SINGULAR COTANGENT BUNDLE REDUCTION & SPIN CALOGERO-MOSER SYSTEMS

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Abstract. We develop a bundle picture for the case that the configuration manifold has only a single isotropy type, and give a formula for the reduced symplectic form in this setting. Furthermore, as an application of this bundle picture we consider Calogero-Moser systems with spin associated to polar representations of compact Lie groups.

1. Introduction

This paper is concerned with symplectic reduction of a cotangent bundle T^*Q with respect to a Hamiltonian action by a compact Lie group K that comes as the cotangent lifted action from the configuration manifold Q. Moreover, we assume that Q is Riemannian and K acts on Q by isometries. The cotangent bundle T^*Q is equipped with its canonical exact symplectic form, and we have a standard momentum map $\mu : T^*Q \to \mathfrak{k}^*$. Consider a coadjoint orbit $\mathcal{O}$ that lies in the image of μ. The goal is now to understand the symplectically reduced space

$$\mu^{-1}(\mathcal{O})/K =: T^*Q/\!/_{\mathcal{O}}K.$$

Several difficulties arise at this point. First of all the action by K on the base Q is *not* assumed to be free. So we will get only a stratified symplectic space. Its strata will be of the form

$$(\mu^{-1}(\mathcal{O}) \cap (T^*Q)_{(L)})/K =: (T^*Q/\!/_{\mathcal{O}}K)_{(L)}$$

where (L) is an element of the isotropy lattice of the K-action on T^*Q. This is follows from the theory of singular symplectic reduction as developed in Sjamaar and Lerman [35], Bates and Lerman [6], and Ortega and Ratiu [24]. See also Theorem 3.1.

One of the aspects of cotangent bundle reduction is to relate the reduced space $(T^*Q/\!/_{\mathcal{O}}K)_{(L)}$ to the cotangent bundle of the reduced configuration space, i.e. to $T^*(Q/K)$. However, in this generality Q/K will not be a smooth manifold, and, worse, the mapping $(T^*Q/\!/_{\mathcal{O}}K)_{(L)} \to T^*(Q/K)$ (which one constructs canonically – see Section 5) does not have locally constant fiber type. To remedy this mess we have to assume that the base manifold is of single isotropy type, that is $Q = Q_{(H)}$ for a subgroup H of K. Assuming this we get a first result that says that

$$\mathcal{O}/\!/_0 H \hookrightarrow T^*Q/\!/_{\mathcal{O}}K \longrightarrow T^*(Q/K)$$

is a symplectic fiber bundle, and this is Theorem 4.4. This result is obtained by applying the Palais Slice Theorem to the action on the base space Q, and then

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using the Singular Commuting Reduction Theorem of Section 3. This is an inroad that was also taken by Schmah [33] to get a local description of $T^*Q/\!/_{\mathcal{O}}K$.

However, one can also give a global symplectic description of the reduced space, and this is done in Section 5. This follows an approach that is generally called gauged cotangent bundle reduction or Weinstein construction ([40]) or also Sternberg construction. In the case that the action by K on the configuration space is free this global description was first given by Marsden and Perlmutter [18]. Their result says that the symplectic quotient $T^*Q/\!/_{\mathcal{O}}K$ can be realized as the fibered product

$$T^*(Q/K) \times_{Q/K} (Q \times_K \mathcal{O})$$

and they compute the reduced symplectic structure in terms of data intrinsic to this realization – [18, Theorem 4.3].

In the presence of a single non-trivial isotropy on the configuration space one gets a non-trivial isotropy lattice on T^*Q and thus has to use stratified symplectic reduction. The result is then the following: Each symplectic stratum $(T^*Q/\!/_{\mathcal{O}}K)_{(L)}$ of the reduced space can be globally realized as

$$(\mathcal{W}/\!/_{\mathcal{O}}K)_{(L)} = T^*(Q/K) \times_{Q/K} (\textstyle\bigsqcup_{q\in Q}\mathcal{O} \cap \operatorname{Ann}\mathfrak{k}_q)_{(L)}/K$$

where

$$\mathcal{W} := (Q \times_{Q/K} T^*(Q/K)) \times_Q \textstyle\bigsqcup_{q\in Q}\operatorname{Ann}\mathfrak{k}_q \cong T^*Q$$

as symplectic manifolds with a Hamiltonian K-action. Moreover, we compute the reduced symplectic structure in terms intrinsic to this realization. This is the content of Theorem 5.5.

It is rather surprising that the subject of cotangent bundle reduction, albeit so important to Hamiltonian mechanics, still is very untouched. Even in the case of a free action on the base the results are rather new, and there is not much to be found about singular cotangent bundle reduction in the literature. One of the first to study this subject is Schmah [33]. The other important paper on singular cotangent bundle reduction is the one by Perlmutter, Rodriguez-Olmos and Sousa-Diaz [29]. By restricting to do reduction at fully isotropic values of the momentum map $\mu : T^*Q \to \mathfrak{k}^*$ they are able to drop all assumptions on the isotropy lattice of the K-action on Q, and give a very complete description of the reduced symplectic space.

As an application we consider Calogero-Moser systems with spin in Section 6. In fact, it was an idea of Alekseevsky, Kriegl, Losik, Michor [2] to consider polar representations of compact Lie groups G on a Euclidean vector space V to obtain new versions of Calogero-Moser systems. We make these ideas precise by using the singular cotangent bundle reduction machinery. Thus let Σ be a section for the G-action in V, let C be a Weyl chamber in this section, and put $M := Z_G(\Sigma)$. Under a strong but not impossible condition on a chosen coadjoint orbit in $\mathfrak{g}^*$ we get

$$T^*V/\!/_{\mathcal{O}}G = T^*C_r \times \mathcal{O}/\!/_0 M$$

from the general theory, where C_r denotes the sub-manifold of regular elements in C. This is the effective phase space of the Spin Calogero-Moser system. The corresponding Calogero-Moser function is obtained as a reduced Hamiltonian from the free Hamiltonian on T^*V. The resulting formula is

$$H_{\mathrm{CM}}(q,p,[Z]) = \tfrac{1}{2}\sum_{i=1}^{l} p_i^2 + \tfrac{1}{2}\sum_{\lambda\in R} \frac{\sum_{i=1}^{k_\lambda} z_\lambda^i z_\lambda^i}{\lambda(q)^2}.$$

This is made precise with the necessary notation in Section 6.

Finally we use a result on non-commutative integrability from Zung [43, Theorem 2.3] to show that this Calogero-Moser system is integrable in the non-commutative sense.

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2. Singular spaces and smooth structure

2.1. **Whitney conditions.** In this subsections we introduce the Whitney conditions which will be necessary in the definition of Whitney stratified spaces – see Definition 2.11. We follow the approach of Mather [19].

Definition 2.1. Let M be a manifold and X, Y sub-manifolds such that $X \cap Y = \emptyset$. The pair (X, Y) is said to satisfy CONDITION (a) AT A POINT $y \in Y$ if the following holds. Consider an arbitrary sequence of points $(x_i)_i$ in X tending to y such that $T_{x_i}X$ converges to some r-plane $\tau \subseteq T_yM$ in the Grassmanian bundle of r-planes in TM. Then it is true that $T_yY \subseteq \tau$. The pair (X, Y) is said to satisfy condition (a) if it does so at every point $y \in Y$.

Example 2.2. Let $M = \mathbb{C}^3 = \{(x, y, z)\}$, and consider the complex analytic sub-manifolds $X := \{(x, y, z) : zx^2 - y^2 = 0\} \setminus \{(0, 0, z)\}$ and $Y := \{(0, 0, z)\}$. The pair (X, Y) satisfies condition (a) at all points of Y except at the origin:

However, Y can be further decomposed: consider $Y_0 = Y \setminus \{0\}$ and $Z = \{0\}$. Now the pairs (X, Y_0), (X, Z), and (Y_0, Z) obviously do satisfy condition (a). □

Definition 2.3 (Whitney condition (b) in $\mathbb{R}^n$)**.** Let X, Y be disjoint sub-manifolds of $\mathbb{R}^n$ with $\dim X = r$. The pair (X, Y) is said to satisfy CONDITION (b) AT $y \in Y$ if the following is true. Consider sequences $(x_i)_i$, $(y_i)_i$ in X, Y, respectively, such that $x_i \to y$, $y_i \to y$, and $x_i \neq y_i$. Assume that $T_{x_i}X$ converges to some r-plane $\tau \subseteq T_y\mathbb{R}^n = \mathbb{R}^n$, and that the lines spanned by the vectors $y_i - x_i$ converge – in $\mathbb{R}P^{n-1}$ – to some line $l \subseteq \mathbb{R}^n = T_y\mathbb{R}^n$. Then $l \subseteq \tau$. The pair (X, Y) satisfies CONDITION (b) if it does so at every $y \in Y$.

Obviously condition (b) behaves well under diffeomorphisms in the following sense: for $i = 1, 2$ consider pairs (X_i, Y_i) in $\mathbb{R}^n$, points $y_i \in Y_i$, open neighborhoods $U_i \subseteq \mathbb{R}^n$ of y_i, and a diffeomorphism $\phi : U_1 \to U_2$ sending y_1 to y_2 and satisfying $\phi(U_1 \cap X_1) = U_2 \cap X_2$ as well as $\phi(U_1 \cap Y_1) = U_2 \cap Y_2$. Thus it makes sense to formulate this condition for manifolds.

Definition 2.4 (Whitney condition (b))**.** Let M be a manifold and X, Y disjoint sub-manifolds. Now (X, Y) is said to satisfy CONDITION (b) if the following holds for all $y \in Y$. Let (U, ϕ) be a chart around y. Then the pair $(\phi(X \cap U), \phi(Y \cap U))$ satisfies condition (b) at $\phi(y)$.

By the above this definition is independent of the chosen chart in the formulation.

Example 2.5. Consider $M = \mathbb{C}^3 = \{(x, y, z)\}$ with Y the z-axis, and $X = \{(x, y, z) : y^2 + x^3 - z^2x^2 = 0\} \setminus Y$. Then the pair (X, Y) satisfies condition (a). It satisfies condition (b) at all points in Y except at $y = 0$. □

Proposition 2.6. *Let M be a manifold and (X, Y) a pair of disjoint sub-manifolds of M.*

(i) *If* (X,Y) *satisfies condition* (b) *at* $y \in Y$ *then it also satisfies condition* (a) *at* y.
(ii) *If* (X,Y) *satisfies condition* (b) *at* $y \in Y \cap \overline{X}$ *then* $\dim X > \dim Y$.

Notice that the assumption $\overline{X} \cap Y \neq \emptyset$ is necessary for the second statement in the proposition.

Proof. Since both assertions are of local nature it suffices to consider the case $M = \mathbb{R}^n$.
First assertion: Let $(x_i)_i$ be a sequence in X such that $x_i \to y \in Y$. Suppose $T_{x_i}X \to \tau \subseteq T_y\mathbb{R}^n = \mathbb{R}^n$. By contradiction we assume that $T_yY \subseteq \mathbb{R}^n$ is not contained in τ. Thus there is a line $l \subseteq T_yY$ which intersects τ at the origin only. Now we choose a sequence $(y_i)_i$ in Y so that the difference $y_i - x_i$ spans a line converging to the line l which lies in T_yY. This, however, contradicts condition (b).
Second assertion: Let $(x_i)_i$ be a sequence in X such that $x_i \to y \in Y$. By compactness of the Grassmanian we can (passing to a subsequence if necessary) assume that $T_{x_i}X$ converges to some plane τ. From the above we know that (b) implies (a) and hence $T_yY \subseteq \tau$. If x_i is close enough to Y, i.e., for i large enough, we can find $y_i \in Y$ minimizing the distance to x_i and so that $y_i \to y$. It follows that $y_i - x_i$ is orthogonal to T_yY. Let l_i denote the line spanned by $y_i - x_i$. Passing to a subsequence if necessary, the l_i converge to a line l in $\mathbb{R}P^{n-1}$ and l is still orthogonal to T_yY. By condition (b) we have $l \subseteq \tau$, and clearly $\dim X = \dim \tau \geq \dim Y + \dim l > \dim Y$. □

2.2. **Stratified spaces.** Let X be a paracompact and second countable topological Hausdorff space, and let $(I, \leq)$ be a partially ordered set.

Definition 2.7 (Decomposed space). An I-DECOMPOSITION of X is a locally finite partition of X into smooth manifolds S_i, $i \in I$ which are disjoint (but may consist of finitely many connected components with differing dimension), and satisfy:

(i) Each S_i is locally closed in X;
(ii) $X = \bigcup_{i \in I} S_i$;
(iii) $S_j \cap \overline{S_i} \neq \emptyset \iff S_j \subseteq \overline{S_i} \iff j \leq i$.

The third condition is called CONDITION OF THE FRONTIER. The manifolds S_i are called STRATA OR PIECES. In the case that $j < i$ one often writes $S_j < S_i$ and calls S_j INCIDENT to S_i or says S_j is a BOUNDARY PIECE of S_i.

We define the dimension of a manifold consisting of finitely many connected components to be the maximum of the dimensions of the manifold's components.
The DIMENSION of the decomposed space X is defined as

$$\dim X := \sup_{i \in I} \dim S_i$$

and we will only be concerned with spaces where this supremum is attained.
The DEPTH OF THE STRATUM S_i of the decomposed space X is defined as

$$\operatorname{depth} S_i := \sup\{l \in \mathbb{N} : \text{ there are strata } S_{i_0} = S_i, S_{i_1}, \ldots, S_{i_l} \\ \text{such that } S_{i_0} < \ldots < S_{i_l}\},$$

Notice that $\operatorname{depth} S_i$ is always finite; indeed, else there would be an infinite family $(S_j)_{j \in J}$ with $S_j > S_i$ thus making any neighborhood of any point in S_i meet all of the S_j which contradicts local finiteness of the decomposition. The DEPTH of X is

$$\operatorname{depth} X := \sup\{\operatorname{depth} S_i : i \in I\}.$$

Thus, if X consists of one just stratum then $\operatorname{depth} X = 0$. From the frontier condition we have that $\operatorname{depth} S_i \leq \dim X - \dim S_i$, and also $\operatorname{depth} X \leq \dim X$.

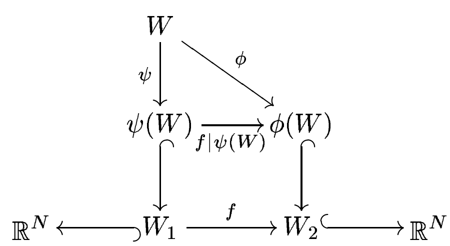
A simple example for a decomposed space is a manifold with boundary with big stratum the interior and small stratum the boundary. Also manifolds with corners are decomposed spaces in the obvious way. Likewise the cone $CM := (M \times [0,\infty))/(M \times \{0\})$ over a manifold M is a decomposed space, the partition being that into cusp and cylinder $M \times (0,\infty)$.

The following definition of singular charts and smooth structures on singular spaces is due to Pflaum [30, Section 2].

Definition 2.8 (Singular charts). Let $X = \bigcup_{i\in I} S_i$ be a decomposed space. A SINGULAR CHART (U,ψ) with patch U an open subset of X is to satisfy the following.

(i) $\psi(U)$ is locally closed in $\mathbb{R}^n$;
(ii) $\psi : U \to \psi(U)$ is a homeomorphism;
(iii) For every stratum S_i that meets U the restriction $\psi|S_i\cap U : S_i\cap U \to \psi(S_i\cap U)$ is a diffeomorphism onto a smooth sub-manifold of $\mathbb{R}^n$.

