ENDOMORPHISM RINGS OF PERMUTATION MODULES OVER MAXIMAL YOUNG SUBGROUPS

STEPHEN DOTY, KARIN ERDMANN AND ANNE HENKE

ABSTRACT. Let K be a field of characteristic two, and let λ be a two-part partition of some natural number r. Denote the permutation module corresponding to a Young subgroup Σ_λ in Σ_r by M^λ. We construct a full set of orthogonal primitive idempotents of the centraliser subalgebra $S(\lambda) = 1_\lambda S(2,r)1_\lambda = \mathrm{End}_{K\Sigma_r}(M^\lambda)$ of the Schur algebra $S(2,r)$; these idempotents are naturally in one-to-one correspondence with the two-Kostka numbers.

1. Introduction

Permutation modules of symmetric groups coming from actions on set partitions are of central interest in the representation theory of symmetric groups and provide also a natural link with the representation theory of general linear groups, via Schur algebras.

Assume K is a field of prime characteristic p. Fix natural numbers n and r and n-part partitions λ and μ of r. The permutation module M^λ over Σ_r is the module obtained by inducing the trivial representation from the Young subgroup Σ_λ to the symmetric group Σ_r. The indecomposable direct summands of M^λ are the Young modules. By James' submodule theorem [7, 7.1.7] there is a unique indecomposable summand of M^λ, defined to be the Young module Y^λ, which contains the Specht module S^λ. The module M^λ is then a direct sum of Young modules Y^μ, and if Y^μ occurs then $\mu \geq \lambda$. The p-Kostka number $[M^\lambda : Y^\mu]$ is defined to be the multiplicity of the Young module Y^μ occuring, up to isomorphism, in a direct sum decomposition of the permutation module M^λ. Thus we have:

$$M^\lambda \quad = \quad \bigoplus_{\mu \geq \lambda} [M^\lambda : Y^\mu] Y^\mu$$

Let $S(n,r)$ be the Schur algebra of degree r, defined by

$$S(n,r) = \mathrm{End}_{K\Sigma_r}(E^{\otimes r}) = \mathrm{End}_{K\Sigma_r}\Big(\bigoplus_{\lambda \in \Lambda(n,r)} M^\lambda\Big)$$

where E is an n-dimensional K-vector space. For the connection of Schur algebras with general linear groups, see Green [3]. The idempotent $1_\lambda \in S(n,r)$ is defined to be the

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projection onto M^λ with kernel $\oplus_{\mu\neq\lambda}M^\mu$. We define the centraliser subalgebra $S(\lambda)$ of $S(n,r)$ by

$$S(\lambda) = 1_\lambda S(n,r)1_\lambda \cong \mathrm{End}_{K\Sigma_r}(M^\lambda).$$

We study these algebras when λ is a two-part partition, that is, the associated Young subgroup is maximal. Then the ordinary character of M^λ is multiplicity-free. Hence the algebra $S(\lambda)$ is commutative, and any given Young module Y^μ occurs at most once as a direct summand of M^λ. All idempotents of $S(\lambda)$ are central, and hence there are finitely many primitive idempotents, and they are in 1-1 correspondence with the indecomposable summands of M^λ. The blocks of $S(\lambda)$ are therefore precisely the endomorphism rings of the Young modules Y^μ which are direct summands of M^λ.

In this paper, we will construct a full set of orthogonal primitive idempotents of the algebra $S(\lambda)$ where λ is a two-part partition and $\mathrm{char}(K) = 2$. These idempotents are naturally in one-to-one correspondence with the 2-Kostka numbers. The philosophy is to consider an infinite family of algebras at the same time, as it was done in [1]. This is possible, by exploiting the presentation obtained in [2] of the Schur algebra in terms of the universal enveloping algebra. It allows one to keep $\lambda_1 - \lambda_2$ fixed and let r vary arbitrarily. This can be thought of as an algebraic analogue of the fact that the 2-Kostka matrix is 'quarter-infinite'. We describe the main results of this paper in more detail in the next section.

2. The Idempotent Theorem

The p-Kostka matrix. Let $\mathrm{char}(K) = p$ be prime, r a natural number and $\lambda = (r-k, k)$ and $\mu = (r-s, s)$ be two-part partitions. The p-Kostka matrix for a given parity of r is a quarter-infinite matrix. Its rows are labelled by $m := r - 2k$ and its columns are labelled by $g := k - s$, and then r varies over the integers $m \geq 2g$ of this parity; the (m, g)th entry of the matrix is given by

$$B(m,g) = \binom{r-2s}{k-s} = \binom{m+2g}{g}.$$

By [4, 5] we have that the Young module $Y^{(r-s,s)}$ is a direct summand of the permutation module $M^{(r-k,k)}$ if and only if $\binom{r-2s}{k-s} \neq 0$ modulo p.

Given a natural number a, write $a = \sum_i a_i p^i$ for the p-adic expansion of a. It is well-known that

$$B(m,g) \equiv \prod_i \binom{(m+2g)_i}{g_i} \text{ modulo } p.$$

We call $\Pi_i\binom{(m+2g)_i}{g_i}$ modulo p the binomial expansion of $B(m,g)$ and we write $B(m,g)_i$ for the i-th factor. Sometimes we will also write the binomial coefficient (modulo p) as matrix where the i-th column is the i-th factor of the product, for $i \geq 0$:

$$B(m,g) = \begin{pmatrix} (m+2g)_0 & (m+2g)_1 & \dots & (m+2g)_i & \dots \\ g_0 & g_1 & \dots & g_i & \dots \end{pmatrix}.$$

We say $B(m,g)_i$ is zero if the ith column in the matrix of $B(m,g)$ is zero.

Let $m \geq 0$. We consider the infinite family of algebras $S(\lambda)$ where λ runs through all partitions $\lambda = (\lambda_1, \lambda_2)$ such that $\lambda_1 - \lambda_2 = m$. The presentation from [2] (see §3) provides $S(\lambda)$ with a basis $\{b(a) : 0 \leq a \leq \lambda_2\}$ with nice properties. The products $b(i)b(j)$ depend only on m (and not on the degree), where terms $b(s)$ appearing with $s > \lambda_2$ are set zero.

Construction of the idempotents. We henceforth take $\text{char}(K) = 2$ and keep $m \geq 0$ fixed. We work in an algebra $S(\lambda)$ of large enough degree r (of the right parity). For any $m \geq 0$ and $g \geq 0$ such that $B(m, g)$ is non-zero modulo 2 and the degree r is large enough (that is $r \geq m + 2g$) we will now define elements in the algebra $S(\lambda)$. First we introduce two index sets: let

$$\begin{aligned} I_{m,g} &:= \{u : g_u = 0 \text{ and } (m + 2g)_u = 1\}, \\ J_{m,g} &:= \{u : g_u = 1 \text{ and } (m + 2g)_u = 1\}. \end{aligned}$$

Then for a natural number t define elements in the algebra $S(\lambda)$ by

$$e_{m,g} := \prod_{u \in J_{m,g}} b(2^u) \prod_{u \in I_{m,g}} (1 - b(2^u)). \tag{1}$$

$$(e_{m,g})_{\leq t} := \prod_{u \in J_{m,g}, u \leq t} b(2^u) \prod_{u \in I_{m,g}, u \leq t} (1 - b(2^u)).$$

Remark. We can associate to each factor of the binary expansion of $B(m, g)$ a factor of an element $e_{m,g}$ by the following rule:

$B(m, g)_u$	$\binom{1}{1}$	$\binom{1}{0}$	$\binom{0}{0}$	$\binom{0}{1}$
factor of $e_{m,g}$	$b(2^u)$	$(1 - b(2^u))$	1	0

In particular, an element $e_{m,g}$ defined in this way would be zero if $\binom{0}{1}$ occurs in the binary expansion of $B(m, g)$, that is if $B(m, g) = 0$ modulo two.

