ON THE DEFORMATION CHIRALITY
OF REAL CUBIC FOURFOLDS

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ABSTRACT. According to our previous results, the conjugacy class of the involution induced by the complex conjugation in the homology of a real non-singular cubic fourfold determines the fourfold up to projective equivalence and deformation. Here, we show how to eliminate the projective equivalence and to obtain a pure deformation classification, that is how to respond to the chirality question: which cubics are not deformation equivalent to their image under a mirror reflection. We provide an arithmetical criterion of chirality, in terms of the eigen-sublattices of the complex conjugation involution in homology, and show how this criterion can be effectively applied taking as examples M-cubics (that is those for which the real locus has the richest topology) and $(M-1)$-cubics (the next case with respect to complexity of the real locus). It happens that there is one chiral class of M-cubics and three chiral classes of $(M-1)$-cubics, contrary to two achiral classes of M-cubics and three achiral classes of $(M-1)$-cubics.

L'univers est un ensemble dissymétrique, et je suis persuadé que la vie, telle qu'elle manifeste à nous, est fonction de la dissymétrie de l'univers ou des conséquences qu'elle entraîne.

Louis Pasteur

Observations sur les forces dissymétriques, CRAS, 78 (1874), 1515–1518

§1. Introduction

Recall that the projective nonsingular cubic fourfolds form the complement in a projective space $P_{4,3} = P(Sym^3(\mathbb{C}^6))$ of dimension $\binom{5+3}{3} - 1 = 55$ to the so-called *discriminant hypersurface*. The discriminant hypersurface, which we denote by $\Delta_{4,3}$, is defined over reals and its real part $\Delta_{4,3}(\mathbb{R})$ is represented by real singular cubics, so that the space under our study is nothing but $P_{4,3}(\mathbb{R}) \setminus \Delta_{4,3}(\mathbb{R})$. (Such a notation specifies the dimension, $n = 4$, and the degree, $d = 3$, of the hypersurfaces under consideration; we make use of it in Section 8 for arbitrary n and d).

The space $\mathcal{C} = P_{4,3}(\mathbb{C}) \setminus \Delta_{4,3}(\mathbb{C})$ is connected, while $\mathcal{C}_\mathbb{R} = P_{4,3}(\mathbb{R}) \setminus \Delta_{4,3}(\mathbb{R})$ is not. Understanding the nature of the connected components of the latter space is a natural, and classical task, it can be rephrased as a *deformation classification* of real

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projective nonsingular cubic fourfolds. In our previous paper [FK] we performed a classification with respect to a weaker *coarse deformation equivalence*: we call two real projective nonsingular hypersurfaces *coarse deformation equivalent* if one hypersurface is deformation equivalent to a projective transformation of the other.

The difference between these two equivalence relations shows up in the case of subvarieties of real projective spaces of odd dimension. It is due to the orientability of real projective spaces of odd dimension, which implies that the group $PGL(n+2, \mathbb{R})$ of real projective transformations of P^{n+1} has two connected components if n is even. In our case, $n = 4$, so some of the coarse deformation classes of real projective nonsingular cubic fourfolds may apriori consist of two deformation classes.

This leads us to a study of the following chirality phenomenon. We say that a real nonsingular cubic $X \subset P^5$ and its coarse deformation class are *chiral* if X and its mirror image X' (that is the image of X under a reflection in a hyperplane) belong to different connected components of $\mathcal{C}_\mathbb{R}$, and *achiral* if they belong to the same component (that is if X and X' can be connected by a continuous family of real non-singular cubics). Clearly, a coarse deformation class consists of two deformation classes if and only if it is chiral.

In the present paper we reduce the chirality problem to a specific problem of the arithmetics of lattices and use this reduction to show that certain real cubic fourfolds are chiral, while certain other real cubic fourfolds are achiral. We pay a special attention to real cubic fourfolds with extremal values of the sum of the Betti numbers. Namely, we consider in details the cases of *M-cubics*, in which $\dim H_*(X(\mathbb{R}); \mathbb{Z}/2) = \dim H_*(X(\mathbb{C}); \mathbb{Z}/2)$ (the maximal value), and the cases of $(M-1)$-*cubics*, in which $\dim H_*(X(\mathbb{R}); \mathbb{Z}/2) = \dim H_*(X(\mathbb{C}); \mathbb{Z}/2) - 2$ (the next value). As is shown in [FK], the M-cubics form three and the $(M-1)$-cubics form six coarse projective classes. In the present paper we prove that one coarse class of M-cubics and three coarse classes of $(M-1)$-cubics are achiral, while the other coarse classes of M- and $(M-1)$-cubics are chiral.

As a by-product, we give a new proof (in a sense, more natural and more direct) of the *homological quasi-simplicity* of cubic fourfolds, where the latter means that two real nonsingular cubic hypersurfaces X_1, X_2 in P^5 are coarse deformation equivalent iff the involutions induced by the complex conjugation on $H_4(X_i(\mathbb{C})), i = 1, 2$, regarded as a lattice via the intersection index form, are isomorphic (cf. Theorem 1.1 in [FK] and Proposition 4.1.2 below).

In our previous paper [FK], we were using a relation between the nodal cubics in P^5 and the complete intersections of bi-degree $(2, 3)$ in P^4. Since these complete intersections are the 6-polarized K3-surfaces, it had allowed us to apply Nikulin's coarse deformation classification of real 6-polarized K3-surfaces in terms of involutions on the K3-lattice and his results on the arithmetics of such involutions, see [N1],[N2].

Such a roundabout approach was imposed by a lack of sufficiently complete understanding of the moduli of cubic hypersurfaces, contrary to that of K3-surfaces. In particular, in the case of K3-surfaces one had in one's hands the surjectivity of the period map, while for cubic fourfolds the characterization of the image of the period map remained unknown. The situation has changed recently, after R. Laza [La] and E. Looijenga [Lo] established a suitable surjectivity statement for cubic fourfolds.

In our opinion, the two approaches are complementary and both deserve to be

developed further. Combined together they should give us a better understanding of the topology of the moduli space of real cubic fourfolds on one hand, and of the topology of the discriminant of cubic fourfolds on the other hand. Note that already in [FK] not only the coarse deformation classes but also their adjacencies were found. Now, via the period map, the deformation classes become endowed with a certain polyhedral structure. This opens a way for a full understanding of some natural stratifications of the moduli and the coefficient spaces of real cubic fourfolds.

Topological study of nonsingular real cubic hypersurfaces has a long history, see [FK] for a brief account. In addition, we would add a reference to the recent investigation of the moduli space of real cubic surfaces performed by D. Allcock, J. Carlson, and D. Toledo [ACT].

Let us recall that according to Klein's classification of real cubic surfaces, see [K] (the classification statement is reproduced in [FK]), all the real nonsingular cubic surfaces are achiral. It may be worth mentioning that Klein's achirality argument in [K] contained a mistake, which was corrected by Klein in his Collected Papers, see [K2].

The paper is organized as follows. In Section 2, we review some properties of the period map for complex cubic fourfolds. In Section 3, we introduce the real period spaces with the real period map and derive the properties of the latter from the corresponding properties of the complex period map. The results of Section 3 are applied then in Section 4 to reduce the chirality problem to some arithmetics of hyperbolic integer lattices and their reflection groups. Section 5 collects necessary information about Vinberg's algorithm for finding the fundamental domains of the arithmetical reflection groups. The technique developed in Sections $3-5$ is applied in Sections 6 and 7 to treat the chirality of M- and $(M-1)$-cubic fourfolds. Section 8 is devoted to concluding remarks. We mention some other cases which were studied using similar methods, and mention some other related results and possible directions of their development. In particular we discuss a notion of reversibility, which is closely related to chirality.

§2. Period map for complex cubic fourfolds

2.1. The period domain for marked cubic fourfolds. Consider a non-singular cubic fourfold $X \subset P^5$. It is well known that its non-zero Hodge numbers in dimension four are $h^{3,1} = h^{1,3} = 1$ and $h^{2,2} = 21$. The lattice $\mathbb{M}(X) = H^4(X)$ is odd with signature $(21, 2)$. The *polarization class* $h(X) \in \mathbb{M}(X)$, that is the square of the hyperplane section, is a characteristic element of $\mathbb{M}(X)$ with $h^2 = 3$, and so the primitive sublattice $\mathbb{M}_0(X) = \{x \in \mathbb{M}(X) \,|\, xh = 0\}$ is even and has discriminant group $\mathbb{Z}_3$. This implies that there is a lattice isomorphism between $\mathbb{M}(X)$ and $\mathbb{M} = 3I + 2U + 2E_8$, which sends $h(X)$ to $h = (1, 1, 1) \in 3I$, so that

$\mathbb{M}_0(X)$ is identified with $\mathbb{M}_0 = A_2 + 2U + 2E_8$. A particular choice of such an isomorphism $\phi : (\mathbb{M}(X), h(X)) \to (\mathbb{M}, h)$ will be called a *marking of X*. We restrict the choice of markings as is specified below.

The complex line $\phi(H^{3,1}(X)) \subset \mathbb{M}_0 \otimes \mathbb{C}$ is isotropic and has negative pairing with the conjugate (and thus, also isotropic) line $\phi(H^{1,3}(X)) = \overline{\phi(H^{3,1}(X))}$, that is to say, $w^2 = 0$, and $w\overline{w} < 0$, (and thus $\overline{w}^2 = 0$) for all $w \in \phi(H^{3,1}(X))$. Writing $w = u + iv$, $u, v \in \mathbb{M}_0 \otimes \mathbb{R}$, we can reformulate it as $u^2 = v^2 < 0$ and $uv = 0$, which implies that the real plane $\langle u, v \rangle \subset \mathbb{M}_0 \otimes \mathbb{R}$ spanned by u and v is negative definite and bears a natural orientation given by $u = \operatorname{Re} w, v = \operatorname{Im} w$. Note that the orientation determined similarly by the complex line $\phi(H^{1,3}(X)) \subset \mathbb{M}_0 \otimes \mathbb{C}$ is the opposite one.

The line $\phi(H^{3,1}(X)) \subset \mathbb{M}_0 \otimes \mathbb{C}$ specifies a point $\Omega(X) \in P(\mathbb{M}_0 \otimes \mathbb{C})$ (as usual, P states for the projectivization) called the *period point of X*. This period point belongs to the quadric $Q = \{w^2 = 0\} \subset P(\mathbb{M}_0 \otimes \mathbb{C})$, and more precisely, to its open subset, $\widehat{\mathcal{D}} = \{w \in Q \,|\, w\overline{w} < 0\}$. This subset has two connected components, which are exchanged by the complex conjugation (this reflects also switching from the given complex structure on X to the complex conjugate one).

The orthogonal projection of a negative definite real plane in $\mathbb{M}_0 \otimes \mathbb{R}$ to another one is non-degenerate. Thus, to select one of the two connected components of $\widehat{\mathcal{D}}$ we fix an orientation of negative definite real planes in $\mathbb{M}_0 \otimes \mathbb{R}$ so that the orthogonal projection preserves it. We call it the *prescribed orientation* and restrict the choice of markings to those for which the orientation of $\phi(H^{1,3}(X))$ defined by the pairs $u = \operatorname{Re} w, v = \operatorname{Im} w$ for $w \in \phi(H^{1,3}(X))$ is the prescribed one. We denote this component by $\mathcal{D}$ and call it the *period domain*. By $\operatorname{Aut}^+(\mathbb{M}_0)$ we denote the group of those automorphisms of $\mathbb{M}_0$ which preserve the prescribed orientation (and thus preserve $\mathcal{D}$). We put $\operatorname{Aut}^-(\mathbb{M}_0) = \operatorname{Aut}(\mathbb{M}_0) \backslash \operatorname{Aut}^+(\mathbb{M}_0)$. This complementary coset consists of automorphisms exchanging the connected components of $\widehat{\mathcal{D}}$.

The projective space $P_{4,3}$ formed by all cubic fourfolds splits into the *discriminant hypersurface* $\Delta_{4,3}$ formed by singular cubics and its complement, $\mathcal{C}$. Let $\mathcal{C}^\sharp$ denote the space of marked non-singular cubics. The natural projection $\mathcal{C}^\sharp \to \mathcal{C}$ is obviously a covering with the deck transformation group $\operatorname{Aut}^+(\mathbb{M}_0)$. The above conventions define the *period map* $\operatorname{per} : \mathcal{C}^\sharp \to \mathcal{D}$, $(X, \phi) \mapsto \phi(H^{1,3}(X))$.

2.2. Principal properties of the period map. The global Torelli theorem for cubic fourfolds proved in [V] claims injectivity of the period map. We need the following version of this theorem.

