

# A variant of the induction theorem for Springer representations

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**ABSTRACT.** Let  $G$  be a simple algebraic group over  $\mathbf{C}$  with the Weyl group  $W$ . For a unipotent element  $u \in G$ , let  $\mathcal{B}_u$  be the variety of Borel subgroups of  $G$  containing  $u$ . Let  $L$  be a Levi subgroup of a parabolic subgroup of  $G$  with the Weyl subgroup  $W_L$  of  $W$ . Assume that  $u \in L$  and let  $\mathcal{B}_u^L$  be a similar variety as  $\mathcal{B}_u$  for  $L$ . For a certain choice of  $L$ ,  $u \in L$  and  $e \geq 1$ , we describe the  $W$ -modules  $\bigoplus_{n \equiv k \pmod{e}} H^{2n}(\mathcal{B}_u)$  for  $k = 0, \dots, e-1$ , in terms of the  $W_L$ -module  $H^*(\mathcal{B}_u^L)$  with some additional data, which is a refinement of the induction theorem due to Lusztig. As an application, we give an explicit formula for the values of Green functions at root of unity, in the case where  $u$  is a regular unipotent element in  $L$ .

## 0. INTRODUCTION

Let  $G$  be a connected reductive group over an algebraically closed field  $k$ , and  $W$  the Weyl group of  $G$ . For a unipotent element  $u \in G$ , let  $\mathcal{B}_u$  be the variety of Borel subgroups containing  $u$ . According to Springer [Sp2], Lusztig [L1],  $W$  acts naturally on the  $l$ -adic cohomology group  $H^n(\mathcal{B}_u) = H^n(\mathcal{B}_u, \mathbf{Q}_l)$ , the so-called Springer representations of  $W$ . Assume that  $k = \mathbf{C}$ , or the characteristic  $p$  of  $k$  is good. Then it is known that  $H^{\text{odd}}(\mathcal{B}_u) = 0$ . We consider the graded  $W$ -module  $H^*(\mathcal{B}_u) = \bigoplus_{n \geq 0} H^{2n}(\mathcal{B}_u)$ . Let  $L$  be a Levi subgroup of a parabolic subgroup of  $G$ . Let  $W_L$  be the Weyl group of  $L$ , which is naturally a subgroup of  $W$ . If  $u \in L$ , the variety  $\mathcal{B}_u^L$  is defined by replacing  $G$  by  $L$ , and we have a graded  $W_L$ -module  $H^*(\mathcal{B}_u^L)$ .

Lusztig proved in [L3] an induction theorem for Springer representations, which describes the  $W$ -module structure of  $H^*(\mathcal{B}_u)$  in terms of the  $W_L$ -module structure of  $H^*(\mathcal{B}_u^L)$ , in the case where  $u \in L$ . However in this theorem, the information on the graded  $W$ -module structure is eliminated. In this paper, we try to recover partly the graded  $W$ -module structure, i.e., for a fixed positive integer  $e$ , we consider the  $W$ -modules  $V_{e,k} = \bigoplus_{n \equiv k \pmod{e}} H^{2n}(\mathcal{B}_u)$  for  $k = 0, \dots, e-1$ . Let  $G$  be a simple group modulo center defined over  $\mathbf{C}$ . We show, under a certain choice of  $L$ ,  $u$  and  $e$ , that the  $W$ -module  $V_{e,k}$  can be described in terms of the graded  $W_L$ -module  $H^*(\mathcal{B}_u^L)$ .

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with some additional data. In particular, we see that  $\dim V_{e,k}$  is independent of the choice of  $k$ .

In the case where  $u = 1$ ,  $H^*(\mathcal{B}_u)$  is isomorphic, as a graded  $W$ -module, to the coinvariant algebra of  $W$ . In this case  $V_{e,k}$  has been studied by many authors, by Stembridge [St] for  $e$  corresponding to the regular elements in  $W$ , by Morita and Nakajima [MN1] for  $W = \mathfrak{S}_n$  with  $e$  such that  $1 \leq e \leq n$ , and by Bonnafé, Lehrer and Michel [BLM] for complex reflection groups  $W$  in the most general framework. Our result partly covers the result of [BLM]. For general  $u \neq 1$ , Morita and Nakajima [MN2] considered certain types of unipotent elements for  $G = GL_n$ , which is a special case of ours.

The proof of the induction theorem in [L3] is done by passing to the finite field  $\mathbf{F}_q$ , and using a certain specialization argument  $q \mapsto 1$  together with the properties of Deligne-Lusztig's virtual character  $R_T(1)$ . Our argument is a variant of that in [L3]. We use a specialization  $q \mapsto \zeta$ , where  $\zeta$  is a primitive  $e$ -th root of unity. Thus our argument is closely related to the values of Green functions at root of unity. In the case where  $u$  is a regular unipotent element in  $L$ , we obtain an explicit formula for such values, which is regarded as a generalization of the result by Lascoux, Leclerc and Thibon [LLT] for the case of Green polynomials of  $GL_n$ .

## 1. THE STATEMENT OF THE MAIN RESULT

**1.1.** Let  $k$  be an algebraic closure of a finite field with  $\text{ch}(k) = p > 0$  or the complex number field  $\mathbf{C}$ . Let  $G$  be a connected reductive group  $G$  over  $k$ . Let  $\mathcal{B}$  be the variety of Borel subgroups of  $G$ , and  $W$  the Weyl group of  $G$ . For any  $g \in G$ , put  $\mathcal{B}_g = \{B' \in \mathcal{B} \mid g \in B'\}$ . We consider the Springer representations of  $W$  on  $H^n(\mathcal{B}_g, \bar{\mathbf{Q}}_l)$  (or on  $H^n(\mathcal{B}_g, \mathbf{C})$  in the case where  $k = \mathbf{C}$ ).

Let  $L$  be a Levi subgroup of a parabolic subgroup  $P$  of  $G$ . The Weyl group  $W_L$  of  $L$  is naturally identified with a subgroup of  $W$ . Let  $\mathcal{B}^L$  be the variety of Borel subgroups of  $L$ . For a unipotent element  $u \in L$ , we consider  $\mathcal{B}_u^L = \{B' \in \mathcal{B}^L \mid u \in B'\}$ . Thus we have a  $W_L$ -module  $H^n(\mathcal{B}_u^L, \bar{\mathbf{Q}}_l)$ , and a  $W$ -module  $H^n(\mathcal{B}_u, \bar{\mathbf{Q}}_l)$ . The induction theorem for Springer representations asserts that

$$(1.1.1) \quad \sum_{n \geq 0} (-1)^n H^n(\mathcal{B}_u, \bar{\mathbf{Q}}_l) = \text{Ind}_{W_L}^W \left( \sum_{n \geq 0} (-1)^n H^n(\mathcal{B}_u^L, \bar{\mathbf{Q}}_l) \right)$$

as virtual  $W$ -modules.

**Remark 1.2.** The induction theorem was stated in [AL], with a brief indication of the proof, in the case where  $k = \mathbf{C}$ , and was proved in [L3] for any  $k$ . Note that if  $p$  is good, the unipotent classes in  $G$  are parametrized in the same way as the case of  $k = \mathbf{C}$ , independent of  $p$ . Moreover in that case, it is known that  $H^n(\mathcal{B}_u, \bar{\mathbf{Q}}_l) = 0$  for odd  $n$ . Then the algorithm of computing Green functions implies that the  $W$ -module structure of  $H^n(\mathcal{B}_u, \bar{\mathbf{Q}}_l)$  is independent of  $p$ . Thus by a general principle  $H^n(\mathcal{B}_u, \bar{\mathbf{Q}}_l)$  is isomorphic to the  $W$ -module  $H^n(\mathcal{B}_{u'}, \mathbf{C})$ , where  $u', \mathcal{B}_{u'}$  are the corresponding objects in the algebraic group  $G_{\mathbf{C}}$  over  $\mathbf{C}$ . In what follows, we express  $H^n(\mathcal{B}_u, \bar{\mathbf{Q}}_l)$  or  $H^n(\mathcal{B}_{u'}, \mathbf{C})$  by  $H^n(\mathcal{B}_u)$  by abbreviation.

**1.3.** Assume that  $k = \mathbf{C}$ . We consider the following variant of the induction theorem. Let  $\Gamma$  be a cyclic group of order  $e$  generated by  $a$ . Let  $\zeta$  be a primitive  $e$ -th root of unity in  $\mathbf{C}$ . Let  $V = \bigoplus_{n \geq 0} V_n$  be a graded  $W$ -module. Then  $V$  turns out to be a  $\Gamma \times W$ -module by defining the action of  $\Gamma$  on  $V$  by  $ax = \zeta^n x$  for  $x \in V_n$ . We denote by  $V^{(\zeta)}$  the thus obtained  $\Gamma \times W$ -module  $V$ .

For  $u \in L$ , we consider the graded  $W_L$ -module  $H^*(\mathcal{B}_u^L) = \bigoplus_{n \geq 0} H^{2n}(\mathcal{B}_u^L)$ , where the degree  $n$  part is given by  $H^{2n}(\mathcal{B}_u^L)$ , and similarly we consider the graded  $W$ -module  $H^*(\mathcal{B}_u) = \bigoplus_{n \geq 0} H^{2n}(\mathcal{B}_u)$ . Let  $\Gamma$  be as before. We choose  $\Gamma$  such that  $\Gamma \subset N_W(W_L)$ , and consider the semidirect product  $\widetilde{W}_L = \Gamma \ltimes W_L$ . We assume that the  $W_L$ -module  $H^n(\mathcal{B}_u^L)$  can be extended to a  $\widetilde{W}_L$ -module for each  $n$ . (In the case where  $a \in Z_W(W_L)$ , we have  $\widetilde{W}_L = \Gamma \times W_L$ . In this case, one can choose a trivial extension to  $\widetilde{W}_L$ , i.e., we may assume that  $\Gamma(\subset \widetilde{W}_L)$  acts trivially on  $H^*(\mathcal{B}_u^L)$ .) Then one can define a  $\Gamma \times \widetilde{W}_L$ -module  $H^*(\mathcal{B}_u^L)$  as above, replacing  $W_L$  by  $\widetilde{W}_L$ , which we denote by  $H^*(\mathcal{B}_u^L)^{(\zeta)}$ . (When we need to distinguish the group  $\Gamma$  as the first factor of  $\Gamma \times \widetilde{W}_L$  from the subgroup of  $\widetilde{W}_L$ , we write the latter as  $\Gamma_0$ .)  $\Gamma \times W$ -module  $H^*(\mathcal{B}_u)^{(\zeta)}$  is defined as before. Put  $V^{(\zeta)} = H^*(\mathcal{B}_u^L)^{(\zeta)}$ , and let  $V_n^{(\zeta)}$  be the degree  $n$ -part of  $V^{(\zeta)}$ . Let us consider the induced  $W$ -module

$$\text{Ind}_{W_L}^W V^{(\zeta)} = \bigoplus_{w \in W/W_L} w \otimes V^{(\zeta)}.$$

Then  $\text{Ind}_{W_L}^W V^{(\zeta)}$  turns out to be a  $\Gamma \times W$ -module by defining the action of  $\Gamma$  by  $b(w \otimes x) = \zeta^n (wb^{-1} \otimes bx)$  for  $b \in \Gamma_0, x \in V_n^{(\zeta)}$ , which we denote by  $\Gamma\text{-Ind}_{W_L}^W V^{(\zeta)}$ .

**1.4.** In the remainder of this paper, we assume that  $G$  is simple modulo center. Let  $T \subset B$  be a pair of maximal torus and a Borel subgroup of  $G$ . Put  $W = N_G(T)/T$ . Let  $L$  be a Levi subgroup of a parabolic subgroup  $P$  of  $G$  containing  $B$  such that  $L \supset T$ . We have  $W_L = N_L(T)/T$ . Let  $\Phi \subset X(T)$  be a root system for  $G$  with respect to  $T$ , with a simple root system  $\Pi$  (with respect to  $B$ ), where  $X(T)$  is the character group of  $T$ . We denote by  $\Phi_L$  the sub system of  $\Phi$  corresponding to  $L$  with the simple root system  $\Pi_L \subset \Pi$ . Let  $\Pi'$  be the set of simple roots which are orthogonal to  $\Pi_L$  with respect to the standard inner product on  $V = \mathbf{R} \otimes_{\mathbf{Z}} X(T)$ . We denote by  $L'$  the Levi subgroup containing  $T$  corresponding to  $\Pi'$ . Let  $W_{L'} = N_{L'}(T)/T$  be the Weyl group of  $L'$ . Then we have  $W \supset W_L \times W_{L'}$ , and so  $W_{L'} \subset N_W(W_L)$ .

We recall here the notion of regular elements of reflection groups due to Springer [Sp1]. Let  $W$  be a reflection group in  $GL(V)$ . A vector  $v \in V$  is called regular if  $v$  is not contained in any reflecting hyperplane in  $V$ . An element  $a \in W$  is called regular if  $a$  has an eigenvector  $v$  which is a regular element in  $V$ . If  $av = \zeta v$ , with  $\zeta$  a primitive  $e$ -th root of unity, then the order of  $a$  is equal to  $e$  ([Sp1, 4.2]). In particular, if  $a$  is regular of order  $e$ , there exists an eigenvalue  $\zeta$  which is a primitive  $e$ -th root of unity.

The regular elements  $a \in W$  in the case of classical groups are given as follows (cf. [Sp1]).

**Type  $A_{n-1}$ .** In this case  $W = \mathfrak{S}_n$  and there are two types of regular elements.

(a)  $e$  is a divisor of  $n$ , and  $a$  is an  $n/e$ -product of (disjoint)  $e$ -cycles in  $\mathfrak{S}_n$ .

(b)  $e$  is a divisor of  $n - 1$ , and  $a$  is an  $(n - 1)/e$ -product of  $e$ -cycles in  $\mathfrak{S}_n$ .

**Type  $B_n$ .** There are two types of regular elements.

(a)  $e$  is an odd divisor of  $n$ , and  $a$  is an  $n/e$ -product of positive cycles of length  $e$ .

(b)  $e$  is an even divisor of  $2n$ , and  $a$  is a  $2n/e$ -product of negative cycles of length  $e/2$ .

**Type  $D_n$ .** In this case there are 4 types of regular elements.

(a)  $e$  is an odd divisor of  $n$ , and  $a$  is a product of positive cycles of length  $e$ .

(b)  $e$  is an odd divisor of  $n - 1$ , and  $a$  is a product of positive cycle of length 1 and  $(n - 1)/e$  positive cycles of length  $e$ .

(c)  $n$  is even, and  $e$  is an even divisor of  $n$ .  $a$  is a product of negative cycles of length  $e/2$ .

(d)  $e$  is an even divisor of  $2n - 2$ , and  $a$  is a product of  $(n - 1)/e$  negative cycles of length  $e/2$  and one cycle of length 1, which is positive or negative according as  $(2n - 2)/e$  is even or odd.

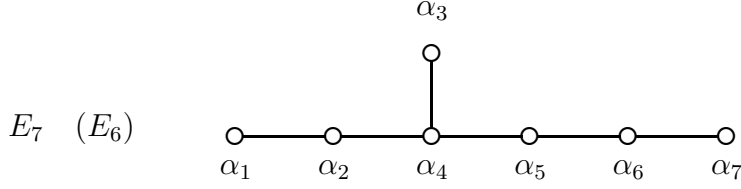
Regular elements in the exceptional Weyl groups are listed in [Sp1].

Returning to the original setting, we consider the subgroups  $W_L, W_{L'}$  of  $W$ . Let  $V'$  be the subspace of  $V$  generated by  $\Pi_{L'}$ .  $W_{L'}$  is realized as a reflection group on  $V'$ . Assume that  $a$  is a regular element of  $W_{L'}$  of order  $e$ . Let  $\zeta$  be a primitive  $e$ -th root of unity, and  $V(a, \zeta)$  the eigensubspace of  $a$  in  $V$  with eigenvalue  $\zeta$ . Since  $a$  is regular,  $V(a, \zeta)$  is not contained in any reflecting hyperplane  $H_\alpha$  for  $\alpha \in \Phi_{L'}$ . We say that  $a$  is  $L$ -regular if  $V(a, \zeta)$  is not contained in any  $H_\alpha$  for  $\alpha \in \Phi - \Phi_L$ . If  $L$  is the torus  $T$ , all the regular elements are  $L$ -regular. But if  $L \neq T$ , regular elements are not necessarily  $L$ -regular. For example, if  $L$  is not simple modulo center, regular elements in  $W_{L'}$  are not  $L$ -regular in many cases. In the case where  $L$  is simple modulo center,  $L$ -regular elements are classified as follows.

**Lemma 1.5.** *Assume that  $L$  is simple modulo center.*

- (i) *If  $W$  is of type  $A_n, B_n, D_n$ , take  $L$  such that  $W_L$  is of the same type as  $W$  of rank  $m$ , and  $W_{L'}$  is of type  $A_{n-m-1}$ . Then a regular element of  $W_{L'}$  of type (a) in 1.4 is  $L$ -regular.*
- (ii) *If  $W$  is of type  $G_2, F_4$  or  $E_8$ , there does not exist  $L$ -regular elements for any  $L \neq T$ .*
- (iii) *Assume that  $W$  is of type  $E_6$  or  $E_7$ . Let  $\Pi = \{\alpha_1, \dots, \alpha_7\}$  (resp.  $\{\alpha_1, \dots, \alpha_6\}$ ) be the set of simple roots in  $E_7$  (resp. in  $E_6$ ) as in the figure. Take  $\Pi_L = \{\alpha_k, \alpha_{k+1}, \dots, \alpha_7\}$  (resp.  $\{\alpha_k, \alpha_{k+1}, \dots, \alpha_6\}$ ) for  $k \geq 3$ . Then  $W_{L'}$  is of type  $A_j$  or of type  $A_j + A_1$  for some  $j$  except the case where  $W$  is of type  $E_7$  and  $\Pi_L = \{\alpha_7\}$ , in which case  $\Pi_{L'}$  is of type  $D_5$ . In the former case, we choose  $a$  a regular element of type (a) for type  $A$ , and in the latter case, we choose  $a$  a regular element of type (a) for type  $D$  in 1.4, respectively. Then  $a$  is  $L$ -regular.*

*Proof.* If there exists  $\beta \in \Phi - \Phi_L$  such that  $\beta$  is orthogonal to  $V_{L'}$ , then any regular element in  $W_{L'}$  cannot be  $L$ -regular. By direct inspections, one can find such  $\beta$  unless  $L$  is the type given in (i), (iii) of the lemma. Assume that  $L$  is as in the



lemma, and let  $a$  be a regular element in  $W_{L'}$ . If  $W_{L'}$  is of type  $A_j$  or type  $A_j + A_1$ , then a regular vector  $v \in V'$  can be written explicitly, and one can check the  $L$ -regularity by direct inspections. If  $W_{L'}$  is of type  $D_5$  (in the case where  $W$  is of type  $E_7$ ),  $a$  must be of type (a) (otherwise it is easy to see that  $a$  is not  $L$ -regular). But this element is nothing but the regular element in  $A_4$ , and the checking is reduced to the previous case. The details are omitted.  $\square$

**1.6.** In what follows we consider a specific cyclic group  $\Gamma \in N_W(W_L)$ , and  $u \in L$  according to the following two cases.

Case (a):  $W_{L'} \neq \{1\}$ .

In this case, we assume that  $L$  is simple modulo center. We choose an  $L$ -regular element  $a \in W_{L'}$ , and put  $\Gamma = \langle a \rangle$ . Let  $e$  be the order of  $\Gamma$ . Thus  $\Gamma \subset W_{L'}$  and we have  $\Gamma \times W_L \subset W$ . We take any unipotent element  $u \in L$ .

Case (b):  $W_{L'} = \{1\}$ .

In this case, we assume that  $L$  is of type  $X_0 + e(A_{n_1-1} + \cdots + A_{n_r-1})$  with  $X_0$  irreducible. We further assume that any  $\beta \in \Phi - \Phi_L$  is not orthogonal to the root system  $e(A_{n_1-1} + \cdots + A_{n_r-1})$ . (Note: since  $W_{L'} = \{1\}$ , any irreducible component of the Dynkin diagram corresponding to  $\Pi - \Pi_L$  consists of 1 or 2 nodes. The latter condition is satisfied for type  $B_n$  if all the irreducible components consist of one node, and for type  $A_n, D_n$  if the number of irreducible components having two nodes is at most 1.)

We choose  $a \in W$  so that  $a$  permutes each component  $A_{n_i-1}$  in a cyclic way, and acts trivially on  $X_0$ . Thus  $a \in \mathfrak{S}_{en_1} \times \cdots \times \mathfrak{S}_{en_r}$ , and  $a$  is a product of disjoint cycles of length  $e$ . In particular,  $\Gamma = \langle a \rangle \subset N_W(W_L)$ , and the subgroup of  $W$  generated by  $\Gamma$  and  $W_L$  coincides with the semidirect product  $\Gamma \ltimes W_L$ . Now  $L$  is isogenic to  $G_0 \times G_1 \times \cdots \times G_r$  modulo center, where  $G_0$  is of type  $X_0$ , and  $G_i \simeq GL_{n_i} \times \cdots \times GL_{n_i}$  ( $e$ -factors). We choose a unipotent element  $u \in L$  so that  $u$  corresponds to  $(u_0, u_1, \dots, u_r)$ , where  $u_0 \in G_0$  is arbitrary, and  $u_i$  is a diagonal element in  $G_i$ , i.e.,  $u_i = (v_i, \dots, v_i)$  with  $v_i \in GL_{n_i}$  for  $i = 1, \dots, r$ .

We can state our main theorem, whose proof will be given in the next section.

**Theorem 1.7.** Assume that  $G$  is defined over  $\mathbf{C}$ . Let  $L$  be a Levi subgroup in  $G$ . Assume that a cyclic subgroup  $\Gamma$  of order  $e$  in  $N_W(W_L)$  and  $u \in L$  are given as in 1.4. Put  $\widetilde{W}_L = \Gamma \ltimes W_L$ . Then the followings hold.

- (i)  $W_L$ -module  $H^*(\mathcal{B}_u^L)$  can be extended to a  $\widetilde{W}_L$ -module so that  $\Gamma \times \widetilde{W}_L$ -module  $H^*(\mathcal{B}_u^L)^{(\zeta')}$  is defined for any  $e$ -th root of unity  $\zeta'$ .
- (ii) There exists a primitive  $e$ -th root of unity  $\zeta$  such that

$$(1.7.1) \quad \Gamma\text{-Ind}_{W_L}^W (H^*(\mathcal{B}_u^L)^{(\zeta)}) \simeq H^*(\mathcal{B}_u)^{(\zeta)}$$

as  $\Gamma \times W$ -modules.

**Remarks 1.8.** (i) The extension of  $W_L$ -module  $H^*(\mathcal{B}_u^L)$  to  $\widetilde{W}_L$ -module is not unique. The theorem asserts that the statement (ii) holds for some choice of extension.

(ii) The theorem asserts that (1.7.1) holds for some choice of primitive  $e$ -th root of unity  $\zeta$ , but then it holds for any choice of primitive root of unity  $\zeta'$ . In fact, we can write  $\zeta' = \zeta^j$  for some  $j$  prime to  $e$ , and we have an automorphism  $\tau$  on  $\Gamma$  such that  $\tau(a) = a^j$ . It follows from (1.7.1) that we have an isomorphism of  $\Gamma \times W$  modules, where the action of  $\Gamma$  is twisted by  $\tau$ . It is easy to check that the twisted  $\Gamma \times W$ -module  $\Gamma\text{-Ind}_{W_L}^W(H^*(\mathcal{B}_u^L)^{(\zeta)})$  is isomorphic to  $\Gamma\text{-Ind}_{W_L}^W(H^*(\mathcal{B}_u^L)^{(\zeta')})$ , and similarly the twisted  $H^*(\mathcal{B}_u)^{(\zeta)}$  is isomorphic to  $H^*(\mathcal{B}_u)^{(\zeta')}$ . Thus (1.7.1) holds also for  $\zeta'$ .

## 2. PROOF OF THEOREM 1.7

**2.1.** In the case where  $e = 1$ , Theorem 1.7 is nothing but the original induction theorem. So we assume that  $e \geq 2$  in what follows. Since the structure of the  $W$ -module  $H^n(\mathcal{B}_u)$  is independent of  $p$  provided that  $p$  is a good prime, it is enough to show the corresponding formula for an appropriate  $p$ . So, we assume that  $G$  is defined over  $\mathbf{F}_p$ , of split type, with Frobenius map  $F$ . We assume that  $T \subset B$  are both  $F$ -stable, and that  $L \subset P$  are  $F$ -stable. Thus  $F$  acts trivially on  $W$  and on  $W_L$ . We first note that

**Lemma 2.2.** *Let  $a \in N_W(W_L)$  and choose  $\dot{a} \in N_G(T) \cap N_G(L)$ . Assume that  $\dot{a} \in Z_G(u)$ . Then  $\text{ad } \dot{a}$  stabilizes  $\mathcal{B}_u^L$ , and acts on  $H^*(\mathcal{B}_u^L)$  in such a way that  $\text{ad } \dot{a}(w) = awa^{-1}$  for  $w \in W_L$ .*

*Proof.* Since  $\dot{a} \in N_G(L)$ ,  $\dot{a}$  acts on  $\mathcal{B}^L$  by the adjoint action  $\text{ad } \dot{a}$ , which stabilizes  $\mathcal{B}_u^L$  since  $\dot{a} \in Z_G(u)$ . Hence  $\dot{a}$  acts naturally on  $H^*(\mathcal{B}_u^L)$ . In order to compare this action with the action of  $W_L$ , we shall recall the construction of Springer representations of  $W_L$ . Let

$$\widetilde{L} = \{(x, gB) \in L \times \mathcal{B}^L \mid g^{-1}xg \in B\},$$

and  $\pi : \widetilde{L} \rightarrow L$  be the first projection. Let  $L_r$  be the set of regular semisimple elements in  $L$ . Then  $\pi^{-1}(L_r)$  is isomorphic to

$$\widetilde{L}_r = T_r \times L/T,$$

where  $T_r = T \cap L_r$ . Let  $\pi_0 : \widetilde{L}_r \rightarrow L_r$  be the map defined by  $\pi_0 : (t, gT) \mapsto g^{-1}tg$ , which coincides with the restriction of  $\pi$  on  $\widetilde{L}_r$  under the identification  $\pi^{-1}(L_r) \simeq \widetilde{L}_r$ . Then  $\pi_0$  is an unramified Galois covering with group  $W_L$ , and for a constant sheaf  $\bar{\mathbf{Q}}_l$  on  $\widetilde{L}_r$ ,  $\mathcal{L} = \pi_* \bar{\mathbf{Q}}_l$  is a  $W_L$ -equivariant local system on  $L_r$ . Thus  $K = \text{IC}(L, \mathcal{L})$  is a  $W_L$ -equivariant complex on  $L$ , and it is known by Lusztig that  $K \simeq \pi_* \bar{\mathbf{Q}}_l$ . Thus for each  $u \in L$ , the stalk  $\mathcal{H}_u^i(K)$  at  $u$  of the  $i$ -th cohomology sheaf of  $K$  gives rise to a  $W_L$ -module  $H^i(\mathcal{B}_u^L)$ .

Now  $\dot{a}$  acts on  $\tilde{L}_r$  (resp. on  $L_r$ ) by  $\text{ad } \dot{a} : (t, gT) \mapsto (\dot{a}t\dot{a}^{-1}, \dot{a}g\dot{a}^{-1}T)$  (resp.  $\text{ad } \dot{a} : x \mapsto \dot{a}x\dot{a}^{-1}$ ), and  $\pi_0$  commutes with  $\text{ad } \dot{a}$ . Hence  $\mathcal{L}$  becomes an  $\dot{a}$ -equivariant local system. Since  $\pi_0^{-1}(t) = \{(wtw^{-1}, wT) \mid w \in W_L\}$  for  $t \in T_r$ , the stalk  $\mathcal{L}_t$  has a natural structure of the regular  $W_L$ -module. Then the isomorphism  $\mathcal{L}_{\dot{a}t\dot{a}^{-1}} \rightarrow \mathcal{L}_t$  is given by  $\text{ad } \dot{a}^{-1}$  under the identification  $\mathcal{L}_x \simeq \bar{\mathbf{Q}}_l[W_L]$  for  $x \in L_r$ . It follows that  $\mathcal{L}$  is  $\langle \dot{a} \rangle \rtimes W_L$ -equivariant, where  $\langle \dot{a} \rangle$  is a cyclic group generated by  $\dot{a}$ , and  $\dot{a}$  acts on  $W_L$  by  $\text{ad } \dot{a}(w) = awa^{-1}$ . By the functoriality of IC functor,  $K$  turns out to be a  $\dot{a}$ -equivariant complex on  $L$  under the adjoint action of  $\dot{a}$ , which is regarded as a  $\langle \dot{a} \rangle \rtimes W_L$ -equivariant complex on  $L$ . Hence for  $u \in L$  such that  $\dot{a}u\dot{a}^{-1} = u$ ,  $\mathcal{H}_u^i(K)$  has a structure of  $\langle \dot{a} \rangle \rtimes W_L$ -module.

On the other hand,  $\dot{a}$  acts naturally on  $\tilde{L}$  and on  $L$  by the adjoint action, which commute with  $\pi$ . Thus  $\pi_* \bar{\mathbf{Q}}_l$  is  $\dot{a}$ -equivariant, which is isomorphic to  $K$  as the complex with  $\dot{a}$ -action. Hence the action of  $\dot{a}$  on  $\mathcal{H}_u^i(K)$  coincides with the action on  $H^i(\mathcal{B}_u^L)$  induced from the adjoint action of  $\dot{a}$  on  $\mathcal{B}_u^L$ . The lemma follows from this.  $\square$

Next we show the following lemma.

**Lemma 2.3.** *There exists a representative  $\dot{a} \in N_G(T) \cap N_G(L) \cap Z_G(u)$  such that  $\dot{a}$  acts trivially on  $H^*(\mathcal{B}_u)$  and that  $\dot{a}^e$  acts trivially on  $H^*(\mathcal{B}_u^L)$ . In particular,  $H^*(\mathcal{B}_u^L)$  has a structure of  $\tilde{W}_L$ -module.*

*Proof.* First consider the case (a) in 1.6. Let  $H$  be the subgroup of  $G$  generated by  $U_\alpha$  with  $\alpha \in \Phi_{L'}$ , where  $U_\alpha$  is the root subgroup corresponding to  $\alpha$ . Then  $H$  is a connected reductive subgroup of  $L'$  whose Weyl group coincides with  $W_{L'}$ . Since  $H \subset Z_G(u)$ , we have  $H \subset Z_G^0(u)$ . One can choose a representative  $\dot{a} \in N_H(T_1)$  of  $a \in W_{L'}$ , where  $T_1$  is a maximal torus of  $H$  contained in  $T$ . Then  $\dot{a} \in Z_G^0(u) \cap N_G(L)$  and  $\dot{a}^e \in T_1$ . Since  $T_1 \subset Z_G(u)$ , we see that  $T_1 \subset Z_L^0(u)$ . Thus,  $\dot{a}^e \in Z_L^0(u)$ . Hence  $\dot{a}$  satisfies the condition.

Next consider the case (b) in 1.6. Let  $L_1$  be the Levi subgroup containing  $L$  of type  $X_{n_0} + A_{en_1-1} + \cdots + A_{en_r-1}$ . We have a natural projection  $\pi : L_1 \rightarrow \bar{L}_1 = L_1/Z^0(L_1)$ , and an isogeny map  $\theta : \tilde{L}_1 = G_0 \times SL_{en_1} \times \cdots \times SL_{en_r} \rightarrow \bar{L}_1$ , where  $G_0$  is the simply connected semisimple group of type  $X_0$ . Put  $\bar{u} = \pi(u) \in \bar{L}_1$ . Now  $Z_{L_1}(u)$  acts on  $H^*(\mathcal{B}_u)$ . Since  $Z^0(L_1)$  acts trivially on  $H^*(\mathcal{B}_u)$ , we have an action of  $Z_{L_1}(u)/Z^0(L_1) = Z_{\bar{L}_1}(\bar{u})$  on  $H^*(\mathcal{B}_u)$ . Let  $\tilde{u}$  be an element in  $\tilde{L}_1$  such that  $\theta(\tilde{u}) = \bar{u}$ .  $\tilde{u} = (u_0, u_1, \dots, u_r)$  can be chosen as given in 1.4. We choose  $\tilde{a} \in \tilde{L}_1$  as follows; put  $\tilde{a} = (a_0, a_1, \dots, a_r)$  with  $a_0 \in G_0$ , and  $a_i \in SL_{en_i}$  for  $1 \leq i \leq r$ . We put  $a_0 = 1$  and choose  $a_1, \dots, a_r$  so that  $a_i \in Z_{SL_{en_i}}^0(u_i)$  and that  $a_i^e \in Z(SL_{en_i})$ . Such a choice is always possible for  $u_i$  of type  $(n_i, \dots, n_i)$ . Thus  $\tilde{a} \in Z_{\tilde{L}_1}^0(\tilde{u})$ . It follows that  $\theta(\tilde{a})$  is contained in a connected subgroup of  $Z_{\bar{L}_1}(\bar{u}_1)$ , and by the previous remark,  $\theta(\tilde{a})$  acts trivially on  $H^*(\mathcal{B}_u)$ . Now take  $\dot{a} \in Z_{L_1}(u)$  such that  $\pi(\dot{a}) = \theta(\tilde{a})$ . Then  $\dot{a} \in N_G(T) \cap N_G(L)$ , and acts trivially on  $H^*(\mathcal{B}_u)$ . On the other hand, similar to  $\pi, \theta$ , we have a map  $\pi' : L \rightarrow \bar{L} = L/Z^0(L)$  and  $\theta' : \tilde{L} = G_0 \times (SL_{n_1})^e \times \cdots \times (SL_{n_r})^e \rightarrow \bar{L}$ . Let  $\bar{u} = \pi'(u) \in \bar{L}$ , and  $\tilde{u} \in \tilde{L}$  such that  $\bar{u} = \theta'(\tilde{u})$ . Then we have an isomorphism  $H^*(\mathcal{B}_u^L) \simeq H^*(\mathcal{B}_{\bar{u}}^{\bar{L}}) \simeq H^*(\mathcal{B}_{\tilde{u}}^{\tilde{L}})$  compatible with the actions of  $Z_L(u)$ ,  $Z_{\bar{L}}(\bar{u})$  and  $Z_{\tilde{L}}(\tilde{u})$  with respect to  $\pi', \theta'$ . We have  $\tilde{a}^e \in Z(SL_{n_1})^e \times Z(SL_{n_2})^e \times \cdots$ . Since the action of

$Z(SL_{n_1})^e \times Z(SL_{n_2})^e \times \cdots$  can be extended to an action of  $Z(GL_{n_1})^e \times Z(GL_{n_2})^e \times \cdots$  on  $H^*(\mathcal{B}_u^{\tilde{L}})$ ,  $\dot{a}^e$  acts trivially on  $H^*(\mathcal{B}_u^{\tilde{L}})$ , and so  $\dot{a}^e$  acts trivially on  $H^*(\mathcal{B}_u^L)$ .  $\square$

**2.4.** Let  $\mathcal{Z} = Z_L^0$  be the identity component of the center of  $L$ . Put  $\mathcal{B}_{\mathcal{Z}} = \{B' \in \mathcal{B} \mid \mathcal{Z} \subset B'\}$ . Then  $\mathcal{B}_{\mathcal{Z}}$  is decomposed into connected components

$$\mathcal{B}_{\mathcal{Z}} = \coprod_{d \in W_L \setminus W} \mathcal{B}_{\mathcal{Z},d},$$

where  $\mathcal{B}_{\mathcal{Z},d} = \{x^d B \mid x \in L\}$ , which is isomorphic to  $\mathcal{B}^L$  under the map  $B' \mapsto B' \cap L$ . Put

$$\mathcal{Z}_{\text{reg}} = \{z \in \mathcal{Z} \mid Z_G^0(z) = L\}.$$

Then for any  $t \in \mathcal{Z}_{\text{reg}}$ , we have  $\mathcal{B}_t = \mathcal{B}_{\mathcal{Z}}$  by Lemma 2.2 (c) in [L3], and so  $\mathcal{B}_{tu} = \mathcal{B}_u \cap \mathcal{B}_t = \mathcal{B}_u \cap \mathcal{B}_{\mathcal{Z}}$ . It follows that

$$\mathcal{B}_{tu} = \coprod_{d \in W_L \setminus W} (\mathcal{B}_{\mathcal{Z},d} \cap \mathcal{B}_u),$$

where  $\mathcal{B}_{\mathcal{Z},d} \cap \mathcal{B}_u$  is isomorphic to  $\mathcal{B}_u^L$  under the map  $B' \mapsto B' \cap L$ . This implies that

$$(2.4.1) \quad H^{2n}(\mathcal{B}_{tu}) \simeq \bigoplus_{d^{-1} \in W/W_L} H^{2n}(\mathcal{B}_{\mathcal{Z},d} \cap \mathcal{B}_u).$$

The right hand side of (2.4.1) has a natural structure of the induced  $W$ -module  $\text{Ind}_{W_L}^W H^{2n}(\mathcal{B}_u^L)$ . It is proved in [L3, Proposition 1.4] that (2.4.1) is actually an isomorphism of  $W$ -modules. Let  $a \in W$  be as in the theorem. Since  $\dot{a} \in N_G(L)$ , it stabilizes  $\mathcal{Z}$ , and so  $\dot{a}$  acts on  $\mathcal{B}_{\mathcal{Z}}$  via  $\text{ad } \dot{a}$ . It is easy to see that  $\dot{a}$  induces a permutation action on the components of  $\mathcal{B}_{\mathcal{Z}}$ ;  $\dot{a} : \mathcal{B}_{\mathcal{Z},d} \mapsto \mathcal{B}_{\mathcal{Z},ad}$ . It follows that  $\dot{a}$  induces an automorphism on  $H^{2n}(\mathcal{B}_{tu})$ , which maps the factor corresponding to  $d^{-1} \in W/W_L$  to  $d^{-1}a^{-1} \in W/W_L$ . Under the isomorphism  $H^{2n}(\mathcal{B}_{\mathcal{Z},d} \cap \mathcal{B}_u) \simeq H^{2n}(\mathcal{B}_u^L)$ , the factor corresponding to  $d^{-1} \in W/W_L$  is written as  $d^{-1} \otimes H^{2n}(\mathcal{B}_u^L)$ , and  $\dot{a}$  maps  $d^{-1} \otimes H^{2n}(\mathcal{B}_u^L) \rightarrow d^{-1}a^{-1} \otimes H^{2n}(\mathcal{B}_u^L)$ . On the other hand, by Lemma 2.3,  $\dot{a}^e$  acts trivially on  $H^{2n}(\mathcal{B}_u^L)$ , and induces an action of  $\widetilde{W}_L$  on it. Hence  $\dot{a}$  induces an action of  $\Gamma_0$  on  $H^{2n}(\mathcal{B}_{tu}) \simeq \text{Ind}_{W_L}^W H^{2n}(\mathcal{B}_u^L)$ , which is given by  $\dot{a} : d^{-1} \otimes x \mapsto d^{-1}a^{-1} \otimes \dot{a}x$  for each factor  $d^{-1} \otimes H^{2n}(\mathcal{B}_u^L)$ .

Now we define an action of  $\Gamma$  on  $H^*(\mathcal{B}_{tu})$  by  $a : x \mapsto \zeta^n \dot{a}x$  for  $x \in H^{2n}(\mathcal{B}_{tu})$ , where  $\dot{a}x$  is the action of  $\Gamma_0$  on  $H^{2n}(\mathcal{B}_{tu})$  given as above. Since the action of  $\dot{a} \in G$  commutes with that of  $W$ ,  $H^*(\mathcal{B}_{tu})$  turns out to be a  $\Gamma \times W$ -module, which we denote by  $H^*(\mathcal{B}_{tu})^{[\zeta]}$ . The following lemma is immediate from the above discussion.

**Lemma 2.5.** *There exists an isomorphism of  $\Gamma \times W$ -modules*

$$H^*(\mathcal{B}_{tu})^{[\zeta]} \simeq \Gamma\text{-Ind}_{W_L}^W H^*(\mathcal{B}_u^L)^{(\zeta)}.$$

In view of Lemma 2.5, in order to prove the theorem it is enough to show the following proposition.



**Proposition 2.6.** *Under an appropriate choice of (a good prime)  $p$ , there exists an isomorphism of  $\Gamma \times W$ -modules for any  $t \in \mathcal{Z}_r$ ,*

$$H^*(\mathcal{B}_u)^{(\zeta)} \simeq H^*(\mathcal{B}_{tu})^{[\zeta]}.$$

**2.7.** The remainder of this section is devoted to the proof of the proposition. We shall prove it by modifying the arguments in [L3]. By [Sh1], [Sh2], [BS], the following fact is known; assume that  $G$  is simple modulo center. Then for each unipotent class  $C$  of  $G$ , there exists  $u_1 \in C^F$ , called a split unipotent element, such that  $F$  acts on  $H^{2n}(\mathcal{B}_{u_1})$  as a scalar multiplication by  $p^n$ . (In the case where  $G$  is of type  $E_8$ , we assume that  $p \equiv 1 \pmod{4}$ ). Since the component group  $A_G(u_1) = Z_G(u_1)/Z_G^0(u_1)$  is isomorphic to  $S_3, S_4, S_5$  or  $(\mathbf{Z}/2\mathbf{Z})^k$  for some  $k$ , there exists a positive integer  $s_0$  (independent of  $p$ ) such that  $F^{s_0}$  acts on  $H^{2n}(\mathcal{B}_u)$  by a scalar multiplication by  $p^{s_0 n}$  for any unipotent element  $u$  of  $G^F$  (e.g., one can take  $s_0 = |S_5|$ .) Similarly,  $F^{s_0}$  acts on  $H^{2n}(\mathcal{B}_u^L)$  by a scalar multiplication by  $p^{s_0 n}$  for any unipotent element  $u \in L^F$ . Note that the isomorphism in (2.4.1) is  $F$ -equivariant. Hence  $F^{s_0}$  acts also as a scalar multiplication by  $p^{s_0 n}$  for  $H^{2n}(\mathcal{B}_{tu})$ .

Note that  $\dot{a}$  acts trivially on  $H^{2n}(\mathcal{B}_u)$  by Lemma 2.3. It follows that one can write

$$(2.7.1) \quad \text{Tr}((F^s \dot{a})^i w, H^*(\mathcal{B}_u)) = \sum_{n \geq 0} a_n(w) p^{i s n},$$

$$(2.7.2) \quad \text{Tr}((w, a^i), H^*(\mathcal{B}_u)^{(\zeta)}) = \sum_{n \geq 0} a_n(w) \zeta^{i n},$$

for any  $w \in W, 0 \leq i \leq e-1$  and for any positive integer  $s$  divisible by  $s_0$ , where  $a_n(w) = \text{Tr}(w, H^n(\mathcal{B}_u))$  are integers for each  $n \geq 0$ .

On the other hand, by the description of the action of  $F$  and of  $\dot{a}$  on  $H^n(\mathcal{B}_{tu})$  in 2.4, together with Lemma 2.5, one can write

$$(2.7.3) \quad \text{Tr}((F^s \dot{a})^i w, H^*(\mathcal{B}_{tu})) = \sum_{n \geq 0} b_{n,i}(w) p^{i s n},$$

$$(2.7.4) \quad \text{Tr}((w, a^i), H^*(\mathcal{B}_{tu})^{[\zeta]}) = \sum_{n \geq 0} b_{n,i}(w) \zeta^{i n},$$

for  $w, i, s$  as above, where  $b_{n,i}(w)$  are certain integers.

For an integer  $x$  and a prime number  $l$ , we denote by  $m_l(x)$  the multiplicative order of  $x$  in  $\mathbf{Z}/l\mathbf{Z}$ , i.e., the smallest positive integer  $m$  such that  $x^m \equiv 1 \pmod{l}$ . The following is a key for the proof of Proposition 2.6.

**Lemma 2.8.** *Assume that  $p \equiv 1 \pmod{4}$ . Let  $s_0, e$  be fixed positive integers coprime to  $p$ . Then there exist infinitely many prime numbers  $l$  satisfying the following properties.*

- (i)  $m_l(p^s) = e$  for a certain integer  $s$  divisible by  $s_0$ .
- (ii)  $l-1$  is divisible by  $e$ .

*Proof.* By our assumption, the image of  $s_0e$  on  $\mathbf{F}_p = \mathbf{Z}/p\mathbf{Z}$  is non-zero. Hence the map  $x \mapsto s_0ex + 1$  induces a bijective map on  $\mathbf{F}_p$ . Thus there exists  $c \in \mathbf{Z}$  such that the image of  $s_0ec + 1$  in  $\mathbf{F}_p$  is contained in  $\mathbf{F}_p^* - (\mathbf{F}_p^*)^2$ . Put  $\alpha = s_0ec + 1$ . Then  $\alpha$  is prime to  $p$ , and so  $(\alpha - 1)p$  and  $\alpha$  are coprime each other. Then by Dirichlet's theorem on arithmetic progression, there exist infinitely many prime numbers  $l$  of the form  $l = n(\alpha - 1)p + \alpha$  for some positive integer  $n$ . It is enough to show that these  $l \geq 3$  satisfy the assertion of the lemma. For an integer  $a$  and a prime number  $p$ , let  $\left(\frac{a}{p}\right)$  be the Legendre symbol, i.e.,

$$\left(\frac{a}{p}\right) = \begin{cases} 1 & \text{if } x^2 \equiv a \pmod{p} \text{ for some } x \in \mathbf{Z}, \\ -1 & \text{otherwise.} \end{cases}$$

We show that

$$(2.8.1) \quad \left(\frac{p}{l}\right) = -1.$$

In fact, by the quadratic reciprocity law (e.g., [Se]), we have

$$\left(\frac{p}{l}\right)\left(\frac{l}{p}\right) = (-1)^{\frac{p-1}{2} \cdot \frac{l-1}{2}} = 1.$$

The second equality follows from the assumption that  $p \equiv 1 \pmod{4}$ . Hence we have  $\left(\frac{p}{l}\right) = \left(\frac{l}{p}\right)$ . But  $l \equiv \alpha \pmod{p}$ , and so  $\left(\frac{l}{p}\right) = \left(\frac{\alpha}{p}\right) = -1$  since the image of  $\alpha$  is not contained in  $\mathbf{F}_p^2$  by our choice of  $\alpha$ . Hence (2.8.1) holds.

Now (2.8.1) is equivalent to  $p^{(l-1)/2} \equiv -1 \pmod{l}$ . It follows that  $m_l(p) = l - 1$ . Since  $l - 1 = s_0ec(np + 1)$ , we see that  $m_l(p^s) = e$  for  $s = s_0c(np + 1)$  and that  $l - 1$  is divisible by  $e$ . Thus this  $l$  satisfies the assertion of the lemma. The lemma is proved.  $\square$

**2.9** For given integers  $s_0 \geq 1, e \geq 2$ , we choose a prime number  $p$  such that  $p$  is not a factor of  $e, s_0$  and that  $p \equiv 1 \pmod{4}$ , and fix it once and for all. For a multiple  $s$  of  $s_0$ , put  $F' = F^s a$  and  $q = p^s$ . Under the setting in 1.6, we shall describe the set  $\mathcal{Z}_{\text{reg}}$  more precisely. As in [L3, Lemma 2.2],  $\mathcal{Z}_{\text{reg}}$  can be written as  $\mathcal{Z}_{\text{reg}} = \mathcal{Z} - \bigcup_{\beta} \ker(\beta|_{\mathcal{Z}})$ , where  $\beta$  runs over all the roots in  $\Phi - \Phi_L$ . ( $\beta|_{\mathcal{Z}}$  gives a non-trivial character of  $\mathcal{Z}$  for  $\beta \in \Phi - \Phi_L$ ).

First consider the case (a). Let  $L'_{\text{der}}$  be the derived subgroup of  $L'$ , and  $S'$  be the split maximal torus of  $L'_{\text{der}}$  contained in  $T$ . Then  $S' \subset \mathcal{Z}$ . Put  $S'_{\text{reg}} = S' \cap \mathcal{Z}_{\text{reg}}$ . Now  $W_{L'}$  leaves the set  $\Phi - \Phi_L$  invariant. For each  $\beta \in \Phi - \Phi_L$ , put  $H_{\beta} = \bigcap_{x \in \Gamma} \ker(x(\beta)|_{S'})$ . Then  $H_{\beta}$  is an  $F'$ -stable subgroup of  $S'$ , and we see that

$$(2.9.1) \quad S'^{F'}_{\text{reg}} = S'^{F'} - \bigcup_{\beta \in \Phi - \Phi_L} H_{\beta}^{F'}.$$

$H_{\beta}$  is a closed subgroup of  $S'$ , and we put  $e_{\beta} = |H_{\beta}/H_{\beta}^0|$  for each  $\beta \in \Phi - \Phi_L$ .

Let  $\mathcal{P}'$  be the set of all prime numbers  $l$  satisfying the condition in Lemma 2.8. Thus  $\mathcal{P}'$  is an infinite set. We denote by  $\mathcal{P}$  the subset of  $\mathcal{P}'$  consisting of  $l$  such that  $l > |\Phi - \Phi_L|$  and that  $l$  does not divide  $e_\beta$  ( $\beta \in \Phi - \Phi_L$ ). Thus  $\mathcal{P}$  is an infinite set also.

Next we consider the case (b). We may assume that  $G$  has a connected center of dimension 1, and that the derived subgroup of  $G$  is simply connected, almost simple. Let  $k$  be an algebraic closure of  $\mathbf{F}_q$ . We see that there exists a subtorus  $S$  of  $\mathcal{Z}$  such that  $S \simeq (k^*)^c$ , where  $c$  is the number of irreducible components of  $\Phi_L$ . Since  $a$  permutes the factors  $k^*$  in  $S$ , we see that  $S^{F'} \simeq (\mathbf{F}_{q^e}^*)^r \times (\mathbf{F}_q^*)^{r'}$ , where  $r'$  is equal to 1 or 0 according to the cases where  $X_0$  is non-empty or empty. Since  $\Gamma \subset N_W(W_L)$ ,  $\Gamma$  preserves the set  $\Phi - \Phi_L$ . For each  $\beta \in \Phi - \Phi_L$ , put  $K_\beta = \bigcap_{x \in \Gamma} \ker(x(\beta)|_S)$ . Then  $K_\beta$  is an  $F'$ -stable subgroup of  $S$ , and we have

$$(2.9.2) \quad S_{\text{reg}}^{F'} = S^{F'} - \bigcup_{\beta \in \Phi - \Phi_L} K_\beta^{F'},$$

where  $S_{\text{reg}} = S \cap \mathcal{Z}_{\text{reg}}$ .  $K_\beta$  is a closed subgroup of  $S$ , and put  $e_\beta = |K_\beta/K_\beta^0|$  for each  $\beta \in \Phi - \Phi_L$ . Under the identification  $S^{F'} \simeq (\mathbf{F}_{q^e}^*)^r \times (\mathbf{F}_q^*)^{r'}$ , we see that  $K_\beta^{0F'} \simeq (\mathbf{F}_{q^e}^*)^{r-1} \times (\mathbf{F}_q^*)^{r'}$  or  $K_\beta^{0F'} \simeq \mathbf{F}_{q^{e'}}^* \times (\mathbf{F}_{q^e}^*)^{r-1} \times (\mathbf{F}_q^*)^{r'}$ , where  $e'$  is a proper divisor of  $e$ . (Let  $S_i$  be the subtorus of  $S$  corresponding to the factor  $eA_{n_i-1}$  for  $i = 1, \dots, r$ . Then the former case occurs if  $\beta|_{S_i}, \beta|_{S_j}$  are non-trivial for some  $i \neq j$ , and the latter case occurs if  $\beta|_{S_i}$  is non-trivial for only one  $i$ . Note that by our assumption in 1.4,  $\beta$  is non-trivial on  $S_1 \times \dots \times S_r$ .)

Let  $\mathcal{P}'$  be as in the case (a). We define a subset  $\mathcal{P}$  of  $\mathcal{P}'$  as the set of prime numbers  $l \in \mathcal{P}'$  such that  $l > |\Phi - \Phi_L|$  and that  $l$  does not divide  $e_\beta$ .

The next lemma is a variant of Lemma 3.4 in [L3].

**Lemma 2.10.** *Assume that  $l \in \mathcal{P}$ , and let  $s$  be a multiple of  $s_0$  such that  $m_l(p^s) = e$  (see Lemma 2.8). Put  $F' = F^s \dot{a}$ . Then there exists  $t \in \mathcal{Z}_{\text{reg}}$  such that  $F'(t) = t$  and that  $t^l = 1$ .*

*Proof.* First consider the case (a) in 1.6. It is enough to show, for each  $l \in \mathcal{P}$ , that there exists  $t \in S'^{F'}_{\text{reg}}$  such that  $t^l = 1$ . Note that  $a$  is a regular element of order  $e$  in  $W_{L'}$ . Put  $V = \mathbf{R} \otimes_{\mathbf{Z}} X(S')$ . Thus  $W_{L'}$  acts on  $V$  as a reflection group. Let  $\zeta$  be a primitive  $e$ -th root of unity, and let  $a(e)$  be the dimension of the eigenspace  $V(a, \zeta) \subset V$  of  $a$  with eigenvalue  $\zeta$ . We show that

$$(2.10.1) \quad \#\{t \in S'^{F'}_{\text{reg}} \mid t^l = 1\} = l^{a(e)}.$$

By a general formula, we have  $|S'^{F'}_{\text{reg}}| = |\det_V(qI - a)| = P_a(q)$ , where  $P_a(x)$  is the characteristic polynomial of  $a \in W_{L'}$ . Since  $a$  is regular  $P_a(x)$  can be written, by [Sp1, 4.2], as

$$P_a(x) = \Phi_e(x)^{a(e)} \Phi'(x),$$

where  $\Phi_e(x)$  is the cyclotomic polynomial of degree  $e$ , and  $\Phi'(x)$  is a product of cyclotomic polynomials  $\Phi_{e'}(x)$  with  $e' < e$ . By our assumption  $m_l(q) = e$ ,  $\Phi_e(q)$  is divisible by  $l$ , and  $\Phi'(q)$  is not divisible by  $l$ . This means that each minimal  $F'$ -stable

torus  $M$  of  $S'$  corresponding to the factor  $\Phi_e(x)$  contains an element of order  $l$ . Since  $\{t \in M^{F'} \mid t^l = 1\} \subset \mathbf{F}_{q^e}^*$ ,  $M^{F'}$  contains exactly  $l$  elements  $t$  such that  $t^l = 1$ . Thus (2.10.1) is proved.

For  $\beta \in \Phi - \Phi_L$ , let  $V_\beta$  be the subspace of  $V$  which is orthogonal to  $x(\beta)$  for all  $x \in \Gamma$ . Then  $V_\beta$  can be identified with  $\mathbf{R} \otimes_{\mathbf{Z}} X(H_\beta^0)$ .  $\Gamma$  stabilizes  $V_\beta$ , and let  $V_\beta(a, \zeta)$  be the eigenspace of  $a$  on  $V_\beta$  with eigenvalue  $\zeta$ . Since  $a$  is  $L$ -regular, we have  $\dim V_\beta(a, \zeta) < \dim V(a, \zeta) = a(e)$ . It follows that the characteristic polynomial  $P'_a(x)$  of  $a$  on  $V_\beta$  contains the factor  $\Phi_e(x)$  with multiplicity less than  $a(e)$ . By a similar argument as above, minimal  $F'$ -stable subtori of  $H_\beta^0$  corresponding to  $\Phi_e(x)$  only contain elements of order  $l$ . This implies that

$$\#\{t \in H_\alpha^{F'} \mid t^l = 1\} = \#\{t \in H_\alpha^{0F'} \mid t^l = 1\} \leq l^{a(e)-1}.$$

It follows, by (2.9.1), that

$$\begin{aligned} \#\{t \in S'^{F'}_{\text{reg}} \mid t^l = 1\} &= \#\{t \in S'^{F'} \mid t^l = 1, t \notin \bigcup_{\beta \in \Phi - \Phi_L} H_\beta^{F'}\} \\ &\geq l^{a(e)} - Nl^{a(e)-1} = l^{a(e)-1}(l - N), \end{aligned}$$

where  $N = |\Phi - \Phi_L|$ . Since  $l > N$  by our assumption, there exists  $t \in S'^{F'}_{\text{reg}}$  such that  $t^l = 1$ . This proves the lemma in the case (a).

Next consider the case (b) in 1.6. It is enough to show, for each  $l \in \mathcal{P}$ , that there exists  $t \in S'^{F'}_{\text{reg}}$  such that  $t^l = 1$ . We note that  $q^{e'} - 1$  is not divisible by  $l$  for any divisor  $e' < e$  of  $e$  by the assumption  $m_l(q) = e$ . Since  $S^{F'} \simeq (\mathbf{F}_{q^e}^*)^r \times (\mathbf{F}_q^*)^{r'}$  (cf. 2.9), we have

$$\#\{t \in S^{F'} \mid t^l = 1\} = l^r.$$

We consider  $K_\beta$  given in 2.9. By the discussion in 2.9, we have

$$\#\{t \in K_\beta^{F'} \mid t^l = 1\} = \#\{t \in K_\beta^{0F'} \mid t^l = 1\} = l^{r-1}.$$

It follows, by (2.9.2), that

$$\begin{aligned} \#\{t \in S'^{F'}_{\text{reg}} \mid t^l = 1\} &= \#\{t \in S^{F'} \mid t^l = 1, t \notin \bigcup_{\beta \in \Phi - \Phi_L} K_\beta^{F'}\} \\ &\geq l^r - Nl^{r-1} = l^{r-1}(l - N), \end{aligned}$$

where  $N$  is as before. Since  $l > N$  by our assumption, the lemma holds also for the case (b).  $\square$

We need the following lemma due to Lusztig.

**Lemma 2.11** ([L3, Lemma 3.2]). *Let  $H$  be a finite group, and  $\phi$  a virtual character of  $H$  (over a field of characteristic 0). Assume that  $\phi$  is integral valued. Let  $x, y \in H$  be such that  $xy = yx$  and  $y^l = 1$  for a prime number  $l$ . Then  $\phi(xy) - \phi(x) \in l\mathbf{Z}$ .*

**2.12.** Let  $s_0$  be as in 2.7, and  $\mathcal{P}$  be as in 2.9. Let  $F' = F^s \dot{a}$  be as in Lemma 2.10 for a fixed  $l \in \mathcal{P}$ . Let  $R_{w,i} = R_{T_w}(1)$  be the Deligne-Lusztig's virtual character of  $G^{F'^i}$  for  $i = 1, \dots, e$ , where  $T_w$  is an  $F'^i$ -stable maximal torus of  $G$  corresponding to  $w \in W \simeq W(T_1)$  (here  $W(T_1) = N_G(T_1)/T_1$  for an  $F'$ -stable pair  $T_1 \subset B_1$ ). Let us choose  $t \in \mathcal{Z}_{\text{reg}}$  as in Lemma 2.10. Then we have

$$(2.12.1) \quad \begin{aligned} \text{Tr}(F'^i w, H^*(\mathcal{B}_u)) &= \text{Tr}(u, R_{w,i}), \\ \text{Tr}(F'^i w, H^*(\mathcal{B}_{tu})) &= \text{Tr}(tu, R_{w,i}). \end{aligned}$$

We remark that (2.12.1) was proved in [L2] under the assumption that  $p^s$  is large enough (which is determined only by the data of the Dynkin diagram of  $G$ ). Thus if we replace  $s_0$  in 2.7 by a suitable large number, the result in [L2] is applicable. One can also apply [Sh3, Theorem 2.2] instead of [L2], where the restriction on  $p^s$  is removed.

Since  $R_{w,i}$  are integral valued, one can apply Lemma 2.11 for  $H = G^{F'^i}$  and  $x = u, y = t$ . Hence we have

$$\text{Tr}(u, R_{w,i}) = \text{Tr}(tu, R_{w,i}) \pmod{l\mathbf{Z}}.$$

It follows from (2.12.1) that

$$(2.12.2) \quad \text{Tr}(F'^i w, H^*(\mathcal{B}_u)) = \text{Tr}(F'^i w, H^*(\mathcal{B}_{tu})) \pmod{l\mathbf{Z}}.$$

Let  $\zeta_0$  be a fixed primitive  $e$ -th root of unity in  $\mathbf{C}$ , and  $R$  the ring of integers of the cyclotomic field  $\mathbf{Q}(\zeta_0)$ . Let  $\mathcal{I}$  be the set of non-zero prime ideals  $\mathfrak{p}$  in  $R$  such that  $\mathfrak{p}$  contains one of the numbers  $1 - \zeta_0^i$  for  $i = 1, \dots, e-1$  and  $\zeta_0$ . Let  $\bar{\mathcal{I}}$  be the set of prime numbers  $l$  such that  $\mathfrak{p} \cap \mathbf{Z} = l\mathbf{Z}$  for  $\mathfrak{p} \in \mathcal{I}$ . Since  $\mathcal{I}$  is a finite set,  $\bar{\mathcal{I}}$  is a finite set. So,  $\mathcal{P} - \bar{\mathcal{I}}$  is an infinite set. Let  $\mathcal{J}$  be the set of prime ideals  $\mathfrak{p}$  of  $R$  such that  $\mathfrak{p} \cap \mathbf{Z} = l\mathbf{Z}$  with  $l \in \mathcal{P} - \bar{\mathcal{I}}$ . Then  $\mathcal{J}$  is an infinite set. Now  $R/\mathfrak{p}$  is a finite extension of  $\mathbf{F}_l$ . Let  $\bar{\zeta}_0$  be the image of  $\zeta_0$  in  $R/\mathfrak{p}$ . Since  $l \in \mathcal{P}$ ,  $l-1$  is divisible by  $e$ . Hence  $\bar{\zeta}_0 \in \mathbf{F}_l^*$ , which has order  $e$  by our choice of  $\mathfrak{p}$ . Since  $m_l(p^s) = e$ , the image of  $p^s$  in  $\mathbf{Z}/l\mathbf{Z}$  has order  $e$ . Hence there exists  $j$  such that

$$(2.12.3) \quad p^s - \zeta_0^j \in \mathfrak{p}.$$

Note that the number  $j$  is determined by the choice of  $\mathfrak{p}$ , which we denote by  $j(\mathfrak{p})$ . For  $j = 1, \dots, e-1$ , let  $\mathcal{J}_j$  be the set of prime ideals  $\mathfrak{p}$  in  $\mathcal{J}$  such that  $j(\mathfrak{p}) = j$ . Thus  $\mathcal{J} = \bigcup_j \mathcal{J}_j$ , and so there exists  $j_0$  such that  $\mathcal{J}_0 = \mathcal{J}_{j_0}$  is an infinite set. We put  $\zeta = \zeta_0^{j_0}$ . By (2.12.3),  $\zeta$  is a primitive  $e$ -th root of unity.

We remark that  $H^*(\mathcal{B}_{tu}) = H^*(\mathcal{B}_u \cap \mathcal{B}_{\mathcal{Z}})$  is independent of the choice of  $t \in \mathcal{Z}_{\text{reg}}$ . Then in view of (2.7.1)  $\sim$  (2.7.4), together with (2.12.3), we see that

$$\begin{aligned} \text{Tr}((F^s \dot{a})^i w, H^*(\mathcal{B}_u)) &= \text{Tr}((w, a^i), H^*(\mathcal{B}_u)^{(\zeta)}) \pmod{\mathfrak{p}}, \\ \text{Tr}((F^s \dot{a})^i w, H^*(\mathcal{B}_u \cap \mathcal{B}_{\mathcal{Z}})) &= \text{Tr}((w, a^i), H^*(\mathcal{B}_u \cap \mathcal{B}_{\mathcal{Z}})^{[\zeta]}) \pmod{\mathfrak{p}} \end{aligned}$$

for any  $\mathfrak{p} \in \mathcal{J}_0$ . Combined with (2.12.2), we have

$$\mathrm{Tr}((w, a^i), H^*(\mathcal{B}_u)^{(\zeta)}) = \mathrm{Tr}((w, a^i), H^*(\mathcal{B}_u \cap \mathcal{B}_Z)^{[\zeta]}) \pmod{\mathfrak{p}}$$

for  $\mathfrak{p} \in \mathcal{J}_0$ . Since  $\mathcal{J}_0$  is an infinite set, we conclude that

$$\mathrm{Tr}((w, a^i), H^*(\mathcal{B}_u)^{(\zeta)}) = \mathrm{Tr}((w, a^i), H^*(\mathcal{B}_u \cap \mathcal{B}_Z)^{[\zeta]}).$$

Hence Proposition 2.6 is proved, and the theorem follows.

### 3. APPLICATIONS

**3.1.** Let  $W_L$  be the subgroup of  $W$ , and  $\Gamma$  the subgroup of  $W$  generated by  $a \in N_W(W_L)$  such that  $\Gamma$  and  $W_L$  generate the semidirect product group  $\widetilde{W}_L = \Gamma \ltimes W_L$ . Let  $V = V^{(\zeta)}$  be the  $\Gamma \times \widetilde{W}_L$ -module as in 1.3. (We write  $\Gamma$  as  $\Gamma_0$  if it is regarded as a subgroup of  $\widetilde{W}_L$ , cf. 1.3.) Then  $V$  can be decomposed as  $V = \bigoplus_{i \in \mathbf{Z}/e\mathbf{Z}} V^{(i)}$ , where  $V^{(i)}$  is the eigenspace of  $a \in \Gamma$  with eigenvalue  $\zeta^i$ , which is a  $\widetilde{W}_L$ -submodule of  $V$ . Then we have

$$\begin{aligned} \mathrm{Ind}_{W_L}^W V &= \bigoplus_i \bigoplus_{w \in W/\widetilde{W}_L} \bigoplus_j w a^j \otimes V^{(i)} \\ &= \bigoplus_i \bigoplus_{w \in W/\widetilde{W}_L} \bigoplus_k w b_k \otimes V^{(i)}, \end{aligned}$$

where  $b_k = \sum_j \zeta^{jk} a^j \in \mathbf{C}[\Gamma]$  (the group ring of  $\Gamma$ ). For each  $i \in \mathbf{Z}$ , let  $\psi^{(i)}$  the linear character of  $\Gamma$  defined by  $\psi^{(i)}(a) = \zeta^i$ . Then  $\Gamma$ -module  $\mathbf{C}b_k$  is afforded by  $\psi^{(-k)}$ . Let  $V_n^{(i)}$  be the eigenspace of  $a \in \Gamma_0$  on the  $\widetilde{W}_L$ -module  $V^{(i)}$  with eigenvalue  $\zeta^n$ . Let  $(\Gamma\text{-Ind}_{W_L}^W V)^{(k)}$  be the eigenspace of  $a \in \Gamma$  with eigenvalue  $\zeta^k$ . Then we have the following lemma.

**Lemma 3.2.** (i) *Let the notations be as above. We have*

$$(3.2.1) \quad (\Gamma\text{-Ind}_{W_L}^W V)^{(k)} \simeq \bigoplus_{w \in W/\widetilde{W}_L} \bigoplus_{j \in \mathbf{Z}/e\mathbf{Z}} \bigoplus_{0 \leq n < e} w b_{k-n-j} \otimes V_n^{(j)}$$

*as vector spaces. In particular,  $\dim(\Gamma\text{-Ind}_{W_L}^W V^{(\zeta)})^{(k)}$  is independent of the choice of  $k \in \mathbf{Z}/e\mathbf{Z}$ , which is given by*

$$(3.2.2) \quad \dim(\Gamma\text{-Ind}_{W_L}^W V)^{(k)} = [W : \widetilde{W}_L] \dim V.$$

(ii) *Assume that  $\Gamma$  commutes with  $W_L$ . Then we have*

$$(3.2.3) \quad (\Gamma\text{-Ind}_{W_L}^W V)^{(k)} \simeq \bigoplus_{j \in \mathbf{Z}/e\mathbf{Z}} \mathrm{Ind}_{\Gamma \times W_L}^W (\psi^{(-k+j)} \otimes V^{(j)})$$

as  $W$ -modules.

*Proof.* Under the action of  $\Gamma$  on  $\Gamma\text{-Ind}_{W_L}^W V$ ,  $wb_k \otimes V_n^{(i)}$  is contained in an eigenspace of  $a$  with eigenvalue  $\zeta^{k+n+j}$ . Then (i) follows easily from the discussion in 3.1. Now assume that  $\Gamma$  commutes with  $W_L$ . Then  $b_k \otimes V^{(i)}$  has a structure of  $\Gamma \times W_L$ -module given by  $\psi^{(-k)} \otimes V^{(i)}$ . (ii) follows from the formula (3.2.1) by noticing that  $V^{(i)} = V_0^{(i)}$ . The lemma is proved.  $\square$

We consider a Levi subgroup  $L \subset G$  and a unipotent element  $u \in L$ , and take  $\Gamma = \langle a \rangle \subset N_W(W_L)$  satisfying the condition in 1.6. We apply the preceding argument to the situation  $V^{(\zeta)} = H^*(\mathcal{B}_u^L)^{(\zeta)}$ . Then as a corollary to Theorem 1.7, we have

**Proposition 3.3.** *Under the setting in Theorem 1.7, we have, for  $0 \leq k \leq e-1$ ,*

$$(3.3.1) \quad \bigoplus_{n \equiv k \pmod{e}} H^{2n}(\mathcal{B}_u) \simeq \bigoplus_{w \in W/\widetilde{W}_L} \bigoplus_{j \in \mathbf{Z}/e\mathbf{Z}} \bigoplus_{0 \leq n < e} wb_{k-n-j} \otimes H^{2j}(\mathcal{B}_u^L)_n$$

as vector spaces, where  $b_i \in \mathbf{C}[\Gamma]$  and  $H^{2n}(\mathcal{B}_u^L)_n$  is the eigenspace of  $a \in \Gamma_0$  with eigenvalue  $\zeta^n$ . In particular,  $\dim(\bigoplus_{n \equiv k \pmod{e}} H^{2n}(\mathcal{B}_u))$  is independent of the choice of  $k$ . In the case (a) in 1.6, (3.3.1) can be made more precise as follows;

$$(3.3.2) \quad \bigoplus_{n \equiv k \pmod{e}} H^{2n}(\mathcal{B}_u) \simeq \text{Ind}_{\Gamma \times W_L}^W \left( \bigoplus_{j \in \mathbf{Z}/e\mathbf{Z}} \psi^{(-k+j)} \otimes H^{2j}(\mathcal{B}_u^L) \right)$$

as  $W$ -modules.

*Proof.* By Theorem 1.7  $\Gamma\text{-Ind}_{W_L}^W H^*(\mathcal{B}_u^L)^{(\zeta)}$  is isomorphic to  $H^*(\mathcal{B}_u)^{(\zeta)}$  as  $\Gamma \times W$ -modules. Since  $(H^*(\mathcal{B}_u)^{(\zeta)})^{(k)} = \bigoplus_{n \equiv k \pmod{e}} H^{2n}(\mathcal{B}_u)$ , the corollary follows from Lemma 3.2.  $\square$

**Remarks 3.4.** (i) In the case where  $u = 1$ , the cohomology ring  $H^*(\mathcal{B}_u) = H^*(\mathcal{B})$  coincides with the coinvariant algebra  $R$  of  $W$ . In the special case where  $G$  is of type  $A_{n-1}$ , i.e.,  $W \simeq \mathfrak{S}_n$ , we consider  $W_L \simeq \mathfrak{S}_{n-re}$  for  $1 \leq e \leq n$ . Then  $W_{L'} \simeq \mathfrak{S}_{re}$ , and if we choose a regular element  $a \in W_{L'}$  as a product of disjoint cycles of length  $e$ , Proposition 3.3 can be applied. This recovers the formula obtained by Morita and Nakajima [MN1].

More generally, consider the Weyl group  $W$  acting on the real vector space  $V$  as the reflection module. For  $v \in V$ , let  $W_v$  be the stabilizer of  $v$  in  $W$ , and  $N_v$  the stabilizer of the line  $\mathbf{R}v$  in  $W$ . Note that  $W_v$  is normal in  $N_v$ , and  $W_v$  coincides with  $W_L$  for a certain Levi subgroup in  $G$ . Then for any  $\Gamma = \langle a \rangle$  such that  $\Gamma \subset N_v$ , Bonnafé, Lehrer and Michel [BLM] have proved a similar formula as in Proposition 3.3. So our formula (3.3.2) can be regarded as a special case of theirs. (Note that they treat a more general case, where  $W$  is a complex reflection group and  $\Gamma$  is not necessarily cyclic, in a framework of coinvariant algebras.)

(ii) We consider a unipotent element  $u \in L$  in the case where  $G = GL_n$ .  $u \in G$  can be written as  $u = u_\mu$  by a partition  $\mu = (1^{m_1}, 2^{m_2}, \dots, n^{m_n})$  of  $n$ . Take a positive integer  $e \geq 2$ , and let  $I$  be a subset of  $\{1, \dots, n\}$  such that  $e \leq m_i$  for  $i \in I$ .

We consider a Levi subgroup  $L$  of type  $X_0 + e \sum_{i \in I} A_{i-1}$ , where  $X_0 = A_k$  with  $k = \sum_{i \notin I} i m_i + \sum_{i \in I} i(m_i - e) - 1$ . Then we have  $W_{L'} = \{1\}$ , and one can choose  $u \in L$  so that it satisfies the assumption of the case (b) in 1.6. Thus Proposition 3.3 can be applied. This covers the results on the stability of dimensions obtained in [MN2], [MN3], where they considered the case  $|I| = 1$  or the case all the  $m_i$  are divisible by  $e$ .

Returning to the general setup, we consider the case where  $u$  is a regular unipotent element in  $L$ . Then  $H^*(\mathcal{B}_u^L) = H^0(\mathcal{B}_u^L) \simeq \mathbf{C}$  is a trivial  $W_L$ -module. Thus Proposition 3.3 implies the following.

**Corollary 3.5.** *Let  $G$  be a simple algebraic group modulo center, and  $L$  a Levi subgroup in  $G$ . Let  $u$  be a regular unipotent element in  $L$ . Let  $\Gamma = \langle a \rangle$  be a subgroup of  $N_W(W_L)$  of order  $e$  satisfying the conditions in 1.6. Then for  $k = 0, \dots, e-1$ , we have*

$$\bigoplus_{n \equiv k \pmod{e}} H^{2n}(\mathcal{B}_u) \simeq \text{Ind}_{\Gamma \rtimes W_L}^W \tilde{\psi}^{(-k)}$$

as  $W$ -modules, where  $\tilde{\psi}^{(-k)}$  is the character of  $\Gamma \rtimes W_L$  obtained as the pull back of  $\psi^{(-k)}$  under the projection  $\Gamma \rtimes W_L \rightarrow \Gamma$ .

*Proof.* In the case (a), the assertion follows from (3.3.2). So we consider the case (b). In the setup of 3.1,  $V^{(i)}$  is a trivial  $W_L$ -module  $\mathbf{C}$  for  $i = 0$  and zero otherwise. Then we see that  $V^{(0)} = V_0^{(0)}$ , and  $b_k \otimes V^{(0)}$  has a structure of  $\widetilde{W}_L$ -module  $\tilde{\psi}^{(-k)}$ . The assertion follows from the formula in 3.1.  $\square$

**3.6.** Let  $G$  be a simple algebraic group defined over  $\mathbf{F}_q$  with Frobenius map  $F$ . We assume that  $G^F$  is of split type. The Green function  $Q_{T_w}$  is defined as the restriction of the Deligne-Lusztig's virtual character  $R_{T_w}(1)$  to the set of unipotent elements in  $G^F$ . We assume that  $p = \text{ch } \mathbf{F}_q$  is good, and in the case where  $G$  is of type  $E_8$ , we further assume that  $q \equiv 1 \pmod{4}$ . Then as explained in 2.7, for each unipotent class  $C$  of  $G$ , there exists a split element  $u \in C^F$ . As in 2.12, we have

$$(3.6.1) \quad Q_{T_w}(u) = \sum_{n \geq 0} \text{Tr}(w, H^{2n}(\mathcal{B}_u)) q^n.$$

Hence there exists a polynomial  $\mathbf{Q}_{w,C}(x) \in \mathbf{Z}[x]$  such that  $Q_{T_w}(u) = \mathbf{Q}_{w,C}(q)$ . Concerning the values of Green functions at root of unity, we have the following.

**Proposition 3.7.** *Suppose that  $G, L$  and  $u \in L$  are as in Corollary 3.5. Then we have*

$$(3.7.1) \quad \mathbf{Q}_{w,C}(\zeta^j) = |W_L|^{-1} \sharp \{x \in W \mid x^{-1}wx \in a^j W_L\}$$

for  $j = 0, \dots, e-1$ . In particular, the value  $\mathbf{Q}_{w,C}(\zeta')$  is independent of the choice of a primitive  $e$ -th root of unity  $\zeta'$ .



*Proof.* Put  $c_i(w) = \#\{x \in W \mid x^{-1}wx \in a^i W_L\}$  for  $i = 0, \dots, e-1$ . Then

$$\begin{aligned} (\text{Ind}_{\Gamma \ltimes W_L}^W \tilde{\psi}^{(-k)})(w) &= |\Gamma \ltimes W_L|^{-1} \sum_{i=0}^{e-1} \sum_{\substack{x \in W \\ x^{-1}wx \in a^i W_L}} \tilde{\psi}^{(-k)}(x^{-1}wx) \\ &= |\Gamma \ltimes W_L|^{-1} \sum_{i=0}^{e-1} c_i(w) \zeta^{-ki}. \end{aligned}$$

It follows, by (3.6.1) together with Corollary 3.5, that

$$\begin{aligned} \mathbf{Q}_{w,C}(\zeta^j) &= \sum_{k=0}^{e-1} \zeta^{kj} \sum_{\substack{n \equiv k \pmod{e}}} \text{Tr}(w, H^{2n}(\mathcal{B}_u)) \\ &= |\Gamma \ltimes W_L|^{-1} \sum_{i=0}^{e-1} c_i(w) \sum_{k=0}^{e-1} \zeta^{(j-i)k} \\ &= |W_L|^{-1} c_j(w). \end{aligned}$$

Hence we obtain the formula (3.7.1). Let  $\zeta^j$  be a primitive  $e$ -th root of unity. There exists an element  $\tau \in \text{Gal}(\mathbf{Q}(\zeta)/\mathbf{Q})$  such that  $\tau(\zeta) = \zeta^j$ . By (3.6.1), we see that  $\mathbf{Q}_{w,C}(\zeta) \in \mathbf{Q}(\zeta)$  and that  $\tau(\mathbf{Q}_{w,C}(\zeta)) = \mathbf{Q}_{w,C}(\zeta^j)$ . But since  $\mathbf{Q}_{w,C}(\zeta) \in \mathbf{Z}$  by (3.7.1), we conclude that  $\mathbf{Q}_{w,C}(\zeta) = \mathbf{Q}_{w,C}(\zeta^j)$ . This proves the proposition.  $\square$

**Remark 3.8.** In the case where  $G = GL_n$  and  $L$  is of type  $A_{m-1} + \dots + A_{m-1}$  ( $e$ -times) with  $n = em$ , take a regular unipotent element  $u$  in  $L$ . Then  $u = u_\mu \in G$  with  $\mu = (m^e)$ . For  $w \in W = \mathfrak{S}_n$ , let  $\lambda(w) = (1^{l_1}, 2^{l_2}, \dots)$  be the partition of  $n$  corresponding to the cycle decomposition of  $w$ . Then one can show by a direct computation (cf. [M, (6.2)]) that

$$|W_L|^{-1} \#\{x \in W \mid x^{-1}wx \in aW_L\} = \begin{cases} e^{l(\lambda(w))} & \text{if } e \mid l_i \text{ for all } i, \\ 0 & \text{otherwise,} \end{cases}$$

where  $l(\lambda)$  is the number of parts for a partition  $\lambda$ . Thus we recover the formula in [LLT, Theorem 3.2, Theorem 3.4] concerning the values of Green polynomials of  $GL_n$  at roots of unity.

**3.9.** We give some more examples where Proposition 3.3 can be applied.

(i) Assume that  $G$  is of type  $B_n$  and  $L$  is a Levi subgroup of type  $B_m$  with  $m < n$ . Then  $L'$  is of type  $A_{n-m-1}$ . For any  $u \in L$  and a divisor  $e$  of  $n-m$ , the proposition can be applied. Similar results hold also for  $C_n$  or  $D_n$ .

(ii) Assume that  $G = Sp_{2n}$ . Then a unipotent element  $u \in G$  can be written as  $u = u_\mu$  as an element of  $GL_{2n}$ , where  $\mu = (1^{m_1}, 2^{m_2}, \dots)$  is a partition of  $2n$  such that  $m_i$  is even for odd  $i$ . Take an even integer  $e \geq 2$ , and let  $I$  be a subset of odd integers  $\{1, 3, \dots, 2n-1\}$  such that  $e \leq m_i$  for  $i \in I$ . We consider a Levi subgroup  $L$  of type  $X_0 + e \sum_{i \in I} A_{i-1}$ , where  $X_0$  is of type  $C_k$  with  $2k = \sum_{i \notin I} im_i + \sum_{i \in I} i(m_i - e)$ .

Then  $W_{L'} = \{1\}$ , and as in Remarks 3.4 (ii), one can find  $u \in L$  so that the case (b) in 1.6 can be applied. Similar results hold also for type  $B_n$  and  $D_n$ .

(iii) Assume that  $G$  is of type  $E_7$ , and choose  $L$  of type  $A_2$  so that  $L'$  is of type  $A_4$ . Take any unipotent element  $u \in L$ . Then the proposition can be applied with  $e = 5$ .

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