infty$. We obtain asymptotic profiles of solutions corresponding to initial conditions which are bounded (this condition is imposed, however, for simplicity of the exposition) and which decay like a function $|x|^{-\alpha}$ as $|x| \to \infty$ for some $\alpha \in (1,n)$. This behavior of initial conditions for large $|x|$ will be expressed in this paper by the assumption

$$u_0 \in L^{n/\alpha,\infty}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) \quad \text{for some} \quad \alpha \in (1,n), \tag{2}$$

where $L^{p,\infty}(\mathbb{R}^n)$ is the weak L^p-space; see the definition below. We discover the critical order of decay, as $|x| \to \infty$, of the initial datum equal to $1 + n/2$ in the following sense: for $\alpha > 1 + n/2$, the Fujigaki-Miyakawa-Gallay-Wayne expansion is still valid, however, a new asymptotic profile appears for $\alpha < 1 + n/2$.

Note that our initial data are not integrable, in general. Moreover, they do not necessarily belong to $L^2(\mathbb{R}^n)$ for $\alpha < n/2$. Here, one should note the imbedding

$$L^{n/\alpha,\infty}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) \subseteq L^p(\mathbb{R}^n) \quad \text{for every} \quad p > n/\alpha \quad \text{and} \quad \alpha \in (1,n), \tag{3}$$

being the consequence of a standard interpolation argument.

We conclude this introduction by recalling that analogous asymptotic expansions of slowly decaying solutions for the convection diffusion equation $u_t - u_{xx} + (u^q)_x = 0$ were obtained in [17]. On the other hand, corresponding asymptotic expansions, as $|x| \to \infty$, of solutions to (1) were recently obtained in [6].

Notation. The L^p-norm of a Lebesgue measurable, real-valued function v defined on $\mathbb{R}^n$ is denoted by $\|v\|_p$ and, in the case of a vector field $v = (v_1, ..., v_n)$, we write $\|v\|_p \equiv \max\{\|v_1\|_p, ..., \|v_n\|_p\}$. In the following, we also use the weak L^p space (also called the Marcinkiewicz space) denoted by $L^{p,\infty}(\mathbb{R}^n)$ and consisting of all measurable functions v satisfying $\|v\|_{p,\infty} = \sup_{t>0} t|\{x \in \mathbb{R}^n : |v(x)| > t\}|^{\frac{1}{p}} < \infty$, where $|E|$ is the Lebesgue measure of a measurable set E. The solution to the linear heat equation $u_t = \Delta u$ supplemented with the initial datum u_0 is given by $S(t)u_0 \equiv E(t) * u_0$, with the Gauss-Weierstrass kernel $E(x,t) = (4\pi t)^{-n/2} \exp(-|x|^2/4t)$. Moreover, we use the matrix valued function

$$V(x,t) = \big(V_{jk}(x,t)\big)_{j,k=1,\ldots,n} \quad \text{with} \quad V_{jk}(x,t) = E(x,t)\delta_{jk} + \int_0^\infty \partial_j \partial_k E(x,t+\tau)\, d\tau,$$

because this is the kernel of the operator $\mathbb{P}S(t)$ with the Leray projection $\mathbb{P}$ onto solenoidal vector fields (cf. [9, Section 2] for more details). It allows us to rewrite the initial value problem (1) as a well-known integral equation for mild solutions, see (22) below.

2. Results and comments. We derive an asymptotic expansion as $t \to \infty$ of a global-in-time solution $u = u(x,t)$ to the initial value problem (1) which satisfies the following conditions. First of all, we assume the existence of a constant $\alpha \in (1,n)$ such that

$$\sup_{t>0} \|u(t)\|_{n/\alpha,\infty} < \infty. \tag{4}$$

Moreover, we suppose that the following estimates hold true

$$\|u(t)\|_p \le C(1+t)^{-\frac{n}{2}(\frac{\alpha}{n}-\frac{1}{p})} \tag{5}$$

and

$$\|\nabla u(t)\|_p \le Ct^{-\frac{1}{2}}(1+t)^{-\frac{n}{2}(\frac{\alpha}{n}-\frac{1}{p})} \tag{6}$$

for every $p \in (n/\alpha, \infty]$, all $t > 0$, and C independent of t. For the completeness of the exposition, we recall (in Theorem 9, below) a result on the existence of global-in-time solutions corresponding to sufficiently small initial conditions and satisfying (4)- (6). Our assumptions (4)- (6), however, are not surprising because they are valid for solutions to the linear heat equation $S(t)u_0 = E(t) * u_0$ with an initial datum from (2). Indeed, we have the well-known algebraic decay rates estimates

$$\|S(t)u_0\|_p \le \|u_0\|_p \quad \text{for every} \quad p \in [1,\infty] \tag{7}$$

and

$$\|S(t)u_0\|_p \le Ct^{-\frac{n}{2}(\frac{\alpha}{n}-\frac{1}{p})}\|u_0\|_{n/\alpha,\infty}, \tag{8}$$

$$\|\nabla S(t)u_0\|_p \le Ct^{-\frac{1}{2}-\frac{n}{2}(\frac{\alpha}{n}-\frac{1}{p})}\|u_0\|_{n/\alpha,\infty}, \tag{9}$$

for every $p \in (n/\alpha, \infty]$, all $t > 0$, and C independent of t and of u_0.