2.2.1. Global Torelli Theorem [V]. *Assume that (X, ϕ) and (X', ϕ') are non-singular marked cubic fourfolds such that $\operatorname{per}(X, \phi) = \operatorname{per}(X', \phi')$. Then there exists one and only one isomorphism $f : X' \to X$ such that $\phi' \circ f^* = \phi$.* $\square$

The existence statement is explicit in [V]. The uniqueness statement is implicit there. It follows easily from two well known observations: first, each automorphism of a nonsingular cubic fourfold is induced by a projective transformation, and, second, if a projective transformation acts trivially in the cohomology then it is trivial.

2.2.2. Construction of (anti-)isomorphisms. *Let X and X' be non-singular cubic fourfolds and $F : H^4(X; \mathbb{Z}) \to H^4(X'; \mathbb{Z})$ an isometry such that $F(h) = h'$.*

  (1) *If $F(H^{3,1}(X)) = H^{3,1}(X')$, then there exists one and only one isomorphism $f : X' \to X$ such that $f^* = F$.*

(2) *If $F(H^{1,3}(X)) = H^{3,1}(X')$, then there exists one and only one isomorphism $f\colon X' \to \overline{X}$ such that $f^* = F$.*

Here and in what follows we denote by $\overline{X}$ the variety complex conjugate to X. If $X \subset P^5$ is given by a polynomial, then $\overline{X} \subset P^5$ can be seen as the variety given by the polynomial with the complex conjugate coefficients.

Proof of 2.2.2. The first statement is nothing but an equivalent version of Theorem 2.2.1. The second statement follows from the first one or directly from Theorem 2.2.1 applied to (X', ϕ') and $(\overline{X}, \phi' \circ F)$, where ϕ' is any marking of X'. □

Consider the reflection R_v in $\mathbb{M}_0 \otimes \mathbb{C}$ across the mirror-hyperplane $H_v = \{x \in \mathbb{M}_0 \otimes \mathbb{C} \,|\, xv = 0\}$ defined as $x \mapsto x - 2\frac{xv}{v^2}v$, and note that it preserves the lattice $\mathbb{M}_0$ invariant if $v \in \mathbb{M}_0$ is such that $v^2 = 2$, or such that $v^2 = 6$ and xv is divisible by 3 for all $x \in \mathbb{M}_0$. We call these two types of lattice elements 2-*roots* and 6-*roots* respectively, and denote their sets by V_2 and V_6. Note that $R_v \in \mathrm{Aut}^+(\mathbb{M}_0)$ for any $v \in V_2 \cup V_6$. If $v \in V_2$, then the reflection R_v extends (as a reflection) to $\mathbb{M}$ and h is preserved by this extension. By contrary if $v \in V_6$, the reflection R_v does not extend to a reflection in $\mathbb{M}$, and moreover, the unique extension of R_v to $\mathbb{M}$ maps h to $-h$ (cf. Lemma 4.3.2 below). On the other hand, if $v \in V_6$ then the *anti-reflection* $-R_v$ extends to an isometry of $\mathbb{M}$ preserving h. This extension is the anti-reflection with respect to the 2-plane generated by h and v. In particular, it represent also an element of $\mathrm{Aut}^+(\mathbb{M}_0)$.

The union of the mirrors H_v for all $v \in V_2$ gives after projectivization a union of hyperplanes $\mathcal{H}_\Delta \subset P(\mathbb{M}_0 \otimes \mathbb{C})$, and a similar union for all $v \in V_6$ gives another union of hyperplanes, $\mathcal{H}_\infty \subset P(\mathbb{M}_0 \otimes \mathbb{C})$.

2.2.3. Surjectivity of the period map [Lo],[La]**.** *The image of the period map* $\mathrm{per}\colon \mathcal{C}^\sharp \to \mathcal{D}$ *is the complement of* $\mathcal{H}_\Delta \cup \mathcal{H}_\infty$. □

According to the Griffiths theory, for any nonsingular cubic $X \subset P^5$ the line $H^{3,1}(X)$ is spanned by the class $[\omega_p] \in H^4(X; \mathbb{C})$ of the 4-form $\omega_p = \mathrm{Res}(\mathcal{E}/p^2)$. Here $\mathcal{E}$ stands for the Euler 5-form in $\mathbb{C}^6$, $\mathcal{E} = \sum_{i=0}^5 (-1)^i x_i dx_0 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_5$, and p for a polynomial defining X (as usual, a hat over x_i means that this term is omitted). The ratio $\mathcal{E}/p^2$ is a well-defined meromorphic 5-form in P^5, with a second order pole along X. The residue ω_p of this form is a 4-form on X, which is a linear combination of $(3,1)$ and $(4,0)$-forms. Its class $[\omega_p]$ is known to be non-trivial, thus, it spans $H^{3,1}(X)$.

§3. Periods in the real setting

3.1. Geometric involutions. Consider a non-singular cubic fourfold X defined by a real polynomial p, and let $\mathrm{conj}\colon X \to X$ denote the complex conjugation map, which will be called also the *real structure on X*. The latter map induces a lattice involution $\mathrm{conj}^*\colon \mathbb{M}(X) \to \mathbb{M}(X)$ such that $\mathrm{conj}^*(h) = h$ and, hence, induces also a lattice involution in $\mathbb{M}_0(X)$. Denote by $\mathbb{M}_0^\pm(X)$ and $\mathbb{M}^\pm(X)$ the eigen-sublattices $\{x \in \mathbb{M}_0(X) \,|\, \mathrm{conj}^*(x) = \pm x\}$ and $\{x \in \mathbb{M}(X) \,|\, \mathrm{conj}^*(x) = \pm x\}$, respectively. We have obviously $\mathbb{M}^- = \mathbb{M}_0^-$ and $\sigma_-(\mathbb{M}^+(X)) = \sigma_-(\mathbb{M}_0^+(X))$, where σ_- denotes the negative index of inertia.

3.1.1. Lemma. *One has $\sigma_-(\mathbb{M}_0^\pm(X)) = 1$.*

Proof. The map $w \mapsto \overline{\mathrm{conj}^* w}$ gives an anti-linear involution in $H^{3,1}(X)$. Thus, there exist non zero elements $w \in H^{3,1}(X)$ such that $\mathrm{conj}^*(w) = \overline{w}$. In terms of the real and imaginary components of $w = u_+ + iu_-$, this identity means that $u_\pm \in \mathbb{M}^\pm(X) \otimes \mathbb{R}$. These components satisfy the relations $u_+^2 = u_-^2 = \frac{1}{2} w\overline{w} < 0$. They belong to $\mathbb{M}_0^\pm(X)$, since $wh = 0$. It remains to notice that the intersection form is positive definite on $H^{2,2}(X)$. $\square$

We call a lattice involution $c : \mathbb{M} \to \mathbb{M}$ *geometric* if $c(h) = h$ and $\sigma_-(\mathbb{M}_0^\pm(c)) = 1$, where $\mathbb{M}_0^\pm(c)$ denotes the eigen-sublattices $\{x \in \mathbb{M}_0 \,|\, c(x) = \pm x\}$. Let us note that all geometric involutions preserve $\mathbb{M}_0$ and the involutions induced in $\mathbb{M}_0$ belong to $\mathrm{Aut}^-(\mathbb{M}_0)$.

According to Lemma 3.1.1, all lattice involutions $c : \mathbb{M} \to \mathbb{M}$ isomorphic to an involution $\mathrm{conj}^* : \mathbb{M}(X) \to \mathbb{M}(X)$ for a non-singular real cubic X are geometric. A pair $(c : \mathbb{M} \to \mathbb{M}, h \in \mathbb{M})$ isomorphic to $(\mathrm{conj}^* : \mathbb{M}(X) \to \mathbb{M}(X), h(X))$ is called the *homological type* of X. By a *real c-marked cubic fourfold* we understand a real non-singular cubic fourfold equipped with a marking ϕ such that $\phi \circ \mathrm{conj}^* = c \circ \phi$.

3.1.2. Theorem. *For any geometric involution $c : \mathbb{M} \to \mathbb{M}$ the pair (c, h) is the homological type of some non-singular real cubic fourfold.*

This theorem is one of the results obtained in [FK]. After fixing some notation, we will give below (at the end of subsection 3.3) an independent proof based on the surjectivity of the period map and the global Torelli theorem.

The number of isometry classes of geometric involutions is finite. Their list can be found in [FK] (see also tables 8,9 in §8).

Up to the end of this section we suppose that c is a geometric involution.

3.2. Real parameter space $\mathcal{C}_{\mathbb{R}}^c$. We denote by $\mathcal{C}_{\mathbb{R}}^c \subset \mathcal{C}_{\mathbb{R}}$ the set of real cubic fourfolds of homological type c, and by $\mathcal{C}_{\mathbb{R}}^{c\sharp}$ the set of c-marked real cubic fourfolds. The former consists of some number of connected components of $\mathcal{C}_{\mathbb{R}}$. The latter can be seen as the real part of $\mathcal{C}^\sharp$ with respect to the involution $\mathrm{conj}^{c\sharp} : \mathcal{C}^\sharp \to \mathcal{C}^\sharp$, which send $(X, \phi) \in \mathcal{C}^\sharp$ to $(\mathrm{conj}(X), c \circ \phi \circ \mathrm{conj}^*)$. The forgetful map $(X, \phi) \to X$ defines a (multi-component) covering $\mathcal{C}_{\mathbb{R}}^{c\sharp} \to \mathcal{C}_{\mathbb{R}}^c$ with the deck transformation group $\mathrm{Aut}^+(\mathbb{M}_0)$.

3.3. Real period domain $\mathcal{D}_{\mathbb{R}}^c$. Let us extend c to a complex linear involution on $\mathbb{M} \otimes \mathbb{C}$ and denote also by c the induced involutions on $\mathbb{M}_0 \otimes \mathbb{C}$, $P = P(\mathbb{M}_0 \otimes \mathbb{C})$, and $\widehat{\mathcal{D}}$. Note that c permutes the two components $\mathcal{D}$ and $\overline{\mathcal{D}}$ of $\widehat{\mathcal{D}}$, and thus, $\overline{c}(\mathcal{D}) = \mathcal{D}$, where $\overline{c} : \mathbb{M}_0 \otimes \mathbb{C} \to \mathbb{M}_0 \otimes \mathbb{C}$ is the composition of c with the complex conjugation in $\mathbb{M}_0 \otimes \mathbb{C}$.

Let $\widehat{\mathcal{D}}_{\mathbb{R}}^c$ and $\mathcal{D}_{\mathbb{R}}^c$ denote the fixed point set of $\overline{c}$ restricted to $\widehat{\mathcal{D}}$ and $\mathcal{D}$. The latter consists of the lines generated by $w = u_+ + iu_-$ such that $u_\pm \in \mathbb{M}_0^\pm(c) \otimes \mathbb{R}$, $u_+^2 = u_-^2 < 0$, and the orientation u_+, u_- is the prescribed one. Since c is geometric, both $\mathcal{D}_{\mathbb{R}}^c$ and its (trivial) double covering $\widehat{\mathcal{D}}_{\mathbb{R}}^c$ are nonempty.

As it follows from definitions, the period of a c-marked real cubic fourfold belongs to $\mathcal{D}_{\mathbb{R}}^c = \{x \in \mathcal{D} \,|\, c(x) = \overline{x}\}$. Therefore, we call $\mathcal{D}_{\mathbb{R}}^c$ the *real period domain of real c-marked cubic fourfolds.*

Proof of Theorem 3.1.2. Pick up a generic point $[w] \in \mathcal{D}_{\mathbb{R}}^c$ (so that there is no vector $v \in V_2 \cup V_6$ orthogonal to w) and apply Theorem 2.2.3. This gives a non-singular cubic fourfold X and a marking ϕ such that $\mathrm{per}(X, \phi) = [w]$. The triple

$(X, X' = \overline{X}, F = \phi^{-1}c\phi)$ satisfies the assumptions of Theorem 2.2.2, which gives an antiholomorphic involution conj : $X \to X$ such that conj$^* = \phi^{-1}c\phi$. Clearly, $(\mathbb{M}, c)$ is the homological type of (X, conj), and it remains to notice that Pic $X = \mathbb{Z}$, $X(\mathbb{R})$ is non empty (as it is for any real hypersurface of odd degree), and therefore any antiholomorphic involution of X is induced by the complex conjugation in suitable projective coordinates of $P^5 = P(\mathcal{O}_X(1))$. $\square$

3.4. Refined real period map. Consider the quadratic cones $\Upsilon_\pm(c) = \{u \in \mathbb{M}_0^\pm(c) \otimes \mathbb{R} \colon u^2 < 0\}$ and the associated Lobachevsky (one- and two-component, respectively) spaces $\Lambda_\pm(c) = \Upsilon_\pm(c)/\mathbb{R}^*$ and $\Lambda_\pm^\sharp(c) = \Upsilon_\pm(c)/\mathbb{R}_+$, where $\mathbb{R}^* = \mathbb{R} \smallsetminus \{0\}$ and $\mathbb{R}_+ = (0, \infty)$.