The Idempotent Theorem. Let the notation be as above. We will prove in this paper the following theorem:

Idempotent Theorem *For any fixed $m \geq 0$, the set with elements $e_{m,g}$ with $B(m, g) \neq 0$ modulo two and $m + 2g \leq r$ is a complete set of primitive orthogonal idempotents for the algebra $S(\lambda)$.*

This theorem will be proved at the end of Section 6. In fact parts of the proof of this result are not so difficult to see. Observe the following:

(i) The element $e_{m,g}$ is non-zero. By Proposition 3.6 or Lemma 3.7 one can even express it explicitly as a linear combination of the basis elements.

(ii) If $g \neq d$ and $B(m, g)$ and $B(m, d)$ are both non-zero modulo two then

$$e_{m,g}^2 \cdot e_{m,d}^2 = 0.$$

Proof. Let i be minimal such that $B(m, g)_i \neq B(m, d)_i$. Since columns $< i$ are the same, and both binomial coefficients are non-zero, the i-th columns cannot be zero: Suppose that one is zero, the other not. Then $(m + 2d)_i \not\equiv (m + 2g)_i$.

However, in column i the carry overs from the previous coulumns are the same, say x, and $d_{i-1} = g_{i-1}$. This implies a contradiction:

$$(m+2d)_i = m_i + d_{i-1} + x = m_i + g_{i-1} + x = (m+2g)_i \qquad \text{mod } 2.$$

Hence one of them is $\binom{1}{1}$ and the other is $\binom{1}{0}$. So the squares of the elements in the algebra have factors $b(2^i)^2$ and $(1 - b(2^i)^2)$ respectively. In Section 4 we will show that the elements $b(2^i)^2$ are idempotents (see Corollary 4.2), and this implies that the product is zero. □

Furthermore the number of primitive idempotents of $S(\lambda)$ is equal to the number of non-zero binomial coefficients, by [5]. So if we have established that the elements defined are idempotents then the theorem is proved. The latter will follow from an orthogonality result:

Orthogonality Lemma. *Suppose $B(m,g)_s$ is zero, then $e_{m,g}^2 \cdot b(2^s)^2 = 0$.*

Blocks of the algebra $S(\lambda)$. By the Idempotent Theorem, a block of the algebra $S(\lambda)$ has the form $S(\lambda)e_{m,g}$. Then this block has basis

$$\{e_{m,g}b(a) \ : a = [a_0, a_1, \ldots] \text{ where } a_s = 1 \text{ only for } (m+2g)_s = 0\}.$$

By Lemma 3.7, the block is (minimally) generated as an algebra by all

$$\{e_{m,g}b(2^s) : \ s \geq 0 \text{ and } (m+2g)_s = 0\}.$$

By the Orthogonality Lemma, the block hence has a set of generators with square zero. Hence for a general degree r, this block is isomorphic to a quotient of an algebra of the form

$$\bigotimes K[x_i]/\langle x_i^2 \rangle.$$

a tensor product of finitely many local 2-dimensional algebras.

3. Basis and multiplication structure in $S(\lambda)$

In this section we take $\lambda = (\lambda_1, \lambda_2)$ to be a two-part partition and we study the multiplicative structure of $S(\lambda)$ over a field K of characteristic $p \geq 0$. The results from this section will then be used to obtain in characteristic two a reduction formula for $b(2^s)^2$, see Section 4.

We describe briefly some results from [2]. Over $\mathbb{Q}$, the Schur algebra $S(2,r)$ is isomorphic to the quotient of the universal enveloping algebra $U(\mathsf{gl}_2)$ modulo the ideal generated by

$$H_1(H_1 - 1) \ldots \cdot (H_1 - d)$$

Here, as a basis for the lie algebra gl_2 one takes e, f as usual and H_1, H_2 the diagonal matrices e_{11} and $-e_{22}$. This quotient algebra is defined over $\mathbb{Z}$ using the usual divided powers. In this presentation, the idempotent 1_λ (which we defined as projection corresponding to λ) is equal to the image of

$$1_\lambda = \binom{H_1}{\lambda_1}\binom{H_2}{\lambda_2}$$

where $H_1 - H_2$ is the commutator of e and f in the Lie algebra, (See [1, Lemma 5.3]. Let

$$b(i) := 1_\lambda f^{(i)} e^{(i)} 1_\lambda.$$

Then $S_{\mathbb{Z}}(\lambda)$ is the subalgebra with basis $\{b(1), b(1), \ldots, b(\lambda_2)\}$.

In [2] another basis is given, namely

$$\{b'(0), \ldots, b'(\lambda_2)\}, \qquad \text{with} \qquad b'(i) = 1_\lambda e^{(i)} f^{(i)} 1_\lambda.$$

Remark [2] A basis of $S_{\mathbf{Z}}(\lambda)$ can be labelled by the 2×2 matrices of the form

$$A(i) = \begin{pmatrix} \lambda_1 - i & i \\ i & \lambda_2 - i \end{pmatrix}$$

for $i = 0, \ldots, \lambda_2$. These are all the 2×2 matrices of nonnegative integers whose row and column sums are λ. Such matrices (also more generally) are known to label double cosets of Young subgroups in symmetric groups; this might be reassuring when working with permutation modules.

We next study the multiplicative structure of $S_{\mathbb{Z}}(\lambda)$. Most importantly for us is a multiplication formula for the basis elements $b(i)$, given in Proposition 3.6, which also proves again that the $b(i)$ generate a $\mathbb{Z}$-fomr of $S(\lambda)$.

Proposition 3.1. *Over $\mathbb{Q}$, the algebra $S(\lambda)$ is semisimple and generated by $b(1)$ (respectively $b'(1)$).*

It is enough to prove the statement about $b(1)$ since a similar argument will obtain the statement about $b'(1)$.

The proof of the proposition can be obtained by a series of lemmas, of which the first is due to Kostant [9], or see also Humphreys [6, Lemma 26.2]. Let $U(\mathfrak{gl}_2)$ be the universal envelopping algebra of the Lie algebra $\mathfrak{gl}_2$, defined over $\mathbb{Q}$.

Lemma 3.2. *Let $a \geq 0$, $c \geq 0$ be natural numbers. Then we have in $U(\mathfrak{gl}_2)$:*

$$e^{(c)} f^{(a)} = \sum_{j=0}^{\min(a,c)} f^{(a-j)} \binom{h - a - c + 2j}{j} e^{(c-j)}$$

where e, f and h are the standard generators in $\mathfrak{sl}_2$.

