Dirac Structures and Generalized Complex Structures on $TM \times \mathbb{R}^h$

by

Izu Vaisman

ABSTRACT. We consider Courant and Courant-Jacobi brackets on the stable tangent bundle $TM \times \mathbb{R}^h$ of a differentiable manifold and corresponding Dirac, Dirac-Jacobi and generalized complex structures. We prove that Dirac and Dirac-Jacobi structures on $TM \times \mathbb{R}^h$ can be prolonged to $TM \times \mathbb{R}^k$, $k > h$, by means of commuting infinitesimal automorphisms. Some of the stable, generalized, complex structures are a natural generalization of the normal, almost contact structures; they are expressible by a system of tensors $(P, \theta, F, Z_a, \xi^a)$ $(a = 1, ..., h)$, where P is a bivector field, θ is a 2-form, F is a $(1,1)$-tensor field, Z_a are vector fields and ξ^a are 1-forms, which satisfy conditions that generalize the conditions satisfied by a normal, almost contact structure (F, Z, ξ). We prove that such a generalized structure projects to a generalized, complex structure of a space of leaves and characterize the structure by means of the projected structure and of a normal bundle of the foliation. Like in the Boothby-Wang theorem about contact manifolds, principal torus bundles with a connection over a generalized, complex manifold provide examples of this kind of generalized, normal, almost contact structures.

1 Brackets on stable tangent bundles

All the manifolds and mappings of the present note are assumed of the C^∞ class. In differential topology, the vector bundle $TM \times \mathbb{R}^h$, where M is a

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differentiable manifold, is called the *stable tangent bundle*. The name comes from the fact that h may be arbitrary and the main interest is for objects defined up to an equivalence that involves a change of h (e.g., [10], Appendix D). On the other hand, interesting differential-geometric structures of M may be defined via a stable tangent bundle; for instance, an almost contact structure is equivalent with an almost complex structure on $TM \times \mathbb{R}$ and the integrability of the latter characterizes the normality of the former [1]. The aim of the present paper is to extend the notions of a Dirac structure [2, 3], a Dirac-Jacobi structure [21, 8] and a generalized complex structure [11, 9] to $TM \times \mathbb{R}^h$. We will give some results concerning the prolongation of a structure from $TM \times \mathbb{R}^h$ to $TM \times \mathbb{R}^k$, $k > h$ and discuss structures defined on M by generalized complex structures in $TM \times \mathbb{R}^h$.

Our general notation is $\chi^k(M)$ for the space of k-vector fields, $\Omega^k(M)$ for the space of differential k-forms, Γ for the space of global cross sections of a vector bundle, $X, Y, ..$ for contravariant vectors (or vector fields), $\alpha, \beta, ...$ for covariant vectors (or 1-forms), $u, v, ...$ for vectors of $\mathbb{R}^h$ and a dot for the natural scalar product of $\mathbb{R}^h$. Furthermore, we define the *big tangent bundle* $T^{big}M = TM \oplus T^*M$ and the *stable big tangent bundle of index h*

$$\mathbf{T}^{big}_h M = (TM \times \mathbb{R}^h) \oplus (T^*M \times \mathbb{R}^h), \tag{1.1}$$

where h is any non negative integer, which, usually, we shall not mention (on the other hand, notice the boldface character, except for $\mathbf{T}^{big}_0 M = T^{big}M$).

The bundle $\mathbf{T}^{big}M$ has a natural, neutral metric (i.e., non degenerate and of signature zero)

$$\begin{aligned} G(&(X_1, u_1) \oplus (\alpha_1, v_1), (X_2, u_2) \oplus (\alpha_2, v_2)) \\ &= \frac{1}{2}(\alpha_2(X_1) + \alpha_1(X_2) + u_1 \cdot v_2 + u_2 \cdot v_1), \end{aligned} \tag{1.2}$$

where $\oplus$ denotes the addition of elements in different terms of a direct sum of vector spaces. Similarly, and with the generic notation

$$\mathcal{X} = (X, u) \oplus (\alpha, v), \tag{1.3}$$

one has the non degenerate 2-form

$$\Omega(\mathcal{X}_1, \mathcal{X}_2) = \frac{1}{2}(\alpha_2(X_1) - \alpha_1(X_2) + u_1 \cdot v_2 - u_2 \cdot v_1). \tag{1.4}$$

The usual, twisted, Courant bracket is an operation on $\Gamma(T^{big}M)$ (i.e., $h=0$) defined by [2, 3, 17]

$$(1.5)\qquad [X\oplus\alpha, Y\oplus\beta]_\Phi = [X,Y]\oplus[L_X\beta - L_Y\alpha$$

$$-d(\Omega(X\oplus\alpha,Y\oplus\beta)) + i(X\wedge Y)\Phi],$$

where L denotes the Lie derivative and Φ is a closed 3-form; if $\Phi=0$ the bracket is untwisted.

We shall also need the notion of a *conformal change* of the Courant bracket, which is a particular case of the deformation by a 1-cocycle defined by Grabowski and Marmo [8] and a generalization of a change used by Wade [22] and by Petalidou and Nunes da Costa [16]. An automorphism $\mathcal{C}_\tau : T^{big}M\rightarrow T^{big}M$ defined by

$$(1.6)\qquad \mathcal{C}_\tau(X\oplus\alpha) = X\oplus e^\tau\alpha,\quad \tau\in C^\infty(M),$$

will be called a conformal change of $T^{big}M$, because it produces a conformal change of the metric G in the sense that

$$(1.7)\qquad G(\mathcal{C}_\tau(X\oplus\alpha),\mathcal{C}_\tau(Y\oplus\beta)) = e^\tau G(X\oplus\alpha,Y\oplus\beta).$$

Correspondingly, the bracket

$$(1.8)\qquad [X\oplus\alpha,Y\oplus\beta]_{\tau,\Phi} = \mathcal{C}_{-\tau}[\mathcal{C}_\tau(X\oplus\alpha),\mathcal{C}_\tau(Y\oplus\beta)]_\Phi,$$

where the right hand side is defined by means of (1.5), will be called a *conformal-Courant bracket*. Using (1.5) we get

$$(1.9)\qquad [X\oplus\alpha,Y\oplus\beta]_{\tau,\Phi} = [X\oplus\alpha,Y\oplus\beta]_{e^{-\tau}\Phi}$$

$$+(0\oplus[(X\tau)\beta-(Y\tau)\alpha-\Omega(X\oplus\alpha,Y\oplus\beta)d\tau]).$$

Remark 1.1. From (1.8), it follows that the conformal Courant bracket satisfies the following among the axioms of a Courant algebroid of anchor $\rho = pr_{TM}$ [15]:

i) $\rho[X\oplus\alpha,Y\oplus\beta]_{\tau,\Phi} = [\rho(X\oplus\alpha),\rho(Y\oplus\beta)]$,

ii) $\rho(\partial f) = 0\quad (f\in C^\infty(M), \partial f = (0,df))$,

iii) $[X\oplus\alpha,f(Y\oplus\beta)]_{\tau,\Phi}=f[X\oplus\alpha,Y\oplus\beta]_{\tau,\Phi}+(Xf)(Y\oplus\beta)$
$-G(X\oplus\alpha,Y\oplus\beta)\partial f.$

But, the other axioms of Courant algebroids are not satisfied. In particular, formula (1.8) yields

$$\sum_{Cycl(1,2,3)}[[X_1\oplus\alpha_1,X_2\oplus\alpha_2]_{\tau,\Phi},X_3\oplus\alpha_3]_{\tau,\Phi}=\frac{1}{3}\partial\sum_{Cycl(1,2,3)}G([X_1\oplus\alpha_1, X_2\oplus\alpha_2]_{\tau,\Phi},X_3\oplus\alpha_3)+\frac{1}{3}\sum_{Cycl(1,2,3)}G([X_1\oplus\alpha_1,X_2\oplus\alpha_2]_{\tau,\Phi},X_3\oplus\alpha_3)\partial\tau. \tag{1.10}$$

We define bracket operations on the stable big tangent bundle from the brackets (1.5) and (1.8) on the manifold $M\times\mathbb{R}^h$. First, a vector field $\tilde X\in\chi^1(M\times\mathbb{R}^h)$, a 1-form $\tilde\alpha\in\Omega^1(M\times\mathbb{R}^h)$, etc. will be called *translation invariant* if they are preserved by the natural action of the group of translations of $\mathbb{R}^h$ on $M\times\mathbb{R}^h$, equivalently, if

$$L_{\frac{\partial}{\partial t^a}}\tilde X=0,\,L_{\frac{\partial}{\partial t^a}}\tilde\alpha=0,\quad\text{etc.}, \tag{1.11}$$

where t^a $(a=1,...,h)$ are the natural coordinates of $\mathbb{R}^h$. Obviously, the space $\Gamma\mathbf{T}^{big}_hM$ is naturally isomorphic with the space of translation invariant cross sections of $T^{big}(M\times\mathbb{R}^h)$ by the identification

$$(X,u)\oplus(\alpha,v)\leftrightarrow(X+\sum_{a=1}^h u^a\frac{\partial}{\partial t^a})\oplus(\alpha+\sum_{a=1}^h v_a dt^a). \tag{1.12}$$

The restriction of the Courant bracket (1.5) of $M\times\mathbb{R}^h$ with a twist form

$$\Phi+\sum_{a=1}^h dt^a\wedge\Psi^a,\qquad \Phi\in\Omega^3(M),\,\Psi^a\in\Omega^2(M),\,d\Phi=0,d\Psi^a=0 \tag{1.13}$$

to translation invariant cross sections, composed by (1.12), will be called the *stable Courant bracket of index h on M*.

