ENDOMORPHISM RINGS OF PERMUTATION MODULES OVER MAXIMAL YOUNG SUBGROUPS

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Abstract. Let $K$ be a field of characteristic two, and let $\lambda$ be a two-part partition of some natural number $r$. Denote the permutation module corresponding to a Young subgroup $\Sigma_\lambda$ in $\Sigma_r$ by $M_\lambda$. We construct a full set of orthogonal primitive idempotents of the centraliser subalgebra $S(\lambda) = 1_\lambda S(2, r) 1_\lambda = \text{End}_{K\Sigma_r}(M^\lambda)$ of the Schur algebra $S(2, r)$; these idempotents are naturally in one-to-one correspondence with the two-Kostka numbers.

1. Introduction

Permutation modules of symmetric groups coming from actions on set partitions are of central interest in the representation theory of symmetric groups and provide also a natural link with the representation theory of general linear groups, via Schur algebras.

Assume $K$ is a field of prime characteristic $p$. Fix natural numbers $n$ and $r$ and $n$-part partitions $\lambda$ and $\mu$ of $r$. The permutation module $M^\lambda$ over $\Sigma_r$ is the module obtained by inducing the trivial representation from the Young subgroup $\Sigma_\lambda$ to the symmetric group $\Sigma_r$. The indecomposable direct summands of $M^\lambda$ are the Young modules. By James’ submodule theorem [7, 7.1.7] there is a unique indecomposable summand of $M^\lambda$, defined to be the Young module $Y^\lambda$, which contains the Specht module $S^\lambda$. The module $M^\lambda$ is then a direct sum of Young modules $Y^\mu$, and if $Y^\mu$ occurs then $\mu \geq \lambda$.

The $p$-Kostka number $[M^\lambda : Y^\mu]$ is defined to be the multiplicity of the Young module $Y^\mu$ occurring, up to isomorphism, in a direct sum decomposition of the permutation module $M^\lambda$. Thus we have:

$$M^\lambda = \bigoplus_{\mu \geq \lambda} [M^\lambda : Y^\mu] Y^\mu$$

Let $S(n, r)$ be the Schur algebra of degree $r$, defined by

$$S(n, r) = \text{End}_{K\Sigma_r}(E^\otimes r) = \text{End}_{K\Sigma_r}(\bigoplus_{\lambda \in \Lambda(n, r)} M^\lambda)$$

where $E$ is an $n$-dimensional $K$-vector space. For the connection of Schur algebras with general linear groups, see Green [3]. The idempotent $1_\lambda \in S(n, r)$ is defined to be the
projection onto $M^\lambda$ with kernel $\bigoplus_{\mu \neq \lambda} M^\mu$. We define the centraliser subalgebra $S(\lambda)$ of $S(n, r)$ by

$$S(\lambda) = 1_{\lambda}S(n, r)1_{\lambda} \cong \text{End}_{K\Sigma_r}(M^\lambda).$$

We study these algebras when $\lambda$ is a two-part partition, that is, the associated Young subgroup is maximal. Then the ordinary character of $M^\lambda$ is multiplicity-free. Hence the algebra $S(\lambda)$ is commutative, and any given Young module $Y^\mu$ occurs at most once as a direct summand of $M^\lambda$. All idempotents of $S(\lambda)$ are central, and hence there are finitely many primitive idempotents, and they are in 1-1 correspondence with the indecomposable summands of $M^\lambda$. The blocks of $S(\lambda)$ are therefore precisely the endomorphism rings of the Young modules $Y^\mu$ which are direct summands of $M^\lambda$.

In this paper, we will construct a full set of orthogonal primitive idempotents of the algebra $S(\lambda)$ where $\lambda$ is a two-part partition and $\text{char}(K) = 2$. These idempotents are naturally in one-to-one correspondence with the 2-Kostka numbers. The philosophy is to consider an infinite family of algebras at the same time, as it was done in [1]. This is possible, by exploiting the presentation obtained in [2] of the Schur algebra in terms of the universal enveloping algebra. It allows one to keep $\lambda_1 - \lambda_2$ fixed and let $r$ vary arbitrarily. This can be thought of as an algebraic analogue of the fact that the 2-Kostka matrix is ‘quarter-infinite’. We describe the main results of this paper in more detail in the next section.

2. The Idempotent Theorem

The $p$-Kostka matrix. Let $\text{char}(K) = p$ be prime, $r$ a natural number and $\lambda = (r - k, k)$ and $\mu = (r - s, s)$ be two-part partitions. The $p$-Kostka matrix for a given parity of $r$ is a quarter-infinite matrix. Its rows are labelled by $m := r - 2k$ and its columns are labelled by $g := k - s$, and then $r$ varies over the integers $m \geq 2g$ of this parity; the $(m, g)$th entry of the matrix is given by

$$B(m, g) = \binom{r - 2s}{k - s} = \binom{m + 2g}{g}.$$

By [4, 5] we have that the Young module $Y^{(r-s,s)}$ is a direct summand of the permutation module $M^{(r-k,k)}$ if and only if $\binom{r-2s}{k-s} \neq 0$ modulo $p$.

Given a natural number $a$, write $a = \sum_i a_ip^i$ for the $p$-adic expansion of $a$. It is well-known that

$$B(m, g) \equiv \prod_i \binom{(m + 2g)i}{gi} \pmod{p}.$$

We call $\Pi_i \binom{(m+2g)i}{gi}$ modulo $p$ the binomial expansion of $B(m, g)$ and we write $B(m, g)_i$ for the $i$-th factor. Sometimes we will also write the binomial coefficient (modulo $p$) as matrix where the $i$-th column is the $i$-th factor of the product, for $i \geq 0$:

$$B(m, g) = \begin{pmatrix} (m + 2g)_0 & (m + 2g)_1 & \cdots & (m + 2g)_i & \cdots \\ g_0 & g_1 & \cdots & g_i & \cdots \end{pmatrix}.$$ 

We say $B(m, g)_i$ is zero if the $i$-th column in the matrix of $B(m, g)$ is zero.
Let $m \geq 0$. We consider the infinite family of algebras $S(\lambda)$ where $\lambda$ runs through all partitions $\lambda = (\lambda_1, \lambda_2)$ such that $\lambda_1 - \lambda_2 = m$. The presentation from [2] (see §3) provides $S(\lambda)$ with a basis $\{b(a) : 0 \leq a \leq \lambda_2\}$ with nice properties. The products $b(i)b(j)$ depend only on $m$ (and not on the degree), where terms $b(s)$ appearing with $s > \lambda_2$ are set zero.

