**Large character heights,$Qd(p)$, and the Ordinary Weight Conjecture**

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**Dedicated to the memory of Walter Feit**

## Introduction

When $p$ is an odd prime, the role of the group $Qd(p)$ in obstructing the control of $p$-fusion by a single local subgroup is well established- see, for example, Glauberman [7]. A somewhat analogous result for blocks was proved by Kessar, Linckelmann and the present author in [9].

On the other hand, for control of fusion and other locally controlled phenomona, Alperin-Goldschmidt subpairs play a crucial role. We take this opportunity to remind the reader that when $B$ is a $p$-block, an Alperin-Goldschmidt $B$-subpair $(U, b_U)$ is a $B$-subpair such that $b_U$ has defect group $Z(U)$ and $O_p(N_G(U, b_U)/UC_G(U)) = 1$. When $B$ is the principal $p$-block, the Alperin-Goldschmidt $B$-subpairs may be identified with their $p$-subgroup components, and those contained in a fixed Sylow $p$-subgroup form a conjugation family.

In the past 20 years or so, conjectures have emerged which attempt to make precise how various block-theoretic invariants are $p$-locally controlled. This began in [2] with Alperin's Weight Conjecture (AWC)- a conjectural $p$-locally determined formula for the number of (characteristic $p$) simple modules in a $p$-block of positive defect. This was later reformulated by R. Knörr and the present author in [10] to a logically equivalent statement which gave a conjectural $p$-locally determined formula for the number of ordinary irreducible characters in a $p$-block of positive defect. This reformulation was in turn refined in a number of ways by E.C. Dade [3], the key new ingredient being the involvement of defects of irreducible characters, and the conjecture which the others were ultimately directed towards proving being Dade's Projective Conjecture (DPC), though it appeared that strengthened versions would be necessary to provide the desired inductive reductions. We remind the reader that the defect of an irreducible character $\chi$ of the finite group $G$ is that integer $d(\chi)$ for which $p^{d(\chi)}\chi(1)_p = |G|_p$, and that when $\chi$ lies in a block $B$ of defect $d$, then the defect of $\chi$ and the height, $h(\chi)$, of $\chi$, are related by the formula $d(\chi) = d - h(\chi)$.

In a series of papers, the most recent being [14], the current author has developed the Ordinary Weight Conjecture (OWC), which has been proved by C.W. Eaton [5] to be logically equivalent to DPC, in the now usual sense for this type of conjecture- a minimal counterexample to one is a minimal counterexample to the other, though it is not a priori clear whether one is true for a given block if the other is. A key result of [14] is that the local computations necessary for OWC for a given block $B$ can all be performed within normalizers of Alperin-Goldschmidt $B$-subpairs ( it is by no means clear a priori that this should be the case for DPC). A consequence of this is that, if OWC is correct, then defects of irreducible characters in the $p$-block $B$ should be defects of irreducible characters of subgroup components of Alperin-Goldschmidt $B$-subpairs. We remind the reader that OWC is known to hold for $p$-blocks of $p$-solvable groups, for nilpotent blocks, for blocks whose defect group is cyclic, dihedral, semi-dihedral, or (generalized) quaternion.

A theme which we wish to emphasize throughout this paper is that "generically, but not generally", Alperin-Goldschmidt $B$-subpairs are large compared to maximal $B$-subpairs. That is, it is usually ( but not always) the case that, given an Alperin-Goldschmidt $B$-subpair $(U, b_U)$, the size of a defect group,$D$, of $B$ is bounded in terms of $|U|$. While it is difficult to make this statement entirely precise, it appears that in the relatively rare circumstances when $|D|$ can't be bounded in terms of $|U|$, the structure of $D$ is quite strongly restricted. In any case, a consequence should be that when $B$ satisfies OWC, the existence of an irreducible character in $B$ of a given defect $e$ implies the existence of an Alperin-Goldschmidt $B$-subpair $(U, b_U)$ with $|U|$ bounded in terms of $p^e$, which should, in turn put restrictions on the structure of the defect group, $D$, of $B$. Another consequence of the results of [14] is that, for a given Alperin-Goldschmidt $B$-subpair $(U, b_U)$, the contributions to the formulae of OWC can be calculated not only entirely within $N_G(U, b_U)$, but that $N_G(U, b_U)$ can be replaced by a $p'$-central extension $\tilde{L_U}$ of a certain extension $L_U$ of $N_G(U, b_U)/UC_G(U)$ by $U$. The 2-cocycle associated to $L_U$ is obtained from the action of $N_G(U, b_U)/UC_G(U)$ on the unique simple $b_U$-module. The necessary extensions date back to the paper [12] of Külshammer-Puig. Once the possibilities for $L_U$ and the associated 2-cocycle have been pinned down, the computations need make no further reference to the block $B$, and the same $\tilde{L_U}$ may occur for many different blocks.

In the first section, we will show that for odd $p$, the involvement of $Qd(p)$ is necessary if a $p$-block $B$ which satisfies OWC is to contain irreducible characters of height large compared to $d$. In the second section, we prove for such a block $B$ that if every Alperin-Goldschmidt $B$-subpair is normalized by a maximal $B$-subpair then it is still the case that character defects can't get "too small" in comparison to the size of the defect group in the above sense. This result is motivated by the case of blocks of finite groups of Lie type in characteristic $p$, where the hypotheses are satisfied, despite the presence of $Qd(p)$. We remark that for $p$-solvable groups, it is known from the results of Haggarty [8] that the height of an irreducible character in a block of defect $d$ is at most $\frac{3d-4}{4}$, a bound which can be ( and, in [8], is) substantially improved for odd $p$.

While it is an open question whether all $p$-blocks $B$ of finite groups of Lie type in characteristic $p$ satisfy OWC, it is of interest to note that it follows from a result of M. Geck [6] that for "good" characteristics $p$- in particular, for primes $p > 5$ in all types- defects and heights of irreducible characters of $p$-blocks of finite groups of Lie type in characteristic $p$ do satisfy the bound given in Theorem 2. Geck's result verified directly that (for "good" primes $p$) defects of irreducible characters in $p$-blocks of finite groups of Lie type in characteristic $p$ are defects of irreducible characters of unipotent radicals of parabolics, as the present author had previously observed should follow (for all primes $p$) from OWC.

It is worth pointing out explicitly that this type of bound can't be achieved for every block. For example, by consideration of principal 2-blocks of $SL(2, q)$ for suitable choices of odd prime power $q$, it can be seen that there are, for arbitrarily large $d > 3$, blocks with generalized quaternion defect groups of order $2^d$ and which contain irreducible characters of defect 2 (or height $d - 2$). Furthermore, any defect group of such a block has a cyclic normal subgroup of order $2^{d-1}$, yet the block contains irreducible characters of defect less than $d - 1$. Such blocks do, however, satisfy OWC. For $d > 4$, these blocks contain Alperin-Goldschmidt subpairs which are not normalized by any maximal subpair. Nevertheless, we will see later that it is the presence of these small Alperin-Goldschmidt $B$-subpairs, and associated irreducible characters of small defect, which forces the block to be of tame type.

In the later sections, we examine the circumstances under which such a $B$-subpair may occur when $U$ is metacyclic or ( for $p$ odd ) extra-special of exponent $p$ and order $p^3$. We determine all possible structures for $U$, and point out the restrictions placed on the structure of the defect group $D$ when such a subpair exists. It turns out that we can determine all possible fusion patterns for the block, except when $p$ is odd and $U$ is elementary Abelian of order $p^2$ or extra-special of exponent $p$ and order $p^3$, in which case we can still show that the defect group must have maximal class. One reason, apart from tractability, for considering these cases is that they are cases in which the 2-cocycle associated to $L_U$ is guaranteed to be trivial.

We are able to calculate the contribution to the formula of OWC for $k_e(B)$ for every integer $e$ from chains of $B$-subpairs beginning with our Alperin-Goldschmidt $B$-subpair $(U, b_U)$ with $U$ metacyclic ( or extra-special of exponent $p$ and order $p^3$). In particular, we will show that, according to OWC, the tame case is typical in the sense that, if OWC holds, then when $d > 2$, characters of defect 2 ( height $d - 2$) can only occur for odd $p$ when the defect group has maximal class ( though is not metacyclic). A key case is when $U$ is elementary Abelian of order $p^2$, and the calculations in that case were performed ( for odd $p$) by A. Alghamdi in his Birmingham Ph.D. thesis [1].

As a by-product of our investigations, we are able to prove that if the defect group $D$ is itself metacyclic, then fusion of $B$-subpairs is controlled by $N_G(D, b_D)$ for $p$ odd, and for $p = 2$, except when $D$ is dihedral, semi-dihedral or generalized quaternion. This means that, apart from the tame case, where OWC is already known to hold, verifying OWC for such blocks reduces to checking that there is a defect preserving bijection between irreducible characters in $B$ and irreducible characters of its unique Brauer correspondent for $N_G(D)$. We also note that when $p = 2$ and $B$ is a 2-block of $G$ with metacyclic defect group

$D$ such that $N_G(D, b_D)$ controls fusion of $B$-subpairs, then there are just three possibilities: $B$ is nilpotent, $D$ is quaternion of order 8, or $D$ is Abelian of type $(2^n, 2^n)$ for some $n$. In the first two cases, OWC is known to hold. We point out that ( for odd $p$), R. Stancu [15] has recently proved a more general result ( for general fusion systems on a metacyclic $p$-group, rather than the Frobenius category of a block with metacyclic defect group). We thank R. Kessar for bringing this paper to our attention.

### Section 1: Some remarks on $p$-stability and quadratic actions

We first give ( in a series of lemmas) a general criterion for the involvement of $Qd(p)$ in a group $G$, which is well-known. We take the opportunity, though, to develop a slightly different approach to these results. The first lemma is a variant of H. Blichfeldt's "two-eigenvalue argument".

**LEMMA 1.1:** *Let $F$ be an algebraically closed field, and let $V$ be a finite-dimensional $FG$-module, where $G = \langle x, y \rangle$ is a finite group such that $x$ and $y$ each have minimum polynomial of degree at most 2 on $V$. Then every composition factor of $V$ as $FG$-module is 1 or 2-dimensional.*

**PROOF:** Let $U$ be such a composition factor. It suffices to consider the case that $x$ and $y$ both act quadratically on $U$. Since $F$ is algebraically closed, there is a vector $u \in U$ and there is a scalar $\lambda$ such that $u(x - y) = \lambda u$. Then $\mathrm{span}\{u, ux\} = \mathrm{span}\{u, uy\}$ is invariant under $\langle x, y \rangle$ since $x$ and $y$ each have quadratic action on $U$. Since $G$ acts irreducibly on $U$, we see that $U$ is 2-dimensional.

**LEMMA 1.2:** *Let $F$ be an algebraically closed field of prime characteristic $p$. Let $G$ be a finite group which is generated by a pair of $p$-elements $x$ and $y$. Suppose that $V$ is a 2-dimensional irreducible $FG$=module. Then there is a scalar $\lambda \in F$ and a choice of basis for $V$ such that with respect to this basis, $x$ acts as $\begin{pmatrix} 1 & 0 \\ -\lambda & 1 \end{pmatrix}$ and $y$ acts as $\begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}$.*

**PROOF:** We may, and do, suppose that $G$ acts faithfully on $V$. As in Lemma 1, there is a scalar $\lambda \in F$ and a non-zero vector $v \in V$ such that $v(x - y) = \lambda v$. We set $u = v(1 - x), w = v(1 - y)$. Then $ux = u, wy = w$, so that $\{u, w\}$ is an $F$-basis for $V$, as $G$ acts irreducibly on $V$. Now $v(1 - y) - v(1 - x) = \lambda v$, so that $w = \lambda v + u$. Hence $u(1 - x) = 0$ and $w(1 - x) = \lambda v(1 - x) = \lambda u$. Thus $ux = u$ and $wx = -\lambda u + w$.

