Euler Equations of Incompressible Ideal Fluids

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Abstract

This article is a survey concerning the state-of-the-art mathematical theory of the Euler equations of incompressible homogenous ideal fluid. Emphasis is put on the different types of emerging instability, and how they may be related to the description of turbulence.

1 Introduction

This contribution is mostly devoted to the time dependent analysis of the $2d$ and $3d$ Euler equations

$$\partial_t u + \nabla \cdot (u \otimes u) + \nabla p = 0\,,\ \nabla \cdot u = 0\,, \tag{1}$$

of incompressible homogenous ideal fluid. We intend to connect several known (and maybe less known) points of view concerning this very classical problem. Furthermore, we will investigate the conditions under which one can consider the above problem as the limit of the incompressible Navier–Stokes equations:

$$\partial_t u_\nu + \nabla \cdot (u_\nu \otimes u_\nu) - \nu \Delta u_\nu + \nabla p_\nu = 0\,,\ \ \nabla \cdot u_\nu = 0\,, \tag{2}$$

when the viscosity $\nu \to 0$, i.e., as the Reynolds number goes to infinity.

At the macroscopic level the Reynolds number, Re, corresponds to the ratio of *the strength of the nonlinear effects* and *the strength of the linear viscous effects*. Therefore, with the introduction of a characteristic velocity, U, and a characteristic length scale, L, of the flow one has the dimensionless parameter:

$$Re = \frac{UL}{\nu}\,. \tag{3}$$

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With the introduction of the characteristic time scale $T = L/U$ and the dimensionless variables:

$$x' = \frac{x}{L}\,, t' = \frac{t}{T} \text{ and } u' = \frac{u'}{U}\,,$$

the Navier–Stokes equations (2) take the non-dimensional form:

$$\partial_t u' + \nabla_{x'} \cdot (u' \otimes u') - \frac{1}{Re}\Delta_{x'} u' + \nabla_{x'} p' = 0\,,\ \nabla \cdot u' = 0\,. \tag{4}$$

These are the equations to be considered in the sequel, omitting the $'$ and returning to the notation ν for Re^{-1}.

In the presence of physical boundary the problems (1) and (2) will be considered in the open domain $\Omega \subset \mathbf{R}^d\,, d = 2\,,\ d = 3\,,$ with a piecewise smooth boundary $\partial\Omega$.

There are several good reason to focus at present on the "mathematical analysis" of the Euler equations rather than on the Navier–Stokes equations.

1. Turbulence applications involving the Navier–Stokes equations (4) often correspond to very large Reynolds numbers; and a theorem which is valid for any finite, but very large, Reynolds number is expected to be compatible with results concerning infinite Reynolds numbers. In fact, this is the case when $Re = \infty$ which drives other results and we will give several examples of this fact.

2. Many nontrivial and sharp results obtained for the incompressible Navier–Stokes equations rely on the smoothing effect of the Laplacian, with viscosity $\nu > 0$, and on the invariance of the set of solutions under the scaling:

$$u(x,t) \mapsto \lambda u(\lambda x, \lambda^2 t)\,. \tag{5}$$

However, simple examples with the same scalings, but without a conservation law of energy may exhibit very different behavior concerning regularity and stability.

1. With ϕ being a scalar function, the viscous Hamilton–Jacobi type or Burgers equation

$$\begin{aligned} &\partial_t \phi - \nu\Delta\phi + \frac{1}{2}|\nabla\phi|^2 = 0 \,\text{in}\, \Omega \times \mathbf{R}_t^+\,, \\ &\phi(x,t) = 0 \text{ for } x \in \partial\Omega\,,\ \text{and } \phi(\cdot,0) = \phi_0(\cdot) \in L^\infty(\Omega)\,, \end{aligned} \tag{6}$$

has (because of the maximum principle) a global smooth solution, for $\nu > 0$. However, for $\nu = 0$, it is well known that certain solutions of the inviscid Burgers equation (6) will become singular (with shocks) in finite time.

2. Denote by $|\nabla|$ the square root of the operator $-\Delta$, defined in Ω with Dirichlet homogeneous boundary conditions. Consider the solution $u(x,t)$ of the equation

$$\partial_t u - \nu\Delta u + \frac{1}{2}|\nabla|(u^2) = 0 \text{ in } \Omega \times \mathbf{R}_t^+\,, \tag{7}$$

$$u(x,t) = 0 \text{ for } x \in \partial\Omega\,,\ \text{and } u(\cdot,0) = u_0(\cdot) \in L^\infty(\Omega)\,. \tag{8}$$

Then one has the following proposition.

Proposition 1.1 *Assume that the initial data u_0 satisfies the relation:*

$$\int_\Omega u_0(x)\phi_1(x)dx = -M < 0\,, \tag{9}$$

where $\phi_1(x) \geq 0$ denotes the first eigenfunction of the operator $-\Delta$ (with Dirichlet boundary condition), $-\Delta\phi_1 = \lambda_1\phi_1$. Then if M is large enough, the corresponding solution $u(x,t)$ of the system (7) (8) blows up in a finite time.

Proof. The L^2 scalar product of the equation (7) with $\phi_1(x)$ gives

$$\frac{d}{dt}\int_\Omega u(x,t)\phi_1(x)dx + \nu\lambda_1\int_\Omega u(x,t)\phi_1(x)dx$$
$$= -\frac{\sqrt{\lambda_1}}{2}\int_\Omega u(x,t)^2\phi_1(x)dx\,.$$

Since $\phi_1(x) \geq 0$ then the Cauchy–Schwarz inequality implies

$$\left(\int u(x,t)\phi_1(x)dx\right)^2 \leq \left(\int_\Omega u(x,t)^2\phi_1(x)dx\right)\left(\int_\Omega \phi_1(x)dx\right)\,.$$

As a result of the above the quantity $m(t) = -\int_\Omega u(x,t)\phi_1(x)dx$ satisfies the relation:

$$\frac{dm}{dt} + \lambda_1 m \geq \frac{\sqrt{\lambda_1}\int_\Omega \phi_1(x)dx}{2}m^2\,,\ \text{with}\, m(0) = M\,;$$

and the conclusion of the proposition follows.

Remark 1.1 *The above example has been introduced with $\Omega = \mathbf{R}^3$ by Montgomery–Smith [55] under the name "cheap Navier–Stokes equations" with the purpose of underlying the role of the conservation of energy (which is not present in the above examples) in the Navier–Stokes dynamics. His proof shows that the same blow up property may appear in any space dimension for the solution of the "cheap hyper-viscosity equations"*

$$\partial_t u + \nu(-\Delta)^m u + \frac{1}{2}|\nabla|(u^2) = 0\,.$$

On the other hand, one should observe that the above argument does not apply to the Kuramoto–Sivashinsky-like equations

$$\partial_t\phi + \nu(-\Delta)^m\phi + \alpha\Delta\phi + \frac{1}{2}|\nabla\phi|^2 = 0, \tag{10}$$

for $m \geq 2$. Without a maximum principle or without the control of some sort of energy the question of global existence of smooth solution, or finite time blow up of some solution, to the above equation is an open problem in $\mathbf{R}^n$, for $n \geq 2$ and for $m \geq 2$. However, if in (10) the term $|\nabla\phi|^2$ is replaced by $|\nabla\phi|^{2+\gamma}, \gamma > 0$ one can prove the blow up of some solutions (cf. [7] and references therein).

In conclusion, the above examples indicate that the conservation of some sort of energy, which is guaranteed by the structure of the equation, is essential in the analysis of the dynamics of the underlying problem. In particular, this very basic fact plays an essential role in the dynamics of the Euler equations.

Taking into account the above simple examples, the rest of the paper is organized as follows. In section 2 classical existence and regularity results for the time dependent Euler equations are presented. Section 3 provides more examples concerning the pathological behavior of solutions of the Euler equations. The fact that the solutions of the Euler equations may exhibit oscillatory behavior implies similar behavior for the solutions of the Navier–Stokes equations, as the viscosity tends to zero. The existence of (or lack thereof) strong convergence is analyzed in section 4 with the introduction of the Reynolds stresses tensor, and the notion of *dissipative solution*. A standard and very important problem, for both theoretical study and applications, is the vanishing viscosity limit of solutions of the Navier–Stokes equations subject to the no-slip Dirichlet boundary condition, in domains with physical boundaries. Very few mathematical results are available for this very unstable situation. One of the most striking results is a theorem of Kato [38], which is presented in section 5. Section 6 is again devoted to the Reynolds stresses tensor. We show that with the introduction of the Wigner measure the notion of Reynolds stresses tensor, deduced from the defect in strong convergence as the viscosity tends to zero, plays the same role as the one originally introduced in the statistical theory of turbulence. When the zero viscosity limit of solutions of the Navier–Stokes equations agrees with the solution of the Euler equations the main difference is confined in a boundary layer which is described by the Prandtl equations. These equations are briefly described in section 7. It is also recalled how the mathematical results are in agreement with the instability of the physical problem. The Kelvin–Helmholtz problem exhibits also some basic similar instabilities, but it is in some sense simpler. This is explained at the end of section 7, where it is also shown that some recent results of [44] , [71] and [72], on the regularity of the vortex sheet (interface), do contribute to the understanding of the instabilities of the original problem.

2 Classical existence and regularity results

2.1 Introduction

The Euler equations correspond, formally, to the limit case when the viscosity is 0, or the Reynolds number is infinite:

$$\partial_t u + \nabla \cdot (u \otimes u) + \nabla p = 0\,,\ \nabla \cdot u = 0\,,\ \text{in } \Omega\,. \tag{11}$$

In the presence of physical boundaries, the above system is supplemented with the standard, no-normal flow, boundary condition:

$$u \cdot \vec{n} = 0 \ \text{ on } \partial\Omega\,, \tag{12}$$

where $\vec{n}$ denotes the outwards normal vector to the boundary $\partial\Omega$. It turns out that the vorticity, $\omega = \nabla \wedge u$, is "the basic quantity", from both the physical and mathematical analysis points of view. Therefore, equations (11) and (12), written in terms of the vorticity, are equivalent to the system:

$$\partial_t \omega + u \cdot \nabla \omega = \omega \cdot \nabla u \text{ in } \Omega \tag{13}$$

$$\nabla \cdot u = 0\,, \nabla \wedge u = \omega \text{ in } \Omega\,, \text{ and } u \cdot \vec{n} = 0 \text{ on } \partial\Omega\,. \tag{14}$$

That is, system (14) fully determines u in terms of ω, which makes the above system "closed". More precisely, the operator $K : \omega \mapsto u$ defined by relation (14) is a linear continuous map from $C^{\alpha}(\Omega)$ with values in $C^{\alpha+1}(\Omega)$ (with $\alpha > 0$); and from $H^s(\Omega)$ with values in $H^{s+1}(\Omega)$.

Furthermore, for $2d$ flows, the vorticity is perpendicular to the plane of motion and therefore equation (13) is reduced (this can also be checked directly) to the advection equation

$$\partial_t \omega + u \cdot \nabla \omega = 0\,. \tag{15}$$

The structure of the quadratic nonlinearity in (13) has the following consequences, which are described below. We will be presenting only the essence of the essential arguments and not the full details of the proofs (see, e.g., [50] or [51] for the details).

2.2 General results in $3d$

The short time existence of a smooth solution for the $3d$ incompressible Euler equations has been obtained already a long time ago, provided the initial data are smooth enough. To the best of our knowledge the original proof goes back to Lichtenstein [45]. The proof is based on a nonlinear Gronwall estimate of the following type:

$$y' \leq C y^{\frac{3}{2}} \Rightarrow y(t) \leq \frac{y(0)}{(1 - 2tCy^{\frac{1}{2}}(0))^2}\,. \tag{16}$$

Therefore, the value of $y(t)$, which represents an adequate norm of the solution, is finite for a finite interval of time; which depends on the size of the initial value of $y(0)$, i.e. the initial data of the solution of Euler. These initial data have to be chosen from an appropriate space of regular enough functions. In particular, if we consider the solution in the Sobolev space H^s, with $s > \frac{5}{2}$, then by taking the scalar product, in the Sobolev space H^s, of the Euler equations with the solution u, and by using the appropriate Sobolev estimates we obtain:

$$\frac{1}{2}\frac{d\|u\|_{H^s}^2}{dt} = -(\nabla \cdot (u \otimes u), u)_{H^s} \leq C_s \|u\|_{H^s}^2 \|\nabla u\|_{L^\infty} \leq C\|u\|_{H^s}^3\,. \tag{17}$$

As a result of (16) and (17) we obtain the local, in time, existence of a smooth solution.

As standard in many nonlinear time dependent problems local regularity of smooth (strong) solutions implies local uniqueness and local stability (i.e, continuous dependence on initial data). Furthermore, one may exhibit a threshold

for this existence, uniqueness, and propagation of the regularity of the intial data (including analyticity Bardos and Benachour [3]). More precisely one uses the following theorem.

Theorem 2.1 Beale–Kato–Majda *[5] Let $u(t)$ be a solution of the $3d$ incompressible Euler equations which is regular for $0 \leq t < T$; that is,*

$$\textit{for all } t \in [0,T]\,, \;\; u(t) \in H^s(\Omega)\,, \textit{ for some } s > \frac{5}{3}\,.$$

Assume that

$$\int_0^T \|\nabla \wedge u(.,t)\|_{L^\infty} dt < \infty\,, \tag{18}$$

then $u(t)$ can be uniquely extended up to a time $T+\delta$ ($\delta > 0$) as a smooth solution of the Euler equations.

The main interest of this statement is the fact that it shows that if one starts with smooth initial data, then instabilities appears only if the size of the vorticity becomes arbitrary large.

Remark 2.1 *The Beale–Kato–Majda theorem was first proven in the whole space in [5]. Extension to a periodic "box" is easy. For a bounded domain with the boundary condition $u \cdot \vec{n} = 0$ it was established by Ferrari [26]. By combining arguments form [3] and [26] one can show, as in the Beale–Kato–Majda theorem, that the solution of $3d$ Euler equations, with real analytic initial data, remains real analytic as long as (18) holds.*

The Beale–Kato–Majda result has been slightly improved by Kozono [40] who proved that on the left-hand side of (18), the $\|\cdot\|_{L^\infty}$–norm can be replaced by the norm in the BMO space. This generalization is interesting because it adapts harmonic analysis (or Fourier modes decomposition) techniques which is an important tool for the study of "turbulent" solutions; indeed, the space BMO, as the dual space of the Hardy space $\mathcal{H}^1$, is well defined in the frequency (Fourier) space. In fact, cf. [52], BMO is the smallest space containing L^∞, which is also invariant under the action of a zero order pseudodifferential operators. The idea behind the Beale–Kato–Majda theorem, and its improvement, is the fact the solution u of the elliptic equations (14) satisfies the following estimate, for $1 < p < \infty$,

$$\|\nabla u\|_{W^{s,p}} \leq C_{s,p}\left(\|u\|_{W^{s,p}} + \|\omega\|_{W^{s,p}}\right)\,. \tag{19}$$

This relation could also be phrased in the context of Hölder spaces $C^{k,\alpha}$, $\alpha > 0$. We stress, however, that the estimate (19) ceases to be true for $p = \infty$ (or $\alpha = 0$.) This is due to the nature of the singularity, which is of the form $|x-y|^{2-d}$, in the kernel of the operator K and which leads (for $s > d/2+1$) to the estimate:

$$\|\nabla u\|_{L^\infty} \leq C\left(\|\omega\|_{L^\infty} \log(1 + \|u\|^2_{H^s})\right)\,, \tag{20}$$

$$\text{or sharper } \|\nabla u\|_{L^\infty} \leq C\left(\|\omega\|_{BMO} \log(1 + \|u\|^2_{H^s})\right)\,. \tag{21}$$

With $z = 1 + \|u\|_{H^s}^2$ and thanks to (21) the inequality (17) becomes:

$$\frac{d}{dt} z \leq C \|\omega\|_{BMO}\, z \log z\,.$$

This yields:

$$(1 + \|u(t)\|_{H^s}^2) \leq (1 + \|u(0)\|_{H^s}^2)^{e^{C \int_0^t \|\omega(s)\|_{BMO} ds}}\,,$$

which proves the statement. The uniqueness of solutions can be proven along the same lines, as long as

$$\int_0^t \|\omega(s)\|_{BMO}$$

remains finite.