Below in Proposition 10, we show that, in fact, the solutions to the Navier-Stokes system (1) behaves at the first approximation as $S(t)u_0$. This property of solutions to (1) is known, however, we state it (and prove) in the form which is the most suitable for our applications.

The main goal of this work is to derive the second order term of the asymptotic expansion of solutions as $t \to \infty$. We discover the critical exponent equal to $\alpha = 1 + n/2$ and the asymptotic expansion of solutions differs for $\alpha \in (1 + n/2)$, $\alpha = 1 + n/2$, and $\alpha \in (1 + n/2, n)$. It is worth of noting that the interval $(1 + n/2, n)$ is nonempty for $n \ge 3$, only.

We begin with $\alpha > 1 + n/2$.

Theorem 1. *Assume that $1 + n/2 < \alpha < n$. Let u be the solution to (1) satisfying (4)- (6). Then*

$$\lim_{t\to\infty} t^{\frac{n}{2}(1-\frac{1}{p})+\frac{1}{2}} \Big\|u(t) - S(t)u_0 + \nabla V(\cdot,t)\int_0^\infty \int_{\mathbb{R}^n} \big(u\otimes u\big)(y,\tau)\, dy d\tau\Big\|_p = 0 \tag{10}$$

for any $p \in [1,\infty]$.

Note that $\nabla V(x,t) = t^{-\frac{n}{2}-\frac{1}{2}}(\nabla V)(x/\sqrt{t},1)$, hence, $\|\nabla V(t)\|_p = t^{-\frac{n}{2}(1-\frac{1}{p})-\frac{1}{2}}\|\nabla V(1)\|_p$. Relation (10) says that the L^p-norm of $u(t) - S(t)u_0 + \nabla V(t)\int_0^\infty \int_{\mathbb{R}^n} u\otimes u\, dy d\tau$ decays faster.

If $\alpha > 1+n/2$, the decay estimate (5) with $p = 2$ implies $u\otimes u \in L^1(\mathbb{R}^n \times [0,\infty))$, hence, the integral $\int_0^\infty \int_{\mathbb{R}^n} \big(u\otimes u\big)(y,\tau)\, dy d\tau$ in (10) is convergent. In this case, the asymptotic formula (10) agrees with that obtained by Fujigaki and Miyakawa [9, Thm. 2.1.i] with the only difference that we do not develop asymptotically the linear part $S(t)u_0$. In fact, as we shall see below, the essential part of the proof of Theorem 1 can be just copied from [9].

Remark 2. It follows from the proof of Theorem 1 and from [9, Thm. 2.1.i] that the expansion (10) holds true for all solutions satisfying $\|u(t)\|_2^2 \le C(1+t)^{-b}$ with some $b > 1$.

Next, we study the case $\alpha \in (1, 1 + n/2]$.

Theorem 3. *Assume that $1 < \alpha \le 1 + n/2$. Let u be the solution to (1) satisfying (4)- (6). Then, for every $p \in [1,\infty]$ satisfying $p > n/(2\alpha)$ we have*

$$\lim_{t\to\infty} t^{\frac{n}{2}(\frac{\alpha}{n}-\frac{1}{p})+\frac{\alpha-1}{2}} \left\|u(t) - S(t)u_0 + \int_0^t \mathbb{P}\nabla S(t-\tau)\left(\tilde{u}\otimes\tilde{u}\right)(\tau)\, d\tau\right\|_p = 0 \tag{11}$$

for all $t > 0$, where, to shorten the notation, we write $\tilde{u}(t) = S(t)u_0$.

Our proof of Theorem 3 provides an explicit decay rate of $t^{\frac{n}{2}(\frac{\alpha}{n}-\frac{1}{p})+\frac{\alpha-1}{2}}\|\ldots\|_p$ in (11), which we do not state here for simplicity of the exposition.

Remark 4. Note that for $\alpha \in (1, 1+n/2]$, the decay rates in expansions (10) and (11) satisfy

$$\frac{n}{2}\left(\frac{\alpha}{n}-\frac{1}{p}\right)+\frac{\alpha-1}{2} \le \frac{n}{2}\left(1-\frac{1}{p}\right)+\frac{1}{2}$$

with the equality for $\alpha = 1+n/2$.

Under an additional assumption on the initial datum, we can replace the second order term in the asymptotic expansion (11) by a self-similar function and we obtain an asymptotic term in the critical case $\alpha = 1+n/2$, as well.