Similarly, $w(1 - y) = 0$ and $u(1 - y) = -\lambda w$, so that $x$ and $y$ act in the manner described.

**LEMMA 1.3:** *Let $F$ be the algebraic closure of $GF(p)$ for some odd prime $p$. Let $H$ be a subgroup of $GL(2, F)$ which contains $p$-elements $x, y$ with $[x, y] \neq 1$. Then $H$ has a subgroup isomorphic to $SL(2, p)$.*

**PROOF:** We may, and do, suppose that $H = \langle x, y \rangle$ is finite, and that all $p$-elements commute in any proper subgroup of $H$. Hence $H$ is irreducible, but all of its proper subgroups of order divisible by $p$ are reducible. In particular, $O_p(H) = 1$. We note that for each $p$-element $z \in H$, $C_H(z)$ has a unique (Abelian) Sylow $p$-subgroup, and that, consequently, distinct Sylow $p$-subgroups of $H$ have trivial intersection.

Consider a non-identity $p$-element $z \in H$. There is certainly some Sylow $p$-subgroup, $U$, of $H$ which does not contain $z$. Then $U \cap U^z = 1$, so we may suppose that $y \in U$, and then we can replace $x$ by $z^{-1}yz$. By Lemma 3, we may suppose that $x = \begin{pmatrix} 1 & 0 \\ -\lambda & 1 \end{pmatrix}$ and $y = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}$ for some $\lambda \in F$. It follows readily that ( in order to perform the necessary conjugation and have determinant 1) $z$ must have the form $\begin{pmatrix} \alpha & \beta \\ -\beta & 0 \end{pmatrix}$ for some $\alpha \in F$ and $\beta = \pm 1$. Since $z$ is a $p$-element, it has trace 2, so that $\alpha = 2$. However, $H$ is normalized ( in $GL(n,F)$) by the matrix $t = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ (as $t$ inverts both of the given generators), so that $H$ contains *both* $\begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}$ *and* $\begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}$. Now

$$\begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix} \neq \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix},$$

so that the hypotheses on $H$ tell us that

$$H = \langle \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix} \rangle,$$

and that $H$ must contain at least $p+1$ Sylow $p$-subgroups. However, $H$ is now a subgroup of (the natural copy of) $SL(2,p)$. Since $SL(2,p)$ has only $p+1$ Sylow $p$-subgroups, we see that $H$ contains all Sylow $p$-subgroups of $SL(2,p)$. Hence $H = SL(2,p)$ since $SL(2,p)$ is well-known (and easily checked) to be generated by its Sylow $p$-subgroups.

**LEMMA 1.4:** *Let $p$ be an odd prime and let $H$ be a finite group of the form $UX$, where $U \lhd H$ is a $p$-group with $C_H(U) \leq U = O_p(H) \neq H$ and where $X = \langle x, y \rangle$, for $p$-elements $x$ and $y$ with $[U,x,x] = [U,y,y] = 1$. Then $H$ involves $Qd(p)$.*

**PROOF:** We may replace $H$ by $H/\Phi(U)$ if necessary, and assume that $U$ is elementary Abelian. Similarly, we may assume that $Z(H) = 1$ ( otherwise passing to $H/Z(H)$). Let $V$ be a minimal normal subgroup of $H$ contained in $U$. Then $X$ acts irreducibly (but non-trivially) on $V$. Extending scalars to a splitting field for $X$, Lemma 1 tells us that all all composition factors of $X$ on $V$ are 2-dimensional, and all such composition factors are Galois conjugate. Furthermore, since $X$ is generated by a pair of $p$-elements, the action of $X$ on each composition factor is unimodular. Thus no $p$-regular element of $X$ has a non-trivial fixed-point on $V$. Then $C_H(V) = UC_X(V)$ is a (normal) $p$-subgroup of $U$, so that $C_H(V) = U$. Now,using Lemma 1.3, $H/U$ has even order and $X$ contains an involution $t$ whose image is central in $H/U$ ( as it acts by inversion on $V$). Thus $H = UC_H(t)$ by a Frattini argument. Now $C_U(t) \lhd H$. If $C_U(t)$ is non-trivial, let $W$ be a minimal normal subgroup of $H$ contained in $C_U(t)$. Since $t$ has a fixed-point on $W$, the argument above shows that $W$ must be trivial, contrary to assumption. Hence $C_U(t) = 1$, and the product $UC_H(t)$ is semi-direct. Thus we may suppose that $H = VC_H(t)$, as $C_H(t)$ acts faithfully on $V$ ( and $C_H(t) \cong H/U$ is generated by a pair of $p$-elements which act quadratically on $V$). Now $C_H(t)$ has a subgroup $Y$ which is isomorphic to $SL(2,p)$ by Lemma 1.3, and $V$ is isomorphic to a direct sum of copies of the natural module for $Y$, so the result follows easily.

**LEMMA 1.5:** *Let $p$ be an odd prime, and let $G$ be a $p$-constrained group with $O_{p'}(G) = 1$. Suppose that there is a $p$-element $x \in G \backslash O_p(G)$ with $[O_p(G), x, x] = 1$. Then $G$ involves $Qd(p)$.*

**PROOF:** By the Baer-Suzuki theorem, there is a conjugate $y$ of $x$ such that $\langle x, y\rangle$ is not a $p$-group. Then we may apply Lemma 1.4 with $H = O_p(G)\langle x, y\rangle$, as we also have $[O_p(G), y, y] = 1$.

## Section 2: Large character heights require $Qd(p)$

We first require a crude approximation, though more precise formulae are well-known.

**LEMMA 2.1:** *Let $D$ be a non-trivial $p$-group and let $A$ be a maximal Abelian normal subgroup of $D$. Then $|D| \leq |A|^{log_p(|A|)}$.*

**PROOF:** Let $r$ be the minimal number of generators for $A$ and note that $r \leq log_p(|A|)$. Since $C_D(A) = A$, we know that $D/A$ is isomorphic to a $p$-subgroup of $\mathrm{Aut}(A)$. Now $|\mathrm{Aut}(A)|$ has the form $n_1 n_2 \dots n_r$, where $n_i$ is the number of choices for the image of the $i$-th generator (of a fixed chosen set of $r$ generators), and each $n_i < |A|$. Hence the power of $p$ dividing $|\mathrm{Aut}(A)|$ is at most $(\frac{|A|}{p})^r$. Thus we have $|D| \leq (\frac{|A|}{p})^{log_p(|A|)}|A|$, and the result follows.

We next remind the reader that (though we modify notation of [8] here somewhat), OWC predicts that if $B$ is $p$-block of positive defect of the finite group $G$, then for each non-negative integer $e$, we should have

$$k_e(B) = \sum_{\sigma \in \mathcal{N}(B)/G} (-1)^{|\sigma|+1} \sum_{\mu \in \mathrm{Irr_e}(V_\sigma)/G_\sigma} f_0(\frac{I_{G_\sigma}(\mu)}{V_\sigma}, b^\sigma).$$

Here $k_e(B)$ denotes the number of (complex) irreducible characters in $B$ which have defect $e$, and $\mathcal{N}(B)$ denotes the set of normal chains of $B$-subpairs of the form $\sigma = (V_1, b_1) < \ldots < (V_n, b_n)$ where each $(V_i, b_i) \lhd (V_n, b_n)$ ( the empty chain is excluded in this formulation). We do not need to insist that the subgroup $V_1$ is non-trivial, since if $V_1$ is trivial, then (as $B$ has positive defect), chains beginning with $(V_1, b_1)$ will make zero contribution anyway. In fact, if we allow chains in which $V_1$ is trivial, the statement of OWC makes sense (and is vacuously satisfied) for $p$-blocks of defect zero.

The length, $|\sigma|$ of the chain $\sigma$ is defined to be the number of subgroups which appear in $\sigma$. The subgroup $V_\sigma$ is the subgroup which appears in the first subpair of the chain. We let $\mathrm{Irr}_{\mathrm{e}}(V_\sigma)$ denote the set of (complex) irreducible characters of defect $e$ of $V_\sigma$. The stabilizer of $\sigma$ ( in the obvious conjugation action of $G$ on such chains) is denoted by $G_\sigma$, and $I_{G_\sigma}(\mu)$ denotes the inertial subgroup of the irreducible character $\mu$ within $G_\sigma$. Finally, for $\sigma$ as above, $b^\sigma$ denotes the block component of the last $B$-subpair in the chain, and

$$f_0(\frac{I_{G_\sigma}(\mu)}{V_\sigma}, b^\sigma)$$

denotes the number of $p$-blocks of defect zero of $\frac{I_{G_\sigma}(\mu)}{V_\sigma}$ which are not annihilated by $1_{b^\sigma}$ when viewed as $I_{G_\sigma}(\mu)$-modules.

A number of equivalent restatements of, or simplifications of, the conjecture are possible. It is *not* usually possible to only consider chains whose subgroup components are always elementary Abelian, but it *is* possible to restrict to normal chains $\sigma = (V_1, b_1) < \ldots < (V_n, b_n)$ such that $V_n/V_1$ is elementary Abelian ( the former is usually not possible because the usual contractions disturb the first subpair of the chain, while the necessary contractions for the second reduction do not). It is also only necessary ( for a general normal chain $\sigma = (V_1, b_1) < \ldots < (V_n, b_n)$) to consider irreducible characters $\mu$ of $V_1$ which induce irreducibly to $V_n$. This is because we are only concerned with the case when $\frac{I_{G_\sigma}(\mu)}{V_\sigma}$ has a $p$-block of defect 0, while

$$\frac{I_{V_n}(\mu)}{V_\sigma} \lhd \frac{I_{G_\sigma}(\mu)}{V_\sigma},$$

so there will be no contribution unless $I_{V_n}(\mu) = V_\sigma$, i.e. unless $\mu$ induces irreducibly to $V_n$. We also note that if $\mu$ does induce irreducibly to $V_n$, and we set $\gamma = Ind_{V_1}^{V_n}(\mu)$, then $I_{G_\sigma}(\gamma) = V_n I_{G_\sigma}(\mu)$ and $\frac{I_{G_\sigma}(\mu)}{V_\sigma} \cong \frac{I_{G_\sigma}(\gamma)}{V_n}$, since $V_n$ transitively permutes the irreducible constituents of $Res_{V_1}^{V_n}(\gamma)$.

For the first two sections, we do not need to be unduly concerned about the precise statement of the conjecture. The most relevant facts that we will be using here are:

a) ( From the results of [14]): we only need to consider chains $\sigma$ whose initial term $(V_1, b_1)$ is an Alperin-Goldschmidt $B$-subpair.
b) If $B$ satisfies OWC and contains an irreducible character of defect $e$, then $e$ must occur as the defect of an irreducible character of some $V_\sigma$ which is the $p$-group component of some Alperin-Goldschmidt $B$-subpair.

As a matter of interest, we remark that if the block $B$ has defect group $D$ of order $p^d$ and satisfies OWC, then only a singleton chain $(D, b_D)$ whose unique term is a maximal $B$-subpair can contribute to the alternating of OWC, and then only via irreducible characters $\mu$ which are linear. For this defect, the formula reduces to the equality of the Alperin-McKay conjecture for $B$.