Remark 2.2 *The vorticity ω can be represented by the anti-symmetric part of the deformation tensor ∇u. However, in the estimates (20) or (21) this anti-symmetric part, i.e. ω, can be replaced by the symmetric part of the deformation tensor*

$$S(u) = \frac{1}{2}(\nabla u + (\nabla u)^t)\,. \tag{22}$$

Therefore, the theorems of Beale–Kato–Majda and Kozono can be rephrased in terms of this symmetric tensor [40].

In fact, the above deformation tensor $S(u)$ (or $\tilde{S}(\omega)$ when expressed in term of the vorticity), plays an important role in a complementary result of Constantin, Fefferman and Majda [16], which shows that it is mostly the variations in the direction of the vorticity that may produce singularities.

Proposition 2.1 *[16] Let u, which is defined in $Q = \Omega \times (0, T)$, be a smooth solution of the Euler equations. Introduce the quantities $k_1(t)$ and $k_2(t)$ (which are well defined for $t < T$):*

$$k_1(t) = \sup_{x \in \Omega} |u(x,t)|\,,$$

which measures the size of the velocity, and

$$k_2(t) = 4\pi \sup_{x,y \in \Omega\,,\, x \neq y} \frac{|\xi(x,t) - \xi(y,t)|}{|x-y|}\,,$$

which measures the Lipschitz regularity of the direction

$$\xi(x,t) = \frac{\omega(x,t)}{|\omega(x,t)|}$$

of the vorticity. Then under the assumptions

$$\int_0^T (k_1(t) + k_2(t))dt < \infty \;\; \textit{and} \;\; \int_0^T k_1(t)k_2(t)dt < \infty\,, \tag{23}$$

the solution u exists, and is as smooth as the initial data up to a time $T + \delta$ for some $\delta > 0$.

Proof. As before we only present here the basic ideas, and for simplicity we will focus on the case when $\Omega = \mathbf{R}^3$. First, since

$$S(u) = \frac{1}{2}\left(\nabla u + (\nabla u)^t\right)(x,t) = \tilde{S}(\omega)(x,t)\,, \tag{24}$$

we have

$$\frac{1}{2}(\partial_t|\omega|^2 + u\cdot\nabla|\omega|^2) = (\omega\cdot\nabla u,\omega) = (\tilde{S}(\omega)\omega,\omega)\,, \tag{25}$$

which gives

$$\frac{d\|\omega\|_\infty}{dt} \le \sup_x(|\tilde{S}(\omega)|)\|\omega\|_\infty\,. \tag{26}$$

Next, we consider only the singular part of the operator $\omega \mapsto \tilde{S}(\omega)$. The Biot–Savart law reproduces the velocity field from the vorticity according to the formula:

$$u(x,t) = \frac{1}{4\pi}\int \frac{(x-y)\wedge\omega(y)}{|x-y|^3}dy\,. \tag{27}$$

For the essential part of this kernel, we introduce two smooth nonnegative radial functions β_1^δ and β_2^δ with

$$\beta_1^\delta + \beta_2^\delta = 1\,, \beta_\delta^1 = 0 \text{ for } |x| > 2\delta \text{ and } \beta_\delta^2 = 0 \text{ for } |x| < \delta\,. \tag{28}$$

Then we have

$$|\tilde{S}(\omega)| \le |\int(\frac{y}{|y|}\cdot\xi(x))(\mathrm{Det}(\frac{y}{|y|},\xi(x+y),\xi(x))\beta_\delta^1(|y|)|\omega(x+y)|\frac{dy}{|y|^3}| +$$
$$|\int(\frac{y}{|y|}\cdot\xi(x))(\mathrm{Det}(\frac{y}{|y|},\xi(x+y),\xi(x))\beta_\delta^2(|y|))|\omega(x+y)|\frac{dy}{|y|^3}|\,. \tag{29}$$

For the first term we use the bound

$$(\mathrm{Det}(\frac{y}{|y|},\xi(x+y),\xi(x))\beta_\delta^1(|y|)) \le \frac{k_2(t)}{4\pi}|y|\,. \tag{30}$$

to obtain:

$$|\int(\frac{y}{|y|}\cdot\xi(x))(\mathrm{Det}(\frac{y}{|y|},\xi(x+y),\xi(x))\beta_\delta^1(|y|)|\omega(x+y)|\frac{dy}{|y|^3}|$$
$$\le k_2(t)\delta\|\omega\|_\infty \tag{31}$$

Next, we write the second term as

$$\int(\frac{y}{|y|}\cdot\xi(x))(\mathrm{Det}(\frac{y}{|y|},\xi(x+y),\xi(x))\beta_\delta^2(|y|))(\xi(x+y)\cdot(\nabla_y\wedge u(x+y)))\frac{dy}{|y|^3}$$

and integrate by parts with respect to y. With the Lipschitz regularity of ξ one has

$$|\nabla_y\left((\frac{y}{|y|}\cdot\xi(x))(\mathrm{Det}(\frac{y}{|y|},\xi(x+y),\xi(x))\right)| \le Ck_2(t)\,.$$

Therefore, one has (observing that the terms coming from large values of $|y|$ and the terms coming from the derivatives of $\beta_\delta^2(|y|)$ give more regular contributions.)

$$
\begin{aligned}
&|\int (\frac{y}{|y|}\cdot\xi(x))(\mathrm{Det}(\frac{y}{|y|},\xi(x+y),\xi(x))\beta_\delta^2(|y|))|\omega(x+y)|\frac{dy}{|y|^3}| \\
&\leq \int |\nabla_y(\frac{y}{|y|}\cdot\xi(x))(\mathrm{Det}(\frac{y}{|y|},\xi(x+y),\xi(x)))|\beta_\delta^2(|y|))\frac{dy}{|y|^3}\|u\|_\infty \\
&\leq Ck_2(t)|\log(\delta)|\|u\|_\infty \\
&\leq Ck_1(t)k_2(t)|\log\delta|\,.
\end{aligned}
$$

Finally, inserting the above estimates in (26) one obtains for $\|\omega\|_\infty > 1$ and $\delta = \|\omega\|_\infty^{-1}$

$$\frac{d\|\omega\|_\infty}{dt} \leq Ck_2(t)(1+k_1(t))\|\omega\|_\infty \log\|\omega\|_\infty\,,$$

and the conclusion follows as in the case of the Beale–Kato–Majda Theorem.

The reader is referred, for instance, to the book of Majda and Bertozzi [50] and the recent review of Constantin [14] for addition relevant material.

2.3 About the two-dimensional case

In $2d$ case the vorticity $\omega = \nabla\wedge u$ obeys the equation

$$\partial_t(\nabla\wedge u) + (u\cdot\nabla)(\nabla\wedge u) = 0\,. \tag{32}$$

This evolution equation guarantees the persistence of any L^p norm ($1\leq p\leq\infty$) of the vorticity. Taking advantage of this observation Youdovitch proved in his remarkable paper [74] the existence, uniqueness, and global regularity for all solutions with initial vorticity in L^∞. If the vorticity is in L^p, for $1<p\leq\infty$, then one can prove the existence of weak solutions. The same results hold also for $p=1$ and for vorticity being a finite measure with "simple" changes of sign. The proof is more delicate in this limit case, cf. Delort [21] and the section 7.2 below.

3 Pathological behavior of solutions

Continuing with the comments of the previous section one should recall the following facts.

• First, in the three-dimensional case.

i) There is no result concerning the global, in time existence of smooth solution. More precisely, it is not known whether the solution of the Euler dynamics defined with initial velocity, say in H^s, for $s>\frac{3}{2}+1$, on a finite time interval can be extended as a regular, or even as a weak, solution for all positive time.

ii) There is no result concerning the existence, even for a small time, of a weak solution for initial data less regular than in the above case.

iii) Due to the scaling property of the Euler equations in $\mathbf{R}^3$, the problem of global, in time existence, for small initial data, is equivalent to the global existence for all initial data and for all $t \in \mathbf{R}$.

• Second, both in the $2d$ and the $3d$ cases, the fact that a function $u \in L^2([0,T]; L^2(\mathbf{R}^d))$ is a weak solution, i.e., that it satisfies the following relations in the sense of distributions

$$\partial_t u + \nabla \cdot (u \otimes u) + \nabla p = 0\,,\ \nabla \cdot u = 0, u(x,0) = u_0(x)\,, \tag{33}$$

is not enough to define it uniquely in terms of the initial data (except in $2d$ with the additional regularity assumption that $\nabla \wedge u_0 \in L^\infty$.) More precisely, one can construct, following Scheffer [64] and Shnirelman [65], both in $2d$ and $3d$, nontrivial solutions $u \in L^2(\mathbf{R}_t; L^2(\mathbf{R}^d))$ of (33) that are of compact support in space and time.

The following examples may contribute to the understanding of the underlying difficulties. First, one can exhibit (cf. Constantin [9], Gibbon and Ohkitani [31], and references therein) blow up for smooth solutions, with infinite energy, of the $3d$ Euler equations. Such solutions can be constructed as follows. The solution u is (x_1, x_2) periodic on a lattice $(\mathbf{R}/L\mathbf{Z})^2$ and is defined for all $x_3 \in \mathbf{R}$ according to the formula

$$u = (u_1(x_1,x_2,t), u_2(x_1,x_2,t), x_3\gamma(x_1,x_2,t)) = (\tilde{u}, x_3\gamma)\,,$$

which is determined by the following equations:

To maintain the divergence free condition, it is required that

$$\nabla \cdot \tilde{u} + \gamma = 0\,,$$

and to enforce the Euler dynamics, it is required that

$$\begin{aligned} &\partial_t(\nabla \wedge \tilde{u}) + (\tilde{u} \cdot \nabla)(\nabla \wedge \tilde{u}) = \gamma \tilde{u} \\ &\partial_t \gamma + (\tilde{u} \cdot \nabla)\gamma = -\gamma^2 + I(t)\,, \end{aligned}$$

and finally to enforce the (x_1, x_2) periodicity it is required that

$$I(t) = -\frac{2}{L^2} \int_{[0,L]^2} (\gamma(x_1,x_2,t))^2 dx_1 dx_2\,.$$

Therefore, the scalar function γ satisfies an integrodifferential Ricatti equation of the following form

$$\partial_t \gamma + \tilde{u}\nabla\gamma = -\gamma^2 - \frac{2}{L^2} \int_{[0,L]^2} (\gamma(x_1,x_2,t))^2 dx_1 dx_2\,,$$

from which the proof of the blow up, including explicit nature of this blow up, follows.

The above example can be considered as non-physical because the initial energy

$$\int_{(\mathbf{R}^2/L)^2 \times \mathbf{R}} |u(x_1,x_2,x_3,0)|^2 dx_1 dx_2 dx_3$$

is infinite. On the other hand, it is instructive because it shows that the conservation of energy, in the Euler equations, may play a crucial role in the absence of singularity. Furthermore, an approximation of the above solution, by a family of finite energy solutions, would probably be possible but to the best of our knowledge this has not yet been done. Such an approximation procedure would lead to the idea that no uniform bound can be obtained for the stability or regularity of $3d$ Euler equations. Along these lines, one has the following proposition.

Proposition 3.1 *For $1 < p < \infty$ there is no continuous function $\tau \mapsto \phi(\tau)$ such that for any smooth solution of the Euler equations the following estimate*

$$\|u(\cdot,t)\|_{W^{1,p}(\Omega)} \leq \phi(\|u(\cdot,0)\|_{W^{1,p}(\Omega)})\,,$$

is true.

Observe that the above statement is not in contradiction with the local stability results, which produce local control of higher norm at time t in term of higher norm at time 0 as done in (17) according to the formula

$$\text{for } s > \frac{5}{2}\,, \quad \|u(t)\|_{H^s(\Omega)} \leq \frac{\|u(0)\|_{H^s(\Omega)}}{1 - Ct\|u(0)\|_{H^s(\Omega)}}\,.$$

Proof. The proof is done by inspection of a pressureless solution, defined on a periodic box $(\mathbf{R}/\mathbf{Z})^3$ of the form

$$u(x,t) = (u_1(x_2), 0, u_3(x_1 - tu_1(x_2), x_2)),$$

which satisfies

$$\nabla \cdot u = 0\,, \quad \partial_t u + u \cdot \nabla u = 0\,.$$

Therefore, the initial data satisfies the relation

$$\|u(\cdot,0)\|^p_{W^{1,p}(\Omega)} \simeq \int_0^1 |\partial_{x_2} u_1(x_2)|^p dx_1 + \int_0^1\int_0^1 (|\partial_{x_1} u_3(x_1,x_2)|^p dx_1 dx_2 + |\partial_{x_2} u_3(x_1,x_2)|^p) dx_1 dx_2\,. \tag{34}$$

And for $t > 0$

$$\|u(\cdot,t)\|^p_{W^{1,p}(\Omega)} \simeq \int |\partial_{x_2} u_1(x_2)|^p dx_1 dx_2 dx_3 + \int_0^1\int_0^1 (|\partial_{x_1} u_3(x_1,x_2)|^p dx_1 dx_2 + |\partial_{x_2} u_3(x_1,x_2)|^p) dx_1 dx_2 + t^p \int_0^1\int_0^1 |\partial_{x_2} u_1(x_2)|^p |\partial_{x_1} u_3(x_1,x_2)|^p dx_1 dx_2\,. \tag{35}$$

Then a convenient choice of u_1 and u_3 makes the left-hand side of (34) bounded and the term

$$t^p \int_0^1\int_0^1 |\partial_{x_2} u_1(x_2)|^p |\partial_{x_1} u_3(x_1,x_2)|^p dx_1 dx_2\,,$$

on the right-hand side of (35) grows to infinity as $t \to \infty$. The proof is then completed by a regularization argument.

