Simulating viscoplastic avalanches

Martin Rentschler, Christophe Ancey

July 4, 2008

Viscoplastic material

- Occurence of a yield-stress: Threashold τ_0, for the stress to shear the material.
- Examples:
  - Concrete
  - Clay,Moodflow
  - Snow,Avalanche
  - Nutrions,Cosmetics
- Focus: Rapid flows.

Viscoplastic material

- Occurence of a yield-stress: Threashold τ_0, for the stress to shear the material.
- Examples:
  - Concrete
  - Clay,Moodflow
  - Snow,Avalanche
  - Nutrions,Cosmetics
- Focus:
  Rapid flows.

Viscoplastic material

- Occurence of a yield-stress: Threashold τ_0, for the stress to shear the material.
- Examples:
  - Concrete
  - Clay,Moodflow
  - Snow,Avalanche
  - Nutrions,Cosmetics
- Focus: Rapid flows.

Viscoplastic material

- Occurence of a yield-stress: Threashold τ_0, for the stress to shear the material.
- Examples:
  - Concrete
  - Clay,Moodflow
  - Snow,Avalanche
  - Nutrions,Cosmetics
- Focus:
  Rapid flows.

Flow down an inclined plane

Outline

Stress Tensor

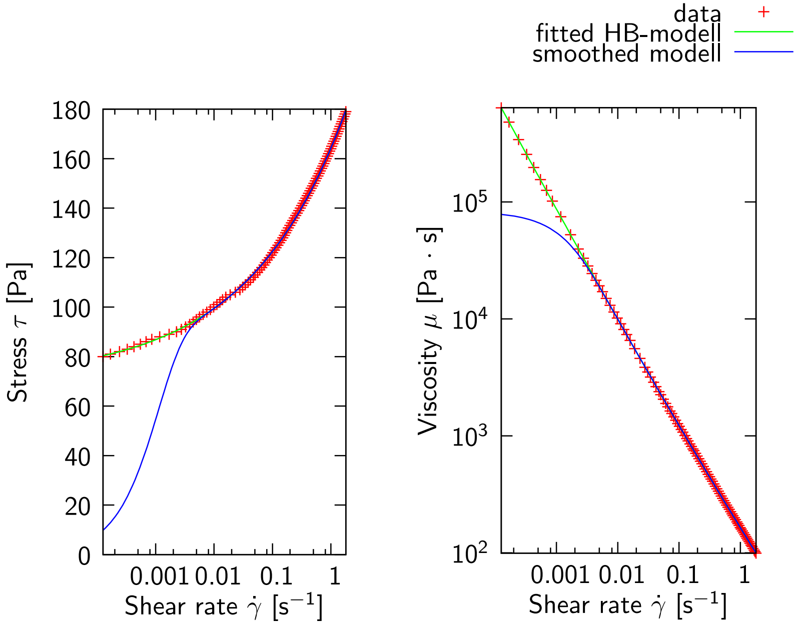
"Stress"	$\boldsymbol{\tau} = \mu \dot{\boldsymbol{\gamma}}$
"Shear-rate"	$\dot{\boldsymbol{\gamma}} = \left(\nabla \vec{u} + (\nabla \vec{u})^T\right)$

- Herschel-Bulkley fluid

$$\mu(\dot{\gamma}) = \frac{\tau_0 + K\dot{\gamma}^{\alpha}}{\dot{\gamma}}$$

  - isotrope
  - no thinxotropy
  - measurable quantity

Viscosity function



Governing equations

Momentum equation:

$$\frac{\partial \vec{u}}{\partial t} = -\nabla p + \frac{1}{\rho} \nabla \cdot \boldsymbol{\tau} \tag{1}$$

Continuity/Incompressibility:

$$\nabla \vec{u} = 0 \tag{2}$$

Equation of state:

$$\boldsymbol{\tau} = \nu(\dot{\gamma})\dot{\boldsymbol{\gamma}} \tag{3}$$

(4)

Level-Set formulation

- Signed distance function: $\Phi(\vec{x}, t)$

$$\begin{aligned} \Rightarrow \Phi = 0 &\qquad \text{Location of the interface} \\ \Phi > 0 &\qquad \text{Liquid phase} \\ \Phi < 0 &\qquad \text{Gas phase} \end{aligned}$$