Condition a) above, is one reason, among others, we find that some theoretical consequences of either conjecture are easier to see directly using OWC, and the main results of this paper are examples of this. Also, since the formulation of OWC has its origins in the theory of relatively projective modules ( see Külshammer-Robinson [13]), it may offer at least a hint of a structural explanation for the expected numerical coincidences.

**THEOREM 1:** *Let $G$ be a finite group and let $B$ be a $p$-block of $G$ with defect group $D > 1$ which satisfies OWC. Let $A$ be an Abelian normal subgroup of $D$ of maximal order.*

*Suppose that* $\mathrm{Irr}(B)$ *contains a character $\chi$ whose defect $d(\chi)$ is either less than $log_p(|A|)$ or less than $\sqrt{log_p(|D|)}$. Then there is an Alperin-Goldschmidt $B$-subpair $(U, b_U)$ such that the Külshammer-Puig extension*

$$1 \to U \to L \to N_G(U, b_U)/UC_G(U) \to 1$$

*involves $Qd(p)$.*

**PROOF:** Since $B$ satisifies OWC, it follows that there is an Alperin-Goldschmidt $B$-subpair $(U, b_U)$ such that $U$ has an irreducible character $\mu$ with $d(\mu) = d(\chi)$ and we may assume that $N_D(U)$ is a defect group for the unique block of $N_G(U, b_U)$ covering $b_U$. If $A \leq U$, then we have $\mu(1) \leq [U : A]$, which yields $p^{d(\mu)} \geq |A|$. Also, we have $p^{d(\mu)^2} \geq |A|^{\log_p(|A|)} \geq |D|$. This contradicts the assumptions of the Theorem, so we conclude that $A$ is not contained in $U$. Now $UA > U$, so that $N_{UA}(U) = UN_A(U) > U$, and $N_A(U)$ is not contained in $U$. Hence we may choose $x \in N_A(U)\backslash U$. Then $[U, x, x] \in [U, A, A] = 1$ as $A \lhd D$ is Abelian. Now there is an element of $L$ corresponding to $x$ which induces the same automorphism on $U$, lies outside $U$, and has $p$-power order. Since $L$ is $p$-constrained with $O_{p'}(L) = 1$, and with $U = O_p(L)$, we deduce that $L$ involves $Qd(p)$.

**Section 2: Normal Alperin-Goldschmidt subpairs yield small heights**

In this section, we prove:

**THEOREM 2:** *Let $B$ be a $p$-block of positive defect with the property that each Alperin-Goldschmidt $B$-subpair $(U, b_U)$ is normalized by a maximal $B$-subpair ( so, up to conjugacy, we may assume that $(U, b_U) \lhd (D, b_D)$ where $D$ is a defect group of $B$ ( of order $p^d$, say)). Suppose further that $B$ satisfies OWC. Then each irreducible character $\chi \in \mathrm{Irr}(B)$ has defect $d(\chi) > \sqrt{\frac{d}{2}}$.*

**PROOF:** Suppose that there is an irreducible character $\chi \in \mathrm{Irr}(B)$ with defect $d(\chi) \leq \sqrt{\frac{d}{2}}$. Then, since $B$ satisfies the Ordinary Weight Conjecture, there is an Alperin-Goldschmidt $B$-subpair $(U, b_U)$ such that $U$ has an irreducible character $\mu$ with $d(\mu) = d(\chi)$. Let $|U| = p^r$. Now we certainly have $\mu(1)^2 \leq [U : Z(U)]$, so that

$$p^{d(\mu)} = \frac{|U|}{\mu(1)} \geq |U|^{\frac{1}{2}}|Z(U)|^{\frac{1}{2}} \geq p^{\frac{r+1}{2}}.$$

Now $N_G(U, b_U)/UC_G(U)$ acts faithfully on $U/\Phi(U)$, so we certainly have $[D : U] \leq p^{\frac{r(r-1)}{2}}$, and hence $|D| = p^d \leq p^{\frac{r(r+1)}{2}}$. But now we have

$$p^{d(\mu)^2} \geq p^{(\frac{r+1}{2})^2} > p^{\frac{r^2+r}{4}} \geq p^{\frac{d}{2}},$$

a contradiction.

## Section 3: On "small" Alperin-Goldschmidt subpairs

In this section, we determine the contribution to the alternating sum of OWC from chains beginning with Alperin-Goldschmidt $B$-subpairs $(U, b_U)$ for which $U$ is metacyclic or $U$ is extra-special of order $p^3$ and exponent $p$. In the metacyclic case, the contribution will usually turn out to be zero unless $(U, b_U)$ is maximal, though we explicitly determine all contributions in the exceptional cases.

**LEMMA 3.1:** *Let $p$ be an odd prime. Then:*
*i) $GL(n, \mathbb{Q}_p)$ contains no element of order $p$ if $n < p - 1$.*
*ii) Let $m > 1, n < p-1$, and suppose that $h \in GL(n, \mathbb{Z}/p^m\mathbb{Z})$ has order $p$. Then (entrywise) $h \equiv I$ ( mod $p$).*

**PROOF:** i) The usual proof via Eisenstein's criterion shows that $\Phi_p(x) = \frac{x^p-1}{x-1}$ is irreducible in $\mathbb{Q}_p[x]$. Hence if $n < p$ and $y \in GL(n, \mathbb{Q}_p)$ has order $p$, then the characteristic polynomial of $y$ must be $(1 - x)^n$, which is impossible ( over a field of characteristic 0) for an element of finite order greater than 1.
ii) Set $G = GL(n, \mathbb{Z}/p^m\mathbb{Z})$. Then $G/O_p(G) \cong GL(n, p)$ and $O_p(G) = \{g \in G : g \equiv I \pmod{p}\}$. We are first required to prove that $O_p(G)$ contains every element of order $p$ in $G$. Suppose that $h \in G \backslash O_p(G)$ has order $p$. Then there is an element $u \in GL(n, \mathbb{Z}_p)$ with $u^p \equiv I \pmod{p^m}$. and $u \not\equiv I$ ( mod $p$). We define a sequence of elements $\{u_s\}$ via $u_0 = u$, and for $s > 0$, $u_{s+1} = u_s - \left(\frac{u_s^{1-p}(u_s^p-I)}{p}\right)$. An easy induction argument shows that $u_s^p \equiv I \pmod{p^{1+2^s(m-1)}}$ and that $u_s \equiv u$ ( mod $p$). It follows that the sequence$\{u_s\}$ converges to a limit $v \in GL(n, \mathbb{Z}_p)$ with $v \equiv u \pmod{p}$ and $v^p = I \neq v$. But there is no such element of order $p$ in $GL(n, \mathbb{Z}_p)$ by part i), a contradiction.

**COROLLARY 3.2:** *Let $p$ be a prime greater than 3, let $m$ be an integer greater than 1. Then $G = GL(2, \mathbb{Z}/p^m\mathbb{Z})$ has no subgroup isomorphic to $SL(2, p)$, and if $H$ is any subgroup of order divisible by $p$ of $G$, then we have $O_p(H) \neq 1$.*

**PROOF:** For $H$ as in the statement, any element of order $p$ in $H$ must lie in $O_p(G) \cap H \leq O_p(H)$ by the previous result, so that $O_p(H) \neq 1$ when $p$ divides $|H|$, by Cauchy's theorem. Since $O_p(SL(2, p)) = 1$ for all primes $p$, we are done.

**LEMMA 3.3:** *Let $P$ be a metacyclic 2-group which admits a non-trivial automorphism of odd order. Then $P$ is either quaternion of order 8, or else is Abelian of type $(2^n, 2^n)$ for some $n$.*

**PROOF:** By standard results on coprime automorphisms, $P$ has a characteristic subgroup $Q$ with all characteristic subgroups central ( so of class at most 2) and of exponent dividing 4, such that all non-trivial automorphims of $P$ restrict non-trivially to $Q$. Furthermore, $Q$ may be assumed to be non-Abelian if it has exponent 4. Now $Q$ is also metacyclic, so that $|Q| \leq 16$. If $Q$ is non-Abelian of order 16, then $Q$ has the form $XY$, where $X, Y$ are cyclic subgroups of order 4 and $X \lhd Q$. Now $Q' \leq \Phi(Q) = \Phi(X)\Phi(Y)$ in that case, while also $Q' \leq X$. Hence $Q' = \Phi(X)$ and $Q/Q' \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$. However, $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$ admits no non-trivial automorphism of odd order, a contradiction, since al automorphisms of $Q$ of odd order act non-trivially on $Q/Q'$. Since the dihedral group of order 8 admits no non-trivial automorphism of odd order, we conclude that if $Q$ is non-Abelian, then it is quaternion of order 8. In that case, we next claim that $C_P(Q) = Z(Q)$. For otherwise, we may choose a subgoup $S = Q\langle x \rangle$ of order 16 of $QC_P(Q)$ with $x \in C_P(Q)$. Now $x^2 \in Z(Q)$, so $x$ has order 2 or 4. In either case, $S/Z(Q)$ is elementary Abelian of order 8, contrary to the fact that $S$ is metacyclic.

However, if we have $C_P(Q) = Z(Q)$ and $Q < P$, then $[P : Q] = 2$, since $\mathrm{Aut}(Q) \cong S_4$. Also, any non-trivial automorphism, $\alpha$, say, of $P$ of odd order necessarily has order 3. Now $\alpha$ normalizes one at least one maximal subgroup of $P$, namely $Q$, and $P$ has exactly three maximal subgroups, since it is 2-generated. Hence $\alpha$ normalizes all maximal subgroups of $P$. Since $[M : \Phi(P)] = 2$ for each such maximal subgroup $M$, we deduce that $\alpha$ acts trivially on $P/\Phi(P)$, a contradiction. Hence $P$ is quaternion of order 8 if $Q$ is non-Abelian.

Suppose then that $Q$ is a Klein 4-group. Then $M = C_P(Q)$ has $[P : M] \leq 2$. Since allk non-trivial automorphisms of odd order act trivially on $P/M$, all such automorphisms must at faithfully on $M$. If $M$ has a cyclic characteristic subgroup $C \neq 1$, then $\Omega_1(C) \leq Z(M)$ and $\Omega_1(C) \leq Q$ since $Q\Omega_1(C)$ is metacycclic. However, now any automorphism of odd order of $Q$ acts trivially on $\Omega_1(C)$ and on $Q/\Omega_1(C)$, a contradiction. Hence $M$ has no non-trivial cyclic characteristic subgroup, so that $M$ is Abelian, since $M'$ is certainly cyclic. In fact $M$ must be homocyclic of rank 2, otherwise it clearly has a non-trivial cyclic characteristic subgroup. If $M = P$, we are done, so suppose that $[P : M] = 2$. Since $\mathrm{Aut}(Q) \cong SL(2,2)$, we see that every non-trivial automorphism of odd order of $P$ has order 3, stabilizes $M$ char $P$, and hence stabilizes each maximal subgroup of $P$. Thus every such automorphism acts trivially on $P/\Phi(P)$, a contradiction. Hence $M = P$ is homocyclic of rank 2 when $Q$ is Abelian, and the proof is complete.

The following Lemma is presumably well-known.