It follows that the stable Courant bracket is given by

$$[(X_1,u_1)\oplus(\alpha_1,v_1),(X_2,u_2)\oplus(\alpha_2,v_2)]_C=([X_1,X_2],X_1u_2-X_2u_1)\oplus(L_{X_1}\alpha_2-L_{X_2}\alpha_1+\frac{1}{2}d(\alpha_1(X_2)-\alpha_2(X_1))+i(X_1\wedge X_2)\Phi+u_1\cdot(i(X_2)\Psi) \tag{1.14}$$

$$-u_2\cdot(i(X_1)\Psi)+\frac{1}{2}(u_2\cdot dv_1+v_2\cdot du_1-u_1\cdot dv_2-v_1\cdot du_2), X_1v_2-X_2v_1+\Psi(X_1,X_2)),$$

where Ψ is the $\mathbb{R}^h$-valued form of components Ψ^a. The index C will be omitted if there is no danger of confusion.

Remark 1.2. The bundle $\mathbf{T}^{big}_h M$ with the metric G, the bracket (1.14) and the anchor $\rho = pr_{TM}$ is a transitive Courant algebroid. Thus, the bracket (1.14) could have been derived from the general formulas of [19].

Furthermore, if we replace the Courant bracket by the conformally changed bracket (1.8) of $M\times\mathbb{R}^h$ with the change function $\tau=\tau(t^1,...,t^h)$, restricted to translation invariant sections and with the result taken at $t^a=0$, we get a new bracket, which has the following expression

$$\begin{aligned}(1.15)\quad &[(X_1,u_1)\oplus(\alpha_1,v_1),(X_2,u_2)\oplus(\alpha_2,v_2)]_W=([X_1,X_2],X_1u_2-X_2u_1)\\ &\oplus(L_{X_1}\alpha_2-L_{X_2}\alpha_1+\frac{1}{2}d(\alpha_1(X_2)-\alpha_2(X_1))+d_0\tau(u_1)\alpha_2-d_0\tau(u_2)\alpha_1\\ &+e^{-\tau(0)}i(X_1\wedge X_2)\Phi+e^{-\tau(0)}u_1\cdot(i(X_2)\Psi)-e^{-\tau(0)}u_2\cdot(i(X_1)\Psi)\\ &+\frac{1}{2}(u_2\cdot dv_1+v_2\cdot du_1-u_1\cdot dv_2-v_1\cdot du_2), X_1v_2-X_2v_1+e^{-\tau(0)}\Psi(X_1,X_2)\\ &+\frac{1}{2}(\alpha_1(X_2)-\alpha_2(X_1)+u_2\cdot v_1-u_1\cdot v_2)d_0\tau+d_0\tau(u_1)v_2-d_0\tau(u_2)v_1).\end{aligned}$$

The bracket (1.15) will be called the *Wade (Courant-Jacobi) bracket of index h* because it was first discovered by Wade [21] in the case $h=1,\tau=t$. The index W will be omitted if there is no danger of confusion.

Remark 1.3. In the untwisted case, and if $\partial\tau/\partial t^a = const.$, the bracket (1.15) with an arbitrary t is translation invariant. Hence, in this case the translation invariant section defined by $[(X_1,u_1)\oplus(\alpha_1,v_1),(X_2,u_2)\oplus(\alpha_2,v_2)]_W$ is equal to the conformal-Courant bracket

$$[(X_1+\sum_{a=1}^h u_1^a\frac{\partial}{\partial t^a})\oplus(\alpha_1+\sum_{a=1}^h v_{1,a}dt^a),(X_2+\sum_{a=1}^h u_2^a\frac{\partial}{\partial t^a})\oplus(\alpha_2+\sum_{a=1}^h v_{2,a}dt^a]_\tau.$$

2 Stable Dirac and Dirac-Jacobi Structures

A twisted, respectively untwisted, *stable Dirac structure of index h* is a maximal, G-isotropic subbundle $L \subseteq \mathbf{T}^{big}_h M$, which satisfies the integrability condition of being closed by the twisted, respectively untwisted, bracket (1.14). If the integrability condition is not satisfied the structure is *stable almost Dirac*. In the case $h = 0$ we have the usual Dirac structures of [2, 3].

We can define an equivalence relation that justifies the "stable" terminology. A stable almost Dirac structure of index h, $L \subseteq \mathbf{T}^{big}_h M$, extends to the following structures of index $h + k$:

$$\begin{aligned} \hat{L}_1 &= \{(X, u, w) \oplus (\alpha, v, 0) \,/\, (X, u) \oplus (\alpha, v) \in L,\ w \in \mathbb{R}^k\}, \\ \hat{L}_2 &= \{(X, u, 0) \oplus (\alpha, v, w) \,/\, (X, u) \oplus (\alpha, v) \in L,\ w \in \mathbb{R}^k\}. \end{aligned} \tag{2.1}$$

If the $\mathbb{R}^h$-valued 2-form Ψ of (1.13) is extended by k zero components to a $\mathbb{R}^{h+k}$-valued form, the structures $L, \hat{L}_1, \hat{L}_2$ simultaneously are integrable or not. Notice also the existence of the metric preserving automorphism $F : \mathbf{T}^{big}_{h+k} M = \mathbf{T}^{big}_h M \oplus \mathbb{R}^{2k} \to \mathbf{T}^{big}_{h+k} M$ defined by $F|_{\mathbf{T}^{big}_h M} = Id, F|_{\mathbb{R}^{2k}}(w, 0) = (0, w)$, which sends $\hat{L}_1$ onto $\hat{L}_2$ and conversely.

Definition 2.1. The (almost) Dirac structures $L \subseteq \mathbf{T}^{big}_h M$, $L' \subseteq \mathbf{T}^{big}_{h'} M$ are called *stably equivalent* if there are non negative integers k, k' such that $h + k = h' + k'$, and there exists a metric preserving, bundle automorphism $\varphi : (\mathbf{T}^{big}_{h+k} M, G) \to (\mathbf{T}^{big}_{h'+k'} M, G)$ that sends the prolongation $\hat{L}_1$ defined by (2.1) to the similar prolongation $\hat{L}'_1$.

It is trivial to see that stable equivalence is an equivalence relation. Moreover, if φ exists then $F' \circ \varphi \circ F$, where F was defined above and F' is defined similarly, sends $\hat{L}_2$ onto $\hat{L}'_2$ and conversely.

Example 2.1. Let M^m be a differentiable manifold and N^n be a submanifold with a trivial normal bundle $T_N M / TN$. Then, there exists a non canonical isomorphism

$$T_N M \approx TN \oplus \mathbb{R}^{m-n}. \tag{2.2}$$

The isomorphism (2.2) yields a G-preserving isomorphism

$$I : T^{big}_N M \overset{\approx}{\to} \mathbf{T}^{big}_{m-n} N, \tag{2.3}$$

and if D is an almost Dirac structure of M then $I(D|_N)$ is a stable almost Dirac structure of N. Since the isomorphism (2.3) is not unique, it is rather the corresponding equivalence class of stable almost Dirac structures that is well defined. Generally, the integrability of D does not imply the integrability of $I(D|_N)$.

The meaning of the notion of a stable Dirac structure of index h is given by the following simple proposition.

Proposition 2.1. *A stable, almost Dirac structures L of M may be identified with a translation invariant, almost Dirac structure $\tilde{L}$of the manifold $M \times \mathbb{R}^h$. The structures L and $\tilde{L}$ are simultaneously integrable or not.*

Proof. The invariance of $\tilde{L}$ by translations means that $\forall s \in \mathbb{R}^h, \forall (x,t) \in M \times \mathbb{R}^h$ the translation $\tau_s(x,t) = (x, t+s)$ satisfies the condition

$$\tilde{L}_{(x,t+s)} = (\tau_s)_*(\tilde{L}_{(x,t)}) = \{(\tau_s)_{*(x,t)}\tilde{X} \oplus (\tau_{-s})^*_{(x,t)}\tilde{\alpha} \,/\, \tilde{X} \oplus \tilde{\alpha} \in \tilde{L}_{(x,t)}\}.$$

This condition is equivalent with

$$L_{\frac{\partial}{\partial t^a}}\tilde{\mathcal{X}} \in \Gamma\tilde{L}, \quad \forall \tilde{\mathcal{X}} \in \Gamma\tilde{L},$$

where the Lie derivative is applied to each component of $\tilde{\mathcal{X}}$. Furthermore, the translation invariance condition is also equivalent with the fact that $\Gamma\tilde{L}$ has local bases that consist of translation invariant cross sections $Z_i \oplus \theta_i$ of the form (1.12); these bases are obtained by translating local bases of cross sections of $\tilde{L}|_{t=0}$.