Two singular charts $\psi : U \to \mathbb{R}^n$ and $\phi : V \to \mathbb{R}^m$ are called COMPATIBLE AT $x \in U\cap V$ if there is an open neighborhood W of x in $U\cap V$, a number $N \geq \max\{n,m\}$, and a diffeomorphism $f : W_1 \to W_2$ between open subsets of $\mathbb{R}^N$ such that:



It follows that $f|\psi(W) : \psi(W) \to \phi(W)$ is a homeomorphism. Further, for all strata S that meet W the restriction $f|\psi(W\cap S) : \psi(W\cap S) \to \phi(W\cap S)$ is a diffeomorphism of sub-manifolds of $\mathbb{R}^N$. The charts (U,ψ) and (V,ϕ) are called COMPATIBLE if they are so at every point of the intersection $U\cap V$. It is straightforward to check that compatibility of charts defines an equivalence relation.

A family of compatible singular charts on X such that the union of patches covers all of X is called a SINGULAR ATLAS. Two singular atlases are said to be compatible if all charts of the first are compatible with all charts of the second. Again it is clear that compatibility of atlases forms an equivalence relation.

Let $\mathfrak{A}$ be a singular atlas on X. Then we can consider the family of all singular charts that belong to some atlas compatible with $\mathfrak{A}$ to obtain a maximal atlas $\mathfrak{A}_{\max}$.

Definition 2.9 (Smooth structure). Let $X = \bigcup_{i\in I} S_i$ be a decomposed space. A maximal atlas $\mathfrak{A}$ on X is called a SMOOTH STRUCTURE on the singular space X. A continuous function $f : X \to \mathbb{R}$ is said to be SMOOTH if the following holds. For all charts $\psi : U \to \mathbb{R}^n$ of the atlas $\mathfrak{A}$ there is a smooth function $F : \mathbb{R}^n \to \mathbb{R}$ such that $f|U = F \circ \psi$. The set of all smooth functions on X is denoted by $C^\infty(X)$.

A continuous map $f : X \to Y$ between decomposed spaces with smooth structures is called SMOOTH if $f^*C^\infty(Y) \subseteq C^\infty(X)$. An ISOMORPHISM $F : X \to Y$ between decomposed spaces is a homeomorphism that is smooth in both directions and maps strata of X diffeomorphically onto strata of Y.

The smooth structure thus defined on decomposed spaces is in no way intrinsic but is a structure that is additionally defined to do analysis on decomposed spaces. Also a smooth map $f : X \to Y$ between decomposed spaces need not at all be strata preserving.

Definition 2.10 (Cone space)**.** A decomposed space $X = \bigcup_{i\in I} S_i$ is called a CONE SPACE if the following is true. Let $x_0 \in X$ arbitrary and S the stratum passing through x_0. Then there is an open neighborhood U of x_0 in X, there is a decomposed space L with global chart $\psi : L \to S^{l-1} \subseteq \mathbb{R}^l$, and furthermore there is an isomorphism of decomposed spaces

$$F : U \to (U \cap S) \times CL$$

such that $F(x) = (x, c)$ for all $x \in U \cap S$. Here $CL = (L \times [0, \infty))/(L \times \{0\})$ is decomposed into the cusp c on the one hand, while the other pieces are of the form stratum of L times $(0, \infty)$. Thus we can take $\Psi : CL \to \mathbb{R}^l$, $[(z, t)] \mapsto t\psi(z)$ as a global chart on CL thereby defining a smooth structure on CL whence also on the product $(U \cap S) \times CL$.

The space L is called a LINK, and the chart F is referred to as a CONE CHART or also LINK CHART. Of course, the link L depends on the chosen point $x_0 \in X$.

An example for a cone space is the quadrant $Q := \{(x, y) \in \mathbb{R}^2 : x \geq 0 \text{ and } y \geq 0\}$. A typical neighborhood of $0 \in Q$ is of the form $\{(x, y) : 0 \leq x < r \text{ and } 0 \leq y < r\}$. The link with respect to the point 0 then is the arc $L := \{(\cos\varphi, \sin\varphi) : 0 \leq \varphi \leq \frac{\pi}{2}\}$. More generally manifolds with corners carry the structure of cone spaces.

Definition 2.11 (Stratified spaces)**.** Let $X \subseteq \mathbb{R}^m$ be a subset and assume that X is a decomposed space, i.e. $X = \bigcup_{i\in I} S_i$, and that the strata S_i be sub-manifolds of $\mathbb{R}^m$. The I-decomposed space X is said to be (WHITNEY) STRATIFIED if all pairs (S_i, S_j) with $i > j$ satisfy condition (b) – see Definition 2.4. For sake of convenience we will simply say stratified instead of Whitney stratified.

Theorem 2.12. *Let $X \subseteq \mathbb{R}^m$ be a subset of a Euclidean space and assume that $X = \bigcup_{i\in I} S_i$ is decomposed. Then X is stratified if and only if X is a cone space.*

Proof. It is proved in Pflaum [31] that every (Whitney) stratified space is also a cone space.

An outline of the converse direction is given in Sjamaar and Lerman [35, Section 6], and also in Goresky and MacPherson [13, Section 1.4]. This argument makes use of Mather's control theory as introduced in Mather [19] as well as Thom's First Isotopy Lemma. □

The above theorem depends crucially on the fact that the decomposed space X can be regarded as a subspace of some Euclidean space. As this assumption will always be satisfied in the present context we will take the words cone space and stratified space to be synonymous. In fact, Sjamaar and Lerman [35] take cone space to be the definition of stratified space.

Example 2.13. As an example consider a compact Lie group K acting by isometries on a smooth Riemannian manifold M. We are concerned with the orbit projection $\pi : M \to M/K$ and endow the orbit space with the final topology with respect to the projection map. For notation and basics on compact transformation groups see Section 7. Fix a point $x_0 \in M$ with isotropy group $K_{x_0} = H$. The slice representation is then the action by H on $\mathrm{Nor}_{x_0}(K.x_0)$. By the Tube Theorem there is a K-invariant open neighborhood U of the orbit $K.x_0$ such that $K \times_H V \cong U$ as smooth K-spaces where V is an H-invariant open neighborhood of 0 in $\mathrm{Nor}_{x_0}(K.x_0)$ – see Section 7.

Now let $p = (p_1, \ldots, p_k)$ be a Hilbert basis for the algebra $\mathrm{Poly}(V)^H$ of H-invariant polynomials on V. That is, $p_1, \ldots, p_k$ is a finite system of generators for $\mathrm{Poly}(V)^H$. The Theorem of Schwarz [34, Theorem 1] now says that $p^* : C^\infty(\mathbb{R}^k) \to C^\infty(V)^H$ is surjective. Moreover, the induced mapping $q : V/H \to \mathbb{R}^k$ is continuous, injective, and proper. See also Michor [20].

As in Section 7 consider the isotropy type sub-manifolds $M_{(H)}$. These give a K-invariant decomposition of M as $M = \bigcup_{(H)} M_{(H)}$ where (H) runs through the isotropy lattice of the K-action on M. We thus get a decomposition of the orbit space

$$M/K = \bigcup_{(H)} M_{(H)}/K$$

where again (H) runs through the isotropy lattice of the K-action on M. By the results of Section 7 this decomposition clearly renders M/K a decomposed space.

Now a theorem of Pflaum [30, Theorem 5.9] says that the induced mapping $\psi : U/K \to \mathbb{R}^k$ as defined in the diagram

$$\begin{array}{ccccc} U & \xrightarrow{\simeq} & K \times_H V & & \\ \downarrow & & \downarrow & & \\ U/K & \xrightarrow[\phi]{\simeq} & V/H & \twoheadleftarrow & V \\ & \overset{\psi}{\searrow} & \downarrow q & \swarrow p & \\ & & \mathbb{R}^k & & \end{array}$$

is a typical singular chart around the point $K.x_0$ in the orbit space. Furthermore, the smooth functions with respect to this smooth structure are $C^\infty(M/K) = C^\infty(M)^K$, i.e. none other than the K-invariant smooth functions on M: indeed, by Schwarz' Theorem we have

$$\psi^* C^\infty(\mathbb{R}^k) = \phi^* q^* C^\infty(\mathbb{R}^k) = \phi^* C^\infty(V)^H = C^\infty(U)^K$$

whence $C^\infty(U)^K = C^\infty(U/K)$. Finally, the decomposition of M/K by orbit types turns the orbit space into a stratified space with smooth structure. □

Lemma 2.14. *The quotient smooth structure $C^\infty(M/K) = C^\infty(M)^K$ separates points in M/K.*

Proof. Let $\mathcal{O}_1$ and $\mathcal{O}_2$ be arbitrary orbits of the K-action. Consider $X = \mathcal{O}_1 \cup \mathcal{O}_2$, and define $f : X \to \mathbb{R}$ to be 1 on $\mathcal{O}_1$ and 2 on $\mathcal{O}_2$. Then f may be extended to a smooth function on M by the Tietze-Urysohn Lemma. Thus it may be extended K-invariantly by averaging since K was assumed compact. □

3. Singular symplectic reduction

Let (M, ω) be a connected symplectic manifold, and K a compact Lie group that acts on (M, ω) in a Hamiltonian fashion such that there is an equivariant momentum map $J : M \to \mathfrak{k}^*$.

The very strong machinery of singular symplectic reduction is due to Sjamaar and Lerman [35] who prove that the singular symplectic quotient is a Whitney stratified space that has symplectic manifolds as its strata. This result which is the Singular Reduction Theorem was then generalized to the case of proper actions by Bates and Lerman [6], Ortega and Ratiu [24], and also others.

Theorem 3.1 (Singular symplectic reduction)**.** *Let (H) be in the isotropy lattice of the K-action on M, and suppose that $J^{-1}(\mathcal{O}) \cap M_{(H)} \neq \emptyset$ for a coadjoint orbit $\mathcal{O} \subseteq \mathfrak{k}^*$. Then the following are true.*

- *The subset $J^{-1}(\mathcal{O}) \cap M_{(H)}$ is an initial sub-manifold of M.*

- *The topological quotient $J^{-1}(\mathcal{O}) \cap M_{(H)}/K$ has a unique smooth structure such that the projection map*
$$J^{-1}(\mathcal{O}) \cap M_{(H)} \xrightarrow{\pi} J^{-1}(\mathcal{O}) \cap M_{(H)}/K$$
is a smooth surjective submersion.
- *Let $\iota : J^{-1}(\mathcal{O}) \cap M_{(H)} \hookrightarrow M$ denote the inclusion mapping. Then $J^{-1}(\mathcal{O}) \cap M_{(H)}/K$ carries a symplectic structure ω_0 which is uniquely characterized by the formula*
$$\pi^*\omega_0 = \iota^*\omega - (J|(J^{-1}(\mathcal{O}) \cap M_H))^*\Omega^{\mathcal{O}}$$
where $\Omega^{\mathcal{O}}$ is the canonical (positive Kirillov-Kostant-Souriau) symplectic form on $\mathcal{O}$.
- *Consider a K-invariant function $H \in C^\infty(M)^K$. Then the flow to the Hamiltonian vector field ∇^ω_H leaves the connected components of $J^{-1}(\mathcal{O}) \cap M_{(H)}$ invariant. Moreover, H factors to a smooth function h on the quotient $J^{-1}(\mathcal{O}) \cap M_{(H)}/K$. Finally, ∇^ω_H and the Hamiltonian vector field to h are related via the canonical projection π, whence the flow of the former projects to the flow of the latter.*
- *The collection of all strata of the form $J^{-1}(\mathcal{O}) \cap M_{(H)}/K$ constitutes a Whitney stratification of the topological space $J^{-1}(\mathcal{O})/K$.*

Proof. This theorem is the content of Ortega and Ratiu [24, Section 8]. See also Bates and Lerman [6, Corollary 14] □

As a matter of convention we write shorthand $M/\!/_{\mathcal{O}}K := J^{-1}(\mathcal{O})/K$ for the reduced space of M with respect to the Hamiltonian action by K. If $\mathcal{O}$ is the coadjoint orbit passing through α then we shall also abbreviate $J^{-1}(\alpha)/K_\alpha = M/\!/_\alpha K = M/\!/_{\mathcal{O}}K$.

Theorem 3.2 (Singular commuting reduction)**.** *Let G and K be compact Lie groups that act by symplectomorphisms on (M,ω) with momentum maps J_G and J_K respectively. Assume that the actions commute, that J_G is K-invariant, and that J_K is G-invariant. Let $\alpha \in \mathfrak{g}^*$ be in the image of J_G and $\beta \in \mathfrak{k}^*$ in the image of J_K.*
Then the G action drops to a Poisson action on $M/\!/_\beta K$ and J_G factors to a momentum map j_G for the induced action. Likewise, the K action drops to a Poisson action on $M/\!/_\alpha G$ and J_K factors to a momentum map j_K for the induced action. Furthermore, we have
$$(M/\!/_\alpha G)/\!/_\beta K \cong M/\!/_{(\alpha,\beta)}(G \times K) \cong (M/\!/_\beta K)/\!/_\alpha G$$
as symplectically stratified spaces.

Proof. This is proved and discussed (in greater generality) in [16, Section 10.4]. □

4. The bundle picture

Let us introduce the basic assumptions of the paper. That is Q is a Riemannian manifold, K is a compact Lie group which acts on Q by isometries. The K action then induces a Hamiltonian action on the cotangent bundle T^*Q by cotangent lifts. This means that the lifted action respects the canonical symplectic form $\Omega = -d\theta$ on T^*Q where θ is the Liouville form on T^*Q, and, moreover, there is a momentum map $\mu : T^*Q \to \mathfrak{k}^*$ given by $\langle \mu(q,p), X\rangle = \theta(\zeta_X^{T^*Q})(q,p) = \langle p, \zeta_X(q)\rangle$ where $(q,p) \in T^*Q$, $X \in \mathfrak{k}$, ζ_X is the fundamental vector field associated to the

K-action on Q, and $\zeta_X^{T^*Q} \in \mathfrak{X}(T^*Q)$ is the fundamental vector field associated to the cotangent lifted action.

In this section we want to apply the slice theorem of Section 7 to the action of K on Q to get a local model of the singularly symplectic reduced space $T^*Q//_{\mathcal{O}}K = \mu^{-1}(\mathcal{O})/K$ where $\mathcal{O}$ is a coadjoint orbit in the image of μ.

Thus we consider a tube U in Q around an orbit $K.q$ with $K_q = H$. And we denote the slice at q by S such that

$$U \cong K \times_H S.$$

In particular it follows that $U/K \cong S/H$.

Assume for a moment that the action by K on U is free, that is $U \cong K \times S$. Let $\mu : T^*U \to \mathfrak{k}^*$ be the canonical momentum mapping, $\lambda \in \mathfrak{k}^*$ a regular value in the image of μ, and $\mathcal{O}$ the coadjoint orbit passing through λ. Then we have

$$\begin{aligned}(T^*U)//_{\mathcal{O}}K = (T^*U)//_{\lambda}K &= (T^*K \times T^*S)//_{\lambda}K \\ &= (T^*K)//_{\lambda}K \times T^*S \\ &= \mathcal{O} \times T^*(U/K)\end{aligned}$$

as symplectic spaces; since $T^*K//_{\lambda}K = \mathcal{O}$. The aim of this section is to drop the freeness assumption. To do so we will take the same approach as Schmah [33] and use singular commuting reduction.

Now we return to the case where $U = K \times_H S$ as introduced above. On $K \times S$ we will be concerned with two commuting actions. These are

$$\begin{aligned}\lambda &: K \times K \times S \longrightarrow K \times S, \quad \lambda_g(k,s) = (gk, s) \\ \tau &: H \times K \times S \longrightarrow K \times S, \quad \tau_h(k,s) = (kh^{-1}, h.s).\end{aligned}$$

These actions obviously commute. The latter, i.e. τ is called the twisted action by H on $K \times S$. We can cotangent lift λ and τ to give Hamiltonian transformations on $T^*(K \times S)$ with momentum mappings J^λ and J^τ, respectively. By left translation we trivialize $T^*(K \times S) = (K \times \mathfrak{k}^*) \times T^*S$.

To facilitate the notation we will denote the cotangent lifted action of λ, τ again by λ, τ respectively.