Remark 3.1 *With smooth initial data, the above construction gives an example of global, in time, smooth solution with vorticity growing (here only linearly) for $t \to \infty$.*

As in the case of the Riccati differential inequality $y' \leq Cy^2$, one can obtain sufficient conditions for the existence of a smooth solution during a finite interval of time (say $0 \leq t < T$). On the other hand, this gives no indication on the possible appearance of blow up after such time. Complicated phenomena that appear in the fluid, due to strong nonlinearities, may later interact in such a way that they balance each other and bring back the fluid to a smooth regime. Such phenomena is called *singularity depletion.*

An example which seems to illustrate such cancelation has been constructed by Hou and Li [35]. It is concerning axi-symmetric solutions of the $3d$ Euler equations form $rf(z)$, which obviously possess infinite energy.

Specifically, let us start with the following system of integro-differential equations with solutions that are defined for $(z,t) \in (\mathbf{R}/\mathbf{Z}) \times \mathbf{R}^+$

$$u_t + 2\psi u_z = -2vu\,,\ v_t + 2\psi v_z = u^2 - v^2 + c(t) \tag{36}$$

$$\psi_z = v\,,\ \int_0^1 v(z,t)dz = 0\,. \tag{37}$$

In (36), the z independent function $c(t)$ is chosen to enforce the second relation of (37), which in turn makes the function $\psi(z,t)$ $1-$periodic in the z direction. As a result one has the following:

Lemma 3.1 *For any initial data $(u(z,0), v(z,0)) \in C^m(\mathbf{R}/\mathbf{Z})$, with $m \geq 1$, the system (36) and (37) has a unique global, in time, smooth solution.*

Proof. The proof relies on a global *a priori* estimate. Taking the derivative with respect to z variable gives (using the notation $(u_z, v_z) = (u', v')$) :

$$u'_t + 2\psi u'_z - 2\psi_z u' = -2v'u - 2vu'$$
$$v'_t + 2\psi v'_z - 2\psi_z v' = 2uu' - 2vv'\,.$$

Next, one uses the relation $\psi_z = -v$, multiplies the first equation by u, multiplies the second equation by v, and adds them to obtain

$$\frac{1}{2}(u_z^2 + v_z^2)_t + \psi(u_z^2 + v_z^2)_z = 0\,. \tag{38}$$

The relation (38) provides a uniform L^∞ bound on the $z-$derivatives of u and v. A uniform L^∞ bound for v follows from the Poincaré inequality, and finally one uses for u the following Gronwall estimate

$$\|u(z,t)\|_{L^\infty} \leq \|u(z,0)\|_{L^\infty} e^{t\|(u(z,0))^2 + (v(z,0))^2\|_{L^\infty}}\,. \tag{39}$$

Remark 3.2 *The global existence for solution of the system (36) and (37), with no restriction on the size of the initial data, is a result of delicate balance/cancelation, which depends on the coefficients of the system. Any modification of these coefficients may lead to a blow up in a finite time of the solutions to the modified system. On the other hand, the solutions of the system (36)–(37) may grow exponentially in time. Numerical simulations performed by [35] indicate that the exponential growth rate in (39) may get saturated.*

The special structure of the system (36)–(37) is related to *the* $3d$ *axi-symmetric Euler equations with swirl* as follows. Introduce the orthogonal basis

$$e_r = (\frac{x}{r}, \frac{y}{r}, 0)\,,\ e_\theta = (-\frac{y}{r}, \frac{x}{r}, 0)\,,\ e_z = (0,0,1)\,,$$

and with the solution of the system (36) and (37), construct solutions of the $3d$ $(2+1/2)$ Euler equation according to the following proposition.

Proposition 3.1 *Assume that $u(z,t)$ and $\psi(z,t)$ are solutions of the systems (36) and (37), then the function*

$$U(z,t) = -r\frac{\partial \psi(z,t)}{\partial z} e_r + ru(z,t)e_\theta + 2r\psi(z,t)e_z$$

is a smooth solution of the $3d$ *Euler, but of an infinite energy. Moreover, this solution is defined for all time and without any smallness assumption on the size of the initial data.*

4 Weak limit of solutions of the Navier–Stokes Dynamics

As we have already remarked in the introduction, for both practical problems, as well as for mathematical analysis, it is feasible to consider the Euler dynamics as the limit of the Navier–Stokes dynamics, when the viscosity tends to zero. Therefore, this section is devoted to the analysis of the *weak limit,* as $\nu \to 0$, of Leray–Hopf type solutions of the Navier–Stokes equations in $2d$ and $3d$. We will consider only convergence over finite intervals of time $0 < t < T < \infty$. We also recall that ν denotes the dimensionless viscosity, i.e., the inverse of the Reynolds number.

4.1 Reynolds stresses tensor and dissipative solutions

As above, we denote by Ω an open set in $\mathbf{R}^d$. For any initial data, $u_\nu(x,0) = u_0(x) \in L^2(\Omega)$, and any given viscosity, $\nu > 0$, the pioneer works of Leray [44] and Hopf [34] (see also Ladyzhenskaya [41] for a detailed survey), which were later generalized by Scheffer [64], and by Caffarelli, Kohn and Nirenberg [8], showed the existence of functions u_ν and p_ν with the following properties

$$u_\nu \in L^\infty((0,T); L^2(\Omega)) \cap L^2((0,T); H^1_0(\Omega)), \quad \text{for every } T \in (0,\infty). \tag{40}$$

In addition, they satisfy the Navier–Stokes equations

$$\partial_t u_\nu + \nabla \cdot (u_\nu \otimes u_\nu) - \nu \Delta u_\nu + \nabla p_\nu = 0\,, \tag{41}$$

$$\nabla \cdot u_\nu = 0\,,\ u_\nu = 0 \text{ on } \partial\Omega\,, \tag{42}$$

in the sense of distributions. Moreover, such solutions satisfy the "pointwise" energy inequality

$$\frac{1}{2}\partial_t |u_\nu(x,t)|^2 + \nu |\nabla u_\nu(x,t)|^2 + \\ \nabla \cdot ((u_\nu \otimes u_\nu)(x,t) - \nu \nabla \frac{|u_\nu(x,t)|^2}{2}) + \nabla \cdot (p_\nu(x,t) u_\nu(x,t)) \leq 0 \tag{43}$$

or in integrated form

$$\frac{1}{2}\partial_t \int_\Omega |u_\nu(x,t)|^2 dx + \nu \int_\Omega |\nabla u_\nu(x,t)|^2 dx \leq 0\,. \tag{44}$$

A pair $\{u_\nu, p_\nu\}$ which satisfies (40),(42) and (43) is called a suitable weak solution of the Navier–Stokes equations, in the sense of Caffarelli–Kohn–Nirenberg. If it satisfies, however, the integrated version of the energy inequality (44) instead of the pointwise energy inequality (43) it will then be called a Leray–Hopf weak solution of the Navier–Stokes equations.

In two-dimensions (or in any dimension but with stronger hypothesis on the smallness of the size of the initial data with respect to the viscosity) these solutions are shown to be smooth, unique, and depend continuously on the initial data. Furthermore, in this case, one has equality in the relations (43) and (44) instead of inequality.

Therefore, as a result of the above, and in particular the energy inequality (44), one concludes that, modulo the extraction of a subsequence, the sequence $\{u_\nu\}$ converges in the weak$-*$ topology of $L^\infty(\mathbf{R}_t^+, L^2(\Omega))$ to a limit $\overline{u}$; and the sequence $\{\nabla p_\nu\}$ converges to a distribution $\nabla \overline{p}$, as $\nu \to 0$, for which the following holds

$$\overline{u} \in L^\infty(\mathbf{R}_t^+, L^2(\Omega)), \nabla \cdot \overline{u} = 0 \text{ in } \Omega\,,\ \overline{u} \cdot \vec{n} = 0 \text{ on } \partial\Omega\,,$$

$$\int_\Omega |\overline{u}(x,t)|^2 dx + 2\nu \int_0^t \int_\Omega |\nabla u|^2 dx dt \leq \int_\Omega |\overline{u}_0(x)|^2 dx\,,$$

$$\lim_{\nu \to 0} (u_\nu \otimes u_\nu) = \overline{u} \otimes \overline{u} + \lim_{\nu \to 0} ((u_\nu - \overline{u}) \otimes (u_\nu - \overline{u}))\,, \tag{45}$$

$$\partial_t \overline{u} + \nabla \cdot (\overline{u} \otimes \overline{u}) + \lim_{\nu \to 0} \nabla \cdot \Bigg(\overline{u} - u_\nu) \otimes (\overline{u} - u_\nu) \Bigg) + \nabla \overline{p} = 0\,. \tag{46}$$

Observe that the term

$$RT(x,t) = \lim_{\nu \to 0} (\overline{u}(x,t) - u_\nu(x,t)) \otimes (\overline{u}(x,t) - u_\nu(x,t)) \tag{47}$$

is a positive, symmetric, measure-valued tensor. In analogy with (see below) the statistical theory of turbulence, this tensor may carry the name of Reynolds

stresses tensor or turbulence tensor. In particular, certain turbulent regions will correspond to the support of this tensor.

This approach leads to the following questions.

1. What are the basic properties (if any) of the tensor $RT(x,t)$?

2. When does the tensor $RT(x,t)$ identically equal zero? Or, what is equivalent, when does the limit pair $\{\overline{u},\overline{p}\}$ satisfies the Euler equations?

3. When does the energy dissipation

$$\nu \int_0^T \int_\Omega |\nabla u_\nu(x,t)|^2 dxdt$$

tend to zero as $\nu \to 0$?

4. Assuming that $\{\overline{u},\overline{p}\}$ is a solution of the Euler equations is such a solution regular enough to imply the conservation energy?

Hereafter, we will use the following notation for the L^2-norm

$$|\Phi| = \left(\int_\Omega |\Phi(x)|^2 dx \right)^{1/2} .$$

Remark 4.1 *The tensor $RT(x,t)$ is generated by the high frequency spatial oscillations of the solution. This feature will be explained in more details in section 6.1. Therefore, such behavior should be intrinsic and, in particular, independent of orthogonal (rotation) change of coordinates. For instance, in the $2d$ case, assuming that the function $\overline{u}$ is regular, the invariance under rotation implies the relation*

$$RT(x,t) = \alpha(x,t) Id + \frac{1}{2}\beta(x,t)(\nabla\overline{u} + (\nabla\overline{u})^T) ,$$

where $\alpha(x,t)$ and $\beta(x,t)$ are some scalar valued (unknown) functions. Thus, the equation (46) becomes

$$\partial_t \overline{u} + \nabla\cdot(\overline{u}\otimes\overline{u}) + \nabla\cdot(\beta(x,t)\frac{1}{2}(\nabla\overline{u} + (\nabla\overline{u})^T)) + \nabla(\overline{p} + \alpha(x,t)) = 0 . \quad (48)$$

Of course, this "soft information" does not indicate whether $\beta(x,t)$ is zero or not. It also does not indicate whether this coefficient is positive, nor how to compute it. But this turns out to be the turbulent eddy diffusion coefficient that is present in classical engineering turbulence models like the Smagorinsky or the $k\epsilon$ models (see, e.g., [43], [54], [61], and [63].

Remark 4.2 *Assume that the limit $\{\overline{u},\overline{p}\}$ is a solution of the Euler equations which is regular enough to ensure the conservation of energy, i.e. $|\overline{u}(t)|^2 = |u_0|^2$. Then by virtue of the energy relation (44) we have*

$$\frac{1}{2}|u_\nu(t)|^2 + \nu \int_0^t |\nabla u_\nu(s)|^2 ds \le \frac{1}{2}|u_0|^2 ,$$

and by the weak limit relation

$$\liminf_{\nu\to 0} \frac{1}{2}|u_\nu(t)|^2 \geq \frac{1}{2}|\overline{u}(t)|^2 \,,$$

one has that the strong convergence and the relation

$$\liminf_{\nu\to 0} \nu \int_0^t |\nabla u_\nu(s)|^2 ds = 0$$

hold. The following question was then raised by Onsager [59]: "What is the minimal regularity needed to be satisfied by the solutions of the $2d$ *or* $3d$ *Euler equations that would imply conservation of energy?". The question was pursued by several authors up to the contribution of Eynik [25], and Constantin, E and Titi [15]. Basically in* $3d$ *it is shown that if* u *is bounded in* $L^\infty(\mathbf{R}_t^+, H^\beta(\Omega))$ *, with* $\beta > 1/3$ *, the energy*

$$\frac{1}{2}\int_\Omega |u(x,t)|^2 dx$$

is constant. On the other hand, arguments borrowed from statistical theory of turbulence (cf. section 6.2), show that the sequence u_ν *will be, in general, bounded in* $L^\infty(\mathbf{R}_t^+, H^{\frac{1}{3}}(\Omega))$ *and one should observe that such a statement does not contradict the possibility of decay of energy in the limit as* $\nu \to 0$ *.*

To study the weak limit of Leray–Hopf solutions of the Navier-Stokes dynamics, P.L. Lions and R. Di Perna [46] introduced the notion of *Dissipative Solution* of the Euler equations. To motivate this notion, let $w(x,t)$ be a divergence free test function, which satisfies $w \cdot \vec{n} = 0$ on the boundary $\partial\Omega$. Let

$$E(w) = \partial_t w + P(w \cdot \nabla w) \,, \tag{49}$$

where P is the Leray–Helmholtz projector (see, e.g., [17]). Then for any smooth, divergence free, solution of the Euler equations $u(x,t)$ in Ω , which satisfies the boundary condition $u \cdot \vec{n} = 0$ on $\partial\Omega$, one has:

$$\begin{gathered}
\partial_t u + \nabla \cdot (u \otimes u) + \nabla p = 0 \,, \\
\partial_t w + \nabla \cdot (w \otimes w) + \nabla q = E(w) \,, \\
\tfrac{d|u-w|^2}{dt} + 2(S(w)(u-w),(u-w)) = 2(E(w),u-w) \,,
\end{gathered}$$

where $S(w)$ denotes, as before, the symmetric tensor

$$S(w) = \frac{1}{2}(\nabla w + (\nabla w)^T) \,.$$

By integration in time this gives

$$\begin{aligned}
&|u(t) - w(t)|^2 \leq e^{\int_0^t 2\|S(w)(s)\|_\infty ds}|u(0) - w(0)|^2 \\
&+2\int_0^t e^{\int_s^t 2\|S(w)(\tau)\|_\infty d\tau}(E(w)(s),(u-w)(s))ds \,.
\end{aligned} \tag{50}$$

The above observation leads to the following definition

Definition 4.1 *A divergence free vector field*

$$u \in w - C(\mathbf{R}_t; (L^2(\Omega))^d),$$

which satisfies the boundary condition $u \cdot \vec{n} = 0$ *on* $\partial\Omega$*, is called a dissipative solution of the Euler equations (11), if for any smooth divergence free vector field* w *, with* $w \cdot \vec{n} = 0$ *on* $\partial\Omega$ *, the inequality (50) holds.*

The following statement is easy to verify, but we mention it here for the sake of clarity.