**LEMMA 3.4** *Let $P$ be a $p$-group of maximal class for some odd prime $p$. Let $H = Out(P)$ and let $X = H/O_p(H)$. Then $X$ is isomorphic to a group of the form $A \times C$, where $A$ is an Abelian $p'$-group and $C$ is isomorphic to a subgroup of $GL(2,p)$. Furthermore, if $X$ has a subgroup isomorphic to $SL(2,p)$, then $P$ is extra-special of order $p^3$ and exponent $p$.*

**PROOF:** Since $P$ has maximal class, the lower central series of $P$ shows that $P$ has a chief series in which the first factor is elementary Abelian of order at most $p^2$, and all remaining factors are cyclic of order $p$. Since $X$ acts faithfully on the direct sum of these chief factors, the first claim follows.

Suppose that $H$ has a subgroup isomorphic to $SL(2,p)$. Let $Q$ be a characteristic subgroup of $P$ which has all characteristic Abelian subgroups central, and exponent $p$, on which $X$ acts faithfully. Then $[P:Q] \leq p$, for otherwise $Q$ is the unique normal subgroup of $P$ of its order ( since $P$ has maximal class), and is hence a term of the lower central series of $P$. Then $Q > [Q,P] > [Q,P,P] > \ldots$ gives a chain of $X$-invariant subgroups in which successive quotients have order $p$. Since $O_p(X) = 1$, $X$ acts faithfully on the direct sum of these cyclic subgroups, and $X$ is Abelian, a contradiction.

However, $[P:Q] = p$ yields that $P > Q > P' > [P',P] > \ldots$ is a chief series for $P$ in which all chief factors have order $p$, again yielding that $X$ is Abelian, a contradiction. Hence $P = Q$ has class 2, order $p^3$ and exponent $p$.

**COROLLARY 3.5:** *Suppose that $(U, b_U)$ is an Alperin-Goldschmidt $B$-subpair for some $p$-block $B$ ($p$ an odd prime) and that $U$ has maximal class. Then $U$ is extra-special of exponent $p$ and order $p^3$.*

**REMARK:** We will see later that $D$, a defect group of $B$ must itself have maximal class under the hypotheses of Corollary 3.5.

**THEOREM 3:** *Suppose that $(U, b_U)$ is an Alperin-Goldschmidt $B$-subpair, and that chains beginning with (conjugates of) $(U, b_U)$ make a non-zero contribution to the alternating sum formula of of OWC for $k_e(B)$ for some non-negative integer $e$. Suppose further that $U$ is metacyclic. Then one of the following occurs:*
i) *$(U, b_U)$ is a maximal $B$-subpair, and ( for each non-negative integer $e$) the contribution is $k_e(\beta_U)$, where $\beta_U$ is the unique block of $N_G(U, b_U)$ covering $b_U$.*
*ii) $U$ is quaternion of order $8, e = 2$ and the contribution is $1$. Furthermore, $D$, the defect group of $B$, is either generalized quaternion or semi-dihedral.*
*iii) $p \leq 3$ and $U$ is Abelian of type $(p^n, p^n)$ for some $n > 1$. Furthermore, in this case, $N_G(U, b_U)$ contains a defect group,$D$ say, for $B$, and $U = J(D)$.*

*Also, in this case, the contribution is $0$ unless $e = 2n$, in which case the contribution is $\frac{p^2-p^{2n}}{p^2-1}$ when $N_G(U, b_U)/UC_G(U) \cong SL(2,p)$ and is $2 - \frac{(3^n-1)(3^n+13)}{16}$ if $N_G(U, b_U)/UC_G(U) \cong GL(2,3)$*

**PROOF:** It is clear that case i) occurs if $(U, b_U)$ is a maximal $B$-subpair, so suppose that this is not the case. We take a maximal $B$-subpair $(D, b_D) > (U, b_U)$. Then $N_D(U)$ stabilizes $b_U$ and $[N_G(U, b_U) : UC_G(U)]$ is divisible by $p$ ( but is not a $p$-group since $O_p(N_G(U, b_U)/UC_G(U)) = 1$).

It follows that $U$ has a characteristic subgroup $Q$ of class at most 2 with all its characteristic Abelian subgroups central (of exponent $p$ if $p$ is odd and of exponent dividing 4 if $p = 2$) such that $N_G(U, b_U)/UC_G(U)$ still acts faithfully on $Q/\Phi(Q)$.

If $p = 2$, then Lemma 3.3 tells us that $U$ is either quaternion of order 8, or else Abelian of type $(2^n, 2^n)$ for some $n$.

If $Q$ is quaternion of order 8, we claim that all Abelian normal subgroups of $D$ are cyclic. For, otherwise, we may choose an elementary Abelian normal subgroup, $E$, say, of $D$, of order 4. Now certainly $E \not\leq Q$, and if $E \leq U$, then $[E, Q] \leq E \cap Q \leq \Phi(Q)$, so that $(EQ)'$ has order 2 and $(EQ)/(EQ)'$ is elementary of order 8, contrary to the fact that $EQ \leq U$ is metacyclic. Hence $E \not\leq U$. But $EU$ is a 2-group, so that $UN_E(U) = N_{EU}(U) > U$, and $N_E(U) \not\leq U$. Hence we may choose $y \in N_E(U) \backslash U$. Now $[Q, y] \leq (Q \cap E) \leq \Omega_1(Q) = \Phi(Q)$, in contradiction to the fact that $N_G(U, b_U)/UC_G(U)$ acts faithfully on $Q/\Phi(Q)$. Hence all Abelian normal subgroups of $D$ are cyclic in this case. Since $D$ has a quaternion subgroup of order 8 this implies that $D$ is generalized quaternion or semi-dihedral.

If $p$ is odd, then $Q$ is elementary of order $p^2$, since $U$ is metacyclic.Suppose then that $p$ is odd, or that $p = 2$ and $Q$ is a Klein 4-group. Then $N_G(U, b_U)/UC_G(U)$ must act faithfully and irreducibly on $Q$ ( since it is isomorphic to a subgroup of $GL(2, p)$ without non-trivial normal $p$-subgroup). Hence $U$ has no non-trivial cyclic characteristic subgroup ( for if $C$ were one such, we must have $Z(U) \geq \Omega_1(C) \leq Q$ as $Q\Omega_1(C)$ is Abelian and metacyclic. This contradicts the irreducibility of the action of $N_G(U, b_U)/UC_G(U)$ on $Q$). However $U'$ is cyclic by assumption, so $U$ must be Abelian of rank 2. If $U$ has type $(p^n, p^m)$ with $n > m$, then $U$ visibly has a non-trivial characteristic cyclic subgroup, so $U$ must be homocyclic. We will deal with the case that $|U| = p^2$ presently.

CASE: U Homocyclic of type $(p^m, p^m), m = 1$ or $m > 1, p \leq 3$.
Now $T = N_G(U, b_U)/UC_G(U)$ is isomorphic to a subgroup of $GL(2, p)$ with more than one Sylow $p$-subgroup, so has a subgroup isomorphic to $SL(2, p)$, while it is also isomorphic to a subgroup of $Out(U) \cong GL(2, \mathbb{Z}/p^m\mathbb{Z})$. By Corollary 3.2, we have $p \leq 3$. We recall that ( as is easily checked) for every prime $p$ and every subgroup $Y$ of $GL(2, p)$ with $SL(2, p) \leq Y \leq GL(2, p)$, $Y$ has trivial Schur multiplier.

Let $L = L(U)$ denote the Külshammer-Puig extension of $T$ by $U$. Then, by the results of [14], the contribution to the alternating sum of OWC may ( for every $e$) be calculated within $L$ using (normal) chains of $p$-subgroups of $L$ beginning with $U$. Suppose first that $m = 1$. Let $T = L/U$. We know that ( with obvious identifications), we have $SL(2, p) \leq T \leq GL(2, p)$. The only value of $e$ for which there could possibly be any non-zero contribution to OWC from chains beginning with $(U, b_U)$ is $e = 2$, since all irreducible characters of $U$ have defect 2. The contribution characters lying over the trivial character of $U$ is $[T : SL(2, p)]$, since this is the number of $p$-blocks of defect 0 of $L/U$. All other linear characters of $U$ are conjugate under the action of $T$ ( already under the action of $SL(2, p)$, in fact). If $\lambda$ is a non-trivial linear character of $U$, then $I_L(\lambda)/U$ has a normal (non-trivial) Sylow $p$-subgroup, so has no $p$-block of defect 0. There is one more $L$ conjugacy class of (normal) chains of $p$-subgroups beginning with $U$, that is to say $U < S$ for $S$ a Sylow $p$-subgroup of $L$. Let $M = N_L(S)/U$. We need (neglecting signs for the moment) to count the number of $p$-blocks of defect 0 of $I_M(\lambda)$ for representatives $\lambda$ of the $B$-orbits on the non-trivial irreducible characters of $U$. Since $M$ is $p$-closed, this is zero unless $I_M(\lambda)$ is a $p'$-group, in which case it is $|I_M(\lambda)|$ as $I_M(\lambda) \cap SL(2,p) = 1$ and $I_M(\lambda)$ is cyclic.

Now $S/U$ fixes $(p-1)$ non-trivial linear characters of $U$ and has $p-1$ orbits of length $p$ on such characters. These last $p-1$ orbits are permuted transitively by $M$, in fact already by $M \cap SL(2,p)$, as no $p$-regular element of $SL(2,p)$ fixes any non-trivial linear character of $U$. For $\lambda$ in the $M$-orbit of size $p^2-p$, we have $M = (M \cap SL(2,p))I_M(\lambda)$, and the product is semi-direct. Hence $|I_M(\lambda)| = [M : M \cap SL(2,p)] = [T : SL(2,p]$ since $T = SL(2,p)M$ by a Frattini argument. Since the chains $U$ and $U < S$ contribute with opposite sign, we find that there is zero contribution to OWC from chains starting with $(U, b_U)$ when $e = 2$ ( and for all other $e$) when $|U| = p^2$.

Now suppose that $U$ is homocyclic of type $(p^m, p^m)$ and $m > 1$. Then we know that $p \leq 3$. Let $(D, b_D)$ be any maximal $B$-subpair containing $(U, b_U)$, and set $D_0 = N_D(U)$. Then $D_0$ is a defect group for the unique block of $N_G(U, b_U)$ covering $b_U$, since $Z(U)$ is a defect group for $b_U$ and $p^2$ does not divide $[N_G(U,b_U) : UC_G(U)]$. Now $C_{D_0}(\Omega_1(U)) \leq C_G(U)$ since $U$ is Alperin-Goldschmidt (and Abelian) and any $p$-regular element of $N_G(U, b_U)$ which acts trivially on $\Omega_1(U)$ must act trivially on $U$. Hence $C_{D_0}(\Omega_1(U)) = U$, since $b_U$ has defect group $U$. Thus $U$ is self-normalizing in $C_D(\Omega_1(U))$, so that $C_D(\Omega_1(U)) = U$. Hence $D_0 = N_D(\Omega_1(U))$, since $\Omega_1(U) char U \lhd D_0$ and $[N_D(\Omega_1(U)) : C_D(\Omega_1(U))] \leq p$.

However, $\Phi(U) \leq \Phi(D_0) \leq U$, so that $\Omega_1(U) = \Omega_1(\Phi(D_0)) char D_0 \lhd N_D(D_0)$. This forces $D_0 = D$. Now for any $x \in D \backslash U$, we have $|C_U(x)| = p|Z(D)| \leq p^{m+1}$. Since $m > 1$, we see that $U$ is the unique Abelian subgroup of $D$ of order $p^{2m}$, so that $U = J(D)$.