**Construction of the idempotents.** We henceforth take $\text{char}(K) = 2$ and keep $m \geq 0$ fixed. We work in an algebra $S(\lambda)$ of large enough degree $r$ (of the right parity). For any $m \geq 0$ and $g \geq 0$ such that $B(m, g)$ is non-zero modulo 2 and the degree $r$ is large enough (that is $r \geq m + 2g$) we will now define elements in the algebra $S(\lambda)$. First we introduce two index sets: let

$$I_{m,g} := \{u : g_u = 0 \text{ and } (m + 2g)_u = 1\},$$

$$J_{m,g} := \{u : g_u = 1 \text{ and } (m + 2g)_u = 1\}.$$

Then for a natural number $t$ define elements in the algebra $S(\lambda)$ by

$$e_{m,g} := \prod_{u \in J_{m,g}} b(2^u) \prod_{u \in I_{m,g}} (1 - b(2^u)).$$

$$\quad \quad \quad (e_{m,g})_{\leq t} := \prod_{u \in J_{m,g}, u \leq t} b(2^u) \prod_{u \in I_{m,g}, u \leq t} (1 - b(2^u)).$$

**Remark.** We can associate to each factor of the binary expansion of $B(m, g)$ a factor of an element $e_{m,g}$ by the following rule:

$$\frac{B(m, g)^u}{\text{factor of } e_{m,g}} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} (1) \\ (0) \\ (1) \\ (0) \\ (0) \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

In particular, an element $e_{m,g}$ defined in this way would be zero if $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ occurs in the binary expansion of $B(m, g)$, that is if $B(m, g) = 0$ modulo two.

**The Idempotent Theorem.** Let the notation be as above. We will prove in this paper the following theorem:

**Idempotent Theorem** For any fixed $m \geq 0$, the set with elements $e_{m,g}$ with $B(m, g) \neq 0$ modulo two and $m + 2g \leq r$ is a complete set of primitive orthogonal idempotents for the algebra $S(\lambda)$.

This theorem will be proved at the end of Section 6. In fact parts of the proof of this result are not so difficult to see. Observe the following:

(i) The element $e_{m,g}$ is non-zero. By Proposition 3.6 or Lemma 3.7 one can even express it explicitly as a linear combination of the basis elements.

(ii) If $g \neq d$ and $B(m, g)$ and $B(m, d)$ are both non-zero modulo two then

$$e_{m,g}^2 = e_{m,d}^2 = 0.$$

**Proof.** Let $i$ be minimal such that $B(m, g)_i \neq B(m, d)_i$. Since columns $< i$ are the same, and both binomial coefficients are non-zero, the $i$-th columns cannot be zero: Suppose that one is zero, the other not. Then $(m + 2d)_i \neq (m + 2g)_i$. 


However, in column \( i \) the carry overs from the previous columns are the same, say \( x \), and 
\[
(m + 2d)_i = m_i + d_{i-1} + x = m_i + g_{i-1} + x = (m + 2g)_i \mod 2.
\]
Hence one of them is \( \binom{1}{1} \) and the other is \( \binom{1}{0} \). So the squares of the elements 
in the algebra have factors \( b(2^i)^2 \) and \( (1 - b(2^i)^2) \) respectively. In Section 4 we 
will show that the elements \( b(2^i)^2 \) are idempotents (see Corollary 4.2), and this 
implies that the product is zero.

Furthermore the number of primitive idempotents of \( S(\lambda) \) is equal to the number of non-
zero binomial coefficients, by [5]. So if we have established that the elements defined are 
idempotents then the theorem is proved. The latter will follow from an orthogonality 
result:

**Orthogonality Lemma.** Suppose \( B(m, g)_s \) is zero, then \( e_{m,g}^2 \cdot b(2^s)^2 = 0 \).

**Blocks of the algebra \( S(\lambda) \).** By the Idempotent Theorem, a block of the algebra \( S(\lambda) \) 
has the form \( S(\lambda)e_{m,g} \). Then this block has basis 
\[
\{e_{m,g}b(a) : a = [a_0, a_1, \ldots] \text{ where } a_s = 1 \text{ only for } (m + 2g)_s = 0\}.
\]
By Lemma 3.7, the block is (minimally) generated as an algebra by all 
\[
\{e_{m,g}b(2^s) : s \geq 0 \text{ and } (m + 2g)_s = 0\}.
\]
By the Orthogonality Lemma, the block hence has a set of generators with square zero. 
Hence for a general degree \( r \), this block is isomorphic to a quotient of an algebra of the 
form 
\[
\bigotimes K[x_i]/\langle x_i^2 \rangle.
\]
a tensor product of finitely many local 2-dimensional algebras.

**3. Basis and multiplication structure in \( S(\lambda) \)**

In this section we take \( \lambda = (\lambda_1, \lambda_2) \) to be a two-part partition and we study the multi-
plicative structure of \( S(\lambda) \) over a field \( K \) of characteristic \( p \geq 0 \). The results from this 
section will then be used to obtain in characteristic two a reduction formula for \( b(2^s)^2 \), 
see Section 4.

We describe briefly some results from [2]. Over \( \mathbb{Q} \), the Schur algebra \( S(2, r) \) is iso-
morphic to the quotient of the universal enveloping algebra \( U(gl_2) \) modulo the ideal 
generated by 
\[
H_1(H_1 - 1) \ldots (H_1 - d)
\]
Here, as a basis for the lie algebra \( gl_2 \) one takes \( e, f \) as usual and \( H_1, H_2 \) the diag-
onal matrices \( e_{11} \) and \( -e_{22} \). This quotient algebra is defined over \( \mathbb{Z} \) using the usual 
divided powers. In this presentation, the idempotent \( 1_\lambda \) (which we defined as projection 
corresponding to \( \lambda \)) is equal to the image of 
\[
1_\lambda = \begin{pmatrix} H_1 \\ \lambda_1 \end{pmatrix} \begin{pmatrix} H_2 \\ \lambda_2 \end{pmatrix}
\]
where $H_1 - H_2$ is the commutator of $e$ and $f$ in the Lie algebra. (See [1, Lemma 5.3].)

Let

$$b(i) := 1_\lambda f^{(i)} e^{(i)} 1_\lambda.$$ 

Then $S_\mathbb{Z}(\lambda)$ is the subalgebra with basis $\{b(1), b(1), \ldots, b(\lambda_2)\}$.

In [2] another basis is given, namely

$$\{b'(0), \ldots, b'(\lambda_2)\}, \quad \text{with} \quad b'(i) = 1_\lambda e^{(i)} f^{(i)} 1_\lambda.$$ 

**Remark** [2] A basis of $S_\mathbb{Z}(\lambda)$ can be labelled by the $2 \times 2$ matrices of the form

$$A(i) = \begin{pmatrix} \lambda_1 - i & i \\ i & \lambda_2 - i \end{pmatrix}$$

for $i = 0, \ldots, \lambda_2$. These are all the $2 \times 2$ matrices of nonnegative integers whose row and column sums are $\lambda$. Such matrices (also more generally) are known to label double cosets of Young subgroups in symmetric groups; this might be reassuring when working with permutation modules.

We next study the multiplicative structure of $S_\mathbb{Z}(\lambda)$. Most importantly for us is a multiplication formula for the basis elements $b(i)$, given in Proposition 3.6, which also proves again that the $b(i)$ generate a $\mathbb{Z}$-form of $S(\lambda)$.

**Proposition 3.1.** Over $\mathbb{Q}$, the algebra $S(\lambda)$ is semisimple and generated by $b(1)$ (respectively $b'(1)$).

It is enough to prove the statement about $b(1)$ since a similar argument will obtain the statement about $b'(1)$.

The proof of the proposition can be obtained by a series of lemmas, of which the first is due to Kostant [9], or see also Humphreys [6, Lemma 26.2]. Let $U(\mathfrak{gl}_2)$ be the universal enveloping algebra of the Lie algebra $\mathfrak{gl}_2$, defined over $\mathbb{Q}$.