Lemma 4.1. *Let $(k, \eta; s, p) \in K \times \mathfrak{k}^* \times T^*S$. Then we have the following formulas*

$$\begin{aligned}J^\lambda(k,\eta;s,p) &= \mathrm{Ad}(k^{-1})^*.\eta =: \mathrm{Ad}^*(k).\eta \in \mathfrak{k}^*, \\ J^\tau(k,\eta;s,p) &= -\eta|\mathfrak{h} + \mu(s,p) \in \mathfrak{h}^*\end{aligned}$$

where μ is the canonical momentum map on T^*S. Moreover, the actions λ and τ commute, and J^λ is H-invariant and J^τ is K-invariant.

Since the canonical momentum map on T^*S is the same as that on T^*U restricted to T^*S the use of the symbol μ for both these maps is unambiguous.

The signs in the formulas for the momentum mappings depend on the choice of sign in the definition of the fundamental vector field map as defined in Section 7.

Proof. We denote the left action by K on itself by L, the right action by R, and the conjugate action by conj. In this notation we then have $\mathrm{Ad}(k).X = T_e\mathrm{conj}_k.X$, and $\mathrm{conj}_k = L_k \circ R^{k^{-1}} = R^{k^{-1}} \circ L_k$. Notice the choice of sign in the definition of the fundamental vector field associated to left actions in Section 7. For right actions we need to choose the opposite sign. It is straightforward to verify that the cotangent lifted actions of L and R on $T^*K = K \times \mathfrak{k}^*$ are given by

$$\begin{aligned}T^*L_g(k,\eta) &= (gk, -\eta) = (gk, \eta \circ \zeta^R(g)) \\ T^*R^g(k,\eta) &= (kg, \mathrm{Ad}(g^{-1})^*.\eta) = (kg, \eta \circ \zeta^L(g))\end{aligned}$$

where ζ^L and ζ^R denote the fundamental vector field mappings associated to L and R respectively.

Thus $\langle J^\lambda(k,\eta;s,p), X\rangle = \langle \eta, \zeta^L_X(k)\rangle = \langle \mathrm{Ad}^*(k).\eta, X\rangle$ for all $X \in \mathfrak{k}$ which shows the first claim. Also, it follows that $\langle J^\tau(k,\eta;s,p), Z\rangle = \langle -\eta, Z\rangle + \langle p, \zeta_X(s)\rangle$ for all $Z \in \mathfrak{h}$. The invariance of J^λ and J^τ is immediate from the formulas of the trivialized cotangent lifted actions. □

Corollary 4.2. *Let $\alpha \in \mathfrak{k}^*$ and $\beta \in \mathfrak{h}^*$ such that α, β is in the image of J^λ, J^τ respectively. Then the following are true.*

(i) *The action λ descends to a Hamiltonian action on the Marsden-Weinstein reduced space $T^*(K \times S)/\!/_\beta H$. Moreover, J^λ factors to a momentum map $j_\lambda : T^*(K \times S)/\!/_\beta H \to \mathfrak{k}^*$ for this action.*

(ii) *The action τ descends to a Hamiltonian action on the Marsden-Weinstein reduced space $T^*(K \times S)/\!/_\alpha K$. Moreover, J^τ factors to a momentum map $j_\tau : T^*(K \times S)/\!/_\alpha K \to \mathfrak{h}^*$ for this action.*

(iii) *The product action $K \times H \times T^*(K \times S) \to T^*(K \times S)$, $(k,h,u) \mapsto \lambda_k.\tau_h.u$ is Hamiltonian with momentum map (J^λ, J^τ). Moreover,*
$$\begin{aligned}(T^*(K \times S)/\!/_\alpha K)/\!/_\beta H &= T^*(K \times S)/\!/_{(\alpha,\beta)}(K \times H)\\ &= (T^*(K \times S)/\!/_\beta H)/\!/_\alpha K\end{aligned}$$
as singular symplectic spaces.

Proof. Since the actions by λ and τ are free the first two assertions can be deduced from the regular commuting reduction theorem with the necessary conditions being verified in the above lemma. Clearly, the product action by $K \times H$ is well-defined and Hamiltonian with asserted momentum map. However, the product action will not be free in general. Thus the last point is a consequence of the singular commuting reduction theorem of Section 3. □

We will only be interested in the case where $\beta = 0$. There are more than one Hamiltonian cotangent lifted actions on T^*K. However, when it comes to reduction we will be only concerned with the lifted action λ. Thus the expression $T^*K/\!/_\alpha K$ unambiguously stands for $(J^\lambda)^{-1}(\alpha)/K_\alpha$.

Proposition 4.3. *Clearly,* 0 *is in the image of J^τ. Therefore,*
$$\begin{aligned}T^*U/\!/_\alpha K &\cong T^*(K \times_H S)/\!/_\alpha K\\ &= T^*(K \times S)/\!/_0 H/\!/_\alpha K\\ &= (T^*K/\!/_\alpha K \times T^*S)/\!/_0 H\\ &= (\mathcal{O} \times T^*S)/\!/_0 H\end{aligned}$$
as stratified symplectic spaces, and where $\mathcal{O} = \mathrm{Ad}^*(K).\alpha$.

Proof. Since the isomorphism $T^*U \xrightarrow{\cong} T^*(K \times_H S)$ comes from an equivariant diffeomorphism $U \xrightarrow{\cong} K \times_H S$ on the base it is an equivariant symplectomorphism that intertwines the respective momentum maps. Now the regular reduction theorem for cotangent bundles at zero momentum says that $T^*(K \times_H S)$ and $T^*(K \times S)/\!/_0 H$ are symplectomorphic. Further it is well-known that $T^*K/\!/_\alpha K = \mathcal{O}$. The rest is a direct consequence of Theorem 3.2 on singular commuting reduction. □

We continue to assume that K acts on the Riemannian manifold Q by isometries. But now we also make the rather strong assumption that
$$Q = Q_{(H)}$$
i.e. all isotropy subgroups of points $q \in Q$ are conjugate within K to H.

Theorem 4.4 (Bundle picture). *Let $Q = Q_H$ and let $\mathcal{O} \subseteq \mathfrak{k}^*$ be a coadjoint orbit in the image of the momentum map $\mu : T^*Q \to \mathfrak{k}^*$. Then we have a symplectic fiber bundle*

$$\mathcal{O}//_0 H \hookrightarrow T^*Q//_{\mathcal{O}} K \longrightarrow T^*(Q/K)$$

with stratified typical fiber $\mathcal{O}//_0 H$ and smooth base $T^*(Q/K)$.

This theorem is to say that the singularities of the reduced phase space are confined to the fiber direction which also will be referred to as the spin direction.

Proof. This follows from the above in the following way. Consider a tube U of the K-action on Q. Then the slice theorem tells us that there is a slice S such that there is a K-equivariant diffeomorphism

$$U \cong K \times_H S = K/H \times S$$

since all points of Q by assumption are regular whence the slice representation is trivial. We can lift this diffeomorphism to a symplectomorphism of cotangent bundles to get

$$T^*U//_{\mathcal{O}} K \cong \mathcal{O}//_0 H \times T^*S$$

as in Proposition 4.3 above. Since T^*S is a typical neighborhood in $T^*(Q/K)$ the result follows. □

5. Gauged cotangent bundle reduction

For this section we continue with the basic assumptions of the paper. That is Q is a Riemannian manifold, K is a compact Lie group which acts on Q by isometries. Moreover, Q is supposed to be of single isotropy type, i.e. $Q = Q_{(H)}$ were H is an isotropy subgroup of K. The K action then induces a Hamiltonian action on the cotangent bundle T^*Q by cotangent lifts. This means that the lifted action respects the canonical symplectic form $\Omega = -d\theta$ on T^*Q and there is a momentum map $\mu : T^*Q \to \mathfrak{k}^*$ given by $\langle \mu(q,p), X\rangle = \theta(\zeta_X^{T^*Q})(q,p) = \langle p, \zeta_X(q)\rangle$ where $(q,p) \in T^*Q$, $X \in \mathfrak{k}$, ζ_X is the fundamental vector field associated to the K-action on Q, and $\zeta_X^{T^*Q} \in \mathfrak{X}(T^*Q)$ is the fundamental vector field associated to the cotangent lifted action.

Since the K action on Q has only a single isotropy type the orbit space Q/K is a smooth manifold, and the projection $\pi : Q \to Q/K$ is a surjective Riemannian submersion with compact fibers. However, the lifted action by K on T^*Q is already much more complicated, and the quotient space $(T^*Q)/K$ is only a stratified space in general. Its strata are of the form $(T^*Q)_{(L)}/K$ where (L) is in the isotropy lattice of T^*Q.

The vertical sub-bundle of TQ with respect to $\pi : Q \to Q/K$ is $\mathrm{Ver} := \ker T\pi$. Via the K-invariant Riemannian metric we obtain the horizontal sub-bundle as $\mathrm{Hor} := \mathrm{Ver}^\perp$. We define the dual horizontal sub-bundle of T^*Q as the sub-bundle Hor^* consisting of those co-vectors that vanish on all vertical vectors. Likewise, we define the dual vertical sub-bundle of T^*Q as the sub-bundle Ver^* consisting of those co-vectors that vanish on all horizontal vectors.

For $X, Y \in \mathfrak{k}$ and $q \in Q$ we define $\mathbb{I}_q(X,Y) := \langle \zeta_X(q), \zeta_Y(q)\rangle$ and call this the INERTIA TENSOR. This gives a non-degenerate pairing on $\mathfrak{k}_q^\perp \times \mathfrak{k}_q^\perp$, whence it gives an identification $\check{\mathbb{I}}_q : \mathfrak{k}_q^\perp \to (\mathfrak{k}_q^\perp)^* = \mathrm{Ann}\,\mathfrak{k}_q$. We use this isomorphism to define a one-form on Q with values in the bundle $\bigsqcup_{q\in Q}\mathfrak{k}_q$ by the following:

$$\begin{array}{ccc} T_q^*Q & \xrightarrow{\mu_q} & \operatorname{Ann}\mathfrak{k}_q \\ \simeq\big\uparrow & & \big\downarrow(\check{\mathbb{I}}_q)^{-1} \\ T_qQ & \overset{A_q}{\dashrightarrow} & \mathfrak{k}_q^{\perp} \end{array}$$

See Smale [36] or Marsden, Montgomery, and Ratiu [17, Section 2]. The form A shall be called the MECHANICAL CONNECTION on $Q \to Q/K$. It has the following properties. It follows from its definition that $TQ \to \bigsqcup_{q\in Q}\mathfrak{k}_q^{\perp}$, $(q,v)\mapsto(q,A_q(v))$ is equivariant, $\ker A_q = T_q(K.q)^{\perp}$, and $A_q(\zeta_X(q)) = X$ for all $X\in\mathfrak{k}_q^{\perp}$.

This means that $A : TQ \to \mathfrak{k}_q^{\perp} \hookrightarrow \mathfrak{k}$, $(q,v)\mapsto A_q(v)$ is a principal connection form on the K-manifold Q in the sense of Alekseevsky and Michor [3, Section 3.1]. According to [3, Section 4.6] the curvature form associated to A is defined by

$$\operatorname{Curv}^A := dA + \tfrac{1}{2}[A,A]^{\wedge}$$

where

$$[\varphi,\psi]^{\wedge} := \tfrac{1}{k!l!}\sum_{\sigma}\operatorname{sign}\sigma[\varphi(v_{\sigma 1},\dots,v_{\sigma l}),\psi(v_{\sigma(l+1)},\dots,v_{\sigma(l+k)})]$$

is the graded Lie bracket on $\Omega(Q;\mathfrak{k}) := \bigoplus_{k=0}^{\infty}\Gamma(\Lambda^kT^*Q\otimes\mathfrak{k})$, and $\varphi\in\Omega^l(Q;\mathfrak{k})$ and $\psi\in\Omega^k(Q;\mathfrak{k})$.

We define a point-wise dual $A_q^* : \operatorname{Ann}\mathfrak{k}_q \to \operatorname{Ver}_q^* \subseteq T_q^*Q$ by the formula $A_q^*(\lambda)(v) = \lambda(A_q(v))$ where $\lambda\in\operatorname{Ann}\mathfrak{k}_q$ and $v\in T_qQ$. Notice that $A_q^*(\mu_q(p)) = p$ for all $p\in\operatorname{Ver}_q^*$ and $\mu_q(A_q^*(\lambda)) = \lambda$ for all $\lambda\in\operatorname{Ann}\mathfrak{k}_q$.

Using the horizontal lift mapping which identifies $\operatorname{Hor}\cong(Q\times_{Q/K}T(Q/K))$ on the one hand and the mechanical connection A on the other hand we obtain an isomorphism

$$TQ = \operatorname{Hor}\oplus\operatorname{Ver} \longrightarrow (Q\times_{Q/K}T(Q/K))\times_Q\bigsqcup\nolimits_{q\in Q}\mathfrak{k}_q^{\perp}$$

of bundles over Q. Via the Riemannian structure there is a dual version to this isomorphism, and to save on typing we will abbreviate

$$\mathcal{W} := (Q\times_{Q/K}T^*(Q/K))\times_Q\bigsqcup\nolimits_{q\in Q}\operatorname{Ann}\mathfrak{k}_q \cong \operatorname{Hor}^*\oplus\operatorname{Ver}^*.$$

To set up some notation for the upcoming proposition, and clarify the picture consider the following stacking of pull-back diagrams.

$$\begin{array}{ccc} \mathcal{W} & \xrightarrow{\rho^*\tilde{\tau}=\tilde{\tilde{\tau}}} & \bigsqcup_q\operatorname{Ann}\mathfrak{k}_q \\ {\scriptstyle \tilde{\tau}^*\rho=\tilde{\rho}}\big\downarrow & & \big\downarrow{\scriptstyle\rho} \\ Q\times_{Q/K}T^*(Q/K) & \xrightarrow{\pi^*\tau=\tilde{\tau}} & Q \\ {\scriptstyle \tau^*\pi=\tilde{\pi}}\big\downarrow & & \big\downarrow{\scriptstyle\pi} \\ T^*(Q/K) & \xrightarrow{\tau} & Q/K \end{array}$$

The upper stars in this diagram are, of course, not pull-back stars. It is in fact the transition functions that are being pulled-back, whence the name.