We need to concern ourselves with $e = 2m$ in the formula of OWC. Let $(D, b_D) \geq (U, b_U)$ be a maximal $B$-subpair, and let $D_0 = N_D(U)$ which is a defect group for the unique block of $N_G(U, b_U)$ covering $b_U$. As before, let $T = N_G(U, b_U)/UC_G(U)$ and let $L$ be the Külshammer-Puig extension of $T$ by $U$. If $p = 2$, then $T \cong SL(2,2)$, while if $p = 3$, we have $T \cong SL(2,3)$ or $T \cong GL(2,3)$.

Suppose first that $T \cong SL(2,p)$. Now all $T$-chief factors contained within $U$ are 2-dimensional, and therefore that no non-identity $p$-regular element of $SL(2,p)$ fixes a non-trivial element of $U$, and no such element fixes a non-trivial linear character of $U$. It follows that $I_L(\lambda)/U$ has a normal Sylow $p$-subgroup whenever $\lambda$ is a non-trivial character of $U$. Hence only the trivial character contributes to the formula of $k_{2n}(B)$ from the singleton chain $(U, b_U)$, and the contribution is the number of $p$-blocks of defect 0 of $L/U$, which is 1.

Since no non-trivial $p$-regular element of $T$ fixes any non-trivial linear character of $U$, we see that each orbit of $T$ on non-trivial linear characters of $U$ has length $p^2 - 1$ or $p(p^2 - 1)$.

We claim that in general, a regular $T$-orbit on linear characters of $U$ contributes $-p$ to the alternating sum of OWC ( for $e = 2n$), while a non-regular $T$-orbit on non-trivial linear characters contributes $-1$ to the alternating sum of OWC. A regular orbit contributes 1 to the contribution from the singleton chain $U$ of $L$ because for $\lambda$ in such an orbit we have $I_L(\lambda)/U = 1$. Now $D_0$ is a Sylow $p$-subgroup of $L$, we see that a regular $L$-orbit on non-trivial linear characters of $U$ breaks into $(p+1)$ regular orbits under $N_L(D_0)$, each contributing $-1$ to the signed contribution from the chain $U < D_0$.

On the other hand, in a $T$-orbit of length $p^2-1$ on linear characters of $U$, there are $p-1$ linear characters fixed by $D_0$ and $p^2-p$ not fixed by $D_0$. Those not fixed by $D_0$ fall into a single $N_L(D_0)$ orbit, which contributes $-1$ so the signed contribution from the chain $U < D_0$ in $L$. On the other hand, for $\lambda$ in the original $T$-orbit, we have $[I_L(\lambda) : U] = p$, so there is no contribution from the orbit of $\lambda$ to the contribution from the singleton chain $U$.

Hence the total contribution to the alternating sum of OWC ( for $e = 2m$) is : $1 - pr - s$, where $T$ has $r$ regular orbits on the non-trivial linear characters of $U$ and $s$ orbits of length $p^2-1$ ( the "1" coming from the trivial character of $U$). However, we have $rp(p^2-1) + s(p^2-1) = p^{2m} - 1$, so that $rp + s = \frac{p^{2m}-1}{p^2-1}$, and the alternating sum of OWC ( for $e = 2m$) reduces to $\frac{p^2-p^{2m}}{p^2-1}$. We note that this is negative when $n > 1$.

Now consider the case $p = 3, U$ of type $(3^m, 3^m)(m > 1)$ and $L(U)/U \cong GL(2,3)$. Again, there will be no contribution from chains beginning with $(U, b_U)$ to $k_e(B)$ unless $e = 2n$. Let $T = L/U$. Then there is 1 central involution of $T$, and 12 involutions $t \in T$ with $|C_U(t)| = 3^m$. Let $x$ be an element of order 3 in $T$. Then $N = N_T(\langle x \rangle) \cong Z(T) \times S_3$. Each involution of $N$ acts on $C = C_U(x)$. Any such involution which centralizes $\Omega_1(C)$ centralizes $C$, and it follows that there are 3 involutions of $N$ which centralize $C$ and 4 involutions of $T$ which act by inversion on $C$.

Let us count the orbits of $T$ on non-identity elements of $U$. The possible orbit lengths of $SL(2,3)$ on such elements are 8 and 24. We have seen above that every non-identity element of $C$ which is fixed by an element of order 3 is also fixed by an involution inverting the element of order 3. It follows that the possible orbit lengths of $T$ on $C^{\#}$ are $8, 24$ and 48. Suppose that there are $\alpha$ orbits of length $8, \beta$ orbits of length 24 and $\gamma$ orbits of length 48. Then we have $\alpha + 3\beta + 6\gamma = \frac{3^{2m}-1}{8}$.

This time, the trivial character of $U$ contributes 2 to the formula for $k_{2n}(B)$. Now $x$ fixes 2 elements in each orbit of length 8 on non-trivial linear characters of $U$, and $N$ permutes the other 6 characters in that orbit transitively. This means that this orbit gives a signed contribution of $-2$ to $k_{2n}(B)$, since for $\lambda$ in such an orbit we have $f_0(I_L(\lambda)/U) = 0$ and $f_0(I_M(\lambda)/U) = 2$, where $M$ is the full pre-image of $N$ in $L$.

For $\lambda$ in an orbit of length 48, we see that the orbit of $\lambda$ breaks into 4 regular orbits under $N$, and it follows that the contribution to $k_{2n}(B)$ from this orbit is $1-4=-3$.

For $\lambda$ in an orbit of length 24, we note that each non-central involution of $N$ fixes 2 elements of that orbit. Hence the orbit breaks up into 3 $N$-orbits, which must have length $6, 6$ and 12. It follows that this orbit contributes $2-5=-3$ to the formula for $k_{2m}(B)$.

Hence the non-trivial linear characters of $U$ contribute $-(2\alpha+3\beta+3\gamma)$ to the formula for $k_{2m}(B)$. Suppose that $|C_U(x)| = 3^a$. Then since $T$ has 4 Sylow 3-subgroups, we see that $4(3^a-1)$ non-identity elements of $U$ are fixed by some subgroup of order 3, and these are precisely the elements which fall into orbits of size 8. Hence $\alpha = \frac{3^a-1}{2}$, and this is also the number of orbits of length 8 of $T'$. Now the total number of orbits of $T$ is

$$\frac{1}{48}((3^{2m}-1)+8(3^a-1)+12(3^m-1)) = (\alpha+\beta+\gamma).$$

We wish to evaluate $2\alpha+3\beta+\gamma) = 3(\alpha+\beta+\gamma)-\alpha$. Since we know that $\alpha = \frac{3^a-1}{2}$, the quantity we seek is

$$\frac{(3^{2m}-1)+12(3^m-1)}{16} = \frac{(3^m-1)(3^m+13)}{16},$$

and the contribution to the formula of OWC for $k_{2m}(B)$ is $2-\frac{(3^m-1)(3^m+13)}{16}$ in this case.

Let us now return to the case when $U$ is quaternion of order 8 ( but not the defect group). The possible values of $e$ for which chains beginning with $(U, b_U)$ might contribute to the alternating sum of OWC are $e=2$ and $e=3$. The contribution when $e=3$ comes from (orbits of ) linear characters of $U$. This contribution may be calculated within $L/Z(U)$, and this returns us to the situation of a Klein 4-group, which we know from above gives zero contribution. Hence there is no contribution for $e=3$.

When $e=2$, the contribution is

$$f_0(I_L(\mu)/U) - f_0(I_N(\mu)/U),$$

where $\mu$ is the unique irreducible character of degree 2 of $U$ and $N = N_L(D_0)$. This is 1, since $I_L(\mu) = L$ and $L/U \cong SL(2,2)$, while $D_0 \lhd I_N(\mu)$ and $D_0 > U$.

**THEOREM 4:** *Suppose that $(U, b_U)$ is an Alperin-Goldschmidt $B$-subpair, and that chains beginning with (conjugates of) $(U, b_U)$ make a non-zero contribution to the alternating sum of OWC for some non-negative integer $e$. Suppose further that $U$ is extra-special of exponent $p$ and order $p^3$. Then one of the following occurs:*
*i) $(U, b_U)$ is a maximal $B$-subpair, and ( for each non-negative integer $e$) the contribution is $k_e(\beta_U)$, where $\beta_U$ is the unique block of $N_G(U, b_U)$ covering $b_U$.*
*ii) $X = N_G(U, b_U)/UC_G(U)$ has a subgroup $Y$ isomorphic to $SL(2,p)$. The*

*contribution to the alternating sum of OWC from chains beginning with* $(U, b_U)$ *is* $\frac{p-1}{[X:Y]}$ *if* $e = 2$ *and* 0 *otherwise. A defect group,*$D$*, for* $B$ *has maximal class, and* $D$ *has a unique normal subgroup of order* $p^2$*.*

**PROOF:** Let $(U, b_U) \leq (D, b_D)$, where $(D, b_D)$ is a maximal $B$-subpair, and suppose that $U \neq D$. Let $D_0 = N_D(U)$. Then $N_G(U, b_U)/UC_G(U)$ does not have a normal Sylow $p$-subgroup, yet is isomorphic to a subgroup of $GL(2, p)$, so contains a subgroup isomorphic to $SL(2, p)$. Hence $|D_0| = p^4$. Suppose that $D_0 \neq D$. Then $U$ is not characteristic in $D_0$.

Now $Z(D) \leq U$, so that $|Z(D)| = p$. Let $A \lhd D$ be of order $p^2$. Suppose that $A \not\leq U$. Then $A \cap U = Z(D)$. Then $N_A(U) \not\leq U$, and for $a \in N_A(U) \backslash U$, we have $[U, a] \leq U \cap A = Z(U) = \Phi(U)$, a contradiction. Hence $A \leq U$. If $D$ has more than one such subgroup, then we have $U = \langle A \lhd D : |A| = p^2 \rangle$ char $D$, a contradiction. Hence $A$ is unique. Let $C = C_D(A)$. Then $[D : C] = p$, and we have $D = \langle u \rangle C$ for any $u \in U \backslash A$. For any such $u$, we have $C_C(u) \leq C_D(U) = Z(U) = Z(D)$, so that $|C_C(u)| = p$ and $|C_D(u)| = p^2$. Hence $[D : D'] = p^2$, and $D$ has maximal class by a Theorem of Blackburn.

If $D_0 = D, |D| = p^4$, then $U \lhd D$. Now $D$ can't have class 2, otherwise $[D, u] \leq Z(D) = Z(U) = \Phi(U)$. Hence $D$ has class 3, which is maximal class in this case, and, in particular, $[D : D'] = p^2$. In this case, every normal subgroup of $D$ of order $p^2$ contains $D'$, so is equal to $D'$.

Thus, in all cases where $(U, b_U)$ is not a maximal $B$-subpair, we see that $D$ has maximal class.

Let us count the contribution to the formula of OWC from chains beginning with $(U, b_U)$. As usual, this may be calculated within the Külshammer-Puig extension $L = L(U)$ of $N_G(U, b_U)/UC_G(U)$ by $U$, since $L/U$ is isomorphic to a subgroup of $GL(2, p)$ which contains $SL(2, p)$ ( so has trivial Schur multiplier). As in the $Q_8$ case of the previous proof, the values of $e$ for which it is possible to get a non-zero contribution are $e = 2$ or $e = 3$. The contribution for $e = 3$ may ( as before) be calculated within $L/Z(U)$, and reduces to the case where the normal $p$-subgroup is of type $(p, p)$, so the contribution for $e = 3$ is 0.

