Siberian Adv. Math. 1999 vol. 9 (4) pp. 125-150

PARABOLICITY OF MANIFOLDS

MARC TROYANOV

ABSTRACT. This Paper is an introduction to the study of an invariant of Riemannian manifolds related to the non-linear potential theory of the p-Laplacian and which is called its parabolic or hyperbolic type. One of our main focus is the relationship between the asymptotic geometry of a manifold and its parabolic type.

1. INTRODUCTION

A Riemann surface is called *parabolic* if it admits no positive Green function, and *hyperbolic* otherwise: one of the classical problems of complex analysis is the so called *type problem* for Riemann surfaces, which can be stated as follow: *Give criteria for a Riemann surface to be parabolic.* This problem began to be systematically investigated in the thirties by Ahlfors, Nevanlinna and Myrberg among others. An extended presentation of this theory can be found in the 1970 book of Sario and Nakai [25].

An analogous theory has been developed for discrete groups and graphs as well as for Riemannian manifolds. In view of applications to the theory of quasi-regular mapping, a conformally invariant theory has also been developed in which the Green function is related to the conformal Laplacian on the manifold.

These theories can be given a uniform treatment by introducing the notions of *p-hyperbolicity* and *p-parabolicity* for a Riemannian manifold or a graph (see [13], [18] and [32]).

The purpose of this paper is to give an introduction to the notion of p-parabolicity of manifolds and to the main geometric criteria for the type problem. We also show that, for a manifold M with bounded geometry, we can define an invariant $d_{par}(M) \in \mathbb{R}$ (the *parabolic dimension* of M) such that M is p-parabolic if $p > d_{par}(M)$ and M is p-hyperbolic if $p < d_{par}(M)$. We have included proofs when they are short, new or hard to find in the literature.

This paper is by no mean a complete exposition of the subject. In particular, relations with probability theory, with the $p-$module of families of curves, with Sobolev inequalities and with quasi-regular mappings have been left aside. One may consult [9], [33], [3] and [26] for expositions of complementary subjects.

Date: 1999.
1991 *Mathematics Subject Classification.* Primary 31C15, 31C12, 31C45; Secondary 53C20.
Key words and phrases. Differential geometry, Potential Theory.

2. Some Examples

A Riemannian manifold is p-hyperbolic ($1 \leq p < \infty$) if it contains a compact set of positive p-capacity and p-parabolic otherwise; we discuss at length this definition in the next two sections. In this section, we informally give a list of examples of p-hyperbolic and p-parabolic manifolds; in each case, their parabolicity/hyperbolicity is an easy consequence of the results contained in the present paper.

To start with, observe that compact manifolds without boundary are trivially p-parabolic for all $1 \leq p < \infty$ (one could call them the *elliptic* objects of the theory).

If a set $E \subset M$ of positive Hausdorff s-dimensional measure is removed from an n-dimensional manifold M; the resulting manifold $M \setminus E$ is p-hyperbolic for all $(n-s) < p < \infty$. In particular, if E contains an open set, then $M \setminus E$ is p-hyperbolic for all $1 \leq p < \infty$ and if $E \subset M$ is *any* non empty subset, then $M \setminus E$ is p-hyperbolic for all $n < p < \infty$.

A complete manifold with finite volume is p-parabolic for all $1 \leq p < \infty$. A complete manifold with polynomial growth of degree d is p-parabolic for all $p \geq d$. For instance a complete n-dimensional manifold with non negative Ricci curvature is p-parabolic for all $p \geq n$.

Conversely, a manifold of isoperimetric dimension d, is p-hyperbolic for all $p < d$, in particular a complete simply connected manifold with sectional curvature $K \leq -1$ is p-hyperbolic for all $p < \infty$.

For manifolds with bounded geometry, we can introduce an invariant $d_{\mathrm{par}} \in [1, \infty]$, called its *parabolic dimension*, such that the manifold M is p-parabolic if $p > d_{\mathrm{par}}(M)$ and p-hyperbolic if $p < d_{\mathrm{par}}(M)$. This is a quasi-isometric invariant. If a manifold M is the universal cover of a compact manifold N, then $d_{\mathrm{par}}(M)$ is the growth degree of the fundamental group $\pi_1(N)$.
For instance the Euclidean space $\mathbb{R}^n$ has parabolic dimension n and the hyperbolic space $\mathbb{H}^n$ has parabolic dimension ∞.

Let us finally stress that n-parabolicity is a quasi-conformaly invariant property of n-dimensional Riemannian manifolds (and is therefore also called *conformal parabolicity*).

3. Capacities

We recall in this section some basic facts about capacities.

Definition Let (M, g) be Riemannian manifold, $\Omega \subset M$ a connected domain in M and $D \subset \Omega$ a compact set. For $1 \leq p < \infty$, the $p-$capacity of D in Ω is defined by:

$$\mathrm{Cap_p}(D, \Omega) := \inf \left\{ \int_\Omega |du|^p : u \in W_0^{1,p}(\Omega) \cap C_0^0(\Omega), \, u \geq 1 \text{ on } D \right\}$$

where the Sobolev space $W_0^{1,p}(\Omega)$ is the closure of $C_0^1(\Omega)$, the space of compactly supported C^1 functions, with respect to the Sobolev norm

$$\|u\|_{1,p} := \|u\|_{L^p} + \||du|\|_{L^p} .$$

Remarks In the above definition, a simple truncation argument shows that one may restrict oneself to functions $u \in W_0^{1,p}(\Omega) \cap C_0^0(\Omega)$ such that $0 \le u \le 1$.

We can extend this definition to arbitrary sets $A \subset \Omega$ by a min-max procedure: first, for an open set $U \subset \Omega$, one defines

$$\mathrm{Cap_p}(U,\Omega) := \sup_{U \supset D \text{ compact}} \mathrm{Cap_p}(D,\Omega)\,;$$

and then, for an arbitrary set A,

$$\mathrm{Cap_p}(A,\Omega) := \inf_{A \subset U \subset \Omega \text{ open}} \mathrm{Cap_p}(U,\Omega)\,.$$

We begin by a very simple observation:

Lemma 3.1. *Suppose $\Omega \subset M$ has finite volume. If $\mathrm{Cap_p}(D,\Omega) = 0$, then $Cap_q(D,\Omega) = 0$ for all $1 \le q < p$.*

Proof This is a direct consequence of Hölder's inequality

$$\int_\Omega |du|^q \;\le\; (\mathrm{Vol}(\Omega))^{(p-q)/p} \left(\int_\Omega |du|^p \right)^{q/p}$$

□

The following properties of capacities are well known (see eg. [12], [21] and [6]).

Theorem 3.2. *Capacities enjoy the following properties.*

i) $\mathrm{Cap_p}(D,\Omega_1) = \inf\{\, \mathrm{Cap_p}(U,\Omega_1) | U$ *is open and* $D \subset U \subset\subset \Omega\}$;
ii) $\Omega_1 \subset \Omega_2 \quad \Rightarrow \quad \mathrm{Cap_p}(D,\Omega_1) \ge \mathrm{Cap_p}(D,\Omega_2)$;
iii) $D_1 \subset D_2 \quad \Rightarrow \quad \mathrm{Cap_p}(D_1,\Omega) \le \mathrm{Cap_p}(D_2,\Omega)$;
iv) $\mathrm{Cap_p}(D_1 \cup D_2,\Omega) \le \mathrm{Cap_p}(D_1,\Omega) + \mathrm{Cap_p}(D_2,\Omega) - \mathrm{Cap_p}(D_1 \cap D_2,\Omega)$;
v) *If* $U \subset\subset \Omega$ *is open, then* $\mathrm{Cap_p}(\bar U,\Omega) = \mathrm{Cap_p}(\partial U,\Omega)$;
vi) *If* $D \subset \Omega_1 \subset \Omega_2 \cdots \subset \cup_i \Omega_i = \Omega$, *then* $\mathrm{Cap_p}(D,\Omega) = \lim_{i\to\infty} \mathrm{Cap_p}(D,\Omega_i)$.

□

We also define a local notion of sets with zero capacity:

Definition A set $E \subset M$ (not necessarily compact) is said to be a *null set* for the $p-$capacity, or a $p-$*polar set*, if for every pair of open balls $B_1 \subset\subset B_2$ we have

$$\mathrm{Cap_p}(E \cap \overline{B_1}, B_2) = 0\,.$$

The set E is said to be of local positive $p-$capacity otherwise.

The property of being a $p-$polar set is local and independent of the choice of a Riemannian metric.
The next lemma is an immediate consequence of Lemma 3.1.