Proposition 5.1 (Symplectic structure on $\mathcal{W}$). *There is a dual isomorphism*

$$\begin{aligned} \psi = \psi(A) : (Q\times_{Q/K}T^*(Q/K))\times_Q\bigsqcup\nolimits_{q\in Q}\operatorname{Ann}\mathfrak{k}_q = \mathcal{W} &\longrightarrow T^*Q, \\ (q,\eta,\lambda) &\longmapsto (q,\eta + A(q)^*\lambda) \end{aligned}$$

where we identify elements in $\{q\}\times T^*_{[q]}(Q/K)$ *with elements in* Hor^*_q *via the dual of the inverse of the horizontal lift.*

This isomorphism can be used to induce a symplectic form on the connection dependent realization of T^*Q*, namely* $\sigma=\psi^*\Omega$ *where* $\Omega=-d\theta$ *is the canonical form on* T^*Q*. Moreover, there is an explicit formula for* σ *in terms of the chosen connection:*

$$\sigma=(\widetilde{\pi}\circ\widetilde{\rho})^*\Omega^{Q/K}-d\widetilde{\widetilde{\tau}}^*B$$

where $\Omega^{Q/K}$ *is the canonical symplectic form on* $T^*(Q/K)$*, and furthermore* $B\in\Omega^1(\bigsqcup_q \mathrm{Ann}\,\mathfrak{k}_q)$ *is given by*

$$B_{(q,\lambda)}(v_1,\lambda_1)=\langle\lambda,A_q(v_1)\rangle.$$

The explicit formula now is

$$\begin{aligned}(dB)_{(q,\lambda)}&((v_1,\lambda_1),(v_2,\lambda_2))\\&=\langle\lambda,\mathrm{Curv}^A_q(v_1,v_2)\rangle+\langle\lambda,[Z_1,Z_2]\rangle-\langle\lambda_2,Z_1\rangle+\langle\lambda_1,Z_2\rangle\end{aligned}$$

where $(q,\lambda)\in\bigsqcup_q\mathrm{Ann}\,\mathfrak{k}_q$*,* $(v_i,\lambda_i)\in T_{(q,\lambda)}(\bigsqcup_q\mathrm{Ann}\,\mathfrak{k}_q)$ *for* $i=1,2$*, and*

$$v_i=\zeta_{Z_i}(q)\oplus v_i^{\mathrm{hor}}\in\mathrm{Ver}_q\oplus\mathrm{Hor}_q$$

is the decomposition into vertical and horizontal part with $Z_i\in\mathfrak{k}$*.*

Furthermore, there clearly is an induced action by K *on* $\mathcal{W}$*. This action is Hamiltonian with momentum mapping*

$$\mu_A=\mu\circ\psi:\mathcal{W}\longrightarrow\mathfrak{k}^*,(q,\eta,\lambda)\longmapsto\lambda,$$

where μ *is the momentum map* $T^*Q\to\mathfrak{k}^*$*, and* ψ *is equivariant.*

Proof. Clearly, the isomorphism ψ does induce a symplectic form $\sigma=\psi^*\Omega$ on $\mathcal{W}$, and it only remains to verify the asserted formula. Let

$$w=(q;[q],\eta;q,\lambda)=(q,\eta,\lambda)\in\mathcal{W},$$

and $\xi_i\in\mathfrak{X}(\mathcal{W})$ for $i=1,2$. We use the notation

$$\xi_i(w)=(v_i(q),\eta_i([q],\eta),\lambda_i(q,\lambda)).$$

That is, $v_i\in\mathfrak{X}(Q)$, $\eta_i\in\mathfrak{X}(T^*(Q/K))$, and $\lambda_i\in\mathfrak{X}(\bigsqcup_q\mathrm{Ann}\,\mathfrak{k}_q)$. By definition of pulling back of forms we have

$$\sigma_w(\xi_1,\xi_2)=\Omega_{(q,\eta+A^*_q(\lambda))}(T_w\psi.\xi_1(w),T_w\psi.\xi_2(w)).$$

Denoting the horizontal lift of $(\mathrm{Fl}^{v_i}_t(q),\mathrm{Fl}^{\eta_i}_t([q],\eta))$ simply by $\mathrm{Fl}^{\eta_i}_t(\eta)$, and considering $\lambda_i(q,\lambda)$ as an element of $\mathrm{Ann}\,\mathfrak{k}_q$ (effectively forgetting the Q-component which is just $v_i(q)$) we compute

$$\begin{aligned}T_w\psi.\xi_i(w)&=\tfrac{\partial}{\partial t}|_0\psi(\mathrm{Fl}^{\xi_i}_t(w))\\&=\tfrac{\partial}{\partial t}|_0(\mathrm{Fl}^{v_i}_t(q),\mathrm{Fl}^{\eta_i}_t(\eta)+A(\mathrm{Fl}^{v_i}_t(q))^*(\mathrm{Fl}^{\lambda_i}_t(q,\lambda)))\\&=(v_i(q),\eta_i(\eta)+A^*_q(\lambda_i(q,\lambda))+(\mathcal{L}_{v_i}A)^*_q(\lambda))\end{aligned}$$

where the last equality is true since:

$$\begin{aligned}\tfrac{\partial}{\partial t}|_0\langle A(\mathrm{Fl}^{v_i}_t(q))^*(\mathrm{Fl}^{\lambda_i}_t(q,\lambda)),X\rangle&=\tfrac{\partial}{\partial t}|_0\langle\mathrm{Fl}^{\lambda_i}_t(q,\lambda),A(\mathrm{Fl}^{v_i}_t(q))(X)\rangle\\&=\langle\lambda_i(q,\lambda),A_q(X)\rangle+\langle\lambda,(\mathcal{L}_{v_i}A)_q(X)\rangle\\&=\langle A^*_q(\lambda_i(q,\lambda))+(\mathcal{L}_{v_i}A)^*_q(\lambda),X\rangle\end{aligned}$$

for $X\in TQ$. Therefore,

$$\begin{aligned}\sigma_w(\xi_1,\xi_2)=\Omega_{(q,\eta+A^*_q(\lambda))}\big(&(v_1(q),\eta_1(\eta)+A^*_q(\lambda_1(q,\lambda))+(\mathcal{L}_{v_1}A)^*_q(\lambda)),\\&(v_2(q),\eta_2(\eta)+A^*_q(\lambda_2(q,\lambda))+(\mathcal{L}_{v_2}A)^*_q(\lambda))\big)\end{aligned}$$

$$\begin{aligned}
&= \langle \eta_2(\eta) + A_q^*(\lambda_2(q,\lambda)) + (\mathcal{L}_{v_2}A)_q^*(\lambda), v_1(q)\rangle \\
&\quad - \langle \eta_1(\eta) + A_q^*(\lambda_1(q,\lambda)) + (\mathcal{L}_{v_1}A)_q^*(\lambda), v_2(q)\rangle \\
&\quad + \langle \eta + A(q)^*(\lambda), [v_1, v_2](q)\rangle \\
&= \langle \eta_2(\eta), v_1^{\mathrm{hor}}(q)\rangle - \langle \eta_1(\eta), v_2^{\mathrm{hor}}(q)\rangle + \langle \eta, [v_1^{\mathrm{hor}}, v_2^{\mathrm{hor}}](q)\rangle \\
&\quad + \langle \lambda, (\mathcal{L}_{v_2}A)_q(v_1(q))\rangle - \langle \lambda, (\mathcal{L}_{v_1}A)_q(v_2(q))\rangle \\
&\quad + \langle \lambda, A(q)[v_1,v_2](q)\rangle + \langle \lambda_2(\lambda), A(q).v_1(q)\rangle - \langle \lambda_1(\lambda), A(q).v_2(q)\rangle \\
&= \Omega^{Q/K}_{([q],\eta)}((v_1^{\mathrm{hor}}, \eta_1), (v_2^{\mathrm{hor}}, \eta_2)) - \langle \lambda, \mathrm{Curv}_q^A(v_1,v_2)\rangle - \langle \lambda, [Z_1, Z_2]\rangle \\
&\quad + \langle \lambda_2(\lambda), Z_1\rangle - \langle \lambda_1(\lambda), Z_2\rangle
\end{aligned}$$

where $v_i(q) = v_i(q)^{\mathrm{hor}} \oplus \zeta_{Z_i}(q) \in \mathrm{Hor}_q \oplus \mathrm{Ver}_q$, and we used that $[\zeta_{Z_1}(q), \zeta_{Z_2}(q)] = -\zeta_{[Z_1,Z_2]}(q)$ – see Section 7. The curvature two-form is given by

$$\mathrm{Curv}_q^A(v_1,v_2) = dA_q(v_1,v_2) + A(q)[v_1,v_2](q) \in \mathfrak{k}.$$

For $B \in \Omega^1(\bigsqcup_q \mathrm{Ann}\,\mathfrak{k}_q)$ given by $B(q,\lambda)(v_1,\lambda_1) = \langle \lambda, A(q).v_1(q)\rangle$ we have

$$\begin{aligned}
&dB(q,\lambda)\big((v_1,\lambda_1),(v_2,\lambda_2)\big) \\
&= (\mathcal{L}_{(v_1,\lambda_1)}B)_{(q,\lambda)}(v_2,\lambda_2) - (\mathcal{L}_{(v_2,\lambda_2)}B)_{(q,\lambda)}(v_1,\lambda_1) - B(q,\lambda)([(v_1,\lambda_1),(v_2,\lambda_2)]) \\
&= \tfrac{\partial}{\partial t}|_0 B(\mathrm{Fl}_t^{(v_1,\lambda_1)}(q,\lambda))(v_2,\lambda_2) - \tfrac{\partial}{\partial t}|_0 B(\mathrm{Fl}_t^{(v_2,\lambda_2)}(q,\lambda))(v_1,\lambda_1) - \langle \lambda, A(q)[v_1,v_2](q)\rangle \\
&= \tfrac{\partial}{\partial t}|_0 \langle \mathrm{Fl}_t^{\lambda_1}(\lambda), A(\mathrm{Fl}_t^{v_1}(q))(v_2)\rangle - \tfrac{\partial}{\partial t}|_0 \langle \mathrm{Fl}_t^{\lambda_2}(\lambda), A(\mathrm{Fl}_t^{v_2}(q))(v_1)\rangle + \langle \lambda, [Z_1,Z_2]\rangle \\
&= \langle \lambda_1(\lambda), Z_2\rangle - \langle \lambda_2(\lambda), Z_1\rangle + \langle \lambda, (\mathcal{L}_{v_1}A)_q(v_2(q)) - (\mathcal{L}_{v_2}A)_q(v_1(q))\rangle + \langle \lambda, [Z_1,Z_2]\rangle.
\end{aligned}$$

Putting this together we find

$$\begin{aligned}
\sigma_w(\xi_1,\xi_2) &= \Omega^{Q/K}_{([q],\eta)}((v_1^{\mathrm{hor}},\eta_1),(v_2^{\mathrm{hor}},\eta_2)) - \big(\langle \lambda_1(\lambda), Z_2\rangle - \langle \lambda_2(\lambda), Z_1\rangle \\
&\quad + \langle \lambda, (\mathcal{L}_{v_1}A)_q(v_2(q))\rangle - \langle \lambda, (\mathcal{L}_{v_2}A)_q(v_1(q))\rangle + \langle \lambda, [Z_1,Z_2]\rangle\big) \\
&= \big((\widetilde{\pi}\circ\widetilde{\rho})^*\Omega^{Q/K} - d\widetilde{\widetilde{\tau}}^* B\big)_w(\xi_1,\xi_2)
\end{aligned}$$

which is the desired formula. Finally, the statement about the K action on $\mathcal{W}$ is obvious since ψ is equivariantly symplectomorphic by construction. ☐

Theorem 5.2 (Poisson structure on Weinstein space). *There are stratified isomorphisms of stratified bundles over Q/K:*

$$\begin{aligned}
\alpha = \alpha(A) : \textstyle\bigsqcup_{(L)}(TQ)_{(L)}/K &\longrightarrow T(Q/K) \times_{Q/K} \textstyle\bigsqcup_{(L)}(\bigsqcup_{q\in Q}\mathfrak{k}_q^{\perp})_{(L)}/K, \\
[(q,v)] &\longmapsto (T\pi(q,v), [(q, A_q v)])
\end{aligned}$$

where (L) runs through the isotropy lattice of TQ. The dual isomorphism is given by

$$\begin{aligned}
\beta = (\alpha^{-1})^* : (T^*Q)/K &\longrightarrow T^*(Q/K) \times_{Q/K} (\textstyle\bigsqcup_{q\in Q}\mathrm{Ann}\,\mathfrak{k}_q)/K =: W, \\
[(q,p)] &\longmapsto (C^*(q,p), [(q,\mu(q,p))])
\end{aligned}$$

where the stratification was suppressed. Here

$$C^* : T^*Q \to \mathrm{Hor}^* \to T^*(Q/K)$$

is constructed as the point wise dual to the horizontal lift mapping $C : T(Q/K)\times_{Q/K} Q \to \mathrm{Hor} \subseteq TQ$, $([q],v;q) \to C_q(v)$.

Moreover, β is an isomorphism of Poisson spaces as follows: we can naturally identify

$$\begin{aligned}
\mathcal{W}/K &\xrightarrow{\ \equiv\ } W, \\
[(q;[q],\eta;q,\lambda)] &\longmapsto ([q],\eta;[(q,\lambda)])
\end{aligned}$$

thus obtaining a quotient Poisson bracket on $C^\infty(W) = C^\infty(\mathcal{W})^K$ *as the quotient Poisson bracket.*

In the case that K acts on Q freely the first assertion of the above theorem can also be found in Cendra, Holm, Marsden, Ratiu [9]. Following Ortega and Ratiu [24, Section 6.6.12] the above constructed interpretation W of $(T^*Q)/K$ is called WEINSTEIN SPACE referring to Weinstein [40] where this universal construction first appeared.

Proof. As already noted above $(TQ)/K$ is a stratified space. Since the base Q is stratified as consisting only of a single stratum, the equivariant foot point projection map $\tau : TQ \to Q$ is trivially a stratified map. Thus, we really get a stratified bundle $(TQ)/K \to Q/K$. In the same spirit $(\bigsqcup_{q\in Q}\mathfrak{k}_q^\perp)/K$ is stratified into orbit types, and the projection onto Q/K is a stratified bundle map. According to Davis [11] pullbacks are well defined in the category of stratified spaces and thus it makes sense to define $T(Q/K) \times_{Q/K} (\bigsqcup_{q\in Q}\mathfrak{k}_q^\perp)/K$.
The map α is well defined: indeed, for $(q,v) \in TQ$ and $k \in K$ we have $T\pi(k.q, k.v) = (\pi(k.q), T_{k.q}\pi(T_q l_k(v))) = (\pi(q), T_q(\pi\circ l_k)(v)) = T_q\pi(v)$, and $[(k.q, A(k.q, k.v))] = [(q, A(q,v))]$ by equivariance of A. It is clearly continuous as a composition of continuous maps.
We claim that α maps strata onto strata, and moreover we have the formula

$$\alpha((TQ)_{(L)}/K) = T(Q/K) \times_{Q/K} (\textstyle\bigsqcup_{q\in Q}\mathfrak{k}_q^\perp)_{(L)}/K.$$

Indeed, consider $(q,v) \in (TQ)_{(L)}$, that is $H \cap K_v = L' \sim L$ where $H = K_q$. The notation $L' \sim L$ means that L' is conjugate to L within K. Now we can decompose v as $v = v_0 \oplus \zeta_X(q) \in \mathrm{Hor}_q \oplus T_q(K.q)$ for some appropriate $X \in \mathfrak{k}$. Since Q consists only of a single isotropy type we have $T_qQ = T_qQ_H + T_q(K.q)$ – which is not a direct sum decomposition. As usual, $Q_H = \{q \in Q : K_q = H\}$. This shows that $v_0 \in T_qQ_H$, and hence $H \subseteq K_{v_0}$. By equivariance of A it follows that

$$K_q \cap K_v = H \cap K_{v_0} \cap K_{\zeta_X(q)} = H \cap K_{\zeta_X(q)} = H \cap K_{A(q,v)}$$

which is independent of the horizontal component. Hence the claim. The restriction of α to any stratum clearly is smooth as a composition of smooth maps.
Since $A_q(\zeta_X(q)) = X$ for $X \in \mathfrak{k}_q^\perp$ we can write down an inverse as

$$\alpha^{-1} : ([q], v; [(q,X)]) \to [(q, C_q(v) + \zeta_X(q))]$$

and again it is an easy matter to notice that this map is well defined, continuous, and smooth on each stratum.
It makes sense to define the dual β of the inverse map α^{-1} in a point wise manner, and it only remains to compute this map.

$$\begin{aligned}
\langle \beta[(q,p)], ([q], v; [(q,X)])\rangle &= \langle [(q,p)], [(q, C_q(v) + \zeta_X(q))]\rangle \\
&= \langle p, C_q(v)\rangle + \langle p, \zeta_X(q)\rangle \\
&= \langle C^*(q,p), v\rangle + \langle \mu(q,p), X\rangle \\
&= \langle (C^*(q,p), [(q, \mu(q,p))]), ([q], v; [(q,X)])\rangle
\end{aligned}$$

where we used the K-invariance of the dual pairing over Q.
Finally, β is an isomorphism of Poisson spaces: note first that the identifying map $\mathcal{W}/K \to W$, $[(q;[q],\eta;q,\lambda)] \mapsto ([q],\eta;[(q,\lambda)])$ is well-defined because K_q acts trivially on $\mathrm{Hor}_q^* = T_{[q]}^*(Q/K) \ni \eta$ which in turn is due to the fact that all points of Q are regular. The quotient Poisson bracket is well-defined since $C^\infty(\mathcal{W})^K \subseteq$

$C^\infty(\mathcal{W})$ is a Poisson sub-algebra. The statement now follows because the diagram

$$\begin{array}{ccccc} T^*Q & \xrightarrow{\psi^{-1}} & \mathcal{W} & & \\ \downarrow & & & \searrow & \\ (T^*Q)/K & \xrightarrow{\beta} & W & = & \mathcal{W}/K \end{array}$$

is commutative, and composition of top and down-right arrow is Poisson and the left vertical arrow is surjective. $\square$

Lemma 5.3. *Let $\mathcal{O} \subseteq \mathfrak{k}^*$ be a coadjoint orbit, $\mu : T^*Q \to \mathfrak{k}^*$ the canonical momentum mapping, and $\mu_q := \mu|(T^*_q Q)$. Then either $\mu_q^{-1}(\mathcal{O}) = \emptyset$ for all $q \in Q$ or $\mu_q^{-1}(\mathcal{O}) \neq \emptyset$ for all $q \in Q$. In the latter case we have*

$$\mu_q^{-1}(\mathcal{O}) = \operatorname{Ann}_q(T_q(K.q)) \times \{A_q^*(\lambda) : \lambda \in \operatorname{Ann}\mathfrak{k}_q \cap \mathcal{O}\}$$

which is an equality of topological spaces and where A_q^* is the adjoint of $A_q : T_q Q \to \mathfrak{k}_q^\perp$.