We need only the special case where $c = 1$ and $a \geq 1$, in which case we get the formula

$$ef^{(a)} = f^{(a)} e + f^{(a-1)}(h - a + 1)$$

which takes the following form if we clear denominators

$$ef^a = f^a e + a f^{a-1}(h - a + 1). \tag{2}$$

Since these formulas hold in the enveloping algebra (over $\mathbb{Q}$), they are valid in the homomorphic image $S_{\mathbb{Q}}(2, r)$. The first part of the next lemma is contained in [2].

Lemma 3.3. *In $S_{\mathbb{Q}}(2, r)$ we have the equality $h 1_\lambda = m 1_\lambda$, where $m = \lambda_1 - \lambda_2$. Moreover,*

$$b(1) \cdot b(k) = (k + 1)^2 b(k + 1) + k(m + k + 1) b(k).$$

Proof. To see this, first calculate using formula (2):

$$\begin{aligned}(k!)^2 \cdot b(1) \cdot b(k) &= fef^k e^k 1_\lambda \\ &= f(f^k e + kf^{k-1}(h-k+1))e^k 1_\lambda \\ &= f^{k+1}e^{k+1}1_\lambda + kf^k(h-k+1)e^k 1_\lambda \\ &= f^{k+1}e^{k+1}1_\lambda + kf^k e^k(h+k+1)1_\lambda\end{aligned}$$

where we have used the fact that $he^k = e^k(h+2k)$. This holds in the enveloping algebra of $\mathfrak{gl}_2$, and hence is valid in $S_{\mathbb{Q}}(2,r)$. Now apply the first statement of this Lemma to obtain the desired formula. □

Lemma 3.4. *Let $x = b(1)$. Then we have in $S_{\mathbb{Q}}(2,r)$ for any $k \geq 1$ the equality*

$$b(k+1) = \frac{1}{(k+1)!^2} x(x-(m+2))(x-2(m+3))\cdots(x-k(m+k+1)).$$

Proof. To see this equation, proceed by induction on k. We define

$$F_{k+1}(x) = x(x-(m+2))(x-2(m+3))\cdots(x-k(m+k+1)).$$

The case $k=1$ in the preceding lemma gives the equality

$$b(2) = \frac{1}{2^2}(x^2-(m+2)x) = \frac{F_2(x)}{2!\,2!}.$$

Thus the formula of the lemma is valid in case $k=1$. Assume that $b(k) = \frac{F_k(x)}{k!\,k!}$. By the preceding lemma and the inductive hypothesis we then have

$$\begin{aligned}b(k+1) &= \frac{1}{(k+1)^2} \cdot (b(1)b(k) - k(m+k+1)b(k)) \\ &= \frac{1}{(k+1)!^2} \cdot (x-k(m+k+1))F_k(x) = \frac{1}{(k+1)!^2} F_{k+1}(x).\end{aligned}$$

The lemma is proved. □

Proof of Proposition 3.1. It follows from Lemma 3.4 that we have the equality

$$b(k) = 1_\lambda f^{(k)} e^{(k)} 1_\lambda = \frac{F_k(x)}{k!\,k!}$$

for all $k \geq 2$. This formula holds in $S_{\mathbb{Q}}(2,r)$ and hence any element in $S_{\mathbb{Q}}(\lambda)$ is generated by $x = b(1)$. □

Proposition 3.5. *The algebra $S_{\mathbb{Q}}(\lambda)$ is isomorphic with $\mathbb{Q}[T]/(F_{\lambda_2+1}(T))$.*

Proof. By commutation formulas appearing in [2] we have

$$b(\lambda_2+1) = 1_\lambda f^{(\lambda_2+1)} e^{(\lambda_2+1)} 1_\lambda = 0 = F_{\lambda_2+1}(x)$$

since $\lambda + (\lambda_2+1)(1,-1) = (\lambda_1+\lambda_2+1,-1)$ is not a polynomial weight belonging to $\Lambda(2,r)$, for any λ. The proposition now follows from Lemma 3.4. □

Proposition 3.6. *A multiplication formula for the basis elements is given by:*

$$b(i) \cdot b(j) = \sum_{k=0}^{i} \binom{j+k}{i} \binom{j+k}{k} \binom{m+j+i}{i-k} b(j+k). \tag{3}$$

When $a > \lambda_2$ then $b(a)$ is zero in this formula.

Proof. The proof is by induction on i. The induction beginning for $i = 1$ is given by Lemma 3.3. Let now $i > 1$. Then the product $P := b(i+1) \cdot b(j)$ equals:

$$\begin{aligned}
P &= \frac{b(i)(b(1) - i(m+i+1))}{(i+1)^2} \cdot b(j) \\
&= \frac{b(1) - (j+k)(m+j+k+1) + (j+k)(m+j+k+1) - i(m+i+1)}{(i+1)^2} \cdot b(i)b(j) \\
&= \frac{b(1) - (j+k)(m+j+k+1) + (j+k-i)(m+j+k+i+1)}{(i+1)^2} \cdot b(i)b(j) \\
&= \sum_{k=0}^{i} \frac{b(1) - (j+k)(m+j+k+1))}{(i+1)^2} \binom{j+k}{i} \binom{j+k}{k} \binom{m+i+j}{i-k} \cdot b(j+k) \\
&\quad + \sum_{k=0}^{i} \frac{(j+k-i)(m+j+k+i+1))}{(i+1)^2} \binom{j+k}{i} \binom{j+k}{k} \binom{m+i+j}{i-k} \cdot b(j+k) \\
&= \sum_{k=0}^{i} \frac{k+1}{i+1} \binom{j+k+1}{i+1} \binom{j+k+1}{k+1} \binom{m+i+j}{i-k} \cdot b(j+k+1) \\
&\quad + \sum_{k=0}^{i} \frac{(m+j+k+i+1))}{(i+1)} \binom{j+k}{i+1} \binom{j+k}{k} \binom{m+i+j}{i-k} \cdot b(j+k) \\
&= \sum_{k=1}^{i+1} \frac{k}{i+1} \binom{j+k}{i+1} \binom{j+k}{k} \binom{m+i+j}{i+1-k} \cdot b(j+k) \\
&\quad + \sum_{k=0}^{i} \frac{(m+j+k+i+1))}{(i+1)} \binom{j+k}{i+1} \binom{j+k}{k} \binom{m+i+j}{i-k} \cdot b(j+k) \\
&= \sum_{k=0}^{i+1} \binom{j+k}{i+1} \binom{j+k}{k} \binom{m+j+i+1}{i+1-k} \cdot b(j+k).
\end{aligned}$$

The last step puts the two sums together as follows: for $1 \leq k \leq i$ we have:

$$\begin{aligned}
& \frac{k}{i+1}\binom{m+i+j}{i+1-k} + \frac{m+j+k+i+1}{i+1}\binom{m+i+j}{i-k} \\
= & \frac{k}{i+1}\binom{m+i+j}{i+1-k} + \frac{i+1-k}{i+1}\binom{m+i+j}{i+1-k} + \binom{m+i+j}{i-k} \\
= & \binom{m+i+j}{i+1-k} + \binom{m+i+j}{i-k} \\
= & \binom{m+i+1+j}{i+1-k}
\end{aligned}$$

□

With the $\mathbb{Z}$-form provided by the previous proposition, we have $S(\lambda) = S_{\mathbb{Z}}(\lambda) \otimes 1_K$, and by abuse of notation we still write $b(i)$ for $1_\lambda f^{(i)} e^{(i)} 1_\lambda \otimes 1_K$. Then the multiplication formula (3) is then also valid in the K-algebra $S(\lambda)$; and again $b(a) = 0$ whenever $a > \lambda_2$.