Lemma 3.3. *If E is a $p-$polar set, then E is also a $q-$polar set for all $q \le p$.*

□

Further properties of null sets are now listed (see [12]):

Theorem 3.4. *Null sets for the $p-$capacity in a Riemannian manifold M of dimension n have the following properties :*

i) *If E is bounded and* $\mathrm{Cap_p}(E,U) = 0$ *for some relatively compact open neighbourhood U, then E is a $p-$polar set;*
ii) *A countable union of $p-$polar sets is a $p-$polar set;*
iii) *$p-$polar sets have zero (n-dimensional) measure;*
iv) *If there is a constant $C < \infty$ such that for all open sets $U \supset E$ we have* $\mathrm{Cap_p}(E,U) \leq C$, *then E is a $p-$polar set.*

□

There is an important relation between capacities and Hausdorff dimension:

Theorem 3.5. *Let $E \subset M$ be a bounded set of Hausdorff dimension s ($0 < s < n-1$). Then E is a $p-$polar set for $1 < p < (n-s)$ and E is a set of local positive $p-$capacity if $p > (n-s)$.*

A proof can be found in [12, pp. 43–48]. □

Proposition 3.1. *A closed set $E \subset M$ is a p-polar set if and only if for every neighbourhood U of E and every $\epsilon > 0$, there exists a function $u \in C^1(M)$ such that*
i) *the support of u is contained in $M \setminus E$;*
ii) $0 \leq u \leq 1$;
iii) $u \equiv 1$ *on* $M \setminus U$;
iv) $\int |du|^p \leq \epsilon$.

Proof Let $E \subset M$ be a p-polar set. We first assume that E is bounded, then for each bounded neighbourhood U of E, we have $\mathrm{Cap_p}(E,U) = 0$.
From the first assertion of Theorem 3.2, we know that there exists a function $v \in C^1_0(U)$ such that $v = 1$ in a neighbourhood of E , $0 \leq v \leq 1$, and $\int |dv|^p \leq \epsilon$.
The function $u \in C^1(M)$ defined by

$$u(x) = \begin{cases} 1 - v(x) & \text{if } x \in U, \\ 1 & \text{if } x \in M \setminus U \end{cases}$$

has the desired properties.

Suppose now that $E \subset M$ is an unbounded p-polar subset, and let $U \supset E$ be some neighbourhood. We can decompose E as a countable union of disjoint bounded sets $E = \cup_{i=1}^{\infty} E_i$. For each i, we can find a bounded neighbourhood U_i such that $E_i \subset U_i \subset U$ and such that the covering $\{U_i\}$ is locally finite (i.e. each compact subset of M meets only finitely many U_i).

We just proved that for each i, there exists $u_i \in C^1(M)$ satisfying
i) the support of u_i is contained in $M \setminus E_i$;
ii) $0 \leq u_i \leq 1$;
iii) $u_i \equiv 1$ on $M \setminus U_i$;
iv) $\int |du_i|^p \leq 2^{-pi}\epsilon$.

The function

$$u = \prod_{i=1}^{\infty} u_i$$

satisfies the conditions of the proposition.

To prove the converse direction, we choose a pair of balls $B_1 \subset B_2 \subset M$ and a smooth function $\varphi : M \to [0,1]$ such that $\varphi = 1$ on B_1 and $\mathrm{supp}(\varphi) \subset B_2$. Let us set

$c := \|d\varphi\|_{L^\infty}$. Now choose a neighbourhood U of E such that $\mathrm{Vol}(U \cap B_2) < \epsilon$ and a function $u : M \to [0,1]$ satisfying (i)–(iv). Let us define the function $w : M \to [0,1]$ by

$$w := \min\{\varphi, 1-u\},$$

then $w = 1$ on a neighbourhood of $E \cap B_1$ and $\mathrm{supp}(w) \subset B_2 \cap U$, therefore

$$\|dw\|_{L^p(B_2)} \leq \|d(1-u)\|_{L^p(B_2\cap U)} + \|d\varphi\|_{L^p(B_2\cap U)}.$$

But $\|d(1-u)\|_{L^p} = \|du\|_{L^p} \leq \epsilon^{1/p}$ and

$$\|d\varphi\|_{L^p(B_2\cap U)} \leq \sup(|d\varphi|)\left(\mathrm{Vol}(U\cap B_2)\right)^{1/p} \leq c\,\epsilon^{1/p},$$

thus

$$\|dw\|_{L^p(B_2)} \leq (1+c)\epsilon^{1/p}.$$

It follows that $\mathrm{Cap}_p(E \cap B_1, B_2) = 0$.

□

Let us also mention that $p-$polar sets are exceptional sets in the theory of Sobolev functions (see [12, th. 2.42]).

If a domain $\Omega \subset M$ is not relatively compact, then the condition $\mathrm{Cap_p}(D,\Omega) = 0$ does not imply that D is a $p-$polar set. In fact, we have the following important result :

Proposition 3.2. *Let Ω be a connected domain in (M,g) and $D_1 \subset\subset D_2 \subset\subset \Omega$ be compact sets. Suppose that D_1 has non empty interior and that* $\mathrm{Cap_p}(D_1,\Omega) = 0$. *Then* $\mathrm{Cap_p}(D_2,\Omega) = 0$.

We sketch below the proof of this proposition in order to illustrate the kind of potential theoretical arguments needed (compare [12, pp. 179–181]).

We will need a few properties of p-harmonic functions. These are continuous functions $u : \Omega \to \mathbb{R}$, which are weak solutions to the equation

$$\Delta_p u = 0.$$

Where Δ_p, the $p-$Laplacian on (M,g), is the Euler-Lagrange operator associated to the functional $\int |du|^p$; that is

$$\Delta_p u = -\mathrm{div}(|\nabla u|^{p-2}\nabla u). \tag{3.1}$$

Proof of Proposition 3.2 By monotonicity of the $p-$capacity, we may suppose that D_1 and D_2 are the closure of smooth domains (otherwise shrink D_1 and enlarge D_2), we may also assume $D_1 \subset\subset D_2$.
Choose an exhaustion of Ω by bounded smooth domains U_i :

$$D_1 \subset\subset D_2 \subset\subset U_1 \subset\subset U_2 \subset\subset \ldots U_k \subset\subset \Omega.$$

Then, by [12, p. 106 and p. 332] there exists a unique continuous solution $u_i : \bar{U}_i \to \mathbb{R}$ to the Dirichlet problem

$$
(3.2) \qquad \begin{cases} u_i = 1 & \text{on } D_1\,; \\ u_i = 0 & \text{on } \partial U_i\,; \\ \Delta_p u = 0 & \text{on } U_i \setminus D_1\,. \end{cases}
$$

The maximum principle [12, p.111] implies the following:

i) $0 \leq u_i \leq 1$;
ii) the infimum $c := \inf_{D_2}(u_1)$ of u_1 on D_2 is > 0;
iii) $u_{i+1} \geq u_i$ on U_i (since $u_{i+1} \geq u_i = 0$ on ∂U_i).

In particular, $u_i \geq c > 0$ on D_2 for all i.
We also know by [12, th. 5.28 and 5.29 pp. 106–107] that u_i is an extremal function, that is

$$
\mathrm{Cap_p}(D_1, U_i) = \int_{U_i} |du_i|^p\,.
$$

Define now the function $v_i : \Omega \to \mathbb{R}$ by $v_i = u_i/c$ on U_i and 0 on $\Omega \setminus U_i$. Then $v_i \geq 1$ on D_2, therefore

$$
\begin{aligned}
\mathrm{Cap_p}(D_2, \Omega) &\leq \lim_{i\to\infty} \int_{\Omega} |dv_i|^p = \frac{1}{c^p} \lim_{i\to\infty} \int_{U_i} |du_i|^p \\
&= \frac{1}{c^p} \lim_{i\to\infty} \mathrm{Cap_p}(D_1, U_i) = \frac{1}{c^p}\,\mathrm{Cap_p}(D_1, \Omega) = 0\,.
\end{aligned}
$$

□

The above Proposition has the following consequence

Corollary 3.1. *If* $\mathrm{Cap_p}(D, \Omega) = 0$ *for some compact subset* $D \subset \Omega$ *with non empty interior, then* $\mathrm{Cap_p}(D', \Omega) = 0$ *for every compact subset* $D' \subset \Omega$.

□

The meaning of this result is that whether or not a ball in (M, g) has positive $p-$capacity is a property of the manifold M and not of the ball. A manifold with this property is said to be $p-$*parabolic*.