However, when $e = 2$, the calculation is a little more subtle. We need to count $U$-projective irreducible characters which lie over faithful irreducible characters of $U$ ( of degree $p$) for both $L$ and $N = N_L(D_0$ ( signed appropriately). Each irreducible character $\mu$ of degree $p$ of $U$ is $D_0$-stable, so that $f_0(I_N(\mu)/U) = 0$. But $f_0(I_L(\mu)/U) = 1$ for each such $\mu$, so there is a contribution of 1 from each orbit of $L/U$ on irreducible characters of degree $p$ of $U$. Hence the contribution to the alternating sum of OWC for $e = 2$ from chains beginning with $(U, b_U)$ is $\frac{p-1}{[GL(2,p):L/U]}$, since ( with the usual obvious identifications) $I_L(\mu)/U \cong SL(2, p)$ for each such $\mu$.

For the sake of completeness, we record:

**LEMMA 3.6:** *Suppose that there is an Alperin-Goldschmidt $B$-subpair $(V, b_V)$, where $V$ is Abelian of type $(p, p)$. Let $(D, b_D)$ be a maximal $B$-subpair strictly containing $(V, b_V)$. Then:*
*i) If $p = 2, B$ has a semi-dihedral or dihedral defect group.*
*ii) If $p$ is odd, then $D$ has maximal class.*

**PROOF:** i) Suppose that $p = 2$. Since $C_D(V) = V$, we may suppose that $|D| > 8$, since $D$ in non-Abelian, but can't be quaternion of order 8. It suffices to prove that each Abelian normal subgroup of $D$ is cyclic. Suppoae that $A \lhd D$ is a Klein 4-group. Then $A \neq V$, otherwise we would have $|D| = 8$ since $C_D(V) = V$. As before, there is an element $a \in N_A(V) \backslash V$. Now $Z(D) \leq V$, and $A = \langle a \rangle Z(D)$, so that $A \leq D_0 = N_D(V)$. Now $D_0$ is dihedral of order 8, so has just two Klein 4-subgroups, which must be $A$ and $V$. However, $D_1 = N_D(D_0) > D_0$ since $|D| \geq 16$. Now $D_1$ does not normalize $V$, but permutes the two Klein 4-subgroups of $D_0$, so $V$ and $A$ are $D_1$-conjugate, contrary to the fact that $A \lhd D$. Hence there is no such $A$, and all Abelian normal subgroups of $D$ are cyclic. Since $D$ has a Klein 4-subgroup, it can't be generalized quaterion, so that $D$ must be dihedral or semi-dihedral, as claimed.

ii) Since $C_D(V) = V$, and $N_G(V, b_V)/VC_G(V)$ has a subgroup isomorphic to $SL(2, p)$, we see that $D_0 = N_D(V)$ is an extra-special group of order $p^3$ and exponent $p$. We may suppose that $D_0 < D$. Now $D$ has an elementary Abelian normal subgroup $A$ of order $p^2$, and by part i), we have $A \leq D_0$. Now $D_1 = N_D(D_0) > D_0$, and $D_1$ does not normalize $V$. Hence $D_1$ has two orbits on the set of $p + 1$ maximal subgroups of $D_0$, one of size 1 ( $\{A\}$), and one of size $p$ containing $V$. But $A$ was an arbitrary elementary Abelian normal subgroup of $D$ of order $p^2$, so $A$ must be unique.

Let $C = C_D(A)$, so that $[D : C] = p$, as $Z(D) \leq C_D(V) = V$. We have $D = \langle v \rangle C$ and $D_0 = \langle v \rangle A$ for any $v \in V \backslash Z(D)$, and for any $x \in C_C(v)$, we have $x \in C_D(D_0) = Z(D)$. Hence $C_C(v)$ has order $p$ and $C_D(v)$ has order $p^2$, so that $D$ has maximal class.

The case $p$ odd of the following Theorem, R. Stancu [15] has proved the analogous ( more general) result for arbitrary fusion systems.

**THEOREM 5:** *Suppose that $(D, b_D)$ is a maximal $B$-subpair, and that $D$ is metacyclic. Then one of the following occurs:*

i) *$N_G(D, b_D)$ controls the fusion of $B$-subpairs, and, setting $\beta$ to be the unique block of $N_G(D, b_D)$ covering $b_D, B$ satisfies OWC if and only if there is a defect preserving bijection between* $\mathrm{Irr}(B)$ *and* $\mathrm{Irr}(\beta)$. *Furthermore, if $p = 2$ then one of three possibilities occur:*
*$D$ is Abelian, $D$ is quaternion of order 8, or $B$ is nilpotent.*

ii) $p = 2$, *$D$ is dihedral, semi-dihedral, or generalized quaternion ( and $B$ satisfies OWC).*

**PROOF:** We may suppose that $D$ is non-Abelian, or else case i) certainly occurs. Suppose that $N_G(D, b_D)$ does not control fusion of $B$-subpairs, there must be an Alperin-Goldschmidt $B$-subpair $(U, b_U)$ with $U < D$ for which $U$ is metacyclic.

Suppose that $p = 2$. Then we have seen that there are three possibilities for $U$:
i) $U$ is quaternion of order 8, which forces $D$ to be semi-dihedral or generalized quaternion.
ii) $U$ is a Klein 4-group , which forces $D$ to be dihedral or semi-dihedral.
iii) $U$ is Abelian of type $(2^n, 2^n)$ for some $n > 1$, and $N_G(U, b_U)/UC_G(U) \cong SL(2, 2)$. Let $D_0 = N_D(U)$. We claim that $D_0$ is not metacyclic in this case. Choose $x \in D_0 \backslash U$. Then $D_0' = [U, x]$ is Abelian of type $(2^a, 2^n)$, where $|C_U(x)| = 2^{n-a}$. Hence, if $D_0$ is metacyclic, then $D_0'$ is cyclic and $a = n$. Let $C$ be a maximal cyclic subgroup of $D_0$ containing $D_0'$. Then $C \cap U$ has order at most $2^n$, so that $C \cap U = D_0'$ and $|C| \leq 2^{n+1}$. However $D_0/D_0'$ has type $(2, 2^n)$, and it follows that $C = D_0'\langle x\rangle$. This in turn forces $x$ to centralize $D_0'$ and $x$ to have order $2^{n+1}$. Hence $D_0' \leq Z(D_0)$ and $D_0$ has class 2. On the other hand, since $D_0$ has cyclic centre, it has a faithful complex irreducible character ( for there is an irreducible character $\mu$ whose kernel does not contain $\Omega_1(Z(D))$, and $\mu$ must therefore be faithful). Since $D_0$ has an Abelian normal subgroup of index 2, we have $\mu(1) \leq 2$, so $\mu(1) = 2$. Since $D_0$ has class 2, $\mu$ must vanish identically on $D_0 \backslash Z(D_0)$, and hence $|D_0| = |Z(D_0)|\mu(1)^2$, so that $[D_0 : Z(D_0)] = 4$. But $Z(D_0) = C_U(x)$ has order $2^n$ and index $2^{n+1}$, which forces $n = 1$, a contradiction.

Suppose next that $p = 2$, $D$ is metacyclic, but neither homocyclic Abelian nor quaternion of order 8, and that $N_G(D, b_D)$ controls fusion of $B$ subpairs. Then by Lemma 3.3, $N_G(D, b_D) = DC_G(D)$, which yields that $N_G(W, b_W)/WC_G(W)$ is a 2-group for each $B$-subpair $(W, b_W)$, and hence $B$ is nilpotent.

Now suppose that $p$ is odd, but that $N_G(D, b_D)$ does not control fusion of $b$-subpairs. Then there is an Alperin Goldschmidt $B$-subpair $(U, b_U) < (D, b_D)$. Since $U$ is metacyclic, we have only the following possibilities ( using Theorem 4):
a) $U$ is Abelian of type $(p, p)$ and $N_G(U, b_U)/UC_G(U)$ is isomorphic to a subgroup of $GL(2, p)$ which contains $SL(2, p)$. This case can be exluded because that Külshammer-Puig extension of $N_G(U, b_U)/UC_G(U)$ by $U$ has a Sylow $p$-subgroup isomorphic to $D_0$ and also has a subgroup which is a split extension $U.SL(2, p)$, so that $D_0$ is extra-special order $p^3$ and exponent $p$, and is not metacyclic.

b) $p = 3$ and $U$ is Abelian of type $(3^n, 3^n)$ for some $n > 1$, and $N_G(U, b_U)/UC_G(U)$ is isomorphic to $SL(2, 3)$ or to $GL(2, 3)$. As before, let $L = L(U)$ be the Külshammer-Puig extension of $N_G(U, b_U)/UC_G(U)$ by $U$. Then $L$ has a Sylow $p$-subgroup isomorphic to $D_0 = N_D(U)$. On the other hand, $L/U$ acts faithfully on $U/\Phi(U)$and $L/\Phi(U)$ contains a split extension of $U/\Phi(U)$ by $SL(2, 3)$ with natural action. Hence $D_0/\Phi(U)$ is extra-special of exponent 3 and order $3^3$, which is not metacyclic, a contradiction.

We conclude with:

**LEMMA 3.7** *Let $B$ be a $p$-block with defect group $D$ of order greater than $p^3$. Then if $k_2(B) \neq 0, D$ must have maximal class (but $D$ is not metacyclic if $p$ is odd, and is not dihedral if $p = 2$). Furthermore, the only Alperin-Goldschmidt $B$-subpairs $(U, b_U)$ which make a non-zero contribution to the formula of OWC for $k_2(B)$ have $U$ extra-special of order $p^3$ ( of exponent $p$ if $p$ is odd, and quaternion if $p = 2$) and $N_G(U, b_U)/UC_G(U)$ must have a subgroup isomorphic to $SL(2, p)$. On the other hand, any Alperin-Goldschmidt $B$-subpair $(U, b_U)$ which has these properties will make a strictly positive contribution to $k_2(B)$.*

**PROOF:** By the results which have gone before, it suffices to prove that $|U| \leq p^3$, and that $U$ is non-Abelian if $|U| = p^3$. But to make a non-zeo contribution, $U$ itself must have an irreducible character $\mu$ of defect 2, so that $\mu(1) = \frac{|U|}{p^2}$. But we have $\mu(1)^2 \leq [U : Z(U)]$, which yields $|U||Z(U)| \leq p^4$, and hence the result.

## SECTION 4: Further Computations

In this section, we remark that when trying to verify OWC, it may be possible to finesse some of the computations in the case when $U$ is homocyclic of type $(p^m, p^m)$ for $m > 1$. When $p = 2$, we will see that we are in the case where the defect group $D$ of our block is $C_{2^m} \wr C_2$. The (ordinary and modular) character-theoretic invariants of the principal block have been determined for this defect group by Brauer and Wong. Also, B. Külshammer [11] has obtained partial results for non-principal blocks with such a wreathed defect groups. For $p = 2$, our results below show that OWC predicts that ( for the three possible fusion patterns) the numbers $k_e(B)$ should be the same as they are for the principal block with the same defect group and fusion pattern.

When $p = 3$, there are a number of different possible fusion patterns to be considered when performing the computations for OWC. Recent work of Díaz,Ruiz and Viruel [2] shows that not all of these 3-fusion patterns can arise for (principal blocks of) finite groups, but it is unclear at present whether the excluded ones can arise as fusion patterns for (non-principal) 3-blocks of finite groups.