Now, the stable almost Dirac structure $L \subseteq \mathbf{T}^{big}M$ produces the translation invariant almost Dirac structure

$$\tilde{L}_{(x,t)} = \{(\tau_t)_{*(x,0)}X \oplus (\tau_{-t})^*_{(x,0)}\alpha \,/\, X \oplus \alpha \in L_x\} \tag{2.4}$$

of $M \times \mathbb{R}^h$. Moreover, all the translation invariant almost Dirac structures of $M \times \mathbb{R}^h$ are produced in this way by $L = \tilde{L}|_{t=0}$; this may be seen by using the local invariant bases of a translation invariant structure. Furthermore, ΓL is identifiable with the space of the translation invariant cross sections of $\tilde{L}$ of (2.4) via (1.12). Hence, the integrability of $\tilde{L}$ implies the integrability of L. The converse follows by expressing the cross sections of $\tilde{L}$ by means of local, translation invariant bases and by using the properties of the Courant bracket and the isotropy of $\tilde{L}$. □

Remark 2.1. If $\tilde{L}$ is an arbitrary almost Dirac structure of $M \times \mathbb{R}^h$, $L = \tilde{L}|_{t=0}$ is a stable almost Dirac structure of M, which may not be integrable even if $\tilde{L}$ is integrable because the restriction to $t = 0$ of the Courant bracket on $M \times \mathbb{R}^h$ is not the stable Courant bracket on M, generally.

Example 2.2. Let

$$\Pi = W + \sum_{a=1}^{h} V_a \wedge \frac{\partial}{\partial t^a}, \quad W \in \chi^2(M), V_a \in \chi^1(M) \tag{2.5}$$

be a Poisson structure on $M \times \mathbb{R}^h$ with the twist form (1.13). The Poisson condition $[\Pi, \Pi] = 0$ (Schouten-Nijenhuis bracket) is equivalent with the conditions

$$[W, W] = 0, \;\; L_{V_a} W = 0, \;\; [V_a, V_b] = 0 \quad (a, b = 1, ..., h), \tag{2.6}$$

i.e., W is a Poisson structure on M and V_a are h commuting, infinitesimal automorphisms of W. The graph of $\sharp_\Pi$ ($\sharp_P$ is defined by $\beta(\sharp_\Pi \alpha) = \Pi(\alpha, \beta)$) is a translation invariant Dirac structure $\tilde{L}$ on $M \times \mathbb{R}^h$ obtained by the translation of the stable Dirac structure

$$L = \tilde{L}|_{t=0} = \{(\sharp_W \alpha - u \cdot V, \alpha(V)) \oplus (\alpha, u)\}, \tag{2.7}$$

where $\alpha \in T^*M, u \in \mathbb{R}^h$, V is the $\mathbb{R}^h$-valued vector field on M with the components V_a (i.e., $V_x f = (V_{a,x} f) \in \mathbb{R}^h, \forall f \in C^\infty(M), \forall x \in M$) and $\alpha(V) \in \mathbb{R}^h$ is calculated on the components V_a of V.

Example 2.3. The graph of $\flat_\Theta$ ($\flat_\Theta(X) = i(X)\Theta$) where

$$\Theta = \sigma + \sum_{a=1}^{h} \theta_a \wedge dt^a \;\; (\sigma \in \Omega^2(M), \theta_a \in \Omega^1(M), d\sigma = 0, d\theta_a = 0) \tag{2.8}$$

is a translation invariant Dirac structure $\tilde{L}$ with a corresponding stable, Dirac structure

$$L = \{(X, u) \oplus (\flat_\sigma X - u \cdot \theta, \theta(X))\} \tag{2.9}$$

on M.

The following prolongation theorem extends Example 2.2.

Theorem 2.1. *Let $L \subseteq \mathbf{T}^{big}_h M$ be an untwisted, stable Dirac structure of index h on M and let V_p $(p = 1, ..., k)$ be commuting infinitesimal automorphisms of L. Then, formula*

$$(2.10) \qquad \hat{L}_x = \{(X - w \cdot V(x), u, \alpha(V(x))) \oplus (\alpha, v, w)$$
$$/ \, (X, u) \oplus (\alpha, v) \in L, \; w \in \mathbb{R}^k\} \;\; (x \in M)$$

defines an untwisted, stable Dirac structure $\hat{L}$ of index $h + k$.

Proof. Notice that the elements and cross sections of $\hat{L}$ may be written as

$$(2.11) \qquad [(X, u, \alpha(V)) \oplus (\alpha, v, 0)] + [(-w \cdot V, 0, 0) \oplus (0, 0, w)],$$

where $(X, u) \oplus (\alpha, v)$ are either elements or cross sections of L and $w \in \mathbb{R}^k$, $w \in C^\infty(M, \mathbb{R}^k)$, respectively. From (2.11), we see that $\hat{L}$ is differentiable and the maximal isotropy of L in $\mathbf{T}^{big}_h M$ implies the maximal isotropy of $\hat{L}$ in $\mathbf{T}^{big}_{h+k} M$. Then, we will check the closure of $\hat{L}$ by the untwisted bracket (1.14) of index $h+k$ for the three possible combinations of components of (2.11). Of course, we shall use the hypotheses on V, which mean that, $\forall p, q = 1, ..., k$, $[V_p, V_q] = 0$ and $\forall (X, u) \oplus (\alpha, v) \in \Gamma L$ one has $(L_{V_p} X, V_p u) \oplus (L_{V_p} \alpha, V_p v) \in \Gamma L$, i.e., the latter cross section is G-orthogonal to any $(X', u') \oplus (\alpha', v') \in L$. On the other hand, we will see $\mathbf{T}^{big}_h M$ as a subbundle of $\mathbf{T}^{big}_{h+k} M$ by

$$(X, u) \oplus (\alpha, v) \mapsto (X, u, 0) \oplus (\alpha, v, 0).$$

First we have

$$[(X_1, u_1, \alpha_1(V)) \oplus (\alpha_1, v_1, 0), (X_2, u_2, \alpha_2(V)) \oplus (\alpha_2, v_2, 0)]_{h+k}$$
$$= [(X_1, u_1) \oplus (\alpha_1, v_1), (X_2, u_2) \oplus (\alpha_2, v_2)]_h$$
$$+ (0, 0, X_1(\alpha_2(V)) - X_2(\alpha_1(V))) \oplus (0, 0, 0),$$

which has the form of a first component (2.11). Indeed, after reductions, we get

$$< L_{X_1}\alpha_2 - L_{X_2}\alpha_1 + \frac{1}{2} d(\alpha_1(X_2) - \alpha_2(X_1)) + \frac{1}{2}(u_2 . dv_1 + v_2 \cdot du_1 - u_1 \cdot dv_2 - v_1 \cdot du_2), V_p >$$
$$= X_1(\alpha_2(V_p)) - X_2(\alpha_1(V_p)) + G((L_{V_p} X_1, V_p u_1) \oplus (L_{V_p} \alpha_1, V_p v_1), (X_2, u_2)(\oplus \alpha_2, v_2))$$
$$- G((L_{V_p} X_2, V_p u_2) \oplus (L_{V_p} \alpha_2, V_p v_2), (X_1, u_1) \oplus (\alpha_1, v_1)) = X_1(\alpha_2(V_p)) - X_2(\alpha_1(V_p)),$$

because the vector fields V_p are infinitesimal automorphisms of L.

Then,

$$[(-w_1 \cdot V, 0, 0) \oplus (0, 0, w_1), (-w_2 \cdot V, 0, 0) \oplus (0, 0, w_2)]$$
$$= ([w_1 \cdot V, w_2 \cdot V], 0, 0) \oplus (0, 0, w_2 \cdot V(w_1) - w_1 \cdot V(w_2))$$

and the commutation of the vector fields V_p shows that this result has the form of a second component (2.11).

Finally,

$$[(X, u, \alpha(V)) \oplus (\alpha, v, 0), (-w \cdot V, 0, 0) \oplus (0, 0, w)]$$
$$= (w \cdot [V, X] - (Xw) \cdot V, (w \cdot V)(u), (w \cdot V)(\alpha(V)))$$
$$\oplus (w \cdot (L_V \alpha), (w \cdot V)(v), Xw),$$

which is the cross section of $\hat{L}$ obtained by using (2.10) with

$$\sum_{p=1}^{k} w^p(([V_p, X], V_p u) \oplus (L_{V_p}\alpha, V_p v)) \in \Gamma L$$

in the role of $(X, u) \oplus (\alpha, v)$ and Xw in the role of w. $\square$

Remark 2.2. The conclusion remains true for a twisted structure L if we ask the following conditions for the vector fields V_p and the twist form (1.13):

$$i(V_p)\Phi = 0, \; i(V_p)\Psi = 0. \tag{2.12}$$

The twist form for $\hat{L}$ is the same as for L with the addition of k forms $\Psi^p = 0$.

Remark 2.3. Assume that $L \subseteq \mathbf{T}^{big}_h M, L' \subseteq \mathbf{T}^{big}_{h'} M$ are stably equivalent Dirac structures with prolongations $\hat{L}_1 \subseteq \mathbf{T}^{big}_{h+u} M, \hat{L}'_1 \subseteq \mathbf{T}^{big}_{h'+u'} M$ $(h + u = h' + u')$ isomorphic by the metric preserving automorphism φ of $\mathbf{T}^{big}_{h+u} M$. Assume also that the commuting vector fields V_p $(p = 1, ..., k)$ are infinitesimal automorphisms for both L and L'. Then the prolongations $\hat{L} \subseteq \mathbf{T}^{big}_{h+u} M, \hat{L}' \subseteq \mathbf{T}^{big}_{h'+u'} M$ given by Theorem 2.1 are stably equivalent again. Indeed, it is easy to see that V_p also are infinitesimal automorphisms of $\hat{L}_1, \hat{L}'_1$, hence, Theorem 2.1 may be applied to these structures and one gets prolongations $\widehat{(\hat{L}_1)}, \widehat{(\hat{L}'_1)} \subseteq \mathbf{T}^{big}_{h+u+k} M$. The latter are isomorphic by the metric preserving automorphism $\hat{\varphi}$ of $\mathbf{T}^{big}_{h+u+k} M = \mathbf{T}^{big}_{h+u} M \times \mathbb{R}^{2k}$ that acts by φ on the $\mathbf{T}^{big}_{h+u} M$-component and by the identity on the $\mathbb{R}^{2k}$-component.