Like in 3.3, we associate with a point in $\mathcal{D}_\mathbb{R}^c$ represented by $w = u_+ + iu_-$ (where $u_\pm \in \mathbb{M}_0^\pm(c) \otimes \mathbb{R}$, $u_+^2 = u_-^2 < 0$, and the orientation u_+, u_- is the prescribed one) the point in $\Lambda_+(c) \times \Lambda_-(c)$ represented by the pair (u_+, u_-). This gives a well-defined analytic isomorphism $\mathcal{D}_\mathbb{R}^c = \Lambda_+(c) \times \Lambda_-(c)$. The ambiguity in the choice of representatives gives rise to a refined real period domain $\mathcal{D}_\mathbb{R}^{c\sharp} \subset \Lambda_+^\sharp(c) \times \Lambda_-^\sharp(c)$, $\mathcal{D}_\mathbb{R}^{c\sharp} = \{(u_+\mathbb{R}_+, u_-\mathbb{R}_+) \in \Lambda_+^\sharp(c) \times \Lambda_-^\sharp(c) \mid$ the orientation u_+, u_- is the prescribed one$\}$.

To define $\text{per}_\mathbb{R}^{c\sharp}(X, \phi) \in \mathcal{D}_\mathbb{R}^{c\sharp}$ for a non-singular real c-marked cubic $(X, \phi) \in \mathcal{C}_\mathbb{R}^{c\sharp}$, we pick up a real polynomial p defining X and consider $w = \phi([\omega_p])$ (see the end of Section 2). As we have seen already, the latter splits as $w = u_+ + iu_-$, where $u_\pm \in \mathbb{M}_0^\pm(c)$, the pair (u_+, u_-) is defined uniquely by X up to a positive factor, and this pair spans a negative definite plane with the prescribed orientation. Thus, we obtain a uniquely defined real period $\text{per}_\mathbb{R}^{c\sharp}(X, \phi) \in \mathcal{D}_\mathbb{R}^{c\sharp}$ and a well defined map $\text{per}_\mathbb{R}^{c\sharp} \colon \mathcal{C}_\mathbb{R}^{c\sharp} \to \mathcal{D}_\mathbb{R}^{c\sharp}$. The above components $u_\pm\mathbb{R}_+ \in \Lambda_\pm^\sharp(c)$ of $\text{per}_\mathbb{R}^{c\sharp}(X, \phi)$ will be denoted $u_\pm^\sharp(X, \phi)$.

3.5. Polyhedral period cells. Denote by $\mathcal{H}_\pm(c) \subset \Lambda_\pm(c)$ and $\mathcal{H}_\pm^\sharp(c) \subset \Lambda_\pm^\sharp(c)$ the union of hyperplanes orthogonal to vectors from $(V_2 \cup V_6) \cap \mathbb{M}_0^\pm(c)$. The connected components of the complement $\Lambda_\pm(c) \setminus \mathcal{H}_\pm(c)$ will be called *the cells of* $\Lambda_\pm(c)$ and the hyperplanes from $\mathcal{H}_\pm(c)$ *the walls*. As is known, these hyperplanes form a locally finite arrangement (the group generated by reflections in these hyperplanes is discrete) so that the above cells are (locally finite) polyhedra. Put

$$\text{Per}_\mathbb{R}^c = \mathcal{D}_\mathbb{R}^{c\sharp} \cap ((\Lambda_+^\sharp(c) \setminus \mathcal{H}_+^\sharp(c)) \times (\Lambda_-^\sharp(c) \setminus \mathcal{H}_-^\sharp(c))).$$

and call *c-cells* the connected components of $\text{Per}_\mathbb{R}^c$. Note that the orientation restriction involved in the definition of $\mathcal{D}_\mathbb{R}^{c\sharp}$ establishes a one-to-one correspondence between the halves of $\Lambda_+^\sharp(c)$ and the halves of $\Lambda_-^\sharp(c)$, and this correspondence commutes with multiplication by -1. Therefore, $\text{Per}_\mathbb{R}^c$ spits into a union of pairs of opposite c-cells. The natural projection $\text{Per}_\mathbb{R}^c \to \Lambda_+(c) \times \Lambda_-(c)$ establishes a one-to-one correspondence between the set of pairs of opposite c-cells and the set of products of the cells of $\Lambda_\pm(c)$.

Given a continuous family of real c-marked cubics (X_t, ϕ_t), $t \in [0, 1]$, they can be defined by a continuous family of polynomials p_t, and hence their real periods $u_\pm^\sharp(X_t, \phi_t)$ belong to the same cells of $\Lambda_\pm^\sharp(c)$. The converse is also true.

3.5.1. Lemma. *Assume that (X_i, ϕ_i), $i = 0, 1$ is a pair of real c-marked cubic fourfolds defined by real polynomials p_i. Then, X_i can be connected by a continuous*

family X_t of real c-marked cubic fourfolds if and only if their periods $u^\sharp_\pm(X_i, \phi_i)$ belong to the same cells of $\Lambda^\sharp_\pm(c)$ (or in other words, if and only if the periods $\mathrm{per}_\mathbb{R}(X_i, \phi_i)$ belong to the same component of $\mathrm{Per}^c_\mathbb{R}$).

Proof. It follows from the description of the periods of cubic fourfolds (and the local Torelli theorem over the reals), because the vectors $v \in (V_2 \cup V_6)$ which are not from $\mathbb{M}^+_0 \cup \mathbb{M}^-_0$ define hyperplanes H_v which have intersection with $\mathbb{M}^\pm_0 \otimes \mathbb{R}$ of codimension less than one. $\square$

§4. DEFORMATIONS AND CHIRALITY

4.1. The mirror pairs of markings. Given a real hypersurface $X \subset P^5$, we can consider its *mirror image*, $X' = R(X)$, obtained from X by a reflection $R \colon P^5 \to P^5$ with respect to some real hyperplane $H \subset P^n$. According to our definitions, X is *chiral* if X and X' belong to different connected components of $\mathcal{C}_\mathbb{R}$, and *achiral* if they belong to the same component.

Assume that (X, ϕ) is a marked non-singular cubic fourfold. Then the isomorphism $R^* \colon \mathbb{M}(X') \to \mathbb{M}(X)$ induced by R respects the Hodge structure and the polarization classes of X and X', and thus yields a marking $\phi \circ R^*$ of X'. We say that the markings ϕ and $\phi' = \phi \circ R^*$ are *mirror images of each other*, or *a mirror pair of markings*.

4.1.1. Lemma. *Assume that a nonsingular real cubic fourfold X is defined by a real polynomial p and its mirror image, X', by a polynomial q. Then the period vectors $\phi[\omega_p]$ and $\phi'[\omega_q]$ are oppositely directed if X and X' are endowed with the mirror pair of markings: ϕ and $\phi' = \phi \circ R^*$.*

Proof. The form $\mathcal{E}/q^2$ representing $[\omega_q]$ changes the direction under the action of R, because $R^*(\mathcal{E}) = -\mathcal{E}$ and $q \circ R$ differs from p by a real factor. $\square$

As an immediate corollary of Lemma 4.1.1 and Lemma 3.5.1 we get a new proof of the following theorem from [FK].

4.1.2. Coarse deformation classification. *One real non-singular cubic fourfold is deformation equivalent to a projective transformation of another real non-singular cubic fourfold if and only if they are of the same homological type.*

Proof. Given a c-marking, we can compose it with lattice reflections $R_v, v \in V_2 \cap \mathbb{M}^0_\pm(c)$, and anti-reflections $-R_v, v \in V_6 \cap \mathbb{M}^0_\pm(c)$, to move the period into any pair of opposite cells of $\mathrm{Per}_\mathbb{R}(c)$ given in advance. When necessary, we can apply Lemma 4.1.1 and move the period into any of these opposite cells. According to Lemma 3.5.1 it means that the real non-singular cubics of homological type c are coarse deformation equivalent to each other. The "only if" part is trivial. $\square$

4.2. Basic criterion of chirality for cubic fourfolds. Let us fix a geometric involution c. Given a nonsingular c-marked real cubic fourfold (X, ϕ), denote by $P^\sharp(X) \subset \mathrm{Per}^c_\mathbb{R}$ the c-cell which contains $\mathrm{per}^{c\sharp}_\mathbb{R}(X, \phi)$ (in other words, the c-cell which contains $w = \phi[\omega_p]$ where, as usual, p is a real polynomial defining X).

4.2.1. Lemma. *The underlying nonsingular real cubic fourfold X of a real c-marked cubic fourfold (X, ϕ) is achiral if and only if there exists a lattice isometry of $\mathbb{M}$ which (1) commutes with c, (2) preserves the polarization class h, (3) induces an automorphism of $\mathbb{M}_0$ which preserves the prescribed orientation, and (4) sends the c-cell $P^\sharp(X)$ to the opposite c-cell, $-P^\sharp(X)$.*

Proof. Let X' denote the mirror image of X with the mirror image marking ϕ'. By Lemma 4.1.1, its period $w' = \phi'[\omega_q]$ belongs to $-P^\sharp(X)$. On the other hand, any continuous family of real nonsingular cubic fourfolds connecting X with X' gives another marking of X', say ϕ'', and according to Lemma 3.5.1 the period $\phi''[\omega_q]$ belongs to $P^\sharp(X)$. Comparing the two markings of X' we obtain a lattice isometry of $\mathbb{M} = \mathbb{M}(X')$ which transforms $P^\sharp(X)$ into $-P^\sharp(X)$; being a difference between two markings, it also preserves the polarization h, induces an automorphism of $\mathbb{M}_0$ which preserves the prescribed orientation, and commutes with c. Conversely, given such a lattice isometry, we can change the mirror image marking of X' and then apply Lemma 3.5.1 to deduce that X and X' belong both to the same component of $\mathcal{C}_\mathbb{R}$. □

4.3. Few lattice gluing lemmas. To simplify the above criterion and to reduce it to a study of $\operatorname{Aut} \mathbb{M}^0_+(c)$ we need the following results involving a technique of discriminant groups. Recall that for any non-degenerate lattice $\mathbb{L}$ of finite rank the discriminant group $\operatorname{discr} \mathbb{L} = \mathbb{L}/\mathbb{L}^*$ is a finite group and that, if the lattice $\mathbb{L}$ is even, this group carries a canonical finite quadratic form $\mathfrak{q}_\mathbb{L} : \operatorname{discr} \mathbb{L} \to \mathbb{Q}/2\mathbb{Z}$ defined via $\mathfrak{q}_\mathbb{L}(x + \mathbb{L}) = x^2 \mod 2\mathbb{Z}$. Note that any isometry, $f \in \operatorname{Aut} \mathbb{L}$, induces an automorphism of $\operatorname{discr} \mathbb{L}$, which preserves $\mathfrak{q}_\mathbb{L}$ if $\mathbb{L}$ is even. This induced automorphism will be denoted by $\delta(f)$.

4.3.1. Nikulin's theorem [N1]. *Assume that $\mathbb{L}$ is an even lattice of signature $(n, 1)$, $n \geqslant 0$, whose discriminant group $\operatorname{discr}(\mathbb{L})$ is 2-periodic. Then any isometry $\delta \colon \operatorname{discr}(\mathbb{L}) \to \operatorname{discr}(\mathbb{L})$ is induced by some isometry $f \colon \mathbb{L} \to \mathbb{L}$.* □

In the present paper we deal with the three lattices: $\mathbb{M}_-(c)$, $\mathbb{M}^0_+(c)$, and the rank 1 lattice $\langle h \rangle \subset \mathbb{M}$ generated by h. The first two lattices are even, and the latter is odd. The discriminant group $\operatorname{discr} \mathbb{M}_-(c)$ is 2-periodic, the discriminant group $\operatorname{discr} \langle h \rangle$ is a cyclic group of order 3, and the discriminant group $\operatorname{discr} \mathbb{M}^0_+(c)$ is canonically isomorphic to the direct sum $\operatorname{discr} \mathbb{M}_-(c) + \operatorname{discr} \langle h \rangle$, so that $\operatorname{discr} \mathbb{M}_-(c)$ is identified with the 2-primary part $\operatorname{discr}_2 \mathbb{M}^0_+(c)$ of $\operatorname{discr} \mathbb{M}^0_+(c)$, and $\operatorname{discr} \langle h \rangle$ with its 3-primary part $\operatorname{discr}_3 \mathbb{M}^0_+(c)$. The canonical isomorphism $\operatorname{discr}_2 \mathbb{M}^0_+(c) \to \operatorname{discr} \mathbb{M}_-(c)$ is an anti-isometry, that is it transforms $-\mathfrak{q}_{\mathbb{M}_-(c)}$ into $\mathfrak{q}_{\mathbb{M}^0_+(c)}$ restricted to $\operatorname{discr}_2 \mathbb{M}^0_+(c)$. (In fact, the lattice $\operatorname{discr} \langle h \rangle$, as any non-degenerate finite rank lattice with a fixed characteristic element, can be also equipped with a quadratic form, and with respect to this quadratic form the canonical isomorphism $\operatorname{discr}_3 \mathbb{M}^0_+(c) \to \operatorname{discr} \langle h \rangle$ is also an anti-isometry.)