**THEOREM 6:** *Suppose that $(U, b_U)$ is an Alperin-Goldschmidt $B$-subpair with $U$ Abelian of type $(p^m, p^m)$ for some $m > 1$ and $U < D$, a defect group for $B$. Then either:*

*i) $p = 2, D \cong C_{2^n} \wr C_2, U = J(D)$, and $B$ satisfies OWC if and only if the following conditions hold ( letting $\beta$ denote the unique Brauer correspondent of $B$ for $N_G(J(D))$ )*

a) *For each defect $e \neq n + 1$, we have $k_e(B) = k_e(\beta)$.*

*b) We have $k_{n+1}(B) = 0$ unless there is an Alperin-Goldschmidt $B$-subpair of the form $(Z(D)Q, b_Q)$, where $Q$ is quaternion of order* 8, *in which case $k_{n+1}(B) = 2^{n-1}$.*

ii) *$p = 3, [D : U] = 3, U = J(D)$, $D$ has maximal class, and $B$ satisfies OWC if and only if the following conditions hold ( letting $\beta$ denote the unique Brauer*

*correspondent of $B$ for $N_G(J(D))$):*
*a) $k_e(B) = k_e(\beta)$ for $e = 2m, 2m+1$, $k_e(B) = 0$ for $e \notin \{2, 2m, 2m+1\}$.*
b) *$k_2(B) \neq 0$ if and only if there is an Alperin Goldschmidt $B$-subpair of the form $(V, b_V) \leq (D, b_D)$, where $V = Q$ and $Q$ is extra-special of exponent 3 and order 27. Furthermore, we have $k_2(B) = (x + 2y)$, where $y$ is the number of $N_G(D, b_D)$ orbits of such subpairs with*
*$N_G(V, b_V)/VC_G(V) \cong SL(2,3)$ and $x$ is the number of $N_G(D, b_D)$ orbits of such subpairs with $N_G(V, b_V)/VC_G(V) \cong GL(2,3)$.*

**COROLLARY 7:** When $D \cong C_{2^m} \wr C_2$, OWC is equivalent to the following, where $U = J(D)$ is the unique Abelian subgroup of $D$ of order $2^{2m}$:
i) If $N_G(U, b_U)/UC_G(U) \cong S_3$, then we have $k_{2m+1}(B) = 2^{m+1}$ and

$$k_{2m}(B) = \left( \frac{2^{2m-1} + 4}{3} - 2^{m-1} \right).$$

Furthermore, in that case, we have $k_{m+1}(B) = 2^{m-1}$ if there is an Alperin-Goldschmidt $B$-subpair $(V, b_V)$ with $V = Z(D)Q$, where $Q$ is quaternion of order 8, and $k_{m+1}(B) = 0$ otherwise. For any other value of $e$ not listed so far, we have $k_e(B) = 0$.

ii) If $N_G(U, b_U)/UC_G(U) \cong C_2$, then we have $k_{2m+1}(B) = 2^{m+1}$ and $k_{2m}(B) = (2^{2m-1} - 2^{m-1})$. Furthermore, in this case, we have $k_{m+1}(B) = 2^{m-1}$ if there is an Alperin-Goldschmidt $B$-subpair $(V, b_V)$ with $V = Z(D)Q$, where $Q$ is quaternion of order 8, and $k_{m+1}(B) = 0$ otherwise. For any other value of $e$ not listed so far, we have $k_e(B) = 0$.
**REMARK:** If $(V, b_V)$ is a B-subpair with $V = Z(D)Q_8$, then ( given the structure of $D$), $(V, b_V)$ is an Alperin-Goldschmidt $B$-subpair if and only if $N_G(V, b_V)/VC_G(V) \cong SL(2,2)$.

**PROOF OF THEOREM 6:** We have seen previously that $[D : U] = p$ for $D$ a defect group of $B$ and that $U = J(D)$. We next note that $Z(D)$ must be cyclic. For otherwise, since $Z(D) \leq C_D(\Omega_1(U)) = U$, and $U$ has rank 2, we have $\Omega_1(U) \leq Z(D)$, which contradicts the fact that $C_D(\Omega_1(U)) = U$ and $D > U$.

For any $x \in D \backslash U$, we know that $C_U(x) = Z(D)$ as $D = U\langle x \rangle$ and $C_D(U) = U$. Hence $|C_D(x)| = p|Z(D)|$ and $C_D(x)$ is Abelian. We also note that it follows from this that $D$ has no subgroup of type $(p, p, p)$, and that $D$ has no Abelian subgroup ( other than subgroups of $U$) of order greater than $p|Z(D)|$).

Suppose first that $N_G(W, b_W)$ controls fusion of $B$-subpairs, where $W = \Omega_1(U) \lhd D$. Now the block $b_W$ has defect group $U = C_D(W)$ as a block of $C = C_G(W)$. For any subgroup, $X$, of $U$, we have $\Omega_1(X) \leq \Omega_1(U) = W \leq Z(C)$. Hence each $p$-regular element of $N_C(X)$ acts trivially on $\Omega_1(X)$, and hence trivially on $X$, ( as $X$ is Abelian). Thus $N_C(X)/C_C(X)$ is a $p$-group. In particular, $b_W$ is a nilpotent block. Since $N_G(W, b_W)/C_G(W)$ is isomorphic to a subgroup of $GL(2,p)$ containing $SL(2,p)$ in this case, we see that he Külshammer-Puig 2-cocycle is trivial. Hence we may replace $N_G(W, b_W)$ by a (split) extension of

$U$ by $Aut_G(U, b_U)$ to perform the necessary calculations. In other words, in this case, $N_G(U, b_U)$ controls fusion of $B$-subpairs, and we obtain $k_e(B) = k_e(\beta)$ for all integers $e$, where $\beta$ is the unique block of $N_G(U, b_U)$ covering $b_U$. Hence we may suppose that $N_G(W, b_W)$ does not control fusion of $B$-subpairs.

Then there is an Alperin-Goldschmidt $B$-subpair $(V, b_V)$ strictly contained in $(D, b_D)$, such that $N_G(V, b_V)$ is not contained in $N_G(W, b_W)$. Then $Z(D) \leq (V \cap U)$ and $[V : V \cap U] = p$ ( we can't have $V \leq U$ as $V \neq U$ and $C_D(V) = Z(V)$).

Now $\Phi(V)$ must be cyclic, since $\Phi(V) \leq U$, for otherwise $W = \Omega_1(U) = \Omega_1(\Phi(V))$ char $V$. If $V \cap U$ is cyclic, then $V$ is metacyclic, which (by Lemma 3.3) yields $V$ quaternion of order 8 or $V$ Abelian of type $(p^n, p^n)$. The former case yields that all Abelian normal subgroups of $D$ are cyclic. Since $V$ has a cyclic subgroup of index $p$ under present assumptions, we must have $n = 1$. Again, $p = 2$ yields a contradiction, since all Abelian normal subgroups of $D$ are cyclic in that case if $V$ is Klein 4-group. The case $p = 3$ yields that $D$ has maximal class, and chains starting with $(V, b_V)$ make zero contribution to the formula of OWC for each non-negative integer value of $e$ in any case. Hence we may suppose that $V \cap U$ is not cyclic.

Now $Z(V) \leq U$, for otherwise, choosing $v \in Z(V) \backslash U$ yields that $C_U(v) = Z(D)$ is cyclic, and hence $U \cap V$ is cyclic, contrary to hypotheses. Then $Z(D) \leq Z(V) \leq Z(D)$, so that $Z(V) = Z(D)$ is cyclic. In particular, $V$ is non-Abelian.

Now $V \cap U$ is not characteristic in $V$, otherwise $\Omega_1(U) = \Omega_1(V \cap U)$ char $V$. Hence $V$ has another Abeliean normal subgroup of index $p$, say $Y$. Then $V = Y(V \cap U)$, so that $V$ has nilpotence class 2.

Since $Z(D)$ is cyclic, $D$ has a faithful irreducible complex character, say $\mu$ ( for there is an irreducible character which does not contain $\Omega_1(Z(D))$ in its kernel, and any such character is faithful, since otherwise its kernel would interesect $Z(D)$ non-trivially). Since $D$ has an Abelian normal subgroup of index $p$, but is not itself Abelian, we have $\mu(1) = p$.

Suppose that $D$ has class 2. Then $\mu$ vanishes identically on $D \backslash Z(D)$, so that $p^2 = \mu(1)^2 = [D : Z(D)]$. Now $|Z(D)| \leq p^m$, since $Z(D) \leq U$ and $Z(D)$ is cyclic. Hence we have $p^{m+2} \geq p^2|Z(D)| = |D| = p^{2m+1}$, yielding $m = 1$, a contradiction. Hence $D$ does not have class 2.

The same argument applied to $V$ allows us to deduce that $|V| = p^2|Z(D)|$. Now $\Omega_1(U)$ is not characteristic in $V$ so that $V$ has another non-central normal subgroup of type $(p, p)$, say $E$. Then $E\Omega_1(U)$ is non-Abelian ( since $D$ contains no subgroup of type $(p, p, p)$), so $Q = E\Omega_1(U)$ is extra-special of order $p^3$ ( of exponent $p$ if $p = 3$, and dihedral if $p = 2$). Hence $V = QC_V(Q)$. Since $E \not\leq U$, we have $C_V(E) = EZ(D)$, so that $V = QZ(D)$.

If $p = 3$, then $Q = \Omega_1(V)$ char $V$. If $p = 2$, then $P = \Omega_2(V) = Q\Omega_2(Z(D))$ char $V$.

Suppose first that $p = 2$. Let $\mu$ be a faithful irreducible character of degree 2 of $V$. Then $\mu$ must restrict irreducibly to $U$, and it follows that $U$ is induced from a linear character of $U$. This gives an injection from $D$ into $C_{2^n} \wr C_2$, so we must have $D \cong C_{2^n} \wr C_2$ by consideration of order. Hence there are $2+2^n$ elements of order 4 in $D$ which have order 4, and determinant 1 in the given representation, exactly two of which lie in $U$. The two such elements in $U$ lie in every quaternion subgroup of order 8 of $D$, and each such subgroup contains 4 such elements of order 4 of $D \backslash U$. Conversely, every element of order 4 and determinant 1 of $D \backslash U$ inverts the two such elements of order 4 in $U$, and, together with these elements, generates a quaternion subgroup of $D$ of order 8. Hence there are $2^{n-2}$ quaternion subgroups of order 8 of $D$.

In particular, applying this argument within the subgroup $T = C_4 \wr C_2$, we deduce that $T$ has a unique quaternion subgroup of order 8, say $R$. Now $Z(D)$ has order $2^n$ by inspection. Now $C_D(R) = Z(D)$ since $R \not\leq U$ and $[N_T(R) : RC_T(R)] = 2$, so that $N_D(R) = N_T(R)C_D(R) = TZ(D)$. Hence $[D : N_D(R)] = 2^{n-2}$, and $D$ has a single conjugacy class of quaternion subgroups of order 8. Thus all subgroups of $D$ of the form $\Omega_2(Z(D))Q_8$ are conjugate to $Q$ within $D$ and all subgroups of $D$ of the form $Z(D)Q_8$ are also conjugate within $D$. For any quaternion subgroup of order 8 of a subgroup like $Z(D)Q_8$ is contained in $\Omega_2(Z(D))Q_8$ as $Z(D)Q_8$ has class 2, and $\Omega_2(Z(D))Q_8$ has a unique quaternion subgroup of order 8.