Proof. Via the K-invariant inner product on $\mathfrak{k}$ we identify $\mathfrak{k}^*$ with $\mathfrak{k}$. Thus $\mathcal{O}$ is an $\operatorname{Ad}(K)$-orbit. Moreover, we identify T^*Q and TQ via the K-invariant metric on Q. The statement in the lemma is now equivalent to the following assertion. Given $q_1, q_2 \in Q$, then $A_{q_1}^{-1}(\mathcal{O}) \neq \emptyset$ if and only if $A_{q_2}^{-1}(\mathcal{O}) \neq \emptyset$. If q_1 and q_2 are in the same K-orbit then this is obvious. Therefore we can assume without loss of generality that $K_{q_1} = K_{q_2}$: all isotropy subgroups are conjugate to each other, and q_2 can be moved around in its orbit.

Let $X = A_{q_1}(v_1) \in \mathcal{O} \cap \mathfrak{k}_{q_1}^\perp$ with $v_1 = \zeta_X(q_1) \in \operatorname{Ver}_{q_1}$. Then also $A_{q_2}(\zeta_X(q_2)) = X \in \mathcal{O} \cap \mathfrak{k}_{q_2}^\perp$ since $\mathfrak{k}_{q_1}^\perp = \mathfrak{k}_{q_2}^\perp$. $\square$

The action of Proposition 5.1 by K on $\mathcal{W}$ is Hamiltonian with momentum map $\mu_A = \mu \circ \psi : \mathcal{W} \to \mathfrak{k}^*$.

Lemma 5.4. *Let $\mathcal{O}$ be a coadjoint orbit in the image of the momentum map $\mu_A : \mathcal{W} \to \mathfrak{k}^*$. Further, let (L) be in the isotropy lattice of the K-action on $\mathcal{W}$ such that $\mu_A^{-1}(\mathcal{O}) \cap \mathcal{W}_{(L)} \neq \emptyset$. Then*

$$\mathcal{W}_{(L)} = (Q \times_{Q/K} T^*(Q/K)) \times_Q (\textstyle\bigsqcup_{q\in Q} \operatorname{Ann}\mathfrak{k}_q)_{(L)}$$

and

$$\mathcal{W}_{(L)} \cap \mu_A^{-1}(\mathcal{O}) = (Q \times_{Q/K} T^*(Q/K)) \times_Q (\textstyle\bigsqcup_{q\in Q} \operatorname{Ann}\mathfrak{k}_q \cap \mathcal{O})_{(L)}$$

are smooth manifolds. Moreover,

$$\mathcal{O}_{(L_0)^H} \cap \operatorname{Ann}\mathfrak{h} \hookrightarrow (\textstyle\bigsqcup_{q\in Q} \mathcal{O} \cap \operatorname{Ann}\mathfrak{k}_q)_{(L)} \longrightarrow Q$$

is a smooth fiber bundle where L_0 is a subgroup of H such that L_0 is conjugate to L within K. The space $\mathcal{O}_{(L_0)^H}$ denotes the isotropy sub-manifold of type L_0 of $\mathcal{O}$ with regard to the $\operatorname{Ad}^*(H)$-action on $\mathcal{O}$.

Notice that $\mathcal{O} \cap \operatorname{Ann}\mathfrak{h}$ is not smooth, in general.

Proof. The statement about $\mathcal{W}_{(L)}$ is clear. Thus also the description of $\mathcal{W}_{(L)} \cap \mu_A^{-1}(\mathcal{O})$ follows from the previous lemma together with Theorem 3.1.

Now to the second assertion. Let $q_0 \in Q$ with $K_{q_0} = H$. Then

$$(q_0, \lambda) \in (\textstyle\bigsqcup_{q\in Q} \mathcal{O} \cap \operatorname{Ann}\mathfrak{k}_q)_{(L)}$$

if and only if

$$\lambda \in \mathcal{O} \cap \operatorname{Ann}\mathfrak{h} \text{ and } H \cap K_\lambda = H_\lambda = L_0 \sim L \text{ within } K$$

which is true if and only if

$$\lambda \in (\mathcal{O} \cap \operatorname{Ann}\mathfrak{h})_{(L_0)^H} = \mathcal{O}_{(L_0)^H} \cap \operatorname{Ann}\mathfrak{h}$$

where L_0 is a subgroup of H conjugate to L within K.

Consider the $\operatorname{Ad}^*(H)$ action on $\mathcal{O}$. This action is a Hamiltonian one with momentum map given by $\rho : \mathcal{O} \to \mathfrak{h}^*$, $\lambda \mapsto \lambda|\mathfrak{h}$, i.e. by restriction. Thus $\mathcal{O} \cap \operatorname{Ann}\mathfrak{h} = \rho^{-1}(0)$ which however is not smooth but only a stratified space in general. Typical smooth strata of this space are of the form $\mathcal{O}_{(L_0)^H} \cap \operatorname{Ann}\mathfrak{h}$ with L_0 a subgroup of H.

To see smooth local triviality we proceed as follows. Let again $q_0 \in Q$ with $K_{q_0} = H$, and let S be a slice at q_0 and U a tube around $K.q_0$. That is, $K/H \times S \cong U$, $(kH, s) \mapsto k.s$. Then we consider the smooth trivializing map

$$\begin{aligned} S \times K \times_H (\mathcal{O} \cap \operatorname{Ann}\mathfrak{h})_{(L_0)^H} &\longrightarrow (\textstyle\bigsqcup_{q\in Q} \mathcal{O} \cap \operatorname{Ann}\mathfrak{k}_q)_{(L)}|U, \\ (s, [(k, \lambda)]) &\longmapsto (k.s, \operatorname{Ad}^*(k).\lambda) \end{aligned}$$

which is well defined since $\operatorname{Ad}^*(k).\lambda \in (\operatorname{Ann}\mathfrak{k}_{k.s} \cap \mathcal{O})_{(kL_0k^{-1})^{K_{k.s}}}$, and the uncertainty coming from the diagonal H-action just cancels out. Clearly this map is smooth with the obvious smooth inverse

$$(q, \lambda) = (k.s, \operatorname{Ad}^*(k).\lambda_0) \longmapsto (s, [(k, \lambda_0)]).$$

In particular this construction constructs smooth bundle charts of the total space $(\bigsqcup_{q\in Q} \mathcal{O} \cap \operatorname{Ann}\mathfrak{k}_q)_{(L)}$. □

The singular reduction diagram of Ortega and Ratiu [24, Theorem 8.4.4] adjoined to the universal reduction procedure of Arms, Cushman, and Gotay [4], see also [24, Section 10.3.2] applied to the Weinstein space has the following form.

$$\begin{array}{ccccc} \mu_A^{-1}(\mathcal{O}) & \hookleftarrow & \mu_A^{-1}(\lambda) & \hookrightarrow & \mathcal{W} & \\ \downarrow & & \downarrow & & \downarrow & \\ \mu_A^{-1}(\mathcal{O})/K & \xleftarrow{\simeq} & \mu_A^{-1}(\lambda)/K & \hookrightarrow & \mathcal{W}/K & = W \end{array}$$

where $\lambda \in \mu_A(\mathcal{W})$ and $\mathcal{O}$ is the coadjoint orbit passing through λ. Therefore it is a sensible generalization of the smooth case to interpret the reduced $\mu_A^{-1}(\mathcal{O})/K = \mathcal{W}/\!/_{\mathcal{O}}K$ as a typical stratified symplectic leaf of the stratified Poisson space W. The following thus generalizes the result of Marsden and Perlmutter [18, Theorem 4.3] to the case of a non-free but single isotropy type action of K on Q.

What now follows is notation for the upcoming theorem. Let $\mathcal{O}$ be a coadjoint orbit in the image of the momentum map $\mu_A : \mathcal{W} \to \mathfrak{k}^*$, and let (L) be in the isotropy lattice of the K-action on $\mathcal{W}$ such that $\mathcal{W}^{\mathcal{O}}_{(L)} := \mu_A^{-1}(\mathcal{O}) \cap \mathcal{W}_{(L)} \neq \emptyset$. Then we have

$$\iota^{\mathcal{O}}_{(L)} : \mathcal{W}^{\mathcal{O}}_{(L)} \hookrightarrow \mathcal{W},$$

the canonical embedding, and the orbit projection mapping

$$\pi^{\mathcal{O}}_{(L)} : \mathcal{W}^{\mathcal{O}}_{(L)} \twoheadrightarrow \mathcal{W}^{\mathcal{O}}_{(L)}/K =: (\mathcal{W}/\!/_{\mathcal{O}}K)_{(L)}.$$

Consider furthermore

$$\rho^{\mathcal{O}}_{(L)} : (\textstyle\bigsqcup_{q\in Q} \mathcal{O} \cap \operatorname{Ann}\mathfrak{k}_q)_{(L)} \twoheadrightarrow (\bigsqcup_{q\in Q} \mathcal{O} \cap \operatorname{Ann}\mathfrak{k}_q)_{(L)}/K$$

and

$$\phi^{\mathcal{O}}_{(L)} : (\textstyle\bigsqcup_{q\in Q} \mathcal{O} \cap \operatorname{Ann}\mathfrak{k}_q)_{(L)} \longrightarrow \mathcal{O}, \quad (q, \lambda) \longmapsto \lambda$$

as well as the embedding

$$j^{\mathcal{O}}_{(L)} : (\textstyle\bigsqcup_{q\in Q} \mathcal{O} \cap \operatorname{Ann}\mathfrak{k}_q)_{(L)} \hookrightarrow \bigsqcup_{q\in Q} \operatorname{Ann}\mathfrak{k}_q.$$

Finally, we denote the Kirillov-Kostant-Souriou symplectic form on $\mathcal{O}$ by $\Omega^{\mathcal{O}}$, that is $\Omega^{\mathcal{O}}(\lambda)(\mathrm{ad}^*(X).\lambda, \mathrm{ad}^*(Y).\lambda) = \langle\lambda, [X, Y]\rangle$. Remember from Proposition 5.1 that the symplectic structure on $\mathcal{W}$ is denoted by σ.

Theorem 5.5 (Gauged symplectic reduction). *Let $Q = Q_{(H)}$, let $\mathcal{O}$ be a coadjoint orbit in the image of the momentum map $\mu_A : \mathcal{W} \to \mathfrak{k}^*$, and let (L) be in the isotropy lattice of the K-action on $\mathcal{W}$ such that $\mathcal{W}^{\mathcal{O}}_{(L)} := \mu_A^{-1}(\mathcal{O}) \cap \mathcal{W}_{(L)} \neq \emptyset$. Then the following are true.*

(i) *The smooth manifolds $(\mathcal{W}/\!/_{\mathcal{O}}K)_{(L)}$ and*

$$(\mathcal{O}/\!/_0 H)_{(L_0)^H} =: (\mathcal{O} \cap \mathrm{Ann}\,\mathfrak{h})_{(L_0)^H}/H$$

are typical symplectic strata of the stratified symplectic spaces $\mathcal{W}/\!/_{\mathcal{O}}K$ and $\mathcal{O}/\!/_0 H$ respectively. Here L_0 is an isotropy subgroup of the induced H-action on $\mathcal{O}$ and $(L_0)^H$ denotes its isotropy class in H.

(ii) *The symplectic stratum $(\mathcal{W}/\!/_{\mathcal{O}}K)_{(L)}$ can be globally described as*

$$(\mathcal{W}/\!/_{\mathcal{O}}K)_{(L)} = T^*(Q/K) \times_{Q/K} (\textstyle\bigsqcup_{q\in Q}\mathcal{O} \cap \mathrm{Ann}\,\mathfrak{k}_q)_{(L)}/K$$

whence it is the total space of the smooth symplectic fiber bundle

$$(\mathcal{O}/\!/_0 H)_{(L_0)^H} \hookrightarrow (\mathcal{W}/\!/_{\mathcal{O}}K)_{(L)} \longrightarrow T^*(Q/K)$$

Hereby L_0 is an isotropy subgroup of the induced H-action on $\mathcal{O}$ which is conjugate in K to L, and $(L_0)^H$ denotes its isotropy class in H.

(iii) *The symplectic structure $\sigma^{\mathcal{O}}_{(L)}$ on $(\mathcal{W}/\!/_{\mathcal{O}}K)_{(L)}$ is uniquely determined and given by the formula*

$$(\pi^{\mathcal{O}}_{(L)})^*\sigma^{\mathcal{O}}_{(L)} = (\iota^{\mathcal{O}}_{(L)})^*\sigma - (\mu_A|\mathcal{W}^{\mathcal{O}}_{(L)})^*\Omega^{\mathcal{O}}.$$

More precisely,

$$\sigma^{\mathcal{O}}_{(L)} = \Omega^{Q/K} - \beta^{\mathcal{O}}_{(L)}$$

where $\beta^{\mathcal{O}}_{(L)} \in \Omega^2((\bigsqcup_{q\in Q}\mathcal{O} \cap \mathrm{Ann}\,\mathfrak{k}_q)_{(L)}/K)$ is defined by

$$(\rho^{\mathcal{O}}_{(L)})^*\beta^{\mathcal{O}}_{(L)} = (j^{\mathcal{O}}_{(L)})^*dB + (\phi^{\mathcal{O}}_{(L)})^*\Omega^{\mathcal{O}}.$$

Finally B is the form that was introduced in Proposition 5.1. Thus for $(q, \lambda) \in (\bigsqcup_{q\in Q}\mathcal{O} \cap \mathrm{Ann}\,\mathfrak{k}_q)_{(L)}$ and

$$(v_i, \mathrm{ad}^*(X_i).\lambda) \in T_{(q,\lambda)}(\textstyle\bigsqcup_{q\in Q}\mathcal{O} \cap \mathrm{Ann}\,\mathfrak{k}_q)_{(L)}$$

where $i = 1, 2$ we have the explicit formulas

$$B_{(q,\lambda)}(v_i, \mathrm{ad}^*(X_i).\lambda) = \langle\lambda, A_q(v_i)\rangle$$

and also

$$\begin{aligned} &dB_{(q,\lambda)}((v_1, \mathrm{ad}^*(X_1).\lambda), (v_2, \mathrm{ad}^*(X_2).\lambda)) \\ &\quad = \langle\lambda, \mathrm{Curv}^A_q(v_1, v_2)\rangle + \langle\lambda, [X_2, Z_1]\rangle - \langle\lambda, [X_1, Z_2]\rangle + \langle\lambda, [Z_1, Z_2]\rangle \end{aligned}$$

where $v_i = \zeta_{Z_i}(q) \oplus v_i^{\mathrm{hor}} \in \mathrm{Ver}_q \oplus \mathrm{Hor}_q$ is the decomposition into vertical and horizontal parts with $Z_i \in \mathfrak{k}$.