Theorem 4.1 *i) Any classical solution* u *of the Euler equations (11) is a dissipative solution.*

ii) Every dissipative solution satisfies the energy inequality relation

$$|u(t)|^2 \leq |u(0)|^2 \,. \tag{51}$$

iii) The dissipative solutions are "stable with respect to classical solutions". More precisely, if w *is a classical solution and* u *is a dissipative solution of the Euler equations, then one has*

$$|u(t) - w(t)|^2 \leq e^{\int_0^t 2\|S(w)(s)\|_\infty ds} |u(0) - w(0)|^2 \,.$$

In particular, if there exists a classical solution for specific initial data, then any dissipative solution with the same initial data coincides with it.

iv) In the absence of physical boundaries, i.e. in the case of periodic boundary conditions or in the whole space $\mathbf{R}^d, d = 2, 3$*, any weak limit, as* $\nu \to 0$*, of Leray–Hopf solutions of the Navier–Stokes equations is a dissipative solution of Euler equations.*

Proof and remarks. The point i) is a direct consequence of the construction. To prove ii) we consider $w \equiv 0$ as a classical solution. As a result, one obtains for any dissipative solution, the relation (51), which justifies the name dissipative. Furthermore, it shows that the pathological examples constructed by Scheffer [64] and Shnirelman [65] are not dissipative solutions of Euler equations.

For the point iii) we use in (50) the fact that w being a classical solution implies that $E(w) \equiv 0$. We also observe that this statement is in the spirit of the "weak with respect to strong" stability result of Dafermos [20] for hyperbolic systems.

Next, we prove iv) in the absence of physical boundaries. Let u_ν be a Leray–Hopf solution of the Navier–Stokes system, which satisfy an energy inequality in (44), and let w be a classical solution of the Euler equations. By subtracting the following two equations from each other

$$\begin{aligned} &\partial_t u_\nu + \nabla \cdot (u_\nu \otimes u_\nu) - \nu \Delta u_\nu + \nabla p_\nu = 0 \\ &\partial_t w + \nabla \cdot (w \otimes w) - \nu \Delta w + \nabla p = -\nu \Delta w, \end{aligned}$$

and taking the L^2 inner product of the difference with $(u_\nu - w)$ one obtains

$$\frac{d|u_\nu - w|^2}{dt} + 2(S(w)(u_\nu - w), (u_\nu - w)) - 2\nu(\Delta(u_\nu - w), (u_\nu - w))$$
$$\leq 2(E(w), u_\nu - w) - (\nu\Delta w, (u_\nu - w))\,. \tag{52}$$

We stress that the above step is formal, and only through rigorous arguments one can see the reason for obtaining an inequality in (52), instead of an equality. However, this should not be a surprise because we are dealing with Leray–Hopf solutions, u_ν, of the Navier–Stokes system which satisfy an energy inequality in (44), instead of an equality.

Now, to conclude our proof we observe that in the absence of physical boundaries one uses the relation

$$-\nu \int \Delta(w - u_\nu)(x,t) \cdot (w - u_\nu)(x,t)dx = \nu \int |\nabla(w - u_\nu)(x,t)|^2 dx \tag{53}$$

and the result follows by letting ν tend to zero.

Remark 4.3 *The above theorem states in particular, and in the absence of physical boundaries, that as long as a smooth solution of the Euler equations does exist, it is the limit, as $\nu \to 0$, of any sequence of Leray–Hopf solutions of the Navier–Stokes equations with the same initial data. In a series of papers, starting with Bardos, Golse and Levermore [4], connections between the notion of Leray–Hopf solutions for the Navier–Stokes equations and renormalized solutions of the Boltzmann equations, as defined by P.L. Lions and Di Perna, were established. In particular, it was ultimately shown by Golse and Saint Raymond [32] that, modulo the extraction of a subsequence, and under a convenient space time scaling, any sequence of such renormalized solutions of the Boltzmann equations converge (in some weak sense) to a Leray–Hopf solution of the Navier–Stokes system. On the other hand, it was shown by Saint Raymond [68] that, under a scaling which reinforces the nonlinear effect (corresponding at the macroscopic level to a Reynolds number going to infinity), any sequence (modulo extraction of a subsequence) of the renormalized solutions of the Boltzmann equations converges to a dissipative solution of the Euler equations. Therefore, such a sequence of normalized solutions of the Boltzmann equations converges to the classical solution of Euler equations, as long as such solution exists. In this situation, one should observe that, with the notion of dissipative solutions of Euler equations, classical solutions of the Euler equations play a similar role for the "Leray–Hopf limit" and the "Boltzmann limit."*

Remark 4.4 *There are at least two situations where the notion of dissipative solution of Euler equations is not helpful.*

The first situation is concerned with the 2d Euler equations. Let $u_\epsilon(x,t)$ be the sequence of solutions of the 2d Euler equations corresponding to the sequence of smooth initial data $u_\epsilon(x,0)$. Suppose that the sequence of initial data $u_\epsilon(x,0)$ converges weakly, but not strongly, in $L^2(\Omega)$ to an initial data $\overline{u}(x,0)$, as $\epsilon \to 0$.

Then for any smooth, divergence free vector field w, one has, thanks to (50), the relation

$$|u_\epsilon(t) - w(t)|^2 \leq e^{\int_0^t 2\|S(w)(s)\|_\infty ds}|u_\epsilon(0) - w(0)|^2$$
$$+2\int_0^t e^{\int_s^t 2\|S(w)(\tau)\|_\infty d\tau}(E(w), u_\epsilon - w)(s)ds\,. \tag{54}$$

However, with the weak convergence as $\epsilon \to 0$, one has only

$$|\overline{u}(0) - w(0)|^2 \leq \liminf_{\epsilon\to 0} |u_\epsilon(0) - w(0)|^2\,,$$

and (54) might not hold at the limit, as $\epsilon \to 0$. To illustrate this situation, we consider a sequence of oscillating solutions of the $2d$ Euler equations of the form

$$u_\epsilon(x,t) = U(x,t,\frac{\phi(x,t)}{\epsilon}) + O(\epsilon)\,,$$

where the map $\theta \to U(x,t,\theta)$ is a nontrivial $1-$periodic function. In Cheverry [13] a specific example was constructed such that

$$\overline{u} = w - \lim_{\epsilon\to 0} u_\epsilon = \int_0^1 U(x,t,\theta)d\theta$$

is no longer a solution of the Euler equations. The obvious reason for that (in comparison to the notion of dissipative solution), is the fact that

$$U(x,0,\frac{\phi(x,0)}{\epsilon})$$

does not converge strongly in $L^2(\Omega)$.

The second situation, which will be discussed at length below, corresponds to the weak limit of solutions of the Navier–Stokes equations in a domain with physical boundary, subject to the no-slip Dirichlet boundary condition.

As we have already indicated in Theorem 4.1, one of the most important features of the above definition of dissipative solution of Euler equations is that it coincides with the classical solution of Euler equations, when the latter exists. This can be accomplished by replacing w in (50) with this classical solution of Euler equations. Therefore, any procedure for approximating dissipative solutions of Euler must lead to, in the limit, to inequality (50). Indeed, in the absence of physical boundaries, we have been successful in showing, in an almost straight forward manner, in Theorem 4.1, that the Leray–Hopf weak solutions of the Navier–Stokes equations converge to dissipative solutions of Euler equations. On the other hand, in the presence of physical boundaries, the proof does not carry on in a smooth manner because of the boundary effects. Specifically, in the case of domains with physical boundaries, inequality (52) leads to

$$\frac{1}{2}\frac{d|u_\nu - w|^2}{dt} + (S(w)(u_\nu - w), (u_\nu - w)) + \nu\int |\nabla(w - u_\nu)|^2 dx$$
$$\leq (E(w), u_\nu - w) - \nu(\Delta w, (u_\nu - w)) + \nu\int_{\partial\Omega} \partial_n u_\nu \cdot w d\sigma\,. \tag{55}$$

The very last term in (55) represents the boundary effect. We will discuss below the subtleties in handling this term.

5 No-slip Dirichlet boundary conditions for the Navier–Stokes dynamics

This section is devoted to the very few available results concerning the limit, as $\nu \to 0$, of solutions of the Navier–Stokes equations in a domain $\Omega \subset \mathbf{R}^d$, $d = 2, 3$ with the homogenous (no-slip) Dirichlet boundary condition $u_\nu = 0$ on $\partial\Omega$. This boundary condition is not the easiest to deal with, as far as the zero viscosity limit is concerned. For instance, the solutions of the of $2d$ Navier–Stokes equations, subject to the boundary conditions $u_\nu \cdot n = 0$ and $\nabla \wedge u_\nu = 0$, are much better understood and much easier to analyze mathematically [2] as the viscosity $\nu \to 0$. However, the no-slip boundary condition is the one which is more suitable to consider physically for the following reasons.

i) It can be deduced in the smooth (laminar) regime, from the Boltzmann kinetic equations when the interaction with the boundary is described by a scattering kernel.

ii) It generates the pathology that is observed in physical experiments, like the Von Karman vortex streets. Moreover, one should keep in mind that almost all high Reynolds number turbulence experiments involve a physical boundary (very often turbulence is generated by a pressure driven flow through a grid!)

The problem emerges first from the boundary layer. This is because for the Navier–Stokes dynamics, the whole velocity field equals zero on the boundary, i.e. $u_\nu = 0$ on $\partial\Omega$, while for the Euler dynamics it is only the normal component of velocity field is equal to zero on the boundary, i.e. $u \cdot \vec{n} = 0$ on $\partial\Omega$. Therefore, in the limit, as the viscosity $\nu \to 0$, the tangential component of the velocity field of the Navier–Stokes dynamics, u_ν, generates, by its "jump", a boundary layer. Then, unlike the situation with linear singular perturbation problems, the nonlinear advection term of the Navier–Stokes equations may propagate this instability inside the domain.

As we have already pointed out, the very last term in (55), i.e. the boundary integral term in the case of no-slip boundary condition,

$$\begin{aligned} \nu \int_{\partial\Omega} \frac{\partial u_\nu}{\partial n} \cdot w d\sigma &= \nu \int_{\partial\Omega} (\frac{\partial u_\nu}{\partial n})^\tau \cdot w^\tau d\sigma \\ &= \nu \int_{\partial\Omega} (\nabla \wedge u_\nu) \cdot (\vec{n} \wedge w) d\sigma \,, \end{aligned} \tag{56}$$

is possibly responsible for the loss of regularity in the limit as $\nu \to 0$. This is stated more precisely in the following.

Proposition 5.1 *Let $u(x,t)$ be a solution of the incompressible Euler equations*

in $\Omega \times (0,T]$ *, with the following regularity assumptions.*

$$S(u) = \frac{1}{2}(\nabla u + (\nabla u)^T) \in L^1((0,T); L^\infty(\Omega)), \text{ and}$$
$$u \in L^2((0,T); H^s(\Omega)), \text{ for } s > \frac{1}{2}\,.$$

Moreover, suppose that the sequence u_ν*, of Leray–Hopf solutions of the Navier–Stokes dynamics (with no-slip boundary condition) with the initial data* $u_\nu(x,0) = u(x,0)$*, satisfies the relation*

$$\lim_{\nu\to 0} \nu \|P_{\partial\Omega}(\nabla \wedge u_\nu)\|_{L^2((0,T);H^{-s+\frac{1}{2}}(\partial\Omega))} = 0\,,$$

where $P_{\partial\Omega}$ *denotes the projection on the tangent plane to* $\partial\Omega$ *according to the formula.*

$$P_{\partial\Omega}(\nabla \wedge u_\nu) = \nabla \wedge u_\nu - ((\nabla \wedge u_\nu)\cdot \vec{n})\vec{n}\,.$$

Then, the sequence u_ν *converges to* u *in* $C((0,T); L^2(\Omega))$*.*

The proof is a direct consequence of (55) and (56), with w being replaced by u. This Proposition can be improved with the following simple and beautiful theorem of Kato which takes into account the vorticity production in the boundary layer $\{x \in \Omega \,|d(x,\partial\Omega) < \nu\}$, where $d(x,y)$ denotes the Euclidean distance between the points x and y.

Theorem 5.1 *Let* $u(x,t) \in W^{1,\infty}((0,T)\times\Omega)$ *be a solution of the Euler dynamics, and let* u_ν *be a sequence of Leray–Hopf solutions of the Navier–Stokes dynamics with no-slip boundary condition.*

$$\partial_t u_\nu - \nu\Delta u_\nu + \nabla\cdot(u_\nu \otimes u_\nu) + \nabla p_\nu = 0\,,\; u_\nu(x,t) = 0 \text{ on } \partial\Omega\,, \tag{57}$$

with initial data $u_\nu(x,0) = u(x,0)$ *. Then, the following facts are equivalent.*

$$(i) \quad \lim_{\nu\to 0} \nu \int_0^T \int_{\partial\Omega} (\nabla \wedge u_\nu)\cdot(\vec{n}\wedge u) d\sigma dt = 0 \tag{58}$$

$$(ii) \quad u_\nu(t) \to u(t) \text{ in } L^2(\Omega) \text{ uniformly in } t \in [0,T] \tag{59}$$

$$(iii) \quad u_\nu(t) \to u(t) \text{ weakly in } L^2(\Omega) \text{ for each } t \in [0,T] \tag{60}$$

$$(iv) \quad \lim_{\nu\to 0} \nu \int_0^T \int_\Omega |\nabla u_\nu(x,t)|^2 dx dt = 0 \tag{61}$$

$$(v) \quad \lim_{\nu\to 0} \nu \int_0^T \int_{\Omega\cap\{d(x,\partial\Omega)<\nu\}} |\nabla u_\nu(x,t)|^2 dx dt = 0\,. \tag{62}$$

Sketch of the proof. The statement (59) is deduced from (58) by replacing w by u in (55) and (56). No proof is needed to deduce (60) from (59), or (61) from (60).

Next, one recalls the energy inequality (44), satisfied by the Leray–Hopf solutions of the the Navier–Stokes dynamics

$$\frac{1}{2}\int_\Omega |u_\nu(x,T)|^2 dx + \nu\int_0^T\int_\Omega |\nabla u_\nu(x,t)|^2 dxdt \le \frac{1}{2}\int_\Omega |u(x,0)|^2 dx\,. \tag{63}$$

By virtue of the weak convergence, as stated in (60), and the fact u is a smooth solution of the Euler dynamics, one has

$$\begin{aligned}\lim_{\nu\to 0}\frac{1}{2}\int_\Omega |u_\nu(x,T)|^2 dx &\ge \frac{1}{2}\int_\Omega |\lim_{\nu\to 0} u_\nu(x,T)|^2 dx\\ &= \frac{1}{2}\int_\Omega |u(x,T)|^2 dx = \frac{1}{2}\int_\Omega |u(x,0)|^2 dx\,. \end{aligned}\tag{64}$$

Together with (63) this shows that (60) implies (61).