Let us contribute the contributions to the alternating sum of OWC from chains beginning with $(V, b_V)$. Both $R$ and $Z(D)$ are characteristic in $V$, and $O^2(N_G(V, b_V)$ centralizes $Z(D)$,so that $Z(D)$ is central in $N_G(V, b_V)$, since $(V, b_V)$ is an Alperin-Goldschmidt $B$-subpair. Hence $N_G(V, b_V)/VC_G(V)$ acts faithfully on $R$, so is isomorphic to $SL(2, 2)$, as it can't have order 2. Hence the Külshammer Puig 2-cocycle is trivial in this case, and we may perform the computation within $Z(D) * GL(2, 3)$. As before, there is no contribution from characters which lie over linear characters of $R$, since $Z(D) * GL(2, 3)/Z(R) \cong C_{2^{n-1}} \times S_4$. The unique irreducible character of degree 2 of $R$ extends in $2^{n-1}$ ways to an irreducible character of $V$ ( of defect $n + 1 = 2 + n - 1$). Each of these extensions in turn has three extensions to $Z(D)SL(2, 3)$, but of these three extensions, only one has determinantal order prime to 3. This extension itself extends in two ways to $Z(D) * GL(2, 3)$. However, the other two extensions ( to $SL(2, 3)$ ) both induce to the same irreducible character of degree 4 of $Z(D)GL(2, 3)$. So the number of $V$-projective irreducible characters of $Z(D) * GL(2, 3)$ which lie over the irrecuible character of degree 2 of $R$ is $2^{n-1}$. However, $Z(D) * S$ has no irreducible character of degree 4, where $Z(D) * S$ is a Sylow 2-subgroup of $Z(D) * GL(2, 3)$, since $Z(D) * S$ has an Abelian subgroup of index 2. Hence a Sylow 2-normalizer in $Z(D) * GL(2, 3)$, ( which is a Sylow 2-subgroup) has no $Q$-projective irreducible characters.

Hence chains beginning with $(V, b_V)$ make a contribution of $2^{n-1}$ to the formula of OWC for $k_{n+1}(B)$, and zero contribution to the formula of OWC for $k_e(B)$ for any other value of $e$.

We will not give full details of the calculations when $p = 3$, but will only point out where the argument diverges from that for $p = 2$. We have seen that the Alperin-Goldschmidt $B$-subpairs ( if any) $(X, b_X)$, which are different from $(U, b_U)$ and $(D, b_D)$ and are not of type $(3,3)$ all have $|X| = 9|Z(D)|$. We will prove that $Z(D)$ has order 3 in this case. We recall that the group $L = L_U$ is a semi-direct product of $U$ with $SL(2,3)$ or $GL(2,3)$, and that $L/U$ is a subgroup of $Y = GL(2, \mathbb{Z}/3^m\mathbb{Z})$, where $U$ has order $3^{2m}$. Now $Y$ has a unique conjugacy class of quaternion subgroups of order 8. Let $Q$ be one of these. Then $C_Y(Q)$ has order $2.3^m$, and the 3-elements in $C_Y(Q)$ are all scalars. Hence a Sylow 3-subgroup of $N_Y(Q)$ is Abelian of type $(3^m, 3)$. Setting $\omega = (1 + 3^{m-1})$ ( in $\mathbb{Z}/3^m\mathbb{Z}$, we see that $1 + \omega + \omega^2 = 3$, and that if $A$ is any matrix of order 3 in $N_Y(Q) \backslash C_Y(Q)$, then $\omega A$ and $\omega^2 A$ are also such matrices. Since $det(A) \in \{1, \omega, \omega^2\}$, we may suppose that $det(A) = 1$.

We have established so far that a subgroup of order 3 of $Y$ which normalizes, but does not centralize, a quaternion subgroup of order 8 of $Y$, and which consists of elements of determinant 1, is unique up to conjugacy.

We may choose an element $x \in \mathbb{Z}/3^m\mathbb{Z}$ which satisfies $x^2 = \frac{3^m-1}{2}(= \frac{-1}{2}$ for ease of notatation). Furthermore, replacing $x$ by $-x$ if necessary, we may suppose that $2x - 1$ is not divisible by 3. Now

$$\begin{pmatrix} \frac{-1}{2} & x + \frac{1}{2} \\ x - \frac{1}{2} & \frac{-1}{2} \end{pmatrix}$$

is a matrix of order 3 whose product with

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

has order 6 ( with 2-part central in $Y$). Either by direct computation, or by using the identification of the group $(2,3,3)$ with $A_4$, these two matrices generate a subgroup of $Y$ isomorphic to $SL(2,3)$. Now

$$\begin{pmatrix} \frac{-1}{2} & x + \frac{1}{2} \\ x - \frac{1}{2} & \frac{-1}{2} \end{pmatrix}$$

is easily checked conjugate within $Y$ to

$$C = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}.$$

Now $C$ fixes exactly 3 elements of $U$. Furthermore, for $\omega$ as above, we see that $\omega C$ and $\omega^2 C$ each have 3 fixed points on $U$. We deduce that there are exactly three inequivalent faithful actions of $SL(2,3)$ on $U$, and that the action is determined by the isomorphism type of its Sylow 3-subgroup. However, we note that only one of these $SL(2,3)$ actions can extend to a $GL(2,3)$ action ( the one in which the 3-elements of $SL(2,3)$ act with determinant 1). We also deduce that in all 3 possible actions, $D$, the Sylow 3-subgroup of the semi-direct product, has maximal class, since, in all cases, $D \backslash U$ contains an element $h$ of order 3 with $|C_U(h)| = 3$. Hence we now have $|Z(D)| = 3$.

Again, we may work with a (split) extension of $V$ by $Out_G(V, b_V)$. But the latter group could be isomorphic to $SL(2,3)$ or $GL(2,3)$. If it is $GL(2,3)$, the action on $Z(D)$ is no longer trivial, as elements of $GL(2,3)\backslash SL(2,3)$ invert $Z(D)$. The characters of $V$ which contain $\Omega_1(Z(D))$ in their kernels make zero contribution to the formula of OWC for $k_e(B)$ for any $e$, by calculations similar to those performed earlier.

We now see that when $Out_G(V, b_V)$ is isomorphic to $SL(2,3)$, each of the two irreducible characters of degree 3 of $Q$ extends uniquely to $V$, and each of those in turn has 5 "extensions" to $O_2(SL(2,3))V$, one of degree 6 and 4 of degree 3. The one of degree 6 must extend to $SL(2,3)V$ ( in three ways) by its uniqueness, so leads to no $V$-projective irreducible characters. Of the 4 extensions of degree 3, only one has a determinant on restriction to $O_2(SL(2,3)$ which is stable under a Sylow 3-subgroup of $SL(2,3)$ ( trivial determinant, in fact). That one must extend to $SL(2,3)V$ ( in three ways), but yields no $V$-projective irreducible character. The other three must each induce to the same irreducible character of degree 9 of $SL(2,3)V$, which is $V$-projective. Hence we get a contribution of 2 to the formula of OWC for $k_2(B)$ in this case.

The argument when $Out_G(U, b_U) \cong GL(2,3)$ is very similar, except that the two irreducible characters of degree 3 of $Q$ are in the same orbit in this case. The inertial subgroup of either one of these is $V.SL(2,3)$ in this case, after which the calculation is just as above. So this time the contribution to the formula of OWC for $k_2(B)$ is just 1.

In either case, $p = 2$ or 3, we see that Alperin–Goldschmidt $B$-subpairs $(V, b_V)$ which have $V$ non Abelian must already be conjugate withn $N_G(D, b_D)$ by the subpair version of Alperin's fusion theorem, since $(V, b_V)$ can't be properly contained in another Alperin-Goldschmidt $B$-subpair, other than $(D, b_D)$, by the structure of $(V, b_V)$.

**PROOF OF COROLLARY 7:** This is now a matter of calculating the relevant number of characters for the correpsonding block of $N_G(U, b_U)$. If $N_G(U, b_U)/UC_G(U) \cong SL(2,2)$, then $(U, b_U)$ is an Alperin-Goldschmidt $B$-subpair and the above analysis goes through, telling us that OWC predicts that $k_e(B) = 0$ for $e \notin \{1, 2n+1, 2n, n+1\}$ and that $k_{2n}(B)$ and $k_{2n+1}(B)$ can be computed within $N_G(U, b_U)$ ( and, in fact, we may work instead with $U.SL(2,2)$).

Thr trivial linear character of $U$ leads to 2 irreducible characters of defect $2n+1$, and one of defect $2n$. A further $3.(2^n - 1)$ linear characters of $U$ are fixed by some ( in fact, exactly one ) involution in the $SL(2,2)$, yielding $2^{n+1} - 2$ irreducible characters of degree 3 of the semi-direct product, so of defect $2n+1$. The remaining $(2^{2n} - 1) - 3.(2^n - 1)$ linear characters fall into orbits of length 6 under $SL(2,2)$, yielding

$$\frac{2^{2n-1} + 1}{3} - 2^{n-1}$$

irreducible characters of degree 6, so of defect $2n$.

Remembering the two irreducible characters of defect $2n + 1$ and the one irreducible character of defect $2n$ from the trivial character, the numbers accord with the statement of the corollary.

In the case that $|Out_G(U, b_U)| = 2$, we get $2.(2^n - 1)$ irreducible characters of defect $2n+1$ from non-trivial linear characters of $U$ and $2^{2n-1} - 2^{n-1}$ irreducible characters of defect $2n$ from non-trivial linear characters of $U$. This time, we only get an additional two irreducible characters ( each of defect $2n + 1$) from the trivial character of $U$.

## Some Concluding Remarks

We summarise some of the results of this paper which may need to be explicitly drawn out from the earlier text. To complete the verification for $p = 2$ for a block $B$ which has an Alperin-Goldschmidt sub-pair of the form $(U, b_U)$ with $U$ metacyclic, it remans only to check the cases when $D = U$ is Abelian of type $(2^m, 2^m)(m > 1)$ ( this has been done by Brauer for principal blocks) and when $D \equiv C_{2^m} \wr C_2$ ( which has been done for principal blocks by Brauer and Wong. Partial results for non-principal blocks have been obtained by Külshammer for this defect group). For this defect group, we have shown that the relevant invariants for these blocks should be the same as for principal blocks with the same pattern, In the case $p$ odd, the results are somewhat less clear-cut. Verifying OWC when the defect group $D$ of the block $B$ is metacyclic has been reduced to checking that there is a defect preserving bijection between irreducible characters in $B$ and its unique Brauer correspondent for $N_G(D)$. In the exceptional case that the group component $U$ of an Alperin-Goldschmidt $B$-subpair is homocyclic of type $(3^m, 3^m)$ for $m > 1$, we have shown that $U = J(D)$, and explicitly identified the discrepancy between the formula of OWC for $B$ and its unique Brauer correspondent for $N_G(J(D))$ in terms of the fusion pattern for $B$-subpairs (and we have shown that OWC predicts that for all defects other than 2, there should be a bijection between irreducible characters of $B$ of that defect, and irreducible characters of the unique Brauer correspondent of $B$ for $N_G(J(D))$.

We close with a few remarks on weights in Alperin's sense. When $(U, b_U)$ is an Alperin-Goldschmidt $B$-subpair and $U$ is 2-generated, but $U \neq D$, we have seen that $X = N_G(U, b_U)/UC_G(U)$ is isomorphic to a subgroup of $GL(2, p)$ containing $SL(2, p)$, so that the number of (Alperin- ie modular) weights contributed to the formula of Alperin's weight conjecture for $B$ from $N_G(U, b_U)$ is $[X : SL(2, p)]$.

**Acknowledgements:** The author is grateful to Antonio Viruel for bringing to his attention the fact that $D$ has maximal class when $U < D$ is homocyclic of type $(3^m, 3^m)$, which simplified some of the computations we had originally performed. The author is grateful to the Centre Bernoulli of the EPFL, Switzerland for its support and hospitality during the preparation of the work.

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