**Lemma 3.2.** Let $a \geq 0$, $c \geq 0$ be natural numbers. Then we have in $U(\mathfrak{gl}_2)$:

$$e^{(c)} f^{(a)} = \sum_{j=0}^{\min(a,c)} f^{(a-j)} \left( h - a - c + 2j \right) e^{(c-j)}$$

where $e$, $f$ and $h$ are the standard generators in $\mathfrak{sl}_2$.

We need only the special case where $c = 1$ and $a \geq 1$, in which case we get the formula

$$ef^{(a)} = f^{(a)} e + f^{(a-1)} (h - a + 1)$$

which takes the following form if we clear denominators

$$ef^a = f^a e + af^{a-1}(h - a + 1).$$

(2)

Since these formulas hold in the enveloping algebra (over $\mathbb{Q}$), they are valid in the homomorphic image $S_\mathbb{Q}(2, r)$. The first part of the next lemma is contained in [2].

**Lemma 3.3.** In $S_\mathbb{Q}(2, r)$ we have the equality $h1_\lambda = m1_\lambda$, where $m = \lambda_1 - \lambda_2$. Moreover,

$$b(1) \cdot b(k) = (k + 1)^2 b(k + 1) + k(m + k + 1)b(k).$$
Proof. To see this, first calculate using formula (2):

\[(k!)^2 \cdot b(1) \cdot b(k) = fe^k e^k 1_\lambda \]
\[= f(f^k e + k f^{k-1}(h-k+1))e^k 1_\lambda \]
\[= f^{k+1} e^{k+1} 1_\lambda + k f^k (h-k+1)e^k 1_\lambda \]
\[= f^{k+1} e^{k+1} 1_\lambda + k f^k e^k (h+k+1) 1_\lambda \]

where we have used the fact that \(he^k = e^k (h+2k)\). This holds in the enveloping algebra of \(\mathfrak{gl}_2\), and hence is valid in \(S_Q(2, r)\). Now apply the first statement of this Lemma to obtain the desired formula.

Lemma 3.4. Let \(x = b(1)\). Then we have in \(S_Q(2, r)\) for any \(k \geq 1\) the equality

\[b(k + 1) = \frac{1}{(k+1)^2} x(x - (m + 2))(x - 2(m + 3)) \cdots (x - k(m + k + 1)).\]

Proof. To see this equation, proceed by induction on \(k\). We define

\[F_{k+1}(x) = x(x - (m + 2))(x - 2(m + 3)) \cdots (x - k(m + k + 1)).\]

The case \(k = 1\) in the preceding lemma gives the equality

\[b(2) = \frac{1}{2^2} (x^2 - (m + 2)x) = \frac{F_2(x)}{2! 2!}.\]

Thus the formula of the lemma is valid in case \(k = 1\). Assume that \(b(k) = \frac{F_k(x)}{k! k!}\). By the preceding lemma and the inductive hypothesis we then have

\[b(k + 1) = \frac{1}{(k+1)^2} \cdot (b(1)b(k) - k(m + k + 1)b(k)) \]
\[= \frac{1}{(k+1)^2} \cdot (x - k(m + k + 1))F_k(x) = \frac{1}{(k+1)^2} F_{k+1}(x).\]

The lemma is proved.

Proof of Proposition 3.1. It follows from Lemma 3.4 that we have the equality

\[b(k) = 1_\lambda f^{(k)} e^{(k)} 1_\lambda = \frac{F_k(x)}{k! k!}\]

for all \(k \geq 2\). This formula holds in \(S_Q(2, r)\) and hence any element in \(S_Q(\lambda)\) is generated by \(x = b(1)\).

Proposition 3.5. The algebra \(S_Q(\lambda)\) is isomorphic with \(\mathbb{Q}[T]/(F_{\lambda_2 + 1}(T))\).

Proof. By commutation formulas appearing in [2] we have

\[b(\lambda_2 + 1) = 1_\lambda f^{(\lambda_2 + 1)} e^{(\lambda_2 + 1)} 1_\lambda = 0 = F_{\lambda_2 + 1}(x)\]

since \(\lambda + (\lambda_2 + 1)(1, -1) = (\lambda_1 + \lambda_2 + 1, -1)\) is not a polynomial weight belonging to \(\Lambda(2, r)\), for any \(\lambda\). The proposition now follows from Lemma 3.4.
Proposition 3.6. A multiplication formula for the basis elements is given by:

\[ b(i) \cdot b(j) = \sum_{k=0}^{i} \binom{j+k}{i} \binom{j+k}{k} \binom{m+j+i}{i-k} b(j+k). \]

When \( a > \lambda_2 \) then \( b(a) \) is zero in this formula.

Proof. The proof is by induction on \( i \). The induction beginning for \( i = 1 \) is given by Lemma 3.3. Let now \( i > 1 \). Then the product \( P := b(i+1) \cdot b(j) \) equals:

\[ P = \frac{b(i)(b(1) - i(m + i + 1))}{(i + 1)^2} \cdot b(j) \]
\[ = \frac{b(1) - (j+k)(m + j + k + 1) + (j+k)(m + j + k + 1) - i(m + i + 1)}{(i + 1)^2} \cdot b(i)b(j) \]
\[ = \frac{b(1) - (j+k)(m + j + k + 1) + (j+k-i)(m + j + k + i + 1)}{(i + 1)^2} \cdot b(i)b(j) \]
\[ = \sum_{k=0}^{i} \frac{b(1) - (j+k)(m + j + k + 1)}{(i + 1)^2} \binom{j+k}{i} \binom{j+k}{k} \binom{m+i+j}{i-k} \cdot b(j+k) \]
\[ + \sum_{k=0}^{i} \frac{(j+k-i)(m + j + k + i + 1)}{(i + 1)^2} \binom{j+k}{i} \binom{j+k}{k} \binom{m+i+j}{i-k} \cdot b(j+k) \]
\[ = \sum_{k=0}^{i} \frac{k+1}{i+1} \binom{j+k+1}{i+1} \binom{j+k+1}{k+1} \binom{m+i+j}{i-k} \cdot b(j+k+1) \]
\[ + \sum_{k=0}^{i} \frac{(m+j+k+i+1)}{(i + 1)} \binom{j+k}{i} \binom{j+k}{k} \binom{m+i+j}{i-k} \cdot b(j+k) \]
\[ = \sum_{k=0}^{i+1} \frac{k}{i+1} \binom{j+k}{i+1} \binom{j+k}{k} \binom{m+i+j}{i+1-k} \cdot b(j+k) \]
\[ + \sum_{k=0}^{i} \frac{(m+j+k+i+1)}{(i + 1)} \binom{j+k}{i} \binom{j+k}{k} \binom{m+i+j}{i-k} \cdot b(j+k) \]
\[ = \sum_{k=0}^{i+1} \binom{j+k}{i} \binom{m+j+i+1}{i+1-k} \cdot b(j+k). \]
The last step puts the two sums together as follows: for $1 \leq k \leq i$ we have:

\[
\begin{align*}
\frac{k}{i+1} \left( \frac{m+i+j}{i+1-k} \right) + \frac{m+j+k+i+1}{i+1} \left( \frac{m+i+j}{i-k} \right) \\
= \frac{k}{i+1} \left( \frac{m+i+j}{i+1-k} \right) + \frac{m+j+i+1}{i+1} \left( \frac{m+i+j}{i-k} \right) \\
= \left( \frac{m+i+j}{i+1-k} \right) + \left( \frac{m+i+j}{i-k} \right) \\
= \left( \frac{m+i+1+j}{i+1-k} \right)
\end{align*}
\]

\[\square\]

With the $Z$-form provided by the previous proposition, we have $S(\lambda) = S_Z(\lambda) \otimes 1_K$, and by abuse of notation we still write $b(i)$ for $1_{\lambda^e(i)}e^{(i)}1_\lambda \otimes 1_K$. Then the multiplication formula (3) is then also valid in the $K$-algebra $S(\lambda)$; and again $b(a) = 0$ whenever $a > \lambda_2$.