Theorem 5. *Let the assumptions of Theorem 3 hold true. Suppose that there exists a function* $U_0 \in L^{n/\alpha,\infty}(\mathbb{R}^n)$*, homogeneous of degree* $-\alpha$ *and divergence free in the sense of distribution such that*

$$\lim_{t\to\infty} \|S(t)(u_0 - U_0)\|_{n/\alpha,\infty} = 0. \tag{12}$$

Denote by $U_\alpha(x,t) = (S(t)U_0)(x) = t^{-\frac{\alpha}{2}} U_\alpha(x/\sqrt{t}, 1)$*, the self-similar solution to the heat equation. If* $1 < \alpha < 1+n/2$*, we have*

$$\lim_{t\to\infty} t^{\frac{n}{2}(\frac{\alpha}{n}-\frac{1}{p})+\frac{\alpha-1}{2}} \left\| u(t) - S(t)u_0 + \int_0^t \mathbb{P}\nabla S(t-\tau)\left(U_\alpha(\tau)\otimes U_\alpha(\tau)\right)\, d\tau \right\|_p = 0 \tag{13}$$

and, in the critical case $\alpha = 1+n/2$*, we obtain*

$$\lim_{t\to\infty} \frac{t^{\frac{n}{2}(\frac{\alpha}{n}-\frac{1}{p})+\frac{\alpha-1}{2}}}{\log(e+t)} \left\| u(t) - S(t)u_0 + (\log(e+t))\nabla V(t) \int_{\mathbb{R}^n} (U_\alpha \otimes U_\alpha)(y,1)\, dy \right\|_p = 0 \tag{14}$$

for any $p \in [1,\infty]$ *such that* $p > n/(2\alpha)$*.*

Remark 6. Using this self-similar form of the kernel ∇V and changing variables, we can easily show that

$$W(x,t) \equiv \int_0^t \mathbb{P}\nabla S(t-\tau)\big(U_\alpha(\tau)\otimes U_\alpha(\tau)\big)\, d\tau = t^{-\alpha+\frac{1}{2}} W\left(\frac{x}{\sqrt{t}}, 1\right)$$

with the profile $W(x,1)$ defined as

$$W(x,1) = \int_0^1 \int_{\mathbb{R}^n} (1-s)^{-\frac{n+1}{2}} \nabla V\left(\frac{x-y}{\sqrt{1-s}}\right) \left(U_\alpha\left(\frac{y}{\sqrt{s}},1\right) \otimes U_\alpha\left(\frac{y}{\sqrt{s}},1\right)\right) s^{-\alpha}\, dyds.$$

This simple observation is important to understand the decay rate in relation (13). Indeed, computing the L^p-norm we have

$$\|W(\cdot,t)\|_p = t^{-\frac{n}{2}(\frac{\alpha}{n}-\frac{1}{p})-\frac{\alpha-1}{2}} \|W(\cdot,1)\|_p.$$

Remark 7. The logarithmic factor $\log(e+t)$ appears in the proof of (14) when one replaces $\tilde{u}(t) = S(t)u_0$ by $\tilde{u}(t) = S(t)U_0$ in the following term $\int_0^t \mathbb{P}\nabla S(t-\tau)\left(\tilde{u}\otimes\tilde{u}\right)(\tau)\, d\tau$ of the asymptotic expansion of solutions. Assuming an explicit decay rate in (12) this logarithmic correction will disappear.

Remark 8. The expansions (11) and (13) are no longer valid for $\alpha = 1$. Indeed, it was proved by Planchon [21] (and improved in [16, 7]) that if a sufficiently small initial condition $u_0 \in L^{n,\infty}(\mathbb{R}^n)$ satisfies (12) with $\alpha = 1$ and a function $U_0 \in L^{n,\infty}(\mathbb{R}^n)$, homogeneous of degree -1, then the large time asymptotics of the corresponding solution is described by the self-similar solution of system (1) with U_0 as the initial datum.

3. Preliminary decay estimates. In our reasoning below, we need the following estimates of the heat semigroup

$$\|\mathbb{P}\nabla S(t) f\|_p \le C t^{-\frac{1}{2}-\frac{n}{2}(\frac{1}{r}-\frac{1}{p})} \|f\|_{r,\infty} \tag{15}$$

for every $1 < r < p \le \infty$, see, e.g. [9, Section 2]. Recall also that the norm $\|f\|_{r,\infty}$ in (15) can be replaced by $\|f\|_r$ due to the inequality $\|f\|_{r,\infty} \le C\|f\|_r$. In this case, $r = 1$ and $r = p$ are allowed, as well.

Here, we also recall the inequality in the limit case

$$\|S(t)f\|_{r,\infty} \le C\|f\|_{r,\infty} \tag{16}$$

valid for each $r \in (1,\infty)$, and weak-type Hölder inequality

$$\|fg\|_{r,\infty} \le C\|g\|_{p,\infty}\|f\|_{q,\infty} \tag{17}$$

for the exponents $1 < r, p, q < \infty$ satisfying $1/r = 1/p + 1/q$.

Our first preliminary result provides global-in-time small solutions to (1) which satisfy the decay estimates (4)-(6). Due to the imbedding (3), the result contained in Theorem 9 below is not new and we only sketch its proof.