4. Parabolicity

Definition Let Ω be a connected domain in a Riemannian manifold (M, g) and p a real number ≥ 1. We say that Ω is $p-$*parabolic* if there exists a compact set $D \subset \Omega$ with non empty interior such that $\mathrm{Cap_p}(D, \Omega) = 0$.

And we say that Ω is $p-$*hyperbolic* if there exists a compact set $D \subset \Omega$ with non empty interior such that $\mathrm{Cap_p}(D, \Omega) > 0$.

Remark This is a dichotomy: every domain is either $p-$parabolic or $p-$hyperbolic. Indeed, Corollary 3.1 says that Ω is $p-$parabolic if and only if $\mathrm{Cap_p}(D', \Omega) = 0$ for all compact subsets $D' \subset \Omega$.

We first observe that hyperbolicity is preserved when passing to a subset.

Lemma 4.1. *If* (N, g) *is a* $p-$*hyperbolic manifold, then every open domain* $\Omega \subset N$ *is also* $p-$*hyperbolic.*

Proof Suppose that Ω is $p-$parabolic. Then there exists a ball $B \subset \Omega$ such that $\mathrm{Cap}_p(B, \Omega) = 0$, hence $\mathrm{Cap}_p(B, N) = 0$ and N is thus $p-$parabolic. □

A domain is p-parabolic if it is possible to approximate the function 1 by functions with compact support and small $p-$energy:

Proposition 4.1. *The domain Ω is $p-$parabolic if and only if there exists a sequence of functions $u_j \in C_0^1(\Omega)$ such that $0 \leq u_j \leq 1$, $u_j \to 1$ uniformly on every compact subsets of Ω and*

$$\int_\Omega |du_j|^p \to 0\,.$$

Proof Suppose $\mathrm{Cap_p}(D, \Omega) = 0$ where $D \subset \Omega$ is compact with non empty interior. Choose an exhaustion

$$D \subset D_1 \subset D_2 \subset \cdots \Omega$$

of Ω by compact subsets. By Proposition 3.2, we know that $\mathrm{Cap_p}(D_j, \Omega) = 0$ for all j; hence we can find a function $u_j \in C_0^1(\Omega)$ such that $u_j \equiv 1$ on D_j and $\int_\Omega |du|^p \leq 1/j$. We have constructed the desired sequence u_j.

Conversely, suppose that there exists a sequence $u_j \in C_0^1(\Omega)$ with the stated properties. Then we can find a ball $B \subset \Omega$ and $j_0 \in \mathbb{N}$ such that $u_j \geq \frac{1}{2}$ on B for all $j \geq j_0$. It follows that $\mathrm{Cap_p}(B, \Omega) = 0$. □

The concept of parabolicity is also related to the existence of a Green function via the following result due to Ilkka Holopainen [13, th. 5.2]. (sees also [18]).

Theorem 4.2. *Let Ω be a domain in a Riemannian manifold (M, g). Then the following are equivalent:*

i) *Ω is $p-$parabolic;*
ii) *there is no non constant positive $p-$superharmonic function on Ω;*
iii) *there is no positive Green function for the $p-$Laplacian Δ_p on Ω.*

(Recall that the $p-$Laplacian is the operator $\Delta_p u = \mathrm{div}(\|\nabla u\|^{p-2})\nabla u$, and that a function u is $p-$superharmonic if $\Delta_p u \leq 0$.)

□

Remark Without the positivity condition, there always exists a Green function on a complete manifold, at least for $p = 2$ (see [19]).

When $p = 2$, we also have the following connection with Brownian motion (Beurling-Deny criterion):

Theorem 4.3. *The Brownian motion on a domain $\Omega \subset (M, g)$ is recurrent if and only if Ω is $2-$parabolic.*

A proof can be found in [1, p.44] or [9, th. 5.1]. □

The next result provides us with a large supply of examples of parabolic and hyperbolic (incomplete) manifolds.

Theorem 4.4. *Let $E \subset M$ be a subset in a Riemannian manifold M.*

(A) If $E \subset M$ has local positive $p-$capacity, then $\Omega := M \setminus E$ is $p-$hyperbolic.

(B) If M is $p-$parabolic and $E \subset M$ is a $p-$polar set, then $\Omega := M \setminus E$ is $p-$parabolic.

Proof We first prove (A). Suppose that Ω is $p-$parabolic. By Proposition 4.1, we can find a sequence of functions $u_j \in C_0^1(\Omega)$ such that $0 \leq u_j \leq 1$, $u_j \to 1$ uniformly on every compact subsets of Ω and $\int_\Omega |du_j|^p \to 0$.
But, by Proposition 3.1, this implies that $E \subset M$ is a p-polar set.

To prove (B), observe that by Proposition 4.1, and because M is $p-$parabolic, there exists a sequence $v_j \in C_0^1(M)$ such that $0 \leq v_j \leq 1$, $v_j \to 1$ uniformly on every compact subsets of M and $\int_M |dv_j|^p \to 0$.
Since E is a $p-$polar set, Proposition 3.1 implies the existence of another sequence $w_j : M \to [0,1]$, of smooth functions with support in $M \setminus E$, such that $w_j \to 1$ uniformly on every compact subsets of $M \setminus E$ and $\int_M |dw_j|^p \to 0$.
Now set $u_j := v_j w_j : \Omega \to \mathbb{R}$. Since the sequence $\{u_j\}$ clearly satisfies all conditions of Proposition 4.1, we deduce that Ω is $p-$parabolic. □

Corollary 4.1. *Let M be a closed Riemannian manifold and $E \subset M$ a set of Hausdorff dimension s ($0 < s < n =$dim(M)). Then $\Omega := M \setminus E$ is $p-$parabolic if $1 < p < n - s$ and $p-$hyperbolic for $p > n - s$.*

Proof This is a direct consequence of the previous Theorem and Theorem 3.5. □

Other criteria for parabolicity are discussed in [7].

5. Geometric Estimates

5.1. **Capacity in term of hypersurface integrals.** In order to give some useful geometric estimates for capacities, we associate to each pair $D \subset\subset \Omega \subset M$ the class $\Lambda(D, \Omega)$ of functions $h : \Omega \to \mathbb{R}$ such that

i) h is continuous, locally Lipschitz, non constant and bounded below;
ii) $D \subset \{x \in \Omega : h(x) = r_0 := \min h\}$;
iii) if $r < \sup h$ then $\{x \in \Omega : h(x) \leq r\}$ is compact.

The $p-$*flux* of a function $h \in \Lambda(D, \Omega)$ is the function $\Phi_{h,p} : [r_0, r_1) \to \mathbb{R}$ defined by

$$\Phi_{h,p}(r) = \int_{\partial\Omega_r} |\nabla h(x)|^{p-1} d\sigma(x)\,,$$

where $\Omega_r := \{x \in \Omega : h(x) < r\}$ and $[r_0, r_1)$ is the range of h (i.e. $r_0 := \min h \in \mathbb{R}$ and $r_1 := \sup h \in \mathbb{R} \cup \{\infty\}$) and $d\sigma$ is the $(n-1)$-dimensional Hausdorff measure.

Note that Ω_r is a bounded domain if $r_0 < r < r_1$. When $p = 1$, then $\Phi_{h,p}$ is simply the area :

$$\Phi_{h,1}(r) = a_h(r) = \int_{\partial\Omega_r} d\sigma(x)\,.$$

For a function $h \in \Lambda(D, \Omega)$ of class C^2, we also have by the divergence formula,

$$\Phi_{h,p}(r) = \int_{\partial\Omega_r} |\nabla h|^{p-2} \langle \nabla h, \mathbf{n} \rangle d\sigma = -\int_{\Omega_r} (\Delta_p h)\, d\,vol\,.$$

We next define $Q_p(h)$ (for $p > 1$) to be the following integral.

$$Q_p(h) = \left(\int_{r_0}^{r_1} \Phi_{h,p}(r)^{\frac{1}{1-p}} dr \right)^{1-p}.$$

The next result is similar to the result of §2.2.2 in [21].

Theorem 5.1. *Let $D \subset\subset \Omega \subset M$ and $p > 1$, then*

$$\mathrm{Cap_p}(D,\Omega) = \inf_{h \in \Lambda(D,\Omega)} Q_p(h)\,.$$

We will need in the proof the following version of the coarea formula (see [6, p. 118] or [21, p. 37]):

$$\int_{\{h>t\}} g(x)\,|\nabla h(x)|\,dx = \int_t^\infty \left(\int_{\{h=s\}} g(x) d\sigma(x) \right) ds$$

which holds if $g : \Omega \to \mathbb{R}$ is integrable and $h : \Omega \to \mathbb{R}$ is locally Lipshitz.
Proof Fix $\epsilon > 0$ and let $u \in C_0^1(\Omega)$ be a function such that $0 \le u \le 1$, $u = 1$ on D and $\mathrm{Cap_p}(D,\Omega) \ge \int_\Omega |du|^p - \epsilon$.