We also want to add a remark of a different nature:

Remark 2.4. Let L be a stable Dirac structure of index h and $\tilde{L}$ the corresponding, translation invariant Dirac structure of $M \times \mathbb{R}^h$. A known result about Dirac structures tells that $d_{\tilde{L}}\tilde{\Omega}(\tilde{\mathcal{X}}_1, \tilde{\mathcal{X}}_2, \tilde{\mathcal{X}}_3)$, where $\tilde{\Omega}$ is the form (1.4) on $M \times \mathbb{R}^h$, $\tilde{\mathcal{X}}_a \in \Gamma\tilde{L}$ $(a = 1, 2, 3)$ and $d_{\tilde{L}}$ is the exterior differential of the Lie algebroid $\tilde{L}$, is given by the twist form computed on $pr_{T(M\times\mathbb{R}^h)}\tilde{\mathcal{X}}_a$. If we use left invariant arguments and the twist form (1.13) we get

$$(d_L\Omega)(\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3) = \Phi(X_1, X_2, X_3) - \sum_{Cycl(1,2,3)} u_1 \cdot \Psi(X_2, X_3).$$

Thus, in the untwisted case, $\Omega|_L$ is d_L-closed and there exists a *fundamental Lie algebroid cohomology class* $[\Omega] \in H^2(L)$.

Now, we proceed to structures that satisfy the Dirac condition modulo a conformal change and we begin by generalities first discussed in [22]. An almost Dirac structure $D \subseteq T^{big}M$ such that $\mathcal{C}_\tau(D)$ is a Dirac structure, for some $\tau \in C^\infty(M)$, i.e., ΓD is closed by the conformal-Courant bracket (1.8) will be called a *conformal-Dirac structure*[1]. Obviously, D and $\mathcal{C}_\tau(D)$ simultaneously are maximally isotropic. Isotropy and Remark 1.1 imply that any conformal-Dirac structure is a Lie algebroid of anchor pr_{TM} with respect to the restriction of the conformal-Dirac bracket.

Formula (1.9) shows that an untwisted, almost Dirac structure D is conformal-Dirac iff $\forall X \oplus \alpha, Y \oplus \beta \in \Gamma D$ one has

$$\begin{aligned}(2.13)\qquad & [X, Y] \oplus (L_X\beta - L_Y\alpha - d(\Omega(X \oplus \alpha, Y \oplus \beta)) \\ & +(X\tau)\beta - (Y\tau)\alpha - (\Omega(X \oplus \alpha, Y \oplus \beta))d\tau) \in \Gamma D.\end{aligned}$$

Let us look at the equivalent pair $(pr_{TM}(D), \theta_D)$ of the almost Dirac structure D [2, 3], where

$$\begin{aligned}(2.14)\qquad & \theta_D(X, Y) = \Omega(X \oplus \alpha, Y \oplus \beta) = \beta(X) = -\alpha(Y) \\ & (X, Y \in pr_{TM}(D), X \oplus \alpha, Y \oplus \beta \in D).\end{aligned}$$

If D is a conformal-Dirac structure then $pr_{TM}(D) = pr_{TM}(\mathcal{C}_\tau(D))$ is a generalized foliation. Furthermore, (2.13) means that the G-scalar product of

[1]This name was used with a different meaning in [21]; on the other hand, the conformal-Dirac structures as defined here are the globally conformal Dirac structures of [22].

its left hand side by any $Z \oplus \gamma \in \Gamma D$ is zero and this is easily seen to be equivalent to Wade's condition [22]

$$d\theta_D = -(d\tau|_{pr_{TM}D}) \wedge \theta_D. \tag{2.15}$$

Thus, a conformal-Dirac structure produces a generalized foliation with leaves that are *conformal-presymplectic manifolds* [22].

More generally, if D is an almost Dirac structure such that $pr_{TM}(D)$ is a generalized foliation and the corresponding 2-form θ_D satisfies the condition

$$d\theta_D = \phi \wedge \theta_D, \tag{2.16}$$

where ϕ is a leaf-wise closed 1-form, D is a *locally conformal-Dirac structure* and the leaves of the corresponding generalized foliation are locally conformal presymplectic manifolds [22].

Example 2.4. [22] Recall the following formula of Gelfand and Dorfman [6]:

$$[P, P](\alpha, \beta, \gamma) = 2\gamma(\sharp_P\{\alpha, \beta\}_P - [\sharp_P\alpha, \sharp_P\beta]),$$

where $P \in \chi^2(M)$, $[P, P]$ is the Schouten-Nijenhuis bracket and

$$\{\alpha, \beta\}_P = L_{\sharp_P\alpha}\beta - L_{\sharp_P\beta}\alpha - d(P(\alpha, \beta)). \tag{2.17}$$

Accordingly, from (1.5), we get

$$\begin{aligned}&[(\sharp_P\alpha) \oplus \alpha, (\sharp_P\beta) \oplus \beta] \\ &= (\sharp_P\{\alpha, \beta\}_P - \frac{1}{2}i(\alpha \wedge \beta)[P, P]) \oplus \{\alpha, \beta\}_P.\end{aligned} \tag{2.18}$$

Now, if we use (1.9) for $X = \sharp_P\alpha, Y = \sharp_P\beta, \Phi = 0$ and ask the result to belong to $graph\,\sharp_P$, we see that $graph\,\sharp_P$ is a conformal-Dirac structure iff there exists a function $f = -2\tau \in C^\infty(M)$ such that

$$[P, P] = (\sharp_P df) \wedge P. \tag{2.19}$$

If this happens we call P a *conformal Poisson bivector field* or *structure*. Accordingly, a *locally conformal Poisson structure* is a pair (P, ϕ) where P is a bivector field, ϕ is a closed 1-form and the following condition holds

$$[P, P] = (\sharp_P\phi) \wedge P. \tag{2.20}$$

For a locally conformal Poisson structure P, $im\,\sharp_P$ defines a generalized foliation by locally conformal symplectic manifolds. Hence, a locally conformal Poisson manifold is a Jacobi manifold (M,P,E), $P\in\chi^2(M), E\in\chi^1(M)$, i.e., [5]

$$[P,P]=2E\wedge P, L_EP=0, \tag{2.21}$$

where $E=(1/2)\sharp_P\phi$, $d\phi=0$. The condition $L_EP=0$ is implied by (2.20) because of the formula [18]

$$[P,P](\alpha,\beta,\phi)=2[d\phi(\sharp_P\alpha,\sharp_P\beta)-(L_{\sharp_P\phi}P)(\alpha,\beta)]. \tag{2.22}$$

Now, we extend the notion of a Dirac-Jacobi structure defined in [21, 8] as follows.

Definition 2.2. A *stable Dirac-Jacobi structure of index h* is a maximally isotropic subbundle $J\subseteq\mathbf{T}^{big}_hM$ which is *integrable* in the sense that J is closed by the bracket (1.15), where $\tau=\sum_{a=1}^h c_at^a$ is a linear function on $\mathbb{R}^h$.

A stable Dirac-Jacobi structure J of index h can be extended to stable Dirac-Jacobi structures $\hat J_1,\hat J_2$ of index $h+k$ $(k\geq 1)$ defined by formula (2.1) and these prolongations provide an equivalence relation that justifies the "stable" terminology. The structures $\hat J_1,\hat J_2$ simultaneously are integrable or not if we always use the same function τ and extend the Ψ part of the twist form by k zero components.

Using (2.4), we can identify J with a translation invariant, almost Dirac structure $\tilde J$ on $M\times\mathbb{R}^h$ and get the following result.

Proposition 2.2. *The untwisted, stable, almost Dirac structure J is Dirac-Jacobi iff the structure $\mathcal{C}_\tau(\tilde J)$ is a Dirac structure on the manifold $M\times\mathbb{R}^h$.*

Proof. The integrability of $\mathcal{C}_\tau(\tilde J)$ obviously implies that of J. For the converse result, we consider a local basis of $\mathcal{C}_\tau(\tilde J)$ consisting of elements of the form $\mathcal{C}_\tau((X+\sum_{a=1}^h u^a(\partial/\partial t^a))\oplus(\alpha+\sum_{a=1}^h v_adt^a))$ where $(X,u)\oplus(\alpha,v)\in J$. Then, since τ is linear, Remark 1.3 shows that, if J is integrable, the Courant brackets of elements of this basis belong to $\mathcal{C}_\tau(\tilde J)$. For arbitrary Courant brackets we get the same result if we express the arguments of the bracket by means of the previous basis and use property iii), Remark 1.1. □

Remark 2.5. The structure $\mathcal{C}_\tau(\tilde{J})$ is not translation invariant. Instead, the vector fields $\partial/\partial t^a$ are infinitesimal conformal automorphisms of $\mathcal{C}_\tau(\tilde{J})$ in the sense of the following definition, which is interesting in its own right. Let D be an almost Dirac structure of an arbitrary manifold M. A vector field $Z \in \chi^1(M)$ is an *infinitesimal conformal automorphisms* of D if there exists a function $f \in C^\infty(M)$ such that

$$X \oplus \alpha \in \Gamma D \Rightarrow ([Z,X] - fX) \oplus (L_Z\alpha) \in \Gamma D. \tag{2.23}$$

This definition is motivated by the fact that if $D = graph\,\sharp_P$ ($P \in \chi^2(M)$) then Z satisfies (2.23) iff $L_Z P = fP$.