Theorem 9. *Assume that*

$$u_0 \in L^{n/\alpha,\infty}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) \quad \text{with} \quad n \ge 2 \text{ and } 1 < \alpha < n. \tag{18}$$

Let the norm $\|u_0\| \equiv \max\{\|u_0\|_{n/\alpha,\infty}, \|u_0\|_\infty\}$ *be sufficiently small. There exists a unique solution* $u = u(x,t)$ *of system (1) satisfying*

$$u \in C_w([0,\infty); L^{\frac{n}{\alpha},\infty}(\mathbb{R}^n) \cap C(0,\infty; L^p(\mathbb{R}^n)) \tag{19}$$

for any $p \in (n/\alpha, \infty]$*. Moreover, the estimates (4)-(6) hold true for all* $t > 0$ *and* C *independent of* t*.*

Proof. It was already observed that $L^{n/\alpha,\infty}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) \subseteq L^n(\mathbb{R}^n)$. Hence, we can apply the Kato result [15] for $\|u_0\|_n$ sufficiently small in order to obtain the global-in-time solution

$$u \in C([0,\infty), L^n(\mathbb{R}^n)) \cap C((0,\infty), L^p(\mathbb{R}^n))$$

for every $p \in (n,\infty)$ together with the decay estimates

$$\|u(t)\|_p \le C(1+t)^{-\frac{n}{2}(\frac{1}{n}-\frac{1}{p})} \quad \|\nabla u(t)\|_p \le Ct^{-\frac{1}{2}}(1+t)^{-\frac{n}{2}(\frac{1}{n}-\frac{1}{p})}$$

for every $p > n$, all $t > 0$, and C independent of t. In order to obtain better decay properties of solutions stated in (4)-(6), it suffices to repeat the Kato algorithm, say, in the space

$$C_w([0,\infty); L^{\frac{n}{\alpha},\infty}(\mathbb{R}^n) \cap \{u \in C(0,\infty; L^n(\mathbb{R}^n)) \;:\; \sup_{t>0} t^{\frac{1}{2}(\alpha-1)}\|u(t)\|_n < \infty\}.$$

This argument, however, is well-known (see *e.g.* [15, Theorems 3 and 4]) and it was applied in several different contexts, hence, we skip other details. □

Next, we show that, under our standing assumptions (4)-(5), solutions to (1) behave at the first approximation as solutions to the heat equation. This is a standard result, which we state it in detail, because we use an optimal decay estimate of $\|u(t) - S(t)u_0\|_p$.

Proposition 10. *Let* $\alpha \in (1,n)$ *and* $p \in [1,\infty]$ *satisfy* $p > n/(2\alpha)$*. Suppose that* u *satisfies (4)-(6). There exists a constant* C *independent of* t *such that*

$$\|u(t) - S(t)u_0\|_p \le \begin{cases} Ct^{-\frac{n}{2}(\frac{\alpha}{n}-\frac{1}{p})-\frac{\alpha-1}{2}} & \text{if} \quad 1 < \alpha < 1 + n/2, \\ Ct^{-\frac{n}{2}(\frac{\alpha}{n}-\frac{1}{p})-\frac{\alpha-1}{2}}\log(e+t) & \text{if} \quad \alpha = 1 + n/2, \\ Ct^{-\frac{n}{2}(1-\frac{1}{p})-\frac{1}{2}} & \text{if} \quad 1 + n/2 < \alpha < n \end{cases} \tag{20}$$

for all $t \ge 1$*. Moreover,*

$$\left\|u(t) - S(t)u_0\right\|_{n/\alpha,\infty} \le \begin{cases} Ct^{-\frac{\alpha-1}{2}} & \text{if} \quad 1 < \alpha < 1 + n/2 \\ Ct^{-\frac{\alpha-1}{2}}\log(e+t) & \text{if} \quad \alpha = 1 + n/2 \\ Ct^{-\frac{n}{2}(1-\frac{\alpha}{n})-\frac{1}{2}} & \text{if} \quad 1 + n/2 < \alpha < n \end{cases} \tag{21}$$

for all $t \ge 1$*.*

Proof. We use the well-known integral representation of solutions to the Cauchy problem (1)

$$\begin{aligned} u(t) &= S(t)u_0 - \int_0^t \mathbb{P}\nabla S(t-\tau)(u\otimes u)(\tau)\, d\tau \\ &= E(t) * u_0 - \int_0^t \nabla V(t-s) * (u\otimes u)(\tau)\, d\tau, \end{aligned} \tag{22}$$

where we decompose the second term on the right-hand side as follows

$$\begin{aligned} u(x,t) - S(t)u_0 &= -\left(\int_0^{t/2} + \int_{t/2}^t\right) \mathbb{P}\nabla S(t-\tau)\big(u\otimes u\big)(\tau)\, d\tau \\ &= I_1(t) + I_2(t). \end{aligned} \tag{23}$$

First, we estimate the term $I_2(t)$ applying inequality (15) (with $r = p$) and (5) (with p replaced by $2p$) as follows

$$\begin{aligned} \|I_2(t)\|_p &\le C\int_{t/2}^t (t-\tau)^{-\frac12}\|u(\tau)\|_{2p}^2\, d\tau \\ &\le C\int_{t/2}^t (t-\tau)^{-\frac12}(1+\tau)^{-n(\frac{\alpha}{n}-\frac{1}{2p})}\, d\tau \le Ct^{-\frac{n}{2}(\frac{\alpha}{n}-\frac1p)-\frac{\alpha-1}{2}} \end{aligned} \tag{24}$$

for any $p \in [1,\infty]$ such that $p > n/(2\alpha)$.