The most subtle part in the proof of this theorem is the fact that (62) implies (58). The first step is the construction of a divergence free function $v_\nu(x,t)$ with support in the region $\{x\in\Omega | d(x,\partial\Omega)\le\nu\}\times[0,T)$, which coincides with u on $\partial\Omega\times[0,T]$, and which satisfies (with K being a constant that is independent of ν) the following estimates

$$\|v_\nu\|_{L^\infty(\Omega\times(0,T))} + \|d(x,\partial\Omega)\nabla v_\nu\|_{L^\infty(\Omega\times(0,T))} \le K\,, \tag{65}$$

$$\|(d(x,\partial\Omega))^2\nabla v_\nu\|_{L^\infty(\Omega\times(0,T))} \le K\nu\,, \tag{66}$$

$$\|v_\nu\|_{L^\infty((0,T);L^2(\Omega))} + \|\partial_t v_\nu\|_{L^\infty((0,T);L^2(\Omega))} \le K\nu^{\frac{1}{2}}\,, \tag{67}$$

$$\|\nabla v_\nu\|_{L^\infty((0,T);L^2(\Omega))} \le K\nu^{-\frac{1}{2}}\,, \tag{68}$$

$$\|\nabla v_\nu\|_{L^\infty(\Omega\times(0,T))} \le K\nu^{-1}\,. \tag{69}$$

Then we multiply the Navier–Stokes equations by v_ν and integrate to obtain

$$\begin{aligned}-\nu\int_0^T\int_{\partial\Omega}(\nabla\wedge u_\nu)\cdot(\vec{n}\wedge u)d\sigma dt &= -\nu\int_0^T\int_{\partial\Omega}\frac{\partial u_\nu}{\partial n}\cdot v_\nu d\sigma dt\\ = -\nu\int_0^T\int_\Omega \Delta u_\nu\cdot v_\nu dxdt - \nu\int_0^T\int_\Omega(\nabla u_\nu : \nabla v_\nu)dxdt \end{aligned}\tag{70}$$

$$= -\int_0^T(\partial_t u_\nu, v_\nu)dt - \int_0^T(\nabla\cdot(u_\nu\otimes u_\nu), v_\nu)dt - \nu\int_0^T(\nabla u_\nu,\nabla v_\nu)dt\,. \tag{71}$$

Eventually, by using the above estimates and (62), one can show that

$$\lim_{\nu\to 0}\left(\int_0^T((\partial_t u_\nu, v_\nu) + (\nabla\cdot(u_\nu\otimes u_\nu), v_\nu) + \nu(\nabla u_\nu,\nabla v_\nu))dt\right) = 0\,, \tag{72}$$

which completes the proof.

Remark 5.1 *In Constantin and Wu [18] the authors study the rate of convergence of solutions of the* 2*d Navier–Stokes equations to the solutions of the*

Euler equations in the absence of physical boundaries, for finite intervals of time. Their main observation is that while the rate of convergence, in the L^2–norm, for smooth initial data is of the order $\mathcal{O}(\nu)$ it is of the order of $\mathcal{O}(\sqrt{\nu})$ for less smooth initial. The order of convergence $\mathcal{O}(\sqrt{\nu})$ is, for instance, attained for the vortex patch data with smooth boundary. In this case the fluid develops an internal boundary layer which is responsible for this reduction in the order of convergence.

In the 2d case and for initial data of finite $W^{1,p}$ norm, with $p > 1$, (and also for initial data with vorticity being a finite measure with a "simple" change of sign [21], [49]) one can prove the existence of "weak solutions of the Euler dynamics." In the absence of physical boundaries, such solutions are limit points of a family of (uniquely determined) Leray–Hopf solutions of the Navier–Stokes dynamics. However, these weak solutions of the, Euler equations are not uniquely determined, and the issue of the conservation of energy for these weak solutions is, to the best of our knowledge, completely open.

On the other hand, in a domain with physical boundary and with smooth initial data, Theorem 5.1 shows a clear cut difference between the following two situations (the same remark being valid locally in time for the 3d problems).

i) The mean rate of dissipation of energy

$$\epsilon = \frac{\nu}{T} \int_0^T \int |\nabla u_\nu(x,t)|^2 dxdt$$

goes to zero as $\nu \to 0$, and the sequence u_ν of Leray–Hopf solutions converges strongly to the regular solution $\overline{u}$ of the Euler dynamics.

ii) The mean rate of dissipation of energy does not go to zero as $\nu \to 0$ (modulo the extraction of a subsequence), so the corresponding weak limit u of u_ν does not conserve energy, i.e.

$$\frac{1}{2}|u(t)|^2 < \frac{1}{2}|u(0)|^2 \,, \quad \textit{for some} \quad t \in (0,T) \,,$$

and one of the following two scenarios may occur at the limit.

a) At the limit one obtains a weak solution (not a strong solution) of the Euler dynamics that exhibits energy decay. Such a scenario is compatible with a uniform estimate for the Fourier spectra

$$E_\nu(k,t) = |\hat{u}_\nu(k,t)|^2 |k|^{d-1} \,,\ \hat{u}_\nu(k,t) = \frac{1}{(2\pi)^d} \int_{\mathbf{R}^d} e^{-ikx} u_\nu(x,t) dx \,,$$

which may satisfy a uniform, in ν, estimate of the following type

$$E_\nu(k,t) \le C|k|^{-\beta} \,, \tag{73}$$

provided $\beta < 5/3$. Otherwise, this would be in contradiction with the results of Onsager [59], Eyink [25], and Constantin, E and Titi [15].

b) No estimate of the type (73) is uniformly (in the viscosity) true, and the limit is not even a solution of the Euler dynamics, rather a solution of a modified system of equations with a term related to turbulence modeling - an "eddy-viscosity" like term.

6 Deterministic and statistical spectra of turbulence

6.1 Deterministic spectra and Wigner transform

The purpose of this section is the introduction of Wigner measures for the analysis (in dimensions $d = 2, 3$) of the Reynolds stresses tensor

$$RT(u_\nu)(x,t) = \lim_{\nu \to 0}((u_\nu - \overline{u}) \otimes (u_\nu - \overline{u}))\,, \tag{74}$$

which appears in the weak limit process of solutions of the Navier–Stokes equations u_ν, as $\nu \to 0$, cf. (45). (Notice that $RT(u_\nu)(x,t)$ is independent of the viscosity ν, but it depends on the sequence $\{u_\nu\}$.) This point of view will be compared below (cf. section 6.2) to ideas emerging from statistical theory of turbulence.

Let $\{u_\nu, p_\nu\}$ be a sequence of Leray–Hopf solutions of the Navier–Stokes equations, subject to no-slip Dirichlet boundary condition in a domain Ω (with a physical boundary). Thanks to the energy inequality (44) (possibly equality in some cases)

$$\frac{1}{2}|u_\nu(\cdot,t)|^2 + \nu \int_0^t |\nabla u_\nu(\cdot,t)|^2 dt \le \frac{1}{2}|u(\cdot,0)|^2\,, \tag{75}$$

the sequence $\{u_\nu\}$ converges (modulo the extraction of a subsequence), as $\nu \to 0$, in the weak$-*$ topology of the Banach space $L^\infty((0,T); L^2(\Omega))$, to a divergence free vector field $\overline{u}$, and the sequence of distributions $\{\nabla p_\nu\}$ converges to a distribution $\nabla \overline{p}$. Moreover, the pair $\{\overline{u}, \overline{p}\}$ satisfies the system of equations

$$\nabla \cdot \overline{u} = 0 \ \text{ in } \Omega\,, \quad \overline{u} \cdot \vec{n} = 0 \ \text{ on } \partial\Omega\,, \tag{76}$$

$$\partial_t \overline{u} + \nabla \cdot (\overline{u} \otimes \overline{u}) + \nabla \cdot RT(u_\nu) + \nabla \overline{p} = 0 \ \text{ in } \Omega\,, \tag{77}$$

(cf. (46)). Being concerned with the behavior of the solution inside the domain, we consider an arbitrary open subset Ω', whose closure is a compact subset of Ω, i.e. $\overline{\Omega'} \subset\subset \Omega$. Assuming the weak$-*$ limit function $\overline{u}$ belongs to the space $L^2((0,T); H^1(\Omega))$ we introduce the function

$$v_\nu = a(x)(u_\nu - \overline{u})\,,$$

with $a(x) \in \mathcal{D}(\Omega)\,, a(x) \equiv 1$ for all $x \in \Omega'$. As a result of (75), and the above assumptions, the sequence v_ν satisfies the uniform estimate

$$\nu \int_0^\infty \int_\Omega |\nabla v_\nu|^2 dx \le C\,. \tag{78}$$

Consequently, the sequence v_ν is (in the sense of Gerard, Mauser, Markowich and Poupaud [30]) $\sqrt{\nu}-$oscillating. Accordingly, we introduce the deterministic correlation spectra, or *Wigner transform*, at the scale $\sqrt{\nu}$:

$$\widehat{RT(v_\nu)}(x,t,k) = \frac{1}{(2\pi)^d} \int_{\mathbf{R}^d_y} e^{ik\cdot y}(v_\nu(x - \frac{\sqrt{\nu}}{2}y) \otimes v_\nu(x + \frac{\sqrt{\nu}}{2}y))dy\,.$$

By means of the inverse Fourier transform, one has

$$v_\nu(x,t)\otimes v_\nu(x,t)=\int_{\mathbf{R}^d_k}\widehat{RT(v_\nu)}(x,t,k)dk\,. \tag{79}$$

The tensor $\widehat{RT(v_\nu)}(x,t,k)$ is the main object of section 1 of [30]. Modulo the extraction of a subsequence, the tensor $\widehat{RT(v_\nu)}(x,t,k)$ converges weakly, as $\nu\to 0$, to a nonnegative symmetric matrix-valued measure $\widehat{RT}(x,t,dk)$, which is called a *Wigner measure or Wigner spectra*. Moreover, inside the open subset Ω' the weak limit $\overline{u}$ is a solution of the equation

$$\partial_t\overline{u}+\nabla\cdot(\overline{u}\otimes\overline{u})+\nabla\cdot\int_{\mathbf{R}^d_k}\widehat{RT}(x,t,dk)+\nabla\overline{p}=0\,.$$

The Wigner spectra has the following properties

i) It is defined by a two points correlation formula

ii) It is an (x,t) locally dependent object. Specifically, for any $\phi\in\mathcal{D}(\Omega)$ one has:

$$\begin{aligned}&\lim_{\nu\to 0}\left(\frac{1}{(2\pi)^d}\int_{\mathbf{R}^d_y}e^{ik\cdot y}((\phi v_\nu)(x-\frac{\sqrt{\nu}}{2}y)\otimes(\phi v_\nu)(x+\frac{\sqrt{\nu}}{2}y))dy\right)\\&=|\phi(\cdot)|^2\widehat{RT}(x,t,dk)\end{aligned} \tag{80}$$

Therefore, the construction of $\widehat{RT}(x,t,dk)$ is independent of the choice of the pair $a(x)$ and the open subset Ω'.

iii) It is a *criteria for turbulence*: Points (x,t) around which the sequence u_ν remains smooth and converges locally strongly to $\overline{u}$, as $\nu\to 0$, are characterized by the relation

$$\mathrm{Trace}(\widehat{RT}(x,t,dk))=0\,. \tag{81}$$

iii) It is a *microlocal object.* In fact, it depends only on the behavior of the Fourier spectra of the sequence $\phi(x)v_\nu$ (or in fact $\phi(x)u_\nu$) in the *frequency band*

$$A\le|k|\le\frac{B}{\sqrt{\nu}}\,.$$

Proposition 6.1 *For any pair of strictly positive constants (A,B) and any test functions $(\psi(k),\phi(x),\theta(t))\in C_0^\infty(\mathbf{R}^d_k)\times C_0^\infty(\mathbf{R}^d_x)\times C_0^\infty(\mathbf{R}^+_t)$ one has*

$$\begin{aligned}&\int_0^\infty\int\psi(k)|\phi(x)|^2\theta(t)\mathrm{Trace}(\widehat{RT(u_\nu)})(x,t,dk)dxdt\\&=\lim_{\nu\to 0}\int_0^\infty\theta(t)\int_{A\le|k|\le\frac{B}{\sqrt{\nu}}}(\psi(\sqrt{\nu}k)\overline{\widehat{(\phi v_\nu)}},\overline{\widehat{(\phi v_\nu)}})dkdt\,.\end{aligned} \tag{82}$$

The only difference between this presentation and what can be found in [30] comes from the fact that the weak limit $\overline{u}$ has been subtracted from the sequence

u_ν. Otherwise the formula (79) together with the energy estimate are the first statement of the Proposition 1.7 of [30]; while the formula (82) is deduced from (1.32) in [30] by observing that the weak convergence of $u_\nu - \overline{u}$ to 0 implies that

$$\lim_{\nu\to 0}\left(\int_0^\infty \theta(t)\int_{|k|\le A}(\psi(\sqrt{\nu}k)\widehat{(\phi v_\nu)},\widehat{(\phi v_\nu)})dkdt\right) = 0\,.$$

6.2 Energy spectrum in statistical theory of turbulence

The Wigner spectra studied in the previous section turns out to be the deterministic version of the "turbulent spectra", which is a classical concept in the statistical theory of turbulence. The two points of view can be connected with the introduction of homogenous random variables. Let $(\mathfrak{M},\mathfrak{F},dm)$ be an underlying probability space. A random variable $u(x,\mu)$ is said to be homogenous if for any function F the expectation of $F(u(x,\mu))$, namely,

$$\langle F(u(x,\cdot))\rangle = \int_{\mathfrak{M}} F(u(x,\mu))dm(\mu)$$

is independent of x, that is

$$\nabla_x(\langle F(u(x,\cdot)\rangle) = 0\,.$$

In particular, if $u(x,\mu)$ is a homogeneous random vector-valued function, one has

$$\langle u(x+r,\cdot)\otimes u(x,\cdot)\rangle = \langle u(x+\frac{r}{2},\cdot)\otimes u(x-\frac{r}{2},\cdot)\rangle\,, \tag{83}$$

which leads to the following proposition.

Proposition 6.1 *Let $u(x,\mu)$ be a homogenous random variable and denote by $\hat{u}(k,\mu)$ its Fourier transform. Then one has:*

$$\langle \hat{u}(k,\cdot)\otimes\overline{\hat{u}(k,\cdot)}\rangle = \frac{1}{(2\pi)^d}\int_{\mathbf{R}^d} e^{-ik\cdot r}\langle u(x+\frac{r}{2},\cdot)\otimes u(x-\frac{r}{2},\cdot)\rangle dr\,. \tag{84}$$

Proof. The proof will be given for a homogenous random variable which is periodic with respect to the variable x, with basic periodic box of size $2\pi L$. The formula (84) is then deduced by letting L go to infinity. From the Fourier series decomposition in $(\mathbf{R}/2\pi L\mathbf{Z})^d$, one has

$$\begin{aligned}
&\hat{u}(k,\mu)\otimes\overline{\hat{u}(k,\mu)} = \\
&\frac{1}{(2\pi L)^{2d}}\int_{(\mathbf{R}/2\pi L\mathbf{Z})^d}\int_{(\mathbf{R}/2\pi L\mathbf{Z})^d}(u(y,\mu)e^{-i\frac{k\cdot y}{L}}\otimes u(y',\mu)e^{i\frac{k\cdot y'}{L}})dy'dy = \\
&\frac{1}{(2\pi L)^{2d}}\int_{(\mathbf{R}/2\pi L\mathbf{Z})^d} e^{-i\frac{k\cdot r}{L}}\int_{(\mathbf{R}/2\pi L\mathbf{Z})^d}(u(y,\mu)\otimes u(y+r,\mu))dydr\,.
\end{aligned}$$

Averaging with respect to the probability measure dm, using the homogeneity of the random variable $u(x,\mu)$ and integrating with respect to dy, gives us the following

$$\langle \hat{u}(k,\cdot)\otimes\overline{\hat{u}(k,\cdot)}\rangle = \frac{1}{(2\pi L)^d}\int_{(\mathbf{R}/2\pi L\mathbf{Z})^d} e^{-i\frac{k\cdot r}{L}}\langle u_\nu(y+\frac{r}{2},\cdot)\otimes u_\nu(y-\frac{r}{2},\cdot)\rangle dr\,. \quad (85)$$

This concludes our proof.