Example 2.5. Take again the bivector field (2.5) on $M \times \mathbb{R}^h$. If the translation invariant, almost Dirac structure $graph\,\sharp_\Pi$ is conformal-Dirac for $\tau = \sum_{a=1}^h c_a t^a$ then, by Proposition 2.2, the stable almost Dirac structure J such that $graph\,\sharp_\Pi = \tilde{J}$ is a stable Dirac-Jacobi structure. The conformal-Dirac condition is equivalent with (2.19), for $P = \Pi$ and $f = -2\sum_{a=1}^h c_a t^a$, which holds iff: i) the vector fields V_a commute, ii) $[W,W] = 2E \wedge W$, iii) $[V_a, W] = E \wedge V_a$, $\forall a = 1, ..., h$, where $E = \sum_{a=1}^h c_a V_a$. Conditions ii) and iii) show that (W, E) is a Jacobi structure.

Theorem 2.1 also yields a prolongation property of Jacobi-Dirac structure.

Proposition 2.3. *Let $J \subseteq \mathbf{T}^{big}_h M$ be an untwisted, stable Dirac-Jacobi structure of index h on M and let V_p $(p = 1, ..., k)$ be commuting vector fields on M such that*

$$(X,u) \oplus (\alpha, v) \in \Gamma J \tag{2.24}$$

$$\Rightarrow \;\; ([V_p, X] + d\tau(u)V_p, V_p(u)) \oplus (L_{V_p}\alpha, V_p(v) - \alpha(V_p)d\tau) \in \Gamma J.$$

Then

$$\hat{J} = \{(X - w \cdot V, u, \alpha(V)) \oplus (\alpha, v, w) \tag{2.25}$$

$$/\,(X,u) \oplus (\alpha, v) \in L,\; w \in \mathbb{R}^k\}$$

is a stable Dirac-Jacobi structure of index $h + k$.

Proof. The integrability of J implies that $\mathcal{C}_\tau(\tilde{J})$ is a Dirac structure on the manifold $M \times \mathbb{R}^h$ (remember that τ is a linear function). Condition (2.24) is equivalent with the fact that the commuting vector fields $e^{-\tau}V_p$ of $M \times \mathbb{R}^h$

are infinitesimal automorphisms of $\mathcal{C}_\tau(\tilde{J})$ (the last term $d\tau$ of (2.24) is to be seen as a vector in $\mathbb{R}^h$). Hence, we can use Theorem 2.1 in order to prolong $\mathcal{C}_\tau(\tilde{J})$ to a stable Dirac structure $\widehat{\mathcal{C}_\tau(\tilde{J})} \subseteq \mathbf{T}^{big}_k(M \times \mathbb{R}^h)$. Moreover, we shall use $e^\tau w$ instead of w in the formula (2.10) that defines the prolongation, which is possible since $w \in \mathbb{R}^k$ was arbitrary. The resulting prolongation can be translated to $M \times \mathbb{R}^h \times \mathbb{R}^k$ and the result is the almost Dirac structure $\mathcal{C}_\tau(\tilde{\hat{J}})$ of $M \times \mathbb{R}^{h+k}$ where $\hat{J}$ is the structure defined by (2.25). Since by Theorem 2.1 this prolongation is integrable we are done. (Notice that the function that defines the conformal change is τ and it does not depend on the coordinates on the factor $\mathbb{R}^k$.) $\square$

If $h = 1$ and J is a Jacobi structure (W, E) seen as a stable Dirac-Jacobi structure of index 1, i.e., as a structure defined like in Example 2.5 with one vector field $V_1 = -E$, a technical calculation shows that hypothesis (2.24) is equivalent to

$$[V_p, W] = E \wedge V_p, \quad [V_p, E] = 0. \tag{2.26}$$

Furthermore, the prolonged structure (2.25) turns out to be the one which is defined by the translation invariant structure *graph* $\sharp_Q$ where

$$Q = W - E \wedge \frac{\partial}{\partial t} + \sum_{p=1}^{k} V_p \wedge \frac{\partial}{\partial s^p}, \tag{2.27}$$

where s^p are the natural coordinates of $\mathbb{R}^k$. Hence, the prolonged structure is of the type described in Example 2.5 with τ having the coefficients $c_1 = -1, c_2 = ... = c_{k+1} = 0$.

3 Generalized almost contact structures

Generalized complex structures recently became a subject of interest for both geometers and physicists [11, 9, 14]. A generalized, almost complex structure can be defined as a complex, almost Dirac structure $L \subseteq T^{big}_c M = T^{big} M \otimes_{\mathbb{R}} \mathbb{C}$ with the property that $L \cap \bar{L} = 0$, where the bar denotes complex conjugation. A necessary condition for the existence of such a structure is the even-dimensionality of M. If L is integrable, i.e., closed by the (twisted) Courant bracket, L is a generalized (twisted) complex structure.

If we give a similar definition in the stable case, where $T^{big}_c M$ is replaced by $\mathbf{T}^{big}_c M = (T_c M \times \mathbb{C}^h) \oplus (T^*_c M \times \mathbb{C}^h)$, we get the notion of a *generalized (almost, twisted) stable complex structure of index* h, which we will denote by a boldface, e.g., $\mathbf{L}$. Such a structure can exist iff $dim\, M + h$ is even.

Like the stable Dirac structures of Section 2, the stable generalized, complex structures of index h of M may be identified with translation invariant, generalized, complex structures on the manifold $M \times \mathbb{R}^h$.

We can define a prolongation of a stable, generalized, almost complex structure $\mathbf{L} \subseteq \mathbf{T}^{big}_{c,h} M$ of index h to a structure of index $h+2k$ by taking the direct sum of the corresponding, translation invariant structure $\tilde{\mathbf{L}}$ of $M \times \mathbb{R}^h$ with a constant complex structure of $\mathbb{R}^{2k}$. If J_0 is the complex structure of $\mathbb{R}^{2k}$ defined by

$$J_0 e_p = f_p,\ J_0 f_p = -e_p, \tag{3.1}$$

where (e_p, f_p) is the canonical orthonormal basis of $\mathbb{R}^{2k}$, the indicated prolongation is defined by

$$\hat{\mathbf{L}} = \{(X, u, w - \sqrt{-1}J_0 w) \oplus (\alpha, v, s - \sqrt{-1}J_0 s) \\ /\, (X, u) \oplus (\alpha, v) \in \mathbf{L},\ w, s \in \mathbb{R}^{2k}\}. \tag{3.2}$$

If we look at the expression (1.14) for cross sections of $\hat{\mathbf{L}}$ and use the compatibility between J_0 and the scalar product in $\mathbb{R}^{2k}$, we see that $\hat{\mathbf{L}}$ is integrable iff $\mathbf{L}$ is integrable. (In the twisted case, we add to Ψ the components $\Psi^p = 0$, $p = 1, ..., 2k$.) The definition of stably equivalent structures via prolongations, given in Section 2 for Dirac structures shall be adapted by the use of the prolongations (3.2).

Remark 3.1. The construction (2.10) of Theorem 2.1 can be used for a stable, generalized complex structure $\mathbf{L}$ and for k commuting, complex, infinitesimal automorphisms V^p of $\mathbf{L}$ and gives a complex Dirac structure of index $h+k$, which is not a generalized complex structure because $\hat{\mathbf{L}} \cap \overline{\hat{\mathbf{L}}} \neq 0$. If we change the construction by using $2k$ commuting, complex, infinitesimal automorphisms V^p and by putting

$$\hat{\mathbf{L}} = \{(X - (w - \sqrt{-1}J_0 w) \cdot V, u, \alpha(V)) \oplus (\alpha, v, w - \sqrt{-1}J_0 w) \\ /\, (X, u) \oplus (\alpha, v) \in \mathbf{L},\ w \in \mathbb{R}^{2k}\},$$

we get a complex isotropic subbundle of $\mathbf{T}^{big}_{c,h+2k}M$ such that $\hat{\mathbf{L}} \cap \overline{\hat{\mathbf{L}}} = 0$. However, $\hat{\mathbf{L}}$ is not a generalized complex structure because it has the complex dimension $h+k$ instead of the required $h+2k$.

Remark 3.2. We could also consider a notion of stable, generalized, complex structure for the stable Wade bracket, i.e., a complex almost Dirac structure $\mathbf{J} \subseteq \mathbf{T}^{big}_{c,h}M$ which is closed by the bracket (1.15). In the untwisted case this is equivalent with asking $\mathcal{C}_\tau(\tilde{\mathbf{J}})$, where τ is like in Definition 2.2 and $\tilde{\mathbf{J}}$ is the corresponding translation invariant structure, to be a generalized, complex structure on the manifold $M \times \mathbb{R}^h$ (see Proposition 2.2). For $h=1$ and $\tau = t$ the structures were discussed in [12] under the name of *generalized almost contact structures*, a name that we will prefer to use differently.