The following lattice gluing lemmas are well known and their proofs are straightforward, see, e.g., [N1].

4.3.2. Lemma. *Any automorphism $f^0_+ \in \operatorname{Aut}(\mathbb{M}^0_+(c))$ can be uniquely extended to $\mathbb{M}_+(c)$. This extension sends the polarization class h to itself if and only if the 3-primary component $\delta_3(f^0_+)$ of $\delta(f^0_+)$ is trivial, that is $\delta_3(f^0_+) = \operatorname{id}$.* □

4.3.3. Lemma. *A pair of automorphisms $f_\pm \in \operatorname{Aut}(\mathbb{M}_\pm(c))$ are induced from $f \in \operatorname{Aut}(\mathbb{M}, c)$ if and only if $\delta(f_+) = \delta(f_-)$.* □

Automorphisms $f_\pm$ satisfying the conditions of Lemma 4.3.3 will be called *compatible*.

4.4. Lattice characterization of chirality. The *reflection group* W_+ generated in $\operatorname{Aut}(\mathbb{M}^0_+(c))$ by reflections R_v, $v \in (V_2 \cup V_6) \cap \mathbb{M}^0_+(c)$ acts transitively on the set of

cells of $\Lambda_+(c)$. If $v \in V_6$, then R_v does not extends to M_+, but anti-reflection $-R_v$ does. So, we consider also the group $W_+^\# \subset \mathrm{Aut}(\mathbb{M}_+^0(c))$ generated by reflections R_v, $v \in V_2 \cap \mathbb{M}_+^0(c)$, and anti-reflections $-R_v$, $v \in V_6 \cap \mathbb{M}_+^0(c)$ (the two groups are isomorphic and induce the same action on Λ_+). Any of the cells $P_+ \subset \Lambda_+(c)$ being fixed, the group $\mathrm{Aut}(\mathbb{M}_+^0(c))$ splits into a semi-direct product $W_+ \rtimes \mathrm{Aut}(P_+)$, where $\mathrm{Aut}(P_+) = \{g \in \mathrm{Aut}(\mathbb{M}_+^0(c)) \mid g(P_+) = P_+\}$ is the stabilizer of P_+.

With $\mathbb{M}_-(c)$ the situation is even simpler: since its discriminant group is of period 2 the intersection $V_6 \cap \mathbb{M}_-(c)$ is empty. Thus, in this case we consider simply the reflection group $W_- \subset \mathrm{Aut}(\mathbb{M}_-(c))$ generated by reflections R_v, $v \in V_2 \cap \mathbb{M}_-^0(c)$. This reflection group acts transitively on the set of cells of $\Lambda_-(c)$ and, therefore, $\mathrm{Aut}(\mathbb{M}_-(c))$ splits into a semi-direct product $W_- \rtimes \mathrm{Aut}(P_-)$, where $\mathrm{Aut}(P_-) = \{g \in \mathrm{Aut}(\mathbb{M}_-(c)) | g(P_-) = P_-\}$ is the stabilizer of a cell P_- of $\Lambda_-(c)$.

The preimage of $P_\pm$ in $\Lambda_\pm^\#$ is the union of a pair of c-cells, $P_\pm^\#$ and $-P_\pm^\#$. Each $g \in \mathrm{Aut}(P_\pm)$ either permutes this pair of cells, and then we say that it is $P_\pm$-*reversing*, or it preserves both $P_\pm^\#$ and $-P_\pm^\#$, and then we call it $P_\pm$-*direct*. The subgroup of $\mathrm{Aut}(P_\pm)$ formed by $P_\pm$-direct elements will be denoted by $\mathrm{Aut}^+(P_\pm)$, while the coset of $P_\pm$-reversing elements will be denoted by $\mathrm{Aut}^-(P_\pm)$. The crucial for our study of chirality observation is that an automorphism $f \in \mathrm{Aut}(\mathbb{M})$ preserving each of $P_\pm$ belongs to $\mathrm{Aut}^+(\mathbb{M})$ if and only if its components $f_+ = f|_{\mathbb{M}_+^0}$, $f_- = f|_{\mathbb{M}_-}$ are both of the same type: either simultaneously $f_\pm \in \mathrm{Aut}^+(P_\pm)$, or simultaneously $f_\pm \in \mathrm{Aut}^-(P_\pm)$.

In the case of lattices $\mathbb{M}_+^0$, an additional characteristic of $g \in \mathrm{Aut}(\mathbb{M}_+^0)$ is its 3-primary component, $\delta_3(g)$, which may be trivial or not. In a bit more general setting, we consider a hyperbolic lattice $\mathbb{L}$ whose discriminant splits as $\mathrm{discr}(\mathbb{L}) = \mathrm{discr}_2(\mathbb{L}) + \mathrm{discr}_3(\mathbb{L})$, where $\mathrm{discr}_2(\mathbb{L})$ is 2-periodic and $\mathrm{discr}_3(\mathbb{L}) = \mathbb{Z}/3$. We say that $g \in \mathrm{Aut}(\mathbb{L})$ is $\mathbb{Z}/3$-*direct* if $\delta_3(g) = \mathrm{id}$, and $\mathbb{Z}/3$-*reversing* if $\delta_3(g) \neq \mathrm{id}$ (certainly, in the later case $\delta_3(g) = -\mathrm{id}$).

4.4.1. Theorem. *A non-singular real cubic fourfold X of homological type c is achiral if and only if the lattice $\mathbb{M}_+^0(c)$ admits an automorphism $g \in \mathrm{Aut}^-(P_+)$ which is $\mathbb{Z}/3$-direct.*

Proof. The "only if" part is a straightforward consequence of the "only if" part of Lemma 4.2.1.

To prove the "if" part, let us pick up a c-marking $\phi \colon \mathbb{M}(X) \to \mathbb{M}$ and choose $f_+^0 \in \mathrm{Aut}^-(P_+(X))$ which is $\mathbb{Z}/3$-direct. From Lemma 4.3.2 it follows that f_+^0 extends to $f_+ \in \mathrm{Aut}\,\mathbb{M}_+$ preserving h. Lemma 4.3.3 and Theorem 4.3.1 imply that we can find $f_- \in \mathrm{Aut}(\mathbb{M}_-)$ compatible with f_+ and $f \in \mathrm{Aut}(\mathbb{M})$ defined by (f_+, f_-). By composing f_- (and f) with a suitable $w_- \in W_-$, the component f_- can be chosen in $\mathrm{Aut}(P_-) \subset \mathrm{Aut}(\mathbb{M}_-)$. If $f \in \mathrm{Aut}(\mathbb{M})$ defined by (f_+, f_-) belongs to $\mathrm{Aut}^+(\mathbb{M})$, then f transforms $P^\sharp(X)$ into $-P^\sharp(X)$ since it preserves the prescribed orientation and $f_+ \in \mathrm{Aut}^-(P_+)$. Therefore, in this case due to Lemma 4.2.1 we are done. If $f \in \mathrm{Aut}^-(\mathbb{M})$, then we replace f_- by $-f_-$, observe that the pair $(f_+, -f_-)$ defines an automorphism $f \circ c$ which belongs to $\mathrm{Aut}^+(\mathbb{M})$, and argue as before. $\square$

§5. AUXILIARY ARITHMETICS

5.1. Root systems and chirality of special hyperbolic lattices. In this section $\mathbb{L}$ is a lattice of signature $(n, 1)$, $n \geqslant 1$. Throughout this section we make

two additional assumptions on $\mathbb{L}$ which are satisfied in the cases of $\mathbb{L} = \mathbb{M}^0_+(c)$ that we are concerned about. The first assumption is that the discriminant $\mathrm{discr}(\mathbb{L})$ splits as $\mathbb{Z}/3 + \mathrm{discr}_2(\mathbb{L})$, where the summand $\mathrm{discr}_2(\mathbb{L})$ is a 2-periodic group. Let $\Phi = V_2 \cup V_6$, where $V_k = \{v \in \mathbb{L} \,|\, v^2 = k, 2\frac{vw}{v^2} \in \mathbb{Z}, \forall w \in \mathbb{L}\}$ (note that for $k = 2$ the condition $2\frac{vw}{v^2} \in \mathbb{Z}$ is always satisfied). Our second assumption is that the rank of Φ equals to the rank of $\mathbb{L}$ (that is maximal possible), and, thus, Φ is a root system in $\mathbb{L}$. This holds for $\mathbb{L} = \mathbb{M}^0_+(c)$ for all geometric involution c except one rather special case $\mathbb{M}^0_+(c) = U(2) + E_6(2)$ in which $\Phi = \varnothing$ (the complete list of $\mathbb{L} = \mathbb{M}^0_+(c)$ is given in Tables 8–9, in section 8, see also [FK] for more details). Vectors $v \in V_k$ will be called k-roots.

We let $\mathbb{L}_{\mathbb{R}} = \mathbb{L} \otimes \mathbb{R}$, and like before, consider $\Upsilon = \{v \in \mathbb{L}_{\mathbb{R}} \,|\, v^2 < 0\}$, and the hyperbolic spaces $\Lambda = \Upsilon/\mathbb{R}^*$, along with $\Lambda^\# = \Upsilon/\mathbb{R}_+$. In this context we use notation H_v for the hyperplane $\{w \in \mathbb{L}_{\mathbb{R}} \,|\, vw = 0\}$ and $H_v^\pm$ for the half-spaces $\{w \in \mathbb{L}_{\mathbb{R}} \,|\, \pm vw \geqslant 0\}$. For $v \in \Upsilon$, H_v, etc., we denote by $[v] \in \Lambda$, $[H_v] \subset \Lambda$, $[v]^\# \in \Lambda^\#$, $[H_v]^\# \subset \Lambda^\#$, etc., the corresponding object after projectivization.

We distinguish the *reflection group* $W \subset \mathrm{Aut}(\mathbb{L})$ generated by the reflections $R_v \in \mathrm{Aut}(\mathbb{L})$, $x \mapsto x - 2\frac{vx}{v^2}v$, $v \in V_2 \cup V_6$, and the group $W^\# \subset \mathrm{Aut}(\mathbb{L})$ generated by the reflections $R_v, v \in V_2$, and the anti-reflections $-R_v, v \in V_6$. Hyperplanes $[H_v]$ (respectively $[H_v]^\#$), $v \in \Phi$, cut Λ (respectively $\Lambda^\#$) into open polyhedra, whose closures are called the *cells*. The cells in Λ are the fundamental chambers of W, and the pairs of opposite cells in $\Lambda^\#$ are the fundamental chambers of $W^\#$.

Let us pick up a cell $P \subset \Lambda$ and fix a covering c-cell $P^\# \subset \Lambda^\#$. Choosing any vector $p \in \Upsilon$ so that $[p]^\#$ lies in the interior of $P^\#$, we let $\Phi^\pm = \{v \in \Phi \,|\, \pm vp > 0\}$. The minimal subset $\Phi^b \subset \Phi^-$ such that $P^\# = \cap_{v \in \Phi^b} [H_v^-]^\#$ is called the *basis of* Φ *defined by* $P^\#$. The hyperplanes $[H_v]$, $v \in \Phi^b$, support n-dimensional faces of P and will be called the *walls of* P. Note that any $v \in \Phi^-$ is a linear combination of the roots in Φ^b with non-negative coefficients.

Theorem 4.4.1 motivates the following definition: $\mathbb{L}$ is called *achiral* if it admits a $\mathbb{Z}/3$-direct automorphism $g \in \mathrm{Aut}^-(P)$, for some cell P. Obviously, if $\mathbb{L}$ is achiral then a $\mathbb{Z}/3$-direct automorphism $g \in \mathrm{Aut}^-(P)$ exists for any cell P. It is also obvious that existence of a $\mathbb{Z}/3$-direct $g \in \mathrm{Aut}^-(P)$ is equivalent to existence of $\mathbb{Z}/3$-reversing $h \in \mathrm{Aut}^+(P)$, since these two kinds of automorphisms just differ by sign.