Set $h(x) := 1 - u(x)$, then $h \in \Lambda(D,\Omega)$. We have $r_0 := \min h = 0$ and $r_1 := \max h = 1$. By the coarea formula

$$\int_\Omega |du|^p = \int_\Omega |dh|^p = \int_0^1 \left(\int_{\partial\Omega_r} |\nabla h(x)|^{p-1} d\sigma(x) \right) dr = \int_0^1 \Phi_{h,p}(r) dr\,.$$

Thus, we have from lemma 5.2 below

$$\mathrm{Cap_p}(D,\Omega) + \epsilon \ge \int_\Omega |du|^p = \int_0^1 \Phi_{h,p}(r)dr \ge \left(\int_{r_0}^{r_1} \Phi_{h,p}(r)^{\frac{1}{1-p}} dr \right)^{1-p} = Q_p(h)\,,$$

and, since ϵ is arbitrary, we have

$$\mathrm{Cap_p}(D,\Omega) \ge \inf_{h \in \Lambda(D,\Omega)} Q_p(h)\,.$$

To see the reverse inequality, we choose some function $h \in \Lambda(D,\Omega)$. We then fix some number $s \in (r_0, r_1]$ and define a function $u_s : \Omega \to \mathbb{R}$ by

$$u_s(x) = \begin{cases} 1 & \text{if } h(x) \le r_0\ ; \\ 1 - \gamma_s \displaystyle\int_{r_0}^{h(x)} \frac{dt}{\Phi_{h,p}(t)^{1/p-1}} & \text{if } r_0 \le h(x) \le s\ ; \\ 0 & \text{if } r(x) \ge s\ ; \end{cases}$$

where

$$\gamma_s := \left(\int_{r_0}^s \Phi_{h,p}(t)^{1/1-p} dt \right)^{-1}\,.$$

Then u_s is Lipschitz with compact support on Ω and we have a.e.

$$|\nabla u_s(x)| = \begin{cases} \gamma_s\, \Phi_{h,p}(h(x))^{1/1-p} |\nabla h(x)| & \text{if } r_0 \le h(x) \le s\ ; \\ 0 & \text{else}\ . \end{cases}$$

We thus have

$$\begin{aligned}\int_\Omega |\nabla u_s|^p dx &= \gamma_s^p \int_{\Omega_s \setminus \Omega_{r_0}} \Phi_{h,p}(h(x))^{p/1-p} |\nabla h(x)|^p dx \\ &= \gamma_s^p \int_{r_0}^{s} \Phi_{h,p}(h(x))^{p/1-p} \left(\int_{\partial\Omega_r} |\nabla h(x)|^{p-1} d\sigma(x) \right) dr \\ &= \gamma_s^p \int_{r_0}^{s} \Phi_{h,p}(r)^{1/1-p} dr = \gamma_s^{p-1}\end{aligned}$$

Thus

$$\mathrm{Cap_p}(D, \Omega) \leq \int_\Omega |du|^p = \gamma_s^{p-1}$$

for every $s < r_1$. By continuity, we have

$$\lim_{s \to r_1} \gamma_s^{p-1} = Q_p(h)\,,$$

and the theorem follows. □

Lemma 5.2. *Let $m(r)$ be any positive bounded function and $p > 1$. Then*

$$\left(\int_0^1 m(r)^{\frac{1}{1-p}} dr \right)^{1-p} \leq \int_0^1 m(r) dr\,.$$

Proof By Hölder inequality, we have

$$1 = \int_0^1 \left(\frac{1}{m(r)} \right)^{\frac{1}{p}} m(r)^{\frac{1}{p}} dr \leq \left(\int_0^1 \left(\frac{1}{m(r)} \right)^{\frac{1}{p-1}} dr \right)^{\frac{p-1}{p}} \left(\int_0^1 m(r) dr \right)^{\frac{1}{p}}\,.$$

Raising this inequality to the power p, one obtains the desired inequality. □

The functional Q_p has the following important invariance property :

Proposition 5.1. *Let $\lambda : \mathbb{R} \cup \{\infty\} \to \mathbb{R} \cup \{\infty\}$ be any locally Lipschitz monotonic function. Then for any $h \in \Lambda(D, \Omega)$, we have*

$$Q_p(\lambda \circ h) = Q_p(h)\,.$$

Proof Let $h \in \Lambda(D, \Omega)$. The function λ defines a homeomorphism $\lambda : [r_0, r_1] \to [t_0, t_1]$ such that $\lambda'(r) > 0$ a.e.
Set $\tilde{h} := \lambda \circ h$, then $\Phi_{\tilde{h},p}(\lambda(r)) = \lambda'(r)^{p-1} \Phi_{h,p}(r)$, whence

$$\int_{t_0}^{t_1} \Phi_{\tilde{h},p}(t)^{\frac{1}{1-p}} dt = \int_{r_0}^{r_1} \Phi_{h,p}(r)^{\frac{1}{1-p}} dr\,.$$

□

Corollary 5.1. *In the computation of $Q_p(h)$, we may assume $|\nabla h(x)| \leq 1$ a.e.*

Proof Choose a smooth positive function $\varphi : \mathbb{R} \to \mathbb{R}$ such that $|\nabla h(x)| \leq \varphi(h(x))$ a.e. Now set $\lambda(r) = \int_{r_0}^{r} \frac{ds}{\varphi(s)}$. Then $\tilde{h} := \lambda \circ h$ satisfies $|\nabla \tilde{h}(x)| = \lambda'(h(x))|\nabla h(x)| = \dfrac{1}{\varphi(s)} |\nabla h(x)| \leq 1$, and $Q_p(\tilde{h}) = Q_p(h)$. □

5.2. **The case** $p = 1$**.** For $p = 1$, Theorem 5.1 must be replaced by the following

Proposition 5.2. *Let* $D \subset\subset \Omega \subset M$, *then*

$$\mathrm{Cap}_1(D, \Omega) = \inf \mathrm{Area}(\partial U)$$

where the infimum is taken over all the sets $U \subset\subset \Omega$ *with finite perimeter containing* D.

Proof Choose a function $u : \Omega \to [0, 1]$ with compact support and such that $u = 1$ on D. By the coarea formula we have

$$\int_\Omega |du| = \int_0^1 \mathrm{Area}(\partial U_t)dt \geq \inf_t \mathrm{Area}(\partial U_t)$$

where $U_t := \{ x \in \Omega | \, u(x) \geq t\}$; hence $\mathrm{Cap}_1(D, \Omega) \geq \inf \ \mathrm{Area}(\partial U)$.
To see the reverse inequality, we consider a regular domain $U \subset\subset \Omega$ containing D. The characteristic function $\chi_U \in BV(\Omega)$ and we can therefore (see [21, p. 298]) find a sequence of functions $u_j \in C_0^\infty(\Omega)$ such that $u_j = 1$ on D and

$$\mathrm{Area}(\partial U) = \langle \chi_U \rangle = \lim_{j\to\infty} \int_\Omega |du_j|$$

hence $\mathrm{Cap}_1(D, \Omega) \leq \inf \ \mathrm{Area}(\partial U)$. □

5.3. **Manifold with warped cylindrical end.** A Riemannian manifold M is said to have a *warped cylindrical end* if there exists a compact Riemannian manifold (N, g_N) and a compact subset $D \subset\subset M$ such that $M \setminus D = N \times_f [1, \infty)$ is the warped product of N and $[1, \infty)$ (i.e the direct product with the Riemannian metric $dt^2 + f^2(t)g_N$).

Let us denote by $D_t \subset M$ the subset $D_t := D \cup \{(y, s) \, | y \in N \text{ and } s < t\}$.