We have to proceed more carefully with the integral $I_1(t)$ and the reasoning depends on the value of α.

Case 1: $1 \le \alpha < n/2$. Since $n/(2\alpha) > 1$, by (15) with $p > n/(2\alpha)$ and (17) with $p = q = n/\alpha$, we obtain

$$\|I_1(t)\|_p \le C\int_0^{t/2} (t-\tau)^{-\frac12-\frac{n}{2}(\frac{2\alpha}{n}-\frac1p)}\|u(\tau)\|_{n/\alpha,\infty}^2\, d\tau. \tag{25}$$

Now, the assumption (4) implies immediately

$$\|I_1(t)\|_p \le Ct^{-\frac{n}{2}(\frac{\alpha}{n}-\frac1p)-\frac{\alpha-1}{2}} \tag{26}$$

for all $t > 0$.

Case 2: $\alpha = n/2$. Here, we fix $q \in (1,p)$ (recall that $p > n/(2\alpha) = 1$) satisfying, moreover, $(n/2)(1-1/q) < 1$. Using (15), the Hölder inequality, and (5) we obtain

$$\begin{aligned} \|I_1(t)\|_p &\le C\int_0^{t/2} (t-\tau)^{-\frac12-\frac{n}{2}(\frac1q-\frac1p)}\|u(\tau)\|_{2q}^2\, d\tau \\ &\le C\int_0^{t/2} (t-\tau)^{-\frac12-\frac{n}{2}(\frac1q-\frac1p)}(1+\tau)^{-n(\frac12-\frac{1}{2q})}\, d\tau \\ &\le Ct^{-\frac{n}{2}(\frac{\alpha}{n}-\frac1p)-\frac{\alpha-1}{2}} \end{aligned} \tag{27}$$

for all $t > 0$.

Case 3: $n/2 < \alpha < n/2+1$. Here, we have $n/\alpha < 2$, hence, by inequalities (15), (17), and (5) with $p = 2$, we get

$$\begin{aligned} \|I_1(t)\|_p &\le C\int_0^{t/2} (t-\tau)^{-\frac12-\frac{n}{2}(1-\frac1p)}\|u(\tau)\|_2^2\, d\tau \\ &\le Ct^{-\frac12-\frac{n}{2}(1-\frac1p)}\int_0^{t/2}(1+\tau)^{-n(\frac{\alpha}{n}-\frac12)}\, d\tau \end{aligned} \tag{28}$$

for any $p \in [1,\infty]$. Since $n(\alpha/n - 1/2) < 1$, computing the integral on the right-hand side of (28) we show (26) for $\alpha \in (n/2, 1+n/2)$.

Case 4: $\alpha = n/2+1$. Now, $n(\alpha/n-1/2) = 1$, hence by inequalities (28), we obtain

$$\|I_1(t)\|_p \le Ct^{-\frac12-\frac{n}{2}(1-\frac1p)}\log(e+t) \tag{29}$$

for any $p \in [1,\infty]$.

Case 5: $n/2+1<\alpha<n$. Since $n(\alpha/n-1/2)>1$, inequality (28) implies

$$\|I_1(t)\|_p \le Ct^{-\frac{1}{2}-\frac{n}{2}(1-\frac{1}{p})} \tag{30}$$

for any $p \in [1,\infty]$. This argument completes the proof of (20).

Now, we proceed with the proof of (21).

First, we deal with $I_2(t)$, where by (15), (17) and the inequality $\|u(\tau)\|_{2n/\alpha,\infty} \le C\|u(\tau)\|_{2n/\alpha}$, we get

$$\begin{aligned}\|I_2(t)\|_{n/\alpha,\infty} &\le C\int_{t/2}^{t}(t-\tau)^{-\frac{1}{2}}\|u(\tau)\|_{2n/\alpha}^2\,d\tau\\ &\le C\int_{t/2}^{t}(t-\tau)^{-\frac{1}{2}}\tau^{-n(\frac{\alpha}{n}-\frac{\alpha}{2n})}\,d\tau = Ct^{-\frac{\alpha-1}{2}}.\end{aligned}$$

Next, we estimate $I_1(t)$. In the case of $1\le\alpha<n/2$ (so $n/(2\alpha)>1$), by (15) and (17), we get

$$\|I_1(t)\|_{n/\alpha,\infty} \le C\int_0^{t/2}(t-\tau)^{-\frac{1}{2}-\frac{n}{2}(\frac{2\alpha}{n}-\frac{\alpha}{n})}\|u(\tau)\|_{n/\alpha,\infty}^2\,d\tau.$$

Using assumption (4) we immediately deduce

$$\|I_1(t)\|_{n/\alpha,\infty} \le Ct^{-\frac{\alpha-1}{2}}. \tag{31}$$

If $\alpha=n/2$, it suffices to use the inequality $\|I_1(t)\|_{2,\infty}\le C\|I_1(t)\|_2$ and to follow the reasoning from inequalities (27).

