The cohomology of line bundles on the three dimensional flag variety

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The purpose of this paper is to give a recursive description of the characters of the cohomology of the line bundles on the three dimensional flag variety over an algebraically closed field k of characteristic $p > 0$. In fact our recursive procedure also involves certain rank 2 bundles and we determine the characters of the cohomology of these bundles at the same time. The paper may be regarded as a substantial worked example of the expansion formula for the character of the cohomology of homogeneous vector bundles, on a generalized flag variety G/B (where G is a reductive group over k and B is a Borel subgroup), given in [**7**].

In Section 1 we set up notation and express the main result of [**7**] in the form most suitable for the application, to the case $G = \mathrm{SL}_3(k)$, given here. Section 2 is the heart of the work and we here work out the precise module structure of the module of invariants of certain tilting modules, under the action of the first infinitesimal subgroup of a maximal unipotent subgroup of G, as well as the invariants of the tensor product of tilting modules and certain two dimensional B-modules. In section 3 we give character formulas for certain homogeneous line bundles and rank 2 bundles on G/B determined by "small" weights. These are the base cases for our recursive procedure for finding the characters of the cohomolgy of all line bundles, and certain rank two bundles on G/B. In Section 4 we apply the results of Section 2 to obtain recursive formulas for the characters of cohomology modules in case $p = 2$. We do this for $p = 3$ in Section 5 and for $p \geq 5$ in Section 6.

In Section 7, we give another smaller application of the main formula of [**7**]. We consider (for any reductive group) the first cohomology group $R\mathrm{Ind}_B^G k_\mu$ of the line bundle determined by $\mu = -p^r\alpha$, where α is a simple root. It is shown in [**12**;II,5.18 Corollary] that this group has simple socle k for $\mu = -p^r\alpha$. We show that in fact $R\mathrm{Ind}_B^G k_\mu$ is equal to k, if α is not isolated. Our route taken to this result gives us an opportunity to add to the number of proofs of Kempf's Vanishing Theorem (see 7.1,(3)) currently available by putting on record a simple argument based on a decomposition space of invariants for the first infinitesimal subgroup B_1 of B acting on a tensor product of a Steinberg module with itself. However, the result on the first cohomology group of a line bundle determined by a multiple of a simple root was already known to Henning Andersen and I am very grateful to him to allowing me to include his short direct proof.

The fact that, for $= \mathrm{SL}_3(k)$, a line bundle on G/B in characteristic $p = 2$ may have non-zero cohomology in two degrees was first noticed by Mumford. Subsequently a precise description (in arbitrary characteristic) of those line bundles and degrees for which there is non-zero cohomology was obtained (see the papers by Griffith [**8**] and Andersen [**1**].)

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1 Preliminaries

1.1 We start by establishing notation and recalling some facts which will be useful in the sequel. We refer the reader to [**12**] for terminology and results not explained here.

Let k denote an algebraically closed field of characteristic $p > 0$. Let G be a semisimple, simply connected, linear algebraic group over k which is defined and split over the prime subfield $\mathbb{F}_p$ of k and let $F : G \to G$ be the corresponding Frobenius morphism. Let B be a Borel subgroup defined and split over $\mathbb{F}_p$ and let U be the unipotent radical of B. Let T be a maximal torus contained in B which is defined and split over $\mathbb{F}_p$.

Let J be an affine group scheme over k. By a left (resp. rignt) J-module we mean a right (resp. left) $k[J]$-comodule and by a morphism of left (resp. right) J-modules we mean a morphism of right (resp. left) J-comodules. A left J-module will usually just be called a J-module. We write $\mathrm{Mod}(J)$ for the category of J-modules and $\mathrm{mod}(J)$ for the category of finite dimensional J-modules. We write $\mathrm{rep}(J)$ for the representation ring of J. We denote the dual module of $V \in \mathrm{mod}(J)$ by V^*. If H is a subgroup scheme of J we write Ind_H^J for the induction functor, from $\mathrm{Mod}(H)$ to $\mathrm{Mod}(J)$. We write J_1 for the first infinitesimal subgroup of J. For $J = G, B, U$

or T the category $\operatorname{mod}(J)$ is identified with the category of rational J-modules and $\operatorname{mod}(J_1)$ is identified with the category of restricted modules for the restricted Lie algebra $\operatorname{Lie}(J)$ of J, in the natural manner. For $V \in \operatorname{mod}(J)$ affording the representation $\pi : J \to \operatorname{GL}(V)$, we write V^F for the vector space V regarded as a G-module via the representation $\pi \circ F$. Recall that if V is a J-module on which J_1 acts trivially then there is a unique J-module Z with underlying vector space V such that $Z^F = V$. We denote Z by $V^{(-1)}$. Recall also that for $V \in \operatorname{mod}(J)$, the space of fixed points $H^0(J_1, V)$ is naturally a J-module on which J_1 acts trivially so that we have available the J-module $H^0(J_1, V)^{(-1)}$.

Let $X(T)$ denote the weight lattice, i.e. the character group of T. The Weyl group $W = N_G(T)/T$ acts naturally on $X(T)$ as group automorphisms so that $\mathbb{R} \otimes_{\mathbb{Z}} X(T)$ has a natural $\mathbb{R}W$-module structure. We choose a positive definite, W-invariant, symmetric, real, bilinear form on $\mathbb{R} \otimes_{\mathbb{Z}} X(T)$. Let Φ denote the set of roots. Let Φ^+ denote the system of positive roots which makes B the negative Borel subgroup. We sometimes write $\alpha > 0$ to indicate that α is a positive root. Let Π denote the set of simple roots. The longest element of W will be denoted w_0. The element $\rho \in X^+(T)$ is defined by $2\rho = \sum_{\alpha>0} \alpha$. The so-called "dot action" of W on $X(T)$ is given by $w \cdot \lambda = w(\lambda + \rho) - \rho$, for $\lambda \in X(T)$, $w \in W$. The weight lattice has a natural partial order: we write $\lambda \leq \mu$ if $\mu - \lambda$ may be expressed as a sum of positive roots (for $\lambda, \mu \in X(T)$). For $\alpha \in \Phi$ we write $\check{\alpha}$ for $2\alpha/(\alpha, \alpha)$. Let $X^+(T) = \{\lambda \in X(T) \,|\, (\lambda, \alpha) \geq 0 \text{ for all } \alpha > 0\}$, the set of dominant weights. Let $X_1(T) = \{\lambda \in X(T) \,|\, 0 \leq (\lambda, \check{\alpha}) < p \text{ for all } \alpha \in \Pi\}$, the set of restricted (dominant) weights.

The integral group ring $\mathbb{Z}X(T)$ has $\mathbb{Z}$-basis $e(\lambda)$, $\lambda \in X(T)$, and multiplication is given by $e(\lambda)e(\mu) = e(\lambda + \mu)$ (for $\lambda, \mu \in X(T)$). Let $M \in \operatorname{mod}(T)$. For $\mu \in X(T)$, we have the weight space $M^\mu = \{v \in M \,|\, tv = \mu(t) \text{ for all } t \in T\}$. Then $M = \oplus_{\mu \in X(T)} M^\mu$ and the character $\operatorname{ch} M \in \mathbb{Z}X(T)$ is defined by $\operatorname{ch} M = \sum_{\mu \in X(T)} (\dim M^\mu) e(\mu)$. For $\phi = \sum_{\mu \in X(T)} a_\mu e(\mu) \in \mathbb{Z}X(T)$ we set $\phi^F = \sum_{\mu \in X(T)} a_\mu e(p\mu)$. We have $\operatorname{ch} M^F = (\operatorname{ch} M)^F$, for $M \in \operatorname{mod}(T)$.

For $\mu \in X(T)$ let k_μ denote the one dimensional B-module on which T acts via μ. Let $\lambda \in X^+(T)$. We write $\nabla(\lambda)$ for the induced module $\operatorname{Ind}_B^G(k_\lambda)$. The socle $L(\lambda)$ of $\nabla(\lambda)$ is simple and $\{L(\lambda) \,|\, \lambda \in X^+(T)\}$ is a complete set of pairwise non-isomorphic simple G-modules. We denote the sign of $w \in W$ by $\operatorname{sgn}(w)$. For $\lambda \in X(T)$ the Weyl character $\chi(\lambda)$ is given by

$$\chi(\lambda) = \Big(\sum_{w \in W} \operatorname{sgn}(w) e(w(\lambda + \rho))\Big) / \Big(\sum_{w \in W} \operatorname{sgn}(w) e(w\rho)\Big).$$

For $\lambda \in X^+(T)$ we have $\chi(\lambda) = \operatorname{ch} \nabla(\lambda)$. We shall also need the orbit sum $s(\mu) = \sum_{\nu \in W\mu} \nu$, for $\mu \in X(T)$.

The character of a finite dimensional G-module belongs to the ring of invariants $A = (\mathbb{Z}X(T))^W$ and A is freely generated, as an abelian group, by the Weyl characters $\chi(\lambda)$, $\lambda \in X^+(T)$. We have on A the natural inner product for which the elements $\chi(\lambda)$, $\lambda \in X^+(T)$, form an orthonormal basis.

Let $A^F = \{\chi^F \mid \chi \in A\}$ then A^F is a subring of A. Moreover, A is free as a module over A^F, on basis $\chi(\lambda)$, $\lambda \in X_1(T)$. We write ζ for $(p-1)\rho$ (often known as the Steinberg weight). We say that A^F bases $(\phi_\lambda)_{\lambda \in X_1(T)}$ and $(\psi_\lambda)_{\lambda \in X_1(T)}$ of A are *p-dual bases* if we have $(\phi_\lambda \eta^F, \chi(\zeta)\psi_\mu \theta^F) = \delta_{\lambda\mu}(\eta, \theta)$, for all $\lambda, \mu \in X_1(T)$ and $\eta, \theta \in A$.

A B-module M gives rise to an induced vector bundle $\mathcal{L}(M)$ on the generalized flag variety G/B. We write simply $\mathcal{L}(\lambda)$ for $\mathcal{L}(k_\lambda)$, $\lambda \in X(T)$. For each $i \geq 0$, the sheaf cohomology group $H^i(G/B, \mathcal{L}(M))$ has a natural G-module structure and the induced module $\nabla(\lambda)$ may be identified with the module of global sections $H^0(G/B, \mathcal{L}(\lambda))$, for $\lambda \in X^+(T)$. For $M \in \operatorname{mod}(B)$ we write $\chi^i(M)$ for $\operatorname{ch} H^i(G/B, \mathcal{L}(M))$ and write simply $\chi^i(\mu)$ for $\chi^i(k_\mu)$, for $\mu \in X(T)$.

The modules $L(\lambda)$, $\lambda \in X_1(T)$, form a complete set of pairwise non-isomorphic irreducible G_1-modules. We denote by $\hat{Q}_1(\lambda)$ the projective cover of $L(\lambda)$ as a G_1T-module, for $\lambda \in X_1(T)$. The modules $\hat{Q}_1(\lambda)$, $\lambda \in X_1(T)$, form a complete set of pairwise non-isomorphic projective indecomposable G_1-modules. The character $\operatorname{ch} \hat{Q}_1(\lambda)$ is divisible by $\chi(\zeta)$, for $\lambda \in X_1(T)$. Moreover, setting $\phi_\lambda = \operatorname{ch} L(\lambda)$, $\psi_\lambda = \operatorname{ch} \hat{Q}_1(\lambda)/\chi(\zeta)$ we have that $(\phi_\lambda)_{\lambda \in X_1(T)}$ and $(\psi_\lambda)_{\lambda \in X_1(T)}$ are p-dual bases (see [**7**;1.2(1) and Proposition 1.4b(ii)]).

By a good filtration of $M \in \operatorname{mod}(G)$ we mean a filtration $0 = M_0 \leq M_1 \leq \cdots \leq M_n = M$ such that for each $0 < i \leq n$ the section M_i/M_{i-1} is either 0 or isomorphic to $\nabla(\lambda_i)$, for some $\lambda_i \in X^+(T)$. For $\lambda \in X^+(T)$ we write $(M : \nabla(\lambda))$ for the number of i such that $1 \leq i \leq n$ and $M_i/M_{i-1} \cong \nabla(\lambda)$.

By a *tilting module* for G we mean a finite dimensional G-module M such that M admits a good filtration and also the dual module M^* admits a good filtration. For each $\lambda \in X^+(T)$ there is an indecomposable tilting module $M(\lambda)$ which has highest weight λ. Every tilting module is a direct sum of copies of $M(\lambda)$, $\lambda \in X^+(T)$.

For $\lambda \in X_1$ the tilting module $M(\zeta + \lambda)$ is projective as a G_1-module. It is conjectured that in general $M(\zeta+\lambda)$ is indecomposable, as a G_1-module (see [**5**;Section 1]). This is known for $p \geq 2h-2$, where h is the Coxeter number of G, and all primes for $G = \mathrm{SL}_3(k)$, see e.g. [**7**; Section 3]. We assume for the rest of 1.1 that this conjecture holds for G. Then $M(\zeta+\lambda)$ is the projective cover of $L(\zeta+w_0\lambda)$, as a G_1T-module, i.e. we have $M(\zeta+\lambda)|_{G_1T} \cong \hat{Q}_1(\zeta+w_0\lambda)$. By [**7**;1.1 Remark 1] we then have

$$\chi^i(Y) = \sum_{\lambda\in X_1(T)} \operatorname{ch} L(\zeta+w_0\lambda)^* \chi^i(H^0(B_1, M(\zeta+\lambda)\otimes Y)^{(-1)})^F$$

for $Y \in \operatorname{mod}(B)$. (This may be seen as a special case of the main result of [**7**] : in the general formulation there is no assumption that the conjecture mentioned above holds.)

For a B-module M and $\mu \in X(T)$ we will often abbreviate $k_\mu \otimes M$ to $\mu\otimes M$ and $M\otimes k_\mu$ to $M\otimes\mu$. We have $L(\zeta+w_0\lambda)^* \cong L(\zeta-\lambda)$ so that, replacing Y by $\zeta\otimes Y$ in the above formula we get:

$$\chi^i(Y\otimes\zeta) = \sum_{\lambda\in X_1(T)} \operatorname{ch} L(\zeta-\lambda)\chi^i(H^0(B_1, M(\zeta+\lambda)\otimes\zeta\otimes Y)^{(-1)})^F.$$

We shall convert this expression to one more suited to our calculations in the later sections. To do this we shall need the following result.

(1) *Let Y be a B-module which is trivial as a U_1-module. Then Y is semisimple as a B_1-module and indeed we have $Y \cong \bigoplus_{\lambda\in X_1(T)} k_\lambda \otimes Y_\lambda^F$, where $Y_\lambda = \operatorname{Hom}_{B_1}(k_\lambda, Y)^{(-1)}$. Moreover, if $0\to Y'\to Y\to Y''\to 0$ is a short exact sequence of B-modules which are trivial as U_1-modules, then we have a short exact sequence of B-modules $0\to Y'_\lambda\to Y_\lambda\to Y''_\lambda\to 0$, for $\lambda\in X_1(T)$.*

Proof. Note that Y is semisimple as a $U_1T = B_1T$-module and hence as a B_1-module. It follows that the natural map $f : \bigoplus_{\lambda\in X_1(T)} k_\lambda\otimes\operatorname{Hom}_{B_1}(k_\lambda, Y)\to Y$ is a linear isomorphism. However, f is a B-module map and thus a B-module isomorphism. The first statement follows. The second statement follows from the fact that $\operatorname{Hom}_{B_1}(k_\lambda, -)$ is exact on semisimple B_1-modules.

The notation Y_λ (for Y a B-module trivial as a U_1-module and $\lambda\in X_1(T)$) as above will be used a great deal in what follows.

Now replacing Y by $Y\otimes k_{-\mu}$ in the above formula we arrive at our preferred formulation:

$$\chi^i(Y\otimes(\zeta-\mu)) = \sum_{\lambda\in X_1(T)} \operatorname{ch} L(\zeta-\lambda)\chi^i(H^0(U_1, M(\zeta+\lambda)\otimes\zeta\otimes Y)_\mu)^F \qquad (*)$$

for $Y\in\operatorname{mod}(B)$, $\mu\in X_1(T)$.

1.2 We recall some further generalities which will be useful in the context of our SL_3 calculation.

(1) *Let Y be a G-module.*
(i) If k_μ occurs in the B-socle of Y then we have $\mu = w_0\lambda$ for some $\lambda \in X^+(T)$.
Assume further that Y is finite dimensional and has a good filtration.
(ii) For $\mu = w_0\lambda$, with $\lambda \in X^+(T)$, we have $\dim \mathrm{Hom}_B(k_\mu, Y) = (Y : \nabla(\lambda))$.
(iii) The dimension of the B-socle of Y is equal to the length (i.e the number of non-zero sections) of a good filtration of Y.

Proof. (i) If k_μ occurs in the B-socle of Y then it occurs in the B-socle of some composition factor of some composition factor $L(\lambda)$, $\lambda \in X^+(T)$. Thus we may assume $Y = L(\lambda)$, which has B-socle $k_{w_0\lambda}$.
(ii) From [**3**;(3.4) Corollary], we get that $\mathrm{Hom}_B(k_\mu, -)$ is exact on modules with a good filtration, for $\mu = w_0\lambda$ with $\lambda \in X^+(T)$. Thus (ii) reduces to the special case in which $Y = \nabla(\tau)$ for some $\tau \in X^+(T)$, and this module has B socle $k_{w_0\tau}$.
(iii) follows from (ii).

(2) *If $Y \in \mathrm{mod}(B_1T)$ is projective as a B_1-module then $\mathrm{ch}\, Y$ is divisible by $\chi(\zeta)$ and writing $\mathrm{ch}\, Y = \chi(\zeta)\psi$, $\psi \in \mathbb{Z}X(T)$, we have*

$$\mathrm{ch}\, H^0(U_1, Y \otimes \zeta) = \psi.$$

Proof. We may assume Y to be indecomposable and then the result follows from [**12**;II, 9.2 Lemma (3) and 9.4 Lemma a)].

(3) *Let $Y \in \mathrm{mod}(G)$ and $\mu \in X(T)$.*
(i) We have $\mathrm{Hom}_B(k_\mu, H^0(U_1, Y \otimes \zeta)) = 0$ if $\zeta - \mu \notin X^+(T)$.
(ii) If Y admits a good filtration and $\zeta - \mu \in X^+(T)$ we have

$$\dim \mathrm{Hom}_B(k_\mu, H^0(U_1, Y \otimes \zeta) = (Y : \nabla(\zeta + w_0\mu)).$$

Proof. We have $\mathrm{Hom}_B(k_\mu, H^0(U_1, Y \otimes \zeta)) = \mathrm{Hom}_B(k_{\mu-\zeta}, Y)$ so that (i) follows from (1)(i). Assuming now that Y admits a good filtration and $\zeta - \mu \in X^+(T)$ we get $\dim \mathrm{Hom}_B(k_{\mu-\zeta}, Y) = (Y : \nabla(w_0(\mu - \zeta))) = (Y : \nabla(\zeta + w_0\mu))$ from (1)(iii).

Let $\theta = \sum_\mu a_\mu e(\mu) \in \mathbb{Z}X(T)$ and suppose $a_\mu = b_\mu - c_\mu$, for non-negative integers b_μ, c_μ, $\mu \in X(T)$. We define $k(\theta)$ to be the virtual module $[M_1] - [M_2] \in \mathrm{rep}(B)$, where M_1 is the semisimple B-module with character $\sum_\mu b_\mu e(\mu)$ and M_2 is the semisimple B-module with character $\sum_\mu c_\mu e(\mu)$. If we have $a_\mu \geq 0$ for all μ we also write $k(\theta)$ for the semisimple B-module with character θ. We also write $k(\mu)$ for $k(e(\mu)) = k_\mu$, $\mu \in X(T)$.

(4) *Suppose that $Y \in \mathrm{mod}(G)$ admits a good filtration. We set $\theta = \sum_\mu a_\mu e(\mu)$, where μ ranges over elements of $X(T)$ such that $\zeta - \mu \in X^+(T)$ and where $a_\mu = (Y : \nabla(\zeta + w_0\mu))$, for such μ. Then the B-socle of $Y \otimes \zeta$ is $k(\theta)$.*

Proof. This follows from (3)(ii).

One can find a discussion of the following properties of Weyl characters in, for example, [**4**; Chapter 2]. These properties will be used in the sequel without further comment.

(5) *(i) For $\lambda \in X^+(T)$ and $\sum_\mu a_\mu e(\mu) \in (\mathbb{Z}X(T))^W$ we have*
$\chi(\lambda) \sum_\mu a_\mu e(\mu) = \sum_\mu a_\mu \chi(\lambda + \mu)$ (Brauer's formula).
(ii) For $\lambda \in X(T)$, $w \in W$, we have $\chi(w \cdot \lambda) = \mathrm{sgn}(w)\chi(\lambda)$.
(iii) For $\lambda \in X(T)$ we have $\chi(\lambda) = 0$ if $(\lambda, \check{\alpha}) = -1$, for some $\alpha \in \Pi$.