Proposition 5.3. *For* $1 \leq r \leq R \leq \infty$ *we have*

$$\mathrm{Cap_p}(\overline{D}_r, D_R) = \mathrm{Vol}_{n-1}(N) \left(\int_r^R f(t)^{\frac{n-1}{1-p}} dt \right)^{1-p}$$

For instance the p-capacity of a spherical ring in $\mathbb{R}^n$ is given by

$$\mathrm{Cap_p}(\overline{B}_r, B_R) = \alpha_{n-1} \left(\frac{R^\nu - r^\nu}{\nu} \right)^{1-p} .$$

where α_{n-1} is the area of the unit sphere S^{n-1} and $\nu = \frac{n-p}{1-p}$ if $p \neq n$ and

$$\mathrm{Cap}_n(\overline{B}_r, B_R) = \alpha_{n-1} \left(\log(R/r)\right)^{1-n} .$$

Corollary 5.2. M *is* p*-parabolic if and only if*

$$\int_1^\infty f(t)^{\frac{n-1}{1-p}} dt = \infty$$

□

For instance the Euclidean space $\mathbb{R}^n$ is p-parabolic for $p \geq n$ and p-hyperbolic for $p < n$.

Proof of the Proposition Let us choose the function $h \in \Lambda(D_r, D_R)$ defined by $h(x) = 1$ if $x \in D$ and $h(x) = t$ if $x = (y,t) \in N \times \{t\}$. Then $|\nabla h(x)| = 1$ for all $x \in M \setminus D$ and the p-flux of h is equal to the area:

$$\Phi_{h,p}(r) = \mathrm{Area}(\partial D_r) = \mathrm{Vol}_{n-1}(N)\, f(t)^{n-1}.$$

One thus obtains from Theorem 5.1

$$\mathrm{Cap_p}(\overline{D}_r, D_R) \leq Q_p(h) = \mathrm{Vol}_{n-1}(N) \left(\int_r^R f(t)^{\frac{n-1}{1-p}} dt \right)^{1-p}.$$

To prove the converse inequality, let us consider an arbitrary function $u \in C_0^1(D_R)$ such that $u \equiv 1$ on D_r. We have for any $y \in N$

$$1 = \left| \int_r^R \frac{\partial u(y,t)}{\partial t} dt \right| \leq \int_r^R |\nabla u(y,t)|\, dt\ ,$$

using Hölder's inequality one gets

$$\begin{aligned} 1 \quad &\leq \quad \int_r^R |\nabla u(y,t)|\, dt = \int_r^R \left(|\nabla u(y,t)|\, f(t)^{(n-1)/p} \right) \left(f(t)^{(1-n)/p} \right) dt \\ &\leq \quad \left(\int_r^R |\nabla u(y,t)|^p\, f(t)^{(n-1)} dt \right)^{1/p} \left(\int_r^R f(t)^{(1-n)/(p-1)} dt \right)^{(p-1)/p}, \end{aligned}$$

i.e.

$$\int_r^R |\nabla u(y,t)|^p\, f(t)^{(n-1)} dt \geq \left(\int_r^R f(t)^{(1-n)/(p-1)} dt \right)^{(1-p)}$$

for all $y \in N$. Integrating this inequality over N gives us

$$\int_{D_R} |\nabla u|^p\, dx \geq \mathrm{Vol}_{n-1}(N) \left(\int_r^R f(t)^{(1-n)/(p-1)} dt \right)^{(1-p)}$$

Since u is an arbitrary test function, we conclude that
$\mathrm{Cap_p}(\overline{D}_r, D_R) \geq \mathrm{Vol}_{n-1}(N) \left(\int_r^R f(t)^{\frac{n-1}{1-p}} dt \right)^{1-p}$. □

5.4. **Isoperimetric Profile.** Let $\Omega \subset (M, g)$ be open and $D \subset \Omega$ compact. Choose a function $h \in \Lambda(D, \Omega)$, and set $\Omega_r := \{x \in \Omega : h(x) < r\}$. Let $[r_0, r_1)$ be the range of h and note $v(r) = v_h(r) := \mathrm{Vol}(\Omega_r)$, $v_0 = v(r_0)$ and $v_1 = v(r_1)$.

The function $v(r)$ defines an absolutely continuous homeomorphism $v : (r_0, r_1) \to (v_0, v_1)$. The inverse function is given by $r(v) = \sup\{r \,|\, \mathrm{Vol}(\Omega_r) < v\}$ and the derivative of v is given by the coarea formula:

$$\frac{dv}{dr} = \int_{\partial \Omega_r} \frac{d\sigma}{|\nabla h|}\,. \tag{5.1}$$

We denote by $a(r) = a_h(r) := \mathrm{Area}(\partial \Omega_r)$. The area may then be expressed in function of the volume v and we denote by $\tilde{a}(v) = a(r(v))$ the corresponding function.

Proposition 5.4. *If $p > 1$, then*

$$Q_p(h) \geq \left(\int_{v_0}^{v_1} \tilde{a}(v)^{\frac{p}{1-p}} \, dv \right)^{1-p} .$$

Proof We have by Hölder inequality

$$a_h(r) = \int_{\partial\Omega_r} d\sigma = \int_{\partial\Omega_r} |\nabla h|^{\frac{p-1}{p}} \, |\nabla h|^{\frac{1-p}{p}} \, d\sigma$$
$$\leq \left(\int_{\partial\Omega_r} |\nabla h|^{p-1} \, d\sigma \right)^{\frac{1}{p}} \cdot \left(\int_{\partial\Omega_r} \frac{d\sigma}{|\nabla h|} \right)^{\frac{p-1}{p}} .$$

Raising this inequality to the power $\frac{p}{1-p}$ and using (5.1), we obtain

$$a_h(r)^{\frac{p}{1-p}} \geq \Phi_{h,p}(r)^{\frac{1}{1-p}} \, \left(\frac{dv}{dr} \right)^{-1} .$$

Integrating the last inequality gives

$$Q_p(h) \geq \left(\int_{r_0}^{r_1} a_h^{\frac{p}{1-p}}(r) \, \left(\frac{dv}{dr} \right) \, dr \right)^{1-p} = \left(\int_{v_0}^{v_1} \tilde{a}(v)^{\frac{p}{1-p}} \, dv \right)^{1-p} .$$

□

Definition A function $P : [0, V) \to \mathbb{R}$ is an *isoperimetric profile* for $\Omega \subset (M, g)$ $(V = \mathrm{Vol}(\Omega))$ if there exist constants $C, \eta > 0$ such that $\eta < V$ and

$$P(\mathrm{Vol}(D)) \leq C \ \mathrm{Area}(\partial D)$$

for all a compact regions $D \subset \Omega$ with $\mathrm{Vol}(D) \geq \eta$.

Remark The reason for the constant η in the above definition is that the isoperimetric ratio of small domains in a Riemannian manifold is comparable to the isoperimetric ratio of small domains in Euclidean space and thus of little interest.

Theorem 5.3. *Suppose that the domain $\Omega \subset (M, g)$ admits an isoperimetric profile $P : [0, V) \to \mathbb{R}$ such that*

$$\int_{\eta}^{V} \frac{dv}{[P(v)]^{p/(p-1)}} < \infty ,$$

($1 < p < \infty$). Then Ω is $p-$hyperbolic.

Proof Let $D \subset \Omega$ be a compact set with volume η. For any function $h \in \Lambda(D, \Omega)$ we define $a(v)$ as in Proposition 5.4. We clearly have $a(v) \geq \frac{1}{C} P(v)$. Thus

$$\int_{v_0}^{v_1} a(v)^{\frac{p}{1-p}} \, dv \leq I := C^{p/(p-1)} \int_{\eta}^{V} \frac{dv}{[P(v)]^{p/(p-1)}} < \infty .$$

From Proposition 5.4 and Theorem 5.1, we obtain

$$\mathrm{Cap_p}(D, \Omega) \geq I^{1-p} > 0 .$$

□

5.5. **Area and Volume growth.** Let us first treat the case of $1-$parabolicity.

Lemma 5.4. *Ω is 1-parabolic if and only if there exists an exhaustion*

$$G_1 \subset G_2 \subset \cdots \subset\subset \Omega$$

by domains such that $\liminf_{i\to\infty} \operatorname{Area}(\partial G_i) = 0.$

Proof By Proposition 5.2, we see that for a ball $B \subset \Omega$,

$$\operatorname{Cap}_1(D, \Omega) = \inf \operatorname{Area}(\partial G)\,,$$

where G runs through all domains such that $D \subset G \subset\subset \Omega$. □

Corollary 5.3. *A complete manifold M with finite volume is 1-parabolic .*

Proof Choose a base point $x_0 \in M$ and define $v(r) = \operatorname{Vol}(B(x_0, r))$ and $a(r) = \operatorname{Area}(\partial B(x_0, r))$. By equation (5.1) we have $\frac{dv}{dr} = a(r)$.
Suppose that M is $1-$hyperbolic. Then, by Lemma 5.4, we have

$$\liminf_{r\to\infty} \frac{dv}{dr} = \liminf_{r\to\infty} a(r) > 0\ ,$$

but this implies $\lim_{r\to\infty} v(r) = \infty$. □

We next see how the growth of area gives us an upper bound for the capacity: Choose a function $h \in \Lambda(D, \Omega)$ (where $D \subset \Omega$ is a compact subset) and note $a_h(r) = \operatorname{Area}(\partial\Omega_r)$.