**Notation.** For a field $K$ of characteristic $p$ we need the $p$-adic expansion of integers. If $n = \sum_{j=0}^{s} n_j p^j$ with $0 \leq n_j \leq p-1$ then we write $n = [n_0, n_1, \ldots, n_s]$. Moreover, we write in the following

\[A_k = \binom{j+k}{i}, \quad B_k = \binom{j+k}{k}, \quad C_k = \binom{m+j+i}{i-k}.\]

**Lemma 3.7.** Write $i = [i_0, i_1, \ldots]$ $p$-adically. Then $b(i) = \prod_{t \geq 0} b(i_t \cdot p^t)$.

**Proof.** This is shown by induction on the length $t$ of the $p$-adic decomposition of $i$. Assume that $j = [i_0, i_1, \ldots, i_{t-1}]$, then by Equation (3):

\[b(j) \cdot b(i_t p^t) = \sum_{k=0}^{j} A_k B_k C_k \cdot b(i_t p^t + k).\]

Here $A_k = \binom{i_t p^t + k}{j}$ is nonzero if and only if $j$ is $p$-contained in $k$, that is $j_s \leq k_s$ for all $s$. Hence $j \leq k$ and by assumption also $k \leq j$. So $A_k \neq 0$ precisely if $k = j$. Hence $b(j) \cdot b(i_t p^t) = b(i_t p^t + k)$. \[\square\]

**Lemma 3.8.** We define the degree of the basis element $b(i)$ to be $i$. Let $1 \leq n \leq p-1$, then

\[b(p^n I) = \prod_{k=1}^{n} \binom{k}{1}^2 b(n \cdot p^t) + \text{terms of lower degree}.\]

**Proof.** This follows by induction on $n$, using the multiplication formula given in Proposition 3.6. More precisely, let $2 \leq c \leq p-1$, then

\[b(p^c) \cdot b((c-1)p^t) = \sum_{k=0}^{p^c-1} A_k \cdot B_k \cdot C_k \cdot b((c-1)p^t + k),\]
where for $k = p^t$ we obtain

$$A_k = \binom{(c-1)p^t + k}{p^t} = \binom{cp^t}{p^t} = \binom{c}{1},$$

$$B_k = \binom{(c-1)p^t + k}{k} = \binom{cp^t}{p^t} = \binom{c}{1},$$

$$C_k = \binom{m + cp^t}{p^t - k} = \binom{m + cp^t}{0} = 1.$$

**Corollary 3.9.** Let $\lambda = (\lambda_1, \lambda_2)$ be a partition of $r$ and assume $t$ is such that $p^t \leq \lambda_2 < p^{t+1}$. Then the algebra $S(\lambda)$ is generated by the elements $b(p^0), b(p^1), \ldots, b(p^t)$.

**Proof.** We know already from Lemma 3.7 a factorisation of a basis element $b(i)$. Write $i = [i_0, i_1, \ldots]$ $p$-adically. Then

$$b(i) = \prod_{t \geq 0} b(i_t \cdot p^t).$$

Hence we need to show that the elements $b(c \cdot p^t)$ for $1 \leq c \leq p - 1$ are generated by the elements $b(p^t)$. This follows by induction on $t$ using Lemma 3.8. \qed

**Remark.** For $c \geq 2$, it seems to be rather hard to find explicit expressions for $b(cp^t)$ in terms of the generators. For $c = 2$ this is done in the next section.

**Example:** Let $m = 0$ and $p = 2$. Then the partitions we look at are the ones of the form $\lambda = (r/2, r/2)$; in this case the algebra $S(\lambda)$ has dimension $r/2 + 1$. It is generated by $b(0), \ldots, b(2^k)$ where $2^k \leq r/2 + 1 < 2^{k+1}$ subject to the relations

$$b(2^i)^2 = 0, \quad 0 \leq i \leq k;$$

$$\prod_{i \in I} b(2^i) = 0, \quad \text{whenever } I \subseteq \{0, 1, \ldots, k\} \text{ and } \sum_{i \in I} 2^i \geq r/2 + 1.$$

It follows that there are no non-zero idempotents except 1, and hence $S(\lambda)$ is indecomposable, that is, the algebra is a block.

### 4. The elements $b(i)^2$ are idempotents

From now we assume that the characteristic of the underlying field is $p = 2$. Then Lemma 3.7 shows that the basis element $b(i)$ is equal to the product of the $b(2^t)$ for which $i_t = 1$. So to understand the multiplication completely we need to understand the squares of the basis elements $b(2^t)$.

**Example.** Let $m$ be fixed with 2-adic expansion $m = [m_0, \ldots, m_t, \ldots]$. Suppose $t = 0, 1$ then we see directly from multiplication formula (3) that

$$b(2^0)^2 = m_0 \cdot b(2^0),$$

$$b(2^1)^2 = b(2^1)[m_1 \cdot 1 + m_0 \cdot b(2^0)].$$

So we can write $b(2^1)^2 = b(2^1)(m_1 + b(2^0)^2)$; and this has the following generalization, which is the reduction we had promised.
Lemma 4.1. Suppose $m = [m_0, \ldots, m_t, \ldots]$ in 2-adic expansion. Let $0 \leq v \leq t$ be maximal such that $m_{v-1} = 0$. Then

$$b(2^t)^2 = b(2^t)[m_t \cdot 1 + \sum_{i=v-1}^{t-1} b(2^i)^2],$$

setting $b(2^i) = 0$.