For $n/2<\alpha<n/2+1$, we have $n/\alpha<2$ and $n(\alpha/n-1/2)<1$, consequently, by (5), (15), and (17), we again obtain

$$\begin{aligned}\|I_1(t)\|_{n/\alpha,\infty} &\le C\int_0^{t/2}(t-\tau)^{-\frac{n}{2}(1-\frac{\alpha}{n})-\frac{1}{2}}\|u(\tau)\|_2^2\,d\tau\\ &\le C(t/2)^{-\frac{n}{2}(1-\frac{\alpha}{n})-\frac{1}{2}}\int_0^{t/2}(1+\tau)^{-n(\frac{\alpha}{n}-\frac{1}{2})}\,d\tau \le Ct^{-\frac{\alpha-1}{2}}.\end{aligned} \tag{32}$$

The proof of (21) for $n/2+1<\alpha<n$ is completed by the same reasoning as in the case of the inequality (28). □

4. Second term of asymptotics.

Proof of Theorem 1. In order to find the second term of asymptotics of solutions to (1), we use the decomposition (23) from the proof of Proposition 10.

We begin by noting that for $\alpha>1+n/2$ the solution u satisfies estimate (5) for any $p\ge 2$. Consequently, in our reasoning below, we have the fundamental inequality

$$\|(u\otimes u)(t)\|_p \le \|u(t)\|_{2p}^2 \le C(1+t)^{-n\left(\frac{\alpha}{n}-\frac{1}{2p}\right)} \quad \text{for all} \quad p\in[1,\infty]. \tag{33}$$

Moreover, since $n(\alpha/n-1/2)>1$, the integral $\int_0^\infty\int_{\mathbb{R}^n}(u\otimes u)(u,\tau)\,dyd\tau$ is convergent by (33) with $p=1$.

Step 1. Let us show that the term $I_2(t)=-\int_{t/2}^{t}\mathbb{P}\nabla S(t-\tau)\big(u\otimes u\big)(\tau)\,d\tau$ does not contribute to the large time asymptotics of solutions. Indeed, we have

$$\lim_{t\to\infty} t^{\frac{n}{2}(1-\frac{1}{p})+\frac{1}{2}}\|I_2(t)\|_p = 0 \quad \text{for any} \quad p\ge 1. \tag{34}$$

This relation follows from (24) combined with (33) due to the inequality

$$\frac{n}{2}\left(\frac{\alpha}{n}-\frac{1}{p}\right)+\frac{\alpha-1}{2} > \frac{n}{2}\left(1-\frac{1}{p}\right)+\frac{1}{2} \quad \text{for} \quad \alpha>1+\frac{n}{2}.$$

Step 2. Now, we prove that the term $I_1(t) = -\int_0^{t/2} \nabla \mathbb{P} S(t-\tau)\big(u\otimes u\big)(\tau)\, d\tau$ determines the second term of the asymptotic expansion from (10), and by this reason, we write it in the form

$$
\begin{aligned}
I_1(t)+&\nabla V(\cdot,t)\int_0^\infty\int_{\mathbb{R}^n} \big(u\otimes u\big)(y,\tau)\; dyd\tau \\
=&\nabla V(\cdot,t)\int_{t/2}^\infty\int_{\mathbb{R}^n} \big(u\otimes u\big)(y,\tau)\; dyd\tau \\
&-\int_0^{t/2}\int_{\mathbb{R}^n} \big(\nabla V(\cdot-y,t-\tau)-\nabla V(\cdot,t-\tau)\big)\big(u\otimes u\big)(y,\tau)\; dyd\tau \\
&-\int_0^{t/2}\int_{\mathbb{R}^n} \big(\nabla V(\cdot,t-\tau)-\nabla V(\cdot,t)\big)\big(u\otimes u\big)(y,\tau)\; dyd\tau \\
=&J_{11}(t)+J_{12}(t)+J_{13}(t).
\end{aligned}
\tag{35}
$$

It is obvious that

$$\lim_{t\to\infty} t^{\frac12+\frac{n}{2}(1-\frac1p)}\, \|J_{11}\|_p = 0$$

since $\|(u\otimes u)(\cdot)\|_1 \in L^1(0,\infty)$ and $\|\nabla V(\cdot,t)\|_p = t^{-\frac12-\frac{n}{2}(1-\frac1p)}\|\nabla V(\cdot,1)\|_p$. In order to obtain estimates for other terms on the right-hand side of (35) (and to complete the proof of (10)) it suffices to copy the corresponding calculations from [9, Proof of Thm. 4.1.i]. Hence, we skip other details of this proof. □

Proof of Theorem 3. Now, we rewrite the integral equation (22) as follows

$$
\begin{aligned}
u(t)-S(t)u_0+&\int_0^t \mathbb{P}\nabla S(t-\tau)\big(\tilde u\otimes\tilde u\big)(\tau)\, d\tau \\
&=-\int_0^t \mathbb{P}\nabla S(t-\tau)\big((u-\tilde u)\otimes u\big)(\tau)\, d\tau \\
&\quad-\int_0^t \mathbb{P}\nabla S(t-\tau)\big(\tilde u\otimes (u-\tilde u)\big)(\tau)\, d\tau \\
&\equiv J_1(t)+J_2(t),
\end{aligned}
\tag{36}
$$

where $\tilde u(\tau) = S(\tau)u_0$, and we estimate each term on the right-hand side.