Our main interest will be in the representation of a stable, generalized, almost complex structure by classical tensor fields. As in [9], the structure $\mathbf{L}$ is equivalent with a G-skew-symmetric endomorphism Φ of $\mathbf{T}^{big}M$ such that $\Phi^2 = -Id$ and the integrability condition is equivalent with the annulation of the Nijenhuis torsion where the brackets are interpreted as stable Courant brackets.

We shall recall the following known results [9, 4, 14, 20]. A generalized, almost complex structure Φ on M is equivalent with a triple of classical tensor fields $(A \in \Gamma(EndTM), \pi \in \chi^2(M), \sigma \in \Omega^2(M))$ obtained from the following matrix representation of Φ

$$\Phi(X \oplus \alpha) = \left(\begin{array}{cc} A & \sharp_\pi \\ \flat_\sigma & -{}^tA \end{array}\right)\left(\begin{array}{c} X \\ \alpha \end{array}\right) = (AX + \sharp_\pi\alpha) \oplus (\flat_\sigma X - \alpha \circ A) \tag{3.3}$$

(the index t denotes transposition). The condition $\Phi^2 = -Id$ is equivalent with

$$A^2 = -Id - \sharp_\pi \circ \flat_\sigma,\ \pi(\alpha \circ A, \beta) = \pi(\alpha, \beta \circ A),\ \sigma(AX, Y) = \sigma(X, AY). \tag{3.4}$$

The second, respectively the third, condition (3.4), are *compatibility* of π, respectively σ, with A.

In terms of the classical tensor fields (A, π, σ), the integrability of Φ is expressed by the following four conditions:

i) the bivector field π defines a Poisson structure on M, i.e., $[\pi, \pi] = 0$;

ii) the *Schouten concomitant* of the pair (π, A) vanishes, i.e. (e.g., [20]),

$$R_{(\pi,A)}(\alpha, X) = \sharp_\pi(L_X(\alpha \circ A) - L_{AX}\alpha) - (L_{\sharp_\pi\alpha}A)(X) = 0; \tag{3.5}$$

iii) the Nijenhuis tensor of A satisfies the condition

$$\mathcal{N}_A(X,Y) = [AX,AY] - A[X,AY] - A[AX,Y] + A^2[X,Y] \tag{3.6}$$
$$= \sharp_\pi[i(X\wedge Y)d\sigma];$$

iv) the *associated* 2*-form* $\sigma_A(X,Y) = \sigma(AX,Y)$ satisfies the condition

$$d\sigma_A(X,Y,Z) = \sum_{Cycl(X,Y,Z)} d\sigma(AX,Y,Z). \tag{3.7}$$

The algebraic part of the previous results extends to stable structures while each entry of the matrix (3.3) will be a $(2,2)$-matrix, corresponding to the two components of the stable tangent bundle. We will look at the case where these $(2,2)$-matrices contain classical tensor fields only. More exactly, we shall assume that

$$A = \begin{pmatrix} F & 0 \\ 0 & 0 \end{pmatrix},\ \sharp_\pi = \begin{pmatrix} \sharp_P & -{}^tZ \\ Z & 0 \end{pmatrix},\ \flat_\sigma = \begin{pmatrix} \flat_\theta & -{}^t\xi \\ \xi & 0 \end{pmatrix}, \tag{3.8}$$

where $F \in End(TM)$, $P \in \chi^2(M), \theta \in \Omega^2(M)$, $Z = (Z_a) : T^*M \to \mathbb{R}^h$ is a sequence of h vector fields and its transposition is ${}^tZ(u) = u\cdot Z$, and $\xi = (\xi^a) : TM \to \mathbb{R}^h$ is a sequence of h 1-forms while ${}^t\xi(u) = u\cdot\xi$.

If (3.8) holds, we will say that the stable, generalized, almost complex structure *has strictly classical components* and a simple calculation shows that the conditions (3.4) are equivalent to

$$\begin{gathered} P(\alpha\circ F,\beta) = P(\alpha,\beta\circ F),\ \theta(FX,Y) = \theta(X,FY), \\ F(Z_a) = 0,\ \xi^a\circ F = 0,\ i(Z_a)\theta = 0,\ i(\xi^a)P = 0,\ \xi^a(Z_b) = \delta^a_b, \\ F^2 = -Id - \sharp_P\circ\flat_\theta + \textstyle\sum_{a=1}^h \xi^a\otimes Z_a. \end{gathered} \tag{3.9}$$

Example 3.1. If $h = 1, P = 0, \theta = 0$ then (F,Z,ξ) is just an almost contact structure of M [1]. Similarly, if $h \geq 1, P = 0, \theta = 0$ then (F,Z_a,ξ^a) is a globally framed f-structure [7], which we prefer to call an almost contact structure of codimension h.

Example 3.1 suggests the following definition.

Definition 3.1. A system of tensor fields (P,θ,F,Z_a,ξ^a) that satisfy conditions (3.9) will be called a *generalized, almost contact structure of codimension h*. If the corresponding, generalized, stable complex structure is integrable the generalized, almost contact structure will be called *normal*.

Theorem 3.1. *A generalized, almost contact structure* $(P, \theta, F, Z_a, \xi^a)$ *of codimension* h *such that* -1 *is not an eigenvalue of* $(\sharp_P \circ \flat_\theta)_x$ $(\forall x \in M)$ *is normal iff*

$$\begin{aligned} &[P,P]=0,\ R_{(P,F)}=0,\\ &L_{Z_a}P=0,\ L_{Z_a}\theta=0,\ L_{\sharp_P\alpha}\xi^a=0,\ [Z_a,Z_b]=0,\\ &\mathcal{N}_F(X,Y)=\sharp_P(i(X\wedge Y)d\theta)-\textstyle\sum_{a=1}^h(d\xi^a(X,Y))Z_a,\\ &d\theta_F(X,Y,Z)=\textstyle\sum_{Cycl(X,Y,Z)}d\theta(FX,Y,Z).\end{aligned} \tag{3.10}$$

Proof. In order to get the integrability conditions for the structure (3.8) we write down the integrability conditions i)-iv) recalled above for the corresponding translation invariant structure of $M \times \mathbb{R}^h$. The classical tensor fields (3.3) of the translation invariant structure are

$$A = F,\ \pi = P + \sum_{a=1}^{h} Z_a \wedge \frac{\partial}{\partial t^a},\ \sigma = \theta + \sum_{a=1}^{h} \xi^a \wedge dt^a. \tag{3.11}$$

Since π is of the form (2.5), we know that condition i) is equivalent with:

i') P is a Poisson bivector field and Z_a are commuting infinitesimal automorphisms of P, i.e.,

$$[Z_a, Z_b] = 0, [P, P] = 0, L_{Z_a}P = 0.$$

Then, if we compute the Schouten concomitant

$$R_{(\pi,A)}(\alpha + \sum_{a=1}^{h} v_a dt^a, X + \sum_{a=1}^{h} u^a(\partial/\partial t^a))$$

for the four possible pairs of terms of the arguments and use (3.9), we see that integrability condition ii) becomes

ii') $\qquad R_{(P,F)} = 0,\ L_{Z_a}F = 0.$

Similarly, if we express integrability condition iii) for the same pairs of terms as above we get iii') $\quad L_{Z_a}\theta = 0,\ L_{Z_b}\xi^a = 0\ L_{\sharp_P\alpha}\xi^a = 0\ \ (\forall\alpha\in\Omega^1(M)),$

$$\mathcal{N}_F(X,Y) = \sharp_P(i(X\wedge Y)d\theta) - \sum_{a=1}^{h}(d\xi^a(X,Y))Z_a.$$

Finally, we have

$$\sigma_A(X+\sum_{a=1}^{h}u^a\frac{\partial}{\partial t^a}, Y+\sum_{a=1}^{h}u'^a\frac{\partial}{\partial t^a}) = \theta(FX,Y) = \theta_F(X,Y)$$

and, if we express (3.7) for arguments like those used to get ii'), we see that integrability condition iv) becomes

iv') $\quad (L_{FX}\xi^a)(Y) - (L_{FY}\xi^a)(X) = 0,$

$$d\theta_F(X,Y,Z) = \sum_{Cycl(X,Y,Z)} d\theta(FX,Y,Z).$$

Thus, i')-iv') are the normality conditions; they include the conditions (3.10) and the supplementary conditions

$$L_{Z_b}\xi^a = 0,\ L_{Z_a}F = 0,\ (L_{FX}\xi^a)(Y) - (L_{FY}\xi^a)(X) = 0. \tag{3.12}$$

We will prove that, if -1 never is an eigenvalue of $\sharp_P\circ\flat_\theta$, (3.12) follow from (3.10).