Notation. For a field K of characteristic p we need the p-adic expansion of integers. If $n = \sum_{j=0}^{s} n_j p^j$ with $0 \leq n_j \leq p-1$ then we write $n = [n_0, n_1, \ldots, n_s]$. Moreover, we write in the following

$$A_k = \binom{j+k}{i}, \quad B_k = \binom{j+k}{k}, \quad C_k = \binom{m+j+i}{i-k}.$$

Lemma 3.7. *Write $i = [i_0, i_1, \ldots]$ p-adically. Then $b(i) = \prod_{t \geq 0} b(i_t \cdot p^t)$.*

Proof. This is shown by induction on the length t of the p-adic decomposition of i. Assume that $j = [i_0, i_1, \ldots, i_{t-1}]$, then by Equation (3):

$$b(j) \cdot b(i_t p^t) = \sum_{k=0}^{j} A_k B_k C_k \;\; b(i_t p^t + k).$$

Here $A_k = \binom{i_t p^t + k}{j} = \binom{k}{j}$ is nonzero if and only if j is p-contained in k, that is $j_s \leq k_s$ for all s. Hence $j \leq k$ and by assumption also $k \leq j$. So $A_k \neq 0$ precisely if $k = j$. Hence $b(j) \cdot b(i_t p^t) = b(i_t p^t + k)$. □

Lemma 3.8. *We define the degree of the basis element $b(i)$ to be i. Let $1 \leq n \leq p-1$, then*

$$b(p^t)^n = \prod_{k=1}^{n} \binom{k}{1}^2 b(n \cdot p^t) + \text{ terms of lower degree.}$$

Proof. This follows by induction on n, using the multiplication formula given in Proposition 3.6. More precisely, let $2 \leq c \leq p-1$, then

$$b(p^t) \cdot b((c-1)p^t) \;\; = \;\; \sum_{k=0}^{p^t} A_k \cdot B_k \cdot C_k \;\; b((c-1)p^t + k),$$

where for $k = p^t$ we obtain

$$\begin{aligned} A_k &= \binom{(c-1)p^t + k}{p^t} = \binom{cp^t}{p^t} = \binom{c}{1}, \\ B_k &= \binom{(c-1)p^t + k}{k} = \binom{cp^t}{p^t} = \binom{c}{1}, \\ C_k &= \binom{m + cp^t}{p^t - k} = \binom{m + cp^t}{0} = 1. \end{aligned}$$

Corollary 3.9. *Let $\lambda = (\lambda_1, \lambda_2)$ be a partition of r and assume t is such that $p^t \leq \lambda_2 < p^{t+1}$. Then the algebra $S(\lambda)$ is generated by the elements $b(p^0), b(p^1), \ldots, b(p^t)$.*

Proof. We know already from Lemma 3.7 a factorisation of a basis element $b(i)$. Write $i = [i_0, i_1, \ldots]$ p-adically. Then

$$b(i) = \prod_{t \geq 0} b(i_t \cdot p^t).$$

Hence we need to show that the elements $b(c \cdot p^t)$ for $1 \leq c \leq p-1$ are generated by the elements $b(p^t)$. This follows by induction on t using Lemma 3.8. □

Remark. For $c \geq 2$, it seems to be rather hard to find explicite expressions for $b(cp^t)$ in terms of the generators. For $c = 2$ this is done in the next section.

Example: Let $m = 0$ and $p = 2$. Then the partitions we look at are the ones of the form $\lambda = (r/2, r/2)$; in this case the algebra $S(\lambda)$ has dimension $r/2 + 1$. It is generated by $b(0), \ldots, b(2^k)$ where $2^k \leq r/2 + 1 < 2^{k+1}$ subject to the relations

$$\begin{aligned} &b(2^i)^2 = 0, && 0 \leq i \leq k; \\ &\textstyle\prod_{i \in I} b(2^i) = 0, && \text{whenever } I \subseteq \{0, 1, \ldots, k\} \text{ and } \textstyle\sum_{i \in I} 2^i \geq r/2 + 1. \end{aligned}$$

It follows that there are no non-zero idempotents except 1, and hence $S(\lambda)$ is indecomposable, that is, the algebra is a block.

4. The elements $b(i)^2$ are idempotents

From now we assume that the characteristic of the underlying field is $p = 2$. Then Lemma 3.7 shows that the basis element $b(i)$ is equal to the product of the $b(2^t)$ for which $i_t = 1$. So to understand the multiplication completely we need to understand the squares of the basis elements $b(2^t)$.

Example. Let m be fixed with 2-adic expansion $m = [m_0, \ldots, m_t, \ldots]$. Suppose $t = 0, 1$ then we see directly from multiplication formula (3) that

$$\begin{aligned} b(2^0)^2 &= m_0 \cdot b(2^0), \\ b(2^1)^2 &= b(2^1)[m_1 \cdot 1 + m_0 \cdot b(2^0)]. \end{aligned}$$

So we can write $b(2^1)^2 = b(2^1)(m_1 + b(2^0)^2)$; and this has the following generalization, which is the reduction we had promised.

Lemma 4.1. *Suppose* $m = [m_0, \ldots, m_t, \ldots]$ *in 2-adic expansion. Let* $0 \leq v \leq t$ *be maximal such that* $m_{v-1} = 0$. *Then*

$$b(2^t)^2 = b(2^t)[m_t \cdot 1 + \sum_{i=v-1}^{t-1} b(2^i)^2],$$

setting $b(2^i) = 0$.

Proof. We make the convention that $m_i = 0$ when $i < 0$. We rewrite the product $b(2^t)^2$ using the multiplication formula given in Equation (3). Note first that the coefficient $A_k = B_k = \binom{2^t+k}{2^t}$ with $k = 2^t$ is zero modulo two, and if $k < 2^t$ it is equivalent to one. Moreover for $k < 2^t$ we have

$$C_k = \binom{m + 2^{t+1}}{2^t - k} \equiv \binom{m}{2^t - k} \mod 2. \tag{4}$$

We will change variables, for this note that $2^t + k = 2^{t+1} - (2^t - k) = 2^{t+1} - l$. Hence by Equation (4) and by changing variables we can write the sum on the right-hand side in Formula (3) as

$$\begin{aligned} b(2^t)^2 &= \sum_{k=0}^{2^t-1} \binom{m}{2^t - k} b(2^t + k) \\ &= \sum_{l=1}^{2^t} \binom{m}{l} b(2^{t+1} - l) \\ &= b(2^t)[\sum_{l=1}^{2^t} \binom{m}{l} b(2^t - l)]. \end{aligned} \tag{5}$$

For the last equality sign note that $2^{t+1} - l = 2^t + (2^t - l)$, and so for $0 \leq 2^t - l < 2^t$ we can factorize $b(2^{t+1} - l) = b(2^t)b(2^t - l)$ by Lemma 3.7. The term with $l = 2^t$ is equal to $m_t b(0) = m_t \cdot 1$. So we can write

$$b(2^t)^2 = b(2^t)[m_t \cdot 1 + \Gamma(t)] \qquad \text{where} \qquad \Gamma(t) := \sum_{l=1}^{2^t-1} \binom{m}{l} b(2^t - l). \tag{6}$$