5.2. Coxeter's graphs and their symmetry. The *Coxeter graph* Γ has Φ^b as the vertex set. The vertices are colored: 2-roots are white and 6-roots are black. The edges are weighted: the weight of an edge connecting vertices $v, w \in \Phi^b$ is $m_{vw} = 4\frac{(vw)^2}{v^2w^2}$, and $m_{vw} = 0$ means absence of an edge. These weights are non-negative integers, because $2\frac{vw}{v^2}, 2\frac{vw}{w^2} \in \mathbb{Z}$, and $v^2, w^2 > 0$ for any $v, w \in \Phi^b$. In the case of $m_{vw} = 1$, the angle between H_v and H_w is $\pi/3$, and $v^2 = w^2$; such edges are not labelled. The case of $m_{vw} = 2$ (corresponds to angle $\pi/4$) cannot happen, since $v^2, w^2 \in \{2, 6\}$. An edge of weight $m_{vw} = 3$ connects always a 2-root with a 6-root; it corresponds to angle $\pi/6$, and will be labelled by 6. The case of $m_{vw} = 4$ corresponds to parallel hyperplanes in Λ, and we sketch a thick edge between v and w. If $m_{vw} > 4$, then the corresponding hyperplanes in Λ are ultra-parallel (diverging), and we sketch a dotted edge.

For a subset $J \subset \Phi^b$ we may consider also the subgraph Γ_J which is formed by the vertex set J and all the edges of Γ connecting these vertices. We say that Γ_J is the *Coxeter graph of* J. If J is finite and ordered, $J = \{v_1, \ldots, v_{|J|}\}$, then we

consider also the *Gram matrix*, G_J, whose (ij)-entry is $v_i v_j$.

A permutation $\sigma \colon J \to J$ will be called a *symmetry of* Γ_J if it preserves the weight of edges and the length of the roots, i.e., $(\sigma(v))^2 = v^2$ and $m_{\sigma(v)\sigma(w)} = m_{vw}$ for all $v, w \in J$.

5.2.1. Existence of symmetries. *Assume that a subset $J \subset \Phi^b$ spans $\mathbb{L}$ over $\mathbb{Z}$. Then any symmetry $\sigma \colon J \to J$ of Γ_J is induced by an automorphism of the lattice $\mathbb{L}$ which preserves the cell $P^\#$ invariant.*

Proof. Such a symmetry preserves the Gram matrix of the vectors from J. Therefore, it is induced by an isometry of $\mathbb{L} \otimes \mathbb{Q}$. Since the vectors from J span $\mathbb{L}$ over $\mathbb{Z}$, this isometry maps $\mathbb{L}$ to $\mathbb{L}$. Assuming that it maps $P^\#$ to another cell, we observe that these two cells have J as a common set of face normal vectors. Pick up a wall separating the two cells and notice that each of the normal root vectors $\pm v \neq 0$ of such a wall has non-negative product with the vectors from J, which is a contradiction, since the vectors from J generate the whole space. $\square$

To recognize $\mathbb{Z}/3$-reversing symmetries of Γ, one can use the following observation. Considering some direct sum decomposition of $\mathbb{L}$, we observe that one of the direct summands, $\mathbb{L}_1$, has $\mathrm{discr}_3(\mathbb{L}_1) = \mathbb{Z}/3$, while the other direct summands have 2-periodic discriminants (because $\mathrm{discr}(\mathbb{L})$ gets an induced direct sum decomposition). For any vertex w of Γ viewed as a vector in $\mathbb{L}$, we can consider its $\mathbb{L}_1$-component. Our simple observation is that σ *is $\mathbb{Z}/3$-direct if for all black vertices, $v \in V_6$, of Γ the $\mathbb{L}_1$-components of v and $\sigma(v)$ are congruent modulo $3\mathbb{L}_1$, and $\mathbb{Z}/3$-reversing if for some $v \in V_6$ we have $v - \sigma(v) \notin 3\mathbb{L}$.*

5.3. Vinberg's algorithm. Vinberg's method [Vin1] of calculation the Coxeter graph of Φ is to pick up a vector $p \in \Upsilon$ so that $[p]^\# \in P^\#$, and then to determine a sequence of roots $v_i \in \Phi^b$, $i = 1, 2, \ldots$, ordered so that the hyperbolic distance from p to the walls $H_i = H_{v_i}$ of P is increasing. Such distance can be characterized by the (non-negative) value $d_i = d_i(p) = 2\frac{(pv_i)^2}{(v_i)^2}$, which will be called *the level of root v_i with respect to p* (the coefficient 2 here is chosen to make d_i integer in the further considerations).

The level zero vectors in Vinberg's sequence form a root basis in the root system $\{v \in V | vp = 0\}$. Since choosing of $[p]$ at a vertex of $[P]$ (rather than in its interior) simplifies calculations, we always try to start with such a choice of p that the system of the level zero roots would be of the maximal rank, namely $\dim \mathbb{L} - 1$.

If Vinberg's sequence, $v_1, \ldots, v_m$, is found up to level r, then the vectors $v \in \Phi$ of higher levels should satisfy the conditions: $pv < 0$ and $vv_i \leqslant 0$ for all v_i, $1 \leqslant i \leqslant m$. If vectors v respecting these conditions do exist, then the next segment of Vinberg's sequence is constituted by all such vectors of the minimal level. Note that the order of Vinberg's roots within the same level is not well-defined (and is inessential).

This process terminates and gives the basis Φ^b, if the latter is finite, otherwise the process enumerates vectors of Φ^b in an infinite sequence. If we found Vinberg's vectors $v_1, \ldots, v_m$ up to some level r, then we can use one of Vinberg's sufficient criteria below for detecting the termination of the process.

5.4. Vinberg's termination criteria. The Gram matrix G_J and the Coxeter graph Γ_J are called *elliptic* (of rank r) if G_J is positive definite (of rank r). As is observed in [Vin1], the elliptic subgraphs of Γ of rank $n - k$ are in one-to-one correspondence with the k-dimensional faces of $[P]$. Namely, an elliptic subgraph

Γ_J corresponds to the face supported by the projectivization of the linear space $H_J = \cap_{v \in J} H_v$.

The connected components of an elliptic graph G_J must belong to the list of the classical elliptic graphs of the root systems. In our case (since $m_{vw} = 2$ do not appear), an elliptic graph cannot be other than A_n, D_n, E_6, E_7, E_8, and G_2.

A connected subgraph Γ_J and its Gram matrix G_J are called *parabolic* if G_J is positive semi-definite matrix of rank $|J| - 1$. In our case, a parabolic connected subgraph should be one of the graphs $\widetilde{A}_n$ (recall that $\tilde{A}_1$ is just a thick edge), $\widetilde{D}_n$, $\widetilde{E}_6, \widetilde{E}_7, \widetilde{E}_8$, and $\widetilde{G}_2$, where the subscript always equals the *rank of parabolic graph*, $|J| - 1$. A disconnected subgraph Γ_J and its Gram matrix are called parabolic if all the connected components of Γ_J are parabolic. The rank of such Γ_J is by definition the sum of the ranks of its components. As is observed in [Vin1], a subgraph Γ_J is parabolic of maximal possible rank, $n - 1$, if and only if the intersection H_J defines a vertex of $[P]$ at infinity (on the Absolute).

Matrix G_J (and its Coxeter graph Γ_J) is called *critical*, if it is not elliptic, but any submatrix $G_{J'}$, $J' \subsetneq J$, is elliptic. Such G_J is parabolic if degenerate. If a critical matrix G_J is non-degenerate, its graph G_J is called *Lannér's diagrams*. The list of Lannér's diagrams can be found, for example, in [Vin1], [Vin2]. Note that the only Lannér's diagram possible under the assumptions of this section is a dotted edge (the other Lannér's diagrams all contain a pair of roots which have the ratio of length different from 1 and 3).

5.4.1. Finite volume criterion [Vin1]. *Vinberg's sequence terminates at $J = \{v_1, \ldots, v_m\}$ if the polyhedron P_J bounded in $\Lambda_{\mathbb{L}}$ by the hyperplanes dual to $v \in J$ has a finite hyperbolic volume.* $\square$

To determine finiteness of the volume, Vinberg gives several criteria. One of them, [Vin2] Proposition 4.2(1), can be formulated (in the form of [D] Proposition 2.4) as follows.

5.4.2. Criterion of finiteness of the volume. *The polyhedron P_J has a finite volume if and only if the following two conditions are satisfied:*

(1) *Γ_J contains an elliptic subdiagram of rank $n - 1$ where $n = \dim \mathbb{L} - 1$,*
(2) *any elliptic subdiagram of rank $n - 1$ of Γ_J can be extended to an elliptic subdiagram of rank n, or to a parabolic subdiagram of rank $n - 1$; and there exist precisely two such extensions.* $\square$

Remark. The second condition in theorem 5.4.2 means just that any edge is adjacent to two vertices: finite, or at infinity.

There is another (more simple, but only sufficient) criterion which can be used if Γ_J does not contain Lannér's schemes (that is, dotted edges in our setting).

5.4.3. Sufficient criterion of finiteness of the volume [Vin1]. *The volume of P_J is finite if the following conditions are satisfied:*

(1) *J has rank $\dim \mathbb{L} = n + 1$;*
(2) *the Coxeter graph, Γ_J, does not contain Lannér's diagrams as subgraphs;*
(3) *every connected parabolic subgraph in Γ_J is a connected component of some parabolic subgraph of rank $n - 1$ in Γ.*

§6. Chirality of M-cubics

6.1. Preliminaries and the main statement. A particular, characteristic, feature of M-cubics is that the lattice $\mathbb{M}$ splits into a direct sum of the eigen-lattices $\mathbb{M}_+$ and $\mathbb{M}_-$. Thus, the eigen-lattices are unimodular in the case of M-cubics, and only in this case. As it follows from the classification in [FK] (or can be easily deduced directly from Theorem 4.1.2, Lemma 3.1.1, and the classification of unimodular lattices), there exists precisely three coarse deformation classes (equivalently, three homological types) of M-cubics. The corresponding three lattices $\mathbb{M}_+$ are $U + 3I = -I + 4I$, $U + 3I + E_8 = -I + 12I$, and $U + 3I + 2E_8 = -I + 20I$. The polarization class $h \in \mathbb{M}_+$ is characteristic, of square 3, and can be identified with $(1, 1, 1) \in 3I$. So, the primitive lattices $\mathbb{M}_+^0$ are even and isomorphic to $U + A_2$, $U + A_2 + E_8$ and $U + A_2 + 2E_8$ respectively. The corresponding lattices $\mathbb{M}_-$ are also even and isomorphic to $U + 2E_8$, $U + E_8$ and U respectively.

6.1.1. Theorem. *Non-singular real cubic fourfold of types $\mathbb{M}_+^0(c) = U + A_2$ and $\mathbb{M}_+^0(c) = U + A_2 + E_8$ are chiral; in particular, the cubic fourfolds of each of these two types form two deformation classes. Non-singular real cubic fourfold of type $\mathbb{M}_+^0(c) = U + A_2 + 2E_8$ are achiral; these cubic fourfold form one deformation class.*

The rest of this Section is devoted to a case by case proof of this theorem.

We fix a basis u_1, u_2 in U and a basis a_1, a_2 in A_2, so that $u_i^2 = 0$ $(i = 1, 2)$, $u_1 u_2 = 1$, $a_i^2 = 2$ $(i = 1, 2)$, and $a_1 a_2 = -1$. The basis $e_1, \ldots, e_8$ in E_8 is chosen as is shown on the Coxeter graph of E_8, see Figure 1 (we use the usual convention: $e_i^2 = 2$ for $i = 1, \ldots, 8$ and $e_i \circ e_j = -\delta_{ij}$). This figure presents also the dual vectors e_i^*, $i = 1, \ldots, 8$, which are also elements of E_8, because the lattice E_8 is unimodular; for example, $e_8^* = 2e_8 + 3e_7 + 4e_6 + 5e_5 + 6e_4 + 4e_3 + 3e_2 + 2e_1$. In the case of $U + A_2 + 2E_8$, the basic vectors of the additional E_8-summand will be denoted by e_i' and their duals by $(e_i')^*$, $i = 1, \ldots, 8$.