Next, assuming that in addition to homogeneity, the expectation of the two points correlation tensor, $\langle u(x+r,\cdot)\otimes u(x,\cdot)\rangle$, is *isotropic* (i.e., it does not depend on the direction of the vector r, but only on its length) one obtains the following formula

$$\langle \hat{u}(k,\cdot)\otimes\overline{\hat{u}(k,\cdot)}\rangle = \frac{1}{(2\pi L)^d}\int_{(\mathbf{R}/2\pi L\mathbf{Z})^d} e^{-i\frac{k\cdot r}{L}}\langle u_\nu(y+\frac{r}{2},\cdot)\otimes u_\nu(y-\frac{r}{2},\cdot)\rangle dr$$
$$= \frac{E(|k|)}{S_{d-1}|k|^{d-1}}(I-\frac{k\otimes k}{|k|^2})\,,$$

with $S_1 = 2\pi\,, S_2 = 4\pi$, which defines the turbulent spectra $E(|k|)\,.$

The notion of homogeneity implies that solutions of the Navier–Stokes equations satisfy a local version of the energy balance, often called the Karman–Howarth relation cf. (86). Specifically, let $\{u_\nu, p_\nu\}$ be solutions of the forced Navier–Stokes equations in Ω (subject to either no-slip Dirichlet boundary condition, in the presence of physical boundary, or in the whole space or in a periodic box)

$$\partial_t u_\nu + \nabla\cdot(u_\nu\otimes u_\nu) - \nu\Delta u_\nu + \nabla p_\nu = f\,.$$

Here u_ν and p_ν are random variables which depend on (x,t). We will drop, below, the explicit dependence on μ when this does not cause any confusion.

Multiplying the Navier–Stokes equations by $u_\nu(x,t,\mu)$ and *assuming* that one has the following equality

$$\frac{1}{2}\partial_t|u_\nu(x,t,\mu)|^2 - \nabla_x\cdot((\nu\nabla_x u_\nu - p_\nu I)u_\nu)(x,t,\mu) +$$
$$\nu|\nabla_x u_\nu(x,t,\mu)|^2 = f(x,t)\cdot u_\nu(x,t,\mu)\,,$$

we observe, that in the $2d$ case, the above relation is a proven fact. However, in the $3d$ case, the class of suitable solutions of the Navier–Stokes equations, in the sense of Caffarelli–Kohn–Nirenberg, are known to satisfy a weaker form of the above relation, involving an inequality instead of equality (cf. (43)).

Thanks to the homogeneity assumption, the quantity

$$\langle((\nu\nabla_x u_\nu - p_\nu I)u_\nu)(x,t,\cdot)\rangle$$

does not depend on x, and therefore

$$\langle\nabla_x\cdot(((\nu\nabla_x u_\nu - p_\nu I)u_\nu))(x,t,\cdot)\rangle = \nabla_x\cdot\langle((\nu\nabla_x u_\nu - p_\nu I)u_\nu)(x,t,\cdot)\rangle = 0\,.$$

Thus, the averaged pointwise energy relation

$$\frac{1}{2}\partial_t\langle|u_\nu(x,t,\cdot)|^2\rangle + \nu\langle|\nabla u_\nu(x,t,\cdot)|^2\rangle = \langle f(x,t)\cdot u_\nu(x,t,\cdot)\rangle \tag{86}$$

is obtained. The above is often called the Karman–Howarth relation, and it implies that the quantities

$$\mathfrak{e} = \langle|u_\nu(x,t,\cdot)|^2\rangle \ \text{ and } \epsilon(\nu) = \frac{1}{t}\int_0^t \nu\langle|\nabla u_\nu(x,s,\cdot)|^2\rangle ds$$

are uniformly bounded in time, under reasonable assumptions on the forcing term.

Finally, assume also that the random process is stationary in time. Then, the equation (86) gives an *a priori* estimate for the mean rate of dissipation of energy

$$\epsilon(\nu) = \nu\langle|\nabla u_\nu(x,t,\cdot)|^2\rangle\,,$$

which is independent of (x,t). Now, with the forcing term f acting only on low Fourier modes, one may assume the existence of a region, called the inertial range, where the turbulence spectra $E(|k|)$ behaves according to a universal power law. *The name inertial refers to the fact that in this range of wave numbers, the energy cascades from low modes to high modes with no leakage of energy. That is, there is no viscous effects in this range and only the inertial term* $(u\cdot\nabla)u$ *is active.* Combining the above hypotheses: homogeneity, isotropy and stationarity of the random process, together with the existence of an inertial range of size

$$A \le |k| \le B\nu^{-\frac{1}{2}}\,,$$

where the spectra behaves according to a power law, one obtains finally, by a dimensional analysis and in the three-dimensional case, the "Kolmogorov law":

$$E(|k|) \simeq \epsilon(\nu)^{\frac{2}{3}}|k|^{-\frac{5}{3}}\,. \tag{87}$$

It is important to keep in mind the fact that this derivation is based on the analysis of a random family of solutions. Therefore, the formula (87) combined with the formula (85) implies that *in the average* the solutions have a spectra which behaves in the turbulent regime according to the prescription.

$$\langle\hat{u}_\nu(k,t,\cdot)\otimes\overline{\hat{u}_\nu(k,t,\cdot)}\rangle = \frac{1}{(2\pi L)^d}\int_{(\mathbf{R}/L\mathbf{Z})^d} e^{-i\frac{k\cdot r}{L}}\langle u_\nu(y+\frac{r}{2},t,\cdot)\otimes u_\nu(y-\frac{r}{2},t,\cdot)\rangle dr$$
$$\simeq \frac{\epsilon(\nu)^{\frac{2}{3}}|k|^{-\frac{5}{3}}}{4\pi|k|^{d-1}}(I-\frac{k\otimes k}{|k|^2})\,.$$

Remark 6.1 *The main difficulty in the full justification of the above derivation is the construction of a probability measure, dm, on the ensemble of solutions of the Navier–Stokes equations that would satisfy the hypotheses of homogeneity, isotropy and stationarity. In particular, the construction of such measure should*

be uniformly valid when the viscosity ν tends to 0. See, for instance, the books of Vishik and Fursikov [69] and Foias et al. [28] for further study and references regarding this challenging problem.

The next difficulty (which is a controversial subject) is the justification for the spectra of an inertial range with a power law. In Foias et al. [28] (see also Foias [27]) it was established, for example, the existence of an inertial range of wave numbers where one has a forward energy cascade. However, we are unaware of a rigorous justification for a power law in this inertial range.

Nevertheless, the construction of a power law of the spectra is often used as a benchmark for validation of numerical computations and experiments. Since, in general, one would have only one run of an experiment, or one run of simulation, a Birkhoff theorem, which corresponds to assuming an ergodicity hypothesis, is then used. This allows for the replacement of the ensemble average by a time average. For instance, one may assume, in presence of forcing term, in addition to stationarity that for almost all solutions, i.e. almost every μ, one has

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T u_\nu(y+\frac{r}{2},t,\mu)\otimes u_\nu(y-\frac{r}{2},t,\mu)dt = \\ \langle u_\nu(y+\frac{r}{2},t,\cdot)\otimes u_\nu(y-\frac{r}{2},t,\cdot)\rangle\,,$$

which would give the following relation, for almost every solution u_ν,

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T u_\nu(k,t,\mu)\otimes\overline{\hat{u}_\nu(k,t,\mu)}dt = \\ \frac{1}{(2\pi)^d}\int_{\mathbf{R}^d} e^{-ik\cdot r}\langle u_\nu(y+\frac{r}{2},t,\cdot)\otimes u_\nu(y-\frac{r}{2},t,\cdot)\rangle dr \\ = \langle\hat{u}(k,t,\cdot)\otimes\overline{\hat{u}(k,t,\cdot)}\rangle \simeq \frac{\epsilon(\nu)^{\frac{2}{3}}|k|^{-\frac{5}{3}}}{4\pi|k|^{d-1}}(I-\frac{k\otimes k}{|k|^2})\rangle\,.$$

6.3 Comparison between deterministic and statistical spectra.

The deterministic point of view considers families of solutions u_ν of the Navier–Stokes dynamics, with viscosity $\nu>0$, and interprets the notion of turbulence in terms of the weak limit behavior (the asymptotic behavior of such sequence as $\nu\to 0$) with the Wigner spectra:

$$\widehat{RT}(x,t,dk) = \\ \lim_{\nu\to 0}\frac{1}{(2\pi)^d}\int_{\mathbf{R}^d_y} e^{ik\cdot y}((u_\nu-\overline{u})(x-\frac{\sqrt{\nu}}{2}y,t)\otimes(u_\nu-\overline{u})(x+\frac{\sqrt{\nu}}{2}y,t))dy.$$

As we have already observed earlier, this is a local object (it takes into account the (x,t) dependence). Moreover, one could define the support of turbulence, for such family of solutions, as the support of the measure Trace$\widehat{RT}(x,t,dk)$. Of course, determining such a support is extremely hard and is a configuration

dependent problem. This is perfectly described in the sentences of Leonardo da Vinci, who is very often quoted, and in particular in page 112 of [29]:

doue la turbolenza dellacqua sigenera
doue la turbolenza dellacqua simantiene plugho
doue la turbolenza dellacqua siposa.

Up to this point nothing much can be said without extra hypotheses, except that the formula (82) indicates the existence of an essential, if not an "inertial", range

$$A \leq |k| \leq \frac{B}{\sqrt{\nu}} \,. \tag{88}$$

On the other hand, the statistical theory of turbulence starts from the hypotheses, that seem difficult to formulate in a rigorous mathematical setting, concerning the existence of statistics (a probability measure) with respect to which the two points correlations for any family of solutions, u_ν, of the Navier–Stokes equations are homogeneous and isotropic. Under these hypotheses, one proves properties on the decay of the spectra of turbulence. Moreover, with all these assumptions one obtains, by simple dimension analysis, *for averages of solutions with respect to the probability measure* $dm(\mu)$, the following formula in the inertial range:

$$\langle |\hat{u}_\nu(k,t,\cdot)|^2 \rangle = \frac{1}{(2\pi)^4} \frac{E(|k|)}{|k|^2} \simeq (\epsilon(\nu))^{\frac{2}{3}} |k|^{-\frac{11}{3}} \,. \tag{89}$$

Finally, by assuming and using the stationarity (in time) of these two points correlations, and by applying, sometimes, the Birkhoff ergodic theorem, one should be able to obtain, for almost every solution, the formula

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T (\hat{u}_\nu(k,t,\mu) \otimes \overline{\hat{u}_\nu(k,t,\mu)}) dt =$$
$$\langle \hat{u}_\nu(k,t,\cdot) \otimes \overline{\hat{u}_\nu(k,t,\cdot)} \rangle \simeq \frac{(\epsilon(\nu))^{\frac{2}{3}} |k|^{-\frac{5}{3}}}{4\pi |k|^{d-1}} (I - \frac{k \otimes k}{|k|^2}) \rangle \,.$$

Remark 6.2 *Some further connections between these two aspects of spectra may be considered.*

i) Assuming that near a point (x_0, t_0) *the Wigner spectra is isotropic. Define the local mean dissipation rate of energy as*

$$\epsilon_{(x_0,t_0)}(\nu) = \nu \int_0^\infty \int_\Omega |\phi(x,t)|^2 |\nabla u_\nu(x,t)|^2 dx dt \,, \tag{90}$$

with ϕ *being a localized function about* (x_0, t_0)*. Then one can prove that, for* $|k|$ *in the range given by (88),*

$$\frac{1}{(2\pi)^d} \int_{\mathbf{R}^d_y} e^{ik\cdot y} ((u_\nu - \overline{u})(x - \frac{\sqrt{\nu}}{2} y, t) \otimes (u_\nu - \overline{u})(x + \frac{\sqrt{\nu}}{2} y, t)) dy$$
$$\sim \epsilon_{(x_0,t_0)}(\nu) |k|^{-\frac{11}{3}} (I - \frac{k \otimes k}{|k|^2}) \,,$$

as ν tends to zero.

ii) Give sufficient conditions that will make the Wigner spectra isotropic. This is in agreement with the fact that this spectra involves only a small scale phenomena; thus, it is a reasonable hypothesis. However, the example constructed by Cheverry [13] shows that this has no chance of always being true.

iii) Another approach for establishing the existence of an inertial range for forward energy cascade in $3d$, and forward enstrophy cascade in $2d$, is presented in [28] (see also references therein). This approach is based on the statistical stationary solutions of the Navier–Stokes equations. These are time independent probability measures which are invariant under the solution operator of the Navier–Stokes equations. Furthermore, in Foias [27] some semi-rigorous arguments are presented to justify the Kolmogorov power law of the energy spectrum.

7 Prandtl and Kelvin–Helmholtz problem

In this section it is assumed that the sequence of solutions $\{u_\nu\}$ of the Navier–Stokes equations with no-slip Dirichlet boundary condition (in the presence of physical boundary) converges to the solution of the Euler equations. According to Theorem 5.1 of Kato, in this situation one has

$$\lim_{\nu\to 0} \nu \int_0^T \int_{\{x\in\Omega:\, d(x,\partial\Omega)<\nu\}} |\nabla u_\nu(x,t)|^2 dxdt = 0\,. \tag{91}$$

However, since the tangential velocity of the solution of the Euler equations is not zero on the boundary, as $\nu \to 0$, a boundary layer is going to appear. On the one hand, the scaling of the boundary layer has to be compatible with the hypothesis (91); and on the other hand the equations that model the behavior in this boundary layer have to reflect the fact that the problem is very unstable. This is because the instabilities (and possible singularities) that occur near the boundary may not remain confined near the boundary, and will in fact propagate inside the domain by the nonlinear advection term of the Navier–Stokes equations. These considerations explain why the Prandtl equations (PE) of the boundary layer are complicated.

There are good reasons to compare the Prandtl equations with the Kelvin–Helmholtz problem (KH):

1. Even though some essential issues remain unsolved for KH, it is much better understood from the mathematical point of view than the PE problem. However, the two problems share similar properties such as instabilities and appearance of singularities.

2. At the level of modeling, in particular for the problem concerning the wake behind an air plane and the vortices generated by the tip of the wings, it is not clear if turbulence should be described by singularities in KH or PE (or both)!

For the sake of simplicity, these problems are considered in the $2d$ case, and for the PE in the half space $x_2 > 0$.