We will now prove a recursion formula for $\Gamma(t)$. We claim that

$$\begin{aligned} \Gamma(1) &= b(2^0)^2, \\ \Gamma(t) &= b(2^{t-1})^2 + m_{t-1}\Gamma(t-1) \quad \text{for } t \geq 2. \end{aligned} \tag{7}$$

First, $\Gamma(1) = \binom{m}{1} b(2^0) = m_0 b(2^0) = b(2^0)^2$. Suppose that $t \geq 2$, then we split $\Gamma(t)$ into two sums:

$$\begin{aligned}
\Gamma(t) &= \sum_{l=1}^{2^{t-1}} \binom{m}{l} b(2^t - l) + \sum_{l=2^{t-1}+1}^{2^t-1} \binom{m}{l} b(2^t - l) && \text{by definition of } \Gamma(t), \\
&= b(2^{t-1})^2 + \sum_{l=2^{t-1}+1}^{2^t-1} \binom{m}{l} b(2^t - l) && \text{by Equation (5), Lemma 3.7,} \\
&= b(2^{t-1})^2 + m_{t-1} \sum_{r=1}^{2^{t-1}-1} \binom{m}{r} b(2^{t-1} - r) \\
&= b(2^{t-1})^2 + m_{t-1}\Gamma(t-1) && \text{by definition of } \Gamma(t-1).
\end{aligned}$$

For the third equality sign in the latter equation, write $l = 2^{t-1} + r$ where $1 \leq r \leq 2^{t-1} - 1$, and note that

$$\binom{m}{2^{t-1}+r} \equiv m_{t-1} \binom{m}{r} \mod 2,$$

and $2^t - l = 2^{t-1} - r$. Hence the recursion formula for $\Gamma(t)$ claimed in Equation (7) is shown. It infact implies the closed formula for $\Gamma(t)$:

$$\Gamma(t) = \sum_{i=v-1}^{t-1} b(2^i)^2$$

where v is as in the statement. Substituting this into (6) completes the proof. □

Corollary 4.2. *For $i \geq 0$, the elements $b(2^i)^2$ are idempotent. Moreover, if $m_j = 0$ for all $j \leq i$ then $b(2^i)^2 = 0$.*

Proof. This follows from Lemma 4.1 by induction. □

5. Analysis of the binomial coefficient $B(m, g)$

We assume throughout that m and g are integers such that the binomial coefficient $B(m, g)$ is non-zero modulo two. We need to relate the binomial expansion of m with that of $B(m, g)$. Note that

m	m_0	m_1	m_2	$\dots$	m_i	$\dots$
$+2g$	0	g_0	g_1	$\dots$	g_{i-1}	$\dots$
$m+2g$	$(m+2g)_0$	$(m+2g)_1$	$(m+2g)_2$	$\dots$	$(m+2g)_i$	$\dots$

In this addition, we need to keep track over the 'carry'overs'. So define integers $x_i \geq 0$ such that

$$m_i + g_{i-1} + x_{i-1} = (m + 2g)_i + 2x_i. \tag{8}$$

So x_i is the carry over from column i to column $i+1$ in the addition of m and $2g$. Most important for the proofs later will be that $(m + 2g)_i = 1$ implies that $x_i = 0$; more precisely we have the following:

Proposition 5.1. *Let $m = [m_0, m_1, \ldots]$ and $g = [g_0, g_1, \ldots]$ be in binary expansion. Assume that $B(m, g)$ is non-zero. Then $(m + 2g)_i + 2x_i < 3$ for all i. In particular, if $(m + 2g)_i = 1$ then $x_i = 0$.*

Proof. Certainly $(m + 2g)_i + 2x_i \leq 3$. Assume for a contradiction that this number is equal to three for some i. Then $x_{i-1} = g_{i-1} = 1$. Since $g_{i-1} = 1$ we must have that $(m+2g)_{i-1} = 1$ as well, since otherwise the binomial coefficient $B(m, g)$ would be zero. But then it follows that $m_{i-1} + g_{i-2} + x_{i-2} = 3$, and then repeating the argument gives $m_1 + g_0 + x_0 = 3$. This implies $x_0 = 1$. On the other hand, $(m + 2g)_0 = m_0$ and hence $x_0 = 0$, a contradiction. □

We will later prove some properties by induction. The elements $e_{m,g}$ are defined as products, and it will be convenient to use factors of these which are already known to be idempotents. The basis for the induction will be the following:

Lemma 5.2 (Splitting Lemma). *Let u be a natural number and define*

$$n := [m_0, m_1, \ldots, m_u] \quad \text{and} \quad d := [g_0, g_1, \ldots, g_{u-1}].$$

Suppose $(m + 2g)_u = 1$. Then the binary expansion of $B(n, d)$ equals the binary expansion of $B(m, g)_{<u}$ extended by one column $\binom{1}{0}$. In particular if $g_u = 0$ then $B(n, d) = B(m, g)_{\leq u}$.

Proof. By Proposition 5.1 we know that $x_u = 0$, and by Equation (8) we hence have $m_u + g_{u-1} + x_{u-1} = 1$; the claim follows. □

Remark. The Splitting Lemma shows that when $g_u = 0$ then the element $e_{n,d}$ is a factor of $e_{m,g}$, when written as in the definition, see Equation (1).

We will have to use the formula from Lemma 4.1. So we need to know the digits of $B(m, g)$, given the binary expansion of m and of g. In the remainder of this section we describe these explicitly.

Lemma 5.3. *Given natural numbers t and a. Suppose $B(m, g)_{\leq\ t+a}$ in binary decomposition is of the form*

$$B(m, g)_{\leq t+a} = \begin{pmatrix} \cdots & 1 & 0 & \cdots & 0 \\ \cdots & g_t & 0 & \cdots & 0 \end{pmatrix}. \tag{9}$$

Then we have:

(a) *Suppose $g_t = 0$, then $m_{t+1} = \ldots = m_{t+a} = 0$ and $x_{t+1} = \ldots = x_{t+a} = 0$.*
(b) *Suppose $g_t = 1$, then $m_{t+1} = \ldots = m_{t+a} = 1$ and $x_{t+1} = \ldots = x_{t+a} = 1$.*

Proof. By Proposition 5.1 we know that $x_t = 0$. By Equation (8) we have:

$$\begin{aligned} m_{t+1} + g_t + 0 &= 0 + 2x_{t+1}, \\ m_{t+2} + 0 + x_{t+1} &= 0 + 2x_{t+2}, \\ \cdots & \quad \cdots \\ m_{t+a} + 0 + x_{t+a-1} &= 0 + 2x_{t+a}. \end{aligned}$$

For (a), assume that $g_t = 0$. Then $x_{t+1} = 0$ and hence $m_{t+1} = 0$. Now the second equation shows that $x_{t+2} = 0$ and hence $m_{t+2} = 0$, and so on. Part (b) is similar. □

We will have to consider sequences of digits such that $m_i = 1$ for $v \le i \le s$ and $m_{v-1} = 0$. For these values of i we need to know the i-th columns of $B(m, g)$.