FIGURE 1. Coxeter graph E_8 and the vectors e_i^*

In all of these three M-cases, to apply Vinberg's algorithm (see Section 5.3) we pick $p = u_1 - u_2$. Then we choose as the set of level-zero vectors the standard bases in each of E_8-components of $\mathbb{M}_+^0(c)$ and complete them by two square-2 vectors $v_1 = u_1 + u_2$ and $v_2 = a_2$, and one square-6 vector $v_3 = a_1 - a_2$. This choice determines uniquely a cell P_+ in $\Lambda_+(c)$. The vectors of higher levels in Vinberg's sequence must have components $x_1 u_1 + x_2 u_2 + y_1 a_1 + y_2 a_2$ in $U + A_2$ satisfying the following relations:

$$x_2 < x_1, \quad x_1 + x_2 \leqslant 0, \quad 2y_2 \leqslant y_1, \quad y_1 \leqslant y_2.$$

Note that the vector $v_4 = -(u_2 + a_1 + a_2)$ satisfies these relation and, thus, appears in the list as a vector of level one in each of the three M-cases.

Certainly, the basic vectors of the E_8-summands impose also restrictions on the vectors of higher levels. Namely, their components in the first (respectively, second) E_8-summand should be linear combinations of $e_1^*, \ldots, e_8^*$ (respectively, $(e_1')^*, \ldots, (e_8')^*$) with non-positive coefficients.

6.2.   The case $\mathbb{M}_+^0(c) = U + A_2$. The Coxeter graph of the vector system $\{v_1, v_2, v_3, v_4\}$ is shown on Figure 2. The only its parabolic subgraph is $\widetilde{G_2}$ (the subgraph generated by v_2 and v_3), and it has rank $2 = \dim \Lambda_+ - 1$. By Vinberg's finite volume criterion, it implies that Vinberg's sequence terminates at $\{v_1, v_2, v_3, v_4\}$, and so the polyhedron P_+ is found. Since the Coxeter graph admits no symmetries, $-\operatorname{id}$ is the only element of $\operatorname{Aut}^-(P_+)$. Thus, applying Theorem 4.4.1 we conclude that the studied cubic fourfolds are chiral.

Table 1

FIGURE 2

	U	A_2
p	$1, -1$	$0, 0$
level 0		
v_1	$1, 1$	$0, 0$
v_2	$0, 0$	$0, 1$
v_3	$0, 0$	$1, -1$
level 1		
v_4	$0, -1$	$-1, -1$

Coxeter's graph for $U + A_2$ Vinberg's vectors for $\mathbb{M}_+^0(c) = U + A_2$

6.3. The case $\mathbb{M}_+^0(c) = U + A_2 + E_8$. Here, the level-zero vectors are $e_1, \ldots, e_8$, v_1, v_2, and v_3. The level-one vectors are v_4 and $v_5 = -u_2 - e_8^*$. This gives the Coxeter graph shown on Figure 3. This graph has only two parabolic subgraphs: $\widetilde{G_2}$ and $\widetilde{E_8}$. Vinberg's finite volume criterion is satisfied because these subgraphs are disjoint and the sum of their ranks is $2 + 8 = \dim \Lambda_{\mathbb{M}_+^0} - 1$. The graph has no symmetries and arguing like in 6.2 we conclude that the studied cubic fourfolds are chiral.

Table 2

FIGURE 3

	U	A_2	E_8
p	$1, -1$	$0, 0$	0
level 0			
v_1	$1, 1$	$0, 0$	0
v_2	$0, 0$	$0, 1$	0
v_3	$0, 0$	$1, -1$	0
level 1			
v_4	$0, -1$	$-1, -1$	0
v_5	$0, -1$	$0, 0$	$-e_8^*$

Coxeter's graph for $U + A_2 + E_8$ Vinberg's vectors for $\mathbb{M}_+^0(c) = U + A_2 + E_8$

6.4. The case $\mathbb{M}_+^0(c) = U + A_2 + 2E_8$**.** Here, the level-zero vectors are $e_1, \ldots, e_8$, $e_1', \ldots, e_8'$, v_1, v_2, and v_3. The level-one consists of three 2-roots v_4, v_5, and $v_5' = -u_2 - (e_8')^*$. On the next level, 16, there is one 2-root $v_6 = 2(u_1 - u_2) - (a_1 + a_2) - e_1^* - (e_1')^*$. Then, on the level 36 there is a pair of 2-roots:

$$v_7 = 3(u_1 - u_2) - (2a_1 + a_2) - e_7^* - (e_2')^*,$$
$$v_7' = 3(u_1 - u_2) - (2a_1 + a_2) - e_2^* - (e_7')^*.$$

Table 3. Vinberg's vectors for $\mathbb{M}_+^0(c) = U + A_2 + 2E_8$

v_i	U	A_2	E_8	E_8
p	$1, -1$	$0, 0$	0	0
level 0				
v_1	$1, 1$	$0, 0$	0	0
v_2	$0, 0$	$0, 1$	0	0
v_3	$0, 0$	$1, -1$	0	0
level 1				
v_4	$0, -1$	$-1, -1$	0	0
v_5	$0, -1$	$0, 0$	$-e_8^*$	0
v_5'	$0, -1$	$0, 0$	0	$-(e_8')^*$
level 16				
v_6	$2, -2$	$-1, -1$	$-e_1^*$	$-(e_1')^*$
level 36				
v_7	$3, -3$	$-2, -1$	$-e_7^*$	$-(e_2')^*$
v_7'	$3, -3$	$-2, -1$	$-e_2^*$	$-(e_7')^*$
level 48				
v_8	$6, -6$	$-4, -2$	$-3e_8^*$	$-3(e_1')^*$
v_8'	$6, -6$	$-4, -2$	$-3e_1^*$	$-3(e_8')^*$

Our list of Vinberg's vectors given in Table 3 below includes also a pair of 6-roots of level 48,

$$v_8 = 6(u_1 - u_2) - (4a_1 + 2a_2) - 3e_8^* - 3(e_1')^*,$$
$$v_8' = 6(u_1 - u_2) - (4a_1 + 2a_2) - 3e_1^* - 3(e_8')^*.$$

FIGURE 4. HEXAGONAL COXETER'S SUBGRAPH FOR $U + A_2 + 2E_8$

The above list contains three 6-roots: v_3, v_8, and v_8'. If we drop them and consider the Coxeter subgraph formed only by the 2-roots, we obtain the hexagonal diagram shown on Figure 4. This diagram has a lot of symmetries. Consider the involution which fixes the vertices e_7, e_4', v_6, e_7' and permutes the vertices v_1, e_4. Since the set of vectors corresponding to the vertices of the diagram generate the lattice $\mathbb{M}_+^0(c)$, this involution is induced by a lattice involution $f : \mathbb{M}_+^0(c) \to \mathbb{M}_+^0(c)$ (see Proposition 5.2.1). Since in the whole Coxeter diagram the 6-root v_3 is connected with the 2-roots v_2, v_7, v_7' and the 6-root v_8' is connected with the 2-roots e_1, v_7, e_8', the automorphism f transforms v_3 into v_8'. The A_2-components of v_3 and v_8' are $(1, -1)$ and $(-4, -2)$, which are not congruent modulo 3. This implies that f is $\mathbb{Z}/3$-reversing. By Proposition 5.2.1, f is P_+-direct, so applying Theorem 4.4.1 we conclude that this homological type is achiral.

Remark. This lattice, its fundamental chamber, and the complete Coxeter graph had appeared already in Vinberg's paper [Vin3] on maximally algebraic $K3$-surfaces. Note that our list contains the full set of 2-roots, and the missing 6-roots can be obtained from the 6-roots in the list by applying the symmetries of the hexagonal subgraph. The same construction is given in [Lo].

§7. CHIRALITY OF $(M - 1)$-CUBICS

7.1. Preliminaries and the main statement. The next after M-cubics by their topological complexity are $(M - 1)$-cubics. The deformation components of the latters, as it follows from [FK], are adjacent to the deformation components of M-cubics. The lattice, $\mathbb{M}$, of an $(M-1)$-cubic contains the direct sum of the eigen-lattices $\mathbb{M}_+$ and $\mathbb{M}_-$ as a sublattice of index 2, and this condition characterizes $(M - 1)$-cubics among all non-singular real cubic fourfolds. In the other words, the characteristic feature of $(M - 1)$-cubics is that $\mathbb{M}_\pm$ have discriminant $\mathbb{Z}/2$. Using the general properties of lattices $\mathbb{M}_\pm$ (namely, that lattice $\mathbb{M}_+$ is odd with a characteristic element $h \in \mathbb{M}_+$ of square $h^2 = 3$, that lattice $\mathbb{M}_-$ is even, and that the both lattices are of index $\sigma_- = 1$), one can deduce that the $(M-1)$-cubics form precisely six homological types, see [FK]. As usual, these types can be distinguished by sublattices M_+, as well as by sublattices M_-. The corresponding six lattices $\mathbb{M}_+^0$ are $U + A_2 + A_1 + kE_8$ and $-A_1 + A_2 + kE_8$, $k = 0, 1, 2$.

7.1.1. Theorem. *Non-singular real cubic fourfolds of types $\mathbb{M}_+^0(c) = -A_1 + A_2, U + A_2 + A_1$, and $-A_1 + A_2 + E_8$, are chiral; in particular, the cubic fourfolds of each of these three types form two deformation classes. Non-singular real cubic fourfolds of types $\mathbb{M}_+^0(c) = U + A_2 + E_8 + A_1, -A_1 + A_2 + 2E_8$, and $U + A_2 + 2E_8 + A_1$ are achiral; the cubic fourfolds of each of these three types form one deformation class.*

7.2. The case $\mathbb{M}_+^0(c) = -A_1 + A_2$. Here, Vinberg's sequence starts from vectors $\{v_1, v_2, v_3\}$ given in Table 4. The Coxeter graph of this sequence of three vectors is shown in Figure 5. It contains a unique parabolic subgraph $\widetilde{A}_1$ (a thick edge connecting v_2 and v_3). Vinberg's criteria 5.4.3 and 5.4.1 can be applied to conclude termination, since the rank of $\widetilde{A}_1$ is $1 = \dim \mathbb{M}_+^0(c) - 2$. The Coxeter graph admits no symmetries. Hence, applying Theorem 4.4.1 we deduce that the studied cubic fourfolds are chiral.

7.3. The case $\mathbb{M}_+^0(c) = U + A_2 + A_1$. Here, Vinberg's sequence starts from four level-zero vectors $\{v_1, v_2, v_3, v_4\}$ and two level-one vectors $\{v_5, v_6\}$ given in Table 5.

FIGURE 5

Table 4

	$-A_1$	A_2
p	1	$0,0$
level 0		
v_1	0	$0,1$
v_2	0	$1,-1$
level 12		
v_3	3	$-4,-2$

Coxeter's graph for $-A_1 + A_2$ Vinberg's vectors for $\mathbb{M}^0_+(c) = -A_1 + A_2$

The Coxeter graph of this sequence of six vectors is shown in Figure 6. It contains precisely two parabolic subgraphs, $\widetilde{G}_2$ (vertices v_3, v_2, v_5) and $\widetilde{A}_1$ (v_4, v_6). Vinberg's criterion is satisfied, since the rank of their union is $2 + 1 = \dim \mathbb{M}^0_+(c) - 2$. The Coxeter graph admits no symmetries. Hence, applying Theorem 4.4.1 we conclude that the studied cubic fourfolds are chiral.

Table 5

FIGURE 6

	U	A_2	A_1
p	$1,-1$	$0,0$	0
level 0			
v_1	$1,1$	$0,0$	0
v_2	$0,0$	$0,1$	0
v_3	$0,0$	$1,-1$	0
v_4	$0,0$	$0,0$	1
level 1			
v_5	$0,-1$	$-1,-1$	0
v_6	$0,-1$	$0,0$	-1

Coxeter's graph for $U + A_2 + A_1$ Vinberg's vectors for $\mathbb{M}^0_+(c) = U + A_2 + A_1$

7.4. The case $\mathbb{M}^0_+(c) = -A_1 + A_2 + E_8$. Here, the level-zero vectors of Vinberg's sequence are $e_1, \ldots, e_8$, v_1, and v_2. They are followed by two vectors of level 4 and one vector of level 12, see Table 6. The Coxeter graph, Γ, of this sequence of thirteen vectors is shown in Figure 7.