(iv) *The stratified symplectic space can be globally described as*

$$\mathcal{W}/\!/_{\mathcal{O}}K = T^*(Q/K) \times_{Q/K} \textstyle\bigsqcup_{q\in Q}\mathcal{O} \cap \mathrm{Ann}\,\mathfrak{k}_q/K$$

whence it is the total space of

$$\mathcal{O}/\!/_0 H \hookrightarrow \mathcal{W}/\!/_{\mathcal{O}}K \longrightarrow T^*(Q/K)$$

which is a stratified symplectic fiber bundle with singularities confined to the fiber direction.

Proof. Assertion (i). This is well-known as a general principle of stratified symplectic reduction – see Ortega and Ratiu [24, Section 8.4] or Section 3.

Assertion (ii). We know from above that all spaces involved in the diagram really are smooth. As in the proof of Lemma 5.4 let $q_0 \in Q$ with $K_{q_0} = H$, S a slice at q_0, and $U \cong K/H \times S$ a tube around the orbit $K.q_0$. Then we get the local description

$$\begin{aligned}(\mathcal{W}//_{\mathcal{O}}K)_{(L)}|U &= T^*S \times_S (\textstyle\bigsqcup_{q\in U}\mathcal{O}\cap \operatorname{Ann}\mathfrak{k}_q)_{(L)}/K \\ &\cong T^*S \times_S S \times (\mathcal{O}\cap \operatorname{Ann}\mathfrak{h})_{(L_0)^H}/H \\ &= T^*S \times (\mathcal{O}//_0 H)_{(L_0)^H}\end{aligned}$$

as claimed.

The bundle is symplectic: This follows from Theorem 4.4.

Assertion (iii). The defining property of the reduced symplectic form $\sigma^{\mathcal{O}}_{(L)}$, namely,

$$(\pi^{\mathcal{O}}_{(L)})^*\sigma^{\mathcal{O}}_{(L)} = (\iota^{\mathcal{O}}_{(L)})^*\sigma - (\mu_A|\mathcal{W}^{\mathcal{O}}_{(L)})^*\Omega^{\mathcal{O}}$$

is a well-established fact, see e.g. Bates and Lerman [6, Proposition 11]. Thus it is clear from Proposition 5.1 that

$$\sigma^{\mathcal{O}}_{(L)} = \Omega^{Q/K} - \beta^{\mathcal{O}}_{(L)}$$

– if $\beta^{\mathcal{O}}_{(L)}$ is a well defined two-form on $(\bigsqcup_{q\in Q}\mathcal{O}\cap \operatorname{Ann}\mathfrak{k}_q)_{(L)}/K$ such that

$$(\rho^{\mathcal{O}}_{(L)})^*\beta^{\mathcal{O}}_{(L)} = (j^{\mathcal{O}}_{(L)})^*dB + (\phi^{\mathcal{O}}_{(L)})^*\Omega^{\mathcal{O}}.$$

To see this notice firstly that

$$\tilde{\beta} := (j^{\mathcal{O}}_{(L)})^*dB + (\phi^{\mathcal{O}}_{(L)})^*\Omega^{\mathcal{O}} \in \Omega^2((\textstyle\bigsqcup_{q\in Q}\mathcal{O}\cap \operatorname{Ann}\mathfrak{k}_q)_{(L)})$$

is K-invariant. Furthermore, we claim that $\tilde{\beta}$ is horizontal, i.e. vanishes upon insertion of a vertical vector field. Indeed, let

$$(q,\lambda) \in (\textstyle\bigsqcup_{q\in Q}\mathcal{O}\cap \operatorname{Ann}\mathfrak{k}_q)_{(L)},$$

and $Z_i \in \mathfrak{k}$ for $i = 1, 2$, and $Y \in \mathfrak{k}$ such that

$$\operatorname{ad}^*(Z_1).\lambda, \operatorname{ad}^*(Y).\lambda \in T_\lambda\mathcal{O}_{(L_0)^{K_q}} \cap \operatorname{Ann}\mathfrak{k}_q,$$

and consider $v_2(q) = v_2^{\mathrm{hor}} \oplus \zeta_{Z_2}(q) \in \mathrm{Hor}_q \oplus \mathrm{Ver}_q$ as in the proof of Proposition 5.1. Then we have

$$\begin{aligned}&\tilde{\beta}_{(q,\lambda)}((\zeta_{Z_1}(q), \operatorname{ad}^*(Z_1).\lambda), (v_2(q), \operatorname{ad}^*(Y).\lambda)) \\ &\quad= 0 + \langle\lambda, [Z_1, Z_2]\rangle - \langle \operatorname{ad}^*(Y).\lambda, Z_1\rangle + \langle \operatorname{ad}^*(Z_1).\lambda, Z_2\rangle + \langle\lambda, [Z_1, Y]\rangle \\ &\quad= \langle\lambda, [Z_1, Z_2]\rangle + \langle\lambda, [Y, Z_1]\rangle - \langle\lambda, [Z_1, Z_2]\rangle + \langle\lambda, [Z_1, Y]\rangle \\ &\quad= 0.\end{aligned}$$

That is $\tilde{\beta}$ is a basic form and thus descends to a form $\beta^{\mathcal{O}}_{(L)}$.

Assertion (iv) is a pasting together of the results in (ii). □

Corollary 5.6. *Let $\mathcal{O}$ be a coadjoint orbit in the image of the momentum map $\mu_A : \mathcal{W} \to \mathfrak{k}^*$, and let (L) be in the isotropy lattice of the K-action on $\mathcal{W}$ such that $\mathcal{W}^{\mathcal{O}}_{(L)} := \mu_A^{-1}(\mathcal{O}) \cap \mathcal{W}_{(L)} \neq \emptyset$. Assume further that there is a global slice S such that $Q \cong K/H \times S$. Then we have the global description*

$$\mathcal{W}//_{\mathcal{O}}K = T^*S \times \mathcal{O}//_0 H.$$

Moreover, the reduced symplectic form $\sigma^{\mathcal{O}}_{(L)}$ on a symplectic stratum $(\mathcal{W}//_{\mathcal{O}}K)_{(L)} = T^*S \times (\mathcal{O}//_0 H)_{(L_0)^H}$ is given by the formula

$$\sigma^{\mathcal{O}}_{(L)} = \Omega^{Q/K} - \Omega^{\mathcal{O}}_{(L_0)^H}$$

where $\Omega^{\mathcal{O}}_{(L_0)^H}$ *is the canonically reduced symplectic form on* $(\mathcal{O}/\!/_0 H)_{(L_0)^H}$*, and* L_0 *is a subgroup of* H *which is conjugate to* L *within* K.

Proof. This is an immediate consequence of Theorems 4.4 and 5.5. □

6. Spin Calogero-Moser systems

6.1. **SL$(m,\mathbb{C})$ by hand.** As an example consider $G = \mathrm{SL}(m,\mathbb{C})$. Here we work along the lines of Kazdhan, Kostant, Sternberg [14, Section 2] who considered the case $G = \mathrm{SU}(m,\mathbb{C})$. See also Alekseevsky, Kriegl, Losik, Michor [2, Section 5.7]. The point to this example is that we try to say as much as possible about the reduced phase space by using an *ad hoc* approach. The result may then be taken as motivation for the general theory of cotangent bundle reduction.

Let $\mathcal{O} = \mathrm{Ad}(G)Z_0$ be an orbit passing through a semisimple element Z_0. Consider $(a,\alpha) \in G_r \times \mathfrak{g}$ with $\alpha - a\alpha a^{-1} = \mu(a,\alpha) = Z$. As usual G_r denotes the set of regular elements, that is G_r consists of those matrices that have m different eigenvalues. Moreover, we let H denote the subgroup of diagonal matrices, and $H_r := H \cap G_r$. Via the $\mathrm{Ad}(G)$-action we can bring a in diagonal form with entries $a_i \neq a_j$ for $i \neq j$. Since $Z_{ij} = \alpha_{ij} - \frac{a_i}{a_j}\alpha_{ij}$ the following are coordinates on $(\mu^{-1}(\mathcal{O}) \cap (G_r \times \mathfrak{g}))/\mathrm{Ad}(G)$.

- a_i for $i = 1,\dots,m$.
- $\alpha_i := \alpha_{ii}$ for $i = 1,\dots,m$.
- $\alpha_{ij} = (1 - \frac{a_i}{a_j})^{-1} Z_{ij}$ for $i \neq j$.

These coordinates give an identification

$$(\mu^{-1}(\mathcal{O}) \cap (G_r \times \mathfrak{g}))/\mathrm{Ad}(G) = (T^*H_r \times (\mathcal{O} \cap \mathfrak{h}^\perp)/\mathrm{Ad}(H))/W$$

where $W = N(H)/H$ is the Weyl group. Claim: *If* $\mathcal{O}$ *is an orbit which is of minimal non-zero dimension then we have that* $\mathcal{O} \cap \mathfrak{h}^\perp/\mathrm{Ad}(H) = \{\text{point}\}$*. Moreover, the reduced phase space can be described as* $(\mu^{-1}(\mathcal{O}) \cap (G_d \times \mathfrak{g}))/\mathrm{Ad}(G) \cong T^*H_r/W$*.* Here G_d denotes the open and dense subset of all diagonable elements in $\mathrm{SL}(m,\mathbb{C})$. Indeed, let $\mu(a,\alpha) = Z \in \mathcal{O} \cap \mathfrak{h}^\perp$ with a in diagonal form. Thus $Z = vw^t - cI$ where $c := \frac{1}{m}\langle v,w\rangle \neq 0$, $v,w \in \mathbb{C}^m$, and w^t is the transposed to the column vector w. Since $Z \in \mathfrak{h}^\perp$ we infer that $v_i w_i = c$. Hence

$$\mathcal{O} \cap \mathfrak{h}^\perp = \{(\tfrac{c}{v_1}v,\dots,\tfrac{c}{v_m}v) - cI : v_i \in \mathbb{C} \setminus \{0\}\}.$$

Take such an $(\frac{c}{v_1}v,\dots,\frac{c}{v_m}v) - cI =: Z_1$. Let $h = \prod_{i=1}^m v_i \cdot \mathrm{diag}(v_1^{-1},\dots,v_m^{-1})$. Then we can bring Z_1 into the normal form $\mathrm{Ad}(h)Z_1 = c(1)_{ij} - cI$ where $(1)_{ij}$ denotes the $m \times m$-matrix with all entries equal to 1. Finally note that $\alpha_{ij} - \frac{a_i}{a_j}\alpha_{ij} = \frac{c}{v_j}v_i \neq 0$ implies that $a = \mathrm{diag}(a_1,\dots,a_m)$ is actually regular.

6.2. **Application: Hermitian matrices.** Consider V the space of complex Hermitian $n \times n$ matrices as the configuration space to start from. This space shall be endowed with the inner product $V \times V \to \mathbb{R}$, $(a,b) \mapsto \mathrm{Tr}(ab)$. Moreover, we let $G = \mathrm{SU}(n,\mathbb{C})$ act on V by conjugation. Clearly this action leaves the trace form invariant. Via the inner product we can trivialize the cotangent bundle as $T^*V = V \times V^* = V \times V$, and the cotangent lifted action of G is simply given by the diagonal action. The canonical symplectic form on T^*V is given by

$$\Omega_{(a,\alpha)}((a_1,\alpha_1),(a_2,\alpha_2)) = \mathrm{Tr}(\alpha_2 a_1) - \mathrm{Tr}(\alpha_1 a_2).$$

The free Hamiltonian on $T^*V = V \times V$ is given by

$$H_{\text{free}} : (a,\alpha) \longmapsto \tfrac{1}{2}\mathrm{Tr}(\alpha\alpha).$$

Trajectories of this Hamiltonian are given by straight lines of the form $t \mapsto a + t\alpha$ in the configuration space V.

Let us further identify $\mathfrak{su}(n)^* = \mathfrak{su}(n)$ via the Killing form. The momentum mapping is then given by

$$\mu : (a, \alpha) \longmapsto [a, \alpha] = \mathrm{ad}(a).\alpha.$$

Consider also an orbit $\mathcal{O}$ together with its canonically induced symplectic structure in the image of the momentum mapping.

Assumption: The orbit $\mathcal{O}$ is such that $\mu^{-1}(\mathcal{O}) \subseteq V_r \times V$. Here V_r denotes the set of regular elements in V with respect to the G action.

This assumption is for example fulfilled if the projection from $\mathcal{O}$ to any root space is non-trivial.

Let Σ denote the subspace of V consisting of diagonal matrices. Then Σ is a section of the G-action on V, see Section 8. Further, we define $\Sigma_r := V_r \cap \Sigma$. Within Σ we choose the positive Weyl chamber $C := \{\mathrm{diag}(q_1, \dots, q_n) : q_1 > \dots > q_n\}$ so that $C = \Sigma/W$ where $W = W(\Sigma) = N_G(\Sigma)/Z_G(\Sigma)$. Thus $C_r := \Sigma_r \cap C$ may be considered as a global slice for the G-action on V_r so that $G/M \times C_r \cong V_r$, $(gM, a) \mapsto g.a$ where $M := Z_G(\Sigma_r) = Z_G(\Sigma)$. That is M is the subgroup of $\mathrm{SU}(n)$ consisting of diagonal matrices only. Now we may apply Corollary 5.6 to get

$$T^*V /\!/_{\mathcal{O}} G = T^*C_r \times \mathcal{O} /\!/_0 M$$

as symplectically stratified spaces. The strata are of the form

$$(T^*V /\!/_{\mathcal{O}} G)_{(L)} = T^*C_r \times (\mathcal{O} /\!/_0 M)_{(L_0)^M}$$

where L_0 is a subgroup of M conjugate to L within G. Moreover, the reduced symplectic structure $\sigma^{\mathcal{O}}_{(L)}$ on $(T^*V /\!/_{\mathcal{O}} G)_{(L)}$ is of product form, i.e.

$$\sigma^{\mathcal{O}}_{(L)} = \Omega^{C_r} - \Omega^{\mathcal{O}}_{(L_0)^M}$$

where $\Omega^{\mathcal{O}}_{(L_0)^M}$ is the canonically reduced symplectic form on $(\mathcal{O} /\!/_0 M)_{(L_0)^M}$.

From the general theory we know that the Hamiltonian H_{free} reduces to a Hamiltonian $H^{(L)}_{\mathrm{CM}}$ on the stratum $(T^*V /\!/_{\mathcal{O}} G)_{(L)}$, and that integral curves of H_{free} project to integral curves of $H^{(L)}_{\mathrm{CM}}$. In particular the dynamics remain confined to the symplectic stratum. The reduced Hamiltonian is thus given by

$$H^{(L)}_{\mathrm{CM}}(q, p, [\lambda]) = H_{\mathrm{free}}(q, p + A^*_q(\lambda))$$

where $[\lambda]$ is the class of λ in $(\mathcal{O} /\!/_0 M)_{(L_0)^M}$ and $A^*_q : \mathfrak{g}^\perp_q = \mathfrak{m}^\perp \to T_q(G.q) = \Sigma^\perp$ is the point wise dual to the mechanical connection as introduced in Section 5. Assume that $q = \mathrm{diag}(q_1 > \dots > q_n)$ and that $\lambda = (\lambda_{ij})_{ij} \in (\mathcal{O} \cap \mathfrak{m}^\perp)_{(L_0)^M}$. Then

$$A^*_q(\lambda)_{ij} = \tfrac{\lambda_{ij}}{q_i - q_j} \text{ for } i \neq j, \text{ and } A^*_q(\lambda)_{ii} = 0.$$

Therefore, for $p = \mathrm{diag}(p_1, \dots, p_n) \in \Sigma$ and $q, [\lambda]$ as introduced we obtain

$$\begin{aligned} H^{(L)}_{\mathrm{CM}}(q, p, [\lambda]) &= \tfrac{1}{2}\mathrm{Tr}(p)^2 + \tfrac{1}{2}\mathrm{Tr}(A^*_q(\lambda))^2 \\ &= \tfrac{1}{2}\sum_{i=1}^{n} p_i^2 + \tfrac{1}{2}\sum_{i \neq j} \tfrac{\lambda_{ij}\lambda_{ji}}{(q_i - q_j)(q_j - q_i)} \\ &= \tfrac{1}{2}\sum_{i=1}^{n} p_i^2 + \sum_{i > j} \tfrac{|\lambda_{ij}|^2}{(q_i - q_j)^2} \end{aligned}$$

since $\lambda_{ji} = -\overline{\lambda_{ij}}$ and $\mathrm{Tr}(pA^*_q(\lambda)) = \mathrm{Tr}(A^*_q(\lambda)p) = 0$. This is the Hamiltonian function of the Calogero-Moser system with spin. Integrability of this system in the non-commutative sense is proved in the next section in a more general context.