Lemma 5.4. *Suppose column s of $B(m, g)$ is zero but column $s-1$ is non-zero. Let $u \ge 0$ be minimal such that $(m+2g)_i = 1$ for $u \le i < s$, and let $0 \le v \le s$ be maximal with $m_{v-1} = 0$. Then $v \ge u$. Moreover:*

(a) *If $m_s = 0$ then $g_i = 0$ for $v-1 \le i \le s-1$.*
(b) *If $m_s = 1$ then $g_{s-1} = 1$ and $g_i = 0$ for $v-1 \le i < s-1$.*

Proof. (i) Suppose $m_u = 0$ or $u = 0$. Then by definition of v we have that $v \ge u$. So assume that $m_u = 1$ and $u > 0$. By definition of u we have that $(m+2g)_u = 1$ and $(m+2g)_{u-1} = 0$. Then Equation (8) for columns u and $u-1$ together with the assumptions and Proposition 5.1 read:

$$\begin{aligned} 1 + g_{u-1} + x_{u-1} &= 1, \\ m_{u-1} + g_{u-2} + x_{u-2} &= 0 + 2x_{u-1}. \end{aligned}$$

So $x_{u-1} = 0 = g_{u-1}$ which implies that $m_{u-1} + g_{u-2} + x_{u-2} = 0$ and hence $m_{u-1} = 0$. This shows that $u \le v$.

(ii) For (a) and (b), use Proposition 5.1 and Equation (8) for columns between v and $s-1$. By assumption and (i) we have that $(m+2g)_i = 1 = m_i$ for $v \le i \le s-1$. This implies $g_{i-1} = 0 = x_{i-1}$ for $v \le i \le s-1$ and $x_{s-1} = 0$. Then Equation (8) for column s becomes

$$m_s + g_{s-1} + 0 = 0 + 2x_s.$$

If $m_s = 0$ then $x_s = 0$ and $g_{s-1} = 0$. On the other hand if $m_s = 1$ then $x_s = 1$ and $g_{s-1} = 1$. □

6. The proofs of the Orthogonality Lemma and the Idempotent theorem

6.1. **Proof of the Orthogonality Lemma.** Suppose the s-th column of $B(m, g)$ is zero. The aim is to show that $e_{m,g}^2 \cdot b(2^s)^2 = 0$. Recall from Lemma 4.1 that $b(2^s)^2 = b(2^s)\psi$ with

$$\psi = \psi_{m,s} = m_s + \sum_{i=v-1}^{s-1} b(2^i)^2 \tag{10}$$

where $0 \le v \le s$ is maximal such that $m_{v-1} = 0$. We will prove that

$$(e_{m,g})^2_{<s} \cdot \psi_{m,s} = 0. \tag{11}$$

Certainly this then implies the Orthogonality Lemma in Section 2. Note that if $s = 0$ then $\psi = m_0 = 0$ since $(m+2g)_0 = m_0 = 0$. So assume $s > 0$. If all columns before column s are zero then $m_i = 0$ for $i \le s$ and then $\psi = 0$ by Corollary 4.2. So assume now that $w < s$ is such that $(m+2g)_w = 1$ and $(m+2g)_i = 0$ for $w+1 \le i \le s$. We use induction on the number of zero columns between w and s to prove Equation (11).

Start of the induction: Suppose column $s-1$ is non-zero. Let $u \geq 0$ be minimal such that $(m+2g)_i = 1$ for $u \leq i < s$. We apply Lemma 5.4, which shows that $v \geq u$. Moreover, suppose $m_s = 0$, then by part (a) of the Lemma we know that $(e_{m,g})_{<s}$ has factors $(1-b(2^i))$ for $v-1 \leq i \leq s-1$. This gives that $(e_{m,g})^2_{<s} \cdot \psi = 0$ by Corollary 4.2.

Similarly, if $m_s = 1$ then part (b) of the Lemma shows that $(e_{m,g})_{<s}$ has factors $(1-b(2^i))$ for $v-1 \leq i < s-1$ and also a factor $b(2^{s-1})$. Then the claim follows again from Corollary 4.2, using that $b(2^{s-1})^2 \cdot (m_s + b(2^{s-1})^2) = 0$.

Inductive step: Suppose now that column $s-1$ is zero. The inductive hypothesis states that

$$(e_{m,g})^2_{<s-1} \cdot \psi_{m,s-1} = 0.$$

If $g_w = 0$ then we have by Lemma 5.3 that $m_i = 0$ for $w+1 \leq i \leq s$. Then $v = s$ and we can write

$$\psi_{m,s} = b(2^{s-1})^2 = \psi_{m,s-1} \cdot b(2^{s-1}),$$

using Lemma 4.1. By the inductive hypothesis we deduce $(e_{m,g})^2_{<s} \cdot \psi_{m,s} = 0$. Now suppose $g_w = 1$, then by Lemma 5.3 we know that $m_i = 1$ for $w+1 \leq i \leq s$. We rewrite and again use Lemma 4.1:

$$\psi_{m,s} = \psi_{m,s-1} + b(2^{s-1})^2 = \psi_{m,s-1} + \psi_{m,s-1} \cdot b(2^{s-1}),$$

and again using the inductive hypothesis we have $(e_{m,g})^2_{<s} \cdot \psi_{m,s} = 0$. This completes the proof of the Orthogonality Lemma. □

6.2. **Proof of the Idempotent Theorem.** This will be done by induction on t, the largest column label of a non-zero column in the binary decomposition of $B(m,g)$, which we call the degree of $e_{m,g}$. In fact, we will prove the following:

Claim: Elements $e_{m,g}$ and $(e_{m,g})_{<t}$ are idempotents.

Assume that $t = 0$ then $B(m,g) = \binom{1}{0}$. In particular $m_0 = 1$ and so $e_{m,g} = (1 - b(2^0))$ is idempotent. Also $(e_{m,g})_{<t} = 1$ is idempotent. We assume the statement holds for all $e_{n,d}$ of degree $< t$. Let $e_{m,g}$ be of degree t and write $e := e_{m,g} = P \cdot (1 - b(2^t))$ where $P = (e_{m,g})_{<t}$. We have

$$e^2 = P^2(1 - b(2^t)^2) = P^2(1 - \psi b(2^t))$$

where $\psi = \psi_{m,g}$ is defined as in Equation (10). We will show that $P^2 \cdot \psi = P^2$, and secondly that $P^2 = P$. This then implies that $e = e_{m,g}$ is idempotent.