Lemma 7.4.1. *Vinberg's criterion 5.4.2 is satisfied for the Coxeter graph Γ on Figure 7.*

Proof. For $S = \{a_1, \ldots, a_n\} \subset \Phi^b$, let $F_S = F_{a_1,\ldots,a_n} \subset P$ denote the face of the cell P supported in the intersection of the walls $[H_v]$, where $v \in \Phi^b \smallsetminus S$. Note that P has two vertices at infinity, F_{v_5,e_8} and F_{v_1,v_3} (because the sets $\Phi^b \smallsetminus S$ span parabolic subgraphs of maximal possible rank $\dim(\mathbb{M}^0_+) - 2 = 9$). The other vertices of P are $F_{a,b,c}$ such that $\Phi^b \smallsetminus \{a, b, c\}$ spans an elliptic subgraph. This subgraph cannot contain the dotted edge connecting v_3 with v_5, so the set $S = \{a, b, c\}$ should contain either v_3 or v_5 (or the both). This set should contain also at least one vertex-root from each of the parabolic subgraphs $\widetilde{E}_7$, $\widetilde{E}_8$, $\widetilde{G}_2$, $\widetilde{A}_1$ of Γ. If the

Table 6

FIGURE 7

	$-A_1$	A_2	E_8
p	1	$0,0$	0
level 0			
v_1	0	$0,1$	0
v_2	0	$1,-1$	0
level 4			
v_3	1	$0,0$	$-e_1^*$
v_4	1	$-1,-1$	$-e_8^*$
level 12			
v_5	3	$-4,-2$	0

Coxeter's graph for $-A_1 + A_2 + E_8$ Vinberg's vectors for $\mathbb{M}_+^0(c) = -A_1 + A_2 + E_8$

both v_3 and v_5 are included in $S = \{a, b, c\}$, then F_S is a vertex of P only for $S = \{v_3, v_4, v_5\}$. If $a = v_5$ and $v_3 \notin S$, then b and c should be chosen from the two disjoint parabolic subgraphs $\widetilde{G_2}$ and $\widetilde{E_7}$, which gives 21 other vertices $F_{v_5, b, c}$, where $b \in \{v_2, v_1, v_4\}$ and $c \in \{e_1, \ldots, e_7\}$. Similarly, if $a = v_3$ and $v_5 \notin S$, then b, c should be chosen from the two disjoint parabolic subgraphs $\widetilde{A_1}$ and $\widetilde{E_8}$, so $b = v_2$ and $c \in \{e_1, \ldots, e_8, v_4\}$, which gives 9 new vertices $F_{a, b, c}$. Totally, Γ contains 31 finite vertices and two vertices at infinity.

The edges of P can be expressed as F_S, $S = \{a, b, c, d\}$, where $\Phi^b \smallsetminus S$ spans an elliptic subgraph. Thus, as above, S should contain at least one of v_3 and v_5. In the edges $F_{v_3, v_5, v_4, d}$, one of the endpoints is F_{v_3, v_5, v_4}. In the cases $d \in \{e_1, \ldots, e_7\}$, the other endpoint is $F_{v_5, v_4, d}$. In the cases $d = v_1$, $d = v_2$, and $d = e_8$, the other endpoint is respectively F_{v_1, v_3}, F_{v_3, v_4, v_2}, and F_{v_5, e_8}. The edges $F_{v_3, v_5, e_8, d}$ must have $d \in \{v_1, v_2\}$ and are incident to F_{v_5, e_8}. Another endpoint is F_{v_3, v_1} for $d = v_1$, and F_{v_3, e_8, v_2} for $d = v_2$. Each of the edges $F_{v_3, v_5, c, d}$, $c \in \{v_1, v_2\}$, $d \in \{e_1, \ldots, e_7\}$ has $F_{v_5, c, d}$ as one of the endpoints. The other endpoint is $F_{v_3, v_2, d}$ if $c = v_2$ and F_{v_3, v_1} if $c = v_1$.

The other edges $F_{a, b, c, d}$ have $a \in \{v_3, v_5\}$ and $b, c, d \notin \{v_3, v_5\}$. If $a = v_3$, then another vertex should be chosen from the subgraph $\widetilde{A_1}$, and we may assume that $b = v_2$ (since the case $b = v_5$ was already considered). This gives edges $F_{v_3, v_2, c, d}$ with $c \in \{e_1, \ldots, e_8, v_4\}$ and $d \in \{e_1, \ldots, e_8, v_4, v_1\}$. If $d \neq v_1$, then the endpoints are $F_{v_3, v_2, c}$ and $F_{v_3, v_2, d}$. The endpoints of F_{v_3, v_2, c, v_1} are $F_{v_3, v_2, c}$ and F_{v_3, v_1}. Finally, if $a = v_5$, then one of b, c, d should be chosen from $\widetilde{G_2}$, say $b \in \{v_2, v_1, v_4\}$ and another from $\widetilde{E_7}$, say $c \in \{e_1, \ldots, e_7\}$. Then $F_{v_5, b, c, d}$ has one endpoint $F_{v_5, b, c}$. Another its endpoint is $F_{v_5, c, d}$ if $d \in \{v_2, v_1, v_4\}$, and $F_{v_5, b, d}$ if $d \in \{e_1, \ldots, e_7\}$. In the remaining case $d = e_8$, the second endpoint is F_{v_5, e_8}. $\square$

The Coxeter graph admits no symmetries. Hence, applying Theorem 4.4.1 we conclude that the studied cubic fourfolds are chiral.

7.5. The case $\mathbb{M}_+^0(c) = U + A_2 + A_1 + E_8$. Here, the level-zero Vinberg's vectors are $e_1, \ldots, e_8$ plus $v_1, \ldots, v_4$ listed in Table 7. Then follow three vectors v_5, v_6, v_7 of level 1 and the vector v_8 of level 48 (see the same Table).

Consider the Coxeter subgraph formed by Vinberg's vectors $e_1, \ldots, e_8, v_1, v_2, v_5$,

Table 7

FIGURE 8

	U	A_2	A_1	E_8
p	$1,-1$	$0,0$	0	0
level 0				
v_1	$1,1$	$0,0$	0	0
v_2	$0,0$	$0,1$	0	0
v_3	$0,0$	$1,-1$	0	0
v_4	$0,0$	$0,0$	1	0
level 1				
v_5	$0,-1$	$-1,-1$	0	0
v_6	$0,-1$	$0,0$	-1	0
v_7	$0,-1$	$0,0$	0	$-e_8^*$
level 48				
v_8	$6,-6$	$-4,-2$	-3	$-3(e_1)^*.$

A symmetric fragment
of Coxeter's graph
for $U + A_2 + A_1 + E_8$

Vinberg's vectors for $\mathbb{M}_+^0(c) = U + A_2 + A_1 + E_8$

v_6, v_7. This subgraph is shown in Figure 8. It has an evident nontrivial involution (which fixes the vertex v_7 and permutes the vertices v_2, e_1). Since the vectors $e_1, \ldots, e_8$, v_1, v_2, v_5, v_6, v_7 generating this subgraph span the lattice $\mathbb{M}_+^0(c)$, this involution is induced by a P_+-direct lattice involution $f : \mathbb{M}_+^0(c) \to \mathbb{M}_+^0(c)$ (see Proposition 5.2.1). In particular, f transforms Vinberg's vector $v_3 = -v_5 - 2v_2 + v_7 + e_8^*$, into another Vinberg's vector $v_3' = -e_3 - 2e_1 + e_5 + (2e_6 + 3e_7 + 4e_8 + 5v_7 + 6v_1 + 4v_5 + 3v_6 + 2v_2)$. The A_2-component of v_3' is $4(-1,-1) + 2(0,1) = (-4,-2)$, while the A_2-component of v_3 is $(1,-1)$. Hence, f is $\mathbb{Z}/3$-reversing and applying Theorem 4.4.1 we conclude that the type considered is achiral.

7.6. The case $\mathbb{M}_+^0(c) = -A_1 + A_2 + 2E_8$. Let us start with a bit more general setting. Namely, assume that $\mathbb{L}$ is a lattice like in §5 (for example, some of the lattices $\mathbb{M}_+^0(c)$), $P \subset \Lambda(\mathbb{L})$ is a cell, and $f \in \mathrm{Aut}^+(P)$ is an automorphism of $\mathbb{L}$ induced by some symmetry of the Coxeter graph, Γ, of $\mathbb{L}$. Suppose that a 2-root v is a vertex of Γ preserved by this symmetry, that is, $f(v) = v$. Then the sublattice $\mathbb{L}_v = \{x \in \mathbb{L} \,|\, xv = 0\}$ is f-invariant and we may consider an induced automorphism $f_v \in \mathrm{Aut}(\mathbb{L}_v)$.

7.6.1. Lemma. *If f is $\mathbb{Z}/3$-reversing, then f_v is also $\mathbb{Z}/3$-reversing, and P_v-direct for some cell P_v of $\mathbb{L}_v$.*

Proof. Since $\mathrm{discr}_3(\mathbb{L}_v) = \mathrm{discr}_3(\mathbb{L}) = \mathbb{Z}/3$, the automorphisms f and f_v are the both $\mathbb{Z}/3$-direct or $\mathbb{Z}/3$-reversing. Furthermore, f preserves the facet $P \cap [H_v]$ of P, since it preserves both P and v. Due to $\mathrm{discr}_3(\mathbb{L}_v) = \mathrm{discr}_3(\mathbb{L})$, each wall in $\Lambda(\mathbb{L}_v)$ is an intersection of $[H_v]$ with a wall $\Lambda(\mathbb{L})$, and thus, the facet $P \cap [H_v]$ is a part of some cell, P_v, of $\Lambda(\mathbb{L}_v)$. Such P_v has to be also invariant. $\square$

7.6.2. Corollary. *Lattice $-A_1 + A_2 + 2E_8$ is achiral.*

Proof. Let $\mathbb{L} = U + A_2 + 2E_8$, then for $v = v_1$ (notation of 6.4) we have $\mathbb{L}_v = -A_1 + A_2 + 2E_8$. An involution of the hexagonal diagram (Figure 4) in 6.4 is conjugate to some involution, $f \in \mathrm{Aut}^+(P)$, preserving v_1. Since f is $\mathbb{Z}/3$-reversing, we can apply Lemma 7.6.1. $\square$

Applying Theorem 4.4.1 we can now conclude that the fourfolds with $\mathbb{M}_+^0(c) = -A_1 + A_2 + 2E_8$ are achiral.

7.7. The case $\mathbb{M}_+^0(c) = U + A_2 + 2E_8 + A_1$. Let $\mathbb{L}$ and $\mathbb{L}_v$ be like in 7.6. Our aim now is to obtain a criterion which is in some sense "converse" to the one in Lemma 7.6.1. Recall that lattice $\mathbb{L}$ either splits into a direct sum of $\mathbb{L}_v$ with a sublattice $A_1 = \mathbb{Z}v$, or contains this direct sum as an index 2 sublattice. We will show that, in the former case, achirality of $\mathbb{L}_v$ implies achirality of $\mathbb{L}$.

7.7.1. Lemma. *Assume that $\mathbb{L} = \mathbb{L}_v + A_1$, where $A_1 = \mathbb{Z}v$, and $\mathbb{L}_v$ is achiral. Then $\mathbb{L}$ is also achiral. In fact, any $\mathbb{Z}/3$-reversing automorphism $f_v \in \mathrm{Aut}^+(P_v)$ for some cell $P_v \subset \Lambda(\mathbb{L}_v)$ can be extended to a $\mathbb{Z}/3$-reversing automorphism $f \in \mathrm{Aut}^+(P)$ for some cell $P \subset \Lambda(\mathbb{L})$.*

Proof. Letting $f(v) = v$ we obtain an extension of f_v to $\mathbb{L}$ which is obviously $\mathbb{Z}/3$-reversing if f_v is.

Like in Lemma 7.6.1, by the same evident reasons, P_v contains the facet $P \cap [H_v]$ of some cell P in $\Lambda(\mathbb{L})$. But now, the relation is stronger: $P \cap [H_v] = P_v$. In fact, the walls of P different from $[H_v]$ are either orthogonal to $[H_v]$ or do not intersect it. To see it, consider any wall $[H_w]$, $w \in V_2 \cup V_6$. Splitting $\mathbb{L} = \mathbb{L}_v + \mathbb{Z}v$ gives a decomposition $w = w_v + kv$, where $w_v \in \mathbb{L}_v$, $k \in \mathbb{Z}$. If $k = 0$, then $[H_w]$ is orthogonal to $[H_v]$, whereas $w_v = 0$ implies $w = v$. Otherwise we observe that $w_v^2 = w^2 - k^2v^2 \leqslant 0$, because $v^2 = 2$, and w^2 is either 2, or 6, but in the latter case k is divisible by 3. Thus, vectors perpendicular to w_v cannot have negative square, which contradicts to $P \cap [H_v] \neq \varnothing$.

The relation $P \cap [H_v] = P_v$ implies that the isometry $f = f_v \oplus \mathrm{id} : \mathbb{L} \to \mathbb{L}$ is P-direct. $\square$

7.7.2. Corollary. *Lattice $U + A_2 + 2E_8 + A_1$ is achiral.*

Proof. According to 6.4, the lattice $\mathbb{L}_v = U + A_2 + 2E_8$ is achiral. It remains to apply Lemma 7.7.1. $\square$

Applying Theorem 4.4.1 we can now conclude that the cubic fourfolds with $\mathbb{M}_+^0(c) = U + A_2 + 2E_8 + A_1$ are achiral.