6.3. Application: Polar representations of compact Lie groups. The idea of considering polar representations of compact Lie groups to obtain new versions of Spin Calogero-Moser systems is due to Alekseevsky, Kriegl, Losik, Michor [2].

As in Section 8 let V be a real Euclidean vector space and G a connected compact Lie group that acts on V via a polar representation. Via the inner product we consider the cotangent bundle of V as a product $T^*V = V \times V$. The canonical symplectic form Ω is thus given by

$$\Omega_{(a,\alpha)}((a_1,\alpha_1),(a_2,\alpha_2)) = \langle \alpha_2, a_1\rangle - \langle \alpha_1, a_2\rangle$$

where $\langle\ ,\ \rangle$ is the inner product on V. The free Hamiltonian on $T^*V = V \times V$ is given by

$$H_{\text{free}} : (a,\alpha) \longmapsto \tfrac{1}{2}\langle \alpha,\alpha\rangle.$$

Trajectories of this Hamiltonian are given by straight lines of the form $t \mapsto a + t\alpha$ in the configuration space V.

Of course, we want to play with the cotangent lifted action of G, and this is just the diagonal action of G on $V \times V$. By Section 8 we may think of the action by G on V as a symmetric space representation and thus consider $\mathfrak{g} \oplus V =: \mathfrak{l}$ as a real semisimple Lie algebra with Cartan decomposition into $\mathfrak{g}$ and V, and with bracket relations $[\mathfrak{g},\mathfrak{g}] \subseteq \mathfrak{g}$, $[\mathfrak{g}, V] \subseteq V$, and $[V,V] \subseteq V$. The momentum mapping corresponding to the G-action on $T^*V = V \times V$ is now given by

$$\mu : V \times V \longrightarrow \mathfrak{g}^* = \mathfrak{g}, \quad (a,\alpha) \longmapsto [a,\alpha] = \operatorname{ad}(a).\alpha$$

where we identify $\mathfrak{g} = \mathfrak{g}^*$ via an $\operatorname{Ad}(G)$-invariant inner product. Consider also an orbit $\mathcal{O}$ together with its canonically induced symplectic structure in the image of the momentum mapping.

Assumption: The orbit $\mathcal{O}$ is such that $\mu^{-1}(\mathcal{O}) \subseteq V_r \times V$. Here V_r denotes the set of regular elements in V with respect to the G action.

We proceed as above, and let Σ denote a fixed section of the G-action on V, consider C a Weyl chamber in Σ, and put $M := Z_G(\Sigma)$. Now we may apply Corollary 5.6 to get

$$T^*V/\!/_{\mathcal{O}}G = T^*C_r \times \mathcal{O}/\!/_0 M$$

as symplectically stratified spaces. The strata are of the form

$$(T^*V/\!/_{\mathcal{O}}G)_{(L)} = T^*C_r \times (\mathcal{O}/\!/_0 M)_{(L_0)^M}$$

where L_0 is a subgroup of M conjugate to L within G. Moreover, the reduced symplectic structure $\sigma^{\mathcal{O}}_{(L)}$ on $(T^*V/\!/_{\mathcal{O}}G)_{(L)}$ is of product form, i.e.

$$\sigma^{\mathcal{O}}_{(L)} = \Omega^{C_r} - \Omega^{\mathcal{O}}_{(L_0)^M}$$

where $\Omega^{\mathcal{O}}_{(L_0)^M}$ is the canonically reduced symplectic form on $(\mathcal{O}/\!/_0 M)_{(L_0)^M}$.

From the general theory we know that the Hamiltonian H_{free} reduces to a Hamiltonian $H^{(L)}_{\text{CM}}$ on the stratum $(T^*V/\!/_{\mathcal{O}}G)_{(L)}$, and that integral curves of H_{free} project to integral curves of $H^{(L)}_{\text{CM}}$. In particular the dynamics remain confined to the symplectic stratum. The reduced Hamiltonian is thus given by

$$H^{(L)}_{\text{CM}}(q,p,[Z]) = H_{\text{free}}(q, p + A^*_q(\lambda))$$

where $[Z]$ is the class of Z in $(\mathcal{O}/\!/_0 M)_{(L_0)^M}$ and $A^*_q : \mathfrak{g}_q^\perp = \mathfrak{m}^\perp \to T_q(G.q) = \Sigma^\perp$ is the point wise dual to the mechanical connection as introduced in Section 5. Let $q \in C_r$, $p = \sum_{i=1}^{l} p_i B_0^i$, and $Z = \sum_{\lambda\in R}\sum_{i=1}^{k_\lambda} z^i_\lambda E^i_\lambda \in (\mathcal{O} \cap \mathfrak{m}^\perp)_{(L_0)^M}$ where $l = \dim\Sigma$ and $k_\lambda = \frac{1}{2}\dim \mathfrak{l}_\lambda$. Here notation is as in Section 8, and $R = R(\mathfrak{l},\Sigma) \subseteq \Sigma^*$

denotes the set of restricted roots, in particular. With these definitions the dual mapping to the mechanical connection is given by

$$A_q^*(Z) = \sum_{\lambda \in R} \sum_{i=1}^{k_\lambda} \frac{z_\lambda^i}{\lambda(q)} B_\lambda^i.$$

Note that $\lambda(q) \neq 0$ for all $\lambda \in R$ since $q \in C_r$ is regular. The reduced Hamiltonian thus computes to

$$H_{\mathrm{CM}}^{(L)}(q,p,[Z]) = \tfrac{1}{2}\langle p + A_q^*(Z), p + A_q^*(Z)\rangle = \tfrac{1}{2}\sum_{i=1}^{l} p_i^2 + \tfrac{1}{2}\sum_{\lambda \in R} \frac{\sum_{i=1}^{k_\lambda} z_\lambda^i z_\lambda^i}{\lambda(q)^2}.$$

The reduced Hamiltonian system $(T^*V/\!/_{\mathcal{O}}G, \sigma^{\mathcal{O}}, H_{\mathrm{CM}})$ is thus a new version of a Calogero-Moser system with spin. It is in fact a stratified Hamiltonian system in the sense that it is a Hamiltonian system on each symplectic stratum $(T^*V/\!/_{\mathcal{O}}G)_{(L)}$, and the dynamics stay confined to these strata.

We now show that the thus obtained Calogero-Moser system is integrable in the non-commutative sense. To do so we will use Theorem 9.3. We start by choosing coordinates $q_1, \ldots, q_n, p_1, \ldots, p_n$ on $T^*V = V \times V$ such that the Poisson bracket of functions $f, g \in C^\infty(V \times V)$ is given by the usual equation $\{f,g\} = \sum_{i=1}^{n}(\frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i} - \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial q_i})$. Moreover we assume that $q_1 \ldots, q_l, p_1, \ldots, p_l$ are coordinates on $\Sigma \times \Sigma \hookrightarrow V \times V$. Let us now consider the map

$$\Phi : V \times V \longrightarrow \Sigma^\perp \times V$$

given by projection, and endow $\Sigma^\perp \times V$ with the inherited Poisson structure. Clearly, $C^\infty(\Sigma^\perp \times V)$ has a center and this is just generated by $p_1, \ldots, p_l$. Thus we may identify $Z(C^\infty(\Sigma^\perp \times V)) = C^\infty(\Sigma)$. Now the set of all first integrals of H_{free}, i.e.

$$\mathcal{F}_{H_{\mathrm{free}}} = \{F \in C^\infty(V \times V) : \{F, H\} = 0\}$$

can be identified with $C^\infty(\Sigma^\perp \times V)$ via Φ since H_{free} factors over the projection onto the second factor and is G-invariant, and thus can be considered as a function on Σ. Therefore,

$$\begin{aligned}\dim V \times V &= \dim \Sigma^\perp \times V + \dim \Sigma \\ &= \operatorname{ddim} \mathcal{F}_{H_{\mathrm{free}}} + \operatorname{ddim} Z(C^\infty(\Sigma^\perp \times V)),\end{aligned}$$

and we are exactly in the situation of Theorem 9.3 to conclude non-commutative integrability of the reduced system.

7. Appendix: Transformation groups

Let K be a compact Lie group that acts by isometries on a Riemannian manifold M, i.e. M is a Riemannian K-space. The action will be written as $l : K \times M \to M$, $(k,x) \mapsto l(k,x) = l_k(x) = l^x(k) = k.x$. Sometimes the action will be lifted to the tangent space TM. That is, we will consider $h.(x,v) := (h.x, h.v) := Tl_h.(x,v) = (l_h(x), T_x l_h.v)$ where $(x,v) \in TM$. As the action is a transformation by a diffeomorphism it may also be lifted to the cotangent bundle. This is the cotangent lifted action which is defined by $h.(x,p) := (h.x, h.p) := T^*l_h.(x,p) = (h.x, T_{h.x}^* l_{h^{-1}}.p)$ where $(x,p) \in T^*M$.

The FUNDAMENTAL VECTOR FIELD is going to repeatedly play an important role. It is defined by

$$\zeta_X(x) := \tfrac{\partial}{\partial t}|_0 l(\exp(+tX), x) = T_e l^x(X)$$

where $X \in \mathfrak{k}$. The fundamental vector field mapping $\mathfrak{k} \to \mathfrak{X}(M)$, $X \mapsto \zeta_X$ is clearly linear. By definition the flow of ζ_X is given by $l_{\exp(tX)}$. Moreover, it intertwines

Lie bracket with the negative of the bracket of vector fields. Indeed, notice first, that

$$T_x l_k.\zeta_X(x) = \tfrac{\partial}{\partial t}|_0 l_k(l(\exp(tX),x)) = T_e l^{k.x}.T_e\mathrm{conj}_k.(X) = \zeta_{\mathrm{Ad}(k)X}(k.x)$$

that is ζ_X and $\zeta_{\mathrm{Ad}(k.X)}$ are l_k-related. Now we use this to compute the bracket to be

$$[\zeta_X,\zeta_Y](x) = \tfrac{\partial}{\partial s}|_0(\mathrm{Fl}^{\zeta_X}_s)^*_x\zeta_Y = \tfrac{\partial}{\partial s}|_0\zeta_{\mathrm{Ad}(\exp(-sX)).Y}(x) = -\zeta_{[X,Y]}(x)$$

where $X,Y\in\mathfrak{k}$ and $x\in M$. Had we chosen the sign negatively in the fundamental vector field mapping then it were Lie algebra homomorphism. However, this choice of sign is the standard one in conjunction with Hamiltonian group actions and momentum maps.

We endow the set of orbits M/K with the quotient topology and call it the orbit space. Let H be a subgroup of K. A point $x\in M$ is said to be of ISOTROPY TYPE H if its isotropy group $K_x=\{k\in K: k.x=x\}$ is conjugate to H within K. If H' is conjugate to H within K we shall also write $H'\sim H$. The family of subgroups of K conjugate to H within K is denoted by (H) and called the CONJUGACY CLASS of H. All conjugacy classes of possible subgroups of K that show up as isotropy groups of points in M taken together constitute the ISOTROPY LATTICE

$$\mathcal{IL}_K(M) := \{(L) : L = K_x \text{ for some } x\in M\}$$

of the action under consideration. If it is clear which action is being looked at we simply write $\mathcal{IL}(M)$. The set of points that have isotropy H is denoted by

$$M_{(H)} := \{x\in M : K_x \text{ is conjugate to } H\},$$

and is called the ISOTROPY TYPE SUB-MANIFOLD of M of type (H). It will be shown below that this terminology is justified, i.e. $M_{(H)}$ indeed is a sub-manifold. Furthermore, there is

$$M_H := \{x\in M : K_x = H\}$$

which is called the set of points that have SYMMETRY H, and the set of points that are fixed by H,

$$\mathrm{Fix}(H) := M^H := \{x\in M : H\subseteq K_x\}$$

There is a natural partial ordering on the isotropy lattice as follows:

$$(H)\leq(L) :\Longleftrightarrow \quad \text{there is } k\in K \text{ such that } H\subseteq kLk^{-1}$$

where (H) and (L) are in the isotropy lattice of M. This relation is anti-symmetric because K was assumed compact. An element $x\in M$ is called REGULAR if it has an open neighborhood such that $(K_x)\leq(K_y)$ for all y in this neighborhood, and the set of regular points is denoted by M_{reg}. If a point is not regular it is said to be SINGULAR.

Definition 7.1 (Slices). A subset $S\subseteq M$ is called a SLICE at $x\in M$ if there are an open K-invariant neighborhood U of $K.x$ in M and a smooth K-equivariant retraction $r: U\to K.x$ such that $S=r^{-1}(x)$.

Proposition 7.2. *Assume S is a slice at $x\in M$ for the K-action, and U an open K-invariant neighborhood of $K.x$ as in the definition. Then the following are true.*

(i) *The slice S is a manifold and K_x acts on S.*
(ii) *$k.S\cap S\neq\emptyset$ if and only if $k\in K_x$.*
(iii) *$K.S=U$.*
(iv) *$K_s\subseteq K_x$ for all $s\in S$*
(v) *If x is regular then there is an open neighborhood V of x in S such that $V\subseteq M_{\mathrm{reg}}$.*

(vi) *If $s_1, s_2 \in S$ have the same isotropy type as with regard to the K_x-action on S then their isotropy types with regard to the K-action coincide. The converse is, however, false.*

(vii) *The topological spaces S/K_x and $(K.S)/K$ are homeomorphic. Moreover, this is a typical open neighborhood of $K.x$ in the orbit space M/K.*

Proof. See Michor [20] or Palais and Terng [26]. □

Theorem 7.3 (Tube Theorem)**.** *Let S be a slice for the K-action at $x \in M$, $H = K_x$, and $U = K.S$. Then there is a K-equivariant diffeomorphism*

$$\begin{aligned} f : K \times_H S &\longrightarrow U \\ [(k,s)] &\longmapsto k.s. \end{aligned}$$

In particular the section $K \times_H \{x\}$ in the associated bundle is mapped to the orbit $K.x$. Here the action by H on $K \times S$ is given by $h.(k,s) = (kh^{-1}, h.s)$. The K action on $K \times_H S$ is induced by left multiplication on the first factor of $K \times S$.

The neighborhood U is called a TUBE around the orbit $K.x$.

Proof. See Michor [20] or Palais and Terng [26]. □

Since M is a Riemannian K-space we can define the normal bundle to an orbit $K.x$ as

$$\mathrm{Nor}(K.x) := T(K.x)^{\perp} = \textstyle\bigsqcup_{y \in K.x} T_y(K.x)^{\perp}.$$

Furthermore, let $\mathrm{Nor}(K.x)_\varepsilon := \{X \in \mathrm{Nor}(K.x) : |X| < \varepsilon\}$.

Theorem 7.4 (Slices)**.** *There is a number $r > 0$ such that the exponential mapping $\exp|\mathrm{Nor}(K.x)_r : \mathrm{Nor}(K.x)_r \to U \subseteq M$, where U is the image, is a diffeomorphism onto U. Moreover, U is an open neighborhood of $K.x$, and*

$$S := \exp_x(\mathrm{Nor}_x(K.x)_r)$$

is a slice at x such that $U = K.S$.