Proof. We make the convention that $m_i = 0$ when $i < 0$. We rewrite the product $b(2^t)^2$ using the multiplication formula given in Equation (3). Note first that the coefficient $A_k = B_k = \binom{2^t+k}{2t}$ with $k = 2^t$ is zero modulo two, and if $k < 2^t$ it is equivalent to one. Moreover for $k < 2^t$ we have

$$C_k = \binom{m + 2^{t+1}}{2^t - k} \equiv \binom{m}{2^t - k} \mod 2. \tag{4}$$

We will change variables, for this note that $2^t + k = 2^{t+1} - (2^t - k) = 2^{t+1} - l$. Hence by Equation (4) and by changing variables we can write the sum on the right-hand side in Formula (3) as

$$b(2^t)^2 = \sum_{k=0}^{2^t-1} \binom{m}{2^t - k} b(2^t + k)$$
$$= \sum_{l=1}^{2^t} \binom{m}{l} b(2^{t+1} - l)$$
$$= b(2^t)[\sum_{l=1}^{2^t} \binom{m}{l} b(2^t - l)]. \tag{5}$$

For the last equality sign note that $2^{t+1} - l = 2^t + (2^t - l)$, and so for $0 \leq 2^t - l < 2^t$ we can factorize $b(2^{t+1} - l) = b(2^t)b(2^t - l)$ by Lemma 3.7. The term with $l = 2^t$ is equal to $m_t b(0) = m_t \cdot 1$. So we can write

$$b(2^t)^2 = b(2^t)[m_t \cdot 1 + \Gamma(t)] \quad \text{where} \quad \Gamma(t) := \sum_{l=1}^{2^t-1} \binom{m}{l} b(2^t - l). \tag{6}$$

We will now prove a recursion formula for $\Gamma(t)$. We claim that

$$\Gamma(1) = b(2^0)^2,$$
$$\Gamma(t) = b(2^{t-1})^2 + m_{t-1} \Gamma(t-1) \quad \text{for} \ t \geq 2. \tag{7}$$
First, $\Gamma(1) = \binom{m}{1}b(2^0) = m_0b(2^0) = b(2^0)^2$. Suppose that $t \geq 2$, then we split $\Gamma(t)$ into two sums:

$$\Gamma(t) = \sum_{l=1}^{2^t-1} \binom{m}{l}b(2^t - l) + \sum_{l=2^t-1+1}^{2^t-1} \binom{m}{l}b(2^t - l)$$

by definition of $\Gamma(t)$,

$$= b(2^{t-1})^2 + \sum_{l=2^t-1+1}^{2^t-1} \binom{m}{l}b(2^t - l)$$

by Equation (5), Lemma 3.7,

$$= b(2^{t-1})^2 + m_{t-1} \sum_{r=1}^{2^t-1} \binom{m}{r}b(2^{t-1} - r)$$

$$= b(2^{t-1})^2 + m_{t-1}\Gamma(t-1)$$

by definition of $\Gamma(t-1)$.

For the third equality sign in the latter equation, write $l = 2^t - 1 + r$ where $1 \leq r \leq 2^t - 1 - 1$, and note that

$$\binom{m}{2^t - 1 + r} \equiv m_{t-1}\binom{m}{r} \mod 2,$$

and $2^t - l = 2^{t-1} - r$. Hence the recursion formula for $\Gamma(t)$ claimed in Equation (7) is shown. It in fact implies the closed formula for $\Gamma(t)$:

$$\Gamma(t) = \sum_{i=v-1}^{t-1} b(2^i)^2$$

where $v$ is as in the statement. Substituting this into (6) completes the proof. □

**Corollary 4.2.** For $i \geq 0$, the elements $b(2^i)^2$ are idempotent. Moreover, if $m_j = 0$ for all $j \leq i$ then $b(2^i)^2 = 0$.

*Proof. This follows from Lemma 4.1 by induction.* □

**5. Analysis of the binomial coefficient $B(m, g)$**

We assume throughout that $m$ and $g$ are integers such that the binomial coefficient $B(m, g)$ is non-zero modulo two. We need to relate the binomial expansion of $m$ with that of $B(m, g)$. Note that

$$\begin{array}{c|ccccccc}
 m & m_0 & m_1 & m_2 & \ldots & m_i & \ldots \\
 +2g & 0 & g_0 & g_1 & \ldots & g_{i-1} & \ldots \\
 m+2g & (m+2g)_0 & (m+2g)_1 & (m+2g)_2 & \ldots & (m+2g)_i & \ldots \\
\end{array}$$

In this addition, we need to keep track over the 'carry'overs'. So define integers $x_i \geq 0$ such that

$$m_i + g_{i-1} + x_{i-1} = (m + 2g)_i + 2x_i.$$

So $x_i$ is the carry over from column $i$ to column $i+1$ in the addition of $m$ and $2g$. Most important for the proofs later will be that $(m + 2g)_i = 1$ implies that $x_i = 0$; more precisely we have the following:
Proposition 5.1. Let \( m = [m_0, m_1, \ldots] \) and \( g = [g_0, g_1, \ldots] \) be in binary expansion. Assume that \( B(m, g) \) is non-zero. Then \( (m + 2g)_i + 2x_i < 3 \) for all \( i \). In particular, if \( (m + 2g)_i = 1 \) then \( x_i = 0 \).

Proof. Certainly \((m + 2g)_i + 2x_i \leq 3\). Assume for a contradiction that this number is equal to three for some \( i \). Then \( x_{i-1} = g_{i-1} = 1 \). Since \( g_{i-1} = 1 \) we must have that \((m + 2g)_{i-1} = 1\) as well, since otherwise the binomial coefficient \( B(m, g) \) would be zero. But then it follows that \( m_{i-1} + g_{i-2} + x_{i-2} = 3 \), and then repeating the argument gives \( m_1 + g_0 + x_0 = 3 \). This implies \( x_0 = 1 \). On the other hand, \((m + 2g)_0 = m_0\) and hence \( x_0 = 0 \), a contradiction. \(\square\)

We will later prove some properties by induction. The elements \( e_{m,g} \) are defined as products, and it will be convenient to use factors of these which are already known to be idempotents. The basis for the induction will be the following:

Lemma 5.2 (Splitting Lemma). Let \( u \) be a natural number and define

\[
n := [m_0, m_1, \ldots, m_u] \quad \text{and} \quad d := [g_0, g_1, \ldots, g_{u-1}].
\]

Suppose \( (m + 2g)_u = 1 \). Then the binary expansion of \( B(n, d) \) equals the binary expansion of \( B(m, g)_{<u} \) extended by one column \( \binom{1}{0} \). In particular if \( g_u = 0 \) then \( B(n, d) = B(m, g)_{\leq u} \).

Proof. By Proposition 5.1 we know that \( x_u = 0 \), and by Equation (8) we hence have \( m_u + g_{u-1} + x_{u-1} = 1 \); the claim follows. \(\square\)

Remark. The Splitting Lemma shows that when \( g_u = 0 \) then the element \( e_{n,d} \) is a factor of \( e_{m,g} \), when written as in the definition, see Equation (1).

We will have to use the formula from Lemma 4.1. So we need to know the digits of \( B(m, g) \), given the binary expansion of \( m \) and of \( g \). In the remainder of this section we describe these explicitly.

Lemma 5.3. Given natural numbers \( t \) and \( a \). Suppose \( B(m, g)_{\leq t+a} \) in binary decomposition is of the form

\[
B(m, g)_{\leq t+a} = \begin{pmatrix}
\ldots & 1 & 0 & \ldots & 0 \\
\ldots & g_t & 0 & \ldots & 0
\end{pmatrix}.
\]

Then we have:

(a) Suppose \( g_t = 0 \), then \( m_{t+1} = \ldots = m_{t+a} = 0 \) and \( x_{t+1} = \ldots = x_{t+a} = 0 \).

(b) Suppose \( g_t = 1 \), then \( m_{t+1} = \ldots = m_{t+a} = 1 \) and \( x_{t+1} = \ldots = x_{t+a} = 1 \).