We begin by showing that (3.9), (3.10) imply the existence of some nice local coordinates and tangent bases. The condition $\xi^a(Z_b) = \delta^a_b$ contained in (3.9) shows that the vector fields Z_a are linearly independent and so are the 1-forms ξ^a. Since the vector fields Z_a commute $span\{Z_a\}$ is tangent to a foliation $\mathcal{Z}$, with parallelizable leaves, with local leaf-wise coordinates z^a such that $Z_a = \partial/\partial z^a$ and with transversal local coordinates y^u such that the leaves of $\mathcal{Z}$ have the local equations $y^u = const.$ $(u = 1, ..., dim\, M - h)$. Furthermore, the 1-forms ξ^a must have local expressions of the form

$$\xi^a = dz^a + \xi^a_u dy^u \tag{3.13}$$

and provide a complementary, normal distribution $\nu\mathcal{Z}$ $(TM = \nu\mathcal{Z}\oplus T\mathcal{Z})$ of equations $\xi^a = 0$ and with local bases

$$Y_u = \frac{\partial}{\partial y^u} - \xi^a_u\frac{\partial}{\partial z^a}. \tag{3.14}$$

Now, if we take $X = Y_u, Y = Z_b$ in the condition (3.10) on $\mathcal{N}_F$, we get

$$F \circ (L_{Z_b}F)(Y_u) = -\sharp_P i(Y_u)i(Z_b)d\theta - \sum_{a=1}^{h} d\xi^a(Y_u, Z_b)Z_a \tag{3.15}$$

$$= -\sharp_P i(Y_u)L_{Z_b}\theta + \sum_{a=1}^{h}(L_{Z_b}\xi^a)(Y_u)Z_a \overset{(3.10)}{=} \sum_{a=1}^{h}(L_{Z_b}\xi^a)(Y_u)Z_a.$$

If we evaluate ξ^c on (3.15) and use (3.9), we get $L_{Z_b}\xi^c(Y_u) = 0$. Since it is also simple to check that $L_{Z_b}\xi^c(Z_a) = 0$, it follows that $L_{Z_b}\xi^c = 0$, which is the first condition (3.12).

Therefore, (3.15) reduces to $F \circ (L_{Z_b}F)(Y_u) = 0$ and if we apply F to the previous equality and use the last condition (3.9) we get

$$(Id + \sharp_P \circ \flat_\theta)((L_{Z_b}F)(Y_u)) = \sum_{b=1}^{h} \xi^a((L_{Z_b}F)(Y_u))Z_a = (L_{Z_b}\xi^a)(FY_u) = 0.$$

Thus, if $\sharp_P \circ \flat_\theta$ never has the eigenvalue -1 on M, we get $(L_{Z_b}F)(Y_u) = 0$. Since in view of (3.9), (3.10) we also have $(L_{Z_a}F)(Z_b) = 0$, we get $L_{Z_b}F = 0$, which is the second condition (3.12).

Finally, we shall derive the last condition (3.12). This condition is trivial for $X = Z_a, Y = Z_b$, because of $L_{Z_a}F = 0$. For $X = Y_u, Y = Z_b$ the condition reduces to $(L_{FY_u}\xi^a)(Z_b) = 0$, which holds since

$$(L_{FY_u}\xi^a)(Z_b) = \xi^a([Z_b, FY_u]) = \xi^a(L_{Z_b}F(Y_u) + F[Z_b, Y_u]) = 0.$$

Finally, for $X = Y_u, Y = Y_v$ one has

$$(L_{FY_u}\xi^a)(Y_v) - (L_{FY_v}\xi^a)(Y_u) = \xi^a([Y_v, FY_u] + [FY_v, Y_u]).$$

The last expression may be calculated from the condition (3.10) on $\mathcal{N}_F$, which gives

$$\xi^a(\mathcal{N}_F(FY_u, Y_v)) = -d\xi^a(FY_u, Y_v),$$

equivalently,

$$\xi^a([FY_v, F^2Y_u] + [FY_u, Y_v]) = 0.$$

Here, we may insert the expression of F^2 given by the last condition (3.9) and the result is

$$\xi^a([Y_v, FY_u] + [FY_v, Y_u]) = \xi^a([\sharp_P \circ \flat_\theta Y_u, Y_v]) = -(L_{\sharp_P \circ \flat_\theta(Y_u)}\xi^a)(Y_v) = 0,$$

because of the condition $L_{\sharp_P \alpha}\xi^a = 0$ contained in (3.10). $\square$

In the classical case $P = 0, \theta = 0$ Theorem 3.1 was known [1, 7]. Theorem 3.1 and some of the facts contained in its proof lead to the following result.

Theorem 3.2. *Any normal, generalized, almost contact structure* $(P, \theta, F, Z_a, \xi^a)$ *of codimension h projects to a generalized complex structure of the space of leaves of the foliation* $\mathcal{Z}$.

Proof. In view of (3.9), the expressions of the tensor fields P, θ, F by means of the local bases (Y_u, Z_a) and cobases (dy^u, ξ^a) are of the form

$$(3.16) \quad P = \frac{1}{2}P^{uv}Y_u \wedge Y_v, \ \theta = \frac{1}{2}\theta_{uv}dy^u \wedge dy^v, \ F(Z_a) = 0, \ F(Y_u) = F^v_u Y_v.$$

Furthermore, the normality conditions (3.10) show that the coefficients P^{uv}, θ_{uv}, F^v_u locally depend on the coordinates y^u alone. Therefore, a projected structure defined by P, θ, F exists. For this projected structure conditions (3.4) and i), ii), iii) follow from the combination of (3.9) with i'), ii'), iii') since the extra terms that appear in the last condition (3.9) and in the last condition iii') are terms in Z_a and have the zero projection. Thus, the projected structure is a generalized, complex structure. □

In Theorem 3.2, if we prefer not to consider a general space of leaves, we may either refer to local transversal submanifolds of $\mathcal{Z}$ or add the hypothesis that $M/\mathcal{Z}$ is a Hausdorff manifold. In what follows, we will give a more complete and precise result.

In the usual way of foliation theory (e.g., [23]), we define a *transversal, generalized, (almost) complex structure* of a foliated manifold $(M, \mathcal{F})$ to be a maximal family $\{U_i, L_i\}$, where $\{U_i\}$ is a covering of M by $\mathcal{F}$-adapted coordinate neighborhoods and L_i is a generalized, (almost) complex structure of the local space of leaves $U_i/(\mathcal{F}|_{U_i})$, such that, $\forall i, j$, L_i and L_j restrict to the same structure on the connected components of the open submanifold $U_i \cap U_j/(\mathcal{F}|_{U_i \cap U_j})$. The structures L_i are equivalent with a triple of tensor fields (A_i, σ_i, π_i) of tensorial type $(1, 1), (0, 2), (2, 0)$, respectively, on $U_i/(\mathcal{F}|_{U_i})$, which satisfy the conditions (3.4). If $\nu\mathcal{F}$ is a normal bundle of $\mathcal{F}$ these tensor fields glue up to corresponding global tensor fields (A, σ, π) of the bundle $\nu\mathcal{F}$, which have a unique extension by 0 to projectable tensor fields of M. Thus, after a choice of $\nu\mathcal{F}$, we may equivalently define a transversal, generalized (almost) complex structure by a projectable triple (A, σ, π).

Now, we shall prove

Theorem 3.3. *A normal, generalized, almost contact structure* $(P, \theta, F, Z_a, \xi^a)$ *of codimension* h *is equivalent with the following triple of objects:* 1) *a foliation* $\mathcal{Z}$ *endowed with a parallelization that consists of commuting vector fields* Z_a, 2) *a* $\mathcal{Z}$*-transversal, generalized complex structure associated with* (F, P, θ), 3) *a normal bundle* $\nu\mathcal{Z}$, *which is invariant by the linear holonomy of* $\mathcal{Z}$ *and by the infinitesimal transformations belonging to im* $\sharp_P$ $(P \in \Gamma \wedge^2 \nu\mathcal{Z})$ *and has an* F*-invariant Ehresmann curvature* $(F \in End(\nu\mathcal{Z}))$.

Proof. The proofs of Theorem 3.1 and Theorem 3.2 show that a normal structure $(F, P, \theta, Z_a, \xi^a)$ yields the three required objects. The foliation $\mathcal{Z}$ and the parallelization (Z_a) were defined there. The $\mathcal{Z}$-transversal, generalized complex structure is given by the projections of the tensor fields (F, P, θ) which have the expression (3.16). The normal bundle $\nu\mathcal{Z}$ has the equations $\xi^a = 0$. The invariance properties required by 3) are ensured by the conditions $L_{Z_b}\xi^a = 0$ and $L_{\sharp_P\alpha}\xi^a = 0$. The Ehresmann curvature is defined as if $\mathcal{Z}$ would be an Ehresmann connection of a fiber bundle, i.e., by the formula

$$R_{\nu\mathcal{Z}}(X, Y) = -pr_{T\mathcal{Z}}[pr_{\nu\mathcal{Z}}X, pr_{\nu\mathcal{Z}}Y] \tag{3.17}$$

and its invariance by F means that

$$R_{\nu\mathcal{Z}}(FX, FY) = R_{\nu\mathcal{Z}}(X, Y). \tag{3.18}$$

If we express the last condition iii') on $X = Y_u, Y = Y_v$ and evaluate ξ^c on the result, we get

$$[FY_u, FY_v] - [Y_u, Y_v] \in \Gamma\nu\mathcal{Z},$$

which is equivalent to (3.18).