A similar theorem also holds for the more general case of a proper action on a smooth manifold, and this was proved by Palais [25], see also Michor [20, Theorems 5.6 & 5.7].

Proof. See Michor [20] or Palais and Terng [26]. □

Locally, a Riemannian action is an orthogonal action on a Euclidean vector space, in the following sense. If $x \in M$ and $H = K_x$ then the representation

$$\begin{aligned} H &\longrightarrow O(\mathrm{Nor}_x(K.x)), \\ h &\longmapsto T_x l_h \end{aligned}$$

is called the SLICE REPRESENTATION of K at x. It is immediate from Proposition 7.2 that a point x is regular if and only if the slice representation at x is trivial.

Theorem 7.5. *Let (H) be in the isotropy lattice of the Riemannian K-action on M. Then the following are true.*

(i) *The subset $\mathrm{Fix}(H) = M^H = \{x \in M : H \subseteq K_x\}$ is a totally geodesic sub-manifold of M.*

(ii) *M_H is an open dense sub-manifold of M^H, and if $M = M_{(H)}$ then $M^H = M_H$. Moreover, $M_H/N(H)$ is a smooth manifold, where $N(H)$ is the normalizer of H in K.*

(iii) *Both $M_{(H)}$ and $M_{(H)}/K$ are smooth manifolds, albeit possibly with countably many connected components that may differ in dimension.*

(iv) *The orbit projection* $M_{(H)} \twoheadrightarrow M_{(H)}/K$ *is a smooth fiber bundle with typical fiber* K/H.
(v) *The orbit projection* $M_H \twoheadrightarrow M_H/N(H)$ *is a smooth fiber bundle with typical fiber* $N(H)/H$. *Moreover,* $M_{(H)}/K$ *and* $M_H/N(H)$ *are diffeomorphic.*

Proof. See Michor [20] or Palais and Terng [26]. □

For completeness sake we also record the following theorem in which we return to a more general setting.

Theorem 7.6. *Let* M *be a proper smooth* G*-manifold. Then the following are true.*

(i) *The set of regular points* M_{reg} *is open and dense in* M.
(ii) *Around any point in* M *there is an open* G*-invariant neighborhood which is only met by finitely many types of orbits.*
(iii) *The set* M_{sing}/G *of all singular orbits does not locally disconnect the orbit space* M/G.

Proof. See Michor [20, Section 6] or Palais and Terng [26]. □

8. Appendix: Polar representations

Let V be a real Euclidean vector space, and G be a connected compact Lie group. Further, let $\rho : G \to \mathrm{SO}(V, \langle\_,\_\rangle)$ be a POLAR REPRESENTATION of G on V. That is, there is subspace $\Sigma \subseteq V$ (a SECTION) such that Σ meets all G-orbits, and does so orthogonally.

The following is due to Dadok [10] and is a consequence of his classification of polar actions.

Proposition 8.1. *There exists a connected Lie group* $\tilde{G}$ *together with a representation* $\tilde{\rho} : \tilde{G} \to \mathrm{SO}(V)$ *such that the following hold. There is a real semisimple Lie algebra* $\mathfrak{l}$ *with a Cartan decomposition* $\mathfrak{l} = \mathfrak{k} \oplus \mathfrak{p}$. *Moreover, there is a Lie algebra isomorphism* $A : \mathrm{Lie}(\tilde{G}) = \tilde{\mathfrak{g}} \to \mathfrak{k}$ *and a linear isomorphism* $B : V \to \mathfrak{p}$ *such that* $B(\tilde{\rho}'(X).v) = [A(X), B(v)]$ *for all* $X \in \tilde{\mathfrak{g}}$ *and* $v \in V$. *Finally, the* G*-orbits coincide with the* $\tilde{G}$*-orbits, that is* $V/G = V/\tilde{G}$.

Proof. See Dadok [10, Proposition 6]. □

Thus, for the purpose of this paper, it suffices to assume that the representation of G on V is a symmetric space representation whence $\mathfrak{l} = \mathfrak{g} \oplus V$ is a Cartan decomposition, and hence $[\mathfrak{g}, \mathfrak{g}] \subseteq \mathfrak{g}$, $[\mathfrak{g}, V] \subseteq V$, and $[V, V] \subseteq \mathfrak{g}$. Therefore, $G \times V \cong L$, $(g, v) \mapsto g \exp(v)$ is a global Cartan decomposition with compact G where $\mathrm{Lie}(L) = \mathfrak{l}$.

An element $v \in V$ is said to be REGULAR (with respect to the G-action) if the orbit $\mathcal{O}(v) = \rho(G).v = G.v$ is of maximal possible dimension. The set of regular elements will be denoted by V_r. The following assertions are well known and/or obvious. See for example Knapp [15, Chapter VI].

Remark 1. Let $v \in V$. Then, by reason of dimension $\mathrm{ad}(v)|Z_{\mathfrak{g}}(v)^\perp : Z_{\mathfrak{g}}(v)^\perp \to Z_V(v)^\perp$ and $\mathrm{ad}(v)|Z_V(v)^\perp : Z_V(v)^\perp \to Z_{\mathfrak{g}}(v)^\perp$ both are linear isomorphisms. □

Remark 2. The set V_r of regular elements is open dense in V. Moreover, $v \in V_r$ if and only if $Z_V(v) =: \Sigma$ is a section in V. This is the case if and only if Σ is maximally abelian. □

Remark 3. Let $\Sigma \in V$ be a section, and put $\mathfrak{m} := Z_{\mathfrak{g}}(\Sigma)$. The set $R = R(\mathfrak{l}, \Sigma) \subseteq \Sigma^*$ shall denote the set of restricted roots. This gives rise to the restricted root space decomposition

$$\mathfrak{l} = \mathfrak{m} \oplus \Sigma \oplus \bigoplus_{\lambda \in R} \mathfrak{l}_\lambda.$$

Any Cartan subalgebra $\mathfrak{h} \subseteq \mathfrak{l}$ of $\mathfrak{l}$ is of the form

$$\mathfrak{h} = \mathfrak{t} \oplus \Sigma$$

where $\mathfrak{t} \subseteq \mathfrak{m}$ is a Cartan subalgebra (Lie algebra to a maximal torus) of $\mathfrak{g}$.
Each restricted root space $\mathfrak{l}_\lambda$ has an orthonormal basis

$$E^i_\lambda \in \mathfrak{g}, \quad B^i_\lambda \in V$$

where $i = 1, \ldots, k_\lambda = \frac{1}{2} \dim \mathfrak{l}_\lambda$, and which is such that $\mathrm{ad}(v) E^i_\lambda = \lambda(v) B^i_\lambda$ and $\mathrm{ad}(v) B^i_\lambda = \lambda(v) E^i_\lambda$ for all $v \in \Sigma$. The vectors

$$E^i_0, i = 1, \ldots, \dim \mathfrak{m}, \quad B^j_0, j = 1, \ldots, \dim \Sigma$$

will denote an orthonormal basis of $\mathfrak{m}$, Σ respectively. □

9. Appendix: Non-commutative integrability

The idea of non-commutative integrability under the name of degenerate integrability is due to Nehorošev [22] who also introduced the appropriate concept of action-angle variables. This section follows mainly the approach of Zung [43, 44]. See also Mishchenko and Fomenko [21]. The following definition is less general than that given in the above cited references but better suited for the applications in this paper.

Definition 9.1. Let $(M, \{\ ,\ \})$ be a Poisson manifold, and consider a Hamiltonian function $H : M \to \mathbb{R}$. We denote the Poisson sub-algebra of all first integrals of H by $\mathcal{F}_H$, that is

$$\mathcal{F}_H := \{F \in C^\infty(M) : \{F, H\} = 0\}.$$

The Hamiltonian system is called NON-COMMUTATIVELY INTEGRABLE if there is a finite dimensional Poisson vector space W and a generalized momentum map $\Phi : M \to W$ which is a Poisson morphism with respect to the Poisson structure on W such that the following are satisfied.

- $\Phi^* : C^\infty(W) \to \mathcal{F}_H$ is an isomorphism of Lie-Poisson algebras.
- $\dim M = \operatorname{ddim} C^\infty(W) + \operatorname{ddim} Z(C^\infty(W))$ where $Z(C^\infty(W))$ denotes the commutative sub-algebra of Casimir functions on W, and $\operatorname{ddim} C^\infty(W) = \dim W$ is the functional dimension of $C^\infty(W)$.

For the following assume that (M, ω, H) is a non-commutatively integrable system on a symplectic manifold with $\dim M = 2n$. Let $l = \operatorname{ddim} Z(C^\infty(W))$ and $m = \operatorname{ddim} C^\infty(W)$. Assume we are given a smooth vector valued function

$$u = (u_1, \ldots, u_l, u_{l+1}, \ldots, u_m) : W \to \mathbb{R}^m = \mathbb{R}^{2n-l}$$

such that $u_1, \ldots, u_l$ generate the center of $C^\infty(W)$ and $u_1, \ldots, u_m$ generate $C^\infty(W)$ as Poisson algebras. Assume further for simplicity that u and Φ both are submersions. If we define

$$\Phi_i := u_i \circ \Phi$$

we get the following assertions.

- $\{\Phi_i, \Phi_j\} = 0$ for all $i \leq l$ and j arbitrary.
- $\{\Phi_j, H\} = 0$ for all j.

- the set $\{d\Phi_1(x), \dots, d\Phi_l(x), d\Phi_{l+1}(x), \dots, d\Phi_{2n-l}(x)\}$ is linearly independent for all $x \in M$.

If $l = n$ this is one of the usual definitions of complete integrability.

Remark 1. Consider the level set $\Phi^{-1}(\Phi(x))$ and the connected component L containing x thereof. By definition all points in M are regular with respect to Φ. Thus L is a closed sub-manifold of dimension l. □

Remark 2. Consider the l-dimensional integrable distribution spanned by the $\nabla^\omega_{\Phi_i}$ where $i \leq l$. Since

$$\tfrac{\partial}{\partial t}|_0(\Phi_j \circ \mathrm{Fl}_t^{\nabla^\omega_{\Phi_i}})(x) = \{\Phi_j, \Phi_i\} = 0,$$

the leaf passing through x is just the connected component $L \subseteq \Phi^{-1}(\Phi(x))$ containing x. Obviously, L is an isotropic submanifold of M, and for the annihilator with respect to ω we have

$$(T_xL)^\omega := \omega\text{-Ann}(T_xL) = \mathrm{Span}\{\nabla^\omega_{\Phi_j}(x) : 1 \leq j \leq 2n-l\}.$$

Moreover, Hamiltonian flow lines of H are parallel to this foliation by isotropic sub-manifolds. Therefore, $\nabla^\omega_H(x) \in \mathrm{Span}\{\nabla^\omega_{\Phi_i}(x) : 1 \leq i \leq l\}$ and applying $\check{\omega}^{-1}$ yields

$$dH(x) = \sum_{i=1}^{l} \tfrac{\partial H}{\partial \Phi_i}(x) d\Phi_i(x).$$

□

Remark 3. As above let L be the leaf through x corresponding to the distribution spanned by $\nabla^\omega_{\Phi_i}$ where $i \leq l$. Since L is at the same time the connected component of $\Phi^{-1}(\Phi(x))$ containing x, the equation

$$dH.\nabla^\omega_{\Phi_i} = \{H, \Phi_i\} = 0$$

where $i \leq l$ implies that H is constant on the fibers of the submersion $\Phi : M \to W$. Thus there is a smooth mapping $h : W \to \mathbb{R}$ such that $\Phi^* h = H$. Moreover, as Φ is a surjective Poisson morphism it follows that h lies in the center of the Poisson algebra $C^\infty(W)$. □

Remark 4. Assume (φ, I, q, p) are GENERALIZED ACTION-ANGLE VARIABLES in the sense of Nehorošev [22] on an open subset $U \subseteq M$. That is

$$\omega|U = \textstyle\sum_{j=1}^{l} dI_j \wedge d\varphi_j + \sum_{j=1}^{n-l} dp_j \wedge dq_j,$$

and thus $\nabla^\omega_{I_j} = \check{\omega}^{-1}(dI_j) = \partial_{\varphi_j}$, for example. Assume furthermore the commutation relations

$$\{I_j, H\} = 0 \qquad \text{and} \qquad \{q_i, H\} = \{p_i, H\} = 0.$$

In these coordinates the flow equations to the Hamiltonian H then assume the canonical form

$$\begin{aligned}
&\tfrac{\partial}{\partial t}|_0(\varphi_i \circ \mathrm{Fl}_t^{\nabla^\omega_H})(x) = d\varphi_i(x).\check{\omega}^{-1}(\sum_{j=1}^{l} \tfrac{\partial H}{\partial I_j}(x) dI_j(x)) = \tfrac{\partial H}{\partial I_i}(x),\\
&\tfrac{\partial}{\partial t}|_0(I_i \circ \mathrm{Fl}_t^{\nabla^\omega_H})(x) = 0,\\
&\tfrac{\partial}{\partial t}|_0(q_j \circ \mathrm{Fl}_t^{\nabla^\omega_H})(x) = 0,\\
&\tfrac{\partial}{\partial t}|_0(q_j \circ \mathrm{Fl}_t^{\nabla^\omega_H})(x) = 0.
\end{aligned}$$

Thus we have $H = H(I)$ in these coordinates. Generalizing in accordance to the harmonic oscillator the numbers $\nu_i(x) = \frac{\partial H}{\partial I_i}(x)$ are called frequencies of the system. They are said to be independent if they are linearly independent over the rationals. □

The following is a generalized Liouville-Arnold theorem, and is the main theorem of Nehorošev [22].

Theorem 9.2. *Let (M, ω) be a symplectic manifold with $\dim M = 2n$. Assume the Hamiltonian system (M, ω, H) is non-commutatively integrable with integrals $\Phi_1, \ldots, \Phi_l, \Phi_{l+1}, \ldots, \Phi_{2n-l}$ defined by the momentum map $\Phi : M \to W$ as above and where $l \leq n$. Then a connected component L of a non-empty level surface $\Phi^{-1}(\Phi(x))$ is an isotropic submanifold of dimension l. On an open neighborhood of L there exist generalized action-angle variables (φ, I, q, p). Moreover $I_j = I_j(\Phi_1, \ldots, \Phi_l)$, and $q_j = q_j(\Phi)$ and $p_j = p_j(\Phi)$.*
The Hamiltonian flow lines are affine in these coordinates, and are given by the set of equations in Remark 4 above. If L is compact it is diffeomorphic to a l-torus,[1] otherwise it is diffeomorphic to the product of a torus by a vector space.

Proof. See Nehorošev [22, Theorem 1]. □

Say we are given a non-commutatively integrable system (M, ω, H) as above which we now additionally assume to be invariant under the Hamiltonian action by a compact Lie group G. That is G acts on M by symplectomorphisms, there is a standard momentum map $J : M \to \mathfrak{g}^*$, and $H \in C^\infty(M)^G$. We know that we can do singular Poisson reduction or singular symplectic reduction with this system to obtain a reduced Hamiltonian system. However, what happens to the integrability of the system? Curiously, this question seems to not have been formally addressed until Zung [43, 44].

Theorem 9.3. *Assume the Hamiltonian system (M, ω, H) is invariant under a Hamiltonian action of a compact Lie group G. If (M, ω, H) is non-commutatively integrable then the reduced system is integrable as well:*

- *The singularly Poisson reduced system is non-commutatively integrable.*
- *The singularly symplectic reduced system is non-commutatively integrable.*

Proof. This theorem is proved by Zung [43, Theorem 2.3]. For material on singular reduction we refer to Ortega and Ratiu [24]. □

It is crucial in the formulation of the above theorem that $\dim M = \operatorname{ddim} \mathcal{F}_H + \operatorname{ddim} Z(\mathcal{F}_H)$, and $\mathcal{F}_H$ is the set of *all* first integrals of H. Thus non-commutative integrability is much better suited in the context of reduction.

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