Proof. By Proposition 5.1 we know that \( x_t = 0 \). By Equation (8) we have:

\[
\begin{align*}
m_{t+1} + g_t + 0 & = 0 + 2x_{t+1}, \\
m_{t+2} + 0 + x_{t+1} & = 0 + 2x_{t+2}, \\
& \quad \ldots \\
m_{t+a} + 0 + x_{t+a-1} & = 0 + 2x_{t+a}.
\end{align*}
\]
For (a), assume that \( g_i = 0 \). Then \( x_{t+1} = 0 \) and hence \( m_{t+1} = 0 \). Now the second equation shows that \( x_{t+2} = 0 \) and hence \( m_{t+2} = 0 \), and so on. Part (b) is similar. □

We will have to consider sequences of digits such that \( m_i = 1 \) for \( v \leq i \leq s \) and \( m_{v-1} = 0 \). For these values of \( i \) we need to know the \( i \)-th columns of \( B(m, g) \).

**Lemma 5.4.** Suppose column \( s \) of \( B(m, g) \) is zero but column \( s - 1 \) is non-zero. Let \( u \geq 0 \) be minimal such that \( (m + 2g)_i = 1 \) for \( u \leq i < s \), and let \( 0 \leq v \leq s \) be maximal with \( m_{v-1} = 0 \). Then \( v \geq u \). Moreover:

(a) If \( m_s = 0 \) then \( g_i = 0 \) for \( v - 1 \leq i \leq s - 1 \).

(b) If \( m_s = 1 \) then \( g_{s-1} = 1 \) and \( g_i = 0 \) for \( v - 1 \leq i < s - 1 \).

**Proof.** (i) Suppose \( m_u = 0 \) or \( u = 0 \). Then by definition of \( v \) we have that \( v \geq u \).

So assume that \( m_u = 1 \) and \( u > 0 \). By definition of \( u \) we have that \( (m + 2g)_u = 1 \) and \( (m + 2g)_{u-1} = 0 \). Then Equation (8) for columns \( u \) and \( u - 1 \) together with the assumptions and Proposition 5.1 read:

\[
\begin{align*}
1 + g_{u-1} + x_{u-1} & = 1, \\
m_{u-1} + g_{u-2} + x_{u-2} & = 0 + 2x_{u-1}.
\end{align*}
\]

So \( x_{u-1} = 0 = g_{u-1} \) which implies that \( m_{u-1} + g_{u-2} + x_{u-2} = 0 \) and hence \( m_{u-1} = 0 \). This shows that \( u \leq v \).

(ii) For (a) and (b), use Proposition 5.1 and Equation (8) for columns between \( v \) and \( s - 1 \). By assumption and (i) we have that \( (m + 2g)_i = 1 = m_i \) for \( v \leq i \leq s - 1 \). This implies \( g_{i-1} = 0 \) for \( v \leq i \leq s - 1 \) and \( x_{s-1} = 0 \). Then Equation (8) for column \( s \) becomes

\[
m_s + g_{s-1} + 0 = 0 + 2x_s.
\]

If \( m_s = 0 \) then \( x_s = 0 \) and \( g_{s-1} = 0 \). On the other hand if \( m_s = 1 \) then \( x_s = 1 \) and \( g_{s-1} = 1 \). □

6. The proofs of the Orthogonality Lemma and the Idempotent theorem

6.1. **Proof of the Orthogonality Lemma.** Suppose the \( s \)-th column of \( B(m, g) \) is zero. The aim is to show that \( e_{m,g}^2 \cdot b(2^s)^2 = 0 \). Recall from Lemma 4.1 that \( b(2^s)^2 = b(2^s)\psi \) with

\[
(10) \quad \psi = \psi_{m,s} = m_s + \sum_{i=v-1}^{s-1} b(2^i)^2
\]

where \( 0 \leq v \leq s \) is maximal such that \( m_{v-1} = 0 \). We will prove that

\[
(11) \quad (e_{m,g})_{v<s}^2 \cdot \psi_{m,s} = 0.
\]

Certainly this then implies the Orthogonality Lemma in Section 2. Note that if \( s = 0 \) then \( \psi = m_0 = 0 \) since \( (m + 2g)_0 = m_0 = 0 \). So assume \( s > 0 \). If all columns before column \( s \) are zero then \( m_i = 0 \) for \( i \leq s \) and then \( \psi = 0 \) by Corollary 4.2. So assume now that \( w < s \) is such that \( (m + 2g)_w = 1 \) and \( (m + 2g)_i = 0 \) for \( w + 1 \leq i \leq s \). We use induction on the number of zero columns between \( w \) and \( s \) to prove Equation (11).
Start of the induction: Suppose column $s - 1$ is non-zero. Let $u \geq 0$ be minimal such that $(m + 2g)_i = 1$ for $u \leq i < s$. We apply Lemma 5.4, which shows that $v \geq u$. Moreover, suppose $m_s = 0$, then by part (a) of the Lemma we know that $(e_{m,g})_{\leq s}$ has factors $(1 - b(2^i))$ for $v - 1 \leq i \leq s - 1$. This gives that $(e_{m,g})_{<s}^2 \cdot \psi = 0$ by Corollary 4.2.

Similarly, if $m_s = 1$ then part (b) of the Lemma shows that $(e_{m,g})_{<s}$ has factors $(1 - b(2^i))$ for $v - 1 \leq i < s - 1$ and also a factor $b(2^{s-1})$. Then the claim follows again from Corollary 4.2, using that $b(2^{s-1}) \cdot (m_s + b(2^{s-1})) = 0$.

Inductive step: Suppose now that column $s - 1$ is zero. The inductive hypothesis states that

$$(e_{m,g})_{<s-1}^2 \cdot \psi_{m,s-1} = 0.$$ 

If $g_w = 0$ then we have by Lemma 5.3 that $m_i = 0$ for $w + 1 \leq i \leq s$. Then $v = s$ and we can write

$$\psi_{m,s} = b(2^{s-1})^2 = \psi_{m,s-1} \cdot b(2^{s-1}),$$

using Lemma 4.1. By the inductive hypothesis we deduce $(e_{m,g})_{<s} \cdot \psi_{m,s} = 0$. Now suppose $g_w = 1$, then by Lemma 5.3 we know that $m_i = 1$ for $w + 1 \leq i \leq s$. We rewrite and again use Lemma 4.1:

$$\psi_{m,s} = \psi_{m,s-1} + b(2^{s-1})^2 = \psi_{m,s-1} + \psi_{m,s-1} \cdot b(2^{s-1}),$$

and again using the inductive hypothesis we have $(e_{m,g})_{<s} \cdot \psi_{m,s} = 0$. This completes the proof of the Orthogonality Lemma.

6.2. Proof of the Idempotent Theorem. This will be done by induction on $t$, the largest column label of a non-zero column in the binary decomposition of $B(m, g)$, which we call the degree of $e_{m,g}$. In fact, we will prove the following:

Claim: Elements $e_{m,g}$ and $(e_{m,g})_{<t}$ are idempotents.

Assume that $t = 0$ then $B(m, g) = \{0\}$. In particular $m_0 = 1$ and so $e_{m,g} = (1 - b(2^0))$ is idempotent. Also $(e_{m,g})_{<t} = 1$ is idempotent. We assume the statement holds for all $e_{n,d}$ of degree $< t$. Let $e_{m,g}$ be of degree $t$ and write $e := e_{m,g} = P \cdot (1 - b(2^t))$ where $P = (e_{m,g})_{<t}$. We have

$$e^2 = P^2(1 - b(2^t))^2 = P^2(1 - \psi b(2^t))$$

where $\psi = \psi_{m,g}$ is defined as in Equation (10). We will show that $P^2 \cdot \psi = P^2$, and secondly that $P^2 = P$. This then implies that $e = e_{m,g}$ is idempotent.