Proposition 5.5.

$$Q_p(h) \leq (\|\nabla h\|_{L_\infty})^{p-1} \left(\int_{r_0}^{r_1} a_h(r)^{\frac{1}{(1-p)}}\, dr \right)^{1-p}\ .$$

Proof Suppose $|\nabla h(x)| \leq c$ for almost all x, then

$$\Phi_{h,p}(r) \leq c^{p-1} \int_{\partial\Omega_r} d\sigma = c^{p-1}\, a_h(r)\,,$$

and thus

$$Q_p(h) \leq c^{p-1} \left(\int_{r_0}^{r_1} a_h(r)^{\frac{1}{(1-p)}}\, dr \right)^{1-p}\ .$$

□

This proposition has the obvious

Corollary 5.4. *Let M be a complete manifold such that*

$$\int^{\infty} \frac{dr}{a(r)^{1/(p-1)}} = \infty$$

where $a(r) = \operatorname{Area}(\partial B(x_0, r))$, Then M is $p-$parabolic.

□

Remarks 1) It follows from Corollary 5.2 that for manifolds with warped cylindrical ends, this condition is not only necessary but also sufficient for M to be p-parabolic.

2) This corollary is false for non complete manifolds. For instance a half plane $M := \{|(x,y)|\, x > 0\} \subset \mathbb{R}^2$ (with its Euclidean metric) is $p-$hyperbolic for all p. Yet in this example,

$$\int^{\infty} \frac{dr}{a(r)} = \infty\,.$$

Corollary 5.5. *Let M be a complete manifold such that*

$$\int^{\infty} \frac{dr}{v(r)^{1/q}} = \infty\,,$$

where $v(r) = \mathrm{Vol}(B(x_0,r))$, then M is p-parabolic for all $p \geq q$.

Proof This is an immediate consequence of the previous corollary and the next lemma together with Equation (**??**). □

Lemma 5.5. *Let $f : [0,\infty) \to \mathbb{R}$ be a positive, strictly monotonic function such that $\lim_{t\to\infty} f(t) = +\infty$. Fix $p \geq q \geq 1$ and $\epsilon > 0$. Then*

$$\int_{\epsilon}^{\infty} (f(t))^{-1/q}\, dt \leq A \left(\int_{\epsilon}^{\infty} (f'(t))^{1/(1-p)}\, dt \right)^{\frac{p-1}{p}}$$

where

$$A = \begin{cases} \left(\frac{q}{p-q}\right)^{\frac{1}{p}} (\, f(\epsilon))^{\frac{1}{p}-\frac{1}{q}} & if \quad q > p \\ \log(\, f(\epsilon)) & if \quad q = p \end{cases}$$

Proof of the lemma Observe that $f'(t) > 0$ for all t. We have by Hölder inequality:

$$\begin{aligned} \int_{\epsilon}^{\infty} (f(t))^{-1/q}\, dt &= \int_{\epsilon}^{\infty} (f(t))^{-1/q} (f'(t))^{-1} \cdot f'(t) dt \\ &\leq \left(\int_{\epsilon}^{\infty} (f(t))^{-p/q} f'(t) dt \right)^{\frac{1}{p}} \left(\int_{\epsilon}^{\infty} (f'(t))^{-p/(p-1)} f'(t) dt \right)^{\frac{p-1}{p}}. \end{aligned}$$

But

$$\int_{\epsilon}^{\infty} (f(t))^{-p/q} f'(t) dt = A.$$

□

Example If M is a complete manifold such that

$$\mathrm{Vol}(B(x_0,r) \leq c\, r^q\,,$$

then M is $p-$parabolic for all $p \geq q$. In particular $\mathbb{R}^n$ is $p-$parabolic for $p \geq n$.

Similarly to Corollary 5.5, we have

Corollary 5.6. *Let M be a complete manifold such that*

$$\int^{\infty} \left(\frac{r}{Vol(B(x_0,r)} \right)^{1/(p-1)} dr = \infty\,,$$

then M is p-parabolic.

A proof can be found in §5.2 of [33].

6. Discretization

6.1. **Potential Theory on Graphs.** We start with a list of definitions from discrete potential theory. For more information, one may consult [5], [26] or [31].

A graph X is a set $V = V(X)$ together with a non reflexive, symmetric relation $\sim$. The elements of $V(X)$ are the *vertices* of the graph $(X, \sim)$. The vertice y is said to be a *neighbour* of x if $x \sim y$.

An unoriented edge is an unordered pair $\{x, y\}$ of neighbour vertices. An oriented edge is an ordered pair $\vec{e} = [x, y]$ of neighbour vertices. The edge $[y, x]$ is called the reversed edge of $\vec{e} = [x, y]$ and is denoted by $-\vec{e}$. We denote by $\alpha(\vec{e}) = x$ the origin of e and $\omega(\vec{e}) = y$ its end.

We may think of an unoriented edge as a segment between two neighbour vertices and of an oriented edge as an arrow.

We will note $S(X)$ the set of unoriented edges of X and $A(X)$ the set of oriented edges. There is an obvious (2 to 1) map $A(X) \to S(X)$. A (global) orientation of the graph is a section $S(X) \to A(X)$ of this map.

A *path* in the graph X is a finite sequence $x_1, x_2, \dots x_n$ of vertices such that $x_i \sim x_{i+1}$. The cardinal n of this sequence is called the length of the path.
The graph X is *connected* if there is a path connecting any pair of vertices.

If $x, y \in V(X)$, we denote by $\rho(x, y)$ the length of the shortest path joining x to y (and $\rho(x, y) = 0$ if $x = y$). Observe that $\rho(x, y) = 1$ iff $x \sim y$. If X is a connected graph, $(V(X), \rho)$ is a metric space.

The *degree* $\deg(x)$ of a vertice is the cardinal of the set of its neighbours.
We say that the graph X has *bounded geometry* if it is connected, $V(X)$ is finite or countable and $\deg(x) \leq N$ for all $x \in V(X)$ where $N = N(X) < \infty$.

A 0-cochain (or a function) on the graph X is simply a function $u : V(X) \to \mathbb{R}$, and 1-cochain (or a 1-form) is a function $\eta : A(x) \to \mathbb{R}$ such that $\eta(-\vec{e}) = -\eta(\vec{e})$. We denote by $\Omega^0(X)$ and $\Omega^1(X)$ the space of 0-cochains and 1-cochains on X.

The *differential* of a 0-cochain $u \in \Omega^0(X)$ is the 1-cochain du defined by

$$du([x, y]) = u(y) - u(x)\,.$$

The codifferential (or divergence) of a 1-cochain η is the 0-cochain $\delta\eta$ defined by

$$\delta\eta(x) = \sum_{y \sim x} \eta([y, x])\,.$$

Observe that if $\eta \in \Omega^1(X)$ has finite support, then

$$\sum_{x \in V(X)} \delta\eta(x) = 0\,.$$

If η and ξ are 1-cochains, their (pointwise) scalar product is the 0-cochain

$$\langle \xi, \eta \rangle(x) = \frac{1}{2} \sum_{y \sim x} \xi([y, x])\eta([y, x])\,.$$

If η or ξ has finite support, then the global (L^2) scalar product is

$$(\xi\,|\eta)_{L^2} := \sum_{x\in V(x)} \langle \xi, \eta\rangle(x) = \frac{1}{2}\sum_{\vec{e}\in A(X)} \xi(\vec{e})\,\eta(\vec{e})\,.$$

Remark If $\xi \in \Omega^1(X)$ and $e \in S(X)$, then $\xi(e)$ is only defined up to multiplication by ± 1. However, if $\xi, \eta \in \Omega^1(X)$ and $e \in S(X)$, then the product $\xi(e)\,\eta(e)$ is well defined, and we may write

$$(\xi\,|\eta)_{L^2} := \sum_{e\in S(X)} \xi(e)\,\eta(e)\,.$$

We may turn $\Omega^1(X)$ into a $\Omega^0(X)$-module where the action is defined by

$$(u\cdot\eta)([x,y]) = \tfrac{u(x)+u(y)}{2}\,\eta([x,y])\,.$$

The operator δ satisfies the Leibniz rule:

$$\delta(u\cdot\eta)(x) = u(x)\delta\eta(x) + \langle du, \eta\rangle(x)\,.$$

A consequence is that if u or η has finite support, a summation by parts shows that

$$\sum_{x\in V(x)} u(x)\delta\eta(x) = -\sum_{x\in V(x)} \langle du, \eta\rangle(x) = -\sum_{\vec{e}\in S(X)} du(e)\eta(e)\,,$$

hence the operators d and $-\delta$ are formal adjoint.