7.1 The Prandtl Boundary Layer

One starts with the $2d$ Navier–Stokes equations in the half plane $x_2 > 0$, with the no-slip boundary condition $u^\nu(x_1, 0, t) \equiv 0$

$$\partial_t u_1^\nu - \nu\Delta u_1^\nu + u_1^\nu \partial_{x_1} u_1^\nu + u_2^\nu \partial_{x_2} u_1^\nu + \partial_{x_1} p^\nu = 0\,, \tag{92}$$

$$\partial_t u_2^\nu - \nu\Delta u_2^\nu + u_1^\nu \partial_{x_1} u_2^\nu + u_2^\nu \partial_{x_2} u_2^\nu + \partial_{x_2} p^\nu = 0\,, \tag{93}$$

$$\partial_{x_1} u_1^\nu + \partial_{x_2} u_2^\nu = 0 \tag{94}$$

$$u_1^\nu(x_1, 0, t) = u_2^\nu(x_1, 0, t) = 0 \text{ on } x_1 \in \mathbf{R}\,, \tag{95}$$

and assumes that inside the domain (away from the boundary) the vector field $u^\nu(x_1, x_2, t)$ converges to the solution $u_{\text{int}}(x_1, x_2, t)$ of the Euler equations with the same initial data. The tangential component of this solution on the boundary, $x_2 = 0$, and of the pressure are denoted by

$$U(x_1, t) = u_1^{\text{int}}(x_1, 0, t)\,, \quad \tilde{P}(x_1, t) = p(x_1, 0, t)\,.$$

Then one introduces the scale $\epsilon = \sqrt{\nu}$. Taking into account that the normal component of the velocity remains 0 on the boundary, one uses the following ansatz, which corresponds to a boundary layer in a parabolic PDE problem

$$\begin{pmatrix} u_1^\nu(x_1, x_2) \\ u_2^\nu(x_1, x_2) \end{pmatrix} = \begin{pmatrix} \tilde{u}_1^\nu(x_1, \frac{x_2}{\epsilon}) \\ \epsilon\tilde{u}_2^\nu(x_1, \frac{x_2}{\epsilon}) \end{pmatrix} + u^{\text{int}}(x_1, x_2)\,. \tag{96}$$

Inserting the right hand side of (96) into the Navier–Stokes equations, and returning to the notation (x_1, x_2) for the variables

$$X_1 = x_1, X_2 = \frac{x_2}{\epsilon}\,,$$

and letting ϵ go to zero, one obtains formally the equations:

$$\tilde{u}_1(x_1, 0, t) + U_1(x_1, 0, t) = 0\,, \tag{97}$$

$$\partial_{x_2}\tilde{p}(x_1, x_2) = 0 \Rightarrow \tilde{p}(x_1, x_2, t) = \tilde{P}(x_1, t) \tag{98}$$

$$\partial_t \tilde{u}_1 - \partial_{x_2}^2 \tilde{u}_1 + \tilde{u}_1 \partial_{x_1} \tilde{u}_1 + \tilde{u}_2 \partial_{x_2} \tilde{u}_1 = \partial_{x_1} \tilde{P}(x_1, t)\,, \tag{99}$$

$$\partial_{x_1} \tilde{u}_1 + \partial_{x_2} \tilde{u}_2 = 0\,,\ \tilde{u}_1(x_1, 0) = \tilde{u}_2(x_1, 0) = 0 \text{ for } x_1 \in \mathbf{R}\,, \tag{100}$$

$$\lim_{x_2 \to \infty} \tilde{u}_1(x_1, x_2) = \lim_{x_2 \to \infty} \tilde{u}_2(x_1, x_2) = 0\,. \tag{101}$$

Remark 7.1 *As an indication of the validity of the Prandtl equations we observe that (96) is consistent with Kato Theorem 5.1. Specifically, thanks to (96) one has*

$$\nu \int_0^T \int_{\Omega \cap \{d(x, \partial\Omega) \le c\nu\}} |\nabla u^\nu(x, s)|^2 dx ds \le C\sqrt{\nu}\,.$$

Remark 7.2 *The following example, constructed by Grenier [33], shows that the Prandtl expansion cannot always be valid. In the case when the solutions are considered in the domain*

$$(\mathbf{R}_{x_1}/\mathbf{Z}) \times \mathbf{R}_{x_2}^+\,.$$

Grenier starts with a solutions u^ν_{ref} of the pressureless Navier–Stokes equations given by

$$u^\nu_{ref} = (u_{ref}(t, y/\sqrt{\nu}), 0)$$
$$\partial_t u_{ref} - \partial_{YY} u_{ref} = 0\,,$$

where $Y = y/\sqrt{\nu}$. Using a convenient and explicit choice of the function u_{ref}, with some sharp results on instabilities, a solution of the Euler equations of the form

$$\tilde{u} = u_{ref} + \delta v + O(\delta^2 e^{2\lambda t}) \quad \text{for} \quad 0 < t < \frac{1}{\lambda \log \delta}$$

is constructed. It is then shown that the vorticity generated by the boundary for the solution Navier–Stokes equations (with the same initial data) is too strong to allow for the convergence of the Prandtl expansion. One should observe, however, that once again this is an example which involves solutions with infinite energy. It would be interesting to see if such an example could be modified to belong to the class of finite energy solutions; and then to analyze how the modified finite energy solution might violate the Kato criteria mentioned in Theorem 5.1.

It is important to observe that in their mathematical properties the PE exhibit the pathology of the situation that they are trying to model. First one can prove the following proposition.

Proposition 7.1 *Let $T > 0$ be a finite positive time, and let $(U(x,t), P(x,t)) \in C^{2+\alpha}(\mathbf{R}_t \times (\mathbf{R}_{x_1} \times \mathbf{R}^+_{x_2}))$ be a smooth solution of the 2d Euler equations satisfying, at time $t = 0$, the compatibility condition $U_1(x_1, 0, t) = U_2(x_1, 0, t) = 0$ (notice that only the boundary condition $U_2(x_1, 0, t) = 0$ is preserved by the Euler dynamics). Then the following statements are equivalent:*

i) With initial data $\tilde{u}(x, 0) = 0$, the boundary condition $\tilde{u}_1(x_1, 0, t) = U_1(x_1, 0, t)$ in (97), the right-hand side in (98) given by $\tilde{P}(x_1, t) = P(x_1, 0, t)$, and the Prandtl equations have a smooth solution $\tilde{u}(x, t)$, for $0 < t < T$.

ii) The solution $u_\nu(x, t)$, of the Navier–Stokes equations with initial data $u_\nu(x, 0) = U(x, 0)$ and with no-slip boundary condition at the boundary $x_1 = 0$, converges in $C^{2+\alpha}$ to the solution of the Euler equations, as $\nu \to 0$.

The fact that statements i) and ii) may be violated for some t is related to the appearance of a detachment zone, and the generation of turbulence. This is well illustrated in the analysis of the Prandtl equations written in the following simplified form

$$\partial_t \tilde{u}_1 - \partial^2_{x_2} \tilde{u}_1 + \tilde{u}_1 \partial_{x_1} \tilde{u}_1 + \tilde{u}_2 \partial_{x_2} \tilde{u}_1 = 0\,, \partial_{x_1} \tilde{u}_1 + \partial_{x_2} \tilde{u}_2 = 0 \tag{102}$$
$$\tilde{u}_1(x_1, 0) = \tilde{u}_2(x_1, 0) = 0 \text{ for } x_1 \in \mathbf{R} \tag{103}$$
$$\lim_{x_2 \to \infty} \tilde{u}_1(x_1, x_2) = 0, \tag{104}$$
$$\tilde{u}_1(x_1, x_2, 0) = \tilde{u}_0(x_1, x_2)\,. \tag{105}$$

Regularity in the absence of detachment corresponds to a theorem of Oleinik [58]. She proved that global smooth solutions of the above system do exist provided the initial profile is monotonic, i.e. for any initial profile satisfying

$$\tilde{u}(x,0) = (\tilde{u}_1(x_1,x_2),0)\,, \quad \partial_{x_2}\tilde{u}_1(x_1,x_2) \neq 0\,.$$

On the other hand, initial conditions with "recirculation properties" leading to a finite time blow up have been constructed by E and Engquist [23] and [24]. An interesting aspect of these examples is that the blow up generally does not occur on the boundary, but rather inside the domain.

The above pathology appears in the fact that the PE is highly unstable. This comes from the determination $\tilde{u}_2$ in term of $\tilde{u}_1$ by the equation

$$\partial_{x_1}\tilde{u}_1 + \partial_{x_2}\tilde{u}_2 = 0\,.$$

Therefore, it is only with analytic initial data (in fact analytic with respect to the tangential variable is enough) that one can obtain (using an abstract version of the Cauchy–Kowalewskya theorem) the existence of a smooth solution of the Prandtl equation for a finite time and the convergence to the solution of the Euler equations during this same time (Asano [1], Caflisch-Sammartino [10], and Cannone-Lombardo-Sammartino [12].)

7.2 The Kelvin–Helmholtz problem

The Kelvin–Helmholtz (KH) problem concerns the evolution of a solution of the $2d$ Euler equations

$$\partial_t u + \nabla\cdot(u\otimes u) + \nabla p = 0\,, \quad \nabla\cdot u = 0 \tag{106}$$

with initial vorticity $\omega(x,0)$ being a measure concentrated on a curve $\Gamma(0)$.

This is already simpler than the PE because the pathology, if any, should in principle be concentrated on a curve. Furthermore, the dynamics in this case inherits the general properties of the $2d$ dynamics. In particular, it will obey the equation

$$\partial_t(\nabla\wedge u) + (u\cdot\nabla)(\nabla\wedge u) = 0\,,$$

for the conservation (for smooth solutions) of the density of the vorticity. Therefore, one can guarantee the existence of a weak solution when the initial vorticity, $\omega(x,0)$, is a Radon measure. This was done first by Delort, assuming that the initial measure has a distinguished sign [21]. Then the result was generalized to situations where the change of sign was simple enough [49]. However, this remarkable positive result is impaired by the non-uniqueness result of Shnirelman [65].

For smooth solutions of the KH, i.e. the ones with vorticity ω - a bounded Radon measure with support contained in curve $\Sigma_t = \{r(\lambda,t),\, \lambda\in\mathbf{R}\}$ - the

velocity field is given, for $x \notin \Sigma_t$, by the so called Biot–Savart law

$$\begin{aligned} u(x,t) &= \frac{1}{2\pi} R_{\frac{\pi}{2}} \int \frac{x-r'}{|x-r'|^2} \omega(r',t) ds' \\ &:= \frac{1}{2\pi} R_{\frac{\pi}{2}} \int \frac{x-r(\lambda',t)}{|x-r(\lambda',t)|^2} \omega(r(\lambda',t),t) \frac{\partial s(\lambda',t)}{\partial \lambda'} d\lambda' \,. \end{aligned} \tag{107}$$

Here $r(\lambda,t)$, for $\lambda \in \mathbf{R}$, is a parametrization of the curve Σ_t, and $s(\lambda,t) = |r(\lambda,t)|$ is the corresponding arc length. $R_{\frac{\pi}{2}}$ denotes the $\frac{\pi}{2}-$counterclockwise rotation matrix

$$R_{\frac{\pi}{2}} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} .$$

Furthermore, as x approaches the curve Σ_t the velocity field u admits the two-sided limits $u_\pm$. By virtue of the incompressibility condition one has the continuity condition for the normal component of the velocity field, i.e.

$$u_- \cdot \vec{n} = u_+ \cdot \vec{n} \,,$$

where hereafter $\vec{\tau}$ and $\vec{n}$ will denote the unit tangent and unit normal vectors to the curve Σ_t, respectively. In addition, the average

$$\langle u \rangle = \frac{u_+ + u_-}{2}$$

is given by the principal value of the singular integral appearing in (107)

$$v = \langle u \rangle = \frac{1}{2\pi} R_{\frac{\pi}{2}} \text{ p.v.} \int \frac{x-r'}{|x-r'|^2} \omega(r',t) ds' \,. \tag{108}$$

Using the calculus of distributions one can show that, as long as the curve Σ_t is smooth, the velocity field u, defined above, being a weak solution of the Euler equations

$$\partial_t u + \nabla \cdot (u \otimes u) + \nabla p = 0 \,, \ \nabla \cdot u = 0 \,,$$

is equivalent to the vorticity density ω and the curve Σ_t satisfying the coupled system of equations

$$\omega_t - \partial_s \Big(\omega (\partial_t r - v) \cdot \vec{\tau} \Big) = 0 \,, \tag{109}$$

$$(r_t - v) \cdot \vec{n} = 0 \,, \tag{110}$$

$$v(r,t) = \frac{1}{2\pi} R_{\frac{\pi}{2}} \text{ p.v.} \int \frac{r-r'}{|r-r'|^2} \omega(r',t) ds' \,. \tag{111}$$

The equations (109), (110) and (111) do not completely determine $r(\lambda,t)$. This is due to the freedom in the choice of the parametrization of the curve Σ_t. Assuming that $\omega \neq 0$ one introduces a new parametrization $\lambda(t,s)$ which reduces the problem to the equation

$$\partial_t r(\lambda,t) = \frac{1}{2\pi} R_{\frac{\pi}{2}} \text{ p.v.} \int \frac{r(\lambda,t) - r(\lambda',t)}{|r(\lambda,t) - r(\lambda',t)|^2} d\lambda' \,, \tag{112}$$

or with the introduction of the complex variable $z = r_1 + ir_2$, where $r = (r_1, r_2)$, one obtains the Birkhoff–Rott equation

$$\partial_t \overline{z}(\lambda, t) = \frac{1}{2\pi i} \text{ p.v.} \int \frac{d\lambda'}{z(\lambda, t) - z(\lambda', t)} \,. \tag{113}$$

Remark 7.3 *The following are certain mathematical similarities of the KH problem with the PE:*

1. As for the PE one has for the evolution equation (113) a local, in time, existence and uniqueness result in the class of analytic initial data. This is done by implementing a version of the Cauchy–Kowalewskya theorem (Bardos, Frisch, Sulem and Sulem [67].)

2. As for the PE one can construct solutions that blow up in finite time.

3. One observes that the singular behavior in the experiments and numerical simulations with the KH problem is very similar to the one that is generated by the no-slip boundary condition when the viscosity is approaching zero.

The best way to understand the structure of the KH is to use the fact that the Euler equations are invariant under both space and time translations, and under space rotations, and to consider a weak solution of the $2d$ Euler dynamics either in the whole plane $\mathbf{R}^2$, satisfying

$$u \in C((-T, T); L^2(\mathbf{R}^2)) \,, \; T > 0$$

or subject to periodic boundary conditions satisfying

$$u \in C((-T, T); L^2((\mathbf{R}/\mathbf{Z})^2)).$$

Assuming that in a small neighborhood $\mathcal{U}$ of the point $(t = 0, z = 0)$ the vorticity is concentrated on a smooth curve in the complex plane which takes the form:

$$z(\lambda, t) = (\alpha t + \beta(\lambda + \epsilon f(\lambda, t)) \,, \; f(0, 0) = \nabla f(0, 0) = 0 \,. \tag{114}$$

Then using the relations $\nabla \cdot u = 0$, $\nabla \wedge u = \omega$ and the Biot–Savart law, one obtains

$$\begin{aligned} &\epsilon |\beta|^2 \partial_t \overline{f}(\lambda, t) = \\ &\quad \frac{1}{2\pi i} \text{ p.v.} \int_{\{z(t,\lambda') \in \mathcal{U}\}} \frac{d\lambda'}{(\lambda - \lambda')(1 - \epsilon \frac{f(\lambda,t) - f(\lambda',t)}{\lambda - \lambda'})} + E(z(\lambda, t)) \end{aligned} \tag{115}$$

where here, and in the sequel, $E(z)$ denotes the "remainder", which is analytic with respect to z. Next, use the expansion

$$\begin{aligned} &\frac{1}{2\pi} \text{ p.v.} \int \frac{d\lambda'}{(\lambda - \lambda')(1 - \epsilon \frac{f(\lambda,t) - f(\lambda',t)}{\lambda - \lambda'})} d\lambda' = \\ &\frac{\epsilon}{2\pi} \text{ p.v.} \int \frac{f(\lambda, t) - f(\lambda', t)}{(\lambda - \lambda')^2} d\lambda' + \sum_{n \geq 2} \frac{\epsilon^n}{2\pi} \text{ p.v.} \int \frac{(f(\lambda, t) - f(\lambda', t))^n}{(\lambda - \lambda')^{(n+1)}} d\lambda' \,, \end{aligned} \tag{116}$$

and implement the following formulas concerning the Hilbert transform

$$\frac{1}{2\pi}\,\text{p.v.}\int \frac{f(\lambda,t)-f(\lambda',t)}{\lambda-\lambda'}d\lambda' = -\frac{i}{2}\,\text{sgn}(D_\lambda)f\,, \tag{117}$$

$$\frac{1}{2\pi}\,\text{p.v.}\int \frac{f(\lambda,t)-f(\lambda',t)}{(\lambda-\lambda')^2}d\lambda' = |D_\lambda|f \tag{118}$$

to deduce, from (115) and (116), that the real and imaginary parts of $f(\lambda,t) = X(\lambda,t)+iY(\lambda,t)$ are local solutions of the system

$$\partial_t X = \frac{1}{2|\beta|^2}|D_\lambda|Y + \epsilon R_1(X,Y) + E_1(X,Y) \tag{119}$$

$$\partial_t Y = \frac{1}{2|\beta|^2}|D_\lambda|X + \epsilon R_2(X,Y) + E_2(X,Y) \tag{120}$$

or

$$\begin{aligned}(\partial_t^2 + \frac{1}{4|\beta|^4}\partial_\lambda^2)Y &= \epsilon(\partial_t R_1(X,Y) - \frac{1}{2|\beta|^2}|D_\lambda|R_2(X,Y))\\ &+\partial_t E_1(X,Y) - \frac{1}{2|\beta|^2}|D_\lambda|E_2(X,Y)\end{aligned} \tag{121}$$

$$\begin{aligned}(\partial_t^2 + \frac{1}{4|\beta|^4}\partial_\lambda^2)Y &= \epsilon(\partial_t R_2(X,Y) - \frac{1}{2|\beta|^2}|D_\lambda|R_1(X,Y))\\ &+\partial_t E_2(X,Y) - \frac{1}{2|\beta|^2}|D_\lambda|E_1(X,Y)\,.\end{aligned} \tag{122}$$

In (121) and (122) the terms

$$\partial_t E_1(X,Y) - \frac{1}{2|\beta|^2}|D_\lambda|E_2(X,Y) \text{ and } \partial_t E_2(X,Y) - \frac{1}{2|\beta|^2}|D_\lambda|E_1(X,Y)$$

are the first order derivatives of analytic functions with respect to (X,Y) while the terms

$$\partial_t R_1(X,Y) - \frac{1}{2|\beta|^2}|D_\lambda|R_2(X,Y)) \text{ and } (\partial_t R_2(X,Y) - \frac{1}{2|\beta|^2}|D_\lambda|R_1(X,Y))$$

are the second order derivative of analytic functions with a small ϵ prefactor. Therefore, one observes that, up to a perturbation, the KH problem behaves like a second order constant coefficient elliptic equation. This fact has several important consequence.

1. It explains why the evolution equation is well-posed only for a short time, and with initial data that belongs to the class of analytic functions. It is like solving an elliptic equation simultaneously with both the Dirichlet and Neumann boundary conditions.

2. It is a tool for the construction of the solutions that blow up in finite time.

3. It explains, by an indirect regularity argument, the very singular behavior of the solution after the first break down of its regularity.

These three points are discussed in further details below.

7.2.1 Local solution

When the curve Σ_t is a graph of a function, say $y = y(x,t)$, the equations (109) and (110) become

$$y_t + y_x v_1 = v_2\,, \quad \partial_t \omega + \partial_x(v_1\omega) = 0 \tag{123}$$

$$v_1(t,x) = -\frac{1}{2\pi}\,\text{p.v.}\!\int_{\mathbf{R}} \frac{y(x,t) - y(x',t)\omega(x',t)}{(x-x')^2 + (y(x,t)-y(x',t))^2}dx' \tag{124}$$

$$v_2(t,x) = \frac{1}{2\pi}\,\text{p.v.}\!\int_{\mathbf{R}} \frac{(x-x')\omega(x',t)}{(x-x')^2 + (y(x,t)-y(x',t))^2}dx'\,, \tag{125}$$

where $(v_1, v_2) = v$ is the average velocity given in (108). Therefore, the above evolution equations involve two unknowns $y(x,t)$ and $\omega(x,t)$, which is also the case for the Birkhoff–Rott equation (113), where the two unknowns are the two components of $r(s,t) = (x(s,t), y(s,t))$, or of $z(s,t) = x(s,t) + iy(s,t)$. In fact, since the Birkhoff–Rott equation has been obtained by choosing the density of vorticity as a parameter, one recovers this vorticity by the formula

$$\omega(s,t) = \frac{1}{|\partial_s z(s,t)|}\,.$$

Since the system is a local perturbation of a second order elliptic equation then imposing two constraints at $t = 0$ is similar to solving this elliptic equation simultaneously with both Neuman and Dirichlet boundary conditions. It is known that in the absence of stringent compatibility conditions (they are related by the so called Dirichlet to Neumann operator) such a problem can be solved only locally and with analytic data. This is the reason why the solution of (123)–(125) is obtained locally in time under the assumption that the functions $y(x,0)$ and $\omega(x,0)$ are analytic.

7.2.2 Singularities

For the construction of singularities one follows the same idea, and furthermore, uses the time reversibility of the $2d$ Euler equations. More precisely, if one constructs solutions which are singular at $t = 0$ and are regular on the interval $(0,T]$ this will imply, just by changing the time variable t into $T-t$, the existence of smooth solutions at $t = 0$ that blow up at $t = T$. The first result in this direction was obtained by Duchon and Robert [22]; the initial condition on the vorticity at $t = 0$ is relaxed, and one assumes that the solution $y(x,t)$ goes to zero as $t \to \infty$. Then one can consider the system (123)–(125) as a two point Dirichlet boundary-value problem with $y(x,t)$ is given for $t = 0$ and is required to tend to zero as $t \to \infty$. Then by a perturbation method one proves the following proposition.

Proposition 7.2 *There exists $\epsilon > 0$ such that for any initial data that satisfies the estimate*

$$y(x,0) = \int e^{ix\xi} g(\xi)d\xi\,,\ \textit{with} \int |g(\xi)|d\xi \le \epsilon\,, \tag{126}$$

the problem (123)–(125) has a unique solution which goes to zero as $t \to \infty$. Furthermore, this solution is analytic (with respect to (x,t)) for all $t > 0$.

As mentioned above, this is a result about singularity formations. It exhibits (by changing the time variable t into $T-t$) an example of solutions which are analytic at some time, but with no more regularity at a later time than what is allowed by the equation (126). In fact, it was observed in some numerical experiments [53] and [56] that the first break down of regularity appears as a cusp on the curve $r(\lambda,t)$. This motivated Caflsich and Orellana [9] to introduce the function

$$f_0(\lambda,t) = (1-i)\{(1-e^{-\frac{t}{2}-i\lambda})^{1+\sigma} - (1-e^{-\frac{t}{2}+i\lambda})^{1+\sigma}\}\,, \tag{127}$$

which enjoys the following properties

i) For any $t > 0$ the mapping $\lambda \mapsto f(\lambda,t)$ is analytic.

ii) For $t = 0$, the mapping $\lambda \mapsto f(0,\lambda)$ does not belong to the Hölder space $C^{1+\sigma}$, but it belongs to every Hölder space $C^{1+\sigma'}$ with $0 < \sigma' < \sigma$.

iii) The function

$$z_0(\lambda,t) = \lambda + \epsilon f_0(\lambda,t)$$

is an exact solution of the linearized Birkhoff–Rott equation. More precisely, one has

$$\partial_t \overline{f(\lambda,t)} = \frac{1}{2\pi}\,\text{p.v.}\!\int \frac{f(\lambda,t)-f(\lambda',t)}{(\lambda-\lambda')^2}d\lambda'\,. \tag{128}$$

Therefore, by using the ellipticity of this linear operator, one can prove by a perturbation method the following proposition.

Proposition 7.3 *For $\epsilon > 0$, small enough, there exist a function $r_\epsilon(\lambda,t)$ with the following properties:*

i) The function $\lambda \mapsto r_\epsilon(\lambda,t)$ is analytic for $t > 0$.

ii) The function $\lambda + \epsilon(f_0(\lambda,t) + r_\epsilon(\lambda,t))$ is a solution of the Birkhoff-Rott equation (113).

ii) The function $\lambda \mapsto r_\epsilon(\lambda,t)$ is (for $\lambda \in \mathbf{R}$, $t \in \mathbf{R}_+$) uniformly bounded in C^2.

As a consequence of Proposition 7.3 (and of the reversibility in time) one can establish the existence of analytic solutions to the Birkhoff–Rott equation (113), say in the interval $0 \le t < T$, such that at time $t = T$ the map $\lambda \mapsto z(\lambda,t)$ does not belongs to $C^{1+\sigma}$ at the point $\lambda = 0$.

7.2.3 Analyticity and pathologic behavior after the break down of regularity

The local reduction of the KH to the equation

$$\epsilon|\beta|^2\partial_t\overline{f}(\lambda,t) = \frac{1}{2\pi i}\,\text{p.v.}\!\int_{z(\lambda',t)\in\mathcal{U}} \frac{d\lambda'}{(\lambda-\lambda')(1-\epsilon\frac{f(\lambda,t)-f(\lambda',t)}{\lambda-\lambda'})} + E(z(\lambda,t)) \tag{129}$$

requires obviously *some hypotheses* on the regularity of the function $z(\lambda, t)$ near the point $(0, 0)$. However, when this reduction is valid it will, thanks to the ellipticity, imply that the solution is C^∞, and even analytic. Therefore, there appears to be a threshold (say $\mathbf{T}$) in the behavior of the solutions of the KH. Existence of such *regularity threshold* is common in the study of free boundary problems. This threshold is characterized by the fact that any function having a regularity stronger than $\mathbf{T}$ is in fact analytic, and that there may exist solutions with less regularity than $\mathbf{T}$. This has the following, practical, important consequence: regularity of the solutions that are smooth for $t < T$ and singular after the time $t = T$ cannot be extended for $t \geq T$ by solutions which are more regular than the threshold $\mathbf{T}$. Otherwise, the above theorem would lead to a contradiction. This fact explains why after the break down of regularity, the solution becomes very singular.

For instance, it was shown by Lebeau [44] (and Kamotski and Lebeau [37] for the local version) that any solution that is near a point belongs to $C_t^\sigma(C_\lambda^{1+\sigma})$ must be analytic. As a consequence, if a solution constructed (by changing the variable t into $T - t$) according to the method of Caflisch and Orellana could be continued after time $t = T$, it would not be in any Hölder space $C^{1+\sigma'}$.

Therefore, the challenge (and an open problem) is the determination of this threshold of regularity that will imply analyticity. Up to now, the best (to the best of our knowledge) known result is due to S. Wu [71] [72]. The hypothesis $C^\alpha_{loc}(\mathbf{R}_t; C^{1+\beta}_{loc}(\mathbf{R}_\lambda))$ is replaced by $H^1_{loc}(\mathbf{R}_t \times \mathbf{R}_\lambda)$. The estimates are done by explicitly using theorems of G. David [20] saying that for all chord arc curves $\Gamma : s \mapsto \xi(s)$, parameterized by their arc length, the Cauchy integral operator

$$C_\Gamma(f) = \text{p.v.} \int \frac{f(s')}{\xi(s) - \xi(s')} d\xi(s')$$

is bounded in $L^2(ds)$.

The importance of this improvement is justified by the numerical experiment of [39]. It is interesting to notice that these results will apply to logarithmic spirals $r = e^\theta$, $\theta \in \mathbf{R}$, but not to infinite length algebraic spirals. What is observed is that from cups singularity the solution evolves into a spiral which behaves like an algebraic spiral, and therefore has an infinite length. *The results of [70] provide an explanation of the fact that the spiral has to be of an infinite length.*

After the appearance of the first singularity the solution becomes very irregular. This leads to the issue of the definition of *weak solutions* (solutions which are less regular than the threshold $\mathbf{T}$) *not of the Euler equations themselves, but of the Birkhoff–Rott equation.* For instance, S. Wu [71] and [72] proposed the following definition:

A weak solution is a function from $\mathbf{R}$ into $\mathbf{C}, \alpha \mapsto z(\alpha, t)$, for which the following relation holds

$$\partial_t \left(\int \overline{z}(\alpha, t)\eta(\alpha) d\alpha \right) = \frac{1}{4\pi i} \iint \frac{\eta(\alpha) - \eta(\beta)}{z(\alpha, t) - z(\beta, t)} d\alpha d\beta \,,$$

for every $\eta \in C_0^\infty(\mathbf{R})$.

However, the problem is basically open because we have no theorem concerning the existence of such a solution. Furthermore, for physical reasons weak solutions of the Birkhoff–Rott equation should provide weak solutions of the incompressible Euler equations, and in fact this is not always the case as it is illustrated by (cf. [48]) the Prandtl–Munk example: Start from the vortex sheet

$$\omega_0(x_1, x_2) = \frac{x_1}{\sqrt{1 - x_1^2}}(\chi_{(-1,1)}(x_1) \otimes \delta(x_2)) \tag{130}$$

where $\chi_{(-1,1)}$ is the characteristic function of the interval $(-1, 1)$. By virtue of the the Biot–Savart law, the velocity v is constant

$$v = (0, -\frac{1}{2}). \tag{131}$$

The solution of the Birkhoff–Rott equation is given by the formula

$$x_1(t) = x_1(0), \;\; x_2(t) = \frac{t}{2}, \;\; \omega(x_1, x_2, t) = \omega_0(x_1, x_2 + \frac{t}{2}).$$

On the other hand, it was observed in [48] that the velocity u associated with this vorticity is *not* even a weak solution of the Euler equations. In fact, one has

$$\nabla \cdot u = 0 \qquad \text{and } \partial_t u + \nabla_x \cdot (u \otimes u) + \nabla p = F, \tag{132}$$

where F is given by the formula

$$F = \frac{\pi}{8}\left((\delta(x_1 + 1, x_2 + \frac{t}{2}) - \delta(x_1 - 1, x_2 + \frac{t}{2})), 0\right). \tag{133}$$

This has led Lopes, Nussenweig and Sochet [48] to propose a weaker definition, which contains more freedom with respect to the parameter, and may be more adapted.

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