(a) We claim that $P^2 \cdot \psi = P^2$, that is $P^2(1-\psi) = 0$. To see this, let $\widetilde{m} := m + 2^t$, then $\widetilde{m}_t = 1 + m_t$ and $\widetilde{m}_i = m_i$ for $i < t$. Hence $B(\widetilde{m}, g)$ differs from $B(m,g)$ in columns t and $t+1$. Therefore

$$(e_{m,g})_{<t} = (e_{\widetilde{m},g})_{<t} = P.$$

Moreover (using $p = 2$) we have $\psi_{\widetilde{m},t} = 1 - \psi_{m,t}$. So we get from the Orthogonality Lemma, see Equation (11):

$$P^2(1 - \psi_{m,t}) = (e_{\widetilde{m},g})^2_{<t} \cdot \psi_{\widetilde{m},t} = 0.$$

(b) We claim that $P^2 = P$. This is clear if $P = 1$. So suppose $P > 1$, then there is some $u < t$ maximal such that $(m + 2g)_u = 1$. If $g_u = 0$ then $P = e_{n,d}$ with d and n as in the Splitting Lemma 5.2. Hence by the inductive hypothesis P is idempotent. If $g_u = 1$, then $P = (e_{m,g})_{<u} \cdot b(2^u)$. Define n and d by

$$e_{n,d} = (e_{m,g})_{<u} \cdot (1 - b(2^u)). \tag{12}$$

By construction $e_{n,d}$ has degree $u < t$ and hence by the inductive hypothesis we get that $e_{n,d}$ and $(e_{m,g})_{<u}$ are idempotents. Since the characteristic of the underlying field is two and by Equation (12), we have that $(e_{m,g})_{<t} = P = (e_{m,g})_{<u} \cdot b(2^u) = e_{n,d} + (e_{m,g})_{<u}$ is idempotent. □

7. The correspondence between idempotents and Young modules.

Fix an integer $g \geq 0$ such that $\binom{m+2g}{g} \neq 0$. Then we have for each $r \geq g$ of the right parity a partition λ with $\lambda_1 - \lambda_2 = m$, and a partition $\mu = (\mu_1, \mu_2)$ with $\mu_1 - \mu_2 = m + 2g$. We also have the primitive idempotent $e_{m,g}$; and we know that Y^μ is a direct summand of M^λ. We will now show that in fact $e_{m,g}$ is the projection of M^λ corresponding to Y^μ.

Theorem 7.1. *Let λ, μ be two-part partitions such that Y^μ is a direct summand of M^λ. Let $\lambda_1 - \lambda_2 = m$, $\mu_1 - \mu_2 = m + 2g$ and $g = \lambda_2 - \mu_2$. Then the idempotent $e_{m,g}$ of $S(\lambda)$ is the projection onto Y^μ.*

The proof of this will take the rest of the chapter. We use induction. on r, starting with the case $\mu_2 = 0$, that is $\mu = (r, 0)$. Then the inductive step will be to show that if the theorem is true for degree r then it is true for degree $r + 2$.

Suppose first that $\mu_2 = 0$. In the special case when $\lambda = \mu$ we have $g = 0$ and $m = r$. So $\lambda_2 = 0$ and the algebra $S(\lambda)$ has dimension 1. Furthermore, $e_{m,0} = 1$ and $M^\lambda = Y^\lambda$, so the theorem is trivially true.

So suppose now that $\mu > \lambda$. We have then $r = \mu_1$ and $\mu_2 = 0$. By the previous case, applied to μ, we know that $e_{r,0} \in S(\mu)$ is the projection corresponding to the summand Y^μ of M^μ. Both idempotents $e_{m,g}$ and $e_{r,0}$ lie in $S(2, r)$. To show that the summand of M^λ corresponding to the projection $e_{m,g}$ is isomorphic to Y^μ we must show that the idempotents $e_{m,g}$ and $e_{r,0}$ are associated in $S(2, r)$.

Proposition 7.2. *The idempotents $e_{m,g}$ and $e_{r,0}$ are associated in $S(2, r)$. Hence the $e_{m,g}M^\lambda$ of M^λ is isomorphic to Y^μ.*

Proof. (a) We first simplify the expressions for the two idempotents. Note that

$$\begin{aligned} e_{m,g} &= \prod_{u \in J_{m,g}} b(2^u) \cdot \prod_{u \in I_{m,g}} (1 - b(2^u)) && \text{by Equation (1),} \\ &= b(g) \cdot \prod_{u \in I_{m,g}} (1 - b(2^u)) && \text{by Lemma 3.7,} \\ &= b(g) \cdot (1 \pm \text{ products of } b(i)\text{'s }) \\ &= b(g) \end{aligned}$$

where this last equality follows as the algebra $S(\lambda)$ has basis $\{b(0), b(1), \ldots, b(g)\}$ and by using Lemma 3.7. Moreover, as $M^{(r,0)} = Y^{(r,0)}$, we have $e_{r,0} = 1_{(r,0)}$.

(b) Let $\alpha = (1, -1)$ and recall from [2], Theorem 2.4) that for any partition ν we have

$$e \cdot 1_\nu = \begin{cases} 1_{\nu+\alpha} \cdot e & \text{if } \nu + \alpha \text{ is a partition,} \\ 0 & \text{otherwise,} \end{cases}$$

and

$$f \cdot 1_\nu = \begin{cases} 1_{\nu-\alpha} \cdot e & \text{if } \nu - \alpha \text{ is a partition,} \\ 0 & \text{otherwise,} \end{cases}$$

Moreover, by [2], Proposition 4.3 we have that $H_i \cdot 1_\lambda = \lambda_i \cdot 1_\lambda$ for $i = 1, 2$, and recall that $h = H_1 - H_2$. These formulas imply that $e \cdot 1_{(r,0)} = 0$ as $(r, 0) + \alpha$ is not a partition. Moreover, with $\lambda = (g + m, g)$ a partition of $r = m + 2g$ we have

$$e^{(g)} \cdot 1_\lambda = 1_{(r,0)} \cdot e^{(g)}, \qquad 1_{(r,0)} \cdot f^{(g)} = f^{(g)} \cdot 1_\lambda, \qquad \binom{h}{g} \cdot 1_{(r,0)} = \binom{r}{g} \cdot 1_{(r,0)}.$$

(c) We next give elements u and v in the Schur algebra $S(2, r)$ such that $e_{m,g} = uv$ and $e_{r,0} = vu$, proving that the two idempotents are associated. More precisely, let

$$u = 1_\lambda f^{(g)} 1_{(r,0)} \qquad \text{and} \qquad v = 1_{(r,0)} e^{(g)} 1_\lambda.$$

Then by repeated use of the Equation in (b) we have

$$u \cdot v = 1_\lambda\, f^{(g)}\, 1_{(r,0)}\, e^{(g)}\, 1_\lambda = 1_\lambda\, f^{(g)}\, e^{(g)}\, 1_\lambda = b(g)$$

and

$$\begin{aligned} v \cdot u &= 1_{(r,0)}\, e^{(g)}\, 1_\lambda\, f^{(g)}\, 1_{(r,0)} \\ &= 1_{(r,0)}\, e^{(g)}\, f^{(g)}\, 1_{(r,0)} \\ &= 1_{(r,0)} \cdot [\sum_{j=0}^{g} f^{(g-j)} \binom{h - 2g + 2j}{j} e^{(g-j)}] \cdot 1_{(r,0)} \\ &= 1_{(r,0)} \cdot [f^{(0)} \binom{h}{g} e^{(0)}] \cdot 1_{(r,0)} \\ &= \binom{r}{g} \cdot 1_{(r,0)} = B(m, g) \cdot 1_{(r,0)} = 1_{(r,0)} \end{aligned}$$

modulo two. Hence $e_{m,g} = b(g)$ and $e_{r,0} = 1_{(r,0)}$ are associated. □

Now it remains to deal with the inductive step, that is to compare M^λ and $M^{\lambda+(1^2)}$. To do so, we will first analyze more closely how the the hyperalgebra actions on $E^{\otimes r}$ and $E^{\otimes r+2}$ are related.