- Advected by the Velocity field:

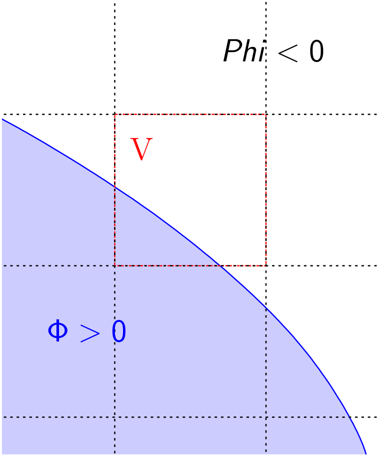
$$\frac{\partial \Phi}{\partial t} + \vec{u} \cdot \nabla \Phi = 0 \tag{5}$$

- Reinitialisation

$$|\nabla \Phi| = sgn\left(\Phi\right) \tag{6}$$

Smoothing

$$H_\epsilon(\Phi) = \begin{cases} 1 & , \Phi > \epsilon \\ 0 & , \Phi < -\epsilon \\ 1 + \sin(\frac{\Phi}{\epsilon}\pi) & \text{else.} \end{cases}$$



$$H_{dx}(\Phi) \approx \text{Volume of fluid}$$

Density:

$$\frac{d\vec{u}}{d\vec{n}} = 0$$

$$\Downarrow$$

$$\bar{\rho} = H(\Phi)\rho_{\mathrm{l}} + (1 - H(\Phi))\rho_{\mathrm{g}}$$

proof

Smoothing

$$H_\epsilon(\Phi) = \begin{cases} 1 & , \Phi > \epsilon \\ 0 & , \Phi < -\epsilon \\ 1 + \sin(\frac{\Phi}{\epsilon}\pi) & \text{else.} \end{cases}$$

$$H_{dx}(\Phi) \approx \text{Volume of fluid}$$

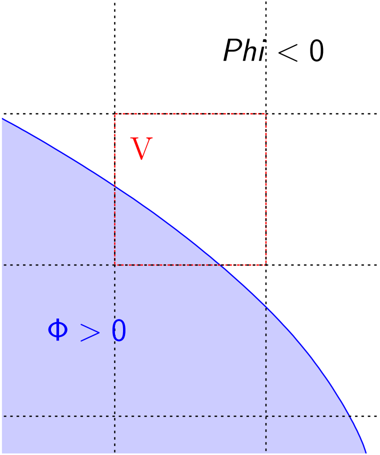
Viscosity:

$$\frac{d\boldsymbol{\tau}}{d\vec{n}} = 0$$

$$\Downarrow$$

$$\frac{1}{\mu(\Phi)} = \frac{H(\Phi)}{\mu_\mathrm{l}(\dot{\gamma}_l)} + \frac{1 - H(\Phi)}{\mu_\mathrm{g}}$$

proof



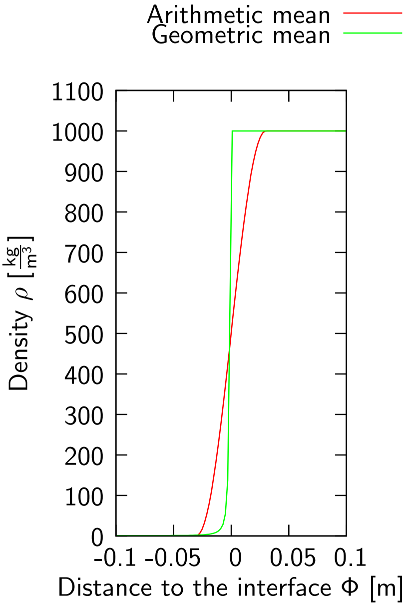
Smoothing near the interface

Geometric mean:

$$\frac{1}{\rho(\Phi)} = \frac{H(\Phi)}{\rho_l} + \frac{1 - H(\Phi)}{\rho_g}$$

Arithmetic mean:

$$\rho(\Phi) = H(\Phi)\rho_l + (1 - H(\Phi))\rho_g$$

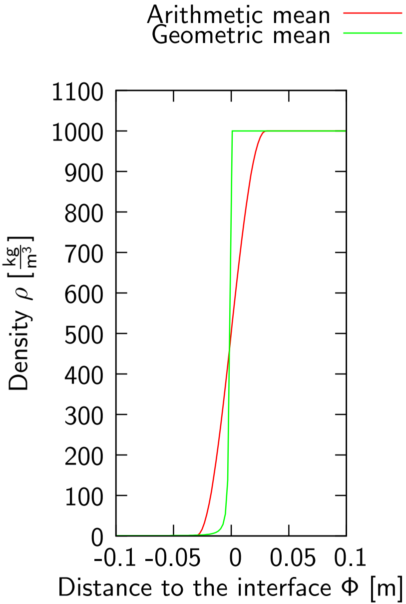


Smoothing near the interface

Geometric mean:

$$\frac{1}{\rho(\Phi)} = \frac{H(\Phi)}{\rho_\mathrm{l}} + \frac{1 - H(\Phi)}{\rho_\mathrm{g}}$$

- Sharp interface
- Physical interpretation: Vapour pressure
- Jump in the density Potential problem for the convergence!
- Numerical results match better the experimental results!



Projection scheme

$$\frac{\partial \vec{u}}{\partial t} = -\frac{1}{\rho}\nabla p - \vec{u}\cdot\nabla\vec{u} + \frac{1}{\rho}\nabla\cdot(\nu(\dot{\gamma})\dot{\boldsymbol{\gamma}}) \tag{7}$$

$$\nabla\vec{u} = 0 \tag{8}$$

- Predictor Step: Solve momentum equation (7)

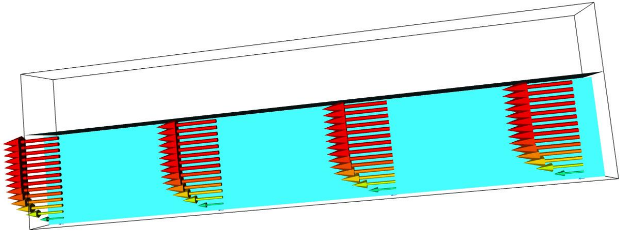
$$\frac{\vec{u}^* - \vec{u}_n}{\delta t} = -\frac{1}{\rho_n}\nabla p_n - \vec{u}_n\cdot\nabla\vec{u}_n + \frac{1}{\rho_n}\nabla\cdot\left(\frac{\nu_n}{2}\left(\dot{\boldsymbol{\gamma}}^* + \dot{\boldsymbol{\gamma}}_{\mathbf{n}}\right)\right)$$

- Corrector Step: Correct the pressure to satisfy (8)

$$\frac{\vec{u}_{n+1} - \vec{u}^*}{\delta t} = -\nabla(p_{n+1} - p^*)$$

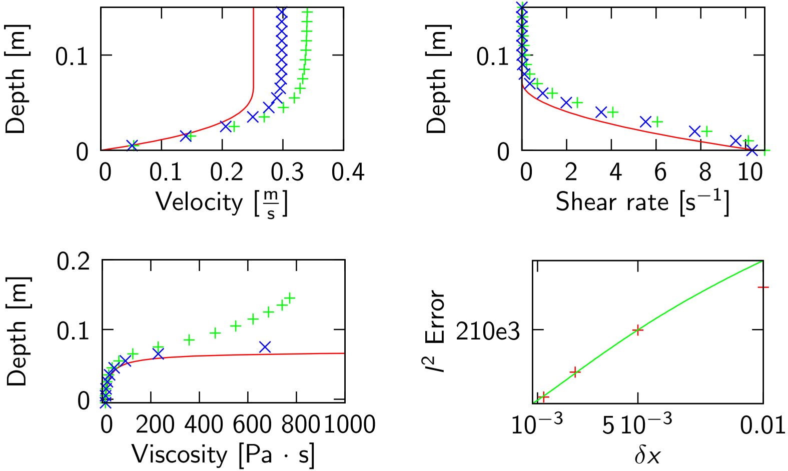
$$\nabla\vec{u}^* = \delta t\,\Delta(p_{n+1} - p^*)$$