We begin with $1<\alpha<n/2$, so $2<n/\alpha$. Applying, first (15) and next (4)-(5) and (20)-(21), we obtain for any $p>n/(2\alpha)$

$$
\begin{aligned}
\|J_1(t)\|_p \le& C\int_0^{t/2} (t-\tau)^{-\frac12-\frac{n}{2}(\frac{2\alpha}{n}-\frac1p)}\|(u-\tilde u)(\tau)\|_{n/\alpha,\infty}\|u(\tau)\|_{n/\alpha,\infty}\, d\tau \\
&+C\int_{t/2}^t (t-\tau)^{-1/2}\|(u-\tilde u)(\tau)\|_{2p}\|u(\tau)\|_{2p}\, d\tau \\
\le& Ct^{-\frac12-\frac{n}{2}(\frac{2\alpha}{n}-\frac1p)}\int_0^{t/2}(1+\tau)^{-\frac{\alpha-1}{2}}\, d\tau + Ct^{-\frac{3\alpha}{2}+\frac{n}{2p}+1}.
\end{aligned}
\tag{37}
$$

Repeating this reasoning in the case of $J_2(t)$ we get

$$\|J_2(t)\|_p \le Ct^{-\frac12-\frac{n}{2}(\frac{2\alpha}{n}-\frac1p)}\int_0^{t/2}(1+\tau)^{-\frac{\alpha-1}{2}}\, d\tau + Ct^{-\frac{3\alpha}{2}+\frac{n}{2p}+1}$$

for any $p>n/(2\alpha)$.

Hence, the proof of (11) for $\alpha\in(1,n/2)$ is complete due to the estimates

$$\int_0^{t/2}(1+\tau)^{-\frac{\alpha-1}{2}}\, d\tau \le \begin{cases} Ct^{-(\alpha-3)/2} & \text{if} \quad 1<\alpha<3, \\ C\log(e+t) & \text{if} \quad \alpha=3, \\ C & \text{if} \quad 3<\alpha \end{cases}$$

and the relation

$$\frac{n}{2}\left(\frac{2\alpha}{n}-\frac1p\right)+\frac12 = \frac{n}{2}\left(\frac{\alpha}{n}-\frac1p\right)+\frac{\alpha-1}{2}+1.$$

For $n/2 < \alpha < 1 + n/2$, one should note that $u(t) \in L^q(\mathbb{R}^n)$ for any $q \in [2, \infty]$ (analogously, $u(t) \in L^q(\mathbb{R}^n)$ for any $q \in (2, \infty]$ if $\alpha = n/2$). Hence, by analogous calculations as those from (37) for every $p \in [1, \infty]$ if $n/2 < \alpha < 1 + n/2$, and for every $p \in (1, \infty]$ if $\alpha = n/2$, we complete the proof of (11) if $\alpha \in (1, 1 + n/2)$.

A similar argument based on inequalities from (37) holds true also in the limit case $\alpha = 1 + n/2$ and Theorem 3 is shown. □

Proof of Theorem 5. Since $U_0 \in L^{n/\alpha,\infty}(\mathbb{R}^n)$, it follows from inequalities (15) and (16) that

$$\|U_\alpha(t)\|_p \le Ct^{-\frac{n}{2}(\frac{\alpha}{n}-\frac{1}{p})}\|U_0\|_{n/\alpha,\infty} \quad \text{and} \quad \|U_\alpha(t)\|_{n/\alpha,\infty} \le C\|U_0\|_{n/\alpha,\infty} \tag{38}$$

for any $p \in (n/\alpha, \infty]$. Now, by estimates (38), (20), (21), and assumption (12), we have

$$\lim_{t\to\infty} t^{\frac{n}{2}(\frac{\alpha}{n}-\frac{1}{p})}\big\|u(t) - U_\alpha(t)\big\|_p = 0 \quad \text{for every} \quad p \in (n/\alpha, \infty]. \tag{39}$$