We fix a basis $\{v_1, v_2\}$ of the K-vector space E. We write briefly $v_{\underline{i}}$ for the tensor product $v_{i_1} \otimes v_{i_2} \otimes \ldots \otimes v_{i_r}$, with $\underline{i}$ the multi-index $\underline{i} = (i_1, \ldots, i_r)$. Define the linear map

$$j : E^{\otimes r} \longrightarrow E^{\otimes r+2} \text{ by } x \mapsto (v_1 \otimes v_2 - v_2 \otimes v_1) \otimes x.$$

Recall that both tensor powers are modules for the hyperalgebra $U_K = U(\mathfrak{gl}_2)_Z \otimes K$. The map j commutes with the action of the divided powers $e^{(a)}$, $f^{(a)} \in U_K$: this is easy to see, noting that the map j is tensoring with $\bigwedge^2 E$, which is trivial under the action of e and f.

Now we restrict j to M^λ; it takes M^λ to $M^{\lambda+(1^2)}$. Since the products $f^{(a)}e^{(a)}$ lie in the zero weight space of U_K, they preserve M^λ and $M^{\lambda+(1^2)}$. The idempotents 1_λ and $1_{\lambda+(1^2)}$ are the projections onto these spaces, and it follows that j intertwines the actions of elements $b(a)$ on M^λ and on $M^{\lambda+(1^2)}$. In particular this implies

$$j(e_{m,g}x) = e_{m,g}j(x), \qquad \text{for all } x \in M^\lambda.$$

Proposition 7.3. *Suppose $e_{m,g}$ is the projection on M^λ corresponding to Y^μ. Then $e_{m,g}$ on $M^{\lambda+(1^2)}$ is the projection corresponding to $Y^{\mu+(1^2)}$.*

Proof. We may assume $m \neq 0$; the case $m = 0$ is understood, see the example at the end of §3. We know that the Specht module S^μ is a submodule of Y^μ. Furthermore, $\mathrm{Hom}_{K\Sigma_r}(S^\mu, M^\lambda)$ is one-dimensional (see [8, 13.13]). So M^λ has a unique submodule isomorphic to Y^μ, which is contained in Y^μ. Similarly $M^{\lambda+(1^2)}$ has a unique submodule isomorphic to $S^{\mu+(1^2)}$ and it is contained in $Y^{\mu+(1^2)}$. It suffices therefore to show the following.

$$\text{If } e_{m,g}(S^\mu) \neq 0 \text{ in } M^\lambda \text{ then } e_{m,g}(S^{\mu+(1^2)}) \neq 0 \text{ in } M^{\lambda+(1^2)}.$$

To do so we use polytabloids, that is the standard generators for Specht modules. Start with standard tableaux of shapes μ and $\mu + (1^2)$ respectively, we take them as follows.

$$t_1 = \begin{matrix} 3\,5\ldots(2u-1)\,(2u+1)\ldots(r+2) \\ 4\,6\ldots(2u), \end{matrix} \qquad t_2 = \begin{matrix} 1\,3\ldots(2u-1)\,(2u+1)\ldots(r+2) \\ 2\,4\ldots(2u). \end{matrix}$$

(Here $u = \mu_2 + 1$). Let R_{t_i} be the row stabilizer of t_i, and C_{t_i} the column stabilizer of t_i. To write down the polytabloid for S^μ in this setup, we must start with an appropriate element $\omega_1 \in M^\lambda$ which is fixed by all elements of R_{t_1} and then the polytabloid is

$$\varepsilon_{t_1} = \omega_1\{C_{t_1}\}^-$$

where $\{C_{t_1}\}^-$ is the alternating sum over all elements in C_{t_1}. We can take

$$\omega_1 = \sum v_{\underline{i}}$$

summing over all $\underline{i}$ such that $i_j = 2$ for j in the second row of t_1. (Note that $\lambda_2 \geq \mu_2$, so this exists. When $\lambda = \mu$ it is just one basis vector.) Similarly one defines the Specht module generator ε_{t_2} from t_2.

Explicitly,

$$\{C_{t_1}\}^- = (1 - (3,4))(1 - (5,6))\ldots(1 - (2u-1, 2u))$$

This shows that $\omega_1\{C_{t_1}\}^- = \tilde{\omega}_1\{C_{t_1}\}^-$ where $\tilde{\omega}_1$ is the sum over all $v_{\underline{i}}$ such that $i_{2t+1} = 1$ and $i_{2t+2} = 2$ for $1 \leq t < u$; which is visibly identifyable with the generator in [7]. We apply the map j to ε_1,

$$j(\varepsilon_{t_1}) = (v_1 \otimes v_2 - v_2 \otimes v_1) \otimes \varepsilon_1 \;=\; (v_1 \otimes v_2 \otimes \tilde{\omega}_1(1 - (1,2)) \cdot \{C_{t_1}\}^-$$

Now, $(1 - (1,2)\{C_{t_1}\}^- = \{C_{t_2}\}^-$ and $v_1 \otimes v_2 \otimes \tilde{\omega}_1 = \tilde{\omega}_2$. This shows that j takes ε_{t_1} precisely to ε_{t_2}.

We can now complete the proof the inductive step. Suppose $e_{m,g}(S^\mu) \neq 0$, then $e_{m,g}(\varepsilon_{t_1}) \neq 0$ since this is a generator of the Specht module (and $e_{m,g}$ is a homomorphism). Then also $j \circ e_{m,g}(\varepsilon_{t_1}) \neq 0$ since j is one-to-one. This is equal to $e_{m,g} \circ j(\varepsilon_{t_1}) = e_{m,g}(\varepsilon_{t_2})$. Hence $e_{m,g}(S^{\mu+(1^2)}) \neq 0$, as required. □

References

1. S. Doty, K. Erdmann, and A. Henke, *A generic algebra associated to certain Hecke algebras*, J. Alg. **278** (2004), 502 – 531.
2. S. Doty and A. Giaquinto, *Presenting Schur algebras*, Internat. Math. Research Notices **36** (2002), 1907–1944.
3. J.A. Green, *Polynomial representations of GL_n*, Springer Lecture Notes in Mathematics, vol. 830, Springer, 1980.
4. A. Henke, *Young modules and Schur subalgebras*, Ph.d. thesis, Linacre College, University of Oxford, 1999.
5. ———, *Combinatorial results on Young modules and p-kostka numbers*, European J. Combinatorics **26** (2005), no. 6, 923 – 942.
6. J.E. Humphreys, *Introduction to Lie algebras and representation theory*, Springer, 1972.
7. G.D. James and A. Kerber, *The representation theory of the symmetric group*, Encyclopedia of mathematics and its applications, no. 16, Addison–Wesley, 1981.
8. G.D. James, *The representation theory of the symmetric groups*, Springer Lecture Notes in Mathematics, vol. 682, Springer, 1978.
9. B. Kostant, *Groups over $\mathbb{Z}$*, Proc. Symposia Pure Math. **9** (1966), 90–98.

Stephen Doty
Mathematics and Statistics
Loyola University Chicago
Chicago, Illinois 60626 USA
doty@math.luc.edu

Karin Erdmann
Dept of Mathematics
University of Oxford
OX1 3LB Oxford, UK
erdmann@math.ox.ac.uk

Anne Henke
Dept of Mathematics
University of Leicester
LE1 7RH Leicester, UK
aeh10@mcs.le.ac.uk