Conversely, from the objects 1) and 3) we get global 1-forms ξ^a by asking that $\xi^a \in ann\,\nu\mathcal{Z}$ and $\xi^a(Z_b) = \delta^a_b$. Then, we have local coordinates where (3.13) and (3.14) hold and we can define tensor fields (F, P, θ) given by (3.16) that produce the given object 2) (this is what we meant by the term "associated" used as a condition in the formulation of 2)). The algebraic conditions (3.9) hold. This is obvious for all but the last of them and the last follows since the fact that (F, P, θ) define a $\mathcal{Z}$-transversal, generalized complex structure implies the existence of coefficients λ^a_u such that

$$F^2 + Id + \sharp_P \circ \flat_\theta(Y_u) = \sum_{a=1}^{h} \lambda^a_u Z_a$$

and $\xi^a(Z_b) = \delta^a_b$ shows that $\lambda^a_u = \xi^a(Y_u)$.

Now, we shall check the integrability conditions (3.10). The holonomy invariance of $\nu\mathcal{Z}$ is equivalent with $[Z_a, Y_u] \in \Gamma\nu\mathcal{Z}$ and, since by (3.14) $[Z_a, Y_u] \in \Gamma T\mathcal{Z}$, we get $[Z_a, Y_u] = 0$, which is equivalent with $L_{Z_b}\xi^a = 0$. These observations and the expressions (3.16) yield $L_{Z_a}P = 0, L_{Z^a}\theta = 0, L_{Z^a}F = 0$.

Furthermore, the Hamiltonian invariance of $\nu\mathcal{Z}$ is equivalent with the existence of coefficients κ^a_b such that $L_{\sharp_P\alpha}\xi^a = \kappa^a_b\xi^b$. If evaluated on Y_u, the former condition gives $L_{\sharp_P\alpha}\xi^a(Y_u) = 0$, while $L_{\sharp_P\alpha}\xi^a(Z_b) = 0$ is a consequence of the algebraic properties. Thus, the integrability condition $L_{\sharp_P\alpha}\xi^a = 0$ holds.

The conditions $[P, P] = 0, R_{(P,F)} = 0$ follow by evaluating $[P, P]$ and $R_{(P,F)}$ on the bases $(Z_a, Y_u), (\xi^a, dy^u)$ while using the formulas (3.16), (2.22) and (3.5). If the pair of arguments that we consider are transverse to $\mathcal{Z}$ the annulation follows because of the integrability of the transverse, generalized, complex structure. For other types of arguments the annulation is either straightforward or a consequence of the holonomy and the P-Hamiltonian invariance of $\nu\mathcal{Z}$ (hypotheses of 3)). A similar procedure checks the last condition (3.10).

Finally, the Nijenhuis tensor condition in (3.10) is checked as follows. For arguments $X = Z_a, Y = Z_b$ and $X = Z_a, Y = Y_u$ the condition can be deduced as in the calculations at the end of the proof of Theorem 3.1. For arguments $Y = Y_u, Y = Y_v$, because of the integrability of the transversal, generalized, complex structure 2), there exist coefficients λ^a_{uv} such that

$$\mathcal{N}_F(Y_u, Y_v) - \sharp_P(i(Y_v)i(Y_u)d\theta) = \sum_{a=1}^{h} \lambda^a_{uv} Z_a.$$

The evaluation of ξ^a on the previous equality gives $\lambda^a_{uv} = \xi^a([FY_u, FY_v])$. Then, since the invariance of the Ehresmann curvature (3.18) is equivalent with

$$[FX, FY] - [X, Y] \in \Gamma\nu\mathcal{Z}, \quad \forall X, Y \in \Gamma\nu\mathcal{Z}, \tag{3.19}$$

we get

$$\lambda^a_{uv} = \xi^a([Y_u, Y_v]) = -d\xi^a(Y_u, Y_v).$$

This finishes the proof. □

Remark 3.3. The properties of $\mathcal{Z}$ required in 1) of Theorem 3.3 are equivalent with the fact that $\mathcal{Z}$ is a foliation of M by the orbits of a locally free action of the additive group $\mathbb{R}^h$.

Example 3.2. Let (N, F, P, θ) be a generalized, complex manifold and M a principal bundle over N with the structure group $\mathbb{T}^h$, the h-dimensional torus. Assume that the principal bundle M has a connection ξ with the connection forms ξ^a and the curvature $\Xi = \{d\xi^a\}$ satisfying the following two conditions: a) $i(\sharp_P\alpha)\Xi = 0$, b) $\Xi(FX, FY) = \Xi(X, Y)$, where P, F are the lifts of the corresponding tensors of N to the horizontal distribution of the connection, extended by 0 to non horizontal arguments. Then, there exists a corresponding, well defined, normal, generalized, almost contact structure of codimension h on M. Indeed, if we denote by Z_a the fundamental, vertical, vector fields defined on M by the natural basis of the Lie algebra $\mathbb{R}^h$ of the structure group $\mathbb{T}^h$ (e.g., see [13]) we get the object 1) required by Theorem 3.3. The horizontal distribution of the connection may play the role of the normal bundle $\nu\mathcal{Z}$ and the connection forms ξ^a satisfy the required algebraic conditions. Conditions a), b) and the property $i(Z_b)\Xi^a = 0$, which holds because Ξ is the curvature of a connection, ensure the fulfillment of the remaining hypotheses of Theorem 3.3 and we are done. For instance, conditions a) and b) hold for a flat connection; this gives a concrete example of a normal, generalized, almost contact structure. Conversely, if we assume that $(P, \theta, F, Z_a, \xi^a)$ is a normal, generalized, almost contact structure of codimension h on a compact manifold M and that suitable regularity conditions hold for the foliations defined by the vector fields Z_a, we should be able to get an extended *Boothby-Wang fibration theorem* [1] telling that M is a principal torus bundle with connection over a manifold endowed with a usual, generalized complex structure.

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References

[1] D. E. Blair, Riemannian geometry of contact and symplectic manifolds, Progress in Math., vol. 203, Birkhäuser Verlag, Basel, 2001.

[2] T. J. Courant and A. Weinstein, Beyond Poisson structures, Actions hamiltoniennes de groupes, Troisième théorème de Lie (Lyon 1986), 39-49, Hermann, Paris, 1988.

[3] T. J. Courant, Dirac Manifolds, Transactions Amer. Math. Soc., 319 (1990), 631-661.

[4] M. Crainic, Generalized complex structures and Lie brackets, arXiv:math.DG/0412097.

[5] P. Dazord, A. Lichnerowicz and Ch.-M. Marle, Structure locale des variétés de Jacobi. J. Math. pure et appl., 70 (1991), 101-152.

[6] I. M. Gelfand and I. Ya. Dorfman, The Schouten bracket and Hamiltonian operators, Funkt. Anal. Prilozhen., 14(3) (1980), 71-74.

[7] S. I. Goldberg and K. Yano, On normal globally framed f-manifolds, Tôhoku Math. J. (2), 22 (1970), 362-370.

[8] J. Grabowski and G. Marmo, The graded Jacobi algebras and (co)homology, J. Phys., A 36 (2003), 161-181.

[9] M. Gualtieri, Generalized complex geometry, Ph.D. thesis, Univ. Oxford, 2003; arXiv:math.DG/0401221.

[10] V. Guillemin, V. Ginzburg and Y. Karshon, Moment Maps, Cobordisms and Hamiltonian Group Actions, AMS, Math. Surveys and Monographs, Vol. 98, Providence RI, 2002.

[11] N. J. Hitchin, Generalized Calabi-Yau manifolds, Quart. J. Math., 54 (2003), 281-308.

[12] D. Iglesias-Ponte and A. Wade, Contact manifolds and generalized complex structures, J. Geom. Phys. 53 (2005), 249-258.

[13] S. Kobayashi and K. Nomizu, Foundations of differential geometry, I, Interscience, New York, 1963.

[14] U. Lindström, R. Minasian, A. Tomasiello and M. Zabzine, Generalized complex manifolds and supersymmetry, Commun. Math. Phys., 257 (2005), 235-256.

[15] Z.-J. Liu, A. Weinstein and P. Xu, Manin triples for Lie bialgebroids, J. Diff. Geom., 45 (1997), 547-574.

[16] F. Petalidou and J. M. Nunes da Costa, Dirac structures for generalized Courant and Courant algebroids, Travaux Math. - Publ. de l'Univ. de Luxembourg (4^{th} Conference on Poisson Geometry; C. Molitor-Braun, N. Poncin and M. Schlichenmaier eds.), 16 (2005), 265-272.

[17] P. Ševera and A. Weinstein, Poisson geometry with a 3-form background, Prog. Theor. Phys. Suppl. 144 (2001), 145-154; arXiv:math.SG/0107133 v2.

[18] I. Vaisman, Coupling Poisson and Jacobi Structures on Foliated Manifolds, Int. J. of Geom. Methods in Modern Phys., 1 (5), (2004), 607-637.

[19] I. Vaisman, Transitive Courant algebroids, Intern. J. of Math. and Math. Sci., 2005(11) (2005), 1737-1758.

[20] I. Vaisman, Reduction and submanifolds of generalized complex manifolds, arXiv:math.DG/0511013.

[21] A. Wade, Conformal Dirac structures, Lett. Math. Phys. 53 (2000), 331-348.

[22] A. Wade, Locally Conformal Dirac Structures and Infinitesimal Automorphisms, Commun. Math. Phys., 246 (2004), 295-310.

[23] R. A. Wolak, Foliated and associated geometric structures on foliated manifolds, Ann. Fac. Sci. Toulouse Math. (5) 10 (1989), no. 3, 337–360.

Department of Mathematics
University of Haifa, Israel
E-mail: vaisman@math.haifa.ac.il