In order to prove Theorem 5, it is sufficient, in view of (11), to derive the asymptotic expansion of $\int_0^t \mathbb{P}\nabla S(t-\tau)(\tilde{u}(\tau) \otimes \tilde{u}(\tau))\, d\tau$ with $\tilde{u}(\tau) = S(\tau)u_0$. So, we split

$$\int_0^t \mathbb{P}\nabla S(t-\tau)(\tilde{u}(\tau) \otimes \tilde{u}(\tau))\, d\tau - \int_0^t \mathbb{P}\nabla S(t-\tau)(U_\alpha(\tau) \otimes U_\alpha(\tau))\, d\tau = K_1(t) + K_2(t)$$

with

$$\begin{aligned} K_1(t) \equiv &\int_{\delta t}^t \mathbb{P}\nabla S(t-\tau)\big((\tilde{u} - U_\alpha) \otimes \tilde{u}\big)(\tau)\, d\tau \\ &+ \int_{\delta t}^t \mathbb{P}\nabla S(t-\tau)\big(U_\alpha \otimes (\tilde{u} - U_\alpha)\big)(\tau)\, d\tau, \\ K_2(t) \equiv &\int_0^{\delta t} \mathbb{P}\nabla S(t-\tau)\big((\tilde{u} - U_\alpha) \otimes \tilde{u}\big)(\tau)\, d\tau \\ &+ \int_0^{\delta t} \mathbb{P}\nabla S(t-\tau)\big(U_\alpha \otimes (\tilde{u} - U_\alpha)\big)(\tau)\, d\tau, \end{aligned}$$

where, $\delta \in (0, 1/2)$ is a parameter to be determined later.

Applying (15), (38) and (39), we get

$$\begin{aligned} \|K_1(t)\|_p &\le C\int_{\delta t}^t (t-\tau)^{-1/2}\|(\tilde{u} - U_\alpha)(\tau)\|_{2p}\left(\|\tilde{u}(\tau)\|_{2p} + \|U_\alpha(\tau)\|_{2p}\right)\, d\tau \\ &\le Cg(\delta)t^{\frac{1}{2}-\alpha+\frac{n}{2p}} \sup_{\tau\in[\delta t,t]} \tau^{\frac{n}{2}(\frac{\alpha}{n}-\frac{1}{2p})}\|(\tilde{u} - U_\alpha)(\tau)\|_{2p} \end{aligned}$$

with $g(\delta) = \int_\delta^1 (1-s)^{-1/2}s^{-n(\alpha/n-1/(2p))}\, ds$. Hence, by (39), we obtain

$$t^{\frac{n}{2}(\frac{\alpha}{n}-\frac{1}{p})+\frac{\alpha-1}{2}}\|K_1(t)\|_p \le Cg(\delta) \sup_{\tau\in[\delta t,t]} \tau^{\frac{n}{2}(\frac{\alpha}{n}-\frac{1}{2p})}\|(\tilde{u} - U_\alpha)(\tau)\|_{2p} \to 0 \quad \text{as} \quad t \to \infty \tag{40}$$

for each fixed $\delta \in (0, 1/2)$ and for any $p \in [1, \infty]$ such that $p > n/(2\alpha)$.

Next we study K_2 and, as usual, we have to distinguish three cases: $1 < \alpha < n/2$, $\alpha = n/2$, and $n/2 < \alpha < n/2 + 1$.

For $1 < \alpha < n/2$, computing its L^p-norm with $p \in [1, \infty]$ and $p > n/(2\alpha)$, by (15) and (17) we have

$$\begin{aligned} \|K_2(t)\|_p \le C\int_0^{\delta t} &(t-\tau)^{-\frac{n}{2}(\frac{2\alpha}{n}-\frac{1}{p})-\frac{1}{2}}\|(\tilde{u} - U_\alpha)(\tau)\|_{n/\alpha,\infty} \\ &\times \left(\|\tilde{u}(\tau)\|_{n/\alpha,\infty} + \|U_\alpha(\tau)\|_{n/\alpha,\infty}\right)\, d\tau. \end{aligned} \tag{41}$$

Hence, by (4) and (38), we conclude

$$t^{\frac{n}{2}(\frac{\alpha}{n}-\frac{1}{p})+\frac{\alpha-1}{2}}\|K_2(t)\|_p \le C\int_0^\delta (1-s)^{-\frac{n}{2}(\frac{2\alpha}{n}-\frac{1}{p})-\frac{1}{2}}\, ds \tag{42}$$

with a constant C independent of δ and for any $p \in [1, \infty]$ such that $p > n/(2\alpha)$. Since $\delta > 0$ can be arbitrarily small, the results stated in (40) and (42) complete the proof of Theorem 5 for $1 < \alpha < n/2$.

The estimate of $\|K_2(t)\|_p$ for $\alpha \in [n/2, 1 + n/2)$ is obtained as in (41) replacing the $L^{n/\alpha,\infty}$-norm by the L^q-norm with the decay as in (39)(with p replaced by q) and with a suitably chosen q analogously as in estimates (27).

If $\alpha = 1 + n/2$, one should follow more-or-less similar reasoning using, moreover, the relation

$$\lim_{t\to\infty} \frac{1}{\log(e+t)} \int_1^{t/2} \int_{\mathbb{R}^n} (U_\alpha \otimes U_\alpha)(y,\tau)\, dyd\tau = \int_{\mathbb{R}^n} (U_\alpha \otimes U_\alpha)(y,1)\, dy,$$

which is an immediate consequence of the self-similar form of U_α (see [17] for a detailed proof of the corresponding result in the case of a convection-diffusion equation). □

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