(a) We claim that $P^2 \cdot \psi = P^2$, that is $P^2(1 - \psi) = 0$. To see this, let $\tilde{m} := m + 2^t$, then $\tilde{m}_i = 1 + m_i$ and $\tilde{m}_i = m_i$ for $i < t$. Hence $B(\tilde{m}, g)$ differs from $B(m, g)$ in columns $t$ and $t + 1$. Therefore

$$(e_{m,g})_{<t} = (e_{\tilde{m},g})_{<t} = P.$$

Moreover (using $p = 2$) we have $\psi_{\tilde{m},t} = 1 - \psi_{m,t}$. So we get from the Orthogonality Lemma, see Equation (11):

$$P^2(1 - \psi_{m,t}) = (e_{\tilde{m},g})_{<t}^2 \cdot \psi_{\tilde{m},t} = 0.$$
(b) We claim that \( P^2 = P \). This is clear if \( P = 1 \). So suppose \( P > 1 \), then there is some \( u < t \) maximal such that \((m + 2g)u = 1\). If \( g_u = 0 \) then \( P = e_{n,d} \) with \( d \) and \( n \) as in the Splitting Lemma 5.2. Hence by the inductive hypothesis \( P \) is idempotent. If \( g_u = 1 \), then \( P = (e_{m,g})_{<u} \cdot b(2^u) \). Define \( n \) and \( d \) by
\[
(12) \quad e_{n,d} = (e_{m,g})_{<u} \cdot (1 - b(2^u)).
\]
By construction \( e_{n,d} \) has degree \( u < t \) and hence by the inductive hypothesis we get that \( e_{n,d} \) and \((e_{m,g})_{<u}\) are idempotents. Since the characteristic of the underlying field is two and by Equation (12), we have that \((e_{m,g})_{<t} = P = (e_{m,g})_{<u} \cdot b(2^u) = e_{n,d} + (e_{m,g})_{<u} \) is idempotent.

7. The correspondence between idempotents and Young modules.

Fix an integer \( g \geq 0 \) such that \((m+2g) \neq 0\). Then we have for each \( r \geq g \) of the right parity a partition \( \lambda \) with \( \lambda_1 = \lambda_2 = m \), and a partition \( \mu = (\mu_1, \mu_2) \) with \( \mu_1 - \mu_2 = m + 2g \). We also have the primitive idempotent \( e_{m,g} \) and we know that \( Y^\mu \) is a direct summand of \( M^\lambda \). We will now show that in fact \( e_{m,g} \) is the projection of \( M^\lambda \) corresponding to \( Y^\mu \).

Theorem 7.1. Let \( \lambda, \mu \) be two-part partitions such that \( Y^\mu \) is a direct summand of \( M^\lambda \). Let \( \lambda_1 - \lambda_2 = m, \mu_1 - \mu_2 = m + 2g \) and \( g = \lambda_2 - \mu_2 \). Then the idempotent \( e_{m,g} \) of \( S(\lambda) \) is the projection onto \( Y^\mu \). 

The proof of this will take the rest of the chapter. We use induction on \( r \), starting with the case \( \mu_2 = 0 \), that is \( \mu = (r,0) \). Then the inductive step will be to show that if the theorem is true for degree \( r \) then it is true for degree \( r + 2 \).

Suppose first that \( \mu_2 = 0 \). In the special case when \( \lambda = \mu \) we have \( g = 0 \) and \( m = r \). So \( \lambda_2 = 0 \) and the algebra \( S(\lambda) \) has dimension 1. Furthermore, \( e_{m,0} = 1 \) and \( M^\lambda = Y^\lambda \), so the theorem is trivially true.

So suppose now that \( \mu > \lambda \). We have then \( r = \mu_1 \) and \( \mu_2 = 0 \). By the previous case, applied to \( \mu \), we know that \( e_{r,0} \in S(\mu) \) is the projection corresponding to the summand \( Y^\mu \) of \( M^\mu \). Both idempotents \( e_{m,g} \) and \( e_{r,0} \) lie in \( S(2,r) \). To show that the summand of \( M^\lambda \) corresponding to the projection \( e_{m,g} \) is isomorphic to \( Y^\mu \) we must show that the idempotents \( e_{m,g} \) and \( e_{r,0} \) are associated in \( S(2,r) \).

Proposition 7.2. The idempotents \( e_{m,g} \) and \( e_{r,0} \) are associated in \( S(2,r) \). Hence the \( e_{m,g}M^\lambda \) of \( M^\lambda \) is isomorphic to \( Y^\mu \).

Proof. (a) We first simplify the expressions for the two idempotents. Note that
\[
e_{m,g} = \prod_{u \in I_{m,g}} b(2^u) \cdot \prod_{u \in I_{m,g}} (1 - b(2^u)) \quad \text{by Equation (1),}
\]
\[
= b(g) \cdot \prod_{u \in I_{m,g}} (1 - b(2^u)) \quad \text{by Lemma 3.7,}
\]
\[
= b(g) \cdot (1 \pm \text{products of } b(i) \text{'s})
\]
\[
= b(g)
\]
where this last equality follows as the algebra $S(\lambda)$ has basis $\{b(0), b(1), \ldots, b(g)\}$ and by using Lemma 3.7. Moreover, as $M^{(r,0)} = Y^{(r,0)}$, we have $e_{r,0} = 1_{(r,0)}$.

(b) Let $\alpha = (1, -1)$ and recall from [2], Theorem 2.4) that for any partition $\nu$ we have

$$e \cdot 1_{\nu} = \begin{cases} 1_{\nu + \alpha} \cdot e & \text{if } \nu + \alpha \text{ is a partition}, \\ 0 & \text{otherwise}, \end{cases}$$

and

$$f \cdot 1_{\nu} = \begin{cases} 1_{\nu - \alpha} \cdot e & \text{if } \nu - \alpha \text{ is a partition}, \\ 0 & \text{otherwise}, \end{cases}$$

Moreover, by [2], Proposition 4.3 we have that $h \cdot 1_{(r,0)} = \lambda \cdot 1_{(r,0)}$ for $i = 1, 2$, and recall that $h = H_1 - H_2$. These formulas imply that $e \cdot 1_{(r,0)} = 0$ as $(r, 0) + \alpha$ is not a partition. Moreover, with $\lambda = (g + m, g)$ a partition of $r = m + 2g$ we have

$$e^{(g)} \cdot 1_{\lambda} = 1_{(r,0)} \cdot e^{(g)}, \quad 1_{(r,0)} \cdot f^{(g)} = f^{(g)} \cdot 1_{\lambda}, \quad (\frac{h}{g}) \cdot 1_{(r,0)} = (\frac{r}{g}) \cdot 1_{(r,0)}.$$

(c) We next give elements $u$ and $v$ in the Schur algebra $S(2, r)$ such that $e_{m,g} = uv$ and $e_{r,0} = vu$, proving that the two idempotents are associated. More precisely, let

$$u = 1_{(r,0)} f^{(g)} \lambda \quad \text{and} \quad v = 1_{(r,0)} e^{(g)} \lambda.$$