§8. Concluding remarks

8.1. Further results. The cases of M-varieties and $(M-1)$-varieties are usually the most interesting and difficult ones, which explains our special interest to them in the context of the chirality problem of the cubic fourfolds. But our methods are applicable as well to the other cases. Our observations concerning the problem of chirality can be summarized as follows.

Let ρ denote the rank of the lattice $\mathbb{M}_+^0$, $r = 22 - \rho$ denote the rank of $\mathbb{M}_-$, and d the discriminant rank, $\mathrm{rk}(\mathrm{discr}_2(\mathbb{M}_+^0)) = \mathrm{rk}(\mathrm{discr}(\mathbb{M}_-))$. In all the cases studied, if $\rho + d \geqslant 14$ then $\mathbb{M}_+^0$ is achiral. In addition, the list of achiral lattices contains $\mathbb{M}_+^0 = U(2) + A_2 + D_4$ and $\mathbb{M}_+^0 = -A_1 + \langle 6 \rangle + kA_1$ with $k = 2, 3$, and 4. The other lattices that we have analyzed are chiral. (In a few cases remaining for analysis, the discriminant form is even and $\rho + d \geqslant 14$. We expect that the corresponding lattices are achiral.)

The lattices $M_+^0(c)$ of cubic fourfolds can be naturally divided into the *principal series*, which contains the most of lattices and is presented in Table 8, and several additional lattices presented in Table 9 (see [FK] for more details).

<div style="display:flex">

Table 8
The principal series of lattices M_+^0

$-A_1$	$+\langle 6\rangle$	$+tA_1,$	$0\leqslant t\leqslant 9$
$-A_1$	$+A_2$	$+tA_1,$	$0\leqslant t\leqslant 9$
U	$+A_2$	$+tA_1,$	$0\leqslant t\leqslant 9$
U	$+A_2+D_4$	$+tA_1,$	$0\leqslant t\leqslant 6$
$-A_1$	$+\langle 6\rangle+E_8$	$+tA_1,$	$0\leqslant t\leqslant 5$
$-A_1$	$+A_2+E_8$	$+tA_1,$	$0\leqslant t\leqslant 5$
U	$+A_2+E_8$	$+tA_1,$	$0\leqslant t\leqslant 5$
U	$+A_2+D_4+E_8$	$+tA_1,$	$0\leqslant t\leqslant 2$
$-A_1$	$+\langle 6\rangle+2E_8$	$+tA_1,$	$0\leqslant t\leqslant 1$
$-A_1$	$+A_2+2E_8$	$+tA_1,$	$0\leqslant t\leqslant 1$
U	$+A_2+2E_8$	$+tA_1,$	$0\leqslant t\leqslant 1$

Table 9
Additional lattices M_+^0

$U(2)+E_6(2)$
$U(2)+A_2$
$U+E_6(2)$
$U(2)+A_2+D_4$
$U(2)+A_2+2D_4$
$U+A_2+2D_4$
$U(2)+A_2+E_8$
$U(2)+A_2+D_4+E_8$
$U(2)+A_2+2E_8$
$U+A_2+E_8(2)$
$U(2)+A_2+E_8(2)$

</div>

The Table 10 describes chirality of the principal series of cubic fourfolds in terms of the ranks r and d.

TABLE 10. CHIRALITY OF CUBIC FOURFOLDS: THE PRINCIPAL SERIES

```
d
11                              a
10                          a       a
9                       a       a       a
8                   a       a       a       a
7               a       a       a       a       a
6           a       a       a       a       a       a
5       a       a       a       a       a       c       a
4       a   a       a       a       a       c       c       a
3       a       a       a       a       a       c       c       c       c
2   a   a       a       a       a       c       c       c       c       c
1 a   a                   a       c                   c       c
0       a                   c                               c
    1  2  3  4  5  6  7  8  9 10 11 12 13 14 15 16 17 18 19 20  r
```

The symbol "c" stands for the chiral deformation classes, and symbol "a" for the achiral ones

8.2. Chirality of singular cubic fourfolds.

Chirality of cubic fourfolds having nodal singularities is an interesting related problem. It is trivial to observe that any perturbation of an achiral nodal cubic provides an achiral non-singular cubic. The non-trivial part of the problem is the converse: if perturbations give only achiral cubics, can we conclude that a nodal cubic is achiral itself ? To solve it, one can use the same approach as in the non-singular case, just taking into account the vanishing cycles. On the other hand, the central projective correspondence discussed in [FK] relates chirality of nodal cubic fourfolds to a certain question about 6-polarized $K3$-surfaces. This relation can be used in the both directions.

A somewhat different kind of observation is chirality of the *discriminant cubic*,
$\det\begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 & x_3 & x_4 \\ x_2 & x_4 & x_5 \end{pmatrix} = 0$, which parameterizes the space of singular conics in the plane. The key observation is chirality of the singular locus of the discriminant cubic (this locus is the image of the Veronese map).

8.3. Explicit equations.

It would be interesting to find explicit (natural) equations for representatives of each of the deformation classes. It can be helpful not

only for proving achirality statements, but also for better understanding of the topology of the cubic hypersurfaces. As an example, let us consider the equations of the following type:

$$(\sum_1^6 x_\alpha)^3 - \sum_1^6 c_\alpha x_\alpha^3 = 0;$$

these equations were proposed in the late 70th by D. Fucks (private communication to the second author) for searching the precise range of the values of the Euler characteristic of real cubic hypersurfaces in each given even dimension (a problem remaining, up to our knowledge still open in its whole generality). Similar equations were used earlier by F. Klein [K], and his student C. Rodenberg [R], to find and to study explicit representatives for each of the five classes of real nonsingular cubic surfaces. In fact, it is by means of these equations that Klein proved in [K2] the achirality of all real nonsingular cubic surfaces (cf., the remark at the end of this subsection).

One can easily check that for c_α having all the same value c, the topology of the hypersurface is changing at $c = 0, 4, 16,$ and 36. For $c < 0$ and $c > 36$ the real part of the hypersurface is diffeomorphic to the real four-dimensional projective space, $\mathbb{R}P^4$. When $c = 36$, there appears a solitary double point, so that for $16 < c < 36$ we observe $S^4 \sqcup \mathbb{R}P^4$. When $c = 16$, our hypersurfaces acquire six double points of Morse index $(1, 4)$ with respect to growing c (the first, respectively second, component of the index is the number of positive, respectively, negative squares), and therefore, for $4 < c < 16$ the real part of the hypersurface is diffeomorphic to the real four-dimensional projective space with five $S^1 \times S^3$-handles, that is $\mathbb{R}P^4 \# 5(S^1 \times S^3)$. Finally, when $c = 4$, one finds that there are fifteen double points of Morse index $(2, 3)$, which implies that the Euler characteristic of our hypersurfaces becomes equal to 21. According to the classification of cubics (see [FK]), there is only one coarse deformation class with this value of Euler characteristic (in fact, it is the class studied above in Section 7.7), and for the cubics of this class the real part has the homological type of $\mathbb{R}P^4 \# 10(S^2 \times S^2)$. (One can also give a direct proof based on the Lefschetz trace formula and the Smith theory, which allow to reconstruct the Betti numbers from the action of the complex conjugation in homology.)

Since for c_α having all the same value the equation is invariant under transposition of the variables, all these hypersurfaces represent achiral classes. In the same manner, one can show that the whole left-hand slanted border of the diagram shown in Table 10 consists exclusively of achiral classes.

8.4. Chirality in lower dimensions. Quartic surfaces. Speaking on the real non-singular hypersurfaces $X_\mathbb{R}$ of dimension n and degree d, it is easy to see their achirality in the trivial cases $n = 0$ (for any d), and $d \leqslant 2$ (for any n). As was pointed out in Introduction, $X_\mathbb{R}$ is also achiral if n is odd. Achirality of cubic surfaces was observed by F. Klein, as we mentioned in 8.3. The next case of quartic surfaces was analyzed in [Kh1],[Kh2] using a technique similar to the one in this paper. It turned out that a non-contractible (in $\mathbb{R}P^3$) quartic $X_\mathbb{R}$ is chiral if and only if it has at least 4 spherical components, and a contractible quartic is chiral if and only if it has at least 3 spherical components and, in addition, a component with at least 3 handles (see Table 11, where r is the rank of the $+1$-eigen-lattice $L_+ = \{x \in H_2(X): \operatorname{conj}_* x = x\}$, d is the discriminant rank of L_+, and symbols a,

TABLE 11. CHIRALITY OF QUARTIC SURFACES

d

d	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
10									a										
9								a		a									
8							a		a		a								
7						a		a		a		a							
6					a		a		a		a			c					
5				a		a		a		a			c		c				
4			a		a		a		a			c		c		c			
3		a		a		a		a			c		c		c		c		
2	a		a		a		a			c		c		c		c		c	
1	a		a							c		c					c		c
0			a							c							c		

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 **r**

Non-contractible case

d

d	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
11										a										
10									a		a									
9								a		a		a								
8							a		a		a		a							
7						a		a		a		a		a						
6					a		a		a		a		a		a					
5				a		a		a		c		a		a		a				
4			a		a		a		c		c		a		a		a			
3		a		a		a		c		c		c		a		a		a		
2	a		a		a		c		c		c		c		a		a		a	
1	a		a							c		c					a		a	
0			a							c								a		

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 **r**

Contractible case

or c stand as in Table 10 for achiral, or respectively, chiral deformation classes).

8.5. Reversibility. In connection with chirality, it may be worth mentioning a different but somehow related notion of *reversibility*, which plays a non-trivial role for instance for odd-dimensional hypersurfaces. Namely, to each deformation class of real non-singular hypersurfaces $X \subset \mathbb{R}\mathrm{P}^{n+1}$ of degree d, that is a connected component $\mathcal{C}$ of $\mathcal{C}_{n,d} = P_{n,d}(\mathbb{R}) \setminus \Delta_{n,g}(\mathbb{R})$, we can associate its pull back $\widetilde{\mathcal{C}}$ into the sphere $\widetilde{P}_{n,d}(\mathbb{R})$ which covers $P_{n,d}(\mathbb{R})$. This pull back is either connected, or splits into a pair of opposite components. We say that C and the corresponding hypersurfaces $X \in C$ are *reversible* in the first case, and *irreversible* in the second one. In other words, X is reversible if its defining homogeneous polynomial, f, can be continuously changed into $-f$ without creating singularities in the process of deformation. One can extend the notion of reversibility to singular varieties replacing non-singular continuous families of equations by equisingular families.

If the degree d is even, then the region in $\mathbb{R}\mathrm{P}^{n+1}$ where $f > 0$ defines a coorientation of $X_{\mathbb{R}}$, and reversibility obviously means possibility to reverse this coorientation by a deformation. If n is odd, then such reversibility for non-singular hypersurfaces is impossible, because the regions where $f > 0$ and $f < 0$ are homologically different: they are distinguished by the highest dimension in which the inclusion homomorphism is nonzero. If n is even, then reversibility is possible: for example, a quadric is reversible if the signature of its equation vanishes and irreversible otherwise. Furthermore, it is not difficult to show that a real non-singular quartic surface $X_{\mathbb{R}}$ is irreversible if it has more than one connected components, as well as if it has a single component which is contractible in $\mathbb{R}\mathrm{P}^3$. Conversely, if

$X_{\mathbb{R}}$ is connected and non-contracible, then the quartic is reversible, at least if the genus of $X_{\mathbb{R}}$ is < 10 (the extremal case, $g = 10$, remains unknown to the authors). Thus we obtain nine reversible cases, more than one hundred irreversible ones, and a unique unresolved case.

If the degree d is odd and n is even, then $X_{\mathbb{R}}$ is reversible for a trivial reason, because $- \mathrm{id}$ and id belong to the same connected component of $GL(n + 2, \mathbb{R})$, and $f(-x) = -x$. If the both d and n are odd, then f determines an orientation of $X_{\mathbb{R}}$ and reversibility obviously means possibility to alternate this orientation. If $X_{\mathbb{R}}$ is symmetric with respect to a mirror reflection, then such an alternation is realizable by a projective transformation, which is one of manifestations of the similarity between the notions of reversibility and achirality. Existence of symmetric models proves in particular reversibility of curves of degree $\leqslant 5$.

In the case of non-singular cubic threefolds the problem of reversibility is already not trivial. The deformation classification of such cubics obtained in [Kr] gives 9 classes. Our analysis has shown that just one of these classes is irreversible, namely, the class denoted $\mathcal{B}(1)'_I$ in [Kr].

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