The (pointwise) norm of a 1-cochain $\eta \in \Omega^1(X)$ is the 0-cochain $\|\eta\|$ defined by

$$\|\eta\|\,(x) = \sqrt{\langle \eta, \eta\rangle}\,(x) = \left(\sum_{y\sim x} \eta([y,x])^2\right)^{1/2}.$$

Let X be a graph and $p \in [1,\infty)$ a real number. The p-energy (or $p-$Dirichlet integral) of a 0-cochain $u \in \Omega^0(X)$ is the sum

$$D_p(u) = \sum_{x\in V(x)} \|du(x)\|^p = \sum_{x\in V(x)} \left(\sum_{y\sim x} (u(y)-u(x))^2\right)^{p/2}.$$

The Euler-Lagrange operator associated to the $p-$energy is the *p-Laplacian* $\Delta_p : \Omega^0(X) \to \Omega^0(X)$ defined by

$$\Delta_p(u) = -\delta(\|du\|^{p-2}du)\,.$$

6.2. **Parabolicity of Graphs.** Let X be connected graph and $A \subset X$ a finite set.

Definition For $p \geq 1$, the $p-$capacity of A is defined by:

$$\mathrm{Cap_p}(A, X) := \inf\{D_p(u) : u \in \Omega^0(X),\ u \geq 1 \text{ on } A, \text{ and } u \text{ has finite support }\}.$$

Remark By truncation, one may restrict oneself to functions $u \in \Omega^0(X)$ such that $0 \leq u \leq 1$.

The proof of the following lemma is not difficult (it can be found in [32, th. 3.1]).

Lemma 6.1. *Let A and A' be two finite non empty subsets of X, then* $\mathrm{Cap_p}(A, X) = 0$ *if and only if* $\mathrm{Cap_p}(A', X) = 0$.

□

Definition A graph X is *$p-$parabolic* if $\mathrm{Cap_p}(A,X)=0$ for all finite subsets $A \subset X$. Otherwise, X is *$p-$hyperbolic*.

Proposition 6.1. *Let X be a graph with bounded geometry. If X is $p-$parabolic, then X is also $q-$parabolic for all $q \geq p$.*

Proof Let $A \subset V(x)$ be any finite set. For all ϵ, we can find a function $u \in \Omega^0(X)$ with finite support, such that $u \geq 1$ on A and $D_p(u) \leq \epsilon$. We may furthermore assume that $0 \leq u(x) \leq 1$ for all $x \in V(X)$.

Let N be such that $\deg(x) \leq N$ for all x, and observe that

$$\frac{\|du(x)\|}{\sqrt{N}} = \frac{1}{\sqrt{N}} \left(\sum_{y \sim x} (u(y)-u(x))^2 \right)^{1/2} \leq 1$$

thus, if $q \geq p$, then $\left(\frac{\|du(x)\|}{\sqrt{N}} \right)^q \leq \left(\frac{\|du(x)\|}{\sqrt{N}} \right)^p$. This implies

$$D_q(u) \leq N^{(q-p)/2} D_p(u)\,,$$

and the result follows. □

Remark Yamasaki has proved the above proposition without the assumption that X has finite geometry (see [32, th. 5.1]).

6.3. **Manifold with bounded geometry.** A manifold M has *bounded geometry* if every ball of radius ρ (= some fixed positive number) is geometrically close to a standard ball in Euclidean space. The exact definition may depend on the actual needs of the theory.

In the present context, it is sufficient to agree on the following

Definition A Riemannian manifold M has *bounded geometry* if it has a positive injectivity radius and its Ricci curvature is bounded below.

For instance compact manifold have bounded geometry, and any Riemannian covering space of a manifold with bounded geometry also has bounded geometry. Observe that a manifold with bounded geometry is necessarily complete.

Let us also recall that two metric spaces X and Y are (roughly) *quasi-isometric* to each other if there exist a map $f : X \to Y$ and constants $\lambda \geq 1$, $c \geq 0$ and $\epsilon \geq 0$ such that

i) $\frac{1}{\lambda} d(x_1,x_2) - c \leq d(f(x_1), f(x_2)) \leq \lambda d(x_1,x_2) + c$ for all $x_1, x_2 \in X$;
ii) every ball of radius ϵ in Y contains a point of $f(X)$.

Definition A *discretization* of Riemannian manifold M is a graph X which is quasi-isometric to M.

By the work of Kanai and Holopainen, we know that $p-$parabolicity is stable under quasi-isometry:

Theorem 6.2. *(A) Let X and Y be two quasi-isometric graphs. Assume that X and Y have bounded geometry. Then X is $p-$parabolic if and only if Y is $p-$parabolic.*

(B) Let the graph X be a discretization of the manifold M. Assume that M and X have bounded geometry. Then M is $p-$parabolic if and only if X is $p-$parabolic.

(C) Every Riemannian manifold M with bounded geometry admits a discretization X, which is a graph with bounded geometry.

(D) Let M and N be two quasi-isometric Riemannian manifolds. Assume that M and N have bounded geometry. Then M is $p-$parabolic if and only if N is $p-$parabolic.

Proof The proof of (A) and (B) can be found in [15] for $p = 2$ and [14, section 5] for all p. The proof of (C) can be found in [15, lemma 2.4 and 2.5]. The last statement is a consequence of the first three. □

We are now able to extend Proposition 6.1 to Riemannian manifolds with bounded geometry.

Theorem 6.3. *Let M be $p-$parabolic Riemannian manifolds with bounded geometry. Then M is also $q-$parabolic for all $q \geq p$.*

Proof Let X be a discretization (with bounded geometry) of M. By part (A) of the previous theorem, we know that X is $p-$parabolic, and by Proposition 6.1, X is $q-$parabolic for all $q \geq p$. Using again statement (A) above, we conclude that M is $q-$parabolic. □

Remark This theorem is false without the condition that M has bounded geometry. For instance, in the example given by Corollary 4.1, the behaviour is opposite to the behaviour of manifolds with bounded geometry described in Theorem 6.3.

7. Dimension at infinity of a manifold

In this section, (M, g) is a complete manifold.

Definition The manifold M is said to satisfy an *isoperimetric inequality* of order k if there exist constants $C, \delta > 0$ ($\delta < \mathrm{Vol}(M)$) such that for all bounded regular domain $D \subset\subset M$ with $\mathrm{Vol}(D) > \delta$, we have

$$\mathrm{Vol}(D)^{(k-1)} \leq C\,\mathrm{Area}(\partial D)^k.$$

And M is said to be *open at infinity* if

$$\mathrm{Vol}(D) \leq C\,\mathrm{Area}(\partial D).$$

The *isoperimetric dimension* of (M, g) is then the number

$$\mathrm{d}_{isop}(M,g) = \sup\{\, k > 0 |\ M \text{ satisfies an isoperimetric inequality of order } k\}\,.$$

Definition The *growth degree* of a complete manifold (M, g) is the number

$$\mathrm{d}_{gr}(M,g) = \inf\{m > 1 \,\big|\, \liminf_{r\to\infty} \frac{v(r)}{r^m} < \infty\}\,,$$

where $v(r) = \mathrm{Vol}(B_{x_0,r})$ is the volume of the sphere centered at some base point $x_0 \in M$.

The invariants $d_{gr}(M,g)$ and $d_{isop}(M,g)$ are known to be stable under quasi-isometries (see [15]).

The results of section 4 imply the next proposition:

Proposition 7.1. *Let (M,g) be a complete manifold. If $p \leq d_{isop}(M)$, then M is $p-$hyperbolic, and if $p \geq d_{gr}(M)$, then M is $p-$parabolic. (If M is open at infinity, then M is $p-$hyperbolic for all p.)*

□

Definition The *parabolic dimension* of M is the number

$$d_{par}(M) = \inf\{p \geq 1 : \ M \text{ is } p-\text{parabolic}\}\,.$$

If M is $p-$hyperbolic for all p, then we set $d_{par}(M,g) = \infty$.

Proposition 7.2. *If M has bounded geometry, then the parabolic dimension can also be defined as*

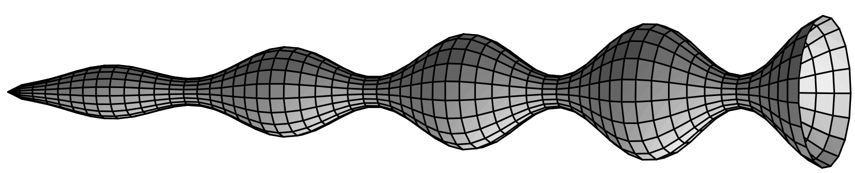
$$d_{par}(M) = \sup\{p \geq 1 : \ M \text{ is } p-\text{hyperbolic}\}\,.$$

Proof This follows from Theorem 6.3. □

If (M,g) be a connected complete Riemannian manifold. Then Proposition 7.1 says that

$$d_{isop}(M,g) \leq d_{par}(M,g) \leq d_{gr}(M,g)\,.$$

It is not difficult to construct a manifold for which these inequalities are strict (for instance a manifold with successive large and small parts as in the figure).



However, this will not be the case if the manifold has a large group of isometries.

Theorem 7.1 (Coulhon, Saloff-Coste)**.** *Let (M,g) be the universal cover of a connected compact manifold N. Then*

$$d_{isop}(M,g) = d_{par}(M,g) = d_{gr}(M,g)\,.$$

The proof can be found in [4]. □

In fact, these numbers are equal to the growth degree of $\pi_1(N)$ (cf. [23]).

The previous theorem also holds for manifolds which satisfy a Poincaré inequality and a volume doubling condition (th. 3 of [4]).

Let us conclude this section by mentioning that there are several other notions of dimension at infinity for manifolds (see [2]).

8. Conformal Parabolicity

Let g and g_0 be two conformally equivalent metrics on a $n-$dimensional manifold M, i.e. $g = \lambda\, g_0$ for some smooth function $\lambda : M \to \mathbb{R}_+$.

The volume forms $d\,vol_g$ and $d\,vol_{g_0}$ are related by

$$d\,vol_g = \lambda^n\, d\,vol_{g_0} \,.$$

If α is a 1-form on M, then its norm for both metrics are related by $|\alpha|_g = \lambda^{-1}|\alpha|_{g_0}$, we thus see that the expression

$$|du|_g^n\, d\,vol_g = |du|_{g_0}^n\, d\,vol_{g_0} \,.$$

is a conformal invariant for all function u on M. This is of course also the case for $n-$capacities and we conclude that

Proposition 8.1. *$n-$parabolicity is a conformally invariant property of manifolds.*

□

A $n-$parabolic (hyperbolic) manifold is also called *conformally parabolic (hyperbolic).*

We may now formulate a version of the so called Ahlfors-Gromov Lemma (see [10, p. 198], [11, p. 85] and [33] for other versions).

Theorem 8.1 (Ahlfors-Gromov Lemma)**.** *Let (M, g_0) be a complete n dimensional Riemannian manifold such that $d_{isop}(M, g_0) > n$. Then $d_{gr}(M, g) \geq n$ for all metric g conformally equivalent to g_0.*

Proof (M, g_0) is $n-$hyperbolic, hence (M, g) also (by conformal invariance of $n-$parabolicity), whence $\mathrm{d}_{cr}(M, g) \geq n$. □

This result is the basis for a description of the possible volume growth of the various metrics within a conformal class, see [8] and [33].

References

[1] A. Ancona *Théorie du Potentiel sur les graphes et les variétés* Ecole d'été de Probabilités de Saint-Flour XVIII - 1988 Springer Lect. Notes in Math. **1427**

[2] T. Coulhon *Dimension at Infinity for Riemannian Manifolds* Potential Analysis **4** (1995) 335–344

[3] T. Coulhon *Heat Kernels on non-compact Riemannian Manifolds : a Partial Survey* preprint Cergy-Pontoise (1997)

[4] T. Coulhon et L. Saloff-Coste *Isopérimétrie pour les groupes et les variétés* Revista Mathemática Iberoamericana **9** (1993) 293–314

[5] J. Dodziuk *Difference Equations, Isoperimetric Inequality and Transience of certain Random Walks* Trans. Amer. Math. Soc **284** (1984), 787–794

[6] L C. Evans and R.F. Gariepy *Measure Theory and Fine properties of Functions* Studies in Advances Mathematics (1992).

[7] V. Gol'dshtein and M. Troyanov *The Kelvin-Nevanlinna-Royden criterion for $p-$hyperbolicity* Math. Zeitschrift (to appear).

[8] R. Grimaldi et P. Pansu *Sur la croissance du volume dans une classe conforme* J. Math. pures et appl. **71** (1992), 1–19.

[9] A. Grigor'ian *Analytic and Geometric Background of Recurence and Non-Explosion of the Brownian Motion on Riemannian Manifolds* preprint, Imperial College 1997

[10] M. Gromov *Hyperbolic Manifolds, Groups and Actions* in *Riemann Surfaces and Related Topics: Proceedings of the 1978 Stony Brooks Conference* Annals of Math. Studies, Princeton University Press **97** (1981), 183–213

[11] M. Gromov, J. Lafontaine et P. Pansu *Structures métriques pour les variétés riemanniennes* Cedic/Fernand Nathan (1981)
[12] J.Heinonen, T.Kilpeläinen, and O.Martio *Non Linear Potential Theory of Degenerate Elliptic Equations* Oxford Math. Monographs (1993)
[13] I.Holopainen *Non linear Potential Theory and Quasiregular Mappings on Riemannian Manifolds* Annales Academiae Scientiarum Fennicae Series A **74** (1990)
[14] I.Holopainen *Rough isometries and $p-$harmonic functions with finite Dirichlet integral* Revista Mathemática Iberoamericana **10** (1994) 143–175
[15] M. Kanai *Rough isometries and combinatorial approximations of geometries of non compact Riemannian manifolds* J. Math.Soc.Japan **37** (1985), 391–413
[16] M. Kanai *Rough isometries and the parabolicity of Riemannian manifolds* J. Math.Soc.Japan **38** (1986), 227–238
[17] M. Kanai *Analytic Inequalities and Rough isometries between non-compact Riemannian Manifolds* J. Math.Soc.Japan **38** (1986), 227–238
[18] V. M. Kesel'man *Riemannian Manifolds of α-parabolic type* Izv. Vyssh. Uchebn. Zaved., Ser. Mat. **4** (1985), 81–83
[19] P. Li and L.F. Tam *Symmetric Green's Functions on Complete Manifolds* Amer. J. Math. **109** (1987), 1129–1154
[20] Lehto-Virtanen *"Quasikonforme Abbildungen"* Springer Grundlehren der Math. **126**
[21] V.G. Maz'ya *Sobolev Spaces* Springer Verlag (1985)
[22] V.G. Maz'ya *Classes of Domain, Measures and Capacities in the Theory of Differentiable Functions* in *Encyclopaedia of Mathematical Sciences, Analysis III* Springer Verlag **26** 141–219 (1991)
[23] J. Milnor *A note on curvature and the fundamental group* J. Diff. Geom **2** (1968), 1–7
[24] S. Rickman *Quasi-regular Mappings* Springer Ergebnisse der Mathematik **26** (1993)
[25] L. Sario and M. Nakai *Classification Theory of Riemann Surfaces* Grundlehren der mathematischen Wissenschaften **164** (1970)
[26] Paolo M. Soardi *Potential Theory on Infinite Networks* Lecture Notes in Mathematics **1590** (1994)
[27] J. Väisälä *Lectures on n-dimensional Quasiconformal Mappings* **229** (1971)
[28] N. Th. Varopoulos *Brownian motion and random walks on manifolds* Ann. Inst. Fourier **34** (1984), 243–269
[29] M. Vuorinen *Conformal Geometry and Quasi-regular Mappings* Springer Lect. Notes in Math. **1319** (1988)
[30] M. Vuorinen *Quasiconformal Space Mappings - a collection of surveys 1960-1990* Springer Lect. Notes in Math. **1508** (1992)
[31] W. Woess *Random Walks on Infinite Graphs and Groups - A Survey of Selected Topics* Bull. London. Math.Soc **26** (1994), 1–60
[32] M. Yamasaki *Parabolic and Hyperbolic Infinite Network* Hiroshima Math. J. **7** (1977), 135–146
[33] V.A. Zorich and V.M. Kesel'man *On the Conformal Type of a Riemannian Manifold* Func. Anal. and Appl. **30** (1996), 106–117

Département de Mathématiques, Ecole Polytechnique Fédérale de Lausanne, 1015 Lausanne Switzerland
E-mail address: `troyanov@math.epfl.ch`