Then by repeated use of the Equation in (b) we have

$$u \cdot v = 1_{(r,0)} f^{(g)} \lambda e^{(g)} \lambda = 1_{(r,0)} f^{(g)} e^{(g)} \lambda = b(g)$$

and

$$v \cdot u = 1_{(r,0)} e^{(g)} \lambda f^{(g)} \lambda = 1_{(r,0)} e^{(g)} f^{(g)} \lambda = 1_{(r,0)} \cdot \left( \sum_{j=0}^{g} f^{(g-j)} \left( \frac{h - 2g + 2j}{j} \right) e^{(g-j)} \right) \cdot 1_{(r,0)}$$

$$= 1_{(r,0)} \cdot \left[ f^{(0)} \left( \frac{h}{g} \right) e^{(0)} \right] \cdot 1_{(r,0)}$$

$$= \left( \frac{r}{g} \right) \cdot 1_{(r,0)} = B(m, g) \cdot 1_{(r,0)} = 1_{(r,0)}$$

modulo two. Hence $e_{m,g} = b(g)$ and $e_{r,0} = 1_{(r,0)}$ are associated. \hfill \Box

Now it remains to deal with the inductive step, that is to compare $M^\lambda$ and $M^{\lambda + (1^2)}$. To do so, we will first analyze more closely how the hyperalgebra actions on $E^{\otimes r}$ and $E^{\otimes r+2}$ are related.

We fix a basis $\{v_1, v_2\}$ of the $K$-vector space $E$. We write briefly $v_i$ for the tensor product $v_{i_1} \otimes v_{i_2} \otimes \ldots \otimes v_{i_r}$, with $i$ the multi-index $i = (i_1, \ldots, i_r)$. Define the linear map

$$j : E^{\otimes r} \longrightarrow E^{\otimes r+2} \text{ by } x \mapsto (v_1 \otimes v_2 - v_2 \otimes v_1) \otimes x.$$ 

Recall that both tensor powers are modules for the hyperalgebra $U_K = U(\mathfrak{gl}_2) \otimes K$. The map $j$ commutes with the action of the divided powers $e^{(a)}$, $f^{(a)} \in U_K$: this is
Suppose $\varepsilon - \eta$. Now, (1) following.

§ end of tableaux of shapes $\mu$ with standard tableaux of shapes $\lambda$ and $1_{\lambda+1(2)}$ are the projections onto these spaces, and it follows that $j$ intertwines the actions of elements $b(a)$ on $M^\lambda$ and on $M^{\lambda+1(2)}$. In particular this implies

$$j(e_{m,g}x) = e_{m,g}j(x), \quad \text{for all } x \in M^\lambda.$$  

**Proposition 7.3.** Suppose $e_{m,g}$ is the projection on $M^\lambda$ corresponding to $Y^\mu$. Then $e_{m,g}$ on $M^{\lambda+1(2)}$ is the projection corresponding to $Y^{\mu+1(2)}$.

**Proof.** We may assume $m \neq 0$; the case $m = 0$ is understood, see the example at the end of §3. We know that the Specht module $S^\mu$ is a submodule of $Y^\mu$. Furthermore, $\text{Hom}_{K\Sigma_\nu}(S^\mu, M^\lambda)$ is one-dimensional (see [8, 13.13]). So $M^\lambda$ has a unique submodule isomorphic to $Y^\mu$, which is contained in $Y^\mu$. Similarly $M^{\lambda+1(2)}$ has a unique submodule isomorphic to $S^{\mu+1(2)}$ and it is contained in $Y^{\mu+1(2)}$. It suffices therefore to show the following.

If $e_{m,g}(S^\mu) \neq 0$ in $M^\lambda$ then $e_{m,g}(S^{\mu+1(2)}) \neq 0$ in $M^{\lambda+1(2)}$.

To do so we use polynomials, that is the standard generators for Specht modules. Start with standard tableaux of shapes $\mu$ and $\mu + (1^2)$ respectively, we take them as follows.

$$t_1 = \frac{3\ 5\ ...\ (2u - 1)\ (2u + 1)\ ...\ (r + 2)}{4\ 6\ ...\ (2u)}, \quad t_2 = \frac{1\ 3\ ...\ (2u - 1)\ (2u + 1)\ ...\ (r + 2)}{2\ 4\ ...\ (2u)}.$$

(Here $u = \mu_2 + 1$). Let $R_{t_1}$ be the row stabilizer of $t_1$, and $C_{t_1}$ the column stabilizer of $t_1$. To write down the polytabloid for $S^\mu$ in this setup, we must start with an appropriate element $\omega_1 \in M^\lambda$ which is fixed by all elements of $R_{t_1}$ and then the polytabloid is

$$\varepsilon_{t_1} = \omega_1 \{C_{t_1}\}^-$$

where $\{C_{t_1}\}^-$ is the alternating sum over all elements in $C_{t_1}$. We can take

$$\omega_1 = \sum v_i$$

summing over all $i$ such that $i_j = 2$ for $j$ in the second row of $t_1$. (Note that $\lambda_2 \geq \mu_2$, so this exists. When $\lambda = \mu$ it is just one basis vector.) Similarly one defines the Specht module generator $\varepsilon_{t_2}$ from $t_2$.

Explicitly,

$$\{C_{t_1}\}^- = (1 - (3, 4))(1 - (5, 6)) \cdots (1 - (2u - 1, 2u))$$

This shows that $\omega_1 \{C_{t_1}\}^- = \tilde{\omega}_1 \{C_{t_1}\}^-$ where $\tilde{\omega}_1$ is the sum over all $v_i$ such that $i_{2t+1} = 1$ and $i_{2t+2} = 2$ for $1 \leq t < u$; which is visibly identifable with the generator in [7]. We apply the map $j$ to $\varepsilon_{t_1}$,

$$j(\varepsilon_{t_1}) = (v_1 \otimes v_2 - v_2 \otimes v_1) \otimes \varepsilon_1 = (v_1 \otimes v_2 \otimes \tilde{\omega}_1 (1 - (1, 2)) \cdot \{C_{t_1}\}^-$$

Now, $(1 - (1, 2) \{C_{t_1}\}^- = \{C_{t_2}\}^-$ and $v_1 \otimes v_2 \otimes \tilde{\omega}_1 = \tilde{\omega}_2$. This shows that $j$ takes $\varepsilon_{t_1}$ precisely to $\varepsilon_{t_2}$.
We can now complete the proof the inductive step. Suppose $e_{m,g}(S^\mu) \neq 0$, then $e_{m,g}(\varepsilon_{t_1}) \neq 0$ since this is a generator of the Specht module (and $e_{m,g}$ is a homomorphism). Then also $j \circ e_{m,g}(\varepsilon_{t_1}) \neq 0$ since $j$ is one-to-one. This is equal to $e_{m,g} \circ j(\varepsilon_{t_1}) = e_{m,g}(\varepsilon_{t_2})$. Hence $e_{m,g}(S^{\mu+(1^2)}) \neq 0$, as required. 

REFERENCES

5. ___, Combinatorial results on Young modules and $p$-kostka numbers, European J. Combinatorics 26 (2005), no. 6, 923 – 942.