Comparing to steady uniform flow



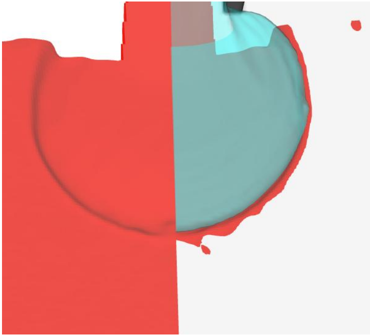
- Analytical solution for 2d one-phase free-surface flow
- Testing of:
  - Convergence of the solver
  - Influence of limiting the maximal viscosity

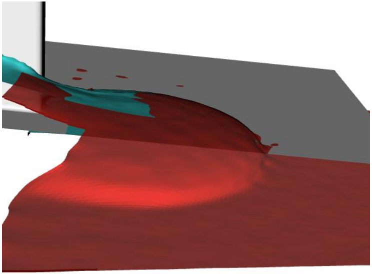
Convergence to analytical solution



Quasi 2d in a channel

Position of the free surface





Yield surface in the flow

Velocity profile

Fluid properties in interface cells

$$H(\Phi) = \begin{cases} 1 & , \Phi > \epsilon \\ 1 + sin(\frac{\Phi}{\epsilon}\pi) & , -\epsilon < \Phi < \epsilon \\ 0 & , \Phi < -\epsilon \end{cases}$$

- Interpreted as volume fraction
  (1st order Approximation)

Fluid properties in interface cells

Density:

$$\int_V \bar{\rho}\vec{u}\,dV = \int_{V\cap\{\Phi<0\}} \rho_l \vec{u}_l + \int_{V\cap\{\Phi<0\}} \rho_g \vec{u}_g \tag{9}$$

$$\bar{\rho}\int_V \vec{u}\,dV = \rho_l \int_{V\cap\{\Phi<0\}} \vec{u}_l\,dV + \rho_g \int_{V\cap\{\Phi<0\}} \vec{u}_g\,dV \tag{10}$$

$$\bar{\rho}\int_V dV = \rho_l \int_{V\cap\{\Phi<0\}} dV + \rho_g \int_{V\cap\{\Phi<0\}} dV \tag{11}$$

$$\bar{\rho} = \rho_l H(\Phi) + \rho_g (1 - H(\Phi)) \tag{12}$$

Fluid properties in interface cells

Viscosity:

$$\int_V \bar{\dot{\gamma}}\, dV = \int_{V\cap\{\Phi<0\}} \dot{\gamma}_l\, dV + \int_{V\cap\{\Phi>0\}} \dot{\gamma}_g\, dV \tag{9}$$

$$\frac{1}{\bar{\mu}} \int_V \bar{\mu}\bar{\dot{\gamma}}\, dV = \frac{1}{\mu_l} \int_{V\cap\{\Phi<0\}} \mu_l \dot{\gamma}_l\, dV + \frac{1}{\mu_g} \int_{V\cap\{\Phi>0\}} \mu_g \dot{\gamma}\, dV \tag{10}$$

$$\frac{1}{\bar{\mu}} \int_V \tau\, dV = \frac{1}{\mu_l} \int_{V\cap\{\Phi<0\}} \tau_l\, dV + \frac{1}{\mu_g} \int_{V\cap\{\Phi>0\}} \tau_g\, dV \tag{11}$$

$$\frac{1}{\bar{\mu}} \int_V dV = \frac{1}{\mu_l} \int_{V\cap\{\Phi<0\}} dV + \frac{1}{\mu_g} \int_{V\cap\{\Phi>0\}} dV \tag{12}$$

$$\frac{1}{\bar{\mu}} \approx \frac{H(\Phi)}{\mu_l} + \frac{1 - H(\Phi)}{\mu_g} \tag{13}$$

back

Shear rate in the liquid phase.
(Essential to calculate the viscosity)

$$\int\limits_{V\cap\{\Phi<0\}} \dot{\gamma}_l \, dV = \int\limits_V \bar{\dot{\gamma}} \, dV - \int\limits_{V\cap\{\Phi>0\}} \dot{\gamma}_g \, dV$$

$$\dot{\gamma}_l \, H(\Phi) = \left(\frac{1}{\bar{\mu}} - \frac{1}{\mu_g}(1 - H(\Phi)) \right) \tau$$

$$\dot{\gamma}_l \, H(\Phi) = \left(1 - \frac{\bar{\mu}}{\mu_g}(1 - H(\Phi)) \right) \bar{\dot{\gamma}}$$