2 Infinitesimal Invariants

2.1 We now specialize to the case $G = \mathrm{SL}_3(k)$. We take $F : G \to G$ to be usual Frobenius morphism, i.e. the map which takes the matrix $(a_{ij}) \in G$ to the matrix (a_{ij}^p). We take B to the subgroup consisting of the lower triangular matrices in G so that U is the subgroup of lower unitriangular matrices. We take T to be the subgroup consisting of the diagonal matrices in G. Then $X(T)$ is freely generated by ω_1, ω_2, where $\omega_1(t) = t_1$ and $\omega_2(t) = t_1t_2$, for $t \in T$ with $(1,1)$-entry t_1 and $(2,2)$-entry t_2. For integers a, b we often abbreviate $a\omega_1 + b\omega_2$ as (a, b). Then $X^+(T) = \{(a, b) \,|\, a, b \geq 0\}$ and $X_1(T) = \{(a, b) \,|\, 0 \leq a, b < p\}$. We have $\Pi = \{\alpha, \beta\}$, where $\alpha = (2, -1)$, $\beta = (-1, 2)$, and the positive roots are α, β, γ, where $\gamma = \alpha + \beta$. For a root θ, we write s_θ for the corresponding reflection (an element of the Weyl group W).

By [**12**;II 5.20 Proposition a)], there is a unique (up to isomorphism) two dimensional indecomposable B-module with character $e(0) + e(-\alpha)$, we denote this by $N(\alpha)$. Thus there is a non-split short exact sequence $0 \to k_{-\alpha} \to N(\alpha) \to k \to 0$. Similarly there is a unique two dimensional indecomposable B-module with character $e(0)+e(-\beta)$ and we denote this $N(\beta)$. Thus there is a non-split exact sequence $0 \to k_{-\beta} \to N(\beta) \to k \to 0$. However, any B-module with character $e(0) + e(-\gamma)$ is necessarilty semisimple. We write $N(\gamma)$ for $k \oplus k_{-\gamma}$.

For $Y \in \mathrm{mod}(B)$ we write $H^0_\alpha(U_1, Y)$ for $H^0(U_1, Y \otimes N(\alpha))$ and $H^0_\beta(U_1, Y)$ for $H^0(U_1, Y \otimes N(\beta))$. In order to give an inductive description of $\chi^i(\lambda)$, for $\lambda \in X(T)$, we shall need, at the same time, similar descriptions of $\chi^i(\lambda \otimes N(\alpha))$ and $\chi^i(\lambda \otimes N(\beta))$. We write $\chi^i_\alpha(\lambda)$ for $\chi^i(\lambda \otimes N(\alpha))$ and $\chi^i_\beta(\lambda)$ for $\chi^i(\lambda \otimes N(\beta)$. We write out explicitly the appropriate versions of the expansion formula (*) of 1.1.

Lemma 1 *For $\mu \in X_1(T)$, $\nu \in X(T)$ we have:*

$$\chi^i((\zeta - \mu) + p\nu) = \sum_{\lambda \in X_1(T)} \mathrm{ch}\, L(\zeta - \lambda)\chi^i(H^0(U_1, M(\zeta + \lambda) \otimes \zeta)_\mu \otimes \nu)^F;$$

$$\chi^i_\alpha((\zeta - \mu) + p\nu) = \sum_{\lambda \in X_1(T)} \mathrm{ch}\, L(\zeta - \lambda)\chi^i(H^0_\alpha(U_1, M(\zeta + \lambda) \otimes \zeta)_\mu \otimes \nu)^F; \quad \textit{and}$$

$$\chi^i_\beta((\zeta - \mu) + p\nu) = \sum_{\lambda \in X_1(T)} \mathrm{ch}\, L(\zeta - \lambda)\chi^i(H^0_\beta(U_1, M(\zeta + \lambda) \otimes \zeta)_\mu \otimes \nu)^F.$$

Let $\phi_\lambda = \mathrm{ch}\, L(\lambda)$, $\lambda \in X_1(T)$. The characters ϕ_λ are given as follows (see e.g. [**11**;p18]).

Lemma 2 *For $\lambda = (a, b) \in X_1(T)$ we have:*

$$\phi_\lambda = \begin{cases} \chi(a,b) - \chi(a-r, b-r), & \text{if } a+b+2 = p+r, r > 0; \\ \chi(a,b), & \text{otherwise.} \end{cases}$$

For $\lambda \in X(T)$ and $Y \in \mathrm{mod}(B)$ we shall often write simply $\lambda{\cdot}Y$ for $\lambda \otimes Y = k_\lambda \otimes Y$. We write E for the natural module for $\mathrm{SL}_3(k)$ and write V for the dual of E. The following is well known and easy to check.

Lemma 3 *We have short exact sequences of B-modules:*

$$\begin{aligned} 0 \to &(-1,1) \cdot N(\beta) \to E \to (1,0) \to 0; \\ 0 \to &(0,-1) \to E \to (1,0) \cdot N(\alpha) \to 0; \\ 0 \to &(1,-1) \cdot N(\alpha) \to V \to (0,1) \to 0; \textit{and} \\ 0 \to &(-1,0) \to V \to (0,1) \cdot N(\beta) \to 0. \end{aligned}$$

Furthermore, for each B-module extension $0 \to K \to L \to M \to 0$ above, the middle term L is, up to isomorphism, the unique non-split extension of M by K.

The following will be used many times in the sequel.

Lemma 4 *For $\mu = (r, s) \in X(T)$ and $M \in \mathrm{mod}(G)$ we have:*
(i) $\mathrm{Ind}_B^G(M \otimes N(\alpha) \otimes \mu) = H^0(B, M \otimes N(\alpha) \otimes \mu) = 0$ *if* $r \leq 0$; *and*
(ii) $\mathrm{Ind}_B^G(M \otimes N(\beta) \otimes \mu) = H^0(B, M \otimes N(\beta) \otimes \mu) = 0$ *if* $s \leq 0$.

Proof. Suppose $r \leq 0$. Then, by Lemma 3, $N(\alpha) \otimes (r, s)$ embeds in $V \otimes (r-1, s+1)$. Hence $\mathrm{Ind}_B^G(M \otimes N(\alpha) \otimes (r, s))$ embeds in $\mathrm{Ind}_B^G(M \otimes V \otimes (r-1, s+1))$ which is isomorphic to $M \otimes V \otimes \mathrm{Ind}_B^G k(r-1, s+1)$ by the tensor identity and hence is 0 since $(r-1, s+1)$ is not dominant. Now, by Frobenius reciprocity, we have

$$H^0(B, M \otimes N(\alpha) \otimes \mu) = H^0(G, \mathrm{Ind}_B^G(M \otimes N(\alpha) \otimes \mu)) = 0.$$

The second assertion is proved similarly.

Recall from 1.1 that the elements $\psi_\lambda = \mathrm{ch}\, \hat{Q}_1(\lambda)/\chi(\zeta)$, $\lambda \in X_1(T)$, form a p-basis of A which is dual to the basis ϕ_λ, $\lambda \in X_1(T)$. The p-basis ψ_λ, $\lambda \in X_1(T)$, may be easily calculated via the duality and the explicit description, Lemma 2, of the basis ϕ_λ, $\lambda \in X_1(T)$. From the fact that $\mathrm{ch}\, M(2\zeta + w_0\lambda) = \mathrm{ch}\, \hat{Q}_1(\lambda)$ (see 1.1) one obtains the following simple explicit description of the characters $\mathrm{ch}\, M(\zeta+\lambda)/\chi(\zeta)$.

Lemma 5 *For $\lambda = (a, b) \in X_1(T)$ we have*

$$\mathrm{ch}\, M(\zeta + \lambda)/\chi(\zeta) = \begin{cases} s(\lambda), & \text{if } a+b \leq p; \\ s(\lambda) + s(\lambda - r\rho), & \text{if } a+b = p+r \text{ with } r > 0. \end{cases}$$

2.2 We calculate the invariants $H^0(U_1, M(\zeta+\lambda)\otimes\zeta)$, for $\lambda \in X_1(T)$.

Lemma 1 *Let $\lambda = (a,b) \in X_1(T)$ with $a+b<p$. Then we have* $H^0(U_1, M(\zeta+\lambda)\otimes\zeta) = k(s(\lambda))$.

Proof. We have $\operatorname{ch} M(\zeta+\lambda) = \chi(\zeta)s(\lambda)$, by 2.1 Lemma 5. It follows that $M(\zeta+\lambda)$ has a filtration with sections $\nabla(\zeta+\xi)$, $\xi \in W\lambda$, and we get the result from 1.2,(4) and 1.2,(2).

Lemma 2 *Let $\lambda = (a,b) \in X_1(T)$ with $a+b=p$. Then we have*

$$\begin{aligned} H^0(U_1, M(\zeta+\lambda)\otimes\zeta) &= (a,b)\cdot N(\gamma)^F \oplus (-a,p)\cdot N(\beta)^F \oplus (p,-b)\cdot N(\alpha)^F \\ &= \lambda\cdot N(\gamma)^F \oplus s_\alpha\lambda\cdot N(\beta)^F \oplus s_\beta\lambda\cdot N(\alpha)^F. \end{aligned}$$

Proof. Let $Y = H^0(U_1, M(\zeta+\lambda)\otimes\zeta)$. Since $\operatorname{ch} M(\zeta+\lambda) = \chi(\zeta)s(\lambda)$ we have $\operatorname{ch} Y = s(\lambda)$. It follows from 1.3,(3),(4) that

$$Y = (a,b)\cdot Y_1^F \oplus (-a,p)\cdot Y_2^F \oplus (p,-b)\cdot Y_3^F$$

where $\operatorname{ch} Y_1 = \operatorname{ch} N(\gamma)$, $\operatorname{ch} Y_2 = \operatorname{ch} N(\beta)$ and $\operatorname{ch} Y_3 = \operatorname{ch} N(\alpha)$. It follows that $Y_1 = N(\gamma)$. Moreover, we have

$$\begin{aligned} \operatorname{ch} M(\zeta+\lambda) &= \chi(\zeta)s(\lambda) \\ &= \chi(\zeta+(a,b)) + \chi(\zeta+(-b,-a)) + \chi(\zeta+(-a,p)) \\ &\quad + \chi(\zeta+(b,-p)) + \chi(\zeta+(p,-b)) + \chi(\zeta+(-p,a)) \\ &= \chi(\zeta+(a,b)) + \chi(\zeta+(-a,p)) + \chi(\zeta+(p,-b)) + \chi(\zeta+(-b,-a)) \end{aligned}$$

so that, by 1.2,(4), Y has 4 dimensional B-socle. It follows that $Y_2 = N(\beta)$ and $Y_3 = N(\alpha)$.

Lemma 3 *Let $\lambda = (a,b) \in X_1(T)$ with $a+b>p$. Then we have*

$$\begin{aligned} H^0(U_1,&M(\zeta+\lambda)\otimes\zeta) \\ &= (a,b)\cdot N(\gamma)^F \oplus (p-a,a+b-p)\cdot[(-1,1)\cdot N(\beta)]^F \\ &\oplus (a+b-p,p-b)\cdot[(1,-1)\cdot N(\alpha)]^F \oplus (b,2p-a-b)\cdot[(-1,0)\cdot N(\beta)]^F \\ &\oplus (2p-a-b,a)\cdot[(0,-1)\cdot N(\alpha)]^F \oplus (p-b,p-a)\cdot N(\gamma)^F. \end{aligned}$$

Proof. Note that $p > 2$. Suppose that $p > 3$. It follows from 1.1(1) that we have a decomposition

$$\begin{aligned} H^0(U_1,&M(\zeta+\lambda)\otimes\zeta) \\ &= (a,b)\cdot Y_1^F \oplus (p-a,a+b-p)\cdot Y_2^F \\ &\oplus (a+b-p,p-b)\cdot Y_3^F \oplus (b,2p-a-b)\cdot Y_4^F \\ &\oplus (2p-a-b,a)\cdot Y_5^F \oplus (p-b,p-a)\cdot Y_6^F \qquad (*) \end{aligned}$$

where $\mathrm{ch}\,Y_1 = \mathrm{ch}\,N(\gamma)$, $\mathrm{ch}\,Y_2 = \mathrm{ch}\,N(\beta)$, $\mathrm{ch}\,Y_3 = \mathrm{ch}\,(1,-1)\cdot N(\alpha)$, $\mathrm{ch}\,Y_4 = \mathrm{ch}\,(-1,0)\cdot N(\beta)$, $\mathrm{ch}\,Y_5 = \mathrm{ch}\,(0,-1)\cdot N(\alpha)$ and $\mathrm{ch}\,Y_6 = \mathrm{ch}\,N(\gamma)$.

Now (by characters) Y_1 and Y_6 must be semisimple. It follows that the dimension of the B-socle of $H^0(U_1, M(\zeta+\lambda)\otimes\zeta)$ is at least 8 and, indeed, if the dimension is precisely 8 then each of Y_2, Y_3, Y_4, Y_5 has a simple socle and we must have $Y_2 = N(\beta)$, $Y_3 = (1,-1)\cdot N(\alpha)$, $Y_4 = (-1,0)\cdot N(\beta)$ and $Y_5 = (0,-1)\cdot N(\alpha)$. Thus it suffices to prove that $M(\zeta+\lambda)\otimes\zeta$ has 8-dimensional B-socle, equivalently that $M(\zeta+\lambda)$ has 8-dimensional B-socle. Hence, by 1.2,(3)(ii), it suffices to prove that $M(\zeta+\lambda)$ has good filtration length 8. We have that $\mathrm{ch}\,M(\zeta+\lambda) = \chi(\zeta)s(\lambda) + \chi(\zeta)s(\mu)$, where $\mu = \lambda - r\rho$, $r = a+b-p$. We leave it to the reader, using 1.2,(5), that $\mathrm{ch}\,M(\zeta+\lambda)$ is the sum of 8 terms of the form $\chi(\xi)$, $\xi \in X^+(T)$.

In the case $p = 3$ we must have $\lambda = (2,2)$ and 1.1,(1) gives a decomposition

$$H^0(U_1, M(\zeta+\lambda)\otimes\zeta) = (2,2)\cdot Z_1^F \oplus (1,1)\cdot Z_2^F$$

where Z_1 has the same character as $N(\gamma)\oplus(-1,0)\cdot N(\beta)\oplus(0,-1)\cdot N(\alpha)$ and where Z_2 has the same character as $(-1.1)\cdot N(\beta)\oplus(1,-1)\cdot N(\alpha)\oplus(1,1)\cdot N(\gamma)$. However, the weights of an indecomposable B-module all lie in the same coset of the root lattice in the weight lattice. It follows that we have decompositions $Z_1 = P_1\oplus Q_1\oplus R_1$, where $\mathrm{ch}\,P_1 = \mathrm{ch}\,N(\gamma)$, $\mathrm{ch}\,Q_1 = \mathrm{ch}\,(-1,0)\cdot N(\beta)$, $\mathrm{ch}\,R_1 = \mathrm{ch}\,(0,-1)\cdot N(\alpha)$, and $Z_2 = P_2\oplus Q_2\oplus R_2$, where $\mathrm{ch}\,P_2 = \mathrm{ch}\,(-1.1)\cdot N(\beta)$, $\mathrm{ch}\,Q_2 = \mathrm{ch}\,(1,-1)\cdot N(\alpha)$, $\mathrm{ch}\,R_2 = \mathrm{ch}\,(1,1)\cdot N(\gamma)$. It follows that (*) is valid too in the case $p = 3$ and one may conclude as in the case $p > 3$.

2.3 We now embark on the more delicate task of calculating the infinitesimal invariants of $M(\zeta+\lambda)\otimes\zeta\otimes N(\alpha)$, for $\lambda = (a,b)\in X_1(T)$. We tackle here the case $\lambda = (a,b)\in X_1(T)$ with $a+b\le p$ and, in 2.4, the case $a+b>p$.

For $\lambda = (a,b)\in X(T)$ the length $\ln(\lambda)$ of λ is defined to by $\ln(\lambda) = a+b$. We have $\ln(\alpha) = \ln(\beta) = 1$ so that $\lambda\le\mu$ implies $\ln(\lambda)\le\ln(\mu)$ (for $\lambda,\mu\in X(T)$).

Lemma 1 *For $\lambda = (a,b)$ with $a+b < p-1$ we have $H^0(U_1, M(\zeta+\lambda)\otimes\zeta\otimes N(\alpha)) = k(\theta)$, where $\theta = s(\lambda)(1+e(-\alpha))$.*

Proof. Let $Y = H^0(U_1, M(\zeta+\lambda)\otimes\zeta\otimes N(\alpha))$. Then $\operatorname{ch} Y = \theta$, so that it suffices to prove that Y is a semisimple B-module. Let $Y' = H^0(U_1, M(\zeta+\lambda)\otimes(\zeta-\alpha))$ and $Y'' = H^0(U_1, M(\zeta+\lambda)\otimes\zeta)$. For each $\mu \in X_1(T)$ we have, by 1.2(1), a short exact sequence $0 \to Y'_\mu \to Y_\mu \to Y''_\mu \to 0$. Each of Y'_μ and Y''_μ is semisimple, by 2.2 Lemma 1. We will therefore be done if we prove that for τ' a weight of Y'_μ and τ'' a weight of Y''_μ we have $\operatorname{Ext}^1_B(k_{\tau''}, k_{\tau'}) = 0$. Assume for a contradiction that this is not the case. Then $\theta = \tau' - \tau''$ is a non-zero element of the root lattice $\mathbb{Z}\Phi$. Moreover, we have $\mu + p\tau' = w\lambda - \alpha$ and $\mu + p\tau'' = y\lambda$, for some $w, y \in W$. Let $z \in W$ be such that $z\theta \in X^+(T)$. We have $p\theta = w\lambda - \alpha - y\lambda$ and so

$$pz\theta = zw\lambda - z\alpha - zy\lambda.$$

Hence we have $p\ln(z\theta) = \ln(zw\lambda) + \ln(-z\alpha) + \ln(-zy\lambda) \leq \ln(\lambda) + \ln(\gamma) + \ln(-w_0\lambda) \leq 2(p-2)+2$. Hence $p\ln(z\theta) < 2p$ and so $\ln(z\theta) = 1$. But then $z\theta = \omega_\alpha$ or ω_β but this contradicts the fact that θ lies in the root lattice.

Lemma 2 *(i) For $\lambda = (p-1, 0)$ we have*

$$\begin{aligned} H^0(U_1, M(\lambda+\zeta)\otimes\zeta\otimes N(\alpha)) = (p-1,0)\cdot N(\alpha)^F \oplus (-p+1, p-1) \\ \oplus(0,-p+1) \oplus (p-3,1) \oplus (-2,-p+2). \end{aligned}$$

(ii) For $\lambda = (0, p-1)$ we have

$$\begin{aligned} H^0(U_1, M(\lambda+\zeta)\otimes\zeta\otimes N(\alpha)) =& (p-1,-p+1)\cdot N(\alpha)^F \\ &\oplus(0,p-1) \oplus (-p+1,0) \oplus (-2,p) \oplus (p-3,-p+2). \end{aligned}$$

Proof. (i) The statement is correct at the level of characters. Moreover, putting $Y = H^0(U_1, M(\lambda+\zeta)\otimes\zeta\otimes N(\alpha))$ we have (in the notation of 1.2, (1)) $\operatorname{ch} Y_{(p-1,0)} = \operatorname{ch} N(\alpha)$ and so if the result is false then $Y_{(p-1,0)}$ contains a copy of k and $H^0(U_1, M(\lambda+\zeta)\otimes\zeta\otimes N(\alpha))$ contains a copy of $(p-1,0)$ and hence

$$\begin{aligned} 0 &\neq H^0(B, (-p+1,0)\otimes H^0(U_1, M(\lambda+\zeta)\otimes\zeta\otimes N(\alpha))) \\ &= H^0(B, M(\lambda+\zeta)\otimes(0,p-1)\otimes N(\alpha)). \end{aligned}$$

But this contradicts 2.1, Lemma 4.

(ii) Once again, it is easy to check that the statement is correct at the level of characters and it only remains to check the B_1 isotypic component labeled by $(p-1,1)$. If the statement is incorrect then this component is the direct sum of $(p-1,-p+1)$ and $(-p-1,1)$. But then we would have

$H^0(B,(-p+1,p-1)\otimes H^0(U_1,M(\lambda+\zeta)\otimes\zeta\otimes N(\alpha))\neq 0$, i.e.
$H^0(B,M(\lambda+\zeta)\otimes(0,2p-2)\otimes N(\alpha))\neq 0$ contradicting 2.1 Lemma 4.

Lemma 3 *For $\lambda=(a,b)$ with $a+b=p-1$ and $a,b\neq 0$ we have:*

$$\begin{aligned} H^0(U_1,M(\lambda+\zeta)\otimes\zeta\otimes N(\alpha)) = {} & (a,b)\oplus(-a,a+b)\oplus(a+b,-b)\cdot N(\alpha)^F \\ & \oplus(b,-a-b)\oplus(-a-b,a)\oplus(-b,-a)\oplus(a-2,b+1) \\ & \oplus(-a-2,a+b+1)\oplus(a+b-2,-b+1) \\ & \oplus(b-2,-a-b+1)\oplus(-b-2,-a+1). \end{aligned}$$

Proof. We write Y for the left hand side of the above and R for the right hand side. Once again we leave it to the reader to check that the result is correct at the level of characters. We have a short exact sequence of B-modules $0\to Y'\to Y\to Y''\to 0$, where $Y'=H^0(U_1,M(\lambda+\zeta)\otimes(\zeta-\alpha))$ and $Y''=H^0(U_1,M(\lambda+\zeta)\otimes\zeta)$. Suppose $\mu\in X_1(T)$ and that the μ-isotypic component (for the action of B_1) is non-split. Then there exists a weight $w\lambda-\alpha$ of Y' and a weight $y\lambda$ of Y'' (with $w,y\in W$) such that $w\lambda-\alpha=\mu+p\tau'$, $y\lambda=\mu+p\tau''$, for some $\tau',\tau''\in X(T)$ and $\mathrm{Ext}^1_B(k_{\tau''},k_{\tau'})\neq 0$. This implies that $\theta=\tau''-\tau'$ is a sum of positive roots. We have

$$y\lambda-w\lambda+\alpha=p\theta.$$

Choosing $z\in W$ such that $z\theta$ is dominant, we get

$$p\ln(z\theta)=\ln(zy\lambda)+\ln(-zw\lambda)+\ln(-z\alpha)\leq 2(p-1)+2=2p.$$

Hence $0<\ln(z\theta)\leq 2$. Since θ and hence $z\theta$ belongs to $\mathbb{Z}\Phi$, the only possibility is that $z\theta=(1,1)$ and θ is a positive root.

Thus $y\lambda-w\lambda+\alpha$ is $p\alpha$, $p\beta$ or $p(\alpha+\beta)$. Now we have $w\lambda=\lambda-r\alpha-s\beta$ and $y\lambda=\lambda-u\alpha-v\beta$ for some $0\leq r,s,u,v<p$. If $y\lambda-w\lambda+\alpha=p\alpha$, we get that $(r-u)\alpha+(s-v)\beta+\alpha$ is $p\alpha,p\beta$ or $p(\alpha+\beta)$. Thus we can only have $y\lambda-w\lambda+\alpha=p\alpha$ and if this occurs then $r=p-1$ and $u=0$. Thus $y=s_\beta$ so that $y\lambda=(p-1,-b)$ and $\mu=(p-1,p-b)$. Hence Y and R agree except possibly at the B_1-isotypic components labelled by $(p-1,p-b)$. If the result is not correct then R contains $(p-1,-b)$ as a B-submodule. We now check that this does not happen. We have $H^0(B,(-p+1,b)\otimes Y)=H^0(B,M(\zeta+\lambda)\otimes(0,p-1+b)\otimes N(\alpha))=0$, by 2.1 Lemma 4.

Lemma 4 *Assume $p \neq 2$ and let $\lambda = (p-1,1)$. Then we have*

$$\begin{aligned} H^0(U_1, M(\zeta+\lambda)\otimes\zeta\otimes N(\alpha)) = {} & (p-1,1)\cdot[(-1,0)\cdot E\oplus(-1,-1)]^F \\ & \oplus(-p+1,p)\cdot N(\beta)^F\oplus(p,-1)\cdot N(\alpha)^F \\ & \oplus(p-3,2)\cdot N(\gamma)^F\oplus(p-2,0)\cdot N(\alpha)^F. \end{aligned}$$

Proof. We write Y for the left hand side of the above and R for the right hand side. Once again we leave it to the reader to check that Y and R have the same character. We put $Y' = H^0(U_1, M(\zeta+\lambda)\otimes(\zeta-\alpha))$ and $Y'' = H^0(U_1, M(\zeta+\lambda)\otimes\zeta)$. From 2.2 Lemma 2 we have

$$Y'' = (p-1,1)\cdot N(\gamma)^F\oplus(-p+1,p)\cdot N(\beta)^F\oplus(p,-1)\cdot N(\alpha)^F$$

and

$$Y' = (p-3,2)\cdot N(\gamma)^F\oplus(p-1,1)\cdot[(-2,1)\cdot N(\beta)]^F\oplus(p-2,0)\cdot N(\alpha)^F.$$

We have a short exact sequence $0\to Y'_\mu\to Y_\mu\to Y''_\mu\to 0$, for all $\mu\in X_1(T)$. Since at least one of the outer terms is 0 in all cases except $\mu=(p-1,1)$, these sequences split and the isotypic component of the B_1-socle labelled by μ is determined for $\mu\neq(p-1,1)$. We now take $\mu=(p-1,1)$. Then we have an extension $0\to(-2,1)\cdot N(\beta)\to Y_\mu\to k\oplus(-1,-1)\to 0$. Moreover we have

$$\mathrm{Ext}^1_B((-1,-1),(-2,1)\cdot N(\beta)) = H^1(B,(-1,2)\cdot N(\beta)) = 0$$

since $H^1(B,k)=H^1(B,(-1,2))=0$. Hence $Y_\mu = X\oplus(-1,-1)$, where X appears in a short exact sequence

$$0\to(-2,1)\cdot N(\beta)\to X\to k\to 0.$$

If this sequence is split then Y contains a copy of k and so $H^0(B, M(\zeta+\lambda)\otimes(0,p-2)\cdot N(\alpha))\neq 0$, contrary to 2.1 Lemmma 2. Hence the sequence is non-split. We claim that $H^1(B,(-2,1)\cdot N(\beta))=k$ and hence there is a unique non-split extension (up to isomorphism). Now $(-2,1)\cdot N(\beta)$ has weights $(-2,1)$ and $(-1,-1)$, each occurring with multiplicity one. Moreover, we have $H^i(B,(-1,-1))=0$ for all $i\geq 0$ so that $H^1(B,(-2,1)\cdot N(\beta)) = H^1(B,(-2,1)) = k$ as a required. Now $(-1,0)\cdot E$ is a non-split extension and hence $X=(-1,0)\cdot E$. Thus we get $Y_\mu = [(-1,-1)\oplus(-1,0)\cdot E]$, which completes the proof.

Lemma 5 *Assume $p \neq 2$ and let $\lambda = (1, p-1)$. Then we have*

$$\begin{aligned} H^0(U_1, M(\zeta+\lambda) \otimes \zeta \otimes N(\alpha)) &= (1, p-1) \cdot N(\gamma)^F \\ &\oplus (p-1, 0) \cdot [(-1,1) \oplus (-1,0) \cdot V]^F \\ &\oplus (0,1) \cdot [(1,-1) \cdot N(\alpha)]^F \oplus (p-3, 1) \cdot [(-1,1) \cdot N(\beta)]^F \\ &\oplus (p-2, 2) \cdot [(0,-1) \cdot N(\alpha)]^F. \end{aligned}$$

Proof. This is similar to the proof of Lemma 4 and we leave it to the reader.

Lemma 6 *For $\lambda = (a, b)$ with $a + b = p$ and $a, b > 1$ we have*

$$H^0(U_1, M(\zeta+\lambda) \otimes \zeta \otimes N(\alpha)) = H^0(U_1, M(\zeta+\lambda) \otimes \zeta) \oplus H^0(U_1, M(\zeta+\lambda) \otimes (\zeta - \alpha)).$$

Proof. We leave it to the reader to check, using 2.2 Lemma 2, that $H^0(U_1, M(\zeta+\lambda) \otimes \zeta)$ and $H^0(U_1, M(\zeta+\lambda) \otimes (\zeta - \alpha))$ have no (non-zero) B_1 isotypic components in common. The result follows.

There remains the case $p = 2$ and $\lambda = \rho$.

Lemma 7 *Let $p = 2$. We have*

$$\begin{aligned} H^0(U_1, M(2\rho) \otimes \rho \otimes N(\alpha)) &= N(\alpha)^F \oplus (1,0) \cdot [(-1,1) \oplus (-1,0) \cdot V]^F \\ &\oplus (0,1) \cdot [(1,-1) \cdot N(\alpha)]^F \oplus (1,1) \cdot [(-1,-1) \oplus (-1,0) \cdot E]^F. \end{aligned}$$

Proof. Let $X = H^0(U_1, M(2\rho) \otimes \rho \otimes N(\alpha))$. Then we have a short exact sequence

$$0 \to H^0(U_1, M(2\rho) \otimes (\rho - \alpha)) \to X \to H^0(U_1, M(2\rho) \otimes \rho) \to 0$$

and hence, by 2.2 Lemma 2, a short exact sequence

$$\begin{aligned} 0 \to (-1,2) \cdot N(\gamma)^F \oplus (-3,3) \cdot N(\beta)^F \oplus N(\alpha)^F \to X \\ \to \rho \cdot N(\gamma)^F \oplus (-1,2) \cdot N(\beta)^F \oplus (2,-1) \cdot N(\alpha)^F \to 0. \end{aligned}$$

It follows that $X_{(0,0)} = N(\alpha)$ and $X_{(0,1)} = (1,-1) \cdot N(\alpha)$. It remains to show that $X_{(1,0)}$ and $X_{(1,1)}$ are as implied by the statement of the Lemma.

We have a short exact sequence $0 \to (-1,1) \cdot N(\gamma) \to X_{(1,0)} \to (-1,1) \cdot N(\beta) \to 0$ and, moreover, $\mathrm{Ext}^1_B((-1,1) \cdot N(\beta), (-1,1)) = \mathrm{Ext}^1_B(N(\beta), k) = 0$ so that $X_{(1,0)} = (-1,1) \oplus Y$, where Y occurs in a short exact sequence

$$0 \to (-2,0) \to Y \to (-1,1) \cdot N(\beta) \to 0.$$

We claim that this sequence is not split. If the sequence were split we would have $H^0(B,(1,-1)\otimes\beta\otimes Y)\neq 0$, i.e $H^0(B,(0,1)\cdot Y)\neq 0$, hence $H^0(B,(0,1)\cdot X_{(1.0)})\neq 0$, i.e. $H^0(B,(-1,1)\otimes(1,0)\otimes X_{(1.0)})\neq 0$ and hence
$H^0(B,M(2\rho)\otimes(\rho+(0,1))\otimes N(\alpha))\neq 0$ which would contradict 2.1 Lemma 4(i).

Now the unique non-split extension of $(-1,1)\cdot N(\beta)$ by $(-2,0)$ is $(-1,0)\cdot V$ and so we get $X_{(1,0)}=(-1,1)\oplus(-1,0)\cdot V$, as required.

We now analyze $X_{(1,1)}$ in a similar fashion. We have a short exact sequence

$$0\to(-2,1)\cdot N(\beta)\to X_{(1,1)}\to N(\gamma)\to 0.$$

Moreover, we have $\mathrm{Ext}^1_B((-1,-1),(-2,1)\cdot N(\beta))=0$ so that $X_{(1,1)}=(-1,-1)\oplus Z$, where Z occurs in a short exact sequence

$$0\to(-2,1)\cdot N(\beta)\to Z\to k\to 0.$$

If this sequence were split we would have $H^0(B,Z)\neq 0$, hence $H^0(B,X_{(1,1)})\neq 0$, in other words, $H^0(B,(-1,-1)\otimes(1,1)\cdot X_{(1,1)})\neq 0$ and hence
$H^0(B,(-1,-1)\otimes M(2\rho)\otimes\rho\otimes N(\alpha))\neq 0$, i.e. $H^0(B,M(2\rho)\otimes N(\alpha))\neq 0$, in contradiction to 2.1, Lemma 4(i). Thus the sequence is not split. There is a unique such extension, which is $(-1,0)\cdot E$ and hence we get $X_{(1,1)}=(-1,-1)\oplus(-1,0)\cdot E$, as required.

2.4 We consider now the case $\lambda=(a,b)$ with $a+b>p$.

Lemma 1 *We have*

$$H^0(U_1,M(\zeta+\lambda)\otimes\zeta\otimes N(\alpha))=H^0(U_1,M(\zeta+\lambda)\otimes(\zeta-\alpha))\oplus H^0(U_1,M(\zeta+\lambda)\otimes\zeta)$$

unless either $a=p-1$ or $b=p-1$.

Proof. We assume $a\neq p-1$, $b\neq p-1$. Note that the condition $a+b>p$ implies $p>3$. We put $Y'=H^0(U_1,M(\zeta+\lambda)\otimes(\zeta-\alpha))$, $Y=H^0(U_1,M(\zeta+\lambda)\otimes\zeta\otimes N(\alpha))$ and $Y''=H^0(U_1,M(\zeta+\lambda)\otimes\zeta)$. Then we have a short exact sequence of B-modules $0\to Y'\to Y\to Y''\to 0$ and, by 1.1 (1), we will be done if we show that

$$0\to Y'_\mu\to Y_\mu\to Y''_\mu\to 0 \qquad (*)$$

splits for every $\mu\in X_1(T)$. Assume, for a contradiction, that μ is such that the sequence does not split. Then $Y''_\mu\neq 0$ and $Y'_\mu\neq 0$. From 2.2 Lemma, we have $\mu\equiv w\lambda \pmod{pX(T)}$ and $\mu\equiv y\lambda-\alpha \pmod{pX(T)}$ for some $w,y\in W$. Thus we have

$$w\lambda-y\lambda+\alpha\in pX(T)\cap\mathbb{Z}\Phi=p\mathbb{Z}\Phi$$

(since $\mathbb{Z}\Phi$ has index 3 in $X(T)$ and $p \neq 3$). Thus we have

$$\lambda - w^{-1}y\lambda + w^{-1}\alpha \in p\mathbb{Z}\Phi.$$

Moreover, $\lambda - w^{-1}y\lambda \in S = \{0, a\alpha, b\beta, a\alpha+(a+b)\beta, (a+b)\alpha+b\beta, (a+b)(\alpha+\beta)\}$. So we have $\theta \equiv -w^{-1}\alpha \pmod{p\mathbb{Z}\Phi}$, for some $\theta \in S$. Since $0 \neq -w^{-1}\alpha = m\alpha + n\beta$ for some $m, n \in \{0, 1, -1\}$ the only possibility is $a+b = p+1$, $\lambda - w^{-1}y = (a+b)(\alpha+\beta)$ and $-w^{-1}\alpha = \alpha + \beta$. This give $w^{-1}y = w_0$, $w = s_\beta w_0$ and hence $y = s_\beta$. Thus $\mu \equiv y\lambda - \alpha \equiv (a+b, -b) - \alpha$ which gives $\mu = (p-1, a)$.

We now get from 2.2 Lemma 3 that the μ isotypic component of Y'' is

$$(2p-a-b, a) \cdot [(0,-1) \cdot N(\alpha)]^F = (p-1, a) \cdot [(0,-1) \cdot N(\alpha)]^F$$

and the μ isotypic component of Y' is

$$((a+b-p, p-b)-\alpha) \cdot [(1,-1) \cdot N(\alpha)]^F = (-1, a) \cdot [(1,-1) \cdot N(\alpha)]^F = (p-1, a) \cdot [(0,-1) \cdot N(\alpha)]^F.$$

Thus the sequence (*) is

$$0 \to (0,-1) \cdot N(\alpha) \to Y_\mu \to (0,-1) \cdot N(\alpha) \to 0.$$

We leave it to the reader to check that $\mathrm{Ext}^1_B(N(\alpha), N(\alpha)) = 0$ and hence $\mathrm{Ext}^1_B((0,-1) \cdot N(\alpha), (0,-1) \cdot N(\alpha)) = 0$ so that (*) is split.

Lemma 2 *For $\lambda = (p-1, b) \in X_1(T)$ with $1 < b < p-1$ we have:*

$$\begin{aligned} H^0(U_1, M(\lambda+\zeta) \otimes \zeta \otimes N(\alpha)) &= (p-1, b) \cdot [(-1,-1) \oplus (-1, 0) \cdot E]^F \\ &\oplus (1, b-1) \cdot [(-1, 1) \cdot N(\beta)]^F \oplus (b-1, p-b) \cdot [(1,-1) \cdot N(\alpha)]^F \\ &\oplus (b, p+1-b) \cdot [(-1, 0) \cdot N(\beta)]^F \\ &\oplus (p+1-b, p-1) \cdot [(0,-1) \cdot N(\alpha)]^F \oplus (p-b, 1) \cdot N(\gamma)]^F \\ &\oplus (p-3, b+1) \cdot N(\gamma)^F \oplus (b-3, p-b+1) \cdot [(1,-1) \cdot N(\alpha)]^F \\ &\oplus (b-2, p+2-b) \cdot [(-1, 0) \cdot N(\beta)]^F \\ &\oplus (p-1-b, p) \cdot [(0,-1) \cdot N(\alpha)]^F \oplus (p-b-2, 2) \cdot N(\gamma)^F \end{aligned}$$

Proof. Let $Y = H^0(U_1, M(\lambda+\zeta) \otimes \zeta \otimes N(\alpha))$, $Y' = H^0(U_1, M(\lambda+\zeta) \otimes (\zeta - \alpha))$ and $Y'' = H^0(U_1, M(\lambda+\zeta) \otimes \zeta)$. We have a short exact sequence

$$0 \to Y' \to Y \to Y'' \to 0$$

of B-modules on which U_1 acts trivially. By 2.2 Lemma 3, the non-zero B_1-isotypic components of Y'' are labelled by the set
$S = \{(p-1, b), (1, b-1), (b-1, p-b), (b, p+1-b), (p+1-b, p-1), (p-b, 1)\}$. The non-zero B_1-isotypic components of Y' are labelled by the elements $\mu \in X_1(T)$ such

that $\mu = \nu - \alpha$, for some $\nu \in S$. We leave it to the reader to check that if $\mu \in X_1(T)$ and both Y' and Y'' have a non-zero component labelled by μ then $\mu = (p-1, b)$ and that the μ isotypic component is $(p-1,b) \otimes [(-1,-1) \oplus (-1,0) \otimes E]^F$, as required.

Lemma 3 *For $\lambda = (a, p-1) \in X_1(T)$ with $1 < a < p-1$ we have:*

$$
\begin{aligned}
H^0(U_1,&M(\zeta+\lambda) \otimes \zeta \otimes N(\alpha)) \\
&= (a, p-1) \cdot N(\gamma)^F \oplus (p-a, a-1) \cdot [(-1,1) \cdot N(\beta)]^F \\
&\oplus (a-1, 1) \cdot [(1,-1) \cdot N(\alpha)]^F \\
&\oplus (p+1-a, a) \cdot [(0,-1) \cdot N(\alpha)]^F \oplus (1, p-a) \cdot N(\gamma)^F \\
&\oplus (a-2, p) \cdot N(\gamma)^F \oplus (p-a-2, a) \cdot [(-1,1) \cdot N(\beta)]^F \\
&\oplus (a-3, 2) \cdot [(1,-1) \cdot N(\alpha)]^F \oplus (p-3, p+2-a) \cdot [(-1,0) \cdot N(\beta)]^F \\
&\oplus (p-1-a, a+1) \cdot [(0,-1) \cdot N(\alpha)]^F \\
&\oplus (p-1, p-a+1) \cdot [(-1,0) \oplus (-1,-1) \cdot V]^F.
\end{aligned}
$$

Proof. Similar to Lemma 2.

Lemma 4 *For $p > 2$ we have*

$$
\begin{aligned}
H^0(U_1,&M(2\zeta) \otimes \zeta \otimes N(\alpha)) \\
&= (p-1, p-1) \cdot [(-1,-1) \oplus (-1,0) \cdot E]^F \oplus (1, p-2) \cdot [(-1,1) \cdot N(\beta)]^F \\
&\oplus (p-2, 1) \cdot [(1,-1) \cdot N(\alpha)]^F \\
&\oplus (2, p-1) \cdot [(0,-1) \cdot N(\alpha)]^F \oplus (1,1) \cdot N(\gamma)^F \\
&\oplus (p-3, p) \cdot N(\gamma)^F \\
&\oplus (p-4, 2) \cdot [(1,-1) \cdot N(\alpha)]^F \oplus (p-3, 3) \cdot [(-1,0) \cdot N(\beta)]^F \\
&\oplus (0, p) \cdot [(0,-1) \cdot N(\alpha)]^F \oplus (p-1, 2) \cdot [(-1,0) \oplus (-1,-1) \cdot V]^F.
\end{aligned}
$$

Proof. Similar to Lemmas 2 and 3. (With an additional step for $p = 3$ as in 2.2, Lemma 3).

2.5 We shall also need the structure of the modules $H^0(U_1, M(\lambda+\zeta) \otimes \zeta \otimes N(\beta))$. However, we can obtain this from the results of Section 2.4 and a certain automorphism of $G = \mathrm{SL}_3(k)$.

Let $n \in \mathrm{GL}_3(k)$ be the element with all anti-diagonal entries 1 and all other entries 0. We write g' for the transpose of $g \in G$. We have an involutory automorphism $\tau : G \to G$ given by $\tau(g) = n(g')^{-1}n$. Note that τ stabilizes B and T. Thus τ induces a group automorphism on $X(T)$ which we extend to a ring automorphism on $\mathbb{Z}X(T)$. For $\phi \in \mathbb{Z}X(T)$, we write ϕ^τ for the image of ϕ under this ring automorphism. If H is a τ-stable closed subgroup of G and $V \in \mathrm{mod}(H)$ affords the representation $\pi : H \to \mathrm{GL}(V)$ we write V^τ for the k-space V viewed as an H-module via the representation $\pi \circ \tau$. If $t \in T$ has $(1,1)$-entry t_1, has $(2,2)$-entry t_2 and has $(3,3)$-entry t_3 then $\tau(t)$ is the element of T which has $(1,1)$-entry t_3, has $(2,2)$-entry t_2 and has $(3,3)$-entry t_1. It follows that $k(a,b)^\tau$ is isomorphic to $k(b,a)$, as B-modules and that we have $e(a,b)^\tau = e(b,a)$, for $a, b \in \mathbb{Z}$. Moreover, $N(\alpha)^\tau$ is an indecomposable two dimensional B-module with weights $0, -\beta$. Hence we have $N(\alpha)^\tau \cong N(\beta)$, $N(\beta)^\tau \cong N(\alpha)$ and $N(\gamma)^\tau = N(\gamma)$. Now, for any $M \in \mathrm{mod}(B)$ and $i \geq 0$ we have $R^i\mathrm{Ind}_B^G M^\tau \cong (R^i\mathrm{Ind}_B^G M)^\tau$, by [**4**;(1.1.14)]. In particular, we have $\chi^i(M^\tau) = \chi^i(M)^\tau$. For $\mu = (a,b) \in X(T)$, we have $\mu^\tau = (b,a)$. For $\lambda \in X^+(T)$ we have $\nabla(\lambda)^\tau \cong \nabla(\lambda^\tau)$ and $L(\lambda)^\tau \cong L(\lambda^\tau)$. In particular we have $E^\tau \cong V$.

We note also that, for $Y \in \mathrm{mod}(B)$, we have $H^0(U_1, Y)^\tau \cong H^0(U_1, Y^\tau)$. (We have $H^0(U_1, Y) \leq Y$ and hence $H^0(U_1, Y)^\tau \leq Y^\tau$ and hence $H^0(U_1, Y)^\tau \leq H^0(U_1, Y^\tau)$. Thus we also have $H^0(U_1, Y^\tau)^\tau \leq H^0(U_1, Y)$ and hence also $H^0(U_1, Y^\tau) \leq H^0(U_1, Y)^\tau$.)

We summarize the properties of τ that we shall need.

Lemma *(i) The automorphism τ of G induces the ring automorphism of $\mathbb{Z}X(T)$ which takes $e(a,b)$ to $e(b,a)$, for $a, b \in \mathbb{Z}$.*
(ii) For $\phi \in A$ we have $\phi^\tau = \phi^*$, $i \geq 0$.
(iii) We have $\chi^i(M^\tau) = \chi^i(M)^\tau$, for $M \in \mathrm{mod}(B)$ and $i \geq 0$.
(iv) $\chi^i(a,b)^\tau = \chi^i(b,a)$, for $a, b \in \mathbb{Z}$, $i \geq 0$.
(v) $\chi^i_\alpha(a,b) = \chi^i_\beta(b,a)$, for $a, b \in \mathbb{Z}$, $i \geq 0$.
(vi) $H^0(U_1, M(\zeta+\lambda) \otimes \zeta)^\tau = H^0(U_1, M(\zeta+\lambda^\tau) \otimes \zeta)$ and $H^0_\alpha(U_1, M(\zeta+\lambda) \otimes \zeta)^\tau = H^0_\beta(U_1, M(\zeta+\lambda^\tau) \otimes \zeta)$.

The first part has been shown already. Part (ii) is clear in the case $\phi = \chi(\lambda)$ for some dominant weight λ, and follows in general by linearity since the elements $\chi(\lambda)$ form a $\mathbb{Z}$-basis of A. Part (iii) has been shown already and gives parts (iv) and (v) by taking $M = (a,b)$ and $M = (a,b) \otimes N(\alpha)$. The final property follows from the discussion preceding the Lemma.

3 Base cases

In this section we give formulas for $\chi^1(r,s)$, $\chi^1_\alpha(r,s)$ and $\chi^1_\beta(r,s)$, without restriction on characteristic, for certain values of r and s. In the next section we take the characteristic of p to be 2 and give expansion formulas for $\chi^1(r,s)$, $\chi^1_\alpha(r,s)$. These formulas, together with the "initial conditions" described in this section recursively determine $\chi^1(r,s)$, $\chi^1_\alpha(r,s)$ $\chi^1_\beta(r,s)$ for all values of r and s. Since G/B has dimension 3 this, together with Serre duality and Weyl's character formula, determines $\chi^i(r,s)$, $\chi^i_\alpha(r,s)$ $\chi^i_\beta(r,s)$ for all i and r,s.

Remarks We first recall that if $\chi^i(\lambda)\neq 0$ and θ is a simple root such that $(\lambda,\check{\theta}) = -n < 0$ then $i>0$ and there exists some $0<m<n$ such that $\chi^i(\lambda+m\theta)\neq 0$, in particular, if $i=1$ then $\lambda+m\theta\in X^+$. Moreover, for $n=1$ we have $\chi^i(\lambda)=0$ for all $i\geq 0$, for $n=2$ we have $\chi^i(\lambda)=\chi^i(\lambda+\theta)$ and for $n=3$ have $\chi^i(\lambda)=\chi^{i-1}(\lambda+2\theta)$ (for $i\geq 0$). References for these fact are [**4**], (5.3) Lemma, (2.1.5),(2.3.1),(5.5) Proposition : they will be used without further reference.

Lemma 1 *For $\lambda\in X^+$ we have $\chi^1(\lambda)=\chi^1_\alpha(\lambda)=\chi^1_\beta(\lambda)=0$.*

Proof. We have $\chi^1(\lambda)=0$ by Kempf's Vanishing Theorem. Morevoer, we have a short exact sequence of B-modules

$$0\to(0,-1)\to E\to(1,0)\cdot N_\alpha\to 0$$

giving a short exact sequence

$$0\to(-1,-1)\to(-1,0)\cdot E\to N_\alpha\to 0$$

and hence a short exact sequence

$$0\to(\lambda_1-1,\lambda_2-1)\to(\lambda_1-1,\lambda_2)\cdot E\to\lambda\cdot N_\alpha\to 0$$

(where $\lambda=(\lambda_1,\lambda_2)$). But now $R\mathrm{Ind}_B^G((\lambda_1-1,\lambda_2)\cdot E)=R^2\mathrm{Ind}_B^G(\lambda_1-1,\lambda_2-1)=0$ and hence $R\mathrm{Ind}_B^G(\lambda\cdot N_\alpha)=0$, i.e. $\chi^1_\alpha(\lambda)=0$. Similarly $\chi^1_\beta(\lambda)=0$.

Lemma 2 *We have have $\chi^1(r,s)=\chi^1_\alpha(r,s)=\chi^1_\beta(r,s)=0$ if $r=0$ or $s=0$.*

Proof. We get $\chi^1(r,s)=0$ from Lemma 1 and the Remarks. Now suppose that $s=0$. We have a short exact sequence

$$0\to(r-1,-1)\to(r-1,0)\cdot E\to(r,0)\cdot N_\alpha\to 0$$

moreover, $\chi^2(r-1,-1)=0=\chi^1(r-1,0)$ so that $\chi^1_\alpha(r,0)=0$.

Now suppose $r = 0$. Then the short exact sequence

$$0 \to (-1, s-1) \to (-1, s) \cdot E \to (0, s) \to 0$$

gives $\chi^1_\alpha(0, s) = 0$.

Similarly we get $\chi^1_\beta(r, s) = 0$ if $r = 0$ or $s = 0$.

Lemma 3 *(i)* $\chi^1(-1, s) = 0 = \chi^1(r, -1)$ *for all* r, s.
(ii) $\chi^1_\alpha(r, -1) = 0 = \chi^1_\beta(-1, s)$ *for all* r, s.
(iii)

$$\chi^1_\alpha(-1, s) = \begin{cases} \chi(1, s-1), & \text{if } s \geq 1; \\ 0, & \text{otherwise} \end{cases}$$

and

$$\chi^1_\beta(r, -1) = \begin{cases} \chi(r-1, 1), & \text{if } r \geq 1; \\ 0, & \text{otherwise.} \end{cases}$$

Proof. (i) Clear.
(ii) We have a short exact sequence $0 \to (r-2, 0) \to (r, -1) \cdot N_\alpha \to (r, -1) \to 0$ and $R^i\mathrm{Ind}_B^G(r, -1) = 0$ for all $i \geq 0$ so that $\chi^1_\alpha(r, -1) = \chi^1(r-2, 0) = 0$, by Lemma 2. Similarly we get $\chi^1_\beta(-1, s) = 0$.
(iii) By the argument of (ii) we get $\chi^1_\alpha(-1, s) = \chi^1(-3, s+1)$ and this is $\chi^0(1, s-1)$, by [**4**; (5.5) Proposition]. Moreover, we have that $\chi^0(1, s-1)$ is $\chi(1, s-1)$ if $s \geq 1$ and is 0 otherwise. Similarly we obtain the other statement.

Lemma 4 *(i)*

$$\chi^1(-2, s) = \begin{cases} \chi(0, s-1), & \text{if } s > 0; \\ 0, & \text{otherwise} \end{cases}$$

and

$$\chi^1(r, -2) = \begin{cases} \chi(r-1, 0), & \text{if } r > 0; \\ 0, & \text{otherwise} \end{cases}$$

(ii)

$$\chi^1_\alpha(r, -2) = \begin{cases} \chi(r-1, 0), & \text{if } r \geq 1; \\ 0, & \text{otherwise} \end{cases}$$

and

$$\chi^1_\beta(-2, s)) = \begin{cases} \chi(0, s-1), & \text{if} s \geq 1; \\ 0, & \text{otherwise.} \end{cases}$$

(iii)

$$\chi^1_\alpha(-2, s) = \begin{cases} \chi(0), & \text{if} s = 1; \\ \chi(2, s-2) + \chi(0, s-1), & \text{if} s \geq 2; \\ 0, & \text{otherwise} \end{cases}$$

and

$$\chi^1_\beta(r,-2) = \begin{cases} \chi(0), & \text{if} r = 1; \\ \chi(r-2,2) + \chi(r-1,0), & \text{if} r \geq 2; \\ 0, & \text{otherwise} \end{cases}$$

Proof. (i) This is true by e.g. [**4**;(2.3.1)].
(ii) There is a short exact sequence

$$0 \to (r-1,-1) \to (r,-2)\cdot N_\alpha \to (r,-2) \to 0$$

and since $R^i\mathrm{Ind}_B^G(r-1,-1) = 0$ for all $i \geq 0$ we get $\chi^1_\alpha(r,-2) = \chi^1(r,-2)$ and hence the required description of $\chi^1_\alpha(r,-2)$ from (i). Similarly we obtain the other statement.
(iii) First note that, in view of the short exact sequence $0 \to (-4,s+1) \to (-2,s)\cdot N_\alpha \to (-2,s) \to 0$, we have that if $\chi^1_\alpha(-2,s) \neq 0$ then either $\chi^1(-4,s) \neq 0$ or $\chi^1(-2,s) \neq 0$. This gives, by the above Remarks, $s-1 \geq 0$, i.e. $s > 0$. We now assume that this is the case.

We have the short exact sequence

$$0 \to (-3,s-1) \to (-3,s)\cdot E \to (-2,s)\cdot N_\alpha \to 0.$$

For $s = 1$ we have $R^i\mathrm{Ind}_B^G(-3,1) = R^{i-1}\mathrm{Ind}_B^G(2,-1) = 0$ for $i > 0$ and hence $R\mathrm{Ind}_B^G((-2,1)\cdot N_\alpha) \cong R^2\mathrm{Ind}_B^G(-3,0) \cong R\mathrm{Ind}_B^G(1,-2) \cong \mathrm{Ind}_B^G k$, which gives $\chi^1_\alpha(-3,1) = \chi(0)$.

Now suppose $s \geq 2$. We get the exact sequence

$$0 \to R\mathrm{Ind}_B^G(-3,s-1) \to E\otimes\mathrm{Ind}_B^G(1,s-2) \to R\mathrm{Ind}_B^G((-2,s)\cdot N_\alpha) \to R^2\mathrm{Ind}_B^G(-3,s-1).$$

Moreover, we get $R^2\mathrm{Ind}_B^G(-3,s-1) = R\mathrm{Ind}_B^G(1,s-3) = 0$ since $s-3 \geq -1$. Hence we get $\chi^1_\alpha(-2,s) = \chi(1,0)\chi(1,s-2) - \chi^0(1,s-3)$. If $s = 2$ this gives $\chi(1,0)\chi(1,0) = \chi(2,0)+\chi(0,1)$ and if $s \geq 3$ this gives $\chi(1,0)\chi(1,s-2)-\chi(1,s-3) = \chi(2,s-2)+\chi(0,s-1)$, as required.

Similarly one obtains the statement involving β.

Lemma 4 *(i)*

$$\chi^1(r,-3) = \begin{cases} \chi(r-2,1), & r \geq 2; \\ 0, & \text{otherwise} \end{cases}$$

and

$$\chi^1(-3,s) = \begin{cases} \chi(1,s-2), & s \geq 2; \\ 0, & \text{otherwise.} \end{cases}$$

Proof. Clear from the Remarks.

Lemma 5 *If $\chi^1(r,s)$,$\chi^1_\alpha(r,s)$ or $\chi^1_\beta(r,s)$ is non-zero then we have $r > 0$ and $s < 0$ or $r < 0$ and $s > 0$.*

Proof. For $\chi^1(r,s)$ this is clear from the Remarks above. Also, if $\chi^1_\alpha(r,s) \neq 0$ then either $r < 0$ or $s < 0$ by Lemma 1.

Suppose $\chi^1_\alpha(-r,s) \neq 0$, with $r > 0$. Then, from the long exact sequence, we have either $\chi^1(-r,s) \neq 0$ or $\chi^1(-r-2,s+1) \neq 0$. From $\chi^1(-r,s) \neq 0$ we deduce $s > 0$. Suppose $\chi^1(-r-2,s+1) \neq 0$. Then we get $s+1 > 0$ and if $s = 0$ we have $\chi^1_\alpha(-r,s) = 0$ by Lemma 1. Hence we have $s > 0$.

Now suppose $\chi^1_\alpha(r,-s) \neq 0$, with $s > 0$. From the long exact sequence, we have $\chi^1(r,-s) \neq 0$ or $\chi^1(r-2,-s+1) \neq 0$. From $\chi^1(r,-s) \neq 0$ we deduce $s > 0$ so we now assume $\chi^1(r-2,-s+1) \neq 0$. From Lemma 1, we have $-s+1 \neq 0$ i.e. $s \neq -1$ so that $-s+1 < 0$ and $r-2 > 0$ and hence $r > 0$.

One similarly obtains the result for $\chi^1_\beta(r,s)$.

Remark In the sections which follow, we shall develop explicit "expansion formulas" of the form $\chi^i(\lambda+p\mu) = \sum_{\nu\in X_1} \mathrm{ch}\, L(\nu)\theta^F_{\nu,\lambda,\mu}$, $\chi^i_\alpha(\lambda+p\mu) = \sum_{\nu\in X_1} \mathrm{ch}\, L(\nu)\phi^F_{\nu,\lambda,\mu}$, and $\chi^i_\beta(\lambda+p\mu) = \sum_{\nu\in X_1} \mathrm{ch}\, L(\nu)\psi^F_{\nu,\lambda,\mu}$, for $\lambda \in X_1$ and $\mu = (a,b) \in X^+$, where each $\theta_{\nu,\lambda,\mu}$, $\phi_{\nu,\lambda,\mu}$, $\psi_{\nu,\lambda,\mu}$ is a sum of terms of the form $\chi^i(c,d)$, $\chi^i_\alpha(c,d)$, $\chi^i_\alpha(c,d)$, with $c \in \{a-1,a,a+1\}$, $d \in \{b-1,b,b+1\}$.

We claim that such expansion formulas, together with the above, recursively determine all $\chi^1(r,s)$, $\chi^1_\alpha(r,s)$, $\chi^1_\beta(r,s)$ (and hence with Weyl's character formula and Serre duality determine $\chi^i(r,s)$,$\chi^i_\alpha(r,s)$, $\chi^i_\beta(r,s)$ for $i \geq 0$) for all r,s.

By Lemma 5 of 3.1, the characters are 0 unless $r > 0$ and $s < 0$ or $r < 0$ and $s > 0$. Moreover, these characters are all described in Lemmas 1,2,3,4 of 3.1 for r and s at least -3. However, for $m \geq 4$, $s > 0$ the expansion formulas give $\chi^1(-m,s)$, $\chi^1_\alpha(-m,s)$, $\chi_\beta(-m,s)$ in terms of $\chi^1(-m',s;)$, $\chi^1_\alpha(-m',s')$, $\chi_\beta(-m',s')$ with $m' < m$, $s' \geq 0$. Similar remarks are valid for expansions of $\chi^1(s,-m)$, $\chi^1_\alpha(s,-m)$, $\chi_\beta(s,-m)$

4 The case $p = 2$.

4.1 We now assume that k has characteristic $p = 2$. By specializing to this case in Section 2.2 we get the following.

Lemma 1 *(i)* $H^0(U_1, M(\rho) \otimes \rho) = k$.
(ii) $H^0(U_1, M(\rho + (1,0)) \otimes \rho) = (1,0) \oplus (0,1) \cdot (0,-1)^F \oplus (1,1) \cdot (-1,0)^F$.
(iii) $H^0(U_1, M(\rho + (0,1)) \otimes \rho) = (1,0) \cdot (-1,0)^F \oplus (0,1) \oplus (1,1) \cdot (0,-1)^F$.
(iv) $H^0(U_1, M(\rho + (1,1)) = (1,0) \cdot [(-1,1) \cdot N(\beta)]^F \oplus (0,1) \cdot [(1,-1) \cdot N(\alpha)]^F \oplus (1,1) \cdot N(\gamma)^F$.

By specializing results in Section 2.4 we get the following.

Lemma 2 *(i)* $H^0_\alpha(U_1, M(\rho) \otimes \rho) = (0,0) \oplus (0,1) \cdot (-1,0)^F$.
(ii) $H^0_\alpha(U_1, M(\rho + (1,0)) \otimes \rho)$

$$= (-1,0)^F \oplus (1,0) \cdot N(\alpha)^F \oplus (0,1) \cdot (0,-1)^F \oplus (1,1) \cdot [2 \times k(-1,0)]^F.$$

(iii) $H^0_\alpha(U_1, M(\rho + (0,1)) \otimes \rho)$

$$= (-1,1)^F \oplus (1,0) \cdot [2 \times k(-1,0)]^F \oplus (0,1) \oplus (1,1) \cdot [(0,-1) \cdot N(\alpha)]^F.$$

(iv) $H^0(U_1, M(\rho + (1,1)) \otimes \rho \otimes N(\alpha))$

$$= N(\alpha)^F \oplus (1,0) \cdot [(-1,1) \oplus (-1,0) \cdot V]^F$$
$$\oplus (0,1) \cdot [(1,-1) \cdot N(\alpha)]^F \oplus (1,1) \cdot [(-1,-1) \oplus (-1,0) \cdot E]^F.$$

4.2 Recall that, since $p = 2$, we have $\phi_\lambda = \chi(\lambda)$, for all $\lambda \in X_1(T)$. Hence, from 3.1 Lemmas 1 and 2 we get the following results.

Lemma 1 *For $i \geq 0$, $r, s \in \mathbb{Z}$ we have:*
(i) $\chi^i(2r, 2s)$

$$= \chi^i(r,s)^F + \chi^i(r-1,s-1)^F + \chi(1,0)\chi^i(r,s-1)^F + \chi(0,1)\chi^i(r-1,s)^F;$$

(ii) $\chi^i(1+2r, 2s)$

$$= \chi^i_\alpha(r+1,s-1)^F + \chi(1,0)\chi^i(r,s)^F + \chi(0,1)\chi^i(r,s-1)^F;$$

(iii) $\chi^i(2r, 1+2s)$

$$= \chi^i_\beta(r-1,s+1)^F + \chi(1,0)\chi^i(r-1,s)^F + \chi(0,1)\chi^i(r,s)^F;$$

(iv) $\chi^i(1+2r, 1+2s) = \chi(1,1)\chi^i(r,s)^F$.

Lemma 2 *For $i \geq 0$, $r, s \in \mathbb{Z}$ we have:*
(i) $\chi^i_\alpha(2r, 2s) = \chi^i(r-1, s-1)^F$
$+\chi(1,0)^F\chi^i(r-1,s)^F + 2\chi(0,1)\chi^i(r-1,s)^F + \chi(1,0)\chi^i_\alpha(r, s-1)^F$;
(ii) $\chi^i_\alpha(2r+1, 2s) = \chi^i_\alpha(r+1, s-1)^F$
$+\chi(1,0)\chi^i(r,s)^F + \chi(0,1)\chi^i(r, s-1)^F + \chi(1,1)\chi^i(r-1,s)^F$;
(iii) $\chi^i_\alpha(2r, 2s+1) = \chi(0,1)^F\chi^i(r-1,s)^F + \chi^i(r-1, s+1)^F$
$+2\chi(1,0)\chi^i(r-1,s)^F + \chi(0,1)\chi^i_\alpha(r,s)^F$;
(iv) $\chi^i_\alpha(2r+1, 2s+1) = \chi^i_\alpha(r,s)^F$
$+\chi(1,0)\chi^i(r-1,s+1)^F + \chi(0,1)\chi^i(r-1,s)^F + \chi(1,1)\chi^i(r,s)^F$.

Applying 2.5,(2),(v) to Lemma 2 we get the following.

Lemma 3 *For $i \geq 0$, $r, s \in \mathbb{Z}$ we have:*
(i) $\chi^i_\beta(2r, 2s) = \chi^i(r-1, s-1)$
$+\chi(0,1)^F\chi^i(r,s-1)^F + 2\chi(1,0)\chi^i(r,s-1)^F + \chi(0,1)\chi^i_\beta(r-1,s)^F$;
(ii) $\chi^i_\beta(2r+1, 2s) = \chi(1,0)^F\chi^i(r, s-1)^F$
$+\chi^i(r+1,s-1)^F + 2\chi(0,1)\chi^i(r,s-1)^F + \chi(1,0)\chi^i_\beta(r,s)^F$;
(iii) $\chi^i_\beta(2r, 2s+1) = \chi^i(r-1, s+1)^F$
$+\chi(0,1)\chi^i(r,s)^F + \chi(1,0)\chi^i(r-1,s)^F + \chi(1,1)\chi^i(r,s-1)^F$;
(iv) $\chi^i_\beta(2r+1, 2s+1) = \chi^i_\beta(r,s)^F$
$+\chi(0,1)\chi^i(r+1,s-1)^F + \chi(1,0)\chi^i(r,s-1)^F + \chi(1,1)\chi^i(r,s)^F$.

Lemma 4 $\quad \chi^1(r,-5) = \begin{cases} \chi(r-4,3)), & \text{if } r \geq 4; \\ 0, & \text{otherwise} \end{cases}$

and

$$\chi^1(-5,s) = \begin{cases} \chi(3, s-4), & \text{if } s \geq 4; \\ 0, & \text{otherwise.} \end{cases}$$

Proof. If $\chi^1(r,-5) \neq 0$ then $s \geq 3$ by the Remarks in 3.1. For r odd we write $r = 2m+1$ and get $\chi^1(r,-5) = \chi^1(2m+1,-5) = \chi(1,1)\chi^1(m,-3)^F$, by Lemma 1, which, together with 3.1 Lemma 4, gives the required formula for $\chi^1(r,-5)$. For $r \geq 4$ even we write $s = 2m$ and then from Lemma 1 we get

$$\begin{aligned} \chi^1(r,-5) = \chi^1(2m,-5) &= \chi^1_\beta(m-1,-2)^F + \chi(1,0)\chi^1(m-1,-3)^F + \chi(0,1)\chi^1(m,-3)^F \\ &= \chi^1_\beta(m-1,-2)^F + \chi(1,0)\chi^1(m-3,1)^F + \chi(0,1)\chi^1(m-2,1)^F. \end{aligned}$$

We leave it to the reader to check, using 3.1, Lemma 4, that this is indeed $\chi(2m-4,3)$.

Lemma 5 (i) $\chi^1_\alpha(r,-3) = \begin{cases} \chi(r-3,0)+\chi(r-2,1), & \text{if } r \geq 3; \\ 0, & \text{otherwise} \end{cases}$

and

$$\chi^1_\beta(-3,s) = \begin{cases} \chi(0,s-3)+\chi(1,s-2), & \text{if } s \geq 3; \\ 0, & \text{otherwise} \end{cases}$$

(ii)
$$\chi^1_\alpha(-3,s) = \begin{cases} \chi(1,0), & \text{if } s = 2; \\ \chi(3,s-3)+\chi(1,s-2), & \text{if } s \geq 2; \\ 0, & \text{otherwise.} \end{cases}$$

and

$$\chi^1_\beta(r,-3) = \begin{cases} \chi(0,1), & \text{if} r = 2; \\ \chi(r-3,3)+\chi(r-2,1), & \text{if } r \geq 2; \\ 0, & \text{otherwise.} \end{cases}$$

Proof. (i) If $\chi^1_\alpha(r,-3) \neq 0$ then $r > 0$. In that case we have $R^2\mathrm{Ind}_B^G(r-2,-2) = R\mathrm{Ind}_B^G(r-3,0) = 0$. Hence the short exact sequence
$0 \to (r-2,-2) \to (r,-3)\cdot N_\alpha \to (r,-3) \to 0$ gives rise to an exact sequence
$0 \to R\mathrm{Ind}_B^G(r-2,-2) \to R\mathrm{Ind}_B^G((r,-1)\cdot N_\alpha) \to R\mathrm{Ind}_B^G(r,-3) \to 0$, which gives the required formula for $\chi^1_\alpha(r,-3)$. Similarly one obtains the formula involving β.
(ii) If $\chi^1_\beta(-3,s) \neq 0$ then we have $s > 0$. We have the usual sequence

$0 \to R\mathrm{Ind}_B^G(-5,s+1) \to R\mathrm{Ind}_B^G((-3,s)\cdot N_\alpha) \to R\mathrm{Ind}_B^G(-3,s) \to R\mathrm{Ind}_B^G(-5,s+1)$

and $R\mathrm{Ind}_B^G(-3,s)$ and $R^2\mathrm{Ind}_B^G(-5,s+1)$ belong to different blocks so the map $R\mathrm{Ind}_B^G(-3,s) \to R^2\mathrm{Ind}_B^G(-5,s+1)$ is zero. Hence we have $\chi^1_\alpha(-3,s) = \chi^1(-5,s+1)+\chi^1(-3,s)$, which gives the desired result.

Thus from the closing Remark of Section 3 we get the following.

Proposition *The characters $\chi^1(r,s)$, $\chi^1_\alpha(r,s)$ and $\chi^1_\beta(r,s)$ are determined recursively, for all r,s, by Lemmas 1,2,3 of Section 3.1 and Lemmas 1,2,3 and 5 above.*

5 The case $p = 3$.

5.1 We shall calculate the recursive formulas to cover the case $p = 3$. We first establish some notation and make a remark in the general context which will facilitate our calculations. Recall that we have (from 1.1,(*)):

$$\chi^i(Y \otimes (\zeta - \mu)) = \sum_{\lambda \in X_1(T)} \operatorname{ch} L(\zeta - \lambda) \chi^i(H^0(U_1, M(\zeta + \lambda) \otimes \zeta \otimes Y)_\mu)^F$$

for $\lambda, \mu \in X_1(T)$, $Y \in \operatorname{mod}(B)$.

We set $\bar{\lambda} = \zeta - \lambda$, for $\lambda \in X_1(T)$. Replacing Y by $p\nu \cdot Y$, $\nu \in X(T)$, we have

$$\chi^i((\bar{\mu} + p\nu) \cdot Y) = \sum_{\lambda \in X_1(T)} \operatorname{ch} L(\bar{\lambda}) \chi^i(\nu \cdot H^0(U_1, M(\zeta + \lambda) \otimes \zeta \otimes Y)_\mu)^F.$$

We define $R_{\lambda\mu}(Y) = H^0(U_1, M(\zeta + \lambda) \otimes Y)_\mu$, for $Y \in \operatorname{mod}(B)$. Thus we have

$$\chi^i((\bar{\mu} + p\nu) \otimes Y) = \sum_{\lambda \in X_1(T)} \operatorname{ch} L(\bar{\lambda}) \chi^i(\nu \cdot R_{\lambda\mu}(Y))^F.$$

We now define $S_{\lambda\mu}(Y) = R_{\bar{\mu}\bar{\lambda}}(Y)$, for $\lambda, \mu \in X_1(T)$, $Y \in \operatorname{mod}(Y)$. Hence we have

$$\chi^i((\lambda + p\nu) \cdot Y) = \sum_{\mu \in X_1(T)} \operatorname{ch} L(\mu) \chi^i(\nu \cdot S_{\lambda\mu}(Y))^F \qquad (*).$$

We shall write simply $R_{\lambda\mu}$ for $R_{\lambda\mu}(k)$ and $S_{\lambda\mu}$ for $S_{\lambda\mu}(k)$, $\lambda, \mu \in X_1(T)$.

The next make a comment about the organization of the recursion formulas (in this section and in section 6). This will be done according to G_1-blocks. We recall the theorem of Humphreys that simple G_1-modules $L(\lambda)$ and $L(\mu)$ are in the same G_1-block if and only if $\mu \in W \cdot \lambda + pX(T)$, see [**9**], or [**12**;II,9.22] for the generalization to G_r, $r \geq 1$.

In this connection we look at the situation in which λ and $\lambda - \alpha$ do not belong to the same G_1-block, by which we mean that $\lambda - \alpha$ is not in $W \cdot \lambda + pX(T)$. Then we have a short exact sequence $0 \to \lambda - \alpha \to \lambda \cdot N(\alpha) \to \lambda \to 0$ of B-modules and hence we set the long exact sequence

$$0 \to \operatorname{Ind}_B^G k(\lambda - \alpha) \to \operatorname{Ind}_B^G(\lambda \cdot N(\alpha)) \to \operatorname{Ind}_B^G k(\lambda)$$

$$\to R\operatorname{Ind}_B^G k(\lambda - \alpha) \to R\operatorname{Ind}_B^G(\lambda \cdot N(\alpha)) \to R\operatorname{Ind}_B^G k(\lambda) \to \cdots$$

However, by the the linkage principle [**12**,II, 6.13 Proposition], we have, for arbitrary $i \geq 0$ and $\mu \in X(T)$, that any composition factor of $R^i\operatorname{Ind}_B^G k_\mu$ has the form $L(\tau)$, $\tau \in X^+(T)$, with $\tau \in W \cdot \lambda + pX(T)$. It follows that each connecting homomorphism $R^i\operatorname{Ind}_B^G k(\lambda) \to R^{i+1}\operatorname{Ind}_B^G k(\lambda - \alpha)$ is 0 so we get an exact sequence

$$0 \to R^i\operatorname{Ind}_B^G k(\lambda - \alpha) \to R^i\operatorname{Ind}_B^G(\lambda \cdot N(\alpha)) \to R^i\operatorname{Ind}_B^G k(\lambda) \to 0$$

for each i, Thus we get the following.

Lemma *Let $\lambda \in X(T)$ and suppose that $\lambda - \alpha \notin W \cdot \lambda + pX(T)$ (resp. $\lambda - \beta \notin W \cdot \lambda + pX(T)$) then we have $\chi^i_\alpha(\lambda) = \chi^i(\lambda) + \chi^i(\lambda - \alpha)$ (resp. $\chi^i_\beta(\lambda) = \chi^i(\lambda) + \chi^i(\lambda - \beta)$).*

5.2 We now take $p = 3$. We first give recursion formulas for the block containing $L(0)$ and $L(\rho)$.

From 2.2, Lemmas 1 and 3 we have:

Lemma 1

(i) $R_{\rho,\rho} = H^0(U_1, M(\zeta + \rho) \otimes \zeta)_\rho = (0,0) \oplus (0,-1) \oplus (-1,0)$

(ii) $R_{\rho,2\rho} = H^0(U_1, M(\zeta + \rho) \otimes \zeta)_{2\rho} = (0,-1) \oplus (-1,0) \oplus (-1,-1)$

(iii) $R_{2\rho,\rho} = H^0(U_1, M(\zeta + 2\rho) \otimes \zeta)_\rho = (-1,1) \cdot N(\beta) \oplus (1,-1) \cdot N(\alpha) \oplus (0,0) \oplus (-1,-1)$.

(iv) $R_{2\rho,2\rho} = (0,0) \oplus (-1,-1) \oplus (-1,0) \cdot N(\beta) \oplus (0,-1) \cdot N(\alpha)$.

From 2.3, Lemma 3 and 2.4 Lemma 4 we have:

Lemma 2

(i) $R_{\rho,\rho}(N(\alpha)) = H^0(U_1, M(\zeta + \rho) \otimes N(\alpha) \otimes \zeta)_\rho = (0,0) \oplus (0,-1) \oplus (-1,0)$.

(ii) $$R_{\rho,2\rho}(N(\alpha)) = H^0((U_1, M(\zeta + \rho) \otimes N(\alpha) \otimes \zeta)_{2\rho} = (-1,0) \oplus (0,-1) \cdot N(\alpha) \oplus 2(-1,-1) \oplus (-1,0)$$

(iii) $$R_{2\rho,\rho}(N(\alpha)) = H^0(U_1, M(\zeta + 2\rho) \otimes N(\alpha) \otimes \zeta)_\rho = (-1,1) \cdot N(\beta) \oplus (1,-1) \cdot N(\alpha) \oplus (0,0) \oplus (-1,-1).$$

(iv) $$R_{2\rho,2\rho}(N(\alpha)) = H^0(U_1, M(\zeta + 2\rho) \otimes N(\alpha) \otimes \zeta)_{2\rho} = (-1,-1) \oplus (-1,0) \cdot E \oplus (0,-1) \cdot N(\alpha) \oplus (-1,0) \oplus (-1,-1) \cdot V.$$

Hence we also have:

Lemma 3

(i) $S_{0,0} = (0,0) \oplus (-1,-1) \oplus (-1,0) \cdot N(\beta) \oplus (0,-1) \cdot N(\alpha)$.

(ii) $S_{0,\rho} = (0,-1) \oplus (-1,0) \oplus (-1,-1)$

(iii) $S_{\rho,0} = (-1,1) \cdot N(\beta) \oplus (1,-1) \cdot N(\alpha) \oplus (0,0) \oplus (-1,-1)$.

(iv) $S_{\rho,\rho} = (0,0) \oplus (0,-1) \oplus (-1,0)$

and also

Lemma 4

(i) $\quad S_{0,0}(N(\alpha)) = (-1,-1) \oplus (-1,0) \cdot E \oplus (0,-1) \cdot N(\alpha) \oplus (-1,0) \oplus (-1,-1) \cdot V$

(ii) $\quad S_{0,\rho}(N(\alpha)) = (-1,0) \oplus (0,-1) \cdot N(\alpha) \oplus 2(-1,-1) \oplus (-1,0)$

(iii) $\quad S_{\rho,0}(N(\alpha)) = (-1,1) \cdot N(\beta) \oplus (1,-1) \cdot N(\alpha) \oplus (0,0) \oplus (-1,-1)$

(iv) $\quad S_{\rho,\rho}(N(\alpha)) = (-1,0) \oplus (0,-1) \cdot N(\alpha) \oplus 2(-1,-1) \oplus (-1,0).$

By 5.1,(*) and Lemmas 3 and 4 we obtain:

Lemma 5

(i) $\chi^i(3r,3s) = [\chi^i(r,s) + \chi^i(r-1,s-1) + \chi^i_\beta(r-1,s) + \chi^i_\alpha(r,s-1)]^F$
$\qquad +\mathrm{ch}\, L(\rho)[\chi^i(r,s-1) + \chi^i(r-1,s) + \chi^i(r-1,s-1)]^F$

(ii) $\chi^i(1+3r,1+3s) = [\chi^i_\beta(r-1,s+1) + \chi^i_\alpha(r+1,s-1) + \chi^i(r,s) + \chi^i(r-1,s-1)]^F$
$\qquad +\mathrm{ch}\, L(\rho)[\chi^i(r,s) + \chi^i(r,s-1) + \chi^i(r-1,s)]^F.$

and

Lemma 6

(i) $\chi^i_\alpha(3r,3s)$
$= [\chi^i(r-1,s-1) + \chi(1,0)\chi^i(r-1,s) + \chi^i_\alpha(r,s-1) + \chi^i(r-1,s) + \chi(0,1)\chi^i(r-1,s-1)]^F$
$\qquad +\mathrm{ch}\, L(\rho)[\chi^i(r-1,s) + \chi^i_\alpha(r,s-1) + 2\chi^i(r-1,s-1) + \chi^i(r-1,0)]^F$

(ii) $\chi^i_\alpha(1+3r,1+3s) = [\chi^i_\beta(r-1,s+1) + \chi^i_\alpha(r,s-1) + \chi^i(r,s) + \chi^i(r-1,s-1)]^F$
$\qquad +\mathrm{ch}\, L(\rho)[\chi^i(r-1,s) + \chi^i_\alpha(r,s-1) + 2\chi^i(r-1,s-1) + \chi^i(r-1,s)]^F.$

5.3 We now consider the expansion over the G_1 block containing $L(1,0), L(0,2), L(2,1)$. From 2.2 we get:

Lemma 1

(i) $R_{(1,2),(1,2)} = (0,0) \oplus (-1,-1)$

(ii) $R_{(1,2),(2,0)} = (-1,1) \cdot N(\beta)$

(iii) $R_{(1,2),(0,1)} = (1,-1) \cdot N(\alpha)$

(iv) $R_{(2,0),(1,2)} = (-1,0)$

(v) $R_{(2,0),(2,0)} = (0.0)$

(vi) $R_{(2,0),(0,1)} = (0,-1)$

(vii) $R_{(0,1),(1,2)} = (0,-1)$

(viii) $R_{(0,1),(2,0)} = (-1,0)$

(ix) $R_{(0,1),(0,1)} = (0,0).$

From 2.3 we get:

Lemma 2

(i) $R_{(1,2),(1,2)}(N(\alpha) = (0,0) \oplus (-1,-1) \oplus (0,-1) \cdot N(\alpha)$

(ii) $R_{(1,2),(2,0)} = (-1,1) \oplus (-1,0) \cdot V$

(iii) $R_{(1,2),(0,1)}(N(\alpha) = (1,-1) \cdot N(\alpha) \oplus (-1,1) \cdot N(\beta)$

(iv) $R_{(2,0),(1,2)}(N(\alpha) = (-1,0) \oplus (-1,-1)$

(v) $R_{(2,0),(2,0)}(N(\alpha) = (0.0) \oplus (-2,1)$

(vi) $R_{(2,0),(0,1)}(N(\alpha) = (0,0) \oplus (0,-1)$

(vii) $R_{(0,1),(1,2)}(N(\alpha)) = (-1,0) \oplus (0,-1)$

(viii) $R_{(0,1),(2,0)}(N(\alpha)) = 2(-1,0)$

(ix) $R_{(0,1),(0,1)}(N(\alpha)) = (0,0) \oplus (-1,0)$.

Hence we also have:

Lemma 3

(i) $S_{(1,0),(1,0)} = (0,0) \oplus (-1,-1)$

(ii) $S_{(1,0),(0,2)} = (-1,0)$

(iii) $S_{(1,0),(2,1)} = (0,-1)$

(iv) $S_{(0,2),(1,0)} = (-1,1) \cdot N(\beta)$

(v) $S_{(0,2),(0,2)} = (0,0)$

(vi) $S_{(0,2),(2,1)} = (-1,0)$

(vii) $S_{(2,1),(1,0)} = (1,-1) \cdot N(\alpha)$

(viii) $S_{(2,1),(0,2)} = (0,-1)$

(ix) $S_{(2,1),(2,1)} = (0,0)$.

and

Lemma 4

(i) $S_{(1,0),(1,0)}(N(\alpha)) = (0,0) \oplus (-1,-1) \oplus (0,-1) \cdot N(\alpha)$

(ii) $S_{(1,0),(0,2)}(N(\alpha)) = (-1,0) \oplus (-1,-1)$

(iii) $S_{(1,0),(2,1)}(N(\alpha)) = (-1,0) \oplus (0,-1)$

(iv) $S_{(0,2),(1,0)}(N(\alpha)) = (-1,1) \oplus (-1,0) \cdot V$

(v) $S_{(0,2),(0,2)}(N(\alpha)) = (0.0) \oplus (-2,1)$

(vi) $S_{(0,2),(2,1)}(N(\alpha)) = 2(-1,0)$

(vii) $S_{(2,1),(1,0)}(N(\alpha)) = (1,-1) \cdot N(\alpha) \oplus (-1,1) \cdot N(\beta)$

(viii) $S_{(2,1),(0,2)}(N(\alpha)) = (0,0) \oplus (0,-1)$

(ix) $S_{(2,1),(2,1)}(N(\alpha)) = (0,0) \oplus (-1,0)$.

Thus, by 5.1,(*), we get the reduction formulas below.

Lemma 5

(i) $\chi^i(1+3r,3s) = \chi(1,0)[\chi^i(r,s)+\chi^i(r-1,s-1)]^F + \chi(0,2)\chi^i(r-1,s)^F$
$+\chi(2,1)\chi^i(r,s-1)^F$.

(ii) $\chi^i(3r,2+3s) = \chi(1,0)\chi^i_\beta(r-1,s+1)^F + \chi(0,2)\chi^i(r,s)^F + \chi(2,1)\chi^i(r-1,s)^F$.

(iii) $\chi^i(2+3r,1+3s) = \chi(1,0)\chi^i_\alpha(r+1,s-1)^F + \chi(0,2)\chi^i(r,s-1)^F + \chi(2,1)\chi^i(r,s)^F$.

Lemma 6

(i) $\chi^i_\alpha(1+3r,3s) = \chi(1,0)[\chi^i(r,s)+\chi(r-1,s-1)+\chi^i_\alpha(r,s-1)]^F$
$+\chi(0,2)[\chi^i(r-1,s)+\chi^i(r-1,s-1)]^F + \chi(2,1)[\chi^i(r-1,s)+\chi(r,s-1)]^F$.

(ii) $\chi^i_\alpha(3r,2+3s) = \chi(1,0)[\chi^i(r-1,s+1)+\chi(0,1)\chi^i(r-1,s)]^F$
$\chi(0,2)[\chi^i(r,s)+\chi^i(r-2,s+1)]^F + 2\chi(2,1)\chi^i(r-1,s)^F$.

(iii) $\chi^i_\alpha(2+3r,1+3s) = \chi(1,0)[\chi^i_\alpha(r+1,s-1)+\chi^i_\beta(r-1,s+1)]^F$
$\chi(0,2)[\chi^i(r,s)+\chi^i(r,s-1)]^F + \chi(2,1)[\chi^i(r,s)+\chi^i(r-1,s)]^F$.

5.4 We now consider the expansion over the G_1 block containing $L(0,1), L(2,0), L(1,2)$. From 2.2 we get:

Lemma 1

(i) $R_{(2,1),(2,1)} = (0,0) \oplus (-1,-1)$

(ii) $R_{(2,1),(0,2)} = (1,-1) \cdot N(\alpha)$

(iii) $R_{(2,1),(1,0)} = (-1.1) \cdot N(\beta)$

(iv) $R_{(0,2),(2,1)} = (0,-1)$

(v) $R_{(0,2),(0,2)} = (0,0)$

(vi) $R_{(0,2),(1,0)} = (-1,0)$

(vii) $R_{(1,0),(2,1)} = (-1,0)$

(viii) $R_{(1,0),(0,2)} = (0,-1)$

(ix) $R_{(1,0),(1,0)} = (0,0)$.

From 2.3 we get:

Lemma 2

(i) $R_{(2,1),(2,1)}(N(\alpha)) = (-1,0) \cdot E \oplus (-1,-1)$
(ii) $R_{(2,1),(0,2)}(N(\alpha)) = (0,0) \oplus (1,-1) \cdot N(\alpha)$
(iii) $R_{(2,1),(1,0)}(N(\alpha)) = (-1,1) \cdot N(\beta) \oplus N(\alpha)$
(iv) $R_{(0,2),(2,1)}(N(\alpha)) = (0,-1) \cdot N(\alpha)$
(v) $R_{(0,2),(0,2)}(N(\alpha)) = (0,0) \oplus (0,-1)$
(vi) $R_{(0,2),(1,0)}(N(\alpha)) = (-1,0) \oplus (-1,-1)$
(vii) $R_{(1,0),(2,1)}(N(\alpha)) = 2(-1,0)$
(viii) $R_{(1,0),(0,2)}(N(\alpha)) = (-1,0) \oplus (0,-1)$
(ix) $R_{(1,0),(1,0)}(N(\alpha)) = (0,0) \oplus (-1,0)$.

Hence we also have:

Lemma 3

(i) $S_{(0,1),(0,1)} == (0,0) \oplus (-1,-1)$
(ii) $S_{(0,1),(2,0)} = (0,-1)$
(iii) $S_{(0,1),(1,2)} = (-1,0)$
(iv) $S_{(2,0),(0,1)} = (1,-1) \cdot N(\alpha)$
(v) $S_{(2,0),(2,0)} = (0,0)$
(vi) $S_{(2,0),(1,2)} = (0,-1)$
(vii) $S_{(1,2),(0,1)} = (-1.1) \cdot N(\beta)$
(viii) $S_{(1,2),(2,0)} = (-1,0)$
(ix) $S_{(1,2),(1,2)} = (0,0)$

and

Lemma 4

(i) $S_{(0,1),(0,1)}(N(\alpha)) = (-1,0) \cdot E \oplus (-1,-1)$
(ii) $S_{(0,1),(2,0)}(N(\alpha)) = (0,-1) \cdot N(\alpha)$
(iii) $S_{(0,1),(1,2)}(N(\alpha)) = 2(-1,0)$
(iv) $S_{(2,0),(0,1)}(N(\alpha)) = (0,0) \oplus (1,-1) \cdot N(\alpha)$
(v) $S_{(2,0),(2,0)}(N(\alpha)) = (0,0) \oplus (0,-1)$
(vi) $S_{(2,0),(1,2)}(N(\alpha)) = (-1,0) \oplus (0,-1)$
(vii) $S_{(1,2),(0,1)}(N(\alpha)) = (-1,1) \cdot N(\beta) \oplus N(\alpha)$
(viii) $S_{(1,2),(2,0)}(N(\alpha)) = (-1,0) \oplus (-1,-1)$
(ix) $S_{(1,2),(1,2)}(N(\alpha)) = (0,0) \oplus (-1,0)$.

Thus we get the reduction formulas below.

Lemma 5

(i) $\chi^i(3r, 1+3s) = \chi(0,1)[\chi^i(r,s) + \chi^i(r-1,s-1)]^F + \chi(2,0)\chi^i(r,s-1)^F$
$+\chi(1,2)\chi^i(r-1,s)^F$

(ii) $\chi^i(2+3r, 3s) = \chi(0,1)\chi^i_\alpha(r+1,s-1)^F + \chi(2,0)\chi^i(r,s)^F$
$+\chi(1,2)\chi^i(r,s-1)^F$

(iii) $\chi^i(1+3r, 2+3s) = \chi(0,1)\chi^i_\beta(r-1,s+1)^F + \chi(2,0)\chi^i(r-1,s)^F$
$+\chi(1,2)\chi^i(r,s)^F$.

Lemma 6

(i) $\chi^i_\alpha(3r, 1+3s) = \chi(0,1)[\chi(r-1,s-1) + \chi(1,0)\chi(r-1,s)]^F$
$+\chi(2,0)\chi^i_\alpha(r,s-1)^F + 2\chi(1,2)\chi^i_\alpha(r-1,s)^F$.

(ii) $\chi^i_\alpha(2+3r, 3s) = \chi(0,1)[\chi^i(r,s) + \chi^i_\alpha(r+1,s-1)]^F$
$+\chi(2,0)[\chi(r,s)+\chi(r,s-1)]^F + \chi(1,2)[\chi^i(r-1,s)+\chi^i(r-1,s)+\chi(r,s-1)]^F$.

(iii) $\chi^i_\alpha(1+3r, 2+3s) = \chi(0,1)[\chi^i_\alpha(r,s) + \chi^i_\beta(r-1,s+1)]^F$
$+\chi(2,0)[\chi^i(r-1,s)+\chi^i(r-1,s-1)]^F + \chi(1,2)[\chi^i(r,s)+\chi^i(r-1,s)]^F$.

5.5 It remains to describe expansion formulas for $\chi^i(2+3r, 2+3s)$ and $\chi^i_\alpha(2+3r, 2+3s)$.

Lemma

(i) $\chi^i(2+3r, 2+3s) = \chi(2,2)\chi^i(r,s)^F$

(ii) $\chi^i_\alpha(2+3r, 2+3s) = \chi^i(2+3r, 2+3s) + \chi^i(3r, 3(s+1))$
$= \chi(2,2)\chi^i(r,s)^F + [\chi^i(r,s+1) + \chi^i(r-1,s) + \chi^i_\beta(r-1,s+1) + \chi^i_\alpha(r,s)]^F$
$+\mathrm{ch}\, L(\rho)[\chi^i(r,s) + \chi^i(r-1,s+1) + \chi^i(r-1,s)]^F$.

Proof. This follows from the Andersen-Haboush identity, [**12**;II,3.19 Proposition]. the fact that $R^i\mathrm{Ind}_B^G(2+3r, 2+3s)$ and $R^i\mathrm{Ind}_B^G(3r, 3(s+1))$ belong to different blocks (see 5.1 Lemma 1) and 5.2 Lemma 5.

Remark This completes the description of the expansions of all $\chi^i(r,s)$ and $\chi^i_\alpha(r,s)$. We now also have expansions for $\chi^i_\beta(r,s)$, for all r, s, thanks for 2.5 Lemma.

As in Section 4 we have the following.

Proposition *The characters* $\chi^1(r,s), \chi^1_\alpha(r,s)$ *and* $\chi^1_\beta(r,s)$ *are determined recursively, for all* r,s *by Lemmas 1,2,3 of Section 3.1, the expansion formulas 5.2, Lemmas 5,6 and 5.3, Lemmas 5,6 and 5.4 Lemmas 5,6 and 5.5 Lemma, together with 2.5 Lemma.*

6 The case $p > 3$.

6.1 We establish reduction formulas for $\chi^i(\lambda)$ and $\chi^i_\alpha(\lambda)$, valid for $p > 3$.

Let $0 \leq a \leq p-2$ and consider the W-orbit $\mathcal{O}$ of $X(T)/pX(T)$ containing $(a+1,0)$ modulo $pX(T)$. Thus $\mathcal{O}$ contains the elements $(1+a,0), (p-1-a,1+a), (0,p-1-a)$ taken modulo $pX(T)$. We first derive expansion formulas over this block.

Lemma 1

(i) $R_{(1+a,0),(1+a,0)} = (0,0)$
(ii) $R_{(1+a,0),(p-1-a,1+a)} = (-1,0)$
(iii) $R_{(1+a,0),(0,p-1-a)} = (0,-1)$
(iv) $R_{(p-1-a,1+a),(1+a,0)} = (-1,1) \cdot N(\beta)$
(v) $R_{(p-1-a,1+a),(p-1-a,1+a)} = (0,0) \oplus (-1,-1)$
(vi) $R_{(p-1-a,1+a),(0,p-1-a)} = (1,-1) \cdot N(\alpha)$
(vii) $R_{(0,p-1-a),(1+a,0)} = (-1,0)$
(viii) $R_{(0,p-1-a),(p-1-a,1+a)} = (0,-1)$
(ix) $R_{(0,p-1-a),(0,p-1-a)} = (0,0)$

Lemma 2

(i) $S_{(p-1,a),(p-1,a)} = (0,0)$
(ii) $S_{(p-1,a),(a,p-2-a)} = (1,-1) \cdot N(\alpha)$
(iii) $S_{(p-1,a),(p-2-a,p-1)} = (0,-1)$
(iv) $S_{(a,p-2-a),(p-1,a)} = (0,-1)$
(v) $S_{(a,p-2-a),(a,p-2-a)} = (0,0) \oplus (-1,-1)$
(vi) $S_{(a,p-2-a),(p-2-a,p-1)} = (-1,0)$
(vii) $S_{(p-2-a,p-1),(p-1,a)} = (-1,0)$
(viii) $S_{(p-2-a,p-1),(a,p-2-a)} = (-1,1) \cdot N(\beta)$
(ix) $S_{(p-2-a,p-1),(p-2-a,p-1)} = (0,0)$

Lemma 3

(i) $$\chi^i(p-1+pr, a+ps) = \chi(p-1,a)\chi^i(r,s)^F + \chi(a,p-2-a)\chi^i_\alpha(r+1,s-1)^F + \chi(p-2-a,p-1)\chi^i(r,s-1)^F$$

(ii) $$\chi^i(a+pr, p-2-a+ps) = \chi(p-1,a)\chi^i(r,s-1)^F + \chi(a,p-2-a)[\chi^i(r,s)+\chi^i(r-1,s-1)]^F + \chi(p-2-a,p-1)\chi^i(r-1,s)^F.$$

(iii) $$\chi^i(p-2-a+pr, p-1+ps) = \chi(p-1,a)\chi^i(r-1,s)^F + \chi(a,p-2-a)\chi^i_\beta(r-1,s+1)^F + \chi(p-2-a,p-1)\chi^i(r,s)^F.$$

6.2 We next produce expansion formulas for $\chi^i(\lambda)$, for $\lambda+\rho+pX(T)$ in a W-orbit of size 6. We fix $1 \leq a,b < p$ with $2a+b, a+2b \leq p$ and put $\mu_1 = (a,b)$, $\mu_2 = (p-a, a+b)$, $\mu_3 = (a+b, p-b)$, $\mu_4 = (b, p-a-b)$, $\mu_5 = (p-a-b, a)$, $\mu_6 = (p-b, p-a)$.

Then we have the following equations.

$$H^0(U_1, M(\zeta+\mu_1)\otimes\zeta) = \mu_1 \oplus \mu_2\cdot(-1,0)^F \oplus \mu_3\cdot(0,-1)^F \\ \oplus \mu_4\cdot[(-1,1)\cdot N(\alpha)]^F \oplus \mu_6\cdot[(0,-1)\cdot N(\alpha)]^F \oplus \mu_5\cdot N(\gamma)^F.$$

$$H^0(U_1, M(\zeta+\mu_2)\otimes\zeta) = \mu_2\cdot N(\gamma)^F \oplus \mu_1\cdot[(-1,1)\cdot N(\beta)]^F \oplus \mu_4\cdot[(1,-1)\cdot N(\alpha)]^F \\ \oplus \mu_3\cdot[(-1,0)\cdot N(\beta)]^F \oplus \mu_6\cdot[(0,-1)\cdot N(\alpha)^{\oplus}\mu_5\cdot N(\gamma)^F.$$

$$H^0(U_1, M(\zeta+\mu_3)\otimes\zeta) = \mu_3\cdot N(\gamma)^F \oplus \mu_5\cdot[(-1,1)\cdot N(\beta)]^F \oplus \mu_1\cdot[(1,-1)\cdot N(\alpha)]^F \\ \oplus \mu_6\cdot[(-1,0)\cdot N(\beta)]^F \oplus \mu_2\cdot[(0,-1)\cdot N(\alpha)]^F \oplus \mu_4\cdot N(\gamma)^F.$$

$$H^0(U_1, M(\zeta+\mu_4)\otimes\zeta) = \mu_4 \oplus \mu_6\cdot(-1,0)^F \oplus \mu_2\cdot(0,-1)^F \\ \mu_5\cdot(0,-1)^F \oplus \mu_1\cdot(-1,0)^F \oplus \mu_3\cdot(-1,1)^F.$$

$$H^0(U_1, M(\zeta+\mu_5)\otimes\zeta) = \mu_5 \oplus \mu_3\cdot(-1,0)^F \oplus \mu_6\cdot(0,-1)^F \\ \mu_1\cdot(0,-1)^F \oplus \mu_4\cdot(-1,0)^F \oplus \mu_2\cdot(-1,-1)^F.$$

$$H^0(U_1, M(\zeta+\mu_6)\otimes\zeta) = \mu_6\cdot N(\gamma)^F \oplus \mu_4\cdot[(-1,1)\cdot N(\beta)]^F \oplus \mu_5\cdot[(1,-1)\cdot N(\alpha)]^F \\ \oplus \mu_2\cdot[(-1,0)\cdot N(\beta)]^F \oplus \mu_3\cdot[(0,-1)\cdot N(\alpha)]^F \oplus \mu_1\cdot N(\gamma)^F.$$

Thus we get the following matrix for $(R_{\mu_i\mu_j})$, $1 \leq i,j \leq 6$.

$$\begin{pmatrix} (0,0) & (-1,0) & (0,-1) & (0,-1) & (-1,0) & (-1,-1) \\ (-1,1)\cdot N(\beta) & N(\gamma) & (-1,0)\cdot N(\beta) & (1,-1)\cdot N(\alpha) & N(\gamma) & (0,-1)\cdot N(\alpha) \\ (1,-1)\cdot N(\alpha) & (0,-1)\cdot N(\alpha) & N(\gamma) & N(\gamma) & (-1,1)\cdot N(\beta) & (-1,0)\cdot N(\beta) \\ (-1,0) & (0,-1) & (-1,1) & (0,0) & (0,-1) & (-1,0) \\ (0,-1) & (-1,-1) & (-1,0) & (-1,0) & (0,0) & (0,-1) \\ N(\gamma) & (-1,0)\cdot N(\beta) & (0,-1)\cdot N(\alpha) & (-1,1)\cdot N(\beta) & (1,-1)\cdot N(\alpha) & N(\gamma) \end{pmatrix}$$

We put $\lambda_i = \zeta - \mu_i$, for $1 \leq i \leq 6$, i.e. $\lambda_1 = (p-1-a, p-1-b)$, $\lambda_2 = (a-1, p-1-a-b)$, $\lambda_3 = (p-1-1-1, b-1)$, $\lambda_4 = (p-1-b, a+b-1)$, $\lambda_5 = (a+b-1, p-1-a)$,
$\lambda_6 = (b-1, a-1)$. We get the transpose matrix for $(S_{\lambda_i\lambda_j})$, $1 \leq i,j \leq 6$, i.e the following.

$$\begin{pmatrix} (0,0) & (-1,1)\cdot N(\beta) & (1,-1)\cdot N(\alpha) & (-1,0) & (0,-1) & N(\gamma) \\ (-1,0) & N(\gamma) & (0,-1)\cdot N(\alpha) & (0,-1) & (-1,-1) & (-1,0)\cdot N(\beta) \\ (0,-1) & (-1,0)\cdot N(\beta) & N(\gamma) & (-1,1) & (-1,0) & (0,-1)\cdot N(\alpha) \\ (0,-1) & (1,-1)\cdot N(\alpha) & N(\gamma) & (0,0) & (-1,0) & (-1,1)\cdot N(\beta) \\ (-1,0) & N(\gamma) & (-1,1)\cdot N(\beta) & (0,-1) & (0,0) & (1,-1)\cdot N(\alpha) \\ (-1,-1) & (0,-1)\cdot N(\alpha) & (-1,0)\cdot N(\beta) & (-1,0) & (0,-1) & N(\gamma) \end{pmatrix}$$

Hence we get the following expansion formulas.

Proposition *Let $1 \le a, b < p$ with $a+2b, 2a+b \le p$. Then we have:*

(i) $\chi^i(p-1-a+pr, p-1-b+ps) = \chi(p-1-a, p-1-b)\chi^i(r,s)^F$
$+\chi(a-1, p-1-a-b)\chi^i_\beta(r-1, s+1)^F + \chi(p-1-a-b, b-1)\chi^i_\alpha(r+1, s-1)^F$
$+\chi(p-1-b, a+b-1)\chi^i(r-1, s)^F + \chi(a+b-1, p-1-a)\chi^i(r, s-1)^F$
$+\chi(b-1, a-1)[\chi^i(r,s) + \chi^i(r-1, s-1)]^F$;

(ii) $\chi^i(a-1+pr, p-1-a-b+ps) = \chi(p-1-a, p-1-b)\chi^i(r-1, s)^F$
$+\chi(a-1, p-1-a-b)[\chi^i(r,s) + \chi^i(r-1, s-1)]^F$
$+\chi(p-1-a-b, b-1)\chi^i_\alpha(r, s-1)^F + \chi(p-1-a-b, b-1)\chi^i(r, s-1)^F$
$+\chi(a+b-1, p-1-a)\chi^i(r-1, s-1)^F + \chi(b-1, a-1)\chi^i_\beta(r-1, s)^F$;

(iii) $\chi^i(p-1-a-b+pr, b-1+ps) = \chi(p-1-a, p-1-b)\chi^i(r, s-1)^F$
$+\chi(a-1, p-1-a-b)\chi^i_\beta(r-1, s)^F + \chi(p-1-a-b, b-1)[\chi^i(r,s)$
$+\chi^i(r-1, s-1)]^F + \chi(p-1-b, a+b-1)\chi^i(r-1, s+1)^F$
$+\chi(a+b-1, p-1-a)\chi^i(r-1, s)^F + \chi(b-1, a-1)\chi^i_\alpha(r, s-1)^F$;

(iv) $\chi^i(p-1-b+pr, p-1-a+ps) = \chi(p-1-a, p-1-b)\chi^i(r, s-1)^F$
$+\chi(a-1, p-1-a-b)\chi^i_\alpha(r+1, s-1)^F$
$+\chi(p-1-a-b, b-1)[\chi^i(r,s) + \chi^i(r-1, s-1)]^F$
$+\chi(p-1-b, a+b-1)\chi^i(r,s)^F + \chi(a+b-1, p-1-a)\chi^i(r-1, s)^F$
$+\chi(b-1, a-1)\chi^i_\beta(r-1, s+1)^F$;

(v) $\chi^i(a+b-1+pr, p-1-a+ps) = \chi(p-1-a, p-1-b)\chi^i(r-1, s)^F$
$+\chi(a-1, p-1-a-b)[\chi^i(r,s) + \chi^i(r-1, s-1)]^F$
$+\chi(p-1-a-b, b-1)\chi^i_\beta(r-1, s+1)$
$+\chi(p-1-b, a+b-1)\chi^i(r, s-1)^F + \chi(a+b-1, p-1-a)\chi^i(r,s)^F$
$+\chi(b-1, a-1)\chi^i_\alpha(r+1, s-1)$;

(vi) $\chi^i(b-1+pr, a-1+ps) = \chi(p-1-a, p-1-b)\chi^i(r-1, s-1)$
$+\chi(a-1, p-1-a-b)\chi^i_\alpha(r, s-1) + \chi(p-1-a-b, b-1)\chi^i_\beta(r-1, s)^F$
$+\chi(p-1-a-b, b-1)\chi^i(r-1, s)^F + \chi(a+b-1, p-1-a)\chi^i(r, s-1)^F$
$+\chi(b-1, a-1)[\chi^i(r,s) + \chi^i(r-1, s-1)]^F$.

6.3 Our next goal is produce reduction formulas for the characters $\chi^i_\alpha(\lambda)$, for $\lambda+\rho+pX(T)$ in a W-orbit of size 3. Thus we fix $0 \le a \le p-2$ and derive expansion formulas for $\chi^i(\lambda+p\mu)$, with $\lambda \in \{(p-1,a),(a,p-2-a),(p-2-a,p-1)\}$, $\mu \in X(T)$. One may easily convince oneself that $\lambda+\rho+pX(T)$ and $\lambda+\rho-\alpha+pX(T)$ belong to different W orbits in $X(T)/pX(T)$, except in case $\lambda=(a,p-2-a)$ with $a=0$. Thus we get $\chi^i_\alpha(\lambda+p\mu)=\chi^i(\lambda)+\chi^i(\lambda-\alpha)$, by 5.1 Lemma, except in the case $\lambda=(a,p-2-a)$ with $a=0$.

So now let $\lambda_1=(0,p-2)$, $\lambda_2=(p-1,0)$, $\lambda_3=(p-2,p-1)$. Then, by block considerations, for $\nu \in X^+(T)$ we have $\chi^i_\alpha(\lambda_1+p\nu)=\chi(\lambda_1)\chi_1^F+\chi(\lambda_2)\chi_2^F+\chi(\lambda_3)^F$ for certain characters χ_1,χ_2,χ_3. Putting $\mu_1=\zeta-\lambda_1=(p-1,1)$, $\mu_2=\zeta-\lambda_2=(0,p-1)$,$\mu_3=\zeta-\lambda_3=(1,0)$ we get:

$$\begin{aligned}\chi^i_\alpha(\lambda_1+p\nu) &= \chi^i(N(\alpha)\otimes(\zeta-\mu_1)+p\nu)\\ &= \sum_{t=1}^{3}\operatorname{ch}L((\zeta-\mu_t))\chi^i(H^0(U_1,M(\zeta+\mu_t)\otimes\zeta\otimes N(\alpha)_{\mu_1}\otimes\nu)^F\\ &= \chi(\lambda_1)\chi^i(H^0(U_1,M(\zeta+(p-1,1))\otimes\zeta\otimes N(\alpha))_{(p-1,1)}\otimes\nu)^F\\ &+\chi(\lambda_2)\chi^i(H^0(U_1,M(\zeta+(0,p-1)\otimes\zeta\otimes N(\alpha))_{(p-1,1)}\otimes\nu)^F\\ &+\chi(\lambda_3)\chi^i(H^0(U_1,M(\zeta+(1,0))\otimes\zeta\otimes N(\alpha)_{(p-1,1)}\otimes\nu)^F.\end{aligned}$$

Using Lemma 4, Lemma 2(ii) and Lemma 1 of 2.3 we see that this is

$$\begin{aligned}\chi(\lambda_1)\chi^i(((-1,0)\cdot E\oplus(-1,-1))\otimes\nu)^F+\chi(\lambda_2)\chi^i((0,-1)\cdot N(\alpha)\otimes\nu)^F\\ +\chi(\lambda_3)\chi^i((-1,0)\cdot\nu)^F.\end{aligned}$$

This completes the proof of the following result.

Lemma 1 *(i)* $\chi^i_\alpha(p-1+pr,a+ps)=\chi^i(p-1+pr,a+ps)+\chi^i(p-3+pr,a+1+ps)$.
(ii) $\chi^i_\alpha(a+pr,p-2-a+ps)=\chi^i(a+pr,p-2-a+ps)+\chi^i(a-2+pr,p-1-a+ps)$, *for* $0<a\le p-2$, *and*

$$\begin{aligned}\chi^i_\alpha(pr,p-2+ps) &= \chi(0,p-2)[\chi(1,0)\chi^i(r-1,s)+\chi^i(r-1,s-1)]^F\\ &+\chi(p-1,0)\chi^i_\alpha(r,s-1)^F+\chi(p-2,p-1)\chi^i(r-1,s)^F.\end{aligned}$$

(iii) $\chi^i_\alpha(p-2-a+pr,p-1+ps)=\chi^i_\alpha(p-2-a+pr,p-1+ps)+\chi^i(p-4-a+pr,p(s+1))$.

6.4 We now deal with the expansion of $\chi^i_\alpha(\lambda)$ for $L(\lambda)$ in a regular G_1-block, i.e. one containing six simple modules. So we fix $1\le a,b<p$ with $2a+b,a+2b\le p$. We set $\lambda_1=(p-1-a,p-1-b)$, $\lambda_2=(a-1,p-1-a-b)$, $\lambda_3=(p-1-a-b,b-1)$, $\lambda_4=(p-1-b,a+b-1)$, $\lambda_5=(a+b-1,p-1-a)$, $\lambda_6=(b-1,a-1)$. Then, for $\lambda=\lambda_i$, $1\le i\le 6$, and $\nu\in X(T)$, we have $\chi^i_\alpha(\lambda+p\nu)=\chi^i(\lambda+p\mu)+\chi^i(\lambda-\alpha+p\mu)$ provided that $\lambda+\rho+pX(T)$ and $\lambda+\rho-\alpha+pX(T)$ belong to different orbits under

W, i.e. provided that we do not have $\lambda + \rho - \alpha \equiv \lambda_j$, modulo $pX(T)$, for any j. Thus we determine those $\mu \in S = \{(-a, -b), (a, -a-b), (-a-b, b), (-b, a+b), (a+b, -a), (b, a)\}$ such that $\mu - \alpha \equiv \tau$ for some $\tau \in S$. Note that it is never that case that $\mu - \alpha \equiv \mu$.

Case (i). $\mu - \alpha = (-a-2, b+1)$

If $(-a-2, -b+1) \equiv (a, -a-b)$ then $a = p-1$ but that is not possible (since $2a + b \leq p$).

If $(-a-2, -b+1) \equiv (-a-b, b)$ then $b = 2$ and so $-1 \equiv 2$, which is not possible.

If $(-a-2, -b+1) \equiv (-b, a+b)$ then $a+2b \equiv 1$, which is not possible as $a+2b \leq p$ and $a, b \geq 1$.

If $(-a-2, -b+1) \equiv (a+b, -a)$ then $2a + b \equiv -2$, $-a + b \equiv 1$ so that $3a \equiv -3$ and $a = p-1$, which is not possible.

If $(-a-2, -b+1) \equiv (b, a)$ then $a + b \equiv -2 \equiv 1$, which is not possible.

Case (ii). $\mu - \alpha = (a-2, -a-b+1)$.

If $(a-2, -a-b+1) \equiv (-a, -b)$ then $a = 1$ and this is possible.

If $(a-2, -a-b+1) \equiv (-a-b, b)$ then $2a + b = 2$, which is not possible.

If $(a-2, -a-b+1) \equiv (a+b, -a)$ then $b \equiv -2$ and $b \equiv 1$, which is not possible.

If $(a-2, -a-b+1) \equiv (b, a)$ then $2a + b = 1$, which is not possible.

Case (iii). $\mu - \alpha = (-a-b-2, b+1)$.

If $(-a-b-2, b+1) \equiv (-a, -b)$ then $b \equiv -2$ and $2b+1 \equiv 0$ which is not possible.

If $(-a-b-2, b+1) \equiv (a, -a-b)$ then $a + 2b = 1$, which is not possible.

If $(-a-b-2, b+1) \equiv (-b, a+b)$ then $a = 1$, and $a \equiv -2$, which is not possible.

If $(-a-b-2, b+1) \equiv (a+b, -a)$ then $a + b \equiv -1$ so that $a + b = p-1$ so $a + 2b = 2a + b = p$ and $a = b = 1$ which is not possible (as $p > 3$).

If $(-a-b-2, b+1) \equiv (b, a)$ then $a + 2b \equiv -2$, $a = b+1$ giving $3b \equiv -3$ wo $b = p-1$, which is not possible.

Case (iv). $\mu - \alpha = (-b-2, a+b+1)$.

If $(-b-2, a+b+1) \equiv (-a, -b)$ then $a - b \equiv 2$, $a + 2b \equiv -1$ so $3a \equiv 3$, $a = 1$, $b = p-1$, which is not possible.

If $(-b-2, a+b+1) \equiv (a, -a-b)$ then $a + b \equiv -2$ and $2(a+b) \equiv -1$, which is not possible.

If $(-b-2, a+b+1) \equiv (-a-b, b)$ then $a \equiv 2$ and $q \equiv -1$, which is not possible.

If $(-b-2, a+b+1) \equiv (a+b, -a)$ then $a + 2b \equiv -2$ and $2a + b \equiv -1$ giving $a + b \equiv -1$ and $a \equiv 0$ so $a = p$, which is not possible.

If $(-b-2, a+b+1) \equiv (b, a)$ then $b \equiv -1$ so $b = p-1$, which is not possible.

Case (v). $\mu - \alpha = (a+b-2, -a+1)$.

If $(a+b-2, -a+1) \equiv (-a, -b)$ then $2a+b = 2$, which is not possible.

If $(a+b-2, -a+1) \equiv (a, -a-b)$ then $b \equiv 2$ and $b \equiv -1$, which is not possible.

If $(a+b-2, -a+1) \equiv (-a-b, b)$ then $a+b = 1$, which is not possible.

If $(a+b-2, -a+1) \equiv (-b, a+b)$ then $2a+b = 1$, which is not possible.

If $(a+b-2, -a+1) \equiv (b, a)$ then $a \equiv 2$ and $2a \equiv 1$, which is not possible.

Case (vi). $\mu - \alpha = (b-2, a+1)$.

If $(b-2, a+1) \equiv (-a, -b)$ then $a+b \equiv 2$ and $a+b \equiv -1$, which is not possible.

If $(b-2, a+1) \equiv (a, -a-b)$ then $a-b \equiv -2$ and $2a+b \equiv -1$ so $a = p-1$, which is not possible.

If $(b-2, a+1) \equiv (-a-b, b)$ then $a+2b = 2$, which is not possible.

If $(b-2, a+1) \equiv (-b, a+b)$ then $b = 1$, and this is possible.

If $(b-2, a+1) \equiv (a+b, -a)$ then $a \equiv -2$ and $2a \equiv -1$, and this is not possible.

Lemma 1 *For $\lambda \in \{(p-1-a, p-1-b), (a-1, p-1-a-b), (p-1-a-b, b-1), (p-1-b, a+b-1), (a+b-1, p-1-a), (b-1, a-1)\}$ and $\nu \in X(T)$ we have*

$$\chi^i_\alpha(\lambda + p\nu) = \chi^i(\lambda + p\nu) + \chi^i(\lambda - \alpha + p\nu)$$

except in the cases $\lambda = (a-1, p-1-a-b)$ with $a = 1$ and $\lambda = (b-1, a-1)$ with $b = 1$.

We now consider the exceptional cases. We write $\mu_i = \zeta - \lambda_i$, for $1 \leq i \leq 6$. Thus $\mu_1 = (a, b)$, $\mu_2 = (p-a, a+b)$, $\mu_3 = (a+b, p-b)$, $\mu_4 = (b, p-a-b)$, $\mu_5 = (p-a-b, a)$, $\mu_6 = (p-b, p-a)$.

We now suppose that either $j = 2$,$a = 1$ or that $j = 6$, $b = 1$ and and consider the expansion of $\chi^i_\alpha(\lambda_j + p\nu)$. Since all composition factor of $R^i\mathrm{Ind}_B^G(\lambda_j + p\nu)$ belong to the same G_1-block, we have

$$\chi^i_\alpha(\lambda_j + p\nu) = \sum_{t=1}^{6} \mathrm{ch}\, L(\lambda_t)\chi_t^F$$

i.e.

$$\chi^i((\zeta - \mu_j) \otimes N(\alpha)) = \sum_{t=1}^{6} \mathrm{ch}\, L(\zeta - \mu_t)\chi_t^F$$

for some characters $\chi_1, \ldots, \chi_6$. Hence, by Section 1.1(*), we have

$$\chi^i_\alpha(\lambda_j + p\nu) = \sum_{t=1}^{6} \mathrm{ch}\, L(\zeta - \mu_t)\chi^i(H^0(U_1, M(\zeta + \mu_t) \otimes \zeta \otimes N(\alpha))_{\mu_j} \otimes \nu)^F.$$

We now take $j = 2, a = 1$. We have $\mu_1 = (1, b)$, $\mu_2 = (p-1, 1+b)$, $\mu_3 = (1+b, p-b)$, $\mu_4 = (b, p-1-b)$, $\mu_5 = (p-1-b, 1)$, $\mu_6 = (p-b, p-1)$.

Now we have

$$\begin{aligned} H^0(U_1, &M(\zeta + (1,b)) \otimes \zeta \otimes N(\alpha))_{(p-1,1+b)} \\ &= H^0(U_1, M(\zeta + (1,b)) \otimes \zeta)_{(p-1,1+b)} \oplus H^0(U_1, M(\zeta + (-1, b+1)) \otimes \zeta)_{(p-1,1+b)} \\ &= (-1,0) \oplus (-1,0) \cdot N(\gamma) \end{aligned}$$

by 2.4 Lemma 1, 2.2 Lemma 1 and 2.2 Lemma 3.

We have

$$H^0(U_1, M(\zeta + (p-1, 1+b)) \otimes \zeta \otimes N(\alpha))_{(p-1,1+b)} = (-1,-1) \oplus (-1,0) \cdot E$$

by 2.4 Lemma 2.

For $b \neq 1$ we have

$$\begin{aligned} H^0(U_1, &M(\zeta + (1+b, p-b)) \otimes \zeta \otimes N(\alpha))_{(p-1,b+1)} \\ &= H^0(U_1, M(\zeta + (1+b, p-b)) \otimes \zeta)_{(p-1,1+b)} \oplus H^0(U_1, M(\zeta + (b-1, p+1-b)) \otimes \zeta)_{(p-1,1+b)} \\ &= (0,-1) \cdot N(\alpha) \end{aligned}$$

by 2.4 Lemma and 2.2 Lemma 3, and we have

$$H^0(U_1, M(\zeta + (2, p-1)) \otimes \zeta \otimes N(\alpha))_{(p-1,2)} = 2(0,-1) \cdot N(\alpha)$$

by 2.4 Lemma 3.

We have

$$H^0(U_1, M(\zeta + (b, p-1-b)) \otimes \zeta N(\alpha))_{(p-1,1+b)} = (0,-1) \cdot N(\alpha)$$

by 2.3 Lemma 3.

We have

$$H^0(U_1, M(\zeta + (p-b, p-1) \otimes \zeta \otimes N(\alpha))_{(p-1,1+b)} = (-1,0) \oplus (-1,-1) \cdot V$$

by 2.4 Lemma 3.

Thus we have the following.

Lemma 2 *(i) We have*

$$\chi^i_\alpha(pr, p-2+ps) = \operatorname{ch} L(p-2,p-2)[2\chi^i(r-1,s)+\chi^i(r-2,s-1)]^F$$
$$+\operatorname{ch} L(0,p-3)[\chi^i(r-1,s-1)+\chi(1,0)\chi^i(r-1,s)]^F + 2\operatorname{ch} L(p-3,2)\chi^i_\alpha(r,s-1)^F$$
$$+\operatorname{ch} L(p-2,1)\chi^i(r,s-1)^F + \operatorname{ch} L(1,p-2)\chi^i(r-1,s-1)^F$$
$$+\operatorname{ch} L(0,0)[\chi^i(r-1,s)+\chi(0,1)\chi^i(r-1,s-1)]^F.$$

(ii) For $b \neq 1$ we have

$$\chi^i_\alpha(pr, p-1-b+ps) = \operatorname{ch} L(p-2,p-1-b)[2\chi^i(r-1,s)+\chi^i(r-2,s-1)]^F$$
$$+\operatorname{ch} L(0,p-2-b)[\chi^i(r-1,s-1)+\chi(1,0)\chi^i(r-1,s)]^F$$
$$+\operatorname{ch} L(p-2-b,b+1)\chi^i_\alpha(r,s-1)^F$$
$$+\operatorname{ch} L(p-1-b,b)\chi^i(r,s-1)^F + \operatorname{ch} L(b,p-2)\chi^i(r-1,s-1)^F$$
$$+\operatorname{ch} L(b-1,0)[\chi^i(r-1,s)+\chi(0,1)\chi^i(r-1,s-1)]^F.$$

We now consider the case $b=1$. Thus we have $\mu_1 = (a,1)$, $\mu_2 = (p-a,1+a)$, $\mu_3 = (1+a,p-1)$, $\mu_4 = (1,p-1-a)$, $\mu_5 = (p-1-a,a)$, $\mu_6 = (p-1,p-a)$.

Now we have

$$\begin{aligned} H^0(U_1, &M(\zeta+(a,1))\otimes\zeta\otimes N(\alpha))_{(p-1,p-a)} \\ &= H^0(U_1, M(\zeta+(a,1))\otimes\zeta)_{(p-1,p-a)} \oplus H^0(U_1, M(\zeta+(a-2,2))\otimes\zeta)_{(p-1,p-a)} \\ &= (-1,-1) \oplus H^0(U_1, M(\zeta+(a-2,2))\otimes\zeta)_{(p-1,p-a)} \end{aligned}$$

moreover, we have

$$H^0(U_1, M(\zeta+(a-2,2))\otimes\zeta)_{(p-1,p-a)} = \begin{cases} (-1,-1)\cdot N(\alpha), & \text{if } a=1; \\ 0, & \text{otherwise.} \end{cases}$$

Hence we have

$$H^0(U_1, M(\zeta+(a,1))\otimes\zeta\otimes N(\alpha))_{(p-1,p-a)} = \begin{cases} (-1,-1)\oplus(-1,-1)\cdot N(\alpha), & \text{if } a=1; \\ (-1,-1), & \text{otherwise.} \end{cases}$$

We now consider $H^0(U_1, M(\zeta+(p-a,1+a)\otimes\zeta\otimes N(\alpha))_{(p-1,p-a)}$. Taking $a=1$, we get

$$H^0(U_1, M(\zeta+(p-1,2)\otimes\zeta\otimes N(\alpha))_{(p-1,p-1)} = 2(0,-1)\cdot N(\alpha).$$

For $a>1$ we get

$$H^0(U_1, M(\zeta+(p-a,1+a)\otimes\zeta\otimes N(\alpha))_{(p-1,p-a)} = (0,-1)\cdot N(\alpha).$$

We have

$$H^0(U_1, M(\zeta+(1+a,p-1)\otimes\zeta\otimes N(\alpha))_{(p-1,p-a)} = (-1,0)\oplus(-1,-1)\cdot V.$$

We have

$$H^0(U_1, M(\zeta + (1, p-1-a) \otimes \zeta \otimes N(\alpha))_{(p-1,p-a)}$$
$$= H^0(U_1, M(\zeta + (1, p-1-a) \otimes \zeta)_{(p-1,p-a)} \oplus H^0(U_1, M(\zeta + (-1, p-a) \otimes \zeta)_{(p-1,p-a)}$$
$$= (-1, 0) \oplus N(\gamma).$$

We have

$$H^0(U_1, M(\zeta + (p-1-a, a) \otimes \zeta \otimes N(\alpha))_{(p-1,p-a)}$$
$$= H^0(U_1, M(\zeta + (p-1-a, a) \otimes \zeta)_{(p-1,p-a)} \oplus H^0(U_1, M(\zeta + (p-3-a, a+1) \otimes \zeta)_{(p-1,p-a)}$$
$$= (0, -1).$$

We have

$$H^0(U_1, M(\zeta + (p-1, p-a) \otimes \zeta \otimes N(\alpha))_{(p-1,p-a)} = (-1, -1) \oplus (-1, 0) \cdot E.$$

Thus we have the following,

Lemma 3 *(i) We have*

$$\chi^i_\alpha(pr, ps) = \mathrm{ch}\, L(p-a-1, p-1)[\chi^i(r-1, s-1) + \chi^i_\alpha(r-1, s-1)]^F$$
$$+2\mathrm{ch}\, L(0, p-3)\chi^i_\alpha(r, s-1)^F + \mathrm{ch}\, L(p-3, 0)[\chi^i(r-1, s) + \chi(0,1)\chi^i(r-1, s-1)]^F$$
$$+\mathrm{ch}\, L(p-2, 1)\chi^i(r-1, s) + \chi^i(r, s) + \chi^i(r-1, s-1)]^F$$
$$+\mathrm{ch}\, L(1, p-1-a)\chi^i(r, s-1)^F + \mathrm{ch}\, L(0,0)[\chi^i(r-1, s-1)$$
$$+\chi(1,0)\chi^i(r-1, 0)]^F.$$

(ii) For $a \neq 1$ we have

$$\chi^i_\alpha(pr, a-1+ps) = \mathrm{ch}\, L(p-a-1, p-1)\chi^i(r-1, s-1)^F$$
$$+\mathrm{ch}\, L(a-1, p-2-a)\chi^i_\alpha(r, s-1)^F + \mathrm{ch}\, L(p-2-a, 0)[\chi^i(r-1, s)$$
$$+\chi(0,1)\chi^i(r-1, s-1)]^F + \mathrm{ch}\, L(p-2, a)\chi^i(r-1, s) + \chi^i(r, s)$$
$$+\chi^i(r-1, s-1)]^F + \mathrm{ch}\, L(a, p-1-a)\chi^i(r, s-1)^F$$
$$+\mathrm{ch}\, L(0, a-1)[\chi^i(r-1, s-1) + \chi(1,0)\chi^i(r-1, 0)]^F.$$

Once more we have, as in Sections 4 and 5, the following.

Proposition *The characters $\chi^1(r,s), \chi^1_\alpha(r,s)$ and $\chi^1_\beta(r,s)$ are determined recursively, for all r, s by Lemmas 1,2,3 of Section 3.1, the expansion formulas 6.1 Lemma 3 and 6.2 Proposition and 6.3 Lemma 1 and 6.4 Lemmas 2,3 together with 2.5 Lemma.*

Remark As remarked in [**7**; 3.3] the expansion formula given there for the character of the cohomology of homogeneous vector bundles is valid also in the quantized case. So one could take G to be, for example, the quantized version of GL_3 considered in [**6**], over a field k, at a primitive lth root of unity, with $l > 1$. Taking B to be the Borel (quantum) subgroup considered in [**6**] one has, for each element λ of $X(2) = \mathbb{Z}^2$, a one dimensional B-module k_λ. The calculations of Section 2 then go through essentially unchanged but should be phrased in terms of the B_1-module socle instead of U_1 invariants. The bases cases, as in Section 3, are also essentially unchanged and the material in Section 2 may be reassembled, as in Sections 4,5,6 to give recursive formulas for $\chi^i(\lambda) = \mathrm{ch}\, R^i\mathrm{Ind}_B^G k_\lambda$ etc. There is however, one additional point that should be noted. For l a multiple of 3, there will be G_1-blocks (equivalently W orbits on $X(T)/lX(T)$) of size 2 so that one will not obtain uniform expansion formulas for l large. However, in the case in which k has characteristic 0, the Frobenius morphism on G has codomain $\bar{G}$, where $\bar{G}$ is the ordinary degree three general linear group over k, viewed as an algebraic group defined over k. In this case the expansion formula of Section 1 takes the form

$$\chi^i(Y) = \sum_{\lambda \in X_1(T)} \mathrm{ch}\, L(\zeta + w_0\lambda)^* \bar{\chi}^i(H^0(B_1, M(\zeta+\lambda) \otimes Y)^{(-1)})^F$$

where we are writing $\bar{\chi}^i(M)$ for the character of $R^i\mathrm{Ind}_{\bar{B}}^{\bar{G}}(M)$, for M a finite dimensional rational module for a Borel subgroup $\bar{B}$ of $\bar{G}$. Thus, in this case, the expansion formulas give, after one step, formulas for the characters of $R^i\mathrm{Ind}_B^G k_\lambda$, etc, in terms of the characters for the ordinary general linear group over a field of characterstic 0, and these are given by the Borel-Weil-Bott Theorem. So in this case one can use the results are methods of this paper to give completely explicit formulas for the cohomology of “quantum line bundles” on G/B. We leave the precise form of these formulas to the interested reader.

7 Remarks on the first cohomogy group of certain multiples of a simple root

Let $G = \mathrm{SL}_3(k)$. One can deduce from the recursive formulas for $\chi^1(\mu)$ described earlier that $\chi^1(-p^r\alpha) = \chi^1(-p^r\beta)) = \chi(0)$, and hence $H^1(G/B, \mathcal{L}(-p^r\alpha)) = H^1(G/B, \mathcal{L}(-p^r\beta)) = k$, for $r \geq 0$. We here show that this is a phenomenon which holds much more generally. In the proof we use a natural B-module decomposition of $H^0(B_1, \mathrm{St} \otimes \mathrm{St})^{(-1)}$. Since this decomposition also has significance in another context we treat it separately in 7.1 and discuss the general result just mentioned in 7.2.

7.1 We now adopt the set-up of 1.1, in particular G is a semisimple, simply connected algebraic group over k.

We recall the result of Koppinen (see [**12**;II,11.11 Remark]) that there exists a G-module lift of the regular G_1-module, i.e. a rational G-module Q such that $Q|_{G_1} \cong k[G_1]$. The module Q may be obtained in the following way. We have $k[G_1] \cong \mathrm{Ind}_1^{G_1} k \cong \mathrm{Ind}_{U_1}^{G_1}(k[U_1])$ by the transitivity of induction. However, we have $\mathrm{St}|_{U_1} \cong k[U_1]$ so, using the tensor identity, we get $k[G_1] \cong \mathrm{Ind}_{U_1}^{G_1}(\mathrm{St} \otimes k) \cong \mathrm{St} \otimes M$, where $M = \mathrm{Ind}_{U^1}^{G_1} k$. Now, let $0 = M_0 < M_1 < \ldots < M_n$ be a G_1-module composition series. We have $M_i/M_{i-1} \cong L(\lambda_i)$ for some $\lambda_i \in X_1(T)$, $1 \leq i \leq n$. Thus we have a G_1-filtration $0 = \mathrm{St} \otimes M_0 < \mathrm{St} \otimes M_1 < \cdots < \mathrm{St} \otimes M_n$. Moreover, the factor $\mathrm{St} \otimes M_i / \mathrm{St} \otimes M_{i-1} \cong \mathrm{St} \otimes (M_i/M_{i-1}) \cong \mathrm{St} \otimes L(\lambda_i)$ is projective as a G_1-module, for $1 \leq i \leq n$. Hence we have $\mathrm{St} \otimes M \cong \bigoplus_{i=1}^n \mathrm{St} \otimes L(\lambda_i)$. Thus we have $k[G_1] \cong \bigoplus_{\lambda \in X_1(T)} \mathrm{St} \otimes C(\lambda)$, as G_1-modules. Here $C(\lambda)$ is the direct sum of c_λ copies of $L(\lambda)$, where c_λ is the G_1-composition multiplicity of $L(\lambda)$ in $\mathrm{Ind}_{U_1}^{G_1} k$, for $\lambda \in X_1(T)$. Since St is projective and self dual as a G_1-module we have

$$c_\zeta = \dim \mathrm{Hom}_{G_1}(\mathrm{St}, \mathrm{Ind}_{U_1}^{G_1} k) = \dim \mathrm{Hom}_{U_1}(\mathrm{St}, k) = \dim H^0(U_1, \mathrm{St}) = 1.$$

Thus we have $k[G_1] \cong Q|_{G_1}$, where $Q = \mathrm{St} \otimes \mathrm{St} \bigoplus \mathrm{St} \otimes N$, for some G-module N.

Now for any B-module Z we have $\dim R^i\mathrm{Ind}_B^G Z = \dim H^0(G, k[G] \otimes R^i\mathrm{Ind}_B^G Z) = \dim H^0(G, R^i\mathrm{Ind}_B^G(k[G] \otimes Z)) = \dim H^i(B, Z \otimes k[G])$. By the transitivity of induction, we have $k[G] \cong \mathrm{Ind}_{G_1}^G k[G_1] \cong \mathrm{Ind}_{G_1}(k \otimes Q) \cong \mathrm{Ind}_{G_1}^G k \otimes Q \cong k[G]^F \otimes Q$.

We get $\dim R^i\mathrm{Ind}_B^G Z = \dim H^i(B, Z \otimes Q \otimes k[G]^F) = \dim H^i(B, H^0(Z \otimes Q)^{(-1)} \otimes k[G]) = \dim R^i\mathrm{Ind}_B^G(H^0(B_1, Z \otimes Q)^{(-1)})$. In paricular, taking $Q = \mathrm{St} \otimes \mathrm{St} \bigoplus \mathrm{St} \otimes N$, as above, we get:

(1) $\dim R^i\mathrm{Ind}_B^G Z \geq \dim R^i\mathrm{Ind}_B^G H^0(B_1, \mathrm{St} \otimes \mathrm{St} \otimes Z)^{(-1)}$.

Now the character of $H^0(U_1, \mathrm{St} \otimes \mathrm{St})$ is $\chi(\zeta)e(-\zeta)$, by 1.2(2). Thus 0 is the unique maximal weight of $H^0(U_1, \mathrm{St} \otimes \mathrm{St})$ and occurs with multiplicity one. Now $H^0(B_1, \mathrm{St} \otimes \mathrm{St}) \geq H^0(G, \mathrm{St} \otimes \mathrm{St}) \cong \mathrm{End}_G(\mathrm{St}) = k$. Hence $H^0(B_1, \mathrm{St} \otimes \mathrm{St}) = H^0(G_1, \mathrm{St} \otimes \mathrm{St}) = L$, say, is one dimensional. Thus 0 is the unique maximal weight of the B-module $H^0(B_1, \mathrm{St} \otimes \mathrm{St})$, and occurs with multiplicity one. Thus the span V of the weights spaces of weight $\nu < 0$ is a submodule of codimension 1 and we have shown the following.

(2) *There is a B-module decomposition $H^0(B_1, \mathrm{St} \otimes \mathrm{St})^{(-1)} = L \bigoplus V$, where $L = H^0(G, \mathrm{St} \otimes \mathrm{St})$ is a one dimensional and all weights of V are negative.*

For any B-module Z we obtain that $H^0(B_1, Z)^{(-1)}$ is a B-module component of $H^0(B_1, \mathrm{St} \otimes \mathrm{St} \otimes Z)^{(-1)}$ and hence, by (1), we have
$\dim R^i\mathrm{Ind}_B^G Z \geq \dim R^i\mathrm{Ind}_B^G(H^0(B_1, Z)^{(-1)})$. Replacing Z by Z^F we get
$\dim R^i\mathrm{Ind}_B^G Z \leq \dim R^i\mathrm{Ind}_B^G Z^F$ and hence

$$\dim R^i\mathrm{Ind}_B^G Z \leq \dim R^i\mathrm{Ind}_B^G Z^{F^r}$$

for any $r \geq 0$. Now suppose $\lambda \in X^+(T)$ is such that $(\lambda, \check{\alpha}) \geq 1$ for each simple root α. Then the line bundle $\mathcal{L}(\lambda)$ is ample. In particular for a fixed $i > 0$ we have $H^i(G/B, \mathcal{L}(n\lambda)) = 0$, for all n sufficiently large. Thus $R^i\mathrm{Ind}_B^G k(p^r\lambda) = 0$ for all r large and hence $R^i\mathrm{Ind}_B^G k(\lambda) = 0$, for $i > 0$. This proves Kempf's vanishing theorem for ample weights. We obtain Kempf's vanishing theorem for general λ by using Serre duality. Let $\lambda \in X^+(T)$. Then we have $\dim R^i\mathrm{Ind}_B^G k(\lambda) = \dim R^{N-i}\mathrm{Ind}_B^G k(-\lambda - 2p\rho)$ where $N = \dim G/B$, by duality, and
$\dim R^{N-i}\mathrm{Ind}_B^G k(-\lambda - 2p\rho) \leq \dim R^{N-i}\mathrm{Ind}_B^G k(-p\lambda - 2p\rho)$, by (*). Moreover, this is $\dim R^i\mathrm{Ind}_B^G k(p\lambda + (2p-2)\rho)$ by duality once more, and this is 0, for $i > 0$, since $p\lambda + (2p-2)\rho$ is ample. Thus we have shown:

(3) *(Kempf's Vanishing Theorem)* *For $\lambda \in X^+(T)$ and $i > 0$ we have $R^i\mathrm{Ind}_B^G k(\lambda) = 0$.*

Remark This proof has elements in common with the usual proof via Frobenius splitting of G/B. Indeed one may see the decomposition $H^0(B_1, \mathrm{St} \otimes \mathrm{St}) = L \bigoplus V$ as a representation-theoretic realization of Frobenius splitting.

7.2 (1) *Assume that G has a connected root system, that $\lambda \in X_1^+(T)$ and that μ is a weight of $H^0(B_1, \mathrm{St} \otimes \nabla(\lambda))^{(-1)}$ such that $R\mathrm{Ind}_B^G k(\mu - p^r\alpha) \neq 0$ for some $r \geq 0$. Then $\lambda = \zeta$ and μ is an integer multiple of α.*

We have that $p\mu$ is a weight of $H^0(U_1, \mathrm{St} \otimes \nabla(\lambda))$, which has character $\chi(\lambda)e(-\zeta)$, by 1.2,(2). Hence $p\mu + \zeta$ is a weight of $\nabla(\lambda)$ and hence $p\mu + \zeta = \lambda - \theta$, where θ is a sum of positive roots. By Section 3 Remarks, we have $-p^r\alpha + \mu + m\beta \in X^+$, for some simple root β and $m > 0$. Let $\tau = -p^r\alpha + \mu + m\beta$. We obtain

$$\lambda - \theta = \zeta + p\tau - pm\beta + p^{r+1}\alpha$$

giving

$$pm\beta = (\zeta - \lambda) + \theta + p\tau + p^{r+1}\alpha \qquad (*).$$

However, any dominant weight ν has the form $\nu = \sum_{i=1}^{l} c_i\alpha_i$, where $\{\alpha_, \ldots, \alpha_l\}$ is the set of simple roots and where c_i is a non-negative rational, for $1 \leq i \leq l$ (see e.g. [**10**;13.2]. Expressing $\zeta - \lambda$ as a linear combination of simple roots and equating coefficients of simple roots in (*) we obtain $\beta = \alpha$. Hence the dominant weights $\zeta - \lambda$ and τ are multiples of the simple root α, which gives $\zeta - \lambda = \tau = 0$. Thus $\lambda = \zeta$ and μ is an integer multiple of α, as required.

(2) *If α is a non-isolated simple root, i.e. a simple root such that $(\alpha, \gamma) \neq 0$ for some simple root $\gamma \neq \alpha$, then we have $R\mathrm{Ind}_B^G k(-p^r\alpha) = k$ for all $r \geq 0$.*

It suffices to prove that $\chi^1(-p^r\alpha) = \chi(0)$ for all $r \geq 0$. The result holds for $r = 0$, e.g. by [**4**;(2.3.1)]. Assume now that $r > 0$. Let $\phi_\lambda = \mathrm{ch}\,\hat{Q}_1(\lambda)/\chi(\zeta)$ and $\psi_\lambda = \mathrm{ch}\,L(\lambda)$, for $\lambda \in X_1(T)$. Then $(\phi_\lambda)_{\lambda \in X_1(T)}$ and $(\psi_\lambda)_{\lambda \in X_1(T)}$ are p-dual bases of A. Hence, by the main result of [**7**] we have

$$\chi^1(-p^r\alpha) = \sum_{\lambda \in X_1(T)} \phi_\lambda^* \chi^i(H^0(B_1, \mathrm{St} \otimes L(\lambda) \otimes (-p^r\alpha))^{(-1)})^F.$$

However, by (2), and left exactness of the B_1 fixed point functor, this reduces to

$$\chi^1(-p^r\alpha) = \phi_\zeta^* \chi^1(H^0(B_1, \mathrm{St} \otimes \mathrm{St})^{(-1)} \otimes (-p^{r-1}\alpha)))^F.$$

However, $\phi_\zeta = \chi(\zeta)/\chi(\zeta) = 1$ and so

$$\chi^1(-p^r\alpha) = \chi^1(H^0(B_1, \mathrm{St} \otimes \mathrm{St})^{(-1)} \otimes (-p^{r-1}\alpha))^F.$$

Now $H^0(B_1, \mathrm{St} \otimes \mathrm{St})^{(-1)} = k \oplus V$, where V has only non-zero negative weights. Let μ be a weight of V. Then $p\mu$ is a weight $H^0(B_1, \mathrm{St} \otimes \mathrm{St})$ and $\zeta + p\mu$ is a weight of St. Hence we get $\mu = -\theta$ for some $\theta > 0$. If $R\mathrm{Ind}_B^G k(\mu) \neq 0$ then we get $-\theta + p^{r-1}\alpha + m\beta = \tau \in X^+(T)$, for some simple root β. But now

$$\tau + \theta + p^{r-1}\alpha = m\beta$$

implies $\alpha = \beta$ and $\tau = 0$. Thus we get $\theta + p^{r-1}\alpha = m\alpha$ and θ is a multiple of α. But $\zeta + pn\alpha$ is only a weight of St if $n = 0$. Hence $\mu = 0$ This is a contradiction since all weights of V are non-zero.

Hence we get $\chi^1(-p^r\alpha) = \chi^1(-p^{r-1}\alpha)$ and the result follows by induction.

Remark In [**12**;II,5.18] it is shown that $R\mathrm{Ind}_B^G k((-p^r\alpha))$ has simple socle k, and conversely that if μ is a weight such that $R\mathrm{Ind}_B^G k((-p^r\alpha))$ has simple socle k then $\mu = -p^r\alpha$, for some simple root α.

7.3 We conclude with the short direct proof of 7.2(2) kindly supplied by Henning Andersen. We first consider modules over $\mathrm{SL}_2(k)$. Recall (e.g. from [**2**]) that we have a short exact sequence

$$0 \to \nabla(a) \otimes \nabla(b)^F \to \nabla(a+pb) \to \nabla(p-2-a) \otimes \nabla(b-1)^F \to 0$$

of $\mathrm{SL}_2(k)$-modules, for $0 \leq a \leq p-2$ and $b \geq 1$. In particular, we have a short exact sequence

$$0 \to \nabla(p-2) \otimes \nabla(2p^{r-1}-1)^F \to \nabla(2p^r-2) \to \nabla(2p^{r-1}-2)^F \to 0$$

for $r \geq 1$. Moreover, $\nabla(2p^{r-1}-1)$ and hence $\nabla(p-2) \otimes \nabla(2p^{r-1}-1)^F$ is simple and hence self-dual so that inclusion $\nabla(p-2) \otimes \nabla(2p^{r-1}-1)^F \to \nabla(2p^r-2)$ gives rise to an epimorphism $\nabla(2p^r-2)^* \to \nabla(p-2) \otimes \nabla(2p^{r-1}-1)^F$. Hence we have an exact sequence

$$\nabla(2p^r-2)^* \to \nabla(2p^r-2) \to \nabla(2p^{r-1}-2)^F \to 0.$$

We now suppose that α is a non-isolated simple root of G and denote by P_α the corresponding minimal parabolic subgroup. For $\lambda \in X(T)$ and $i \geq 0$, we write $H^i(\lambda)$ for $R\mathrm{Ind}_B^G(\lambda)$ and $H^i_\alpha(\lambda)$ for $R^i\mathrm{Ind}_B^{P_\alpha} k(\lambda)$.

Then, from the above, we get a short exact sequence

$$H^0_\alpha((p^r-1)\alpha)^* \to H^0_\alpha((p^r-1)\alpha) \to H^0_\alpha((p^{r-1}-1)\alpha)^F \to 0$$

and, by Serre duality on P_α/B, an exact sequence

$$0 \to H^1_\alpha(-p^{r-1}\alpha)^F \to H^1_\alpha(-p^r\alpha) \to H^0_\alpha((p^r-1)\alpha).$$

Moreover, for $m \geq 0$, the module $H^0_\alpha(p^r-1)\alpha)^{F^m}$ embeds in $H^0_\alpha(p^m(p^r-1)\alpha)$ so we get an exact sequence of P_α-modules

$$0 \to H^1_\alpha(-p^{r-1}\alpha)^{F^{m+1}} \to H^1_\alpha(-p^r\alpha)^{F^m} \to H^0_\alpha(p^m(p^r-1)\alpha).$$

But now, by the transitivity of induction we have

$H^0(H^0_\alpha(p^m(p^r-1)\alpha)) = H^0((p^m(p^r-1)\alpha) = 0$ since $p^m(p^r-1)\alpha$ is not dominant. So exactness of induction gives an isomorphism $\mathrm{Ind}_{P_\alpha}^G(H^1_\alpha(-p^{r-1}\alpha)^{F^{m+1}}) \to \mathrm{Ind}_{P_\alpha}^G(H^1_\alpha(-p^r\alpha)^{F^m})$. Thus we get

$$\mathrm{Ind}_{P_\alpha}^G(H^1_\alpha(-p^r\alpha)) = \mathrm{Ind}_{P_\alpha}^G(H^1_\alpha(-p^{r-1}\alpha)^F) = \cdots = \mathrm{Ind}_{P_\alpha}^G(H^1_\alpha(-\alpha)^{F^r}) = \mathrm{Ind}_{P_\alpha}^G k = k.$$

But now, for a B-module V, we have a spectral sequence with E_2-page $R^i\mathrm{Ind}_{P_\alpha}^G R^j\mathrm{Ind}_B^{P_\alpha} V$ converging to $R^*\mathrm{Ind}_B^G V$. Moreover, we have $R^j\mathrm{Ind}_B^{P_\alpha} = 0$ for $j > 1$ and $\mathrm{Ind}_B^{P_\alpha} k(-p^r\alpha) = 0$ so that $R\mathrm{Ind}_B^G k(-p^r\alpha) = \mathrm{Ind}_{P_\alpha}^G R\mathrm{Ind}_B^{P_\alpha} k(-p^r\alpha) = k$.

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