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Holditch-type theorems under the closed planar homothetic motions. (English summary)
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Back in 1858, H. Holditch stated and proved an interesting theorem. An oval path is given together with a rigid rod of length l. Each end of the rod (say A and B) touches the oval. Whenever the rod completes a full turn while moving along the oval path, a given fixed point in the rod (say at the distance a from A and b from B so that $l=a+b$) also describes a closed curve, but one which is not necessarily convex. The surface part between the center of the oval and the curve traced out by the fixed point on the rod at distance a from A is then removed from the surface bounded by the oval curve. This yields Holditch's ring, the surface of which obeys a simple and elegant formula

$$F=ab\pi.$$

In the paper under review, Holditch-type theorems are presented for closed homothetic motions. For these motions, the linked frame that is attached to the selected moving point in-between A and B distorts according to a homothetic scale $h(t)$ (i.e. a sort of continuous "zooming in and out" motion). Therefore, two moving frames in the Euclidean planes are defined, one for the fixed reference frame $E'=\{O';\mathbf{e}'_1,\mathbf{e}'_2\}$ and one for the moving referential $E=\{O;\mathbf{e}_1,\mathbf{e}_2\}$. For a closed homothetic motion, the moving frame is related to the fixed one through

$$\mathbf{e}_1=\cos(\varphi(t))\mathbf{e}'_1+\sin(\varphi(t))\mathbf{e}'_2,$$
$$\mathbf{e}_2=-\sin(\varphi(t))\mathbf{e}'_1+\cos(\varphi(t))\mathbf{e}'_2,$$

and the point $\mathbf{x}'=(x'_1,x'_2)\in E'$ is related to $\mathbf{x}=(x_1,x_2)\in E$ through

$$\mathbf{x}'=h(t)\mathbf{x}-\begin{pmatrix}u_1(t)\\ u_2(t)\end{pmatrix},$$

where $h(t)$ is the homothetic periodic scale, $\varphi(t)$ is a periodic rotation and $u_1(t)$ and $u_2(t)$ are periodic translation functions, that is, there exists $T>0$ such that $h(t+T)=h(t)$, $\varphi(t+T)=\varphi(t)+2\varphi\nu$, $u_1(t+T)=u_1(t)$ and $u_2(t+T)=u_2(t)$. The integer $\nu\in\mathbb{Z}$ is called the rotation number of the closed planar homothetic motion.

The first theorem states that if the points $A=(0,0)$ and $B=(a,0)\in E$ draw closed-trajectory curves k_A and k_B with the orbit areas F_A and F_B, respectively, then the point $C=(a,c)\in E$ describes a closed-trajectory curve k_C with area

$$F_C=[aF_B+bF_A]/(a+b)$$
$$+(c^2-ab)h^2(t_0)\pi\nu-2h^2(t_0)\nu cL_{AB}/(\oint hd\varphi),$$

where L_{AB} is the length of the rod (segment AB) and t_0 is a certain time instant during the homothetic motion.

The main theorem of the paper states that if three points $A=(0,0)$, $B=(b,0)$ and $C=(c,d)$

$\in E$ describe closed-trajectory curves of areas F_A, F_B and F_C respectively, then a point $X = (x, y) \in E$ describes the area

$$F_X = \left(1 - \frac{x}{b} + \frac{c-b}{bd}y\right) F_A + \left(\frac{x}{b} - \frac{cy}{bd}\right) F_B + \frac{y}{d} F_C + \left(x^2 + y^2 - bx - \frac{c^2+d^2}{d}y + \frac{bc}{d}y\right) h^2(t_0)\pi\nu.$$

Reviewed by *Philippe